Automorphism towers and definability in generalized Baire spaces

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Introduction

The present thesis is a collection of the author's work on *automorphism towers* and *definability in generalized Baire spaces* during the last three years. We give a brief introduction to the contents of this work. Detailed introductions to the individual topics can be found at the beginning of the corresponding chapters.

Automorphism towers. It is a common practice in modern mathematics to study mathematical structures via their groups of automorphisms. The idea behind this approach is summarized by the following motto: *structure is whatever is preserved by automorphisms.*[1] As a consequence of this approach, automorphism groups are interesting objects on their own, as are their groups of automorphisms.

If we take an mathematical structure and form its automorphism group, then this group can have a higher cardinality than the original object and its algebraic structure can be complicated. Moreover, if the original structure is infinite, then its automorphism group can also be complicated in a set-theoretic sense. For example, there are infinite objects with the property that basic algebraic properties of their automorphism groups are independent from the standard axioms of set theory (see, for example, [**SS88**], [**Far11**] and [**LT11**]). In particular, it is necessary to use both algebraic and set-theoretic methods to understand the algebraic structure of these groups.

The construction of *automorphism towers of centreless groups* illustrates this phenomenon by iterating the process of forming automorphism groups transfinitely often. Given a centreless group G, the automorphism group $\mathrm{Aut}(G)$ of G is again centreless and we may view G as a normal subgroup of $\mathrm{Aut}(G)$ by identifying elements of G with the corresponding inner automorphisms. We construct the automorphism tower $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ of G by setting $G_0 = G$, $G_{\alpha+1} = \mathrm{Aut}(G_\alpha)$ and $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$ (at successor levels, we identify G_α with the group of inner automorphisms of G_α to obtain an ascending sequence). As suggested by the above remarks, the resulting sequence of groups $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ can be a very complex object and there are many open questions about them. The first part of this thesis focuses on the analysis of these towers.

[1] This motto was formulated by Wilfried Hodges in [**Hod93**, p. 131]

In [**Tho85**], Simon Thomas proved his celebrated *automorphism tower theorem* stating that the automorphism tower of every centreless group terminates in the sense that there is an ordinal α with $G_\alpha = G_\beta$ for all $\beta \geq \alpha$. Therefore it makes sense to talk about the height of automorphism towers and we can state the *automorphism tower problem*: *construct a model of set theory such that for some infinite cardinal κ in this model, it is possible to compute the least upper bound of the heights of the automorphism towers of all centreless groups of cardinality κ.* This problem is still open.

The first two chapters of this thesis deal with the related question of finding upper bounds for the heights of automorphism towers of infinite centreless groups of cardinality κ that are uniformly definable from the parameter κ. After giving an overview on known upper bounds, we will present a new bound that relies on *admissible set theory* (i.e. *abstract recursion theory*) and improves the existing estimates. The second chapter is devoted to the proof of this result. This proof is based on the combination of admissible set theory with techniques developed by Itay Kaplan and Saharon Shelah in [**KS09**] that allow the representation of automorphism towers as *inductive definitions*.

The third chapter focuses on the non-absoluteness of the computation of the height of automorphism towers. A result of Joel David Hamkins and Simon Thomas in [**HT00**] suggests that there is no nontrivial correlation between the heights of the automorphism towers of a group computed in different models of set theory that is provable for all centreless groups. This result shows that it is consistent that for every cardinal κ and ordinal $\alpha < \kappa$ there is a group whose automorphism tower has height α and this height can be changed to every non-zero ordinal below κ by passing to a forcing extension of the ground model.

In joint work with Gunter Fuchs, this result was extended in two ways. First, it is shown that it is consistent to have a centreless group whose automorphism tower height can be changed again and again by passing to larger and larger forcing extensions of the ground model. Second, it is also possible to drastically change the height of automorphism towers by passing to smaller and smaller inner models. These results are published in [**FLb**] and presented in the first five sections of Chapter 3.

The last section of Chapter 3 presents an example of a group whose automorphism tower is highly non-absolute in another sense. We show that it is consistent to have a centreless group with the property that for every ordinal α there is a cofinality preserving partial order that forces the automorphism tower of this group to be higher than α. The presented proof is a simplification of the original proof published in [**Lüca**].

In Chapter 4, it is shown that, in contrast to the results mentioned above, it is possible to prove nontrivial absoluteness results for the second stages of automorphism towers of countable groups. In particular, if G is a countable centreless group and $G_1 \neq G_2$ holds in a transitive model of set theory that contains G and a bijection witnessing the countability of G, then this inequality will hold in any bigger transitive model of set theory. The proof of this result uses the theory of *Polish groups* and the existence of *unique Polish group topologies* on the automorphism groups of countable centreless groups. This result was published in [**Lücc**].

Chapter 5 is devoted to the notion of *special pair* introduced by Itay Kaplan and Saharon Shelah in [**KS09**]. This notion allows us to view the groups appearing in the automorphism tower of an infinite centreless group of cardinality κ as subsets of the power set of κ and talk about their complexity in this way. It plays a central role in the proofs of Chapter 2.

We will strengthen the notion of special pairs and show that the results of [**KS09**], which establish a connection between automorphism towers and special pairs, also hold for *strongly special pairs*. Then we will use certain actions of groups on Hausdorff spaces to produce various examples of strongly special pairs not induced by automorphism towers. Finally, we will show that the notions of special pairs and strongly special pairs do not coincide. The results of this chapter are contained in [**Lücc**].

Definability in generalized Baire spaces. Let κ be an infinite cardinal. The *generalized Baire space for* κ consists of the set ${}^\kappa\kappa$ of all functions $f : \kappa \longrightarrow \kappa$ equipped with the topology whose basic open sets are of the form $U_s = \{f \in {}^\kappa\kappa \mid s \subseteq f\}$ for some partial function $s : \kappa \xrightarrow{part} \kappa$ of cardinality less than κ. Objects in various areas of mathematics can be represented as subsets of these spaces. We want to study the definable subsets of such spaces and their structural properties in the case where κ is an uncountable regular cardinal with the property that there are only κ-many bounded subsets of κ.

Motivated by the work of the *Helsinki school* on infinitary model theory and logic, a systematic study of these spaces was initiated by Alan Mekler and Jouko Väänänen (see [**Vää91**] and [**MV93**]) and was extended by many others. In addition, a number of publications revealed deep connections to infinite combinatorics, infinitary logic and model theory (see, for example, [**NS78**], [**HV90**], [**TV99**], [**SV02**] and [**FHK**]).

In the second part of the thesis, we will focus on the question whether it is possible to produce *simple* definitions of arbitrary subsets of ${}^\kappa\kappa$ by forcing with set-sized partial orders that preserve cardinalities, cofinalities and the value of 2^κ. In this analysis, we are particularly interested in producing long well-orderings of subsets of ${}^\kappa\kappa$ with simple definitions.

In Chapter 6, we make precise the meaning of "*simple definition*" by generalizing the notion of *projective subset* to the uncountable context. Then we prove that the ground model V might not contain certain *long* well-orderings of subsets of ${}^\kappa\kappa$ with simple definitions; so in such cases it is indeed necessary to pass to a forcing extension in order to construct such well-orderings. We generalize basic structural properties of subsets of the classical Baire space ${}^\omega\omega$ to our uncountable context and show that long well-orderings cannot satisfy all of these properties. Finally, we prove that it is consistent with the axioms of set theory plus large cardinal axioms that all simply definable subsets possess these properties.

In Chapter 7, we will present the main result of [**Lücb**] that provides a positive answer to the above question. This coding result states that every subset of ${}^{\kappa}\kappa$ has a simple definition in a forcing extension of the ground model by a $<\kappa$-closed partial order that satisfies the κ^{+}-chain condition and has cardinality 2^{κ}. In particular, the existence of a simply definable well-ordering of a subset of ${}^{\kappa}\kappa$ of order type greater than 2^{κ} is consistent with the axioms of ZFC plus large cardinal axioms. We will also use the coding to force the existence of well-orderings of ${}^{\kappa}\kappa$ with simple definitions. The work presented in these two chapters is published in [**Lücb**].

The last chapter considers variations of the above question that ask for coding forcings which preserve large cardinal properties of κ. In joint work, Sy-David Friedman and the author showed that this question has a positive answer in the case of supercompact cardinals if we allow class forcing.

In [**FLa**], it is shown that there is a class-sized forcing iteration $\mathbb{P}$ with the property that forcing with $\mathbb{P}$ preserves ZFC, the inaccessibility of inaccessible cardinals, the supercompactness of supercompact cardinals and the value of 2^{α} for every inaccessible cardinal α and, if α is inaccessible and A is an arbitrary subset of ${}^{\alpha}\alpha$, then there is a $\mathbb{P}$-generic extension of the ground model in which A is simply definable. This class forcing can also be used to force the existence of a well-ordering of $\mathrm{H}_{\alpha^{+}}$ that is definable in the structure $\langle \mathrm{H}_{\alpha^{+}}, \in \rangle$ for every inaccessible cardinal α. Chapter 8 consists of a detailed presentation of the results of [**FLa**].

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Contents

Notations and Conventions

We fix some notations and conventions that will be used throughout this thesis. All other notation will be standard, as for example, in [**Kun80**], [**Jec03**] and [**Lan02**].

Metatheory. Unless noted otherwise, the results of this thesis are derived from the axioms of *Zermelo-Fraenkel set theory with the Axiom of Choice* ZFC. If a result does not depend on the *Axiom of Choice* and can be derived from the axioms of *Zermelo-Fraenkel set theory* ZF, then we add a "(ZF)" to its name.

Logic. If $\mathcal{L}$ is a first order language and φ is an $\mathcal{L}$-formula, then we write $\varphi \equiv \varphi(v_0, \ldots, v_{n-1})$ to denote that every free variable of φ is contained in the set $\{v_0, \ldots, v_{n-1}\}$. We use the same notation for terms. Given $\mathcal{L}$-terms $t, t_0, \ldots, t_{n-1}$ and variables $v_0, \ldots, v_{n-1}$, we let $t^{v_0,\ldots,v_{n-1}}_{t_0,\ldots,t_{n-1}}$ denote the term constructed from t by substituting every occurrence of v_i in t by t_i. If $t \equiv t(v_0, \ldots, v_{n-1})$ is an $\mathcal{L}$-term, $\mathcal{M}$ is an $\mathcal{L}$-model with domain M and $x_0, \ldots, x_{n-1} \in M$, then $t^{\mathcal{M}}(x_0, \ldots, x_{n-1})$ denotes the evaluation of t in $\mathcal{M}$ with respect to $x_0, \ldots, x_{n-1}$.

We let $\mathcal{L}_\in$ denote the first-order language of set theory. An $\mathcal{L}_\in$-formula is a Δ_0-*formula* if it is contained in the smallest class of $\mathcal{L}_\in$-formulae that contains every atomic formula and is closed under negation, conjunction and bounded existential quantification. We call an $\mathcal{L}_\in$-formula φ a Σ_1-*formula* if there is a Δ_0-formula $\varphi_0 \equiv \varphi_0(u_0, \ldots, u_{n+m-1})$ such that

$$\varphi \equiv (\exists x_0, \ldots, x_{m-1})\ \varphi_0(v_0, \ldots, v_{n-1}, x_0, \ldots, x_{m-1}).$$

We let $\mathcal{L}_{\mathrm{GT}} = \langle *, {}^{-1}, \mathbb{1} \rangle$ denote the first-order language of group theory and GT denote the axioms of group theory. Finally, let $\mathcal{L}_{\mathrm{NT}^2}$ denote the language of second order number theory.

Given a first order language $\mathcal{L}$, a class $\mathcal{C}$ of $\mathcal{L}$-structures and a subset X of all domains of structures in $\mathcal{C}$, we say that X is *uniformly definable in* $\mathcal{C}$ if there is an $\mathcal{L}$-formula $\varphi(v_0, \ldots, v_n)$ and parameters $x_0, \ldots, x_{n-1}$ contained in all domains of structures in $\mathcal{C}$ that define X in every structure in $\mathcal{C}$.

Set theory. Fix a set x, a class C, a cardinal κ and an ordinal λ. We let $\mathrm{tc}(x)$ denote the *transitive closure* of x, $\mathcal{P}(x)$ denote the *power set* of x and $\mathcal{P}_\kappa(x)$ denote the set of all subsets of κ of cardinality less than κ. If C is a class, then we also use $[C]^{<\kappa}$ to denote the class of all sets x of cardinality less than κ with $x \subseteq C$. The class of all functions f with

$\mathrm{dom}(f) = \lambda$ and $\mathrm{ran}(f) \subseteq C$ is ${}^{\lambda}C$. We also define ${}^{<\lambda}C = \bigcup_{\alpha<\lambda} {}^{\alpha}C$ and let $\kappa^{<\lambda}$ denote the cardinality of ${}^{<\lambda}\kappa$. Given sets a sequence $\langle x_i \mid i \in I\rangle$, we let $\bigsqcup_{i\in I} x_i$ denote the corresponding *disjoint union* $\{\langle i, y\rangle \mid i \in I,\ y \in x_i\}$. We will write $x_0 \sqcup x_1$ instead of $\bigsqcup_{i<2} x_i$.

If f is a function, A is a subset of the domain of f and B is a subset of the range of f, then $f``A$ is the pointwise image of $A$ under $f$ and $f^{-1}``B$ is the preimage of B under f. We denote *composition* of functions by $f \circ g$, i.e. $(f \circ g)(x) = f(g(x))$ for all $x \in \mathrm{dom}(g)$ with $g(x) \in \mathrm{dom}(f)$. A *partial function* $f : A \xrightarrow{part} B$ is a function with $\mathrm{dom}(f) \subseteq A$ and $\mathrm{ran}(f) \subseteq B$. A *partial surjection* is a partial function $f : A \xrightarrow{part} B$ with $\mathrm{ran}(f) = B$.

We let $\prec\cdot,\cdot\succ : \mathrm{On} \times \mathrm{On} \longrightarrow \mathrm{On}$ denote the Gödel-Pairing function.

Given a nonempty set X and $A \subseteq X^{n+1}$, we define

$$\exists^x A = \{\langle x_0, \ldots, x_{n-1}\rangle \in X^n \mid (\exists x_n)\ \langle x_0, \ldots, x_n\rangle \in A\}.$$

If X is a nonempty set, then we call a set T a *tree on* X^n if there is a $\gamma \in \mathrm{On}$ such that $T \subseteq ({}^{<\gamma}X)^n$ and the following statements hold.

(1) If $\langle s_0, \ldots, s_{n-1}\rangle \in T$, then $\mathrm{lh}(s_0) = \cdots = \mathrm{lh}(s_{n-1})$.
(2) If $\langle s_0, \ldots, s_{n-1}\rangle \in T$ and $\alpha < \mathrm{lh}(s_0)$, then

$$\langle s_0 \restriction \alpha, \ldots, s_{n-1} \restriction \alpha\rangle \in T.$$

In the above situation, we call T a *subtree of* ${}^{<\gamma}X$. Given a tuple $t = \langle t_0, \ldots, t_{n-1}\rangle \in T$, we define $\mathrm{lh}(t) = \mathrm{lh}(t_0)$ and call the ordinal $\mathrm{ht}(T) = \mathrm{lub}\{\mathrm{lh}(t) \mid t \in T\}$ the *height of* T. We say that a tree T_0 on X is an *end-extension* of a tree T_1 on X if $T_1 = T_0 \cap {}^{<\mathrm{ht}(T_1)}X$ holds.

Given a tree T on X, a tuple of functions $\langle x_0, \ldots, x_{n-1}\rangle \in \left({}^{\mathrm{ht}(T)}X\right)^n$ is called a *cofinal branch through* T if the tuple $\langle x_0 \restriction \alpha, \ldots, x_{n-1} \restriction \alpha\rangle$ is an element of T for every $\alpha < \mathrm{ht}(T)$. We let $[T]$ denote the set of all cofinal branches through T. If T is a tree on X^{n+1} of height λ, then we define $p[T] = \exists^x[T] \subseteq ({}^{\lambda}X)^n$.

Given a partial order $\mathbb{P}$, we also use the letter $\mathbb{P}$ to denote the domain of $\mathbb{P}$, $\leq_{\mathbb{P}}$ to denote the ordering of $\mathbb{P}$ and $\mathbb{1}_{\mathbb{P}}$ to denote the maximal element of $\mathbb{P}$.

Fix cardinals κ and ν. We let $\mathrm{Add}(\kappa, \nu)$ denote the partial order that adds ν-many Cohen-subsets of κ by forcing with partial function of cardinality less than κ and $\mathrm{Col}(\kappa, {<}\nu)$ denote the corresponding *Levy Collapse*.

Group theory. Given a group G, we will also use the letter G to denote the domain of G and use 1_G to denote the identity element of G. We denote applications of the group operation by $g \cdot h$ if it is clear which group is meant. Otherwise, we write $g \cdot_G h$. We will abbreviate the term $g \cdot h \cdot g^{-1}$ by h^g.

If A is a subset of the domain of G, then we let $\langle A\rangle_G$ denote the subgroup of G generated by A and $\mathrm{C}_G(A)$ to denote the *centralizer* of A in G, i.e. the set $\{g \in G \mid (\forall h \in A)\ h^g = h\}$. The set $\mathrm{C}_G(G)$ is called the centre of G and is also denoted by $\mathrm{Z}(G)$. The *normal closure* of a subset A of G is the intersection of all normal subgroups of G that contain A as a subset.

Given a homomorphism $\varphi : G \longrightarrow H$ of groups, we use $\ker(\varphi)$ to denote the *kernel* of φ.

We let $\mathrm{Sym}(X)$ denote the *symmetric group* of a set X and $\mathrm{Alt}(X)$ denote the corresponding *alternating group* consisting of all finite even permutations of X. If $a, b \in X$, then $(a\ b)$ denotes the *transposition* of the elements a and b.

Graph theory. In the following, *graph* will always mean *undirected graph*, i.e. a pair $\Gamma = \langle V, E \rangle$ consisting of a nonempty set V (the *vertices* of Γ) and an irreflexive, symmetric binary relation E on V (the *edges* of Γ).

Part 1

Automorphism towers

CHAPTER 1

The heights of automorphism towers

In this chapter, we give an introduction to the *automorphism tower problem* and the related problem of finding upper bounds for the heights of automorphism towers of centreless groups of a given cardinality. An extensive account of all aspects of the automorphism tower problem can be found in Simon Thomas' forthcoming monograph [**Tho**].

We start by giving a detailed introduction to the construction of automorphism towers of centreless groups and the *automorphism tower problem* in the first section. Section 1.2 contains an overview on existing upper bounds for the heights of automorphism towers of groups of a given cardinality. We will also present independence results that restrict the class of possible strengthenings of these bounds. In the last section of this chapter, we define the notion of *x-admissible ordinals* to state our new upper bound. Chapter 2 consists of the proof of this result.

1.1. Introduction

Let G be a group. Then composition of functions induces a group structure on the set $\mathrm{Aut}(G)$ of all automorphisms of G. If g is an element of G, then the map

$$\iota_g : G \longrightarrow G;\ h \longmapsto h^g = g \cdot h \cdot g^{-1}.$$

is an automorphism of G and we call ι_g the *inner automorphism of* G corresponding to g. We let $\mathrm{Inn}(G)$ denote the group of all inner automorphisms of G. The map

$$\iota_G : G \longrightarrow \mathrm{Aut}(G);\ g \longmapsto \iota_g$$

is a homomorphism of groups with $\ker(\iota_G) = \mathrm{Z}(G)$. Given $g \in G$ and $\pi \in \mathrm{Aut}(G)$, an easy computation shows that

$$\iota_{\pi(g)} = \pi \circ \iota_g \circ \pi^{-1}$$

holds and this implies that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

If G is a group with trivial centre, then ι_G is an embedding of groups and the above equality implies that

$$\mathrm{C}_{\mathrm{Aut}(G)}(\mathrm{Inn}(G)) = \{\mathrm{id}_G\}$$

holds. In particular, this assumption causes $\mathrm{Aut}(G)$ to be a group with trivial centre. By iterating this process, we construct the automorphism tower of a centreless group G.

DEFINITION 1.1.1. Let G be a group with trivial centre. We call a sequence $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ of groups *an automorphism tower of* G if the following statements hold.

(1) $G = G_0$.
(2) If $\alpha \in \mathrm{On}$, then G_α is a normal subgroup of $G_{\alpha+1}$ and the induced homomorphism
$$\varphi_\alpha : G_{\alpha+1} \longrightarrow \mathrm{Aut}(G_\alpha);\ g \mapsto \iota_g \restriction G_\alpha$$
is an isomorphism.
(3) If $\alpha \in \mathrm{Lim}$, then $G_\alpha = \bigcup\{G_\beta \mid \beta < \alpha\}$.

In this definition, we replaced $\mathrm{Aut}(G_\alpha)$ by an isomorphic copy $G_{\alpha+1}$ that contains G_α as a normal subgroup. This allows us to take unions at limit stages. Without this isomorphic correction, we would have to take direct limits at limit stages. By induction, we can construct such a tower for each centreless group and it is easy to show that each group G_α in such a tower is uniquely determined up to an isomorphism which is the identity on G. We can therefore speak of *the* α-th group G_α in the automorphism tower of a centreless group G.

It is natural to ask whether the automorphism tower of every centreless group eventually *terminates* in the sense that there is an ordinal α with $G_\alpha = G_{\alpha+1}$ and therefore $G_\alpha = G_\beta$ for all $\beta \geq \alpha$. A classical result due to Helmut Wielandt shows that the automorphism tower of every finite centreless group terminates.

THEOREM 1.1.2 ([**Wie39**]). *If G is a finite group with trivial centre, then there is an $n < \omega$ with $G_n = G_{n+1}$.*

In [**Tho85**] and [**Tho98**], Simon Thomas showed that the automorphism tower of every centreless group eventually terminates by proving the following result. An application of Fodor's Lemma (and hence of the *Axiom of Choice*) lies at the heart of the proof of this result.

THEOREM 1.1.3 ([**Tho98**, Theorem 1.3]). *If G is an infinite centreless group of cardinality κ, then there is an $\alpha < (2^\kappa)^+$ with $G_\alpha = G_{\alpha+1}$.*

This result allows us to make the following definitions.

DEFINITION 1.1.4. Given a centreless group G, we let $\tau(G)$ denote the least ordinal α with $G_\alpha = G_{\alpha+1}$. We call this ordinal the *height of the automorphism tower of* G. If κ is an infinite cardinal, then we define
$$\tau_\kappa = \mathrm{lub}\{\tau(G) \mid G \textit{ is a centreless group of cardinality } \kappa\}.$$

We are interested to determine the possible values of these ordinals. The following result of Simon Thomas implies that κ^+ is a lower bound for τ_κ.

THEOREM 1.1.5 ([**Tho85**, Theorem 2]). *If κ is a an infinite cardinal and $\alpha < \kappa^+$, then there is a centreless group G of cardinality κ with $\tau(G) = \alpha$.*

There are only 2^κ-many centreless groups of cardinality κ and $(2^\kappa)^+$ is a regular cardinal. This allows us to combine the above results to conclude that

$$\kappa^+ \leq \tau_\kappa < (2^\kappa)^+$$

holds for every infinite cardinal κ.

Next to nothing is known about the possible values of τ_κ. The following open questions are supposed to illustrate this lack of knowledge.

QUESTION 1.1.6. *Is it consistent with the axioms of set theory that there is an infinite cardinal κ such that τ_κ is a ...*

(1) *... successor ordinal?*
(2) *... limit ordinal?*
(3) *... limit ordinal of cofinality greater than* $\mathrm{cof}(\kappa)$*?*
(4) *... cardinal?*

The *automorphism tower problem* asks for the computation of the actual value of τ_κ in some model of set theory. This open problem motivates the work of the first part of this thesis.

PROBLEM 1.1.7 (The automorphism tower problem). *Find a model $\mathcal{M}$ of* ZFC *and an infinite cardinal κ in $\mathcal{M}$ such that it is possible to* compute *the exact value of τ_κ in $\mathcal{M}$.*

In the above statement, the phrase "*compute the value of τ_κ*" should be interpreted as "*give a set-theoretic characterizations of τ_κ*". Examples of such characterizations would be $\mathcal{M} \models$ "$\tau_\kappa = \kappa^+$" or $\mathcal{M} \models$ "$\tau_\kappa = 2^\kappa$".

1.2. Upper bounds for τ_κ

Our aim is to find upper bounds for τ_κ that are uniformly definable from the parameter κ. This means that we want to find a *set-theoretic characterizations* of an ordinal α_κ from the parameter κ such that the estimate $\tau_\kappa \leq \alpha_\kappa$ follows from the axioms of set theory.

We start by presenting consistency results about the possible cardinalities of τ_κ that restrict the class of possible bounds. By constructing partial orders that force the existence of groups with *long automorphism towers*, Winfried Just, Saharon Shelah and Simon Thomas showed that there cannot be a uniformly definable upper bound for τ_κ (in the above sense) that is always equal to a cardinal smaller than $(2^\kappa)^+$.

THEOREM 1.2.1 ([**JST99**, Theorem 1.4]). *Assume that the* GCH *holds in the ground model* V*. Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$ and ν be a cardinal with $\kappa < \mathrm{cof}(\nu)$. If $\alpha < \nu^+$, then there is a partial order $\mathbb{P}$ with the following properties.*

(1) *$\mathbb{P}$ is $<\kappa$-closed and satisfies the κ^+-chain condition.*
(2) *If F is $\mathbb{P}$-generic over* V*, then $(2^\kappa) = \nu$ and there is a centreless group $G \in \mathrm{V}[F]$ such that $\tau(G) = \alpha$ holds in $\mathrm{V}[F]$.*

In particular, it is consistent with the axioms of ZFC that τ_κ is bigger than 2^κ for some uncountable regular cardinal κ. In contrast, it is not known if $\tau_\omega > \omega_1$ is consistent with the axioms of set theory or if the statement $\tau_\omega = \omega_1$ is the consequence of some extension of ZFC by large cardinal axioms.

In another direction, Simon Thomas showed that the cardinality of τ_κ can consistently be smaller than 2^κ.

Theorem 1.2.2 ([**Tho98**, Theorem 1.8]). *It is consistent with the axioms of* ZFC *that* $\tau_\kappa < 2^\kappa$ *holds for every regular cardinal* κ*.*

A model of the above statement is produced with the help of the following theorem and a class-sized forcing iteration with Easton support.

Theorem 1.2.3 ([**Tho98**, Theorem 4.1]). *Let* κ*,* λ *and* ν *be regular cardinals with* $\kappa = \kappa^{<\kappa}$*,* $\kappa \leq \lambda$*,* $2^\lambda = \lambda^+$*,* $\nu \geq \lambda^{++}$ *and* $\nu = \nu^\lambda$*. If* G *is* $\mathrm{Add}(\kappa, \nu)$*-generic over* V*, then* $\tau_\lambda \leq \lambda^{++}$ *and* $2^\lambda = \nu$ *hold in* $\mathrm{V}[G]$*.*

In [**KS09**], Itay Kaplan and Saharon Shelah analyse automorphism towers in the absence of the *Axiom of Choice*. Their work provides two examples of upper bounds for τ_κ and it motivates the work presented in the first two chapters of this thesis.

Given a centreless group G, it is possible to construct an automorphism tower of G without using the Axiom of Choice and this tower is still uniquely determined up to isomorphisms that induce the identity on G. Since Simon Thomas' proof of Theorem 1.1.3 uses Fodor's Lemma, it is *a priori* not clear whether the axioms of ZF imply that every automorphism tower terminates. The results of [**KS09**] show that this is indeed the case and they also produce an upper bound for the heights of automorphism towers of centreless groups with a given domain.

Theorem 1.2.4 (ZF, [**KS09**, Main Theorem 3.16]). *Let* κ *be an infinite cardinal. There is an ordinal* α *such that there is a surjection of* $\mathcal{P}(\kappa)$ *onto* α *and, if* $\langle G_\gamma \mid \gamma \in \mathrm{On}\rangle$ *is an automorphism tower of a centreless group with domain* κ*, then there is an ordinal* $\beta < \alpha$ *with* $G_\beta = G_{\beta+1}$*.*

Hence, it already follows from the axioms of ZF that the ordinal τ_κ exists for every infinite cardinal κ. Moreover, this result also gives us an uniformly definable upper bound for τ_κ. Given a set A, remember that $\mathrm{L}(A)$ denotes the smallest inner model of set theory that contains A (see [**Jec03**, page 193]).

Definition 1.2.5. Define

$$\theta_A = \mathrm{lub}\{\alpha \in \mathrm{On} \mid (\exists f \in \mathrm{L}(A))\ f : A \longrightarrow \alpha \textit{ is a surjection}\}.$$

If κ is an infinite cardinal, then it is easy to see that $\kappa^+ < \theta_{\mathcal{P}(\kappa)} \leq (2^\kappa)^+$.

Theorem 1.2.6 (ZF, [**KS09**, Theorem 3.18]). *If* κ *is an infinite cardinal, then* $\tau_\kappa < \theta_{\mathcal{P}(\kappa)}$*.*

This estimate is an easy consequence of Theorem 1.2.4 and the following *Absoluteness Lemma*.

THEOREM 1.2.7 (ZF, [**KS09**, Lemma 3.17]). *Let κ be an infinite cardinal and G be a centreless group with domain κ. If $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ is an automorphism tower of G in $\mathrm{L}(\mathcal{P}(\kappa))$, then $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ is an automorphism tower of G.*

The other upper bound developed in [**KS09**] uses the theory of *inductive definitions on first order structures*. In the following, we briefly define the objects of interest in this theory. Extensive introductions to this topic can be found in [**Mos74**] and [**Mos75**].

DEFINITION 1.2.8. Let $\mathcal{L}$ be a finite first-order language, $\mathcal{M}$ be an $\mathcal{L}$-structure with domain M and n be a natural number.

- We let $\mathcal{L}^n_{\mathcal{M}}$ denote the first-order language that extends $\mathcal{L}$ by a new n-ary predicate $\dot{R}$ and a constant symbol $\dot{x}$ for every $x \in M$.
- If X is a subset of M^n, then we define $\mathcal{M}(X)$ to be the unique $\mathcal{L}^n_{\mathcal{M}}$-expansion $\mathcal{N}$ of $\mathcal{M}$ with $\dot{R}^{\mathcal{N}} = X$ and $\dot{x}^{\mathcal{N}} = x$ for every $x \in M$.
- Given an $\mathcal{L}^n_{\mathcal{M}}$-formula $\varphi \equiv \varphi(v_0, \ldots, v_{n-1})$ with n free variables, we let $\langle I^\varphi_\alpha \subseteq M^n \mid \alpha \in \mathrm{On}\rangle$ denote the unique sequence of subsets of M^n that satisfies the following statements for all $\alpha \in \mathrm{On}$.
  (1) $I^\varphi_0 = \{\vec{x} \in M^n \mid \mathcal{M}(\emptyset) \models \varphi(\vec{x})\}$.
  (2) $I^\varphi_{\alpha+1} = I^\varphi_\alpha \cup \{\vec{x} \in M^n \mid \mathcal{M}(I^\varphi_\alpha) \models \varphi(\vec{x})\}$.
  (3) If $\alpha \in \mathrm{Lim}$, then $I^\varphi_\alpha = \bigcup_{\bar{\alpha}<\alpha} I^\varphi_{\bar{\alpha}}$.

It follows from the axioms of ZF that for every such $\mathcal{L}^n_{\mathcal{M}}$-formula $\varphi \equiv \varphi(v_0, \ldots, v_{n-1})$ there is an ordinal α with $I^\varphi_\alpha = I^\varphi_{\alpha+1}$ and therefore $I^\varphi_\alpha = I^\varphi_\beta$ for all $\beta \geq \alpha$. This allows us to make the following definition.

DEFINITION 1.2.9. Let $\mathcal{L}$ be a finite first-order language and $\mathcal{M}$ be an $\mathcal{L}$-structure with domain M.

(1) If $n < \omega$ and $\varphi \equiv \varphi(v_0, \ldots, v_{n-1})$ is an $\mathcal{L}^n_{\mathcal{M}}$-formula, then we define
$$\|\varphi\| = \min\{\alpha \in \mathrm{On} \mid I^\varphi_\alpha = I^\varphi_{\alpha+1}\}$$
and $I_\varphi = I^\varphi_{\|\varphi\|}$.
(2) The *inductive ordinal of* $\mathcal{M}$ is the ordinal
$$\sup\{\|\varphi\| \mid \varphi \equiv \varphi(v_0, \ldots, v_{n-1}) \textit{ is an } \mathcal{L}^n_{\mathcal{M}}\textit{-formula for some } n < \omega\}.$$

We are now ready to present the second bound derived in [**KS09**].

THEOREM 1.2.10 (ZF, [**KS09**, Conclusion 4.4]). *If $\mathcal{A}$ is the standard model of second order number theory and α is the inductive ordinal of $\mathcal{A}$, then $\tau_\omega \leq \alpha$.*

The arguments used in the proof of this result directly generalize to higher cardinalities κ and the corresponding structures with domain $\kappa \sqcup \mathcal{P}(\kappa)$. This generalization and all results presented in this section will be a direct consequence of the results presented in the next section.

1.3. Admissible ordinals as upper bounds

In the next chapter, we will extend methods developed in [**KS09**] to find a better upper bound for τ_κ that is uniformly definable from the parameter κ. To formulate this bound, we need to introduce some notions from the theory of *admissible sets*.

Richard Platek introduced admissible sets in [**Pla66**] as natural domains on which *abstract recursion theory* can be developed. The related notion of *admissible ordinals* was defined by Saul Kripke in [**Kri64**]. In the next chapter, we will present some basic results from the theory of admissible set that will be needed in our proofs. Detailed treatments of *admissible set theory* can be found in [**Bar75**],[**Jen72**, Section 2.3] and [**Mos74**, Section 9D].

Definition 1.3.1. A set M is *admissible* if it satisfies the following statements.

(1) M is nonempty, transitive and closed under forming pairs and unions.
(2) The structure $\langle M, \in\rangle$ satisfies Δ_0-*Separation*, i.e. the sentence
$$(\forall x_0,\ldots,x_n)(\exists y)(\forall z)\ \big[z \in y \longleftrightarrow [y \in x_0 \wedge \varphi(x_0,\ldots,x_n,z)]\big]$$
holds in $\langle M, \in\rangle$ for every Δ_0-formula $\varphi \equiv \varphi(v_0,\ldots,v_{n+1})$.
(3) The structure $\langle M, \in\rangle$ satisfies Δ_0-*Collection*, i.e. the sentence
$$(\forall x_0,\ldots,x_n)\big[(\forall y \in x_0)(\exists z)\ \varphi(x_0,\ldots,x_n,y,z) \longrightarrow (\exists w)(\forall y \in x_0)(\exists z \in w)\ \varphi(x_0,\ldots,x_n,y,z)\big]$$
holds in $\langle M, \in\rangle$ for every Δ_0-formula $\varphi \equiv \varphi(v_0,\ldots,v_{n+3})$.

This means that a nonempty transitive set M is admissible if and only if the structure $\langle M, \in\rangle$ is a model of *Kripke-Platek set theory* KP (see [**Bar75**, Chapter 1, Section 2]).

Definition 1.3.2. Let x be an arbitrary set. We say that an ordinal α is x-admissible if there is an admissible set M with $x \in M$ and $\alpha = M \cap \mathrm{On}$.

Let x be an arbitrary set and κ be the cardinality of $\mathrm{tc}(\{x\})$. Then κ^+ is x-admissible, because $x \in \mathrm{H}_{\kappa^+}$ and H_{κ^+} is an admissible set (see [**Bar75**, Theorem 3.1]). In particular, $(2^\kappa)^+$ is $\mathcal{P}(\kappa)$-admissible and, by forming the Skolem hull of $\mathcal{P}(\kappa)$ in $\mathrm{H}_{(2^\kappa)^+}$ and considering its transitive collapse, we see that there is a $\mathcal{P}(\kappa)$-admissible ordinal smaller than $(2^\kappa)^+$.

It follows directly from Definition 1.3.1 that every x-admissible ordinal is a limit ordinal. Using admissible set theory and codes for well-orderings of an infinite cardinal κ, it is easy to show that every $\mathcal{P}(\kappa)$-admissible ordinal is bigger than κ^+ (this will follow directly from Theorem 1.3.9 and Δ_0-Separation).

We are now ready to formulate our new upper bound for τ_κ. As above, the *Axiom of Choice* is not needed in the proof of this result.

THEOREM 1.3.3 (ZF). *Let κ be an infinite cardinal and α be a $\mathcal{P}(\kappa)$-admissible ordinal. If $\tau_\kappa \neq \alpha + 1$, then $\tau_\kappa < \alpha$.*

In [**Bec11**], Howard Becker independently derived the upper bound for τ_ω produced by the above result. His proof uses *descriptive set theory* and the theory of *positive inductive definitions on the reals*.

Theorem 1.3.3 is proven by refining methods developed in [**KS09**] and combining them with basic techniques from admissible set theory. The next chapter contains a detailed proof of this result and starts with an outline of the idea behind it.

In the remainder of this section, we will discuss the relation of this upper bound and the bounds given by Theorem 1.2.6 and Theorem 1.2.10.

By analysing the *fine structure* of $\mathrm{L}(\mathcal{P}(\kappa))$, it is possible to derive the following statement that directly implies that the first $\mathcal{P}(\kappa)$-admissible ordinal is smaller than $\theta_{\mathcal{P}(\kappa)}$ for every infinite cardinal κ.

THEOREM 1.3.4 (ZF). *Let κ be an infinite cardinal and α be the least $\mathcal{P}(\kappa)$-admissible ordinal. If M is admissible with $\mathcal{P}(\kappa) \in M$ and $\alpha = M \cap \mathrm{On}$, then there is a partial surjection*

$$s : \mathcal{P}(\kappa) \xrightarrow{part} \alpha$$

that is definable in the structure $\langle M, \in \rangle$ by a Σ_1-formula with parameters.

COROLLARY 1.3.5 (ZF). *If κ is an infinite cardinal, then there is a $\mathcal{P}(\kappa)$-admissible ordinal α with $\alpha < \theta_{\mathcal{P}(\kappa)}$.* □

The first section of [**Ste83**] contains a proof of the statement of Theorem 1.3.4 in the case "$\kappa = \omega$". The arguments used in that proof directly generalizes to higher cardinalities.

In the following, we present the classical Barwise-Gandy-Moschovakis theorem. This result connects the theory of inductive definitions with recursion theory on admissible sets. We will use it to show that the first $\mathcal{P}(\omega)$-admissible ordinal is smaller than the inductive ordinal of the standard model of second order number theory. To state this result, we need to introduce some concepts.

DEFINITION 1.3.6. Let $\mathcal{L}$ be a finite first order language and $\mathcal{M}$ be an $\mathcal{L}$-structure with domain M.

(1) We say that an $\mathcal{L}^n_{\mathcal{M}}$-formula φ is *$\dot{R}$-positive* if it is contained in the smallest class of $\mathcal{L}^n_{\mathcal{M}}$-formulae that contains all atomic formulae and all formulae in which $\dot{R}$ does not occur and is closed under conjunction, disjunction, existential quantification and universal quantification.
(2) A subset X of M^n is *inductive on $\mathcal{M}$* if there are $y_0, \dots, y_{m-1} \in M$ and an $\dot{R}$-positive $\mathcal{L}^{n+m}_{\mathcal{M}}$-formula $\varphi \equiv \varphi(v_0, \dots, v_{n+m-1})$ such that

$$\langle x_0, \dots, x_{n-1} \rangle \in X \;\longleftrightarrow\; \langle x_0, \dots, x_{n-1}, y_0, \dots, y_{m-1} \rangle \in I_\varphi.$$

(3) A subset X of M^n is *hyperelementary on* $\mathcal{M}$ if both X and $M^n \setminus X$ are inductive on $\mathcal{M}$.

As usual, we call a function $f : M^n \longrightarrow M^m$ hyperelementary on some structure $\mathcal{M}$ if its graph is hyperelementary on $\mathcal{M}$. It is easy to see that every subset of M^n that is definable in the structure $\mathcal{M}$ is also hyperelementary on $\mathcal{M}$.

The Barwise-Gandy-Moschovakis theorem applies to structures that allow a certain amount of *coding*. This property is made precise by the following definition.

DEFINITION 1.3.7. Let $\mathcal{L}$ be a finite first order language and $\mathcal{M}$ be an $\mathcal{L}$-structure with domain M.

(1) A *coding scheme on* $\mathcal{M}$ is a pair $\langle o, c\rangle$ that consists of injections $o : \omega \longrightarrow M$ and $c : {}^{<\omega}M \longrightarrow M$.
(2) We say that $\mathcal{M}$ *admits a hyperelementary coding scheme* if there is a coding scheme $\langle o, c\rangle$ such that the following objects are hyperelementary on $\mathcal{M}$.
   - $\mathrm{ran}(c)$.
   - $\mathrm{l} : M \longrightarrow M$ with
   $$\mathrm{l}(x) := \begin{cases} o(0), & \text{if } x \notin \mathrm{ran}(c), \\ o(n+1), & \text{if } x = c(s) \text{ and } n = \mathrm{lh}(s). \end{cases}$$
   - $q : M \times M \longrightarrow M$ with
   $$q(x, y) := \begin{cases} s(i), & \text{if } x = c(s),\ y = o(n) \text{ and } i \in \mathrm{dom}(s), \\ o(0), & \text{otherwise.} \end{cases}$$

If $\mathcal{A}$ is the standard model of second order number theory, then $\mathcal{A}$ admits a hyperelementary coding scheme, because there is a coding scheme on $\mathcal{A}$ such that the corresponding objects $\mathrm{ran}(c)$, l and q are definable in $\mathcal{A}$.

We are now ready to state the Barwise-Gandy-Moschovakis theorem.

THEOREM 1.3.8 ([**BGM71**]). *Let $\mathcal{L}$ be a first order language that extends the language of set theory by finitely many relation symbols $\dot{R}_0, \ldots, \dot{R}_{n-1}$ and $\mathcal{N}$ be an $\mathcal{L}$-structure with domain N. Assume that N is transitive, $\dot{\in}^{\mathcal{N}} = \in \restriction (N \times N)$ and $\mathcal{N}$ admits a hyperelementary coding scheme. If we define*

$$\mathcal{N}^+ = \bigcap\{M \mid M \textit{ is admissible and } N, \dot{R}_0^{\mathcal{N}}, \ldots, \dot{R}_{n-1}^{\mathcal{N}} \in M\}$$

and $\alpha = \mathcal{N}^+ \cap \mathrm{On}$, then $\mathcal{N}^+$ is an admissible set and the following statements hold.

(1) *A subset $X \subseteq N^n$ is hyperelementary on $\mathcal{N}$ if and only if X is an element of $\mathcal{N}^+$.*
(2) *A subset $X \subseteq N^n$ is inductive on $\mathcal{N}$ if and only if X is definable in the structure $\langle \mathcal{N}^+, \in\rangle$ by a Σ_1-formula with parameters.*
(3) *Given an $\dot{R}$-positive $\mathcal{L}^n_{\mathcal{M}}$-formula $\varphi \equiv \varphi(v_0, \ldots, v_{n-1})$, we have $\|\varphi\| \leq \alpha$ and $I_\varphi \in \mathcal{N}^+$ if and only if $\|\varphi\| < \alpha$.*

Proof. The above statement follows directly from the combination of [**Mos74**, Theorem 2B.1] and [**Mos74**, Theorem 9F.2]. □

We need one more result from admissible set theory to show that the first $\mathcal{P}(\omega)$-admissible ordinal is smaller than the inductive ordinal of the standard model of second order number theory. The following theorem will also play a central role in the proof of Theorem 1.3.3 in the next chapter. Section 2.4 contains a proof of this statement.

Theorem 1.3.9 (ZF). *Let M be an admissible set with $\alpha = M \cap \mathrm{On}$ and $f : M \longrightarrow \alpha$ be a function that is definable in $\langle M, \in\rangle$ by a Σ_1-formula with parameters. If $X \in M$, then there is an ordinal $\beta < \alpha$ with $f"X \subseteq \beta$.*

Corollary 1.3.10. *If $\mathcal{A}$ is the standard model of second order number theory and α is the least $\mathcal{P}(\omega)$-admissible ordinal, then the inductive ordinal of $\mathcal{A}$ is bigger than $\alpha + 1$.*

Proof. If we define $\mathcal{A}^+$ as in Theorem 1.3.8, then we get $\alpha = \mathcal{A}^+ \cap \mathrm{On}$, because every admissible set that contains $\mathcal{P}(\omega)$ also contains $\mathcal{A}$. We let $s : \mathcal{P}(\omega) \xrightarrow{part} \alpha$ denote the partial surjection given by Theorem 1.3.4. Since $\mathrm{dom}(s)$ is definable in $\langle \mathcal{A}^+, \in\rangle$ by a Σ_1-formula with parameters, Theorem 1.3.8 shows that there is an $\dot{R}$-positive $(\mathcal{L}_{\mathrm{NT}^2})^{n+1}_{\mathcal{A}}$-formula and $y_1, \ldots, y_n$ contained in the domain of $\mathcal{A}$ such that

$$\mathrm{dom}(s) = \{x \in \mathcal{P}(\omega) \mid \langle x, y_1, \ldots, y_n\rangle \in I_\varphi\}.$$

By Theorem 1.3.9, $\mathrm{dom}(s)$ is not an element of $\mathcal{A}^+$ and another application of the Barwise-Gandy-Moschovakis theorem yields $\|\varphi\| = \alpha$.

Since $\mathrm{dom}(s) \notin \mathcal{A}^+$, there is a $y_0 \in \mathcal{P}(\omega) \setminus \mathrm{dom}(s)$ such that the domain of $\mathcal{A}$ is not equal to $\mathrm{dom}(s) \cup \{y_0\}$. Define $\psi \equiv \psi(v_0, \ldots, v_n)$ to be the $(\mathcal{L}_{\mathrm{NT}^2})^{n+1}_{\mathcal{A}}$-formula

$$\begin{aligned}&\varphi(v_0, \ldots, v_n) \vee \dot{R}(\dot{y}_0, \ldots, \dot{y}_n)\\ &\vee\big[(\forall x_0, \ldots, x_n)\ [\varphi(x_0, \ldots, x_n) \rightarrow \dot{R}(x_0, \ldots, x_n)] \wedge v_0 = \dot{y}_0 \wedge \cdots \wedge v_n = \dot{y}_n\big].\end{aligned}$$

An easy induction shows that $I^\varphi_\beta = I^\psi_\beta$ holds for all $\beta \leq \alpha$. Moreover, we have $I^\psi_{\alpha+1} = I^\varphi_\alpha \cup \{\langle y_0, \ldots, y_n\rangle\}$ and $I^\psi_{\alpha+1}$ is equal to the domain of $\mathcal{A}$. In particular, the inductive ordinal of $\mathcal{A}$ is strictly bigger than $\alpha + 1$. □

CHAPTER 2

A new upper bound for τ_κ

This chapter is devoted to the proof of Theorem 1.3.3: *if κ is an infinite cardinal and α is $\mathcal{P}(\kappa)$-admissible, then either $\tau_\kappa = \alpha + 1$ or $\tau_\kappa < \alpha$.*

Section 2.1 is supposed to illustrate the idea behind this proof. We will formulate a theorem that shows how automorphism towers of centreless groups with domain κ can be constructed inside admissible sets which contain the power set of κ and how the actions of automorphism in higher stages on elements in the lower stages can be reconstructed within the admissible set. We will then sketch how this statement implies Theorem 1.3.3.

The next section will deal with *special pairs.* This notion was introduced by Itay Kaplan and Saharon Shelah in [**KS09**] to analyse automorphism towers in the absence of the Axiom of Choice. It allows us to code the automorphisms that appear in the automorphism tower of some centreless group G with domain κ into subsets of κ. In particular, this coding allows us to view automorphism towers as increasing sequences of subsets of $\mathcal{P}(\kappa)$.

In Section 2.3, we will present a strengthening of this result and show that the sequence of subsets associated with an automorphism towers is induced by an inductive definition on some first order structure. This representation was developed by Itay Kaplan and Saharon Shelah in [**KS09**, Section 4]. We will give a detailed presentation of their construction and extend it to show that it is possible to reconstruct the actions of automorphism from their codes inside admissible sets.

This representation will allow us to construct automorphism towers of groups with domain κ inside admissible structures containing $\mathcal{P}(\kappa)$ and prove the theorem stated in Section 2.1. The final proof of Theorem 1.3.3 in Section 2.4 will follow from this result, the Σ_1-Recursion Principle of admissible set theory and the Σ_1-Boundedness Theorem 1.3.9.

2.1. Introduction

As mentioned above, this section is supposed to illustrate the idea behind the proof of Theorem 1.3.3 and explain its structure.

We start with a result of J. Hulse that shows that the automorphisms appearing in the automorphism tower of a centreless group G are uniquely determined by their interactions with elements of G. This result plays a central role in Simon Thomas' proof of Theorem 1.1.3 in [**Tho98**].

THEOREM 2.1.1 ([**Hul70**, Lemma 8.1.1]). *If $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ is the automorphism tower of a centreless group G, then $\mathrm{C}_{G_\alpha}(G) = \{1_G\}$ holds for every $\alpha \in \mathrm{On}$.*

In the next section, we will present a strengthening of this statement using methods developed in [**KS09**]. The following consequence of Theorem 2.1.1 is the starting point of our approach.

PROPOSITION 2.1.2. *Let $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be the automorphism tower of a centreless group G, λ be a limit ordinal with $\mathrm{cof}(\lambda) > \omega$ and $h \in G_{\lambda+1}$. Assume that, for every $\alpha < \lambda$, there is a $\beta < \lambda$ with the property that*

$$\iota_h(g), \iota_{h^{-1}}(g) \in G_\beta$$

for every $g \in G_\alpha$. Then $h \in G_\lambda$.

PROOF. Recursively define a strictly increasing sequence $\langle \alpha_n \mid n < \omega\rangle$ of ordinals in λ by setting $\alpha_0 = 0$ and

$$\alpha_{n+1} = \min\{\beta \in (\alpha_n, \lambda) \mid (\forall g \in G_{\alpha_n})\ \iota_h(g), \iota_{h^{-1}}(g) \in G_\beta\}.$$

By our assumptions, we have $\alpha_* = \sup_{n<\omega} \alpha_n \in \lambda \cap \mathrm{Lim}$. If $g \in G_{\alpha_*}$, then there is an $n < \omega$ with $g \in G_{\alpha_n}$ and

$$\iota_h(g), \iota_{h^{-1}}(h) \subseteq G_{\alpha_{n+1}} \subseteq G_{\alpha_*}.$$

We can conclude $\iota_h \restriction G_{\alpha_*} \in \mathrm{Aut}(G_{\alpha_*})$ and, by the definition of automorphism towers, there is a $h_* \in G_{\alpha_*+1}$ with $\iota_h \restriction G_{\alpha_*} = \iota_{h_*} \restriction G_{\alpha_*}$. Since $G_{\alpha_*+1} \subseteq G_{\lambda+1}$, we have

$$h^{-1} \circ h_* \in \mathrm{C}_{G_{\lambda+1}}(G_{\alpha_*}) \subseteq \mathrm{C}_{G_{\lambda+1}}(G) = \{1_G\}$$

and this means $h = h_* \in G_{\alpha_*+1} \subseteq G_\lambda$. □

Combined with Wielandt's Theorem 1.1.2, this proposition has the following direct consequence.

COROLLARY 2.1.3. *Let $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be the automorphism tower of a centreless group G and κ be an infinite regular cardinal. If $|G_\alpha| < \kappa$ holds for every $\alpha < \kappa$, then $\tau(G) \leq \kappa$.* □

The idea behind the proof of Theorem 1.3.3 is to modify the statement of the above corollary in the following way.

- Replace the regular cardinal by the ordinal height α of some admissible set M.
- Replace the cardinality assumption by the assumption that there is an automorphism tower of G with the property that both the function that lists the groups in this tower up to α and the actions of the automorphisms in the $(\alpha+1)$-th groups are Σ_1-definable in $\langle M, \in\rangle$.
- Use admissible set theory to show that these assumptions cause the automorphism tower to terminate at the some stage less or equal to α.

The main consequence of admissibility used in this argument is the Σ_1-Boundedness Theorem 1.3.9: *given an admissible set M with $\alpha = M \cap \mathrm{On}$, there is no cofinal map from an element of M into α that is Σ_1-definable in the structure $\langle M, \in\rangle$.* Since our proofs will only deal with the Σ_1-*cofinality* of admissible ordinals, this result enables us to replace regularity by admissibility.

The following theorem will allow us to run the argument sketched above if κ is an infinite cardinal, α is $\mathcal{P}(\kappa)$-admissible and G is a centreless group of cardinality κ.

THEOREM 2.1.4 (ZF). *Let κ be an infinite cardinal, $\mathcal{G}_\kappa$ be the set of all centreless groups with domain κ and M be an admissible set with $\mathcal{P}(\kappa) \in M$ and $\alpha = M \cap \mathrm{On}$. Then there are*

- *Σ_1-formulae $\Phi_* \equiv \Phi_*(w_0, \ldots, w_3)$ and $\Psi_* \equiv \Psi_*(w_0, \ldots, w_5)$,*
- *parameters $y, z \in M$, and*
- *an injection $\mathfrak{c} : \mathcal{G}_\kappa \longrightarrow M$ with $\mathrm{ran}(\mathfrak{c}) \in M$*

with the property that, whenever G is an element of $\mathcal{G}_\kappa$, then there is a group G_0 isomorphic to G and an automorphism tower $\langle G_\beta \mid \beta \in \mathrm{On}\rangle$ of G_0 such that the following statements hold.

(1) *For all $\beta \in \mathrm{On}$, the domain of G_β is a subset of M.*
(2) *If $\beta < \alpha$, then the domain of G_β is an element of M and it is the unique $X \in M$ with*
$$\langle M, \in\rangle \models \Phi_*(\beta, X, \mathfrak{c}(G), y).$$
(3) *If $h \in G_{\alpha+1}$ and $g \in G_\alpha$, then $\langle \iota_h(g), \iota_{h^{-1}}(g)\rangle$ is the unique pair $\langle g_0, g_1\rangle$ in $G_\alpha \times G_\alpha$ with*
$$\langle M, \in\rangle \models \Psi_*(g, g_0, g_1, h, \mathfrak{c}(G), z).$$

One way to describe the complexity of a set-theoretic construction is to isolate fragments of ZFC (or some extension of ZFC by large cardinal axioms) with the property that transitive models of this theory containing the necessary parameters compute the outcome of this construction correctly. In combination with Theorem 1.3.3, the above result shows that the complexity of automorphism towers of centreless groups of cardinality κ can be described by the class of admissible sets containing the power set of κ.

The following sections contain the proof of Theorem 2.1.4. It is based on the representation of automorphism towers as inductive definitions developed in [**KS09**]. Given a centreless group G with domain κ, we will use this method to show that there is a first-order structure $\mathcal{N}_\kappa$ and an $\mathcal{L}^4_{\mathcal{N}_\kappa}$-formula $\Phi \equiv \Phi(w_0, \ldots, w_3)$ such that the set I^Φ_α *codes* the group operation of G_α. We will then show that each admissible set M with $\mathcal{P}(\kappa) \in M$ and $\alpha = M \cap \mathrm{On}$ contains such a structure and the actions of automorphisms of $G_{\alpha+1}$ are Σ_1-definable in $\langle M, \in\rangle$ from certain codes contained in the domains of $\mathcal{N}$.

In the remainder of this section, we will sketch how Theorem 1.3.3 can be derived from Theorem 2.1.4 using the axioms of ZFC. Note that, if the

Countable Axiom of Choice $(\mathrm{AC})_\omega$ holds, then Theorem 1.3.4 and Theorem 1.3.9 imply that the least $\mathcal{P}(\kappa)$-admissible ordinal has uncountable cofinality. The final proof of Theorem 1.3.3 in Section 2.4 will work without this extra assumption.

Let κ be an infinite cardinal, α be the least $\mathcal{P}(\kappa)$-admissible ordinal and M be an admissible set with $\mathcal{P}(\kappa) \in M$ and $\alpha = M \cap \mathrm{On}$. Given a centreless group G with domain κ, we let $\langle G_\beta \mid \beta \in \mathrm{On}\rangle$ denote the automorphism tower produced by an application of Theorem 2.1.4. Fix $h \in G_{\alpha+1}$ and $\beta < \alpha$. Let $f_\beta : M \longrightarrow \alpha$ denote the map defined by

$$f_\beta(g) := \begin{cases} \min\{\delta < \alpha \mid \iota_h(g), \iota_{h^{-1}}(g) \in G_{\delta+1}\}, & \text{if } g \in G_\beta, \\ 0, & \text{if } g \notin G_\beta. \end{cases}$$

By the properties of M and the closure properties of the class of Σ_1-definable subsets of admissible sets, the function f_β is definable in $\langle M, \in\rangle$ by a Σ_1-formula with parameters. Since G_β is an element of M, Theorem 1.3.9 gives us an ordinal $\gamma < \alpha$ with f_β"$G_\beta \subseteq \gamma$. This argument shows that, for every $\beta < \alpha$, there we can find a $\gamma < \alpha$ such that

$$\iota_h(g), \iota_{h^{-1}}(g) \in G_\gamma$$

for every $g \in G_\beta$. By Proposition 2.1.2, this implies that h is an element of G_α. We can conclude $\tau_\kappa \leq \alpha + 1$.

Now, assume that $\tau(G) < \alpha$ holds for every G in $\mathcal{G}_\kappa$. If G is an element of $\mathcal{G}_\kappa$, then our assumption and Theorem 2.1.4 imply that $\tau(G)$ is equal to the least ordinal $\beta < \alpha$ with

$$\langle M, \in\rangle \models (\exists X) \ \left[\Phi_*(\beta, X, \mathfrak{c}(G), y) \wedge \Phi_*(\beta+1, X, \mathfrak{c}(G), y)\right].$$

Since $\mathrm{ran}(\mathfrak{c})$ is an element of M and Φ_* is a Σ_1-formula, the closure properties of admissible sets imply that the function

$$c : M \longrightarrow \alpha; \ c(x) := \begin{cases} \tau(G), & \text{if } x = \mathfrak{c}(G) \in \mathrm{ran}(\mathfrak{c}) \\ 0, & \text{otherwise.} \end{cases}$$

is definable in $\langle M, \in\rangle$ by a Σ_1-formula with parameters. Another application of Theorem 1.3.9 yields $\tau_\kappa < \alpha$.

2.2. Special pairs

We introduce the notion of *special pairs* defined by Itay Kaplan and Saharon Shelah in [**KS09**]. This notion allows us to code automorphisms appearing in the automorphism tower of a centreless group of cardinality κ into subsets of κ. In this way, we can identify the stages of an automorphism tower with subsets of $\mathcal{P}(\kappa)$ and talk about the complexity of these groups. Moreover, we will represent the tower as an inductive definition on a structure whose domain contains all such codes.

Definition 2.2.1. Let A be a set and $\mathcal{L}_A$ be the first-order language that expands the language of group theory $\mathcal{L}_{\mathrm{GT}}$ by a constant symbols $\dot{a}$ for each element a of A. If G is a group whose domain contains A as a subset, then we regard G as an $\mathcal{L}_A$-model in the obvious way.

Let $\langle v_n \mid n < \omega\rangle$ be an enumeration of all variables in $\mathcal{L}_A$. We define $\mathcal{T}^n_A$ to be the set of all $\mathcal{L}_A$-terms of the form $t \equiv t(v_0, \ldots, v_{n-1})$, i.e. the set of all terms whose free variables are contained in the set $\{v_0, \ldots, v_{n-1}\}$. If $\vec{g} \in G^n$, then we define

$$\mathsf{qft}_{G,A}(\vec{g}) = \{t(\vec{v}) \in \mathcal{T}^n_A \mid t^G(\vec{g}) = 1_G\}$$

and call this set the *quantifier-free A-type of g*.

Definition 2.2.2. Given a group G and a subset A of the domain of G, the pair $\langle G, A\rangle$ is *special* if the function

$$\mathsf{qft}_{G,A} : G \longrightarrow \mathcal{P}(\mathcal{T}^1_A);\ g \longmapsto \mathsf{qft}_{G,A}(g)$$

is injective.

It is also possible to characterize special pairs by the non-existence of certain local isomorphisms.

Lemma 2.2.3 (ZF, [**KS09**, Remark 3.5 (1)]). *If G is a group and A is a subset of the domain of G, then the following statements are equivalent for all $g_0, g_1 \in G$.*

(1) $\mathsf{qft}_{G,A}(g_0) = \mathsf{qft}_{G,A}(g_1)$.
(2) *There is a group monomorphism $\varphi : \langle A \cup \{g\}\rangle_G \longrightarrow G$ with $\varphi(g_0) = g_1$ and $\varphi \restriction A = \mathrm{id}_A$.*

This characterization directly shows that the computation of quantifier-free types is local.

Corollary 2.2.4 (ZF). *Let $\langle H, A\rangle$ be a special pair and G be a subgroup of H that contains A. Then $\langle G, A\rangle$ is a special pair and $\mathsf{qft}_{G,A}(g) = \mathsf{qft}_{H,A}(g)$ for all $g \in G$.* □

The following theorem due to Itay Kaplan and Saharon Shelah establishes a connection between automorphism towers and special pairs. It may be viewed as a strengthening of Theorem 2.1.1, in the sense that it provides stronger characterizations of elements in the automorphism tower of a centreless group G in terms of their interactions with the elements of G. It also shows that the Axiom of Choice is not needed to derive the statement of Theorem 2.1.1. This result lies at the heart of the proof of Theorem 1.2.4.

Theorem 2.2.5 (ZF, [**KS09**, Conclusion 3.10]). *Let $\langle G, A\rangle$ be a special pair with $\mathrm{C}_G(A) = \{1_G\}$ and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. If $\alpha \in \mathrm{On}$, then $\langle G_\alpha, A\rangle$ is a special pair and $\mathrm{C}_{G_\alpha}(A) = \{1_G\}$ holds.*

Corollary 2.2.6 (ZF). *If $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ is the automorphism tower of a centreless group G with domain A, then $\langle G_\alpha, A\rangle$ is a special pair for every $\alpha \in \mathrm{On}$.* □

In Chapter 5, we will strengthen the notion of special pairs and prove that the above statements about automorphism towers also hold with respect to this stronger notion. These proofs will be almost identical to the ones presented in [**KS09**].

In our representation of the automorphism tower of a centreless group G with domain A as an inductive definition, the set $\mathcal{T}^2_A \cup \mathcal{P}(\mathcal{T}^2_A)$ will form the domain of the structure in which the inductive definition takes place and we will identify the automorphisms that appear in this tower with their quantifier-free A-type.

In the remainder of this section, we will introduce relations and functions on the above domain that will allow us to translate group-theoretic statements into the language of quantifier-free types. Then we will prove some basic facts about these objects. **All statements will be derived from the axioms of** ZF.

Remember that we let $t^v_{t_*}$ denote the term produced by substituting all occurrences of a variable v in a term t by a term t_*.

1. We define

$$\mathrm{E}_A : \mathcal{P}(\mathcal{T}^2_A) \times \mathcal{T}^2_A \longrightarrow \mathcal{P}(\mathcal{T}^1_A);\ \langle z, t_* \rangle \longmapsto \{t \in \mathcal{T}^1_A \mid t^{v_0}_{t_*} \in z\}.$$

Proposition 2.2.7. *Let $\langle G, A\rangle$ be a special pair and $g, h \in G$.*

(1) *If $k \in \langle A \cup \{g,h\}\rangle_G$ and $t_* \in \mathcal{T}^2_A$, then $k = t^G_*(g,h)$ if and only if*
$$\mathsf{qft}_{G,A}(k) = \mathrm{E}_A(\mathsf{qft}_{G,A}(g,h), t_*).$$
(2) *If $k \in \langle A \cup \{g\}\rangle_G$ and $t_* \in \mathcal{T}^1_A$, then $k = t^G_*(g)$ if and only if*
$$\mathsf{qft}_{G,A}(k) = \mathrm{E}_A(\mathsf{qft}_{G,A}(g), t_*).$$

Proof. (1) Assume $k = t^G_*(g,h)$. If $t \in \mathcal{T}^1_A$, then

$$t \in \mathsf{qft}_{G,A}(k) \Leftrightarrow t^G(k) = 1_G \Leftrightarrow (t^{v_0}_{t_*})^G(g,h) = 1_G \Leftrightarrow t^{v_0}_{t_*} \in \mathsf{qft}_{G,A}(g,h).$$

This implies $\mathsf{qft}_{G,A}(k) = \mathrm{E}_A(\mathsf{qft}_{G,A}(g,h), t_*)$. In the other direction, a similar argument shows $\mathrm{E}_A(\mathsf{qft}_{G,A}(g,h), t_*) = \mathsf{qft}_{G,A}(t^G_*(g,h))$ and we can conclude $k = t^G_*(g,h)$, because $\langle G, A\rangle$ is a special pair.

(2) Since $t^{v_0}_{t_*} \in \mathcal{T}^1_A$, $t^G_*(g) = t^G_*(g,h)$ and $\mathsf{qft}_{G,A}(g) = \mathsf{qft}_{G,A}(g,h) \cap \mathcal{T}^1_A$, we have $\mathrm{E}_A(\mathsf{qft}_{G,A}(g), t_*) = \mathrm{E}_A(\mathsf{qft}_{G,A}(g,h), t_*)$ and the statement follows directly from this first part of the proposition. □

2. Let $\approx_A$ be the equivalence relation on $\mathcal{T}^2_A$ defined by

$$t_0 \approx_A t_1 \iff \mathrm{GT} \vdash (\forall x, y)\ t_0(x,y) = t_1(x,y).$$

3. Given $i < 2$, we define N^i_A to be the set of all $z \in \mathcal{P}(\mathcal{T}^i_A)$ with the following properties.

(1) $\mathbb{1} \in z$.
(2) If $t_0, t_1 \in z$ and $t \in \mathcal{T}^i_A$, then $t_0^{-1},\ t_0 * t_1,\ t * t_0 * t^{-1} \in z$.
(3) z is closed under $\approx_A$ in $\mathcal{T}^i_A$.

PROPOSITION 2.2.8. *If G is a group, A is a subset of the domain of G and $g, h \in G$, then $\mathsf{qft}_{G,A}(g) \in \mathrm{N}_A^1$ and $\mathsf{qft}_{G,A}(g,h) \in \mathrm{N}_A^2$.* □

PROPOSITION 2.2.9. *Let $z \in \mathrm{N}_A^2$ and $t_0, t_1 \in \mathcal{T}_A^2$. If $t_0^{-1} * t_1 \in z$, then*

$$\mathrm{E}_A(z, t_0) = \mathrm{E}_A(z, t_1).$$

In particular, $\mathrm{E}_A(z, t) = \mathrm{E}_A(z, \mathbb{1})$ for all $t \in z$ and, if $t_0 \approx_A t_1$, then $\mathrm{E}_A(z, t_0) = \mathrm{E}_A(z, t_1)$.

PROOF. An easy induction shows that

$$(t_{t_0}^{v_0})^{-1} * t_{t_1}^{v_0}, (t_{t_1}^{v_0})^{-1} * t_{t_0}^{v_0} \in z$$

holds for every $t \in \mathcal{T}_A^1$. Hence, we have

$$t \in \mathrm{E}_A(z, t_0) \Leftrightarrow t_{t_0}^{v_0} \in z \Leftrightarrow t_{t_0}^{v_0} * (t_{t_0}^{v_0})^{-1} * t_{t_1}^{v_0} \in z \Leftrightarrow t_{t_1}^{v_0} \in z \Leftrightarrow t \in \mathrm{E}_A(z, t_1)$$

for every $t \in \mathcal{T}_A^1$. □

4. We define N_A to be the intersection of all $z \in \mathrm{N}_A^2$ with $v_1 \in z$ and $\dot{a} \in z$ for all $a \in A$.

5. We define

$$\mathrm{I}_A : \mathcal{P}(\mathcal{T}_A^1) \longrightarrow \mathcal{P}(\mathcal{T}_A^1);\ z \longmapsto \{t \in \mathcal{T}_A^1 \mid t_{v_0^{-1}}^{v_0} \in z\}.$$

PROPOSITION 2.2.10. *If $z \in \mathcal{P}(\mathcal{T}_A^2)$ and $t_* \in \mathcal{T}_A^2$, then*

$$\mathrm{E}_A(z, t_*^{-1}) = (\mathrm{I}_A \circ \mathrm{E}_A)\ \langle z, t_* \rangle.$$

PROOF. If $t \in \mathcal{T}_A^1$, then

$$t \in \mathrm{E}_A(z, t_*^{-1}) \Leftrightarrow ((t_{v_0^{-1}}^{v_0})_{t_*}^{v_0}) \in z \Leftrightarrow t_{v_0^{-1}}^{v_0} \in \mathrm{E}_A(z, t_*) \Leftrightarrow t \in (\mathrm{I}_A \circ \mathrm{E}_A)\ (z, t_*).$$

□

PROPOSITION 2.2.11. *If $\langle G, A \rangle$ is a special pair and $g \in G$, then*

$$\mathsf{qft}_{G,A}(g^{-1}) = \mathrm{I}_A(\mathsf{qft}_{G,A}(g)).$$

PROOF. If $t \in \mathcal{T}_A^1$, then

$$t \in \mathsf{qft}_{G,A}(g^{-1}) \Leftrightarrow (t_{v_0^{-1}}^{v_0}) \in \mathsf{qft}_{G,A}(g) \Leftrightarrow t \in \mathrm{I}_A(\mathsf{qft}_{G,A}(g)).$$

□

We close this section with an argument that shows that the set of all quantifier-free types of identity elements of centreless groups with domain A has an easy definition.

6. Let G_A denote the set of all $x \in \mathcal{P}(\mathcal{T}_A^1)$ that satisfy the following statements.

(1) $x \in \mathrm{N}_A^1$ and $x = \mathrm{E}_A(x, \mathbb{1})$.
(2) If $t \in \mathcal{T}_A^1$, then there is a unique $a \in A$ with $\mathrm{E}_A(x,t) = \mathrm{E}_A(x,\dot{a})$.
(3) If $t_0, t_1 \in \mathcal{T}_A^1$, then $\mathrm{E}_A(x,t_0) = \mathrm{E}_A(x,t_1)$ if and only if $t_0^{-1} * t_1 \in x$.
(4) If $a_0 \in A$, then $\mathrm{E}_A(x, a_0 * a_1) \neq \mathrm{E}_A(x, a_1 * a_0)$ for some $a_1 \in A$.

Proposition 2.2.12. *Let x be an element of $\mathcal{P}(\mathcal{T}_A^1)$. Then x is an element of G_A if and only if $x = \mathsf{qft}_{G(x),A}(1_{G(x)})$ for some centreless group $G(x)$ with domain A.*

Proof. Let $x \in \mathrm{G}_A$. If $t \in \mathcal{T}_A^1$, then we let a_t denote the unique element a of A with $t^{-1} * \dot{a} \in x$. Then $a_{\mathbb{1}} = a_{v_0}$ and $a_{\dot{a}} = a$ for all $a \in A$.

We define an $\mathcal{L}_A$-structure $\mathcal{M}$ with domain A by setting $\mathbb{1}^{\mathcal{M}} = a_{\mathbb{1}}$, $\dot{a}^{\mathcal{M}} = a$, $a^{-1} = a_{\dot{a}^{-1}}$ and $a_0 \cdot_{\mathcal{M}} a_1 = a_{\dot{a}_0 * \dot{a}_1}$ for all $a, a_0, a_1 \in A$. Given $t_0, t_1 \in \mathcal{T}_A^1$, the closure properties of x imply

$$(a_{t_0} * a_{t_1})^{-1} * a_{t_0 * t_1}$$
$$\approx_A (a_{t_1}^{-1} * t_1) * (t_1^{-1} * (a_{t_0}^{-1} * t_0) * t_1) * ((t_0 * t_1)^{-1} * a_{t_0 * t_1}) \in x$$

and therefore $a_{t_0 * t_1} = a_{t_0} \cdot_{\mathcal{M}} a_{t_1}$. This allows us to run an easy induction to show that $t^{\mathcal{M}}(a_{\mathbb{1}}) = a_t$ holds for every $t \in \mathcal{T}_A^1$.

Let $t_0, t_1 \in \mathcal{T}_A^0$ with $\mathrm{GT} \vdash t_0 = t_1$. Then $t_0^{-1} * t_1 \in x$ and

$$\mathrm{E}_A(x, a_{t_0}) \;=\; \mathrm{E}_A(x, t_0) \;=\; \mathrm{E}_A(x, t_1) \;=\; \mathrm{E}_A(x, a_{t_1}).$$

This means $a_{t_0} = a_{t_1}$ and $t_0^{\mathcal{M}} = t_1^{\mathcal{M}}$. We can conclude $\mathcal{M} \models \mathrm{GT}$.

If $G(x)$ denotes the $\mathcal{L}_{\mathrm{GT}}$-reduct of $\mathcal{M}$, then $G(x)$ is a group and the centre of $G(x)$ is trivial by the definition of the group operation and the last clause in the definition of G_A. If $t \in \mathcal{T}_A^1$, then

$$t^{G(x)}(1_{G(x)}) = 1_{G(x)} \Leftrightarrow a_t = a_{\mathbb{1}} \Leftrightarrow \mathrm{E}_A(x,t) = \mathrm{E}_A(x,\mathbb{1}) \Leftrightarrow t \in x.$$

The opposite implication follows directly from Proposition 2.2.7, Proposition 2.2.8 and the assumption $\mathrm{Z}(G(x)) = \{1_{G(x)}\}$. □

2.3. Representing automorphism towers as inductive definitions

In this section, we give a detailed outline of the representation of automorphism towers as inductive definitions on certain structures developed in [**KS09**, Section 4]. Then we show how the actions of automorphisms can be reconstructed from certain codes contained in the domains of these structures. We start by defining the language of this inductive definition and constructing the underlying model. Then we will define the formula used in the inductive definition in several steps. Throughout this section **we derive our result from the axioms of** ZF.

We let $\mathcal{L}_C$ denote the first order language that extends the language $\mathcal{L}_\in$ of set theory by a ternary relation symbol $\dot{C}$. From now on, we assume that A **is a set of ordinals that contains** ω **as a subset and is closed under Gödel Pairing** $\prec\cdot,\cdot\succ$.

To keep our constructions simple, we will equip our model with a *coding relation*. This coding is based on the following assignment of ordinals $\alpha_t \in A$ to $\mathcal{L}_A$-terms t.

- $\alpha_{\mathbb{1}} = 0$ and $\alpha_{v_n} = \prec 1, n \succ$ for all $n < \omega$.
- $\alpha_{\dot{a}} = \prec 2, a \succ$ for all $a \in A$.
- $\alpha_{t^{-1}} = \prec 3, \alpha_t \succ$ for every $\mathcal{L}_A$-term t.
- $\alpha_{t_0 * t_1} = \prec 4, \prec \alpha_{t_0}, \alpha_{t_1} \succ \succ$ for all $\mathcal{L}_A$-terms t_0 and t_1.

The constructions in the proof of Theorem 2.1.4 in Section 2.4 will explain this particular choice of coding.

DEFINITION 2.3.1. We define $\mathcal{M}_A$ to be the unique $\mathcal{L}_C$-model with the following properties.

(1) The domain of $\mathcal{M}_A$ is the set $\mathcal{T}_A^2 \cup \mathcal{P}(\mathcal{T}_A^2)$.
(2) $\in^{\mathcal{M}_A} = \in \restriction (\mathcal{T}_A^2 \times \mathcal{P}(\mathcal{T}_A^2))$.
(3) $\dot{C}^{\mathcal{M}_A} = \{\langle t_0, t_1, t_2 \rangle \in (\mathcal{T}_A^2)^3 \mid (\exists a \in A)\ [t_0 \equiv \dot{a} \wedge a = \prec \alpha_{t_1}, \alpha_{t_2} \succ]\}$.

We let $\mathcal{L}^A$ denote the extended language $(\mathcal{L}_C)^4_{\mathcal{M}_A}$.

By iterating applications of the coding relations and using subsets of $\mathcal{T}_A^2$ as parameters, the following statements follow directly.

PROPOSITION 2.3.2. (1) *Every function and every relation on $\mathcal{T}_A^2$ is uniformly definable in the class of all $\mathcal{L}^A$-models of the form $\mathcal{M}_A(C)$ with $C \subseteq \mathcal{P}(\mathcal{T}_A^1)^4$.*
(2) *The functions E_A and I_A and the relations $\mathcal{P}(\mathcal{T}_A^1)$, $\mathcal{P}(\mathcal{T}_A^2)$, N_A^1, N_A^2 and G_A are uniformly definable in the class of all $\mathcal{L}^A$-models of the form $\mathcal{M}_A(C)$ with $C \subseteq \mathcal{P}(\mathcal{T}_A^1)^4$.* □

Since we will only work with models of the form $\mathcal{M}_A(C)$ with $C \subseteq \mathcal{P}(\mathcal{T}_A^1)^4$, this proposition allows us to include all functions and relations introduced in the last section into our vocabulary by identifying their names with their uniform definitions.

The following definition and the subsequent theorem clarify what we mean by *representing automorphism towers as inductive definitions.*

DEFINITION 2.3.3. Let $\langle G, A \rangle$ be a special pair and C be a set. Then $\langle G, A \rangle$ *is coded by* C if $C \subseteq \mathcal{P}(\mathcal{T}_A^1)^4$ and

$$\begin{aligned}&\{\langle x, y, z \rangle \mid \langle x, y, z, \mathsf{qft}_{G,A}(1_G) \rangle \in C\}\\ &\quad = \{\langle \mathsf{qft}_{G,A}(g), \mathsf{qft}_{G,A}(h), \mathsf{qft}_{G,A}(g \cdot_G h) \rangle \mid g, h \in G\}.\end{aligned}$$

THEOREM 2.3.4. *If A is a set of ordinals that contains ω as a subset and is closed under Gödel Pairing $\prec\cdot,\cdot\succ$, then there are $\mathcal{L}^A$-formulae $\Phi \equiv \Phi(w_0, \ldots, w_3)$ and $\Psi \equiv \Psi(w_0, \ldots, w_4)$ such that the following statements hold whenever $\langle G_\alpha \mid \alpha \in \mathrm{On} \rangle$ is an automorphism tower of a centreless group G with domain A.*

(1) *If $\alpha \in \mathrm{On}$, then the special pair $\langle G_\alpha, A\rangle$ is coded by I^Φ_α for all $\alpha \in \mathrm{On}$.*
(2) *Let $\lambda \in \mathrm{Lim}$, $h \in G_{\lambda+1}$ and $g \in G_\lambda$ with the property there is an $\alpha < \lambda$ such that $t^{G_{\lambda+1}}(h,g) \in G_\alpha$ for every $t \in \mathrm{N}_A$. Then $\langle \iota_h(g), \iota_{h^{-1}}(g)\rangle$ is the unique pair $\langle g_0, g_1\rangle$ in $G_\lambda \times G_\lambda$ such that the statement*

$$\Psi(\mathsf{qft}_{G_\lambda,A}(g), \mathsf{qft}_{G_\lambda,A}(g_0), \mathsf{qft}_{G_\lambda,A}(g_1), \mathsf{qft}_{G_{\lambda+1},A}(h), \mathsf{qft}_{G,A}(1_G))$$

holds in the structure $\mathcal{M}_A(I^\Phi_\alpha)$ for some $\alpha < \lambda$.

By translating the above statement into the language of admissible sets, we will derive Theorem 2.1.4 from this result. To prove Theorem 2.3.4, we construct the formula Φ in seven steps with the help of six auxiliary formulae $\Phi_{\mathrm{I}}, \dots, \Phi_{\mathrm{VI}}$. Let $\langle w_n \mid n < \omega\rangle$ list the free variables in $\mathcal{L}^A$.

1. Define $\Phi_{\mathrm{I}} \equiv \Phi_{\mathrm{I}}(w_0, w_1)$ to be the $\mathcal{L}^A$-formula

$$w_0, w_1 \in \mathcal{P}(\mathcal{T}^1_A) \ \wedge \ \dot{R}(w_0, w_1, w_0, w_1).$$

Proposition 2.3.5. *If C codes the special pair $\langle G, A\rangle$ and x is an element of the domain of $\mathcal{M}_A(C)$, then the following statements are equivalent.*
(1) *$x = \mathsf{qft}_{G,A}(g)$ for some $g \in G$.*
(2) *$\mathcal{M}_A(C) \models \Phi_{\mathrm{I}}(x, \mathsf{qft}_{G,A}(1_G))$.* □

2. Define $\Phi_{\mathrm{II}} \equiv \Phi(w_0, w_1)$ to be the conjunction of the following $\mathcal{L}_A$-statements.

- $w_0 \in \mathrm{N}^2_A$ and $\mathrm{E}_A(w_0, \mathbb{1}) = w_1$.
- If $a \in A$, then $\dot{a} * v_0^{-1} \in \mathrm{E}_A(w_0, \dot{a})$.
- If $t_0, t_1 \in \mathcal{T}^2_A$ with $\mathrm{E}_A(w_0, t_0) = \mathrm{E}_A(w_0, t_1)$, then $t_0^{-1} * t_1 \in w_0$.
- If $t_0, t_1 \in \mathrm{N}_A$, then $\dot{R}(\mathrm{E}_A(w_0, t_0), \mathrm{E}_A(w_0, t_1), \mathrm{E}_A(w_0, t_0 * t_1), w_1)$.

Proposition 2.3.6. *Let $\langle H, A\rangle$ be a special pair, G be a subgroup of H that contains A, $g \in G$ and $h \in H$ with the property that $t^H(h,g) \in G$ for every $t \in \mathrm{N}_A$. If the special pair $\langle G, A\rangle$ is coded by C, then*

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{II}}(\mathsf{qft}_{H,A}(h,g), \mathsf{qft}_{G,A}(1_G)).$$

Proof. Proposition 2.2.8 implies $\mathsf{qft}_{H,A}(h,g) \in \mathrm{N}^2_A$ and we can use Proposition 2.2.7 to get $\mathrm{E}_A(\mathsf{qft}_{H,A}(h,g), \mathbb{1}) = \mathsf{qft}_{G,A}(1_G)$. Given $a \in A$, the same proposition implies $\mathsf{qft}_{G,A}(a) = \mathrm{E}_A(\mathsf{qft}_{H,A}(h,g), \dot{a})$ and, by the definition of quantifier-free types, we have $\dot{a} * v_0^{-1} \in \mathsf{qft}_{G,A}(a)$. If

$$\mathrm{E}_A(\mathsf{qft}_{H,A}(h,g), t_0) = \mathrm{E}_A(\mathsf{qft}_{H,A}(h,g), t_1)$$

for some $t_0, t_1 \in \mathcal{T}^2_A$, then $t_0^H(h,g) = t_1^H(h,g)$ by Proposition 2.2.7 and this means that $t_0^{-1} * t_1 \in \mathsf{qft}_{H,A}(h,g)$. Another application of Proposition 2.2.7 and our assumptions imply that

$$\mathrm{E}_A(\mathsf{qft}_{H,A}(h,g), t_0 * t_1) = \mathsf{qft}_{G,A}((t_0 * t_1)^H(h,g)) = \mathsf{qft}_{G,A}(t_0^H(h,g) \cdot t_1^H(h,g)).$$

for all $t_0, t_1 \in \mathrm{N}_A$. This equality implies that the last clause in the definition of Φ_{II} is also satisfied by $\mathsf{qft}_{G,A}(h,g)$ and $\mathsf{qft}_{G,A}(1_G)$. $\square$

3. Define $\Phi_{\mathrm{III}} \equiv \Phi_{\mathrm{III}}(w_0, \ldots, w_3)$ to be the $\mathcal{L}^A$-formula

$$w_0 \in \mathrm{N}_A^1 \wedge \Phi_{\mathrm{I}}(w_1, w_3) \wedge \Phi_{\mathrm{II}}(w_2, w_3) \wedge \mathrm{E}_A(w_2, v_0) = w_0 \wedge \mathrm{E}_A(w_2, v_1) = w_1.$$

LEMMA 2.3.7. *Let $\langle G, A\rangle$ be a special pair coded by C, $x, z \in \mathcal{P}(\mathcal{T}_A^2)$ and $g \in G$ with*

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{III}}(x, \mathsf{qft}_{G,A}(g), z, \mathsf{qft}_{G,A}(1_G)).$$

Then

$$G_z = \{k \in G \mid (\exists t \in \mathrm{N}_A)\ \mathsf{qft}_{G,A}(k) = \mathrm{E}_A(z,t)\}$$

is a subgroup of G containing $A \cup \{g\}$ and there is a unique automorphism π_z of G_z with

$$\mathsf{qft}_{G,A}(\pi_z(k)) = \mathrm{E}_A(z, v_0 * t * v_0^{-1})$$

for all $k \in G_z$ and $t \in \mathrm{N}_A$ with $\mathsf{qft}_{G,A}(k) = \mathrm{E}_A(z,t)$.

PROOF. Let $k_0, k_1 \in G_z$ and $t_0, t_1 \in \mathrm{N}_A$ with $\mathsf{qft}_{G,A}(k_i) = \mathrm{E}_A(z, t_i)$. By the definition of Φ_{II}, we have

$$\langle \mathsf{qft}_{G,A}(k_0), \mathsf{qft}_{G,A}(k_1), \mathrm{E}_A(z, t_0 * t_1), \mathsf{qft}_{G,A}(1_G)\rangle \in C.$$

This means $\mathsf{qft}_{G,A}(k_0 \cdot k_1) = \mathrm{E}_A(z, t_0 * t_1)$ and $k_0 \cdot k_1 \in G_z$, because $\langle G, A\rangle$ is coded by C and $t_0 * t_1 \in \mathrm{N}_A$. By Proposition 2.2.10 and Proposition 2.2.11, we have

$$\mathsf{qft}_{G,A}(k_0^{-1}) = \mathrm{I}_A(\mathsf{qft}_{G,A}(k_0)) = (\mathrm{I}_A \circ \mathrm{E}_A)\langle z, t_0\rangle = \mathrm{E}_A(z, t_0^{-1})$$

and, since $t_0^{-1} \in \mathrm{N}_A$, this shows $k_0^{-1} \in G_z$. We can use the definition of Φ_{II} to see that $\mathsf{qft}_{G,A}(1_G) = \mathrm{E}_A(z, \mathbb{1})$ and therefore $1_G \in G_z$. This shows that G_z is a subgroup of G.

Given $t \in \mathrm{N}_A$, we have

$$\langle \mathrm{E}_A(z,t), \mathsf{qft}_{G,A}(1_G), \mathrm{E}_A(z, t * \mathbb{1}), \mathsf{qft}_{G,A}(1_G)\rangle \in C \tag{2.1}$$

and there is a unique $k \in G_z$ with $\mathsf{qft}_{G,A}(k) = \mathrm{E}_A(z,t)$. In particular, if $a \in A$, then $\mathsf{qft}_{G,A}(a) = \mathrm{E}_A(z, \dot{a})$, because $\dot{a} \in \mathrm{N}_A$ and $\dot{a} * v_0^{-1} \in \mathrm{E}_A(z, \dot{a})$ by the definition of Φ_{II}. Since $v_1 \in \mathrm{N}_A$ and $\mathsf{qft}_{G,A}(g) = \mathrm{E}_A(z, v_1)$, we can conclude that $A \cup \{g\}$ is contained in G_z.

Pick $k \in G_z$ and $t_0, t_1 \in \mathrm{N}_A$ with $\mathrm{E}_A(z, t_0) = \mathsf{qft}_{G,A}(k) = \mathrm{E}_A(z, t_1)$. Then $t_0^{-1} * t_1 \in z$, because $z \in \mathrm{N}_A^2$. By the definition of N_A^2, this implies $(v_0 * t_0 * v_0^{-1})^{-1} * (v_0 * t_1 * v_0^{-1}) \in z$ and we can use Proposition 2.2.9 to show

$$\mathrm{E}_A(z, v_0 * t_0 * v_0^{-1}) \ = \ \mathrm{E}_A(z, v_0 * t_1 * v_0^{-1}).$$

In combination with (2.1), this argument shows that for each $k \in G_z$ there is a unique $\pi_z(k)$ in G_z with the property that, whenever $t \in \mathrm{N}_A$ with $\mathsf{qft}_{G,A}(k) = \mathrm{E}_A(z,t)$, then $\mathsf{qft}_{G,A}(\pi_z(k)) = \mathrm{E}_A(z, v_0 * t * v_0^{-1})$.

Pick $k_0, k_1 \in G_z$ and $t_0, t_1 \in \mathrm{N}_A$ with $\mathsf{qft}_{G,A}(k_i) = \mathrm{E}_A(z, t_i)$. The above argument shows that $\mathsf{qft}_{G,A}(k_0 \cdot k_1) = \mathrm{E}_A(z, t_0 * t_1)$ and this implies

$$\begin{aligned}\mathsf{qft}_{G,A}(\pi_z(k_0 \cdot k_1)) &= \mathrm{E}_A(z, v_0 * t_0 * t_1 * v_0^{-1}) \\ &= \mathrm{E}_A(z, (v_0 * t_0 * v_0^{-1}) * (v_0 * t_1 * v_0^{-1}))\end{aligned}$$

by Proposition 2.2.9. The definition of Φ_{II} implies that

$$\langle \mathsf{qft}_{G,A}(\pi_z(k_0)), \mathsf{qft}_{G,A}(\pi_z(k_1)), \mathrm{E}_A(z, (v_0 * t_0 * t_1 * v_0^{-1})), \mathsf{qft}_{G,A}(1_G)\rangle$$

is an element of C and therefore

$$\pi_z(k_0 \cdot k_1) = \pi_z(k_0) \cdot \pi_z(k_1).$$

The first part of the definition of Φ_{II} and Proposition 2.2.9 imply $\pi_z(1_G) = 1_G$. We have shown that π_z is a homomorphism.

Let $k \in G_z$ and $t \in \mathrm{N}_A$ with $\mathsf{qft}_{G,A}(k) = \mathrm{E}_A(z,t)$ and $\pi_z(k) = 1_G$. Then

$$\mathrm{E}_A(z, v_0 * t * v_0^{-1}) \ = \ \mathsf{qft}_{G,A}(1_G) \ = \ \mathrm{E}_A(z, \mathbb{1})$$

and this implies $t \in z$ by the definition of Φ_{II} and N^2_A. An application of Proposition 2.2.9 yields $\mathrm{E}_A(z,t) = \mathsf{qft}_{G,A}(\mathbb{1}_G)$ and $k = 1_G$.

Finally, fix $k \in G_z$ with $\mathsf{qft}_{G,A}(k) = \mathrm{E}_A(z,t)$ for some $t \in \mathrm{N}_A$. By (2.1) and the closure properties of N_A, there is a $k_* \in G_z$ with

$$\mathsf{qft}_{G,A}(k_*) = \mathrm{E}_A(z, v_0^{-1} * t * v_0)$$

and hence $\pi_z(k_*) = k$ by Proposition 2.2.9. □

PROPOSITION 2.3.8. *Let $\langle H, A\rangle$ be a special pair, G be a subgroup of H that contains A, $g \in G$ and $h \in H$ with the property that $t^H(h,g) \in G$ for every $t \in \mathrm{N}_A$. If the special pair $\langle G, A\rangle$ is coded by C, then following statements hold.*

(1) $\mathcal{M}_A(C) \models \Phi_{\mathrm{III}}(\mathsf{qft}_{H,A}(h), \mathsf{qft}_{G,A}(g), \mathsf{qft}_{H,A}(h,g), \mathsf{qft}_{G,A}(1_G))$.
(2) $\pi_{\mathsf{qft}_{H,A}(h,g)}(k) = \iota_h(k)$ *for all* $k \in G_{\mathsf{qft}_{H,A}(h,g)}$.

PROOF. The first part of the propositions follow directly from the Propositions 2.2.7, 2.2.8, 2.3.5 and 2.3.6.

If $k \in G_{\mathsf{qft}_{H,A}(h,g)}$ and $t \in \mathrm{N}_A$ with $\mathsf{qft}_{G,A}(k) = \mathrm{E}_A(\mathsf{qft}_{H,A}(h,g), t)$, then

$$\mathsf{qft}_{G,A}(\pi_{\mathsf{qft}_{H,A}(h,g)}(k)) = \mathrm{E}_A(\mathsf{qft}_{H,A}(h,g), v_0 * t * v_0^{-1}) = \mathsf{qft}_{H,A}(\iota_h(k))$$

by Proposition 2.2.7 and Lemma 2.3.7. □

4. Define $\Phi_{\mathrm{IV}} \equiv \Phi_{\mathrm{IV}}(w_0, \ldots, w_3)$ to be the $\mathcal{L}^A$-formula

$$(\forall z) \ \left[\Phi_{\mathrm{III}}(w_1, w_2, z, w_3) \longrightarrow \mathrm{E}_A(z, v_0 * v_1 * v_0^{-1}) = w_0\right].$$

PROPOSITION 2.3.9. *Let $\langle G, A\rangle$ be a special pair coded by C, $g \in G$ and $x \in \mathcal{P}(\mathcal{T}^2_A)$. If*

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{III}}(x, \mathsf{qft}_{G,A}(g), z_0, \mathsf{qft}_{G,A}(1_G)) \tag{2.2}$$

holds for some $z_0 \in \mathcal{P}(\mathcal{T}^2_A)$, then $\pi_{z_0}(g)$ is the unique element k of G with

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{IV}}(\mathsf{qft}_{G,A}(k), x, \mathsf{qft}_{G,A}(g), \mathsf{qft}_{G,A}(1_G)). \tag{2.3}$$

PROOF. Assume $z_1 \in \mathcal{P}(\mathcal{T}_A^2)$ satisfies

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{III}}(x, \mathsf{qft}_{G,A}(g), z_1, \mathsf{qft}_{G,A}(1_G)).$$

By the definition of Φ_{II}, we have

$$z_0 \cap \mathcal{T}_A^1 = \mathrm{E}_A(z_0, v_0) = x = \mathrm{E}_A(z_1, v_0) = z_1 \cap \mathcal{T}_A^1.$$

This directly implies $\mathrm{E}_A(z_0, t) = \mathrm{E}_A(z_1, t)$ for all $t \in \mathcal{T}_A^1$. By the definition of π_{z_i}, we can conclude that $\pi_{z_0}^{-1}(a) = \pi_{z_1}^{-1}(a)$ for all $a \in A$. Therefore

$$\pi_{z_0}^{-1} \text{ `` } \langle A \cup \{\pi_{z_0}(g)\}\rangle_G \subseteq G_{z_1} = \mathrm{dom}(\pi_{z_1})$$

and the map

$$\varphi : \langle A \cup \{\pi_{z_0}(g)\}\rangle_G \longrightarrow G;\ k \longmapsto \pi_{z_1}(\pi_{z_0}^{-1}(k))$$

is a monomorphism with $\varphi \upharpoonright A = \mathrm{id}_A$ and $\varphi(\pi_{z_0}(g)) = \pi_{z_1}(g)$. By Lemma 2.2.3, this implies $\pi_{z_0}(g) = \pi_{z_1}(g)$ and we can conclude

$$\mathrm{E}_A(z_0, v_0 * v_1 * v_0^{-1}) = \mathsf{qft}_{G,A}(\pi_{z_0}(g)) = \mathsf{qft}_{G,A}(\pi_{z_1}(g)) = \mathrm{E}_A(z_1, v_0 * v_1 * v_0^{-1}).$$

This equality implies that (2.3) holds if $k = \pi_{z_0}(g)$. If follows from (2.2) and the definitions of Φ_{IV} and π_{z_0} that $\pi_{z_0}(g)$ is the unique element of G with this property. $\square$

5. Define $\Phi_{\mathrm{V}} \equiv \Phi_{\mathrm{V}}(w_0, w_1)$ to be the conjunction of the following $\mathcal{L}_A$-statements.

- If $\Phi_{\mathrm{I}}(y, w_1)$ holds for some y, then there is a z with $\Phi_{\mathrm{III}}(w_0, y, z, w_1)$.
- $\Phi_{\mathrm{IV}}(w_1, w_0, w_1, w_1)$.
- For all $y_0, \ldots, y_5$ with $\dot{R}(y_0, y_1, y_2, w_1)$ and $\Phi_{\mathrm{IV}}(y_{3+i}, w_0, y_i, w_1)$ for every $i < 3$, we have $\dot{R}(y_3, y_4, y_5, w_1)$.

LEMMA 2.3.10. *Let $\langle G, A\rangle$ be a special pair coded by C and $x \in \mathcal{P}(\mathcal{T}_A^2)$ with*

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{V}}(x, \mathsf{qft}_{G,A}(1_G)).$$

Then there is an endomorphism $\sigma_{C,x} : G \longrightarrow G$ such that the following properties are equivalent for all $g, k \in G$.

(1) $\sigma_{C,x}(g) = k$.
(2) $\mathcal{M}_A(C) \models \Phi_{\mathrm{IV}}(\mathsf{qft}_{G,A}(k), x, \mathsf{qft}_{G,A}(g), \mathsf{qft}_{G,A}(1_G))$.

PROOF. Let $g \in G$. Then there is a $z_0 \in \mathcal{P}(\mathcal{T}_A^2)$ with (2.2) and there is a unique $k \in G$ with (2.3). If we denote this unique element by $\sigma_{C,x}(g)$, then the second and the third clause in the definition of Φ_{V} ensure that the the resulting map $\sigma_{C,x}$ is an endomorphism. $\square$

PROPOSITION 2.3.11. *Let $\langle H, A\rangle$ be a special pair, G be a normal subgroup of H that contains A and $h \in H$. If the special pair $\langle G, A\rangle$ is coded by C, then*

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{V}}(\mathsf{qft}_{H,A}(h), \mathsf{qft}_{G,A}(1_G))$$

and $\sigma_{C,\mathsf{qft}_{H,A}(h)} = \iota_h \upharpoonright G$.

PROOF. Pick $g \in G$ and define

$$z = \{t \in \mathcal{T}^2_A \mid t^H(h,g) \in G\}.$$

Then $z \in \mathrm{N}^2_A$, $v_1 \in z$ and $\dot{a} \in z$ for all $a \in A$. Hence $\mathrm{N}_A \subseteq z$ and $t^H(h,g) \in G$ for all $t \in \mathrm{N}_A$.

Proposition 2.3.8 implies

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{III}}(\mathsf{qft}_{H,A}(h), \mathsf{qft}_{G,A}(g), \mathsf{qft}_{H,A}(h,g), \mathsf{qft}_{G,A}(1_G)).$$

A combination of Proposition 2.3.8 and Proposition 2.3.9 yields

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{IV}}(\mathsf{qft}_{G,A}(\iota_h(g)), \mathsf{qft}_{H,A}(h), \mathsf{qft}_{G,A}(g), \mathsf{qft}_{G,A}(1_G)).$$

The above conclusions directly imply the statement of the proposition. □

6. Define $\Phi_{\mathrm{VI}} \equiv \Phi_{\mathrm{V}}(w_0, w_1)$ to be the conjunction of the following $\mathcal{L}_A$-statements.

- $w_0 \in \mathrm{N}^1_A$ and $\Phi_{\mathrm{V}}(\mathrm{E}_A(w_0,t), w_1)$ for all $t \in \mathcal{T}^1_A$.
- If $t \in \mathcal{T}^1_A$, then

$$t \in w_0 \longleftrightarrow (\forall x \in \mathcal{P}(\mathcal{T}^1_A))\ [\Phi_{\mathrm{I}}(x,w_1) \to \Phi_{\mathrm{IV}}(x, \mathrm{E}_A(w_0,t), x, w_1)]$$

- If $t_0, t_1 \in \mathcal{T}^1_A$ and $x_0, x_1, x_2 \in \mathcal{P}(\mathcal{T}^1_A)$ with $\Phi_{\mathrm{I}}(x_i, w_1)$ for all $i < 3$ and $\Phi_{\mathrm{IV}}(x_{i+1}, \mathrm{E}_A(w_0, t_i), x_i, w_1)$ for all $i < 2$, then

$$\Phi_{\mathrm{IV}}(x_2, \mathrm{E}_A(w_o, t_1 * t_0), x_0, w_1).$$

- If $a \in A$, then

$$\Phi_{\mathrm{IV}}(\mathrm{E}_A(x, \dot{a} * v_0 * \dot{a}^{-1}), \mathrm{E}_A(w_0, \dot{a}), x, w_1)$$

  holds for all x with $\Phi_{\mathrm{I}}(x, w_1)$.

PROPOSITION 2.3.12. *Let $\langle H, A\rangle$ be a special pair with $\mathrm{C}_H(A) = \{1_H\}$ and G be a normal subgroup of H that contains A. If the special pair $\langle G, A\rangle$ is coded by C, then the following statements hold.*

(1) *If $h \in H$, then*

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{VI}}(\mathsf{qft}_{H,A}(h), \mathsf{qft}_{G,A}(1_G)).$$

(2) *If $x \in \mathcal{P}(\mathcal{T}^1_A)$ with $\mathcal{M}_A(C) \models \Phi_{\mathrm{VI}}(x, \mathsf{qft}_{G,A}(1_G))$, then*

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{V}}(x, \mathsf{qft}_{G,A}(1_G))$$

and $\sigma_{C,x} \in \mathrm{Aut}(G)$. Moreover, if $\sigma_{C,x} = \iota_h \restriction G$ for some $h \in H$, then $x = \mathsf{qft}_{H,A}(h)$.

PROOF. (1) Pick $h \in H$. By Proposition 2.2.8, we have $\mathsf{qft}_{H,A}(h) \in \mathrm{N}^1_A$. If $t \in \mathcal{T}^1_A$, then $\mathrm{E}_A(\mathsf{qft}_t(h), t) = \mathsf{qft}_{H,A}(t^H(h))$ by Proposition 2.2.7 and Proposition 2.3.11 implies

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{V}}(\mathrm{E}_A(\mathsf{qft}_{H,A}(h), t), \mathsf{qft}_{G,A}(1_G))$$

and $\sigma_{C,\mathrm{E}_A(\mathsf{qft}_{H,A}(h),t)} = \iota_{t^H(h)} \restriction G$. In particular, if $t \in \mathsf{qft}_{H,A}(h)$, then $\sigma_{C,\mathrm{E}_A(\mathsf{qft}_{H,A}(h),t)} = \mathrm{id}_G$ and

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{IV}}(\mathsf{qft}_{G,A}(g), \mathrm{E}_A(\mathsf{qft}_{H,A}(h),t), \mathsf{qft}_{G,A}(g), \mathsf{qft}_{G,A}(1_G)) \tag{2.4}$$

for all $g \in G$. In the other direction, assume that (2.4) holds for all $g \in G$. Then

$$\mathrm{id}_G = \sigma_{C,\mathrm{E}_A(\mathsf{qft}_{H,A}(h),t)} = \iota_{t^H(h)} \restriction G$$

and this implies

$$t^H(h) \in \mathrm{C}_H(G) \subseteq \mathrm{C}_H(A) = \{1_H\}$$

and $t \in \mathsf{qft}_{H,A}(h)$.

Next, fix $t_0, t_1 \in \mathcal{T}^1_A$ and $g_0, g_1, g_2 \in G$ with

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{IV}}(\mathsf{qft}_{G,A}(g_{i+1}), \mathrm{E}_A(\mathsf{qft}_{H,A}(h), t_i), \mathsf{qft}_{G,A}(g_i), \mathsf{qft}_{G,A}(1_G))$$

for all $i < 2$. Then

$$\sigma_{C,\mathrm{E}_A(\mathsf{qft}_{H,A}(h),t_i)}(g_i) = \iota_{t_i^H(h)}(g_i) = g_{i+1}$$

for all $i < 2$ and therefore

$$\begin{aligned}\sigma_{C,\mathrm{E}_A(\mathsf{qft}_{H,A}(h),t_1 * t_0)}(g_0) &= \sigma_{C,\mathsf{qft}_{H,A}(t_1^H(h)\cdot t_0^H(h))}(g_0) \\ &= (\iota_{t_1^H(h)} \circ \iota_{t_0^H(h)})(g_0) = g_2\end{aligned}$$

and we can conclude

$$\mathcal{M}_A(C) \models \Phi_{\mathrm{IV}}(\mathsf{qft}_{G,A}(g_2), \mathrm{E}_A(\mathsf{qft}_{H,A}(h), t_1 * t_0), \mathsf{qft}_{G,A}(g_0), \mathsf{qft}_{G,A}(1_G)).$$

Finally, if $a \in A$ and $g \in G$, then

$$\mathsf{qft}_{G,A}(\sigma_{C,\mathrm{E}_A(\mathsf{qft}_{H,A}(h),\dot{a})}(g)) = \mathsf{qft}_{G,A}(\iota_a(g)) = \mathrm{E}_A(\mathsf{qft}_{G,A}(g), \dot{a} * v_0 * \dot{a}^{-1})$$

and this shows that the last clause in the definition of Φ_{VI} also holds in this case.

(2) Fix $x \in \mathcal{P}(\mathcal{T}^2_A)$ with $\mathcal{M}_A(C) \models \Phi_{\mathrm{VI}}(x, \mathsf{qft}_{G,A}(1_G))$. Since $x = \mathrm{E}_A(x, v_0)$ and $\mathrm{I}_A(x) = \mathrm{E}_A(x, v_0^{-1})$, the first part of the definition of Φ_{VI} ensures that the functions $\sigma_{C,x}$, $\sigma_{C,\mathrm{I}_A(x)}$ and $\sigma_{C,\mathrm{E}_A(x,\mathbb{1})}$ are defined. By the third clause in the definition of Φ_{VI} and Proposition 2.2.9, we know that

$$\sigma_{C,x} \circ \sigma_{C,\mathrm{I}_A(x)} \;=\; \sigma_{C,\mathrm{E}_A(x,\mathbb{1})} \;=\; \sigma_{C,\mathrm{I}_A(x)} \circ \sigma_{C,x}.$$

Since $\mathbb{1} \in x$, the second clause in the definition of Φ_{VI} implies $\sigma_{C,\mathrm{E}_A(x,\mathbb{1})} = \mathrm{id}_G$ and we can conclude $\sigma_{C,x} \in \mathrm{Aut}(G)$.

Now assume that $h \in H$ satisfies

$$\iota_h \restriction G = \sigma_{C,x} = \sigma_{C,\mathrm{E}_A(x,v_0)}.$$

The last clause in the definition of Φ_{VI} implies that $\sigma_{C,\mathrm{E}_A(x,\dot{a})} = \iota_a \restriction G$ holds for all $a \in A$ and a trivial modification of the above argument shows that $\sigma_{C,\mathrm{E}_A(x,t^{-1})} = \sigma^{-1}_{C,\mathrm{E}_A(x,t)}$ holds for all $t \in \mathcal{T}^1_A$. This allows us to use the third clause in the definition of Φ_{VI} to see that

$$\sigma_{C,\mathrm{E}_A(x,t)} = \iota_{t^H(h)} \restriction G$$

holds for all $t \in \mathcal{T}^1_A$. We can conclude that

$$t \in x \Leftrightarrow \sigma_{C,\mathrm{E}_A(x,t)} = \mathrm{id}_G \Leftrightarrow \iota_{t^H(h)} \restriction G = \mathrm{id}_G \Leftrightarrow t^H(h) = 1_H \Leftrightarrow t \in \mathsf{qft}_{H,A}(h)$$

holds for all $t \in \mathcal{T}^1_A$. $\square$

7. Define $\Phi \equiv \Phi(w_0, \dots, w_3)$ to be the conjunction of the following $\mathcal{L}_A$-statements.

- If $\neg\dot{R}(w_3, \dots, w_3)$, then there are $a_0, a_1, a_2 \in A$ with the property that $\dot{a}_0 * \dot{a}_1 * \dot{a}_2^{-1} \in w_3$ and $x_i = \mathrm{E}_A(w_3, \dot{a}_i)$ for all $i < 3$.
- If $\dot{R}(w_3, \dots, w_3)$, then the following statements hold.
  - $\Phi_{\mathrm{VI}}(w_i, w_3)$ for all $i < 3$.
  - If $y_0, y_1, y_2 \in \mathcal{P}(\mathcal{T}^1_A)$ with $\Phi_{\mathrm{IV}}(y_{2-i}, w_i, y_{1-i}, w_3)$ for all $i < 2$ and $\Phi_{\mathrm{I}}(y_i, w_3)$ for all $i < 3$, then $\Phi_{\mathrm{IV}}(y_2, w_2, y_0, w_3)$.

Proof of Theorem 2.3.4. (1) We prove that the special pair $\langle G_\alpha, A\rangle$ is coded by I^Φ_α by induction on α.

Let $\alpha = 0$ and $g_0, g_1, g_2 \in G$ with $g_0 \cdot g_1 = g_2$. Then

$$\dot{g}_0 * \dot{g}_1 * \dot{g}_2^{-1} \in \mathsf{qft}_{G,A}(1_G)$$

and $\mathsf{qft}_{G,A}(g_i) = \mathrm{E}_A(\mathsf{qft}_{G,A}(1_G), \dot{g}_i)$ for all $i < 3$. This implies

$$\mathcal{M}_A(\emptyset) \models \Phi(\mathsf{qft}_{G,A}(g_0), \mathsf{qft}_{G,A}(g_1), \mathsf{qft}_{G,A}(g_2), \mathsf{qft}_{G,A}(1_G)).$$

In the other direction assume that $\mathcal{M}_A(\emptyset) \models \Phi(x_0, x_1, x_2, \mathsf{qft}_{G,A}(1_G))$. Then there are $a_0, a_1, a_2 \in A$ such that $\dot{a}_0 * \dot{a}_1 * \dot{a}_2^{-1} \in \mathsf{qft}_{G,A}(1_G)$ and

$$x_i = \mathrm{E}_A(\mathsf{qft}_{G,A}(1_G), \dot{a}_i) = \mathsf{qft}_{G,A}(a_i)$$

for all $i < 3$. In particular, $a_0 \cdot_G a_1 = a_2$.

Now, let $\alpha = \beta + 1$ and assume that the special pair $\langle G_\beta, A\rangle$ is coded by the set I^Φ_β. Then $\langle \mathsf{qft}_{G,A}(1_G), \dots, \mathsf{qft}_{G,A}(1_G)\rangle \in I^\Phi_\beta$.

Let $h_0, h_1, h_2 \in G_{\beta+1}$ with $h_0 \cdot_{G_{\beta+1}} h_1 = h_2$. By Corollary 2.2.6 and Proposition 2.3.12, we have $\mathcal{M}_A(I^\Phi_\beta) \models \Phi_{\mathrm{VI}}(\mathsf{qft}_{G_{\beta+1},A}(h_i), \mathsf{qft}_{G,A}(1_G))$ and $\sigma_{I^\Phi_\beta, \mathsf{qft}_{G_{\beta+1},A}(h_i)} = \iota_{h_i} \restriction G_\beta$ for all $i < 3$. In particular,

$$\sigma_{I^\Phi_\beta, \mathsf{qft}_{G_{\beta+1},A}(h_2)} = \sigma_{I^\Phi_\beta, \mathsf{qft}_{G_{\beta+1},A}(h_0)} \circ \sigma_{I^\Phi_\beta, \mathsf{qft}_{G_{\beta+1},A}(h_1)}.$$

Since $\langle \mathsf{qft}_{G,A}(1_G), \dots, \mathsf{qft}_{G,A}(1_G)\rangle \in I^\Phi_\beta$, we can use Lemma 2.3.10 to get

$$\mathcal{M}_A(I^\Phi_\beta) \models \Phi(\mathsf{qft}_{G_{\beta+1},A}(h_0), \mathsf{qft}_{G_{\beta+1},A}(h_1), \mathsf{qft}_{G_{\beta+1},A}(h_2), \mathsf{qft}_{G,A}(1_G)).$$

Pick $x_0, x_1, x_2 \in \mathcal{P}(\mathcal{T}^2_A)$ with $\langle x_0, x_1, x_2, \mathsf{qft}_{G,A}(1_G)\rangle \in I^\Phi_{\beta+1}$. If the tuple is an element of I^Φ_β, then the induction hypothesis gives us $g_0, g_1, g_2 \in G_\beta \subseteq G_{\beta+1}$ with $x_i = \mathsf{qft}_{G_\beta,A}(g_i) = \mathsf{qft}_{G_{\beta+1},A}(g_i)$ for all $i < 3$. We may therefore assume

$$\mathcal{M}_A(I^\Phi_\beta) \models \Phi(x_0, x_1, x_2, \mathsf{qft}_{G,A}(1_G)).$$

By Proposition 2.3.12, the maps $\sigma_{I^\Phi_\beta, x_i}$ are well-defined and elements of $\mathrm{Aut}(G_\beta)$. The construction of automorphism towers gives us $h_0, h_1, h_2 \in G_{\beta+1}$ with $\sigma_{I^\Phi_\beta, x_i} = \iota_{h_i} \restriction G_\beta$ for all $i < 3$. By combining Corollary 2.2.6 and Proposition 2.3.12 we can conclude that $x_i = \mathsf{qft}_{G_{\beta+1},A}(h_i)$ holds for $i < 3$. The last clause in the definition of Φ ensures that $\sigma_{I^\Phi_\beta, x_0} \circ \sigma_{I^\Phi_\beta, x_1} = \sigma_{I^\Phi_\beta, x_2}$ and, by the definition of the successor stage in the definition of automorphism towers, we can conclude $h_0 \cdot_{G_{\beta+1}} h_1 = h_2$.

Finally, if α is a limit ordinal and the special pair $\langle G_\beta, A\rangle$ is coded by I^Φ_β for all $\beta < \alpha$, then the statement follows directly from the induction hypothesis and Corollary 2.2.4.

(2) Define $\Psi \equiv \Psi(w_0, \ldots, w_4)$ to be the $\mathcal{L}^A$-formula

$$(\exists z)\ \big[\Phi_{\mathrm{III}}(w_3, w_0, z, w_4) \wedge w_1 = \mathrm{E}_A(z, v_0{*}v_1{*}v_0^{-1}) \wedge w_2 = \mathrm{E}_A(z, v_0^{-1}{*}v_1{*}v_0)\big].$$

Let $\lambda \in \mathrm{Lim}$, $h \in G_{\lambda+1}$ and $g \in G_\lambda$ such that there is an $\alpha < \lambda$ with $t^{G_{\lambda+1}}(h,g) \in G_\alpha$ for every $t \in \mathrm{N}_A$. Then $g, \iota_h(g), \iota_{h^{-1}}(g) \in G_\alpha$. By Proposition 2.3.8, we have

$$\mathcal{M}_A(I^\Phi_\alpha) \models \Phi_{\mathrm{III}}(\mathsf{qft}_{G_{\lambda+1},A}(h), \mathsf{qft}_{G_\alpha,A}(g), \mathsf{qft}_{G_{\lambda+1},A}(h,g), \mathsf{qft}_{G,A}(1_G))$$

and Proposition 2.2.7 implies

$$\mathrm{E}_A(\mathsf{qft}_{G_{\lambda+1},A}(h,g), v_0 * v_1 * v_0^{-1}) = \mathsf{qft}_{G_\lambda,A}(\iota_h(g))$$

and

$$\mathrm{E}_A(\mathsf{qft}_{G_{\lambda+1},A}(h,g), v_0^{-1} * v_1 * v_0) = \mathsf{qft}_{G_\lambda,A}(\iota_{h^{-1}}(g)).$$

The combination of these statements shows that

$$\Psi(\mathsf{qft}_{G_\lambda,A}(g), \mathsf{qft}_{G_\lambda,A}(\iota_h(g)), \mathsf{qft}_{G_\lambda,A}(\iota_{h^{-1}}(g)), \mathsf{qft}_{G_{\lambda+1},A}(h), \mathsf{qft}_{G,A}(1_G))$$

holds in $\mathcal{M}_A(I^\Phi_\alpha)$.

Assume that $g_0, g_1 \in G_\lambda$, $\bar\alpha < \lambda$ and $z \in \mathcal{P}(\mathcal{T}^2_A)$ with

$$\mathcal{M}_A(I^\Phi_{\bar\alpha}) \models \Phi_{\mathrm{III}}(\mathsf{qft}_{G_{\lambda+1},A}(h), \mathsf{qft}_{G_\lambda}(g), z, \mathsf{qft}_{G,A}(1_G)),$$

$\mathrm{E}_A(z, v_0{*}v_1{*}v_0^{-1}) = \mathsf{qft}_{G_\lambda,A}(g_0)$ and $\mathrm{E}_A(z, v_0^{-1}{*}v_1{*}v_0) = \mathsf{qft}_{G_\lambda,A}(g_1)$. Then $g \in G_{\bar\alpha}$ by the definition of Φ_{III} and

$$z \cap \mathcal{T}^1_A = \mathrm{E}_A(z, v_0) = \mathsf{qft}_{G_{\lambda+1},A}(h) = \mathsf{qft}_{G_{\lambda+1},A}(h,g) \cap \mathcal{T}^1_A.$$

The subgroup G_z and the map π_z, as defined in Lemma 2.3.7, exist and there is a monomorphism

$$\varphi_0 : \langle A \cup \{\pi_z^{-1}(g)\}\rangle_{G_{\bar\alpha}} \longrightarrow G_{\bar\alpha}$$

with $\varphi_0(\pi_z^{-1}(g)) = g$ and

$$\begin{aligned}
\mathsf{qft}_{G_{\bar\alpha},A}(\varphi_0(a)) &= \mathrm{E}_A(z, v_0 * \dot a * v_0^{-1}) \\
&= \{t \in \mathcal{T}^1_A \mid t^{v_0}_{v_0*\dot a*v_0^{-1}} \in z \cap \mathcal{T}^1_A\} \\
&= \{t \in \mathcal{T}^1_A \mid t^{v_0}_{v_0*\dot a*v_0^{-1}} \in \mathsf{qft}_{G_{\lambda+1},A}(h,g) \cap \mathcal{T}^1_A\} \\
&= \mathrm{E}_A(\mathsf{qft}_{G_{\lambda+1},A}(h,g), v_0 * \dot a * v_0^{-1}) \\
&= \mathsf{qft}_{G_\lambda,A}(\iota_h(a))
\end{aligned}$$

for all $a \in A$. If we define

$$\varphi : \langle A \cup \{\pi_z^{-1}(g)\}\rangle_{G_\lambda} \longrightarrow G_\lambda;\ k \longmapsto (\iota_{h^{-1}} \circ \varphi_0)(k),$$

then $\varphi(\pi_z^{-1}(g)) = \iota_{h^{-1}}(g)$ and $\varphi \upharpoonright A = \mathrm{id}_A$. Since $\langle G_\lambda, A\rangle$ is a special pair, we can apply Lemma 2.2.3 to see that $\pi_z^{-1}(g) = \iota_{h^{-1}}(g)$ and Lemma 2.3.7 yields

$$\mathsf{qft}_{G_\lambda,A}(\iota_{h^{-1}}(g)) = \mathsf{qft}_{G_{\bar\alpha},A}(\pi_z^{-1}(g)) = \mathrm{E}_A(z, v_0^{-1} * v_1 * v_0) = \mathsf{qft}_{G_\lambda,A}(g_1).$$

We can conclude $g_1 = \iota_{h^{-1}}(g)$. The equality $g_0 = \iota_h(g)$ can be derived in the same way. □

2.4. Admissible set theory and automorphism towers

In this section, we will apply admissible set theory to the results of the last section to prove Theorem 1.3.3. As above, **all results in this section can be derived from the axioms of** ZF.

In the following, we present three basic results from admissible set theory. The proofs of these results can be found in [**Jen72**, Section 2.3] or [**Mos74**, Section 9D]. The first result lists the closure properties of the class of Σ_1-definable subsets of an admissible set.

PROPOSITION 2.4.1. *Let M be an admissible set.*

(1) *If $\varphi \equiv \varphi(v_0,\dots,v_{n+2})$ is a Δ_0-formula, then*

$$(\forall x_0,\dots,x_n)\ \big[(\forall x_{n+1}\in x_0)(\exists x_{n+2})\ \varphi(x_0,\dots,x_{n+2}) \longleftrightarrow (\exists y)(\forall x_{n+1}\in x_0)(\exists x_{n+2}\in y)\ \varphi(x_0,\dots,x_{n+2})\big]$$

holds in $\langle M,\in\rangle$.

(2) *The class of subsets of finite products of M that are definable in the structure $\langle M,\in\rangle$ by a Σ_1-formula with parameters is closed under intersections, unions, restricted universal quantification and existential quantification over $\langle M,\in\rangle$.*

(3) *If X is a subset of M^n that is definable in $\langle M,\in\rangle$ by a Σ_1-formula with parameters, then X is definable in $\langle M,\in\rangle$ by a Σ_1-formula and a single parameter $y\in M$.*

By collecting witnesses for the validity of Σ_1-statements, it is easy to see that Σ_1-Collection holds in admissible sets.

PROPOSITION 2.4.2 (Σ_1-Collection). *If M is an admissible set, then Σ_1-Collection holds in $\langle M,\in\rangle$, i.e. the sentence*

$$(\forall x_0,\dots,x_n)\big[(\forall y\in x_0)(\exists z)\ \varphi(x_0,\dots,x_n,y,z) \longrightarrow (\exists w)(\forall y\in x_0)(\exists z\in w)\ \varphi(x_0,\dots,x_n,y,z)\big]$$

holds in $\langle M,\in\rangle$ for every Σ_1-formula $\varphi\equiv\varphi(v_0,\dots,v_{n+2})$.

The following *Recursion Theorem* may be viewed as the motivation behind the definition of admissible sets and the axioms of KP.

THEOREM 2.4.3 (Σ_1-Recursion Principle). *Let M be an admissible set with $\alpha = M\cap\mathrm{On}$ and $\prec$ be a well-founded relation on M such that the set*

$\mathrm{prec}_{\prec}(x) = \{y \in M \mid y \prec x\}$ *is an element of* M *for every* $x \in M$ *and the function*

$$r : M \longrightarrow M;\ x \longmapsto \mathrm{prec}_{\prec}(x)$$

is definable in $\langle M, \in\rangle$ *by a* Σ_1*-formula with parameters.*

If $f : M \times M \longrightarrow M$ *is a function that is definable in* $\langle M, \in\rangle$ *by a* Σ_1*-formula with parameters, then there is a function* $F : M \longrightarrow M$ *such that the following statements hold.*

(1) *If* $x \in M$*, then* $F \restriction \mathrm{prec}_{\prec}(x) \in M$ *and* $F(x) = f(F \restriction \mathrm{prec}_{\prec}(x), x)$*.*
(2) F *is definable in the structure* $\langle M, \in\rangle$ *by a* Σ_1*-formula with parameters.*

We derive some consequences from the above results that are important in our context. First, we we show that Theorem 1.3.9 directly follows from Proposition 2.4.2.

Proof of Theorem 1.3.9. Let $f : M \longrightarrow \alpha$ be a function that is defined in $\langle M, \in\rangle$ by the Σ_1-formula $\varphi \equiv \varphi(v_0, v_1, w_0, \dots, w_{m-1})$ and parameters $z_0, \dots, z_{m-1} \in M$. Given $X \in M$, Proposition 2.4.2 shows that there is a $Y \in M$ with

$$\langle M, \in\rangle \models (\forall x \in X)(\exists y \in Y)\ \varphi(x, y, z_0, \dots, z_{m-1}).$$

By Δ_0-Separation, the set $\bar{Y} = Y \cap \mathrm{On}$ is an element of M. Since M is closed under unions, we have $\beta = \bigcup \bar{Y} \in M \cap \mathrm{On} = \alpha$ and $\beta + 1 < \alpha$ contains the image of X under f. □

As a corollary of the Σ_1-Recursion Principle 2.4.3, we can conclude that x-admissible ordinals are closed under the Gödel-Pairing function.

Corollary 2.4.4. *Let* M *be an admissible set and* $\alpha = M \cap \mathrm{On}$*. Then* α *is closed under Gödel-Pairing and the function* $\prec\cdot,\cdot\succ \restriction (A \times A)$ *is an element of* M *for every* $A \in M \cap \mathcal{P}(\alpha)$*.* □

In the following, we use the Σ_1-Recursion Principle to show that the stages of an inductive definition on a structures contained in an admissible set can be computed inside this set up to its ordinal height.

The statement of the following proposition follows from an easy induction and the iterated application of Proposition 2.4.1.

Proposition 2.4.5. *Let* $\mathcal{L}$ *be a finite first-order language,* $\mathcal{N}$ *be an* $\mathcal{L}$*-structure with domain* N *and* $\varphi \equiv \varphi(v_0, \dots, v_{n-1})$ *be an* $\mathcal{L}^n_{\mathcal{N}}$*-formula. If* M *is an admissible set with* $\mathcal{N} \in M$*, then there is a* Δ_0*-formula* $\varphi_* \equiv \varphi_*(v_0, \dots, v_n)$ *and a parameter* $y \in M$ *such that the following statements are equivalent for all* $x_0, \dots, x_{n-1} \in N$ *and* $X \in M \cap \mathcal{P}(N^n)$*.*

(1) $\mathcal{N}(X) \models \varphi(x_0, \dots, x_{n-1})$.
(2) $\langle M, \in\rangle \models \varphi_*(x_0, \dots, x_{n-1}, y, X)$. □

Lemma 2.4.6. *Let* $\mathcal{L}$ *be a finite-first order language,* $\mathcal{N}$ *be an* $\mathcal{L}$*-structure with domain* N *and* $\varphi \equiv \varphi(v_0, \dots, v_{n-1})$ *be an* $\mathcal{L}^n_{\mathcal{N}}$*-formula. If* M *is an* *admissible set with $\mathcal{N} \in M$ and $\alpha = M \cap \mathrm{On}$, then $I^\varphi_\beta \in M$ for all $\beta < \alpha$ and the function*

$$F : \alpha \longrightarrow M;\ \beta \longmapsto I^\varphi_\beta$$

is definable in the structure $\langle M, \in \rangle$ by a Σ_1-formula with parameters.

Proof. Let $\varphi_* \equiv \varphi_*(v_0, \dots, v_n)$ and $y \in M$ be the objects produced by Proposition 2.4.5 with respect to φ and M. Let $f : M \times \alpha \longrightarrow \mathrm{V}$ be the function defined by the following clauses

$$f(g, \beta) := \begin{cases} \{\vec{x} \in N^n \mid \langle M, \in \rangle \models \varphi_*(\vec{x}, y, \emptyset)\}, & \text{if } \beta = 0, \\ g(\bar{\beta}) \cup \{\vec{x} \in N^n \mid \langle M, \in \rangle \models \varphi_*(\vec{x}, y, g(\bar{\beta}))\}, & \text{if } \beta = \bar{\beta} + 1, \\ \bigcup \mathrm{ran}(f), & \text{if } \beta \in \mathrm{Lim}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We have $f(g, \beta) \in M$ for every pair $\langle g, \beta \rangle \in M \times \alpha$, because M is closed under taking unions and satisfies Δ_0-Separation. Moreover, f is definable in the structure $\langle M, \in \rangle$ by a Δ_0-formula with parameters. If $F : \alpha \longrightarrow M$ is the corresponding function produced by Theorem 2.4.3, then an easy induction shows that $F(\beta) = I^\varphi_\beta$ holds for all $\beta < \alpha$. □

We are now ready to translate the statement of Theorem 2.3.4 to the language of admissible sets and prove Theorem 2.1.4.

Proof of Theorem 2.1.4. Fix an infinite cardinal κ and let $\mathcal{G}_\kappa$ denote the set of all centreless groups with domain κ. We let $\Phi \equiv \Phi(w_0, \dots, w_3)$ and $\Psi \equiv \Psi(w_0, \dots, w_4)$ be the $\mathcal{L}^\kappa$-formulae produced by Theorem 2.3.4.

Given $t \in \mathcal{T}^2_\kappa$, we define the ordinal $\alpha_t < \kappa$ as in Section 2.3. Set

$$B = \{\alpha_t < \kappa \mid t \in \mathcal{T}^2_\kappa\}$$

and define a bijection $\mathfrak{b} : \mathcal{T}^2_\kappa \cup \mathcal{P}(\mathcal{T}^2_\kappa) \longrightarrow B \sqcup \mathcal{P}(B)$ by

$$\mathfrak{b}(x) := \begin{cases} \langle 0, \alpha_t \rangle, & \text{if } x \in \mathcal{T}^2_\kappa, \\ \langle 1, \{\alpha_t < \kappa \mid t \in x\} \rangle, & \text{if } x \in \mathcal{P}(\mathcal{T}^2_\kappa). \end{cases}$$

We let $\mathcal{N}_\kappa$ denote the unique $\mathcal{L}_C$-structure with domain $B \sqcup \mathcal{P}(B)$ and the property that $\mathfrak{b}$ is an isomorphism of $\mathcal{M}_\kappa$ and $\mathcal{N}_\kappa$. Let $\Phi_0 \equiv \Phi_0(w_0, \dots, w_3)$ be the $(\mathcal{L}_\kappa)^4_{\mathcal{N}_\kappa}$-formula corresponding to Φ with respect to $\mathfrak{b}$ and $\Psi_0 \equiv \Psi_0(w_0, \dots, w_4)$ be the formula corresponding to Ψ. Finally, define

$$\mathfrak{c} : \mathcal{G}_\kappa \longrightarrow B \sqcup \mathcal{P}(B);\ G \longmapsto \mathfrak{b}(\mathsf{qft}_{G,\kappa}(1_G)).$$

Let M be an admissible set with $\mathcal{P}(\kappa) \in M$ and $\alpha = M \cap \mathrm{On}$. Then the structure $\mathcal{N}_\kappa$ is an element of M, because $B \sqcup \mathcal{P}(B) \in M$,

$$\langle i, x \rangle \in^{\mathcal{N}_\kappa} \langle j, y \rangle \Leftrightarrow [i = 0 \wedge j = 1 \wedge x \in y]$$

for all $\langle i, x \rangle, \langle j, y \rangle \in B \sqcup \mathcal{P}(B)$ and

$$\dot{C}^{\mathcal{N}_\kappa} = \{\langle \langle 0, \alpha_0 \rangle, \langle 0, \alpha_1 \rangle, \langle 0, \alpha_2 \rangle \rangle \in B \sqcup \mathcal{P}(B) \mid \alpha_0 = \prec 2, \prec \alpha_1, \alpha_2 \succ \succ\}.$$

Moreover, we have $\mathrm{ran}(\mathfrak{c}) \in M$, because this set is definable in the structure $\mathcal{N}_\kappa$ by Proposition 2.2.12.

Let $\mathcal{T}$ denote the set of all $\langle \beta, X, x\rangle \in \alpha \times M \times \operatorname{ran}(\mathfrak{c})$ with

$$X = \{y \in B \sqcup \mathcal{P}(B) \mid \langle y, x, y, x\rangle \in I_\beta^{\Phi_0}\}.$$

By Lemma 2.4.6 and Proposition 2.4.1, the set $\mathcal{T}$ is definable in the structure $\langle M, \in\rangle$ by a Σ_1-formula $\Phi_* \equiv \Phi_*(w_0, \dots, w_3)$ and a parameter $y \in M$.

Let $\mathcal{A}$ denote the set of all tuples $\langle x_0, \dots, x_3, x\rangle \in M^5$ such that $x \in \operatorname{ran}(\mathfrak{c})$, $x_0, \dots, x_3 \in B \sqcup \mathcal{P}(B)$ and

$$\mathcal{N}_\kappa(I_\beta^{\Phi_0}) \models \Psi_0(x_0, \dots, x_3, x)$$

holds for some $\beta < \alpha$. By Proposition 2.4.1, Proposition 2.4.5 and Lemma 2.4.6, there is a Σ_1-formula $\Psi_* \equiv \Psi_*(w_0, \dots, w_5)$ and a parameter $z \in M$ that define the set $\mathcal{A}$ in $\langle M, \in\rangle$.

Let G be an element of $\mathcal{G}_\kappa$ and $\langle \bar{G}_\beta \mid \beta \in \mathrm{On}\rangle$ be an automorphism tower of G. By Theorem 2.2.5, the special pair $\langle \bar{G}_\beta, \kappa\rangle$ is coded by I_β^Φ for all $\beta \in \mathrm{On}$. Given $\beta \in \mathrm{On}$, there is a unique group G_β with domain

$$\{\mathfrak{b}(\mathsf{qft}_{\bar{G}_\beta,\kappa}(\bar{g})) \in B \sqcup \mathcal{P}(B) \mid g \in \bar{G}_\beta\}$$

and the property that the function

$$\mathfrak{b}_{G_\beta} : \bar{G}_\beta \longrightarrow G_\beta;\ \bar{g} \longmapsto \mathfrak{b}(\mathsf{qft}_{\bar{G}_\beta,\kappa}(\bar{g}))$$

is an isomorphism of groups. In particular, G is isomorphic to G_0, the sequence $\langle G_\beta \mid \beta \in \mathrm{On}\rangle$ is an automorphism tower of G_0 and Corollary 2.2.4 shows that $\mathfrak{b}_{G_\gamma} \restriction \bar{G}_\beta = \mathfrak{b}_{G_\beta}$ holds for all $\beta < \gamma$.

Pick $\beta < \alpha$. Then Lemma 2.4.6 implies that the set

$$X = \{x \in \kappa \sqcup \mathcal{P}(\kappa) \mid \langle x, \mathfrak{c}(G), x, \mathfrak{c}(G)\rangle \in I_\beta^{\Phi_0}\}$$

is an element of M and X is the unique set in M such that $\Phi_*(\beta, X, \mathfrak{c}(G))$ holds in $\langle M, \in\rangle$. Given an arbitrary $x \in M$, we have

$$\begin{aligned} x \in G_\beta \ &\Leftrightarrow\ (\exists \bar{g} \in \bar{G}_\beta)\ x = \mathfrak{b}(\mathsf{qft}_{\bar{G}_\beta,\kappa}(\bar{g})) \\ &\Leftrightarrow\ (\exists \bar{x} \in \mathcal{P}(\mathcal{T}_\kappa^1))\ [x = \mathfrak{b}(\bar{x}) \wedge \langle \bar{x}, \mathsf{qft}_{G,\kappa}(1_G), \bar{x}, \mathsf{qft}_{G,\kappa}(1_G)\rangle \in I_\beta^\Phi] \\ &\Leftrightarrow\ \langle x, \mathfrak{c}(G), x, \mathfrak{c}(G)\rangle \in I_\beta^{\Phi_0} \\ &\Leftrightarrow\ x \in X. \end{aligned}$$

This shows that the set X is equal to the domain of G_β.

Fix $h \in G_{\alpha+1}$ and $g \in G_\alpha$. Define $\bar{h} = \mathfrak{b}_{G_{\alpha+1}}^{-1}(h)$, $\bar{g} = \mathfrak{b}_{G_\alpha}^{-1}(g)$ and $\bar{z} = \mathsf{qft}_{\bar{G}_{\alpha+1},\kappa}(\bar{h}, \bar{g})$. Let $\Xi_0 \equiv \Xi_(v_0, v_1, v_2)$ be the $(\mathcal{L}_\kappa)^4_{\mathcal{N}_\kappa}$-formula corresponding to the $\mathcal{L}^\kappa$-formula $\Xi(v_0, v_1, v_2) \equiv \Phi_{\mathrm{I}}(\mathrm{E}_\kappa(v_0, v_1), v_2)$.

If we define

$$n = \{t \in \mathcal{T}_\kappa^2 \mid t^{\bar{G}_{\alpha+1}}(\bar{h}, \bar{g}) \in \bar{G}_\alpha\},$$

then $v_1 \in n$, $\dot{a} \in n$ for every $a \in \kappa$ and $n \in \mathrm{N}_\kappa^2$. This shows $\mathrm{N}_\kappa \subseteq n$ and $t^{\bar{G}_{\alpha+1}}(\bar{h}, \bar{g}) \in \bar{G}_\alpha$ for every $t \in \mathrm{N}_\kappa$. In particular, for every $x \in \mathfrak{b}(\mathrm{N}_\kappa)$, there is an ordinal $\beta < \alpha$ such that $\Xi_0(\mathfrak{b}(\bar{z}), x, \mathfrak{c}(G))$ holds in $\mathcal{N}_\kappa(I_\beta^{\Phi_0})$.

Let $f : M \longrightarrow \alpha$ be the function with

$$f(x) := \begin{cases} \min\{\beta < \alpha \mid \mathcal{N}_\kappa(I_\beta^{\Phi_0}) \models \Xi_0(\mathfrak{b}(\bar{z}), x, \mathfrak{c}(G))\}, & \text{if } x \in \mathfrak{b}(\mathrm{N}_\kappa), \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.4.1, Proposition 2.4.5 and Lemma 2.4.6 show that f is definable in the structure $\langle M, \in\rangle$ by a Σ_1-formula with parameters and Theorem 1.3.9 gives us an $\alpha_* < \alpha$ with $f"\mathfrak{b}(\mathrm{N}_\kappa) \subseteq \alpha_*$. We can conclude that $t^{\bar{G}_{\alpha+1}}(\bar{h}, \bar{g}) \in \bar{G}_{\alpha_*}$ for every $t \in \mathrm{N}_\kappa$. By Theorem 2.3.4, this implies that $\langle \iota_{\bar{h}}(\bar{g}), \iota_{\bar{h}^{-1}}(g)\rangle$ is the unique pair $\langle \bar{g}_0, \bar{g}_1\rangle$ in $\bar{G}_\alpha \times \bar{G}_\alpha$ such that there is a $\beta < \alpha$ with

$$\mathcal{M}_\kappa(I_\beta^\Phi) \models \Psi(\mathsf{qft}_{\bar{G}_\alpha,\kappa}(\bar{g}), \mathsf{qft}_{\bar{G}_\alpha,\kappa}(\bar{g}_0), \mathsf{qft}_{\bar{G}_\alpha,\kappa}(\bar{g}_1), \mathsf{qft}_{\bar{G}_{\alpha+1},\kappa}(\bar{h}), \mathsf{qft}_{G,\kappa}(1_G))$$

But this means that $\langle \iota_h(g), \iota_{h^{-1}}\rangle$ is the unique pair $\langle g_0, g_1\rangle \in G_\alpha \times G_\alpha$ such that $\Psi_*(g, g_0, g_1, h, \mathfrak{c}(G), z)$ holds in $\langle M, \in\rangle$. □

We end this chapter with the proof of Theorem 1.3.3. This proof strongly resembles the argument sketched at the end of Section 2.1. We replace the application of the *Countable Axiom of Choice* in that argument by another application of the Σ_1-Recursion Principle 2.4.3 to derive the statement of the theorem from the axioms of ZF.

Proof of Theorem 1.3.3. Let κ be an infinite cardinal, $\mathcal{G}_\kappa$ be the set of all centreless groups with domain κ and M be an admissible set with $\mathcal{P}(\kappa) \in M$ and $\alpha = M \cap \mathrm{On}$. We let Φ_*, Ψ_*, y, z and $\mathfrak{c}$ denote the objects produced by an application of Theorem 2.1.4.

Let G be an element of $\mathcal{G}_\kappa$ and $\langle G_\beta \mid \beta \in \mathrm{On}\rangle$ denote the automorphism tower produced by Theorem 2.1.4 with respect to G. Fix an $h \in G_{\alpha+1}$ and define $\mathcal{A}_h$ to be the set of all pairs $\langle g, \beta\rangle \in G_\alpha \times \alpha$ with the property that $\iota_h(g), \iota_{h^{-1}}(g) \in G_\gamma$. By Theorem 2.1.4, $\mathcal{A}_h$ is definable in $\langle M, \in\rangle$ by a Σ_1-formula with parameters.

For all $\beta < \alpha$, we define

$$f_\beta : G_\beta \longrightarrow \alpha;\ g \longmapsto \min\{\gamma < \alpha \mid \langle g, \gamma\rangle \in \mathcal{A}_h\}.$$

By the above remarks, every function of the form f_β is definable in $\langle M, \in\rangle$ by a Σ_1-formula with parameters and Theorem 1.3.9 implies that for every $\beta < \alpha$ there is a $\gamma < \alpha$ with the property that $\langle g, \gamma\rangle \in \mathcal{A}_h$ holds for every $g \in G_\beta$. This shows that there is a unique function $f_h : \omega \longrightarrow \alpha$ with $f_h(0) = 0$ and

$$f_h(n+1) = \min\{\gamma \in (f_h(n), \alpha) \mid (\forall g \in G_{f_h(n)})\ \langle g, \gamma\rangle \in \mathcal{A}_h\}.$$

By the above computations and the Σ_1-Recursion Principle 2.4.3, the function f_h is definable in $\langle M, \in\rangle$ by a Σ_1-formula with parameters and Theorem 1.3.9 implies

$$\alpha_* = \sup_{n<\omega} f_h(n) \in \alpha \cap \mathrm{Lim}.$$

If $g \in G_{\alpha_*}$, then there is an $n < \omega$ with $g \in G_{f_h(n)}$ and $\iota_h(g), \iota_{h^{-1}}(g) \in G_{f_h(n+1)} \subseteq G_{\alpha_*}$. This shows that $\iota_h \restriction G_{\alpha_*} \in \mathrm{Aut}(G_{\alpha_*})$ and there is an

$h_* \in G_{\alpha_*+1}$ with $\iota_{h_*} \upharpoonright G_{\alpha_*} = \iota_h \upharpoonright G_{\alpha_*}$. By Theorem 2.2.5, this implies $h = h_* \in G_{\alpha_*+1} \subseteq G_\alpha$.

The above argument shows that $\tau_\kappa \leq \alpha + 1$.

Now, assume that $\tau(G) < \alpha$ holds for every G in $\mathcal{G}_\kappa$. If $G \in \mathcal{G}_\kappa$, then this assumption and Theorem 2.1.4 imply that $\tau(G)$ is the minimal ordinal $\beta < \alpha$ with

$$\langle M, \in \rangle \models (\exists X) \ \left[\Phi(\beta, X, \mathfrak{c}(G), y) \wedge \Phi(\beta + 1, X, \mathfrak{c}(G), y)\right].$$

This shows that the function

$$c : \mathrm{ran}(\mathfrak{c}) \longrightarrow \alpha; \ \mathfrak{c}(G) \longmapsto \tau(G)$$

is definable in $\langle M, \in \rangle$ by a Σ_1-formula with parameters. A final application of Theorem 1.3.9 yields $\tau_\kappa < \alpha$. □

CHAPTER 3

Changing the heights of automorphism towers

One of the reasons why it is so difficult to compute the exact value of the ordinal τ_κ for some infinite cardinal κ is that, although the definition of automorphism towers is purely algebraic, there can be groups whose automorphism tower heights depend on the model of set theory in which they are computed. Therefore, you always have to take into account the set-theoretic background in which the computation of τ_κ takes place. We may therefore conclude that the automorphism tower construction contains a *set-theoretic essence.*[1] This phenomenon is illustrated by a result due to Joel David Hamkins and Simon Thomas in [**HT00**] stating that the existence of centreless groups *whose automorphism towers are highly malleable by forcing* is consistent.

In [**FLb**], Gunter Fuchs and the author extended this result by showing that any reasonable sequence of ordinals can be realized as the automorphism tower heights of a certain group in consecutive forcing extensions or ground models. For example, it is possible to increase the height of the automorphism tower by passing to a forcing extension, then increase it further by passing to a ground model, and then decrease it by passing to a further forcing extension, and so on. In the first five sections of this chapter, we will give a detailed presentation of the results of [**FLb**].

In another direction, it is also possible to construct models of set theory that contain a group with *unbounded potential automorphism tower height* in the sense that for every ordinal we can find a partial order that preserves cofinalities and forces the automorphism tower of the given group to be taller than this ordinal. The last section of this chapter contains an argument that derives this result from Theorem 1.2.1 and simplifies the original proof in [**Lüca**].

3.1. Introduction

In [**Tho98**], Simon Thomas showed that the height of the automorphism tower of an infinite centreless group is not absolute between models of set theory by proving the following theorems.

THEOREM 3.1.1 ([**Tho98**, Theorem 2.1]). *There is a partial order $\mathbb{P}$ satisfying the countable chain condition and a centreless group G with $\tau(G) = 0$ and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\tau(\check{G}) \geq 1$ ".*

[1]This formulation is due to Joel David Hamkins, see [**Ham02**].

THEOREM 3.1.2 ([**Tho98**, Theorem 2.4]). *There is a centreless group H with $\tau(H) = 2$ and $\mathbb{1}_{\mathbb{Q}} \Vdash$ "$\tau(\check{H}) = 1$" for every notion of forcing $\mathbb{Q}$ that adds a new real.*

Let M, N be transitive models of ZFC with $M \subseteq N$ and $G \in M$ be a centreless group. By the above, the height of the automorphism tower of G computed in M, $\tau(G)^M$, can be higher or smaller than the height computed in N, $\tau(G)^N$. This leads to the natural question whether the value of $\tau(G)^M$ places any constraints on the value of $\tau(G)^N$, and vice versa. Obviously, $\tau(G)^N = 0$ implies $\tau(G)^M = 0$. The following result by Joel David Hamkins and Simon Thomas suggests that this is the only provable implication that holds for all centreless groups in the above situation. In short, the theorem states that the existence of centreless groups *whose automorphism towers are highly malleable by forcing* is consistent with the axioms of set theory.

THEOREM 3.1.3 ([**HT00**, Theorem 1.4]). *It is consistent with the axioms of* ZFC *that for every infinite cardinal κ and every ordinal $\alpha < \kappa$, there exists a centreless group G with the following properties.*

(1) $\tau(G) = \alpha$.
(2) *If β is any ordinal such that $0 < \beta < \kappa$, then there exists a notion of forcing $\mathbb{P}_\beta$, which preserves cofinalities and cardinalities, such that $\mathbb{1}_{\mathbb{P}_\beta} \Vdash$ "$\tau(\check{G}) = \check{\beta}$".*

The proof of this theorem splits into an algebraic and a set-theoretic part. The following definition features the key concept of both parts of the proof. The terminology is taken from [**FH08**].

DEFINITION 3.1.4. Let κ be a cardinal, $\vec{\Gamma} = \langle \Gamma_\alpha \mid \alpha < \kappa \rangle$ be a sequence of rigid graphs and E be an equivalence relation on κ. We say that a forcing notion $\mathbb{P}$ *is able to realize E on $\vec{\Gamma}$*, if $\mathbb{P}$ forces that all Γ_α are rigid and, that for all $\beta, \gamma < \kappa$, $\Gamma_\beta \cong \Gamma_\gamma \Leftrightarrow \beta E \gamma$.

The following theorem sums up the results of the set-theoretic part of the proof.

THEOREM 3.1.5 ([**HT00**]). *It is consistent that for every infinite regular cardinal, there exists a sequence $\vec{\Gamma} = \langle \Gamma_\alpha \mid \alpha < \kappa^+ \rangle$ of pairwise non-isomorphic connected rigid graphs with the following property: Whenever E is an equivalence relation on κ^+, there exists a notion of forcing $\mathbb{P}_E$ that satisfies the following statements.*

(1) *$\mathbb{P}_E$ preserves cofinalities and adds no new κ-sequences.*
(2) *$\mathbb{P}_E$ is able to realize E on $\vec{\Gamma}$.*

The algebraic part of the proof then shows that the conclusions of Theorem 3.1.3 are a consequence of this theorem. Since we are going to adopt the techniques developed in these proofs, the next section contains an overview of the construction of the groups in the algebraic part of the proof.

The consistency result of the former theorem is obtained by a class-sized forcing over a model of ZFC + GCH. In [**FH08**], Gunter Fuchs and Joel David Hamkins showed that the conclusions of this theorem also hold in the constructible universe L. They deduce these conclusions from combinatorial principles that hold in L and that we will introduce presently.

DEFINITION 3.1.6. Let κ be an infinite cardinal and let Cof_κ denote the set $\{\alpha < \kappa^+ \mid \mathrm{cof}(\alpha) = \kappa\}$. Then $\diamond_{\kappa^+}(\mathrm{Cof}_\kappa)$ is the assertion that there is a sequence $\vec{D} = \langle D_\alpha \mid \alpha \in \mathrm{Cof}_\kappa\rangle$ such that for any $A \subseteq \kappa^+$ the set $\{\alpha \in \mathrm{Cof}_\kappa \mid A \cap \alpha = D_\alpha\}$ is stationary in κ^+.

In L, the hypotheses that $\kappa = \kappa^{<\kappa}$ and $\diamond_{\kappa^+}(\mathrm{Cof}_\kappa)$ are known to hold for every infinite regular cardinal κ. Note that $\diamond_{\kappa^+}(\mathrm{Cof}_\kappa)$ implies that κ is regular, for otherwise Cof_κ is empty.

In the first five sections of this chapter, we **fix a cardinal κ that satisfies the following assumption**.

ASSUMPTION 3.1.7. *κ is an infinite regular cardinal such that $\kappa = \kappa^{<\kappa}$ and $\diamond_{\kappa^+}(\mathrm{Cof}_\kappa)$ holds.*

DEFINITION 3.1.8. Let E be an equivalence relation on κ. If $\gamma < \kappa$, then we let $[\gamma]_E$ denote the E-equivalence class of γ. We call E *bounded*, if there is some $\bar{\kappa} < \kappa$ such that $[\gamma]_E = \{\gamma\}$ for all $\gamma \in [\bar{\kappa}, \kappa)$.

Now we are ready to formulate a modified version of the result mentioned above. This modification follows from the results of [**FH08**] by coding trees into connected graphs as in [**Tho**, Theorem 4.1.8]. If $\Gamma(T)$ denotes the graph coding a tree T, then the following statements hold and are upwards-absolute.

(1) $\mathrm{Aut}(T)$ is isomorphic to $\mathrm{Aut}(\Gamma(T))$ for every tree T.
(2) Given trees T_0 and T_1, T_0 is isomorphic to T_1 if and only if $\Gamma(T_0)$ is isomorphic to $\Gamma(T_1)$.

These absolute properties of the coding allow us to directly conclude the following result from [**FH08**, Theorem 3.1].

THEOREM 3.1.9 ([**FH08**], under Assumption 3.1.7). *There is a sequence $\vec{\Gamma} = \langle \Gamma_\alpha \mid \alpha < \kappa\rangle$ of rigid, pairwise non-isomorphic connected graphs and a sequence $\vec{C} = \langle C_{\alpha,\beta} \mid \alpha < \beta < \kappa\rangle$ of κ^+-Souslin trees with the following property: Whenever E is a bounded equivalence relation on κ, the full support product forcing*

$$C_E = \prod_{\substack{\gamma<\kappa \\ \gamma\neq\min[\gamma]_E}} C_{\min[\gamma]_E,\gamma}$$

has the following properties.

(1) *C_E preserves cofinalities and adds no new κ-sequences.*
(2) *C_E is able to realize E on $\vec{\Gamma}$.*

The aim of the first part of this chapter is to show that this theorem already implies the existence of groups whose automorphism tower is even more malleable by forcing than those of the groups mentioned in Theorem 3.1.3. It gives rise to groups whose automorphism tower heights can be changed multiple times to any non-zero height by passing from one model of set-theory to another, either by always going to a forcing extension, by always passing to a ground model, or by mixing these possibilities. In fact, for the given cardinal κ, we will use Assumption 3.1.7 to construct a *single* group $\mathfrak{G} = \mathfrak{G}_\kappa$ with $\tau(\mathfrak{G}) = 0$ and the property that the height of the automorphism tower of $\mathfrak{G}$ can be changed in each of these ways, repeatedly.

Let us now formulate precisely the three ways in which the height of the automorphism tower of $\mathfrak{G}$ can be changed repeatedly. The first main result addresses the possibility of passing from models to larger and larger forcing extensions in each step.

Theorem 3.1.10 ([**FLb**, Theorem 3.10], under Assumption 3.1.7). *For every function $s : \kappa \longrightarrow (\kappa \setminus \{0\})$, there is a sequence of partial orders $\langle \mathbb{P}^s_\gamma \mid 0 < \gamma < \kappa \rangle$, such that the following statements hold for each $0 < \alpha < \kappa$.*

(1) *$\mathbb{P}^s_\alpha$ preserves cofinalities and adds no new κ-sequences.*
(2) $\mathbb{1}_{\mathbb{P}^s_{\alpha+1}} \Vdash$ *"$\tau(\check{\mathfrak{G}}) = \check{s}(\check{\alpha})$ ".*
(3) *If α is a limit ordinal, then $\mathbb{1}_{\mathbb{P}^s_\alpha} \Vdash$ "$\tau(\check{\mathfrak{G}}) = 1$ ".*
(4) *If $\beta < \alpha$, then $\mathbb{P}^s_\alpha$ extends $\mathbb{P}^s_\beta$ (in the sense that $\mathbb{P}^s_\alpha \cong \mathbb{P}^s_\beta \times \mathbb{Q}$ for some partial order $\mathbb{Q}$).*

Moreover, if $t : \kappa \longrightarrow (\kappa \setminus \{0\})$, and $s \upharpoonright \gamma = t \upharpoonright \gamma$ for some $0 < \gamma < \kappa$, then $\mathbb{P}^s_\gamma = \mathbb{P}^t_\gamma$.

The next main theorem addresses the possibility of producing a model with the property that the height of the automorphism tower of $\mathfrak{G}$ can be changed by passing to smaller and smaller ground models.

Theorem 3.1.11 ([**FLb**, Theorem 4.1], under Assumption 3.1.7). *For every ordinal $\lambda < \kappa$, there is a notion of forcing $\mathbb{Q}_\lambda$ with the following properties.*

(1) *$\mathbb{Q}_\lambda$ preserves cofinalities and adds no new κ-sequences.*
(2) $\mathbb{1}_{\mathbb{Q}_\lambda} \Vdash$ *"$\tau(\check{\mathfrak{G}}) = 1$ ".*
(3) *In every $\mathbb{Q}_\lambda$-generic extension of the ground model the following holds: For every sequence $s : \lambda \longrightarrow (\lambda \setminus \{0\})$ there exists a* decreasing sequence of ground models $\langle M^s_\alpha \mid 0 < \alpha < \lambda \rangle$ *such that for all $0 < \alpha < \lambda$ the following statements hold.*
   (a) *$M^s_{\alpha+1} \models$ "$\tau(\mathfrak{G}) = s(\alpha)$ ".*
   (b) *If α is a limit ordinal, then $M^s_\alpha \models$ "$\tau(\mathfrak{G}) = 1$ ".*

   Moreover, if $t : \lambda \longrightarrow (\lambda \setminus \{0\})$, then $s(\alpha) = t(\alpha)$ implies $M^s_{\alpha+1} = M^s_{\alpha+1}$ for all $\alpha < \lambda$ and $M^s_\nu = M^t_\nu$ for all limit ordinals $\nu < \lambda$.

Section 3.4 contains the proof of this theorem.

Next, the possibilities of passing to a ground model or to a forcing extension can be mixed. In order to make sense of models that are reached by unboundedly often passing to a forcing extension and unboundedly often passing to a ground model, we need a suitable notion of limit. We make this precise and prove in Theorem 3.5.2, vaguely speaking, that all patterns can be realized, provided that the set of $\alpha < \kappa$ at which one passes to a forcing extension contains a club.

In the last section of this chapter, we will construct groups whose automorphism tower is highly malleable by forcing in another way: these groups have *unbounded potential automorphism tower height*.

THEOREM 3.1.12 ([**Lüca**, Theorem 1.8]). *It is consistent with the axioms of* ZFC *that there is a centreless group G of cardinality $\aleph_1$ with the property that for every ordinal α there is a σ-distributive partial order $\mathbb{P}$ that satisfies the $\aleph_2$-chain condition and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\tau(\check{G}) \geq \alpha$ ".*

The proof of this statement presented in Section 3.5 is a simplification of the proof presented in [**Lüca**]. Both proofs rely on the statement of Theorem 1.2.1.

3.2. Preliminaries

In general, it is very difficult to compute the automorphism tower of a given group. We will use a technique developed by Simon Thomas that makes the construction of groups with a certain automorphism tower height easier. The *Normalizer Tower Technique* was developed in [**Tho85**].

DEFINITION 3.2.1. If H is a subgroup of the group G, then *the normalizer tower* $\langle \mathrm{N}^{\alpha}_{G}(H) \mid \alpha \in \mathrm{On}\rangle$ *of H in G* is defined inductively as follows.

(1) $\mathrm{N}^{0}_{G}(H) = H$.
(2) $\mathrm{N}^{\alpha+1}_{G}(H) = \mathrm{N}_{G}(\mathrm{N}^{\alpha}_{G}(H)) = \{g \in G \mid \iota_g'' \mathrm{N}^{\alpha}_{G}(H) = \mathrm{N}^{\alpha}_{G}(H)\}$.
(3) $\mathrm{N}^{\lambda}_{G}(H) = \bigcup\{\mathrm{N}^{\alpha}_{G}(H) \mid \alpha < \lambda\}$, if $\lambda \in \mathrm{Lim}$.

An easy cardinality argument shows that for each group G of cardinality κ and each subgroup H of G there is an $\alpha < \kappa^+$ such that $\mathrm{N}^{\alpha}_{G}(H) = \mathrm{N}^{\alpha+1}_{G}(H)$. The normalizer length $\tau^{nlg}_{G}(H)$ of H in G is the least such α.

The following theorem reduces the problem of manipulating automorphism towers to the problem of manipulating normalizer towers in automorphism groups of first-order structures. It is implicitly proved in [**Tho85**]. A detailed explanation of this result and the absoluteness of the corresponding construction can be found in [**HT00**, Section 2].

THEOREM 3.2.2 ([**Tho85**]). *Let $\mathcal{L}$ be a first-order language, $\mathcal{M}$ be an $\mathcal{L}$-structure and H be a subgroup of* $\mathrm{Aut}(\mathcal{M})$*. Then there exists a centreless group G such that the statement*

$$\tau(G) = \tau^{nlg}_{\mathrm{Aut}(\mathcal{M})}(H)$$

holds and is upwards-absolute between transitive models of ZFC.

We will now summarize the results that we need in order to construct structures whose automorphism groups can be changed by forcing. In the following, we adopt notations from [**HT00**].

We call a pair (G, Ω) a *permutation group*, if G is a subgroup of $\mathrm{Sym}(\Omega)$. Given a family $\langle (G_i, \Omega_i) \mid \in I \rangle$ of permutation groups, the *direct product* of the family is defined to be the permutation group

$$\prod_{i \in I} (G_i, \Omega_i) = \left(\prod_{i \in I} G_i, \bigsqcup_{i \in I} \Omega_i \right),$$

where the direct product of groups acts on the disjoint union of sets in the obvious manner. We say that two permutation groups (G, Ω) and (H, Δ) are *isomorphic*, if there is a bijection $f : \Omega \to \Delta$ such that the induced isomorphism

$$f^* : \mathrm{Sym}(\Omega) \longrightarrow \mathrm{Sym}(\Delta);\ \sigma \longmapsto f \circ \sigma \circ f^{-1}$$

maps G onto H. We write $(H_0, \Omega_0) \times (H_1, \Omega_1)$ instead of $\prod_{i<2} (H_i, \Omega_i)$ and $\tau^{nlg}(H, \Omega)$ instead of $\tau^{nlg}_{\mathrm{Sym}(\Omega)}(H)$.

For each ordinal α, we inductively define permutation groups $(H_\alpha, \Delta_\alpha)$ and $(F_\alpha, \Delta_\alpha)$ in the following way.

(1) $\Delta_0 = \{\emptyset\}$ and $H_0 = F_0 = \{\mathrm{id}_{\Delta_0}\}$.
(2) If $\alpha > 0$, then we define

$$(H_\alpha, \Delta_\alpha) = (H_0, \Delta_0) \times \prod_{\beta < \alpha} (F_\beta, \Delta_\beta)$$

and

$$F_\alpha = \mathrm{N}^\alpha_{\mathrm{Sym}(\Delta_\alpha)}(H_\alpha).$$

Note that the second clause directly implies

$$(H_\alpha, \Delta_\alpha) \cong (H_\beta, \Delta_\beta) \times \prod_{\beta \leq \gamma < \alpha} (F_\gamma, \Delta_\gamma)$$

for all $\beta < \alpha$. In order to keep our calculation clear, we also define

$$(H^*_\alpha, \Delta^*_\alpha) = (H_\alpha, \Delta_\alpha) \times (F_1, \Delta_1) \times (F_1, \Delta_1)$$

for $\alpha > 1$.

An easy induction shows $\max(\{\omega, |\alpha|\})$ is an upper bound for the cardinality of Δ_α and this means that the definitions of $(H_\alpha, \Delta_\alpha)$ and $(F_\alpha, \Delta_\alpha)$ are absolute between models with the same α-sequences of ordinals, because the symmetric group of Δ_β is the same in those models for all $\beta \leq \alpha$.

These permutation groups are the first ingredient in our construction. The following theorem summarizes their important properties deduced in the algebraic part of [**HT00**].

THEOREM 3.2.3 ([**HT00**]). *For each ordinal α, the following statements hold.*

(1) $\tau^{nlg}(H_\alpha, \Delta_\alpha) = \alpha$.

(2) $\tau^{nlg}(F_\alpha, \Delta_\alpha) = 0$.
(3) *If* $\alpha > 1$, *then* $\tau^{nlg}(H^*_\alpha, \Delta^*_\alpha) = 1$.

Proof. The first statement is [**HT00**, Lemma 2.10] and the second statement follows directly from the first, together with the definition of F_α. The third statement is [**HT00**, Lemma 2.14] in the case $\beta = 1$. $\square$

The sequence $\vec{\Gamma} = \langle \Gamma_\alpha \mid \alpha < \kappa \rangle$ of rigid, pairwise non-isomorphic connected graphs and the sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ of κ^+-Souslin trees constructed in Theorem 3.1.9 are the second ingredient in our construction.

If E and F are equivalence relations on κ, then we define

$$E \preceq F \iff E \subseteq F \wedge (\forall \alpha < \kappa)([\alpha]_E \neq \{\alpha\} \rightarrow \min[\alpha]_E = \min[\alpha]_F).$$

Note that, as the notation suggests, $\preceq$ is a reflexive, transitive relation. Moreover, by checking the definition of the forcing C_E in Theorem 3.1.9, we arrive at the following observation.

Observation 3.2.4 (Under Assumption 3.1.7). *If $E \preceq F$ are bounded equivalence relations on κ, then the forcing C_F extends C_E, in the strong sense that there is a partial order $\mathbb{Q}$ such that $C_F \cong C_E \times \mathbb{Q}$.* $\square$

The following construction allows us to combine the two ingredients.

If $\langle \Gamma_i = (X_i, E_i) \mid i \in I \rangle$ is a family of graphs, then we define the *direct sum* of the family to be the graph

$$\bigoplus_{i\in I} \Gamma_i = \left(\bigsqcup_{i\in I} X_i, \bigsqcup_{i \in I} E_i \right)$$

obtained by taking the disjoint unions of the sets of vertices and edges, respectively.

We call a pair (G, Γ) a *graph permutation group*, if Γ is a graph and G is a subgroup of $\mathrm{Aut}(\Gamma)$. As above, if a $\langle (G_i, \Gamma_i) \mid i \in I \rangle$ is a family of graph permutation groups, then we define the *direct product* of the family to be the graph permutation group

$$\prod_{i\in I} (G_i, \Gamma_i) = \left(\prod_{i\in I} G_i, \bigoplus_{i\in I} \Gamma_i \right),$$

where the product of groups acts on the direct sum of graphs in the obvious way. We say that two graph permutation groups are *isomorphic*, if there is an isomorphism of the underlying graphs such that the induced isomorphism of automorphism groups maps the subgroups correctly. Again, we write $(G_0, \Gamma_0) \times (G_1, \Gamma_1)$ instead of $\prod_{i<2}(G_i, \Gamma_i)$ and $\tau^{nlg}(G, \Gamma)$ instead of $\tau^{nlg}_{\mathrm{Aut}(\Gamma)}(G)$.

If Ω is a set and Γ is a graph, then we define

$$\mathcal{G}_\Omega(\Gamma) = \bigoplus_{x \in \Omega} \Gamma$$

to be the graph obtained by replacing each element of Ω by a copy of Γ. We can embed $\mathrm{Sym}(\Omega)$ into $\mathrm{Aut}(\mathcal{G}_\Omega(\Gamma))$ in a natural way and, if Γ is connected and rigid, then it is not hard to show that this embedding is an isomorphism.

If (G, Ω) is a permutation group, then we get a new graph permutation group $(G(\Gamma), \mathcal{G}_\Omega(\Gamma))$, where $G(\Gamma)$ is the image of G under the above embedding of $\mathrm{Sym}(\Omega)$ into $\mathrm{Aut}(\mathcal{G}_\Omega(\Gamma))$.

In the following lemma, we list facts about graph permutation groups used in the algebraic part of [**HT00**]. They will play an important role in our later constructions, because they will enable us to compute normalizer towers in products of graph permutation groups.

LEMMA 3.2.5 ([**HT00**]). *If $\vec{\Gamma} = \langle \Gamma_i \mid i \in I \rangle$ is a sequence of connected rigid graphs and $\langle (G_i, \Omega_i) \mid i \in I \rangle$ is a sequence of permutation groups, then then following statements hold for all $i_0 \in I$.*

(1) $\tau^{nlg}\left(G_{i_0}(\Gamma_{i_0}), \mathcal{G}_{\Omega_{i_0}}(\Gamma_{i_0})\right) = \tau^{nlg}(G_{i_0}, \Omega_{i_0})$.

(2) *If $\vec{\Gamma}$ consists of pairwise non-isomorphic graphs, $\tau^{nlg}(G_{i_0}, \Omega_{i_0}) \geq 1$ and $\tau^{nlg}(G_j, \Omega_j) \leq 1$ holds for all $j \in I \setminus \{i_0\}$, then*

$$\tau^{nlg}\left(\prod_{i \in I} (G_i(\Gamma_i), \mathcal{G}_{\Omega_i}(\Gamma_i))\right) = \tau^{nlg}(G_{i_0}, \Omega_{i_0}).$$

(3) *If $\vec{\Gamma}$ consists of pairwise isomorphic graphs and*

$$(G, \Omega) = \prod_{i \in I} (G_i, \Omega_i),$$

then

$$(G(\Gamma_{i_0}), \mathcal{G}_\Omega(\Gamma_{i_0})) \cong \prod_{i \in I} (G_i(\Gamma_i), \mathcal{G}_{\Omega_i}(\Gamma_i)).$$

PROOF. By the assumption, the canonical embedding of $\mathrm{Sym}(\Omega_i)$ into $\mathrm{Aut}(\mathcal{G}_{\Omega_i}(\Gamma_i))$ is an isomorphism and maps G onto $G(\Gamma_i)$. This proves the first statement.

The set of connected components of $\prod_{i \in I} (G_i(\Gamma_i), \mathcal{G}_{\Omega_i}(\Gamma_i))$ consists of a copy of Γ_i for each element of Ω_i and each $i \in I$. If all Γ_i's are pairwise non-isomorphic, then each subgraph of the form $\mathcal{G}_{\Omega_i}(\Gamma_i)$ is invariant under all automorphisms and therefore each automorphism of the graph is induced by an element of the group $\prod_{i \in I} \mathrm{Aut}(\mathcal{G}_{\Omega_i}(\Gamma_i))$ acting on the graph in the obvious way. By the rigidity of the Γ_i's, this means that the automorphism group of $\bigoplus_{i \in I} \mathcal{G}_{\Omega_i}(\Gamma_i)$ is isomorphic to $\prod_{i \in I} \mathrm{Sym}(\Omega_i)$ and this isomorphism sends $\prod_{i \in I} G_i(\Gamma_i)$ to $\prod_{i \in I} G_i$. An easy induction then shows

$$\mathrm{N}^\alpha_{\prod_{i \in I} \mathrm{Sym}(\Omega_i)}\left(\prod_{i \in I} G_i\right) \cong \mathrm{N}^\alpha_{\mathrm{Sym}(\Omega_{i_0})}(G_{i_0}) \times \prod_{j \in I \setminus \{i_0\}} \mathrm{N}^1_{\mathrm{Sym}(\Omega_j)}(G_j)$$

for all $\alpha > 0$ and, by the existence of the above isomorphism, this proves the second statement.

Each automorphism of $\bigoplus_{i\in I}\mathcal{G}_{\Omega_i}(\Gamma_i)$ that fixes a connected component setwise also fixes it pointwise by rigidity. This shows that the natural isomorphism between $\bigoplus_{i\in I}\mathcal{G}_{\Omega_i}(\Gamma_i)$ and $\bigoplus_{j\in I}\mathcal{G}_{\Omega_j}(\Gamma_{i_0})$ induced by the isomorphisms between Γ_{i_0} and the Γ_i's is also an isomorphism between the graph permutation groups $(G(\Gamma_{i_0}),\mathcal{G}_\Omega(\Gamma_{i_0}))$ and $\prod_{i\in I}(G_i(\Gamma_i),\mathcal{G}_{\Omega_i}(\Gamma_i))$. □

We now introduce the group $\mathfrak{G}$ which is the protagonist of the first five sections of this chapter. For the remainder of this chapter, **fix a sequence** $\langle (G_\alpha,\Omega_\alpha)\mid \alpha<\kappa\rangle$ **of permutation groups** such that each (G_α,Ω_α) is of the form $(F_{\bar\alpha},\Delta_{\bar\alpha})$, for some $\bar\alpha<\kappa$, and such that for every $\beta<\kappa$, the set of $\delta<\kappa$ such that $(G_\delta,\Omega_\delta)=(F_\beta,\Delta_\beta)$ is unbounded in κ. So for example, using the Gödel pairing function, we could let $(G_\gamma,\Omega_\gamma)=(F_\alpha,\Delta_\alpha)$, if $\gamma=\prec\alpha,\beta\succ<\kappa$. We write $\mathcal{G}_\alpha(\Gamma)$ instead of $\mathcal{G}_{\Omega_\alpha}(\Gamma)$.

Definition 3.2.6. If $\vec\Pi=\langle \Pi_\alpha\mid\alpha<\kappa\rangle$ is a sequence of graphs, then we define

$$\mathcal{G}(\vec\Pi)=\prod_{\alpha<\kappa}\left(G_\alpha(\Pi_\alpha),\mathcal{G}_\alpha(\Pi_\alpha)\right).$$

As noted above, the definition of $\mathcal{G}(\vec\Pi)$ is absolute between models with the same κ-sequences of ordinals that contain $\vec\Pi$.

Under Assumption 3.1.7, we also **fix a sequence** $\vec\Gamma=\langle\Gamma_\alpha\mid\alpha<\kappa\rangle$ **of graphs and a sequence** $\vec C=\langle C_{\alpha,\beta}\mid\alpha<\beta<\kappa\rangle$ **of trees** as in Theorem 3.1.9.

Definition 3.2.7. Let $\mathfrak{G}=\mathfrak{G}_\kappa$ be the centreless group the existence of which is postulated in Theorem 3.2.2, with respect to $\mathcal{G}(\vec\Gamma)$.

So by definition, $\tau(\mathfrak{G})=\tau^{nlg}(\mathcal{G}(\vec\Gamma))$ holds and is upwards-absolute. Hence we can change the height of the automorphism tower of $\mathfrak{G}$ by changing the height of the normalizer tower of $\mathcal{G}(\vec\Gamma)$ in the corresponding symmetric group.

Since all Γ_α are rigid and pairwise non-isomorphic and

$$\tau^{nlg}(G_\alpha,\Omega_\alpha)=\tau^{nlg}(F_{\bar\alpha},\Delta_{\bar\alpha})=0,$$

we may use Theorem 3.2.3 and the second part of Lemma 3.2.5 to get the following statement.

Observation 3.2.8 (Under Assumption 3.1.7). $\tau(\mathfrak{G})=\tau^{nlg}(\mathcal{G}(\vec\Gamma))=0$. □

3.3. Consecutive Forcing Extensions

To make the following constructions clearer, we introduce some vocabulary. We would like to remind the reader that we are working under Assumption 3.1.7, and that we have fixed the objects mentioned at the end of the previous section.

DEFINITION 3.3.1. Let X be a subset of κ with monotone enumeration $\langle \gamma_\alpha \mid \alpha < \operatorname{otp}(X,<)\rangle$.

(1) We call X *active* if $\operatorname{otp}(X,<) = \beta + 1 > 2$ for some $\beta < \kappa$ and
   (a) For all $\alpha < \beta$, $(G_{\gamma_\alpha}, \Omega_{\gamma_\alpha}) = (F_\alpha, \Delta_\alpha)$.
   (b) $(G_{\gamma_\beta}, \Omega_{\gamma_\beta}) = (F_0, \Delta_0)$.

(2) We call X *sealed* if $\operatorname{otp}(X,<) = \beta+3$ for some $\beta < \kappa$, $X \cap (\gamma_\beta + 1)$ is active and
$$\left(G_{\gamma_{\beta+1}}, \Omega_{\gamma_{\beta+1}}\right) = \left(G_{\gamma_{\beta+2}}, \Omega_{\gamma_{\beta+2}}\right) = (F_1, \Delta_1).$$

(3) If X is a sealed subset of κ with $\operatorname{otp}(X,<) = \beta+3$ and $1 < \bar\beta \le \beta$, then $\{\gamma_\alpha \mid \alpha < \bar\beta\} \cup \{\gamma_\beta\}$ is *the active segment of X of order type $\bar\beta + 1$*.

(4) We call X *trimmed*, if $\operatorname{otp}(X,<) = 2$ and
$$(G_{\gamma_0}, \Omega_{\gamma_0}) = (G_{\gamma_1}, \Omega_{\gamma_1}) = (F_0, \Delta_0).$$

(5) If Y is either an active subset of κ with monotone enumeration $\langle \delta_\alpha \mid \alpha < \beta + 1\rangle$ or a sealed subset of κ with monotone enumeration $\langle \delta_\alpha \mid \alpha < \beta + 3\rangle$, then $\{\delta_0, \delta_\beta\}$ is *the trimmed segment of* Y.

So the permutation groups associated to a sealed subset X of κ with monotone enumeration $\langle \gamma_\alpha \mid \alpha < \beta + 3\rangle$ look as follows:

$$\begin{array}{ccccccc}
(G_{\gamma_0}, \Omega_{\gamma_0}) & (G_{\gamma_1}, \Omega_{\gamma_1}), & (G_{\gamma_2}, \Omega_{\gamma_2}) & \dots & (G_{\gamma_\beta}, \Omega_{\gamma_\beta}) & (G_{\gamma_{\beta+1}}, \Omega_{\gamma_{\beta+1}}) & (G_{\gamma_{\beta+2}}, \Omega_{\gamma_{\beta+2}}) \\
\parallel & \parallel & \parallel & & \parallel & \parallel & \parallel \\
(F_0, \Delta_0) & (F_1, \Delta_1) & (F_2, \Delta_2) & \dots & (F_0, \Delta_0) & (F_1, \Delta_1) & (F_1, \Delta_1)
\end{array}$$

Note that a sealed subset of κ must have order type at least 5. By definition, the following equation holds for the above set X.

$$\begin{aligned}(3.1)\qquad \prod_{\delta \in X} (G_\delta, \Omega_\delta) &= (F_0, \Delta_0) \times \left(\prod_{\alpha<\beta} (F_\alpha, \Delta_\alpha)\right) \times (F_1, \Delta_1) \times (F_1, \Delta_1) \\ &= \left(H^*_\beta, \Delta^*_\beta\right).\end{aligned}$$

If $\bar\beta \le \beta$ and Y is the active segment of X of order type $\bar\beta + 1$, then the following equation holds.

$$(3.2)\qquad \prod_{\delta \in Y} (G_\delta, \Omega_\delta) = (F_0, \Delta_0) \times \prod_{\alpha<\bar\beta} (F_\alpha, \Delta_\alpha) = \left(H_{\bar\beta}, \Delta_{\bar\beta}\right).$$

Finally, if $Z = \{\xi_0, \xi_1\}$ is a trimmed subset of κ, then the following equation holds.

$$(3.3)\qquad \prod_{\delta \in Z} (G_\delta, \Omega_\delta) = (F_0, \Delta_0) \times (F_0, \Delta_0) = (H_1, \Delta_1).$$

We extend the above definitions to equivalence relations on κ and show how we can use them to change the height of the automorphism tower of $\mathfrak{G}$.

DEFINITION 3.3.2. Let E be a non-trivial equivalence relation on κ.

(1) We call E *inactive*, if every non-trivial equivalence class is either a sealed or a trimmed subset of κ.
(2) We call E *active*, if all non-trivial E-equivalence classes are either active, sealed or trimmed subsets of κ and there is a unique active E-equivalence class.

LEMMA 3.3.3 (Under Assumption 3.1.7). *If E is a bounded, inactive equivalence relation on κ, then $\mathbb{1}_{C_E} \Vdash$ "$\tau(\check{\mathfrak{G}}) = 1$".*

PROOF. Work in a C_E-generic extension of the ground model. As noted after Definition 3.2.6,

$$\mathcal{G}(\vec{\Gamma}) = \prod_{\alpha<\kappa} (G_\alpha(\Gamma_\alpha), \mathcal{G}_\alpha(\Gamma_\alpha))$$

still holds. Let S denote the set of all sealed E-equivalence classes, and for $c \in S$, let $\langle \gamma^c_\alpha \mid \alpha < \beta^c + 3\rangle$ be the monotone enumeration of c. Define T to be the set of all trimmed E-equivalence classes and let $d = \{\xi^d_0, \xi^d_1\}$ for each $d \in T$. Finally, let N denote the union of all trivial E-equivalence classes. Using the third part of Lemma 3.2.5 and the equations (3.1) and (3.3), we can conclude that the following objects are isomorphic.

$$\begin{aligned}
\mathcal{G}(\vec{\Gamma}) &\cong \left(\prod_{\alpha\in N} (G_\alpha(\Gamma_\alpha), \mathcal{G}_\alpha(\Gamma_\alpha))\right) \times \left(\prod_{c\in S}\left(\prod_{\delta\in c}\left(G_\delta(\Gamma_{\gamma^c_0}), \mathcal{G}_\delta(\Gamma_{\gamma^c_0})\right)\right)\right)\\
&\quad\times \prod_{d\in T}\left((G_{\xi^d_0}(\Gamma_{\xi^d_0}), \mathcal{G}_{\xi^d_0}(\Gamma_{\xi^d_0})) \times (G_{\xi^d_1}(\Gamma_{\xi^d_0}), \mathcal{G}_{\xi^d_1}(\Gamma_{\xi^d_0}))\right)\\
&\cong \left(\prod_{\alpha\in N} (G_\alpha(\Gamma_\alpha), \mathcal{G}_\alpha(\Gamma_\alpha))\right) \times \left(\prod_{c\in S}(H^*_{\beta^c}(\Gamma_{\gamma^c_0}), \mathcal{G}_{\Delta^*_{\beta^c}}(\Gamma_{\gamma^c_0}))\right)\\
&\quad\times \prod_{d\in T}(H_1(\Gamma_{\xi^d_0}), \mathcal{G}_{\Delta_1}(\Gamma_{\xi^d_0})).
\end{aligned}$$

By assumption, all graphs appearing in this product are rigid and pairwise non-isomorphic. The first part of Lemma 3.2.5 and Theorem 3.2.3 now yield the following statements.

(1) For all $\alpha \in N$, $\tau^{nlg}\left(G_\alpha(\Gamma_\alpha), \mathcal{G}_\alpha(\Gamma_\alpha)\right) = \tau^{nlg}\left(G_\alpha, \Omega_\alpha\right) = 0$.
(2) For all $c \in S$, $\tau^{nlg}(H^*_{\beta^c}(\Gamma_{\gamma^c_0}), \mathcal{G}_{\Delta^*_{\beta^c}}(\Gamma_{\gamma^c_0})) = \tau^{nlg}(H^*_{\beta^c}, \Delta^*_{\beta^c}) = 1$.
(3) For all $t \in T$, $\tau^{nlg}(H_1(\Gamma_{\xi^d_0}), \mathcal{G}_{\Delta_1}(\Gamma_{\xi^d_0})) = \tau^{nlg}(H_1, \Delta_1) = 1$.

By definition, there is at least one non-trivial equivalence class and we can therefore apply the second part of Lemma 3.2.5 to conclude

$$\tau(\mathfrak{G}) = \tau^{nlg}(\mathcal{G}(\vec{\Gamma})) = 1.$$

□

LEMMA 3.3.4 (Under Assumption 3.1.7). *Let E be a bounded, active equivalence relation on κ. If e is the unique active E-equivalence class, then $\mathbb{1}_{C_E} \Vdash$ "$\tau(\check{\mathfrak{G}}) + 1 = \operatorname{otp}(\check{e}, <)$".*

PROOF. Work in a C_E-generic extension of the ground model. By the definition of active subsets, the monotone enumeration of e is of the form $\langle\gamma_\alpha \mid \alpha < \beta+1\rangle$ for some $1 < \beta < \kappa$. Define N, S, T, γ^c_α and ξ^d_i as in the proof of Lemma 3.3.3. Using the third part of Lemma 3.2.5 and the equations (3.1)-(3.3), we get the following equalities.

$$\begin{aligned}
\mathcal{G}(\vec{\Gamma}) \cong & \left(\prod_{\alpha\in N}(G_\alpha(\Gamma_\alpha),\mathcal{G}_\alpha(\Gamma_\alpha))\right)\times\left(\prod_{c\in S}\left(\prod_{\delta\in c}(G_\delta(\Gamma_{\gamma^d_0}),\mathcal{G}_\delta(\Gamma_{\gamma^d_0}))\right)\right)\\
& \times\left(\prod_{d\in T}\left((G_{\xi^d_0}(\Gamma_{\xi^d_0}),\mathcal{G}_{\xi^d_0}(\Gamma_{\xi^d_0}))\times(G_{\xi^d_1}(\Gamma_{\xi^d_0}),\mathcal{G}_{\xi^d_1}(\Gamma_{\xi^d_0}))\right)\right)\\
& \times\prod_{\delta\in e}(G_\delta(\Gamma_{\gamma_0}),\mathcal{G}_\delta(\Gamma_{\gamma_0}))\\
\cong & \left(\prod_{\alpha\in N}(G_\alpha(\Gamma_\alpha),\mathcal{G}_\alpha(\Gamma_\alpha))\right)\times\left(\prod_{c\in S}(H^*_{\beta^c}(\Gamma_{\gamma^c_0}),\mathcal{G}_{\Delta^*_{\beta^c}}(\Gamma_{\gamma^c_0}))\right)\\
& \times\left(\prod_{d\in T}(H_1(\Gamma_{\xi^d_0}),\mathcal{G}_{\Delta_1}(\Gamma_{\xi^d_0}))\right)\times(H_\beta(\Gamma_{\gamma_0}),\mathcal{G}_{\Delta_\beta}(\Gamma_{\gamma_0})).
\end{aligned}$$

Again, all graphs in this products are rigid and pairwise non-isomorphic and

$$\tau^{nlg}\left(H_\beta(\Gamma_{\gamma_0}),\mathcal{G}_{\Delta_\beta}(\Gamma_{\gamma_0})\right)=\tau^{nlg}\left(H_\beta,\Delta_\beta\right)=\beta>1.$$

By the second part of Lemma 3.2.5 and the computations made in the proof of Lemma 3.3.3, we can conclude

$$\tau(\mathfrak{G})+1=\tau^{nlg}(\mathcal{G}(\vec{\Gamma}))+1=\beta+1=\mathrm{otp}\,(e,<).$$

□

Next, we define a family of functions that allows us the construction of special bounded equivalence relations in our proofs of the theorems. Remember that for each $\alpha < \kappa$ the set

$$\{\beta<\kappa \mid (G_\beta,\Omega_\beta)=(F_\alpha,\Delta_\alpha)\}$$

is unbounded in κ.

LEMMA 3.3.5. *For each function $s : \kappa \longrightarrow (\kappa \setminus \{0,1\})$, there exists a function $s^* : \kappa \longrightarrow [\kappa]^{<\kappa}$ with the following properties.*

(1) *If $\beta < \alpha$, then $s^*(\beta) \subseteq \min(s^*(\alpha))$.*
(2) *For all $\alpha < \kappa$, $s^*(\alpha)$ is a sealed subset of κ with*

$$\mathrm{otp}\,(s^*(\alpha),<)=s(\alpha)+3.^{2}$$

[2]Remember that a sealed subset of κ must have order type at least 5. This is why we require $s(\alpha) > 1$ here.

Proof. Assume $s^* \restriction \alpha$ is already defined for some $\alpha < \kappa$. We define $s^*(\alpha) = \{\gamma^\alpha_\delta \mid \delta < s(\alpha)+3\}$, where $\langle \gamma^\alpha_\delta \mid \delta < s(\alpha)+3 \rangle$ is defined as follows: γ^α_0 is the least $\nu < \kappa$ such that $\bigcup\{s^*(\beta) \mid \beta < \alpha\} \subseteq \nu$ and $(G_\nu, \Omega_\nu) = (F_0, \Delta_0)$. If $0 < \delta < s(\alpha)$ and $\langle \gamma^\alpha_\xi \mid \xi < \delta \rangle$ is already defined, then γ^α_δ is the least $\nu < \kappa$ such that $\nu > \sup\{\gamma^\alpha_\xi \mid \xi < \delta\}$ and $(G_\nu, \Omega_\nu) = (F_\delta, \Delta_\delta)$. Finally, $\gamma^\alpha_{s(\alpha)}$ is the least $\nu < \kappa$ such that $\nu > \sup\{\gamma^\alpha_\delta \mid \delta < s(\alpha)\}$ and $(G_\nu, \Omega_\nu) = (F_0, \Delta_0)$, $\gamma^\alpha_{s(\alpha)+1}$ is the least $\nu < \kappa$ such that $\nu > \gamma^\alpha_{s(\alpha)}$ and $(G_\nu, \Omega_\nu) = (F_1, \Delta_1)$, and $\gamma^\alpha_{s(\alpha)+2}$ is the least $\nu < \kappa$ such that $\nu > \gamma^\alpha_{s(\alpha)+1}$ and $(G_\nu, \Omega_\nu) = (F_1, \Delta_1)$. $\square$

From now on, we fix an operator $[s \mapsto s^*]$ with the above properties. We may also assume that if $s, t : \kappa \longrightarrow (\kappa \setminus \{0,1\})$ are functions with $s \restriction \gamma = t \restriction \gamma$ for some $\gamma < \kappa$, then $s^* \restriction \gamma = t^* \restriction \gamma$. For each $s : \kappa \longrightarrow (\kappa \setminus \{0,1\})$ and each $\alpha < \kappa$, we define a bounded, inactive equivalence relation E^s_α on κ by

$$\gamma E^s_\alpha \delta \iff \gamma = \delta \ \vee \ (\exists \beta < \alpha)\ \gamma, \delta \in s^*(\beta).$$

It is easy to see that $\alpha < \beta < \kappa$ implies $E^s_\alpha \preceq E^s_\beta$.

Definition 3.3.6. Let E be a bounded equivalence relation on κ. If E is active and e is the unique active E-equivalence class, then we define $\mathrm{ht}(E)$ to be the unique ordinal α with $\mathrm{otp}\,(e, <) = \alpha + 1$. If E is inactive, then we define $\mathrm{ht}(E) = 1$.

As an illustration of the concepts introduced above, note the following observation which is a direct consequence of Lemmas 3.3.3 and 3.3.4.

Observation 3.3.7 (Under Assumption 3.1.7). *If E is a bounded equivalence relation on κ and E is either active or inactive, then*

$$\mathbb{1}_{C_E} \Vdash \text{“}\tau(\check{\mathfrak{G}}) = \mathrm{ht}(\check{E})\,\text{”}.$$

$\square$

Next, we want to analyze $\preceq$-ascending and -descending chains of equivalence relations.

Definition 3.3.8. Let $\vec{A} = \langle A_\alpha \mid \alpha < \beta \rangle$ be a sequence of sets. We say that $\vec{A}$ *converges*, if for every x there is an $\alpha < \beta$ such that either $x \in A_\gamma$ for all $\alpha \leq \gamma < \beta$ or $x \notin A_\gamma$ for all $\alpha \leq \gamma < \beta$. If $\vec{A}$ converges, then we define the limit of $\vec{A}$ to be the set

$$\lim_{\alpha \to \beta} A_\alpha = \bigcup_{\alpha < \beta} \ \bigcap_{\alpha \leq \gamma < \beta} A_\gamma.$$

If $\beta = 0$ or $\beta = \alpha + 1$, then $\vec{A}$ automatically converges. Namely, $\lim_{\gamma \to 0} A_\gamma = \emptyset$, and $\lim_{\gamma \to \alpha+1} A_\gamma = A_\alpha$. Trivially, if $\vec{A}$ is increasing (in the inclusion relation), then $\vec{A}$ converges with limit $\bigcup_{\alpha<\beta} A_\alpha$, and if it is decreasing, then it converges with limit $\bigcap_{\alpha<\beta} A_\alpha$. It is easy to see that if

$\vec{A}$ is a convergent sequence of equivalence relations on a set I, then $\lim \vec{A}$ is also an equivalence relation on I.

We will apply the following facts in the proofs of the first two main results. They follow directly from the above remarks and the transitivity of the relation $\preceq$.

OBSERVATION 3.3.9. *Let $\langle E_\alpha \mid \alpha < \kappa\rangle$ be a sequence of equivalence relations on κ.*

(1) *If $E_\gamma \preceq E_\beta$ holds for all $\gamma < \beta < \kappa$, then $\langle E_\beta \mid \beta < \alpha\rangle$ converges for all $\alpha < \kappa$ and $\lim_{\beta\to\bar{\alpha}} E_\beta \preceq \lim_{\beta\to\alpha} E_\beta$ holds for all $\bar{\alpha} \leq \alpha < \kappa$.*
(2) *If $E_\beta \preceq E_\gamma$ holds for all $\gamma < \beta < \kappa$, then $\langle E_\beta \mid \beta < \alpha\rangle$ converges for all $\alpha < \kappa$ and $\lim_{\beta\to\alpha} E_\beta \preceq \lim_{\beta\to\bar{\alpha}} E_\beta$ holds for all $\bar{\alpha} \leq \alpha < \kappa$.* □

We are now ready to apply our methods and constructions to prove Theorem 3.1.10.

PROOF OF THEOREM 3.1.10. For a given $s : \kappa \longrightarrow (\kappa \setminus \{0\})$, let $s{+}1$ be the function with domain κ defined by $(s{+}1)(\alpha) = s(\alpha){+}1$. We construct a sequence $\langle E_\alpha \mid \alpha < \kappa\rangle$ of equivalence relations on κ by defining the nontrivial equivalence classes of each relation. For $\alpha < \kappa$, a subset $Z \subsetneq \kappa$ is a nontrivial equivalence class of E_α if and only if one of the following conditions holds.

- $Z = (s+1)^*(\beta)$, for some $\beta < \alpha$,
- $s(\alpha) = 1$ and $Z = (s+1)^*(\alpha)$,
- $s(\alpha) > 1$ and Z is the active segment of $(s+1)^*(\alpha)$ of order type $s(\alpha) + 1$.

It is easy to check that the following statements hold for all $\alpha < \kappa$.

(1) E_α is bounded and either active or inactive. Moreover, we have $\mathrm{ht}(E_\alpha) = s(\alpha)$.
(2) For all $\beta < \alpha$, $E_\beta \preceq E_\alpha$. In particular, $\langle E_\beta \mid \beta < \alpha\rangle$ converges and $E^*_\alpha = \lim_{\beta\to\alpha} E_\beta$ is a bounded equivalence relation on κ.

For each $\alpha < \kappa$, we define $\mathbb{P}^s_\alpha = C_{E^*_\alpha}$. These partial orders satisfy the first property of the theorem by the first statement of Theorem 3.1.9. By the first part of Observation 3.3.9, if $\beta < \alpha < \kappa$, then $E^*_\beta \preceq E^*_\alpha$ and we can use Observation 3.2.4 to see that the partial orders satisfy the last property of the theorem.

Let $\alpha < \kappa$. We have $E^*_{\alpha+1} = E_\alpha$, $\mathbb{P}^s_{\alpha+1} = C_{E_\alpha}$ and therefore

$$\mathbb{1}_{\mathbb{P}^s_{\alpha+1}} \Vdash \text{“}\tau(\check{\mathfrak{G}}) = \mathrm{ht}(\check{E}_\alpha) = \check{s}(\check{\alpha})\text{”}.$$

If α is a limit ordinal, then it is not hard to show that

$$E^*_\alpha = \lim_{\beta\to\alpha} E_\beta = \bigcup_{\beta<\alpha} E_\beta = E^{s+1}_\alpha$$

and $\mathbb{P}^s_\alpha = C_{E^{s+1}_\alpha}$. This means

$$\mathbb{1}_{\mathbb{P}^s_\alpha} \Vdash \text{“}\tau(\mathfrak{G}) = \mathrm{ht}(E^{s+1}_\alpha) = 1\text{”},$$

because E^{s+1}_α is an inactive bounded equivalence relation on κ.

Finally, if $s \restriction \gamma = t \restriction \gamma$ for $s, t : \kappa \longrightarrow (\kappa \setminus \{0\})$ and $\gamma < \kappa$, then we also have $s^* \restriction \gamma = t^* \restriction \gamma$ and it is easy to check that the above construction yields the same equivalence relations E_δ for all $\delta < \gamma$. Since $E^*_\gamma = \lim_{\delta \to \gamma} E_\delta$, the resulting E^*_γ coincide, and therefore $\mathbb{P}^s_\gamma = \mathbb{P}^t_\gamma$. $\square$

3.4. Consecutive Ground Models

This section is devoted to the proof of Theorem 3.1.11.

Before proving the theorem, we would like to comment on the first order expressibility of its statement. It is by now a well-known fact that every ground model is uniformly definable in a parameter, see [**Lav07**]. Even this *fact*, though, may at first not seem to be first order expressible. But here is a simple way to state it: There is a first order formula $\varphi(x, y)$ in the language of set theory[3] such that the following is provable in ZFC:

$$(\forall \mathbb{P})(\forall z)\ \big[(\mathbb{P} \textit{ is a partial order and } z = \mathcal{P}(|\mathbb{P}|^+)) \\ \longrightarrow \mathbb{1}_{\mathbb{P}} \Vdash \text{“}\check{\mathrm{V}} = \{x \mid \phi(x, z)\}\text{”}\big]$$

Vice versa, given a set z, it is a simple matter to check whether $\{x \mid \phi(x, z)\}$ is a ZFC-model of which the universe is a forcing extension. So point (3) of the theorem can be expressed by saying that for every sequence $s : \lambda \longrightarrow (\lambda \setminus \{0\})$, there is a sequence $\langle z_\alpha \mid 0 < \alpha < \lambda \rangle$ of sets such that, for all $0 < \alpha < \lambda$, the class $M^s_\alpha = \{x \mid \phi(x, z_\alpha)\}$ is a ground model that satisfies the given statements. Formulating the additional requirement in (3) does not pose a problem either. So let us turn to the proof of Theorem 3.1.11.

Proof of Theorem 3.1.11. Let $t : \kappa \longrightarrow \kappa$ denote the function with constant value $\lambda + 2$, and let t^* be the function given by Lemma 3.3.5. We define E to be the bounded, sealed equivalence relation E^t_λ on κ, i.e.

$$\mu E \eta \iff \mu = \eta \ \vee\ (\exists \alpha < \lambda)\ \mu, \eta \in t^*(\alpha).$$

Set $\mathbb{Q}_\lambda = C_E$. By Theorem 3.1.9 and Lemma 3.3.3, $\mathbb{Q}_\lambda$ satisfies the first and the second statement of the theorem.

Let $\mathrm{V}[G]$ be a $\mathbb{Q}_\lambda$-generic extension of the ground model V and let $s : \lambda \longrightarrow (\lambda \setminus \{0\})$ be a sequence in $\mathrm{V}[G]$. By the above remark, s is already an element of V and we can make the following definitions in V.

For $\alpha < \lambda$, we define an equivalence relation E_α on κ by specifying that $Z \subseteq \kappa$ is a nontrivial equivalence class of E_α if and only if one of the following conditions holds.

- $Z = t^*(\beta)$ for some $\alpha < \beta < \lambda$.
- $s(\alpha) = 1$ and $Z = t^*(\alpha)$.
- $s(\alpha) > 1$ and Z is the active segment of $t^*(\alpha)$ of order type $s(\alpha)+1$.

Again, it is easy to see that the following statements hold for all $\alpha < \lambda$.

[3] Of course, this existential quantification can be eliminated by writing down the formula ϕ explicitly, but the details of its definition are irrelevant for our purposes.

(1) E_α is bounded and either active or inactive. Moreover, we have $\operatorname{ht}(E_\alpha) = s(\alpha)$.
(2) For all $\beta < \alpha$, $E_\alpha \preceq E_\beta$. In particular, $\langle E_\beta \mid \beta < \alpha \rangle$ converges and $E^*_\alpha = \lim_{\beta \to \alpha} E_\beta$ is a bounded equivalence relation on κ.

For each $\alpha < \lambda$, we define $\mathbb{P}^s_\alpha = C_{E^*_\alpha}$ and $M^s_\alpha = \mathrm{V}[G \cap \mathbb{P}^s_\alpha]$. By the second part of Observation 3.3.9, if $\beta < \alpha < \lambda$, then $E^*_\alpha \preceq E^*_\beta$ and we can use Observation 3.2.4 to see that the sequence $\langle M^s_\alpha \mid \alpha < \lambda \rangle$ of ground models is decreasing.

Let $\alpha < \lambda$. We have $E^*_{\alpha+1} = E_\alpha$ and $\mathbb{P}^s_{\alpha+1} = C_{E_\alpha}$. Observation 3.3.7 yields

$$\mathbb{1}_{\mathbb{P}^s_{\alpha+1}} \Vdash \text{“}\tau(\check{\mathfrak{G}}) = \operatorname{ht}(\check{E}_\alpha) = \check{s}(\check{\alpha})\text{”}.$$

If α is a limit ordinal, then $E^*_\alpha = \lim_{\beta \to \alpha} E_\beta = \bigcap_{\beta < \alpha} E_\beta$, because the sequence $\langle E_\beta \mid \beta < \alpha \rangle$ is decreasing. As a result, the nontrivial equivalence classes of E^*_α are precisely the sets $\{t^*(\beta) \mid \alpha \leq \beta < \lambda\}$ and this shows that E^*_α is an inactive bounded equivalence relation on κ. By Observation 3.3.7, $\mathbb{P}^s_\alpha = C_{E^*_\alpha}$ and

$$\mathbb{1}_{\mathbb{P}^s_\alpha} \Vdash \text{“}\tau(\check{\mathfrak{G}}) = \operatorname{ht}(\check{E}^*_\alpha) = 1\text{”}.$$

If $s(\alpha) = s'(\alpha)$ for some $s, s' : \lambda \longrightarrow (\lambda \setminus \{0\})$ and $\alpha < \lambda$, then the above construction produces the same equivalence relation E_α for both functions and therefore the same model $M_{\alpha+1} = \mathrm{V}[G \cap C_{E_\alpha}]$. Finally, by the above analysis, the equivalence relation $E^*_\nu = \lim_{\beta \to \nu} E_\beta$ is the same for all functions $s : \lambda \longrightarrow (\lambda \setminus \{0\})$ and every limit ordinal $\nu < \lambda$. $\square$

3.5. The Mix

In this section, we are producing models of set theory, where a given sequence of nonzero ordinals can be realized as the height of the automorphism tower of $\mathfrak{G}$ in consecutive models such that the next one is a forcing extension *or* a ground model of the previous one, as desired. There are some limitations on the possible patterns, and to formalize them precisely, we introduce the notion of a *realizable prescription*.

DEFINITION 3.5.1. A function $s : \kappa \longrightarrow (\kappa \setminus \{0\}) \times 2$ is a *prescription* on κ. It is *realizable* if $(s(0))_1 = 1$ and the set of all $\alpha < \kappa$ such that $(s(\alpha))_1 = 0$ is not stationary in κ.[4]

The interpretation is that the first coordinate of $s(\alpha)$ gives the desired height of the automorphism tower of $\mathfrak{G}$ in the $(\alpha + 1)$-th model, and the second coordinate says whether the $(\alpha + 1)$-th model should be a forcing extension or a ground model of the α-th model.

THEOREM 3.5.2 ([**FLb**, Theorem 5.2], under Assumption 3.1.7). *For every realizable prescription s on κ, there is a sequence $\langle E_\alpha \mid \alpha < \kappa \rangle$ of bounded equivalence relations on κ with the following properties.*

[4]Here, we use the following notation for components of ordered pairs: $(\langle x, y \rangle)_0 = x$, $(\langle x, y \rangle)_1 = y$.

(1) *For every $\alpha \leq \kappa$, the sequence $\langle E_\beta \mid \beta < \alpha\rangle$ of equivalence relations converges with limit E^*_α.*
(2) *If $\alpha < \kappa$, then $\mathbb{1}_{C_{E^*_{\alpha+1}}} \Vdash$ "$\tau(\check{\mathfrak{G}}) = (\check{s}(\check{\alpha}))_0$ ".*
(3) *If $\alpha < \kappa$ is a limit ordinal, then $\mathbb{1}_{C_{E^*_\alpha}} \Vdash$ "$\tau(\check{\mathfrak{G}}) = 1$ ".*
(4) *If $\alpha < \kappa$ and $s(\alpha)_1 = 0$, then $E^*_{\alpha+1} \preceq E^*_\alpha$.*
(5) *If $\alpha < \kappa$ and $s(\alpha)_1 = 0$, then $E^*_\alpha \preceq E^*_{\alpha+1}$.*

Proof. Let a realizable prescription s be given and $C \subseteq \kappa$ be a club of α with $(s(\alpha))_1 = 1$ and $0 \in C$. Let $f_C : \kappa \longrightarrow C$ be the monotone enumeration of C. Given $\beta < \kappa$, let $i(\beta)$ be that ordinal less than κ such that $\beta \in [f_C(i(\beta)), f_C(i(\beta)+1))$. Let t be the function with domain κ defined by setting $t(\alpha) = (s(\alpha))_0 + 1$.

For $\beta < \kappa$, we define an equivalence relation E_β on κ by specifying its nontrivial equivalence classes. Namely, X is a nontrivial equivalence class of E_β if and only if one of the following statements holds.

- There is an $\alpha < \beta$ such that $(s(\alpha+1))_1 = 1$ and $X = t^*(\alpha)$.
- There is an $\alpha < \beta$ such that $(s(\alpha+1))_1 = 0$ and X is the trimmed segment of $t^*(\alpha)$.
- There is an $\alpha \in (\beta, f_C(i(\beta)+1))$ such that $(s(\alpha))_1 = 0$ and $X = t^*(\alpha)$.
- $(s(\beta))_0 > 1$ and X is the active segment of $t^*(\beta)$ of order type $t(\beta)$ (which is $(s(\beta))_0 + 1$), *or* $(s(\beta))_0 = 1$ and $X = t^*(\beta)$.

This defines the sequence $\langle E_\beta \mid \beta < \kappa\rangle$ of equivalence relations. Obviously, each E_β is bounded. If E_β is active, then its active equivalence class is the active segment of $t^*(\beta)$ of order type $(s(\beta))_0 + 1$. In particular, we have

$$\mathbb{1}_{C_{E^*_{\beta+1}}} \Vdash \text{“}\tau(\check{\mathfrak{G}}) = (\check{s}(\check{\beta}))_0\text{”}. \tag{3.4}$$

If E_β is not active, then $(s(\beta))_0 = 1$, E_β is inactive and (3.4) also holds in this case.

We have to show the sequence has the desired properties. To this end, we verify the following claims.

Claim 1. *For every $\alpha \leq \kappa$, the sequence $\langle E_\beta \mid \beta < \alpha\rangle$ converges.*

Proof of the Claim. Fix a limit ordinal $\alpha \leq \kappa$. Let $\gamma, \delta < \kappa$ be given. We have to find $\bar{\alpha} < \alpha$ such that either for all $\beta \in (\bar{\alpha}, \alpha)$, $\gamma E_\beta \delta$ holds, or for all $\beta \in (\bar{\alpha}, \alpha)$, $\gamma E_\beta \delta$ fails. This is trivial if $\gamma = \delta$, and it is also trivial if there is no $\mu < \alpha$ such that $\gamma E_\mu \delta$ holds. But if there is such a μ, then this means that $\gamma, \delta \in t^*(\xi)$, for some $\xi < f_C(i(\mu)+1)$ – this is easily confirmed by looking at the definition of E_μ above. If $\xi < \alpha$, then for all $\beta, \beta' \in (\xi, \alpha)$ we have $\gamma E_\beta \delta$ if and only id $\gamma E_{\beta'} \gamma$ (again, this is easily checked by referring to the clauses defining the equivalence relations). Hence we can let $\bar{\alpha} = \xi$. But if $\xi \geq \alpha$, then this means that $t^*(\xi)$ is a nontrivial equivalence class of E_μ due to the third condition in the definition of E_μ, so $\xi \in (\mu, f_C(i(\mu)+1)$.

But then, for all $\beta \in [\mu, \alpha)$, $i(\beta) = i(\mu)$, and again, by the same condition, $t^*(\xi)$ will be a nontrivial equivalence class of E_β. So in this case, we can set $\bar{\alpha} = \mu$. □

It is also easy to see that if α is a limit ordinal, then E^*_α is inactive and therefore $\mathbb{1}_{C_{E^*_\alpha}} \Vdash$ "$\tau(\check{\mathfrak{G}}) = 1$".

Claim 2. *If $\alpha < \kappa$ with $(s(\alpha))_1 = 0$, then $E_\alpha \preceq E^*_\alpha$.*

Proof of the Claim. Note that if $(s(\alpha))_1 = 0$, then

$$\alpha \in (f_C(i(\alpha)), f_C(i(\alpha) + 1),$$

since $\alpha \notin C$. There are two cases to consider here.

The first case is that α is a limit ordinal. In that case, it follows that the only disagreement between E^*_α and E_α is that the α-th nontrivial equivalence class of E^*_α is $t^*(\alpha)$, while the α-th nontrivial equivalence class of E_α is the active segment of $t^*(\alpha)$ of order type $(s(\alpha))_0 + 1$. So $E_\alpha \preceq E^*_\alpha$.

The second case is that $\alpha = \bar{\alpha} + 1$ is a successor ordinal. In this case, $E^*_\alpha = E_{\bar{\alpha}}$ and we have to show that $E_\alpha \preceq E_{\bar{\alpha}}$. Since

$$\alpha \in (f_C(i(\alpha)), f_C(i(\alpha) + 1),$$

it follows that the α-th nontrivial equivalence class of $E_{\bar{\alpha}}$ is $t^*(\alpha)$, while the α-th nontrivial equivalence class of E_α is the active segment of $t^*(\alpha)$ of order type $(s(\alpha))_0 + 1$ (using the fourth clause in the definition of E_α and the third clause in the definition of $E_{\bar{\alpha}}$). Moreover, the $\bar{\alpha}$-th nontrivial equivalence class of $E_{\bar{\alpha}}$ is the active segment of $t^*(\bar{\alpha})$ (by the fourth clause in the definition of $E_{\bar{\alpha}}$) and the $\bar{\alpha}$-th nontrivial equivalence class of E_α is the trimmed segment of $t^*(\bar{\alpha})$ (by the second clause in the definition of E_α). E_α and $E_{\bar{\alpha}}$ agree about the other nontrivial equivalence classes, so that it follows that $E_\alpha \preceq E_{\bar{\alpha}}$, as desired. □

Claim 3. *If $\alpha < \kappa$ with $(s(\alpha))_1 = 1$, then $E^*_\alpha \preceq E_\alpha$.*

Proof of the Claim. As in the proof of Claim 2, we distinguish two cases.

The first case is that α is a limit ordinal. As before, E_α and E^*_α agree about the γ-th equivalence classes. The α-th equivalence class of E_α is the active segment of $t^*(\alpha)$ of order type $(s(\alpha))_0 + 1$, while for $\gamma \in t^*(\alpha)$, $\{\gamma\} = [\gamma]_{E^*_\alpha}$. E_α and E^*_α agree about the other nontrivial equivalence classes, which are of the form $t^*(\beta)$, for $\beta \in (\alpha, f_C(i(\alpha) + 1))$. So $E^*_\alpha \preceq E_\alpha$, as claimed.

In the second case to consider, $\alpha = \bar{\alpha} + 1$ is a successor ordinal. So $E^*_\alpha = E_{\bar{\alpha}}$ and we have to show that $E_{\bar{\alpha}} \preceq E_\alpha$. The α-th nontrivial equivalence class of E_α is the active segment of $t^*(\alpha)$ of order type $(s(\alpha))_0 + 1$ (using the fourth clause in the definition of E_α) and, for $\gamma \in t^*(\alpha)$, we have $\{\gamma\} = [\gamma]_{E_{\bar{\alpha}}}$. The $\bar{\alpha}$-th nontrivial equivalence class of $E_{\bar{\alpha}}$ is the active segment of $t^*(\bar{\alpha})$ (by the fourth clause in the definition of $E_{\bar{\alpha}}$), and the $\bar{\alpha}$-th nontrivial equivalence class of E_α is $t^*(\bar{\alpha})$ (by the first clause in the definition of E_α). E_α and $E_{\bar{\alpha}}$

agree about the other nontrivial equivalence classes, so that it follows that $E_\alpha \preceq E_{\bar{\alpha}}$, as desired. □

This finishes the proof of the theorem. □

3.6. Unbounded potential automorphism tower heights

This section contains the proof of Theorem 3.1.12. This proof is based on an application of Theorem 1.2.1 and folklore results about splitting forcings into two-step iterations. We start by stating and proving these standard results. Given a boolean algebra $\mathbb{B}$, we let $\mathbb{B}^*$ denote the partial order with domain $\mathbb{B} \setminus \{0_\mathbb{B}\}$ ordered by the restriction of $\leq_\mathbb{B}$ to this set.

LEMMA 3.6.1. *Let κ be an infinite cardinal, $\mathbb{B}$ be a complete boolean algebra and $\dot{x} \in \mathrm{V}^{\mathbb{B}^*}$ with $\mathbb{1}_\mathbb{B} \Vdash$ "$\dot{x} \subseteq \check{\kappa}$". Then there is a κ-generated complete subalgebra $\mathbb{C}$ of $\mathbb{B}$ in V and names $\dot{\mathbb{D}}, \dot{y} \in \mathrm{V}^{\mathbb{C}^*}$ with the following properties.*

1. *We have $\mathbb{1}_\mathbb{C} \Vdash$ "$\dot{\mathbb{D}}$ is a partial order" and there is a dense embedding $i : \mathbb{B}^* \longrightarrow \mathbb{C}^* * \dot{\mathbb{D}}$ such that $i(c) = \langle c, \dot{c}\rangle$ with $c \Vdash$ "$\dot{c} = \check{c}$" for all $c \in \mathbb{C}^*$.*
2. *If $G_0 * G_1$ is $(\mathbb{C}^* * \dot{\mathbb{D}})$-generic over V and G is the preimage of $G_0 * G_1$ under i, then $\dot{x}^G = \dot{y}^{G_0} \in \mathrm{V}[G_0]$.*

PROOF. Given $\alpha < \kappa$, set

$$\mathbb{B}_\alpha = \{b \in \mathbb{B}^* \mid b \Vdash \text{“}\check{\alpha} \in \dot{x}\text{”}\}$$

and $b_\alpha = \sup_\mathbb{B} \mathbb{B}_\alpha$. Let $\mathbb{C}$ be the complete subalgebra of $\mathbb{B}$ generated by the set $\{b_\alpha \mid \alpha < \kappa\}$ and define

$$\dot{y} = \{\langle \check{\alpha}, b_\alpha\rangle \in \mathrm{V}^{\mathbb{C}^*} \times \mathbb{C}^* \mid \alpha < \kappa,\ b_\alpha \neq 0_\mathbb{B}\} \in \mathrm{V}^{\mathbb{C}^*}.$$

Let G be $\mathbb{B}^*$-generic over V. If $\alpha \in \dot{x}^G$, then there is a $b \in G$ with $b \Vdash$ "$\check{\alpha} \in \dot{x}$" and this shows $b_\alpha \in G$ and $\alpha \in \dot{y}^{G \cap \mathbb{C}^*}$. The other direction follows directly from the fact that $b_\alpha \in \mathbb{B}_\alpha$ holds for all $\alpha < \kappa$.

There is a canonical $\mathbb{C}^*$-name $\dot{\mathbb{D}}$ with the property that, whenever G_0 is $\mathbb{C}^*$-generic over V, then $\dot{\mathbb{D}}^{G_0}$ is the partial order whose domain is the set

$$\{b \in \mathbb{B}^* \mid (\forall c \in G_0)\ b \parallel_{\mathbb{B}^*} c\}$$

ordered by the restriction of $\leq_\mathbb{B}$ to this domain.

If $b \in \mathbb{B}^*$ and G is $\mathbb{B}^*$-generic over V with $b \in G$, then $b \in \dot{\mathbb{D}}^{G \cap \mathbb{C}^*}$ and there is a $c \in G \cap \mathbb{C}^*$ with $c \Vdash$ "$\check{b} \in \dot{\mathbb{D}}$". This shows that the function

$$i_0 : \mathbb{B}^* \longrightarrow \mathbb{C}^*;\ b \mapsto \sup_\mathbb{B}\{c \in \mathbb{C}^* \mid c \Vdash \text{“}\check{b} \in \dot{\mathbb{D}}\text{”}\}$$

is well-defined. Pick a function $i_1 : \mathbb{B}^* \longrightarrow \mathrm{V}^{\mathbb{C}^*}$ with $\mathbb{1}_\mathbb{C} \Vdash$ "$i_1(b) \in \dot{\mathbb{D}}$" and $i_0(b) \Vdash$ "$\check{b} = i_1(b)$" for all $b \in \mathbb{B}^*$. Define $i : \mathbb{B}^* \longrightarrow \mathbb{C}^* * \dot{\mathbb{D}}$ by setting $i(b) = \langle i_0(b), i_1(b)\rangle$.

Given $c, c' \in \mathbb{C}^*$, it is easy to see that $c' \Vdash$ "$\check{c} \in \dot{\mathbb{D}}$" is equivalent to $c' \leq_{\mathbb{C}} c$. This shows that $i_0(c) = c$ holds for all $c \in \mathbb{C}^*$. We show that i is a dense embedding.

Let $b_0, b_1 \in \mathbb{B}^*$ with $b_0 \leq_{\mathbb{B}} b_1$. Given $c \in \mathbb{C}^*$, if $c \Vdash$ "$\check{b}_0 \in \dot{\mathbb{D}}$", then $c \Vdash$ "$\check{b}_1 \in \dot{\mathbb{D}}$". This shows that $i_0(b_0) \leq_{\mathbb{C}^* * \dot{\mathbb{D}}} i_0(b_1)$ holds and hence $i(b_0) \leq_{\mathbb{C}^* * \dot{\mathbb{D}}} i(b_1)$. Next, fix $a_0, a_1 \in \mathbb{B}^*$ with $a_0 \perp_{\mathbb{B}^*} a_1$. Assume, toward a contradiction, that there is a $\langle c, \dot{d} \rangle \in \mathbb{C}^* * \dot{\mathbb{D}}$ with $\langle c, \dot{d} \rangle \leq_{\mathbb{C}^* * \dot{\mathbb{D}}} i(a_0), i(a_1)$. We can find a $0_{\mathbb{C}} <_{\mathbb{C}} c_* \leq_{\mathbb{C}} c$ and a condition $d \in \mathbb{B}^*$ with $c_* \Vdash$ "$\dot{d} = \check{d}$". This means $c_* \Vdash$ "$\check{d} \leq_{\dot{\mathbb{D}}} \check{a}_0, \check{a}_1$" and therefore $0_{\mathbb{B}} <_{\mathbb{B}} d \leq_{\mathbb{B}} a_0, a_1$, a contradiction. Finally, fix $\langle c, \dot{d} \rangle \in \mathbb{C}^* * \dot{\mathbb{D}}$. As above, there are $0_{\mathbb{C}} <_{\mathbb{C}} c_* \leq_{\mathbb{C}} c$ and $d \in \mathbb{B}^*$ with $c_* \Vdash$ "$\dot{d} = \check{b}$". Since $c_* \Vdash$ "$\check{c}_* \parallel_{\check{\mathbb{B}}^*} \check{d}$", there is a condition $d_* \in \mathbb{B}^*$ with $d_* \leq_{\mathbb{B}} c_*, d$. By the above computations, $i_0(d_*) \leq_{\mathbb{C}} i_0(c_*) = c_* \leq_{\mathbb{C}} c$ and $i_0(d_*) \Vdash$ "$i_1(d_*) \leq_{\dot{\mathbb{D}}} \dot{d}$". This means $i(d_*) \leq_{\mathbb{C}^* * \dot{\mathbb{D}}} \langle c, \dot{d} \rangle$ and i is a dense embedding.

If $G_0 * G_1$ is $(\mathbb{C}^* * \dot{\mathbb{D}})$-generic over V and G is the preimage of $G_0 * G_1$ under i, then both G_0 and $G \cap \mathbb{C}^*$ are $\mathbb{C}^*$-generic over V. Since $i_0 \restriction \mathbb{C}^* = \mathrm{id}_{\mathbb{C}^*}$, it follows that $G \cap \mathbb{C}^* \subseteq G_0$ and the maximality of generic filters yields $G \cap \mathbb{C}^* = G_0$. By the above calculations, $\dot{x}^G = \dot{y}^{G \cap \mathbb{C}^*} = \dot{y}^{G_0} \in \mathrm{V}[G_0]$. □

If κ is an infinite regular cardinal, $\mathbb{B}$ is a boolean algebra and $\mathbb{C}$ is a subalgebra of $\mathbb{B}$, then $\mathbb{C}$ is called $<\kappa$-complete in $\mathbb{B}$ if $\inf_{\mathbb{B}} X \in \mathbb{C}$ for all $X \in [\mathbb{C}]^{<\kappa}$.

PROPOSITION 3.6.2. *Let κ be an infinite regular cardinal, $\mathbb{B}$ be a complete boolean algebra that satisfies the κ-chain condition and $\mathbb{C}$ be a subalgebra of $\mathbb{B}$. If $\mathbb{C}$ is $<\kappa$-complete in $\mathbb{B}$, then $\mathbb{C}$ is a complete subalgebra of $\mathbb{B}$.*

PROOF. Assume, toward a contradiction, that $\mathbb{C}$ is not a complete subalgebra of $\mathbb{B}$ and let ν be the least cardinal such that there is a sequence $\langle c_\alpha \in \mathbb{C} \mid \alpha < \nu \rangle$ with $\inf_{\mathbb{B}} \{c_\alpha \mid \alpha < \nu\} \notin \mathbb{C}$. By our assumption, $\nu \geq \kappa$ and it is easy to see that ν is a regular cardinal. Given $\alpha < \nu$, we define $b_\alpha = \inf_{\mathbb{B}} \{c_\beta \mid \beta < \alpha\}$. Our assumptions imply $0_{\mathbb{B}} \neq b_\alpha \in \mathbb{C}$ and $b_\beta \leq_{\mathbb{B}} b_\alpha$ for all $\alpha \leq \beta < \nu$. Moreover,

$$\inf_{\mathbb{B}} \{b_\alpha \mid \alpha < \nu\} = \inf_{\mathbb{B}} \{c_\alpha \mid \alpha < \nu\} \notin \mathbb{C}.$$

If we define $a_\alpha = b_\alpha - b_{\alpha+1}$ for all $\alpha < \nu$, then the set

$$A = \{a_\alpha \in \mathbb{B} \mid \alpha < \nu, a_\alpha \neq 0_{\mathbb{B}}\}$$

is an anti-chain in $\mathbb{B}$ and therefore has cardinality less than κ. This means that there is an $\alpha < \nu$ with $a_\beta = 0_{\mathbb{B}}$ for all $\alpha \leq \beta < \nu$ and an easy induction shows that this implies $b_{\alpha+1} = b_\beta$ for all $\alpha < \beta < \nu$. We can conclude $\inf_{\mathbb{B}} \{b_\alpha \mid \alpha < \nu\} = b_{\alpha+1} \in \mathbb{C}$, a contradiction. □

LEMMA 3.6.3. *Let κ be an infinite cardinal, $\mathbb{B}$ be a complete boolean algebra that satisfies the κ^+-chain condition and C be a subset of $\mathbb{B}$ of cardinality at most κ. If $\mathbb{C}$ is the complete subalgebra of $\mathbb{B}$ generated by C, then $\mathbb{C}$ has cardinality at most 2^κ.*

PROOF. It suffices to construct a complete subalgebra $\mathbb{C}^+$ of $\mathbb{B}$ that contains C and has cardinality at most 2^κ. We define an ascending sequence $\langle \mathbb{C}_\alpha \mid \alpha < \kappa^+ \rangle$ of subalgebras of $\mathbb{B}$ in the following way.

(1) $\mathbb{C}_0$ is the subalgebra of $\mathbb{B}$ generated by C.
(2) If $\alpha \in \kappa^+ \cap \mathrm{Lim}$, then $\mathbb{C}_\alpha = \bigcup\{\mathbb{C}_\beta \mid \beta < \alpha\}$.
(3) $\mathbb{C}_{\alpha+1}$ is the subalgebra of $\mathbb{B}$ generated by the set

$$\{\inf_{\mathbb{B}} X \mid X \in [\mathbb{C}_\alpha]^{<\kappa^+}\}$$

for all $\alpha < \kappa^+$.

An easy induction shows that the subalgebra $\mathbb{C}_\alpha$ has cardinality at most 2^κ for all $\alpha < \kappa^+$ and this shows that the subalgebra $\mathbb{C}^+ = \bigcup\{\mathbb{C}_\alpha \mid \alpha < \kappa^+\}$ also has cardinality at most 2^κ. We show that $\mathbb{C}^+$ is a complete subalgebra of $\mathbb{B}$. By Proposition 3.6.2, it suffices to show that $\mathbb{C}^+$ is $<\kappa^+$-complete in $\mathbb{B}$. If $X \in [\mathbb{C}^+]^{<\kappa^+}$, then there is an $\alpha < \kappa^+$ with $X \subseteq \mathbb{C}_\alpha$. But this means $\inf_{\mathbb{B}} X \in \mathbb{C}_{\alpha+1} \subseteq \mathbb{C}^+$. □

With the help of the above results, it is easy to show that the statement of Theorem 3.1.12 is a direct consequence of Theorem 1.2.1.

PROOF OF THEOREM 3.1.12. Assume $\mathrm{V} = \mathrm{L}$. Given an ordinal α, Theorem 1.2.1 shows that there is a σ-distributive complete boolean algebra $\mathbb{B}$ that satisfies the $\aleph_2$-chain condition, a $\mathbb{B}^*$-name $\dot{G}$ with

$$\mathbb{1}_{\mathbb{B}^*} \Vdash \text{“}\dot{G} \textit{ is a centreless group with domain } \omega_1 \textit{ and } \tau(\dot{G}) = \check{\alpha}\text{”}$$

and a $\mathbb{B}^*$-nice name $\dot{x}$ for a subset of ω_1 with

$$\mathbb{1}_{\mathbb{B}^*} \Vdash \text{“}\dot{x} = \{\prec\prec\beta,\gamma\succ,\delta\succ \mid \beta,\gamma,\delta < \omega_1,\ \beta \cdot_{\dot{G}} \gamma = \delta\}\text{”}.$$

We let $\langle \mathbb{B}_\alpha, \dot{G}_\alpha, \dot{x}_\alpha \rangle$ denote the $<_{\mathrm{L}}$-least triple with the above properties. Next, let $\mathbb{C}_\alpha$, $\dot{\mathbb{D}}_\alpha$, $\dot{y}_\alpha$ and i_α denote the $<_{\mathrm{L}}$-least objects satisfying the conclusion of Lemma 3.6.1 with respect to $\mathbb{B}_\alpha$ and $\dot{x}_\alpha$. Since $\mathbb{C}_\alpha$ is a $\aleph_1$-generated complete subalgebra of $\mathbb{B}_\alpha$ and $\mathbb{B}_\alpha$ satisfies the $\aleph_2$-chain condition, we can apply Lemma 3.6.3 to see that $\mathbb{C}_\alpha$ has cardinality at most $2^{\aleph_1} = \aleph_2$.

Up to isomorphism, there are only set-many boolean algebras of cardinality $\aleph_2$ and we can find a complete boolean algebra $\mathbb{C}_C$ such that the class $\{\alpha \in \mathrm{On} \mid \mathbb{C}_\alpha \textit{ is isomorphic to } \mathbb{C}_C\}$ is cofinal in On.

Given $\alpha \in C$, we let $\dot{z}_\alpha$ denote the $<_{\mathrm{L}}$-least $\mathbb{C}^*_C$-nice name for a subset of ω_1 such that $\dot{z}_\alpha$ corresponds to $\dot{y}_\alpha$ with respect to some isomorphism of $\mathbb{C}_C$ and $\mathbb{C}_\alpha$. Again, there are only set-many $\mathbb{C}^*_C$-nice names for subsets of ω_1 and we can find such a name $\dot{z}$ such that the class $D = \{\alpha \in C \mid \dot{z} = \dot{z}_\alpha\}$ is cofinal in On.

Let F be $\mathbb{C}^*_C$-generic over the ground model V. In $\mathrm{V}[F]$, define an $\mathcal{L}_{\mathrm{GT}}$-model G with domain ω_1 by setting

$$\beta \ \cdot_G \ \gamma = \delta \iff \prec\prec\beta,\gamma\succ,\delta\succ \in \dot{z}^F$$

for all $\beta,\gamma,\delta < \omega_1$.

Given $\alpha \in D$, fix an isomorphism $\pi : \mathbb{C}_C \longrightarrow \mathbb{C}_\alpha$ contained in V such that $\dot{z}_\alpha$ corresponds to $\dot{y}_\alpha$ with respect to π and let F_0 denote the filter in $\mathbb{C}^*_\alpha$ induced by F via π. Then F_0 is $\mathbb{C}^*_\alpha$-generic over V, $\mathrm{V}[F] = \mathrm{V}[F_0]$ and $\dot{y}_\alpha^{F_0} = \dot{z}^F$. Let F_1 be $\dot{\mathbb{D}}_\alpha^{F_0}$-generic over $\mathrm{V}[F]$ and let $\bar{F}$ denote the preimage of $F_0 * F_1$ under i_α. Then $\bar{F}$ is $\mathbb{B}^*_\alpha$-generic over V, $\mathrm{V}[\bar{F}] = \mathrm{V}[F_0][F_1]$, $\dot{G}_\alpha^{\bar{F}}$ is a centreless group with domain ω_1 and $\tau(\dot{G}_\alpha^{\bar{F}}) = \alpha$ in $\mathrm{V}[\bar{F}]$ and

$$\beta \cdot_{\dot{G}_\alpha^{\bar{F}}} \gamma = \delta \iff \prec\prec\beta, \gamma\succ, \delta\succ \in \dot{x}_\alpha^{\bar{F}}$$

for all $\beta, \gamma, \delta < \omega_1$. Since $\dot{x}_\alpha^{\bar{F}} = \dot{y}_\alpha^{F_0} = \dot{z}^F$, we can conclude $\dot{G}_\alpha^{\bar{F}} = G \in \mathrm{V}[F]$. In particular, G is a centreless group and

$$\mathbb{1}_{\dot{\mathbb{D}}_\alpha^{F_0}} \Vdash \text{“}\tau(\check{G}) = \check{\alpha}\text{”}$$

holds in $\mathrm{V}[F]$. $\square$

CHAPTER 4

An absoluteness result for countable groups

All groups appearing in the non-absoluteness results of the last chapter have uncountable cardinality. In this short chapter, we show that this is a necessary condition for groups whose automorphism towers are highly malleable by forcing by proving an absoluteness result for the first three stages of the automorphism tower of countable centreless groups. In particular, it is not possible to have results like Theorem 3.1.2 or Theorem 3.1.3 for countable centreless groups. The proof of this absoluteness statement uses results from the theory of *Polish groups* and heavily relies on the notion of *special pairs* and Theorem 2.2.5.

The work presented in this chapter is published in [**Lücc**].

4.1. Unique Polish group topologies

We introduce techniques from the theory of *Polish groups* that will be essential for the proof of the absoluteness result for the automorphism towers of countable, centreless groups mentioned above. Remember that a *topological group* is a pair $\langle G, \tau\rangle$ consisting of a group G and a topology τ on the domain of G such that the map $\left[\langle g, h\rangle \mapsto g \cdot h^{-1}\right]$ is continuous with respect to τ. We call a topological space $\langle X, \tau\rangle$ *Polish* if τ is induced by a complete metric on X and there is a countable subset of X that is dense in τ. Finally, we call a topological group $\langle G, \tau\rangle$ a *Polish group* if the corresponding topological space is Polish. In this case, we call τ a *Polish group topology* on G.

Proposition 4.1.1. *Let $\langle G, \tau\rangle$ be a topological group such that the corresponding topological space is a Hausdorff space. If $\mathrm{t} \in \mathcal{T}^1_G$, then the set $\{g \in G \mid \mathrm{t}^G(g) = 1_G\}$ is closed in τ.*

Proof. An easy induction shows that the map

$$\xi_{\mathrm{t}} : G^n \longrightarrow G;\ \vec{g} \longmapsto \mathrm{t}^G(\vec{g})$$

is continuous with respect to τ for every $\mathcal{L}_G$-term $\mathrm{t} \in \mathcal{T}^n_G$. Since τ is a Hausdorff space, we can conclude that the set

$$\{g \in G \mid \mathrm{t}^G(g) = 1_G\} = \xi_{\mathrm{t}}^{-1}\text{"}\{1_G\}$$

is closed in τ for every $t \in \mathcal{T}^1_G$. $\square$

Next, we consider Polish groups whose topology is completely determined by the algebraic structure of the group.

DEFINITION 4.1.2. Let G be a group. We say that G has a *unique Polish group topology* if there is exactly one topology τ on the domain of G such that $\langle G, \tau\rangle$ is a Polish group.

We state a theorem of George W. Mackey that allows a nice characterization of groups with unique Polish group topologies. Remember that a measurable space $\langle X, \mathcal{S}\rangle$ is a *standard Borel space* if there is a Polish topology τ on X such that $\mathcal{S}$ is equal to the σ-algebra $\mathcal{B}(\tau)$ of all subsets of X that are Borel with respect to τ.

THEOREM 4.1.3 ([**Mac57**, Theorem 3.3]). *Let $\langle X, \mathcal{S}_0\rangle$ and $\langle X, \mathcal{S}_1\rangle$ be standard Borel spaces. If there is a countable point-separating family*[1] *of subsets of X whose members are elements of both $\mathcal{S}_0$ and $\mathcal{S}_1$, then $\mathcal{S}_0 = \mathcal{S}_1$.*

COROLLARY 4.1.4. *The following statements are equivalent for a Polish group $\langle G, \tau\rangle$.*

(1) *τ is the unique Polish group topology on G.*
(2) *There is a countable point-separating family of subsets of the domain of G whose members are Borel with respect to any Polish group topology on G.*

PROOF. If τ is the unique Polish group topology on G and B is a countable basis of τ, then B satisfies the above properties.

In the other direction, assume that $\mathcal{F}$ is a family of subsets with the above properties and $\bar{\tau}$ is a Polish group topologies on G. If we define $\mathcal{B}(\tau)$ and $\mathcal{B}(\bar{\tau})$ as above, then Theorem 4.1.3 and our assumptions imply $\mathcal{B}(\tau) = \mathcal{B}(\bar{\tau})$. Since Borel sets have the Baire Property (see [**Kec95**, Proposition 8.22]), the identity map on G is a Baire-measurable group homomorphism with respect to τ and $\bar{\tau}$. By [**BK96**, Theorem 1.2.6], it is continuous and open with respect to τ and $\bar{\tau}$. This shows $\tau = \bar{\tau}$. □

PROPOSITION 4.1.5. *Let $\langle G, \tau\rangle$ be a Polish group. If there is a countable subset A of the domain of G such that $\langle G, A\rangle$ is a special pair, then τ is the unique Polish group topology on G.*

PROOF. If $\mathrm{t} \equiv \mathrm{t}(v)$ is a term in $\mathcal{T}^1_A$, then we define

$$T^0_{\mathrm{t}} = \{g \in G \mid \mathrm{t}^G(g) = 1_G\}$$

and $T^1_{\mathrm{t}} = G \setminus T_{\mathrm{t}}$. Let $\mathcal{F}$ denote the family consisting of all subsets of the domain of G of the form T^0_{t} or T^1_{t} for some $\mathrm{t} \in \mathcal{T}^1_A$. Then $\mathcal{F}$ is countable and separates points, because $\langle G, A\rangle$ is a special pair. If $\bar{\tau}$ is a Polish group topology on G, then all elements of $\mathcal{F}$ are contained in $\mathcal{B}(\bar{\tau})$ by Proposition 4.1.1. Corollary 4.1.4 implies $\tau = \bar{\tau}$. □

REMARK 4.1.6. *The converse of the above implication is not true: Bojana Pejić and Paul Gartside showed that the group* $\mathrm{SO}(3, \mathbb{R})$ *has a unique*

[1] We call a family $\mathcal{F}$ of subsets of X *separating* if for any pair $\langle x, y\rangle$ of distinct elements in X, there is an $F \in \mathcal{F}$ with $x \in F$ and $y \notin F$.

Polish group topology (see [**GP08**, Theorem 11]*) and there is no countable subset I of $\mathcal{T}_{\mathrm{SO}(3,\mathbb{R})}$ such that the family $\{T^i_{\mathrm{t}} \mid \mathrm{t} \in I,\ i < 2\}$ separates points (see* [**GP08**, Lemma 12]*).*

We close this section by introducing a consequence of the existence of a unique Polish group topology that allows us to deduce the absoluteness result in the next section. This consequence is called *automatic continuity of automorphisms*.

PROPOSITION 4.1.7. *Let G be a group with a unique Polish group topology. Then every group automorphism of G is continuous with respect to the unique Polish group topology on G.*

PROOF. Let τ be the unique Polish group topology on G and assume, toward a contradiction, that there is an automorphism π of G that is not continuous with respect to τ. Define $\bar{\tau}$ to be the collection of all subsets of G of the form π"U, where U is open in τ. It is easy to check that $\bar{\tau}$ is a Polish group topology that is not equal to τ, a contradiction. $\square$

4.2. The absoluteness result

The aim of this section is to prove the following theorem.

THEOREM 4.2.1 ([**Lücc**, Corollary 4.2]). *Let M be a transitive class*[2] *such that $\langle M, \in\rangle$ is a model of* ZFC *and G be a centreless group that is an element of M. If G is countable in $\langle M, \in\rangle$ and $\tau(G) > 1$ holds in $\langle M, \in\rangle$, then $\tau(G) > 1$.*

This result is an easy consequence of the following theorem.

THEOREM 4.2.2 ([**Lücc**, Theorem 4.1]). *Let M be a transitive class such that $\langle M, \in\rangle$ is a model of* ZFC*, G be a centreless group that is an element of M, $\langle G^M_\alpha \mid \alpha \in \mathrm{On} \cap M\rangle$ be an automorphism tower of G in $\langle M, \in\rangle$ and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. If G is countable in M, then there is an embedding $\pi : G^M_2 \longrightarrow G_2$ with $\pi \restriction G = \mathrm{id}_G$.*

PROOF OF THE THEOREM 4.2.1 FROM THEOREM 4.2.2. Assume that G is countable in $\langle M, \in\rangle$ and $G^M_1 \neq G^M_2$. Let $\pi : G^M_2 \longrightarrow G_2$ be the embedding given by Theorem 4.2.2. It suffices to show that π^{-1}"$G_1 \subseteq G^M_1$.

Let $h \in G^M_2$ with $\pi(h) \in G_1$. Given $g \in G$, we have $\iota_{\pi(h)}(g) \in G$ and therefore

$$\pi(\iota_{\pi(h)}(g)) = \iota_{\pi(h)}(g) = \iota_{\pi(h)}(\pi(g)) = \pi(\iota_h(g)).$$

Since π is an embedding, we can conclude that $\iota_h(g) = \iota_{\pi(h)}(g)$ holds for all $g \in G$ and hence $\iota_h \restriction G = \iota_{\pi(h)} \restriction G \in \mathrm{Aut}(G) \cap M$. By the definition of G^M_2, there is an $\bar{h} \in G^M_1$ with $\iota_{\bar{h}} \restriction G = \iota_h \restriction G$ and this shows $h^{-1} \cdot \bar{h} \in \mathrm{C}_{G^M_2}(G)$. An application of Theorem 2.1.1 in $\langle M, \in\rangle$ yields $h = \bar{h} \in G^M_1$. $\square$

[2]Note that M can be set-sized or even countable. In addition, we only need to assume that $\langle M, \in\rangle$ is a model of a "*suitable*" finite fragment of ZFC which enables us to run all the arguments of this section that take place inside of M.

We outline how the results of Section 4.1 can be applied to analyze the first stages of the automorphism tower of a countable, centreless group. If $\mathcal{L}$ is a first-order language and $\mathcal{M}$ is an $\mathcal{L}$-model with domain ω, then $\mathrm{Aut}(\mathcal{M})$ is a subset of Baire space ${}^{\omega}\omega$ and the corresponding subspace topology induces a Polish group topology on $\mathrm{Aut}(\mathcal{M})$ (see [**Kec95**, Example 9.B 7]). If B is the family of subsets of $\mathrm{Aut}(\mathcal{M})$ of the form

$$\{\sigma \in \mathrm{Aut}(\mathcal{M}) \mid \pi \restriction X = \sigma \restriction X\}$$

for some $\pi \in \mathrm{Aut}(\mathcal{M})$ and a finite subset X of ω, then B forms a countable basis of this group topology.

Let G be a countable group and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. Let B denote the family of all subsets of G_1 of the form

$$\{h \in G_1 \mid \iota_g \restriction X = \iota_h \restriction X\}$$

for some $g \in G_1$ and a finite subset X of G. By the above remarks, B is a countable basis of a Polish group topology on G_1. Moreover, Corollary 2.2.6 and Proposition 4.1.5 imply that this is the unique Polish group topology on G_1 and $\iota_\pi \restriction G_1$ is continuous with respect to this topology for every $\pi \in G_2$ by Proposition 4.1.7.

The following folklore result is the last ingredient in our proof of Theorem 4.2.2. A proof of this statement can be found in [**BK96**, page 6].

Proposition 4.2.3. *Let $\langle G, \tau\rangle$ be a Polish group, H be a subgroup of G that is dense in τ and $\varphi : H \longrightarrow G$ be a group homomorphism that is continuous with respect to the subspace topology induced by τ on H and τ. Then there is a unique group homomorphism $\varphi^* : G \longrightarrow G$ that extends φ and is continuous with respect to τ.*

Proof of Theorem 4.2.2. Assume that M is a transitive class such that $\langle M, \in\rangle$ is a model of ZFC, G is a centreless group with domain ω that is an element of M, $\langle G^M_\alpha \mid \alpha \in \mathrm{On} \cap M\rangle$ is an automorphism tower of G in $\langle M, \in\rangle$ and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ is an automorphism tower of G. Since every automorphism of G in M is an automorphism of G, we may replace G_1 by an isomorphic copy and assume that G^M_1 is a subgroup of G_1. We fix the following collections of sets.

(1) Let τ denote the unique Polish group topology on G_1.
(2) Let τ^M denote the unique Polish group topology on G^M_1 in $\langle M, \in\rangle$.
(3) Let $\bar{\tau}$ denote the subspace topology induced by τ on G^M_1.

Note that τ^M is contained in $\bar{\tau}$, because every basic open set in τ^M is an element of $\bar{\tau}$.

Let $U = \{h \in G_1 \mid \iota_g \restriction X = \iota_h \restriction X\}$ be a nonempty basic open set in τ with $g \in G_1$ and X is a finite subset of ω. Then both X and $\iota_g \restriction X$ are elements of M and there is a tree T on $\omega \times \omega$ of height ω in M such that every cofinal branch through T is of the form $\langle x, y\rangle \in {}^{\omega}\omega \times {}^{\omega}\omega$ with $x, y \in \mathrm{Aut}(G)$, $y = x^{-1}$ and $\iota_g \restriction X \subseteq x$. It is easy to see that this property is absolute between transitive ZFC-models. Since U is nonempty, there is a cofinal branch through T and, by Mostowski's Absoluteness Theorem (see [**Jec03**, Theorem 25.4]), there is a branch through T that is an element of M. We can conclude $G_1^M \cap U \neq \emptyset$. This argument shows that G_1^M is dense in τ.

Fix $h \in G_2^M$. Let U be a basic open set in τ defined by $g \in G_1$ and $X \subset \omega$ as above. The above computations show that we may assume $g \in G_1^M$ and

$$U \cap G_1^M = \{k \in G_1^M \mid \iota_g \upharpoonright X = \iota_k \upharpoonright X\}$$

is a basic open set in τ^M. The subset

$$(\iota_h \upharpoonright G_1^M)^{-1}{}``U = (\iota_h \upharpoonright G_1^M)^{-1}{}``(G_1^M \cap U)$$

is an element of τ^M, because $\iota_h \upharpoonright G_1^M$ is continuous with respect to τ^M in M. By the above remarks, the subset is also an element of $\bar{\tau}$. This shows that the map $\iota_h \upharpoonright G_1^M : G_1^M \longrightarrow G_1$ is a group homomorphism that is continuous with respect to $\bar{\tau}$ and τ. By Proposition 4.2.3, there is a unique group homomorphism $h^* : G_1 \longrightarrow G_1$ that extends $\iota_h \upharpoonright G_1^M$ and is continuous with respect to τ.

For all $h \in G_2^M$, the map $(h^{-1})^* \circ h^*$ is the identity on the dense subset G_1^M and is therefore the identity on G_1. This shows $h^* \in \mathrm{Aut}(G_1)$ with $(h^*)^{-1} = (h^{-1})^*$. We let $\pi(h)$ denote the unique element of G_2 with $h^* = \iota_{\pi(h)} \upharpoonright G_1$. This means $\iota_{\pi(h)} \upharpoonright G_1^M = \iota_h \upharpoonright G_1^M$ and π is injective. Moreover, if $g \in G_1^M \subseteq G_1$, then $\iota_{\pi(g)} \upharpoonright G = \iota_g \upharpoonright G$ and this shows $g = \pi(g)$.

Given $h_0, h_1 \in G_2^M$, our definitions imply that $\iota_{\pi(h_0 \cdot h_1)}$ is equal to $\iota_{\pi(h_0)\cdot\pi(h_1)}$ on G_1^M and therefore on G_1. This shows $\pi(h_0 \cdot h_1) = \pi(h_0) \cdot \pi(h_1)$ holds for all $h_0, h_1 \in G_2^M$ and π is a group homomorphism. $\square$

CHAPTER 5

Examples of special pairs

The notion of *special pairs* was introduced by Itay Kaplan and Saharon Shelah in [**KS09**] to analyze automorphism towers of centreless groups. Given a special pair $\langle G, A\rangle$, this notion allows us to measure the complexity of the group G by interpreting it as a set of subsets of $\mathcal{T}^1_A$. For example, if A is countable, then we can easily identify subsets of $\mathcal{T}^1_A$ with elements of *Cantor space* ${}^{\omega}2$ (i.e. *reals*) and talk about the complexity of G in terms of *descriptive set theory* (i.e. as *definable sets of reals*).

The aim of this chapter is to further investigate this notion and a strengthening of it. This work will produce various examples of special pairs that are not of the form $\langle G_\alpha, A\rangle$ for some ordinal α and a centreless group G with domain A.

In the first section, we will introduce the notion of *strongly special pair* and show that the statement of Theorem 2.2.5 also holds if we replace *special pair* by *strongly special pair*. Section 5.2 shows how strongly special pairs can be constructed using groups of autohomeomorphisms of certain Hausdorff spaces. This construction relies on methods and results developed by Robert R. Kallman in [**Kal86**]. In the last section, we will use a result of Manfred Droste, Michèle Giraudet and Rüdiger Göbel from [**DGG01**] to show that there are special pairs that are not strongly special.

The results of this chapter are contained in [**Lücc**].

5.1. Strongly special pairs

This section focuses on the following definition and its connection with automorphism towers.

Definition 5.1.1. Given a group G and a subset A of the domain of G, we call the pair $\langle G, A\rangle$ *strongly special* if $\mathsf{qft}_{G,A}(g) \subseteq \mathsf{qft}_{G,A}(h)$ implies $g = h$ for all $g, h \in G$.

We will show that the statement of Theorem 2.2.5 still holds if we replace *special pair* by *strongly special pair*. We start by generalizing Lemma 2.2.3.

Lemma 5.1.2. *If G is a group and A is a subset of the domain of G, then the following statements are equivalent.*

(1) *$\langle G, A\rangle$ is a strongly special pair.*
(2) *If $g \in G$ and $\varphi : \langle A \cup \{g\}\rangle_G \longrightarrow G$ is a group homomorphism with $\varphi \upharpoonright A = \mathrm{id}_A$, then $\varphi(g) = g$.*

Proof. Assume that $\langle G, A\rangle$ is a strongly special pair, $g \in G$ and

$$\varphi : \langle A \cup \{g\}\rangle_G \longrightarrow G$$

is a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$. An easy induction shows that $\mathrm{t}^G(g) \in \langle A \cup \{g\}\rangle_G$ and $\varphi(\mathrm{t}^G(g)) = \mathrm{t}^G(\varphi(g))$ hold for every term $\mathrm{t}(v) \in \mathcal{T}^1_A$. In particular, $\mathsf{qft}_{G,A}(g) \subseteq \mathsf{qft}_{G,A}(\varphi(g))$ and we can conclude $g = \varphi(g)$.

Assume that the second statement holds. Fix elements $g_0, g_1 \in G$ with $\mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$. Pick $\mathrm{t}_0, \mathrm{t}_1 \in \mathcal{T}^1_A$ with $\mathrm{t}^G_0(g_0) = \mathrm{t}^G_1(g_0)$. Then

$$\mathrm{t}_0 * \mathrm{t}_1^{-1} \in \mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$$

and $\mathrm{t}^G_0(g_1) = \mathrm{t}^G_1(g_1)$. Given $h \in \langle A \cup \{g_0\}\rangle_G$, there is a term $\mathrm{t}(v) \in \mathcal{T}^1_A$ with $\mathrm{t}^G(g_0) = h$ and, if we define $\varphi(h) = \mathrm{t}^G(g_1)$, then the above computations show that $\varphi(h)$ does not depend on the choice of t. Moreover, these computations directly imply that $\varphi : \langle A \cup \{g_0\}\rangle_G \longrightarrow G$ is a group homomorphism with $\varphi(g_0) = g_1$ and $\varphi \restriction A = \mathrm{id}_A$. By our assumption, we have $g_0 = g_1$. $\square$

This characterization allows us to prove a version of [**KS09**, Claim 3.8] for strongly special pairs. Note that the proofs of the two statements are almost identical.

Lemma 5.1.3. *Let $\langle G, A\rangle$ be a strongly special pair and H be a group such that G is a normal subgroup of H and $\mathrm{C}_H(G) = \{1_G\}$. Then $\langle H, A\rangle$ is a strongly special pair.*

Proof. Let $h \in H$ and $\varphi : \langle A \cup \{h\}\rangle_H \longrightarrow H$ be a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$. Pick $a \in A$. Then $a^h \in G$, $\varphi(a^h) = a^{\varphi(h)} \in G$ and, if we define $\psi = \varphi \restriction \langle A \cup \{a^h\}\rangle_G$, then $\psi : \langle A \cup \{a^h\}\rangle_G \longrightarrow G$ is a group homomorphism with $\psi \restriction A = \mathrm{id}_A$. By our assumption, we have $a^h = \psi(a^h) = a^{\varphi(h)}$. This argument shows $h \cdot \varphi(h^{-1}) \in \mathrm{C}_H(A)$.

Now fix $g \in G$ and define $\xi : \langle A \cup \{g\}\rangle_G \longrightarrow G$ by

$$\xi = \iota_{h\cdot\varphi(h^{-1})} \restriction \langle A \cup \{g\}\rangle_G.$$

By the above computations, we have $\xi \restriction A = \mathrm{id}_A$ and this means

$$g = \xi(g) = g^{h\cdot\varphi(h^{-1})}.$$

We can conclude $h \cdot \varphi(h^{-1}) \in \mathrm{C}_H(G) = \{1_G\}$ and $h = \varphi(h)$. $\square$

We are now ready to prove the promised version of Theorem 2.2.5 for strongly special pairs. Again, the proofs of both results are almost identical.

Theorem 5.1.4 ([**Lücc**, Theorem 2.9]). *Let $\langle G, A\rangle$ be a strongly special pair with $\mathrm{C}_G(A) = \{1_G\}$ and $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. If $\alpha \in \mathrm{On}$, then $\langle G_\alpha, A\rangle$ is a strongly special pair.*

Proof. We prove the statement of the theorem by induction.

Assume $\langle G_\alpha, A\rangle$ is a strongly special pair. If $h \in \mathrm{C}_{G_{\alpha+1}}(G_\alpha)$, then $\iota_h \restriction G_\alpha = \mathrm{id}_{G_\alpha}$ and $h = 1_G$. Since G_α is a normal subgroup of $G_{\alpha+1}$, we can apply Lemma 5.1.3 to see that $\langle G_{\alpha+1}, A\rangle$ is also a strongly special pair.

Let α be a limit ordinal and assume that $\langle G_\beta, A\rangle$ is a strongly special pair for every $\beta < \alpha$. Given $g_0, g_1 \in G_\alpha$ with $\mathsf{qft}_{G_\alpha,A}(g_0) \subseteq \mathsf{qft}_{G_\alpha,A}(g_1)$, there is an ordinal $\beta < \alpha$ with $g_0, g_1 \in G_\beta$ and it is easy to see that $\mathsf{qft}_{G_\alpha,A}(g_i) = \mathsf{qft}_{G_\beta,A}(g_i)$. In particular, we have $g_0 = g_1$. □

5.2. Groups of autohomeomorphism

In this section, we produce a variety of examples of strongly special pairs using certain group actions on Hausdorff spaces. Given a group G that consists of autohomeomorphisms of a Hausdorff space and satisfies a *locally movability condition*, we will construct a subset A of the domain of G such that $\langle G, A\rangle$ is strongly special pair and the cardinality of A is equal to the cardinality of a basis of the corresponding Hausdorff space.

Definition 5.2.1. Let G be a group and $\langle X, \tau\rangle$ be a Hausdorff space. We say that G *acts locally mixing on* $\langle X, \tau\rangle$ if the following statements hold.

(1) G is a subgroup of the group $\mathcal{H}(\tau)$ of all autohomeomorphisms of $\langle X, \tau\rangle$.
(2) If U is an element of τ that consists of more than one point, then there is a $g \in G \setminus \{1_G\}$ with $g \restriction (X \setminus U) = \mathrm{id}_{X\setminus U}$.

This condition also appears in the study of topological spaces that can be reconstructed from their autohomeomorphism groups (see [**Rub89**]).

We present some easy examples of autohomeomorphism groups acting locally mixing on the corresponding topological space. Given a topological space $\langle X, \tau\rangle$ and a subset A of X, we let $\bar{A}$ denote the closure of A with respect to τ, δA denote the boundary of A with respect to τ and τ_A denote the corresponding subspace topology on A induced by τ.

Proposition 5.2.2. *Let $\langle X, \tau\rangle$ be a Hausdorff space. Assume that for every subset U in τ with at least two points, there is a $V \subseteq U$ in τ such that $\bar{V} \subseteq U$ and $\langle \bar{V}, \tau_{\bar{V}}\rangle$ has a nontrivial autohomeomorphisms π with $\pi \restriction \delta V = \mathrm{id}_{\delta V}$. Then $\mathcal{H}(\tau)$ acts locally mixing on $\langle X, \tau\rangle$.*

Proof. Let U be an element of τ with more than one point. Pick V and π as above and define $\pi^* = \pi \cup \mathrm{id}_{X\setminus\bar{V}}$. We show that π^* is continuous with respect to τ in every $x \in X$.

If $x \in X\setminus\bar{V}$, then this statement is trivial, because $\pi^* \restriction (X\setminus\bar{V}) = \mathrm{id}_{X\setminus\bar{V}}$ and $X \setminus \bar{V}$ is open. Given $x \in \delta V$ and W_1 open in τ with $x = \pi^*(x) \in W_1$, there is $\tilde{W}_0$ in $\tau_{\bar{V}}$ with $x \in \tilde{W}_0$ and $\tilde{W}_0 \subseteq \pi^{-1}{}''(\bar{V} \cap W_1)$. Pick W_0 in τ with $\tilde{W}_0 = \bar{V} \cap W_0$. Then $x \in W_0 \cap W_1$ and $W_0 \cap W_1 \subseteq \pi^{*-1}{}''W_1$. Finally, if $x \in V$ and W_1 is open in τ with $\pi^*(x) \in W_1$, then $\pi(x) = \pi^*(x) \in V \cap W_1$ and there is $\tilde{W}_0$ in $\tau_{\bar{V}}$ with $x \in \tilde{W}_0$ and $\tilde{W}_0 \subseteq \pi^{-1}{}''(V \cap W_1)$. Pick W_0 in τ with $\tilde{W}_0 = \bar{V} \cap W_0$. Then $x \in V \cap W_0$ and $V \cap W_0 \subseteq \pi^{*-1}{}''W_1$. □

Example 5.2.3. *Let $\langle X, \tau\rangle$ be an n-dimensional topological manifold. If U is an element of τ and $x \in U$, then there is a W in τ with $x \in W$ and $\langle W, \tau_W\rangle$ is homeomorphic to an open Euclidean n-ball. The preimage* *of $U \cap W$ under this homeomorphism is nonempty and therefore contains an open n-ball. This shows that there is a V in τ such that $\bar{V} \subseteq U \cap W \subseteq U$ and there is an homeomorphism of $\langle \bar{V}, \tau_{\bar{V}} \rangle$ and $[-1,1]^n$ that maps δV onto the boundary of $[-1,1]^n$ in $\mathbb{R}^n$. There are nontrivial autohomeomorphisms of $[-1,1]^n$ that map its boundary in $\mathbb{R}^n$ onto itself and, by the above calculations, this shows that $\mathcal{H}(\tau)$ acts locally mixing on $\langle X, \tau \rangle$.*

EXAMPLE 5.2.4. *Remember that a partial order $\mathbb{P} = \langle P, <_{\mathbb{P}} \rangle$ is a* tree *if the set $prec(p) = \{q \in P \mid q <_{\mathbb{P}} p\}$ is a well-ordered by $<_{\mathbb{P}}$ for every $p \in P$. Given a tree $\mathbb{T} = \langle T, <_{\mathbb{T}} \rangle$, we call a subset of T a* branch through $\mathbb{T}$ *if it is linearly ordered by $<_{\mathbb{T}}$ and downwards-closed. We let $[\mathbb{T}]$ denote the set of all maximal branches through T. Let $\tau_{\mathbb{T}}$ denote the topology on $[\mathbb{T}]$ generated by basic open sets of the form $U_t = \{b \in [\mathbb{T}] \mid t \in b\}$ with $t \in T$.*

Let $\mathbb{T} = \langle T, <_{\mathbb{T}} \rangle$ be a tree with the property that for every $t \in T$ there is an automorphism π of $\mathbb{T}$ with $\pi(t) = t$ and $\pi(s) \neq s$ for some $s \in T$ with $t <_{\mathbb{T}} s$. We show that $\mathcal{H}(\tau_{\mathbb{T}})$ acts locally mixing on $\langle [\mathbb{T}], \tau_{\mathbb{T}} \rangle$. By Proposition 5.2.2, it suffices to show that the space $\langle U_t, (\tau_{\mathbb{T}})_{U_t} \rangle$ has a nontrivial autohomeomorphism for every $t \in T$, because

$$[\mathbb{T}] \setminus U_t = \bigcup \{U_s \mid s \textit{ and } t \textit{ are incompatible in } \mathbb{T}\}$$

and this shows that U_t is also closed in $\tau_{\mathbb{T}}$. If $t \in T$ and $\pi \in \mathrm{Aut}(\mathbb{T})$ with $\pi(t) = t$ and $\pi(s) \neq s$ for some $s \in T$ with $t <_{\mathbb{T}} s$, then we define $\pi^*(b) = \pi"b$ for every $b \in U_t$. It is easy to check that $\pi^* : U_t \longrightarrow U_t$ is continuous with respect to $(\tau_{\mathbb{T}})_{U_t}$ and if $s \in b \in U_t$, then $\pi^*(b) \neq b$, because $\pi(s) <_{\mathbb{T}} s$ or $s <_{\mathbb{T}} \pi(s)$ would contradict the well-foundedness of $<_{\mathbb{T}}$ below the element s.

In particular, if α is an ordinal, X is a set with at least two elements and ${}^{<\alpha}X$ is the tree consisting of functions f with $\mathrm{dom}(f) \in \alpha$ and $\mathrm{ran}(f) \subseteq X$ ordered by inclusion, then $[{}^{<\alpha}X]$ can be identified with the set ${}^{\alpha}X$ of all functions from α to X and the group of autohomeomorphisms of the corresponding topological space acts locally mixing on it.

EXAMPLE 5.2.5. *Let $\mathbb{L} = \langle L, <_{\mathbb{L}} \rangle$ be a linear order without end-points that has a nontrivial automorphism and the property that every nonempty, open interval $(a, b) = \{l \in L \mid a <_{\mathbb{L}} l <_{\mathbb{L}} b\}$ is order-isomorphic to $\mathbb{L}$. If $\tau_{\mathbb{L}}$ denotes the order-topology on $\mathbb{L}$, then Proposition 5.2.2 directly implies that $\mathrm{Aut}(\mathbb{L})$ acts locally mixing on $\langle L, \tau_{\mathbb{L}} \rangle$. In particular, the group of order-preserving bijections of the rational numbers $\mathbb{Q}$ acts locally mixing on $\mathbb{Q}$ equipped with the order topology.*

We use methods and computations from Robert R. Kallman's proof of [**Kal86**, Theorem 1.1] to derive the following result.

THEOREM 5.2.6 ([**Lücc**, Theorem 5.6]). *Let G be a group, $\langle X, \tau \rangle$ be a Hausdorff space and B be a basis of τ. If G acts locally mixing on $\langle X, \tau \rangle$ and $\langle X, \tau \rangle$ does not have exactly two isolated points, then there is a subset*

A of the domain of G of cardinality $|\mathrm{B}| + \aleph_0$ such that $\langle G, A\rangle$ is a strongly special pair and $\mathrm{C}_G(A) = \{1_G\}$.

For the rest of this section, we **fix a Hausdorff space $\langle X, \tau\rangle$, a basis** B **of τ and a group G that acts locally mixing on** $\langle X, \tau\rangle$. Given $Y \subseteq X$, we define

$$\mathrm{Sub}_{\mathrm{B}}(Y) = \{U \in \mathrm{B} \mid U \subseteq Y,\ |U| > 1\}.$$

and define $\bar{Y}$ to be the closure of Y with respect to τ. Finally, we fix a sequence $\langle g_U \in G \setminus \{1_G\} \mid U \in \mathrm{Sub}_{\mathrm{B}}(X)\rangle$ such that $g_U \upharpoonright (X \setminus U) = \mathrm{id}_{X\setminus U}$ holds for all $U \in \mathrm{Sub}_{\mathrm{B}}(X)$.

In the following, we adopt the arguments of [**Kal86**, Section 2] to our setting to prove Theorem 5.2.6.

LEMMA 5.2.7. *Let U be open in τ such that U contains either no points isolated in τ or more than two points isolated in τ. The following statements are equivalent for all $h \in G$.*

(1) $h \upharpoonright \bar{U} = \mathrm{id}_{\bar{U}}$.
(2) $g_{U'}^h = g_{U'}$ *holds for all* $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$.

PROOF. Assume $h \upharpoonright \bar{U} = \mathrm{id}_{\bar{U}}$ and fix $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$. Then

$$h \circ g_{U'} = g_{U'} \circ h$$

holds, because we have $g_{U'} \upharpoonright (X \setminus \bar{U}) = \mathrm{id}_{\bar{U}}$.

Now, assume that $g_{U'}^h = g_{U'}$ holds for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$. By the continuity of h, it suffices to show $h \upharpoonright U = \mathrm{id}_U$. Let I_U denote the set of all points in U that are isolated in τ. We start by showing $h \upharpoonright I_U = \mathrm{id}_{I_U}$. If U contains no isolated points, then this is trivial. We may therefore assume $|I_U| > 2$.

Assume, toward a contradiction, that there is an $a \in I_U$ with $h(a) \neq a$. We can find distinct $b_0, b_1 \in I_U$ with $a \notin \{b_0, b_1\}$. Then $\{a, b_i\} \in \mathrm{Sub}_{\mathrm{B}}(U)$ and $g_{\{a,b_i\}} = (a\ b_i)$. Our first assumption yields $(a\ b_i)^h = (a\ b_i)$ and this implies $h``\{a, b_i\} = \{a, b_i\}$. We can conclude $b_0 = h(a) = b_1$, a contradiction. This shows $h \upharpoonright I_U = \mathrm{id}_{I_U}$.

Assume, toward a contradiction, that there is an $x \in U$ with $h(x) \neq x$. Since x is not isolated in τ and $\langle X, \tau\rangle$ is a Hausdorff space, we can find $V \in \mathrm{Sub}_{\mathrm{B}}(U)$ with $V \cap (h``V) = \emptyset$. If $y \in V$ with $g_V(y) \neq y$, then $g_V^h = g_V$, $g_V(h(y)) = h(y)$ and therefore

$$h(y) = (g_V \circ h)(y) = (h \circ g_V)(y) \neq h(y),$$

a contradiction. □

Set $A = \{g_U \mid U \in \mathrm{Sub}_{\mathrm{B}}(X)\}$ and, for all $U, V \in \mathrm{Sub}_{\mathrm{B}}(X)$, we define

$$\mathrm{t}_{U,V}(v) \equiv v * \dot{g}_U * v^{-1} * \dot{g}_V * v * \dot{g}_U^{-1} * v^{-1} * \dot{g}_V^{-1} \in \mathcal{T}_A^1.$$

LEMMA 5.2.8. *Let U and V be open subsets in τ. Assume that both U and $X \setminus \bar{V}$ contain either no points isolated in τ or more than two points isolated in τ. Then the following statements are equivalent for all $h \in G$.*

(1) $\mathrm{t}^G_{U',V'}(h) = 1_G$ *for all* $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ *and* $V' \in \mathrm{Sub}_{\mathrm{B}}(X \setminus \bar{V})$.
(2) $h"\bar{U} \subseteq \bar{V}$.

Proof. The first statement is equivalent to $g^h_{U'} \circ g_{V'} = g_{V'} \circ g^h_{U'}$ for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ and $V' \in \mathrm{Sub}_{\mathrm{B}}(X \setminus \bar{V})$. By Lemma 5.2.7, this is equivalent to $g^h_{U'} \restriction (X \setminus \bar{V}) = \mathrm{id}_{X \setminus \bar{V}}$ for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ and we can reformulate this to

(1)* $(g_{U'} \circ h^{-1}) \restriction (X \setminus \bar{V}) = h^{-1} \restriction (X \setminus \bar{V})$ for all $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$.

By our assumptions, the set of all points which are moved by some $g_{U'}$ with $U' \in \mathrm{Sub}_{\mathrm{B}}(U)$ is dense in U with respect to τ. This shows that (1)* is equivalent to $U \cap h^{-1}"(X \setminus \bar{V}) = \emptyset$. This statement holds if and only if $h"U \subseteq \bar{V}$ and this is equivalent to the second statement of the lemma. □

Proof of Theorem 5.2.6. We may assume that B is closed under finite unions. By our assumptions, there are not exactly two points in X which are isolated in τ. If there is exactly one point $x_0 \in X$ which is isolated in τ, then it is easy to check that there is a group isomorphic to G that acts locally mixing on $\langle X \setminus \{x_0\}, \tau^* \rangle$, where τ^* is the subspace topology induced by τ. We may therefore assume that there are either no points isolated in τ or more than two.

Pick $g_0, g_1 \in G$ with $\mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$ and assume, toward a contradiction, that $g_0 \neq g_1$ holds. Then $U = \{x \in X \mid g_0(x) \neq g_1(x)\}$ is nonempty and open in τ. Let I_U denote the set of all points in U that are isolated in τ.

First, assume that there is an $x \in U \setminus I_U$. We can find disjoint subsets V_0 and V_1 in B such that $g_i(x) \in V_i$ for $i < 2$ and $X \setminus \bar{V}_0$ contains either no points isolated in τ or more than two. Now we can find $U' \in \mathrm{B}$ with $x \in U'$, $g_i"U' \subseteq V_i$ and U' contains either no points isolated in τ or more than two. This means $g_0"\bar{U}' \subseteq \bar{V}_0$ and we can apply Lemma 5.2.8 to conclude

$$\mathrm{t}_{U'',V'} \in \mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1)$$

for all $U'' \in \mathrm{Sub}_{\mathrm{B}}(U')$ and $V' \in \mathrm{Sub}_{\mathrm{B}}(X \setminus \bar{V}_0)$. Another application of the lemma yields $g_1"\bar{U}' \subseteq \bar{V}_0$ and this means $g_1(x) \in \bar{V}_0 \subseteq X \setminus V_1$, a contradiction.

This shows $I_U = U \neq \emptyset$. Pick $x \in I_U$. By the above assumptions, we can find distinct $y_0, y_1 \in X$ isolated in τ with $x \notin \{y_0, y_1\}$. For all $i < 2$, we have $\{x, y_i\}, \{g_0(x), g_0(y_i)\} \in \mathrm{B}$, $g_{\{x,y_i\}} = (x\ y_i)$ and

$$g^{g_0}_{\{x,y_i\}} = (g_0(x)\ g_0(y_i)) = g_{\{g_0(x),g_0(y_i)\}}.$$

The above equalities allow us to conclude

$$v * \dot{g}_{\{x,y_i\}} * v^{-1} * \dot{g}_{\{g_0(x),g_0(y_i)\}} \in \mathsf{qft}_{G,A}(g_0) \subseteq \mathsf{qft}_{G,A}(g_1).$$

In particular, $g_1"\{x, y_i\} = \{g_0(x), g_0(y_i)\}$ and this shows $g_1(x) = g_0(y_i)$, because $g_1(x) \neq g_0(x)$. We can conclude $g_0(y_0) = g_1(x) = g_0(y_1)$ and therefore $y_0 = y_1$, a contradiction.

If $h \in \mathrm{C}_G(A)$, then $g_U^h = g_U$ holds for all $U \in \mathrm{Sub}_{\mathrm{B}}(X)$. By our assumptions and the above remark, we can apply Lemma 5.2.7 to conclude $h = \mathrm{id}_X = 1_G$. $\square$

5.3. Special pairs that are not strongly special

In this section, we construct special pairs that are not strongly special using simple groups as building blocks. A theorem of Manfred Droste, Michèle Giraudet and Rüdiger Göbel will allow us to prove the following result.

THEOREM 5.3.1 ([**Lücc**, Theorem 6.1]). *If κ is an uncountable regular cardinal, then there is a special pair $\langle G, A\rangle$ such that G has cardinality 2^κ, A has cardinality κ, $\mathrm{C}_G(A) = \{1_G\}$ and $\langle G, A\rangle$ is not strongly special.*

We start with a simple statement about normal subgroups of automorphism groups of centreless groups.

PROPOSITION 5.3.2. *Let G be a centreless group and N be a normal subgroup of $\mathrm{Aut}(G)$. Then $N \neq \{\mathrm{id}_G\}$ if and only if $\mathrm{Inn}(G) \cap N \neq \{\mathrm{id}_G\}$.*

PROOF. Assume $\mathrm{Inn}(G) \cap N = \{\mathrm{id}_G\}$. Given $\pi \in N$, we have

$$\iota_{\pi(g)\cdot g^{-1}} = \pi \circ \iota_g \circ \pi^{-1} \circ \iota_g^{-1} \in \mathrm{Inn}(G) \cap N$$

and therefore $\pi(g) = g$ for all $g \in G$. This shows $N = \{\mathrm{id}_G\}$. $\square$

In the proof of Theorem 5.3.1, we start by constructing a special pair $\langle G, A\rangle$ with $|G| = |A|$ that is not strongly special. The following proposition will allow us to replace G by a group of higher cardinality.

PROPOSITION 5.3.3. *Let G and H be groups, A be a subset of the domain of G and $A^* = A \times \{1_H\} \cup \{1_G\} \times H \subseteq G \times H$.*

(1) *If $\langle G, A\rangle$ is a special pair and $\mathrm{Z}(H) = \{1_H\}$, then $\langle G \times H, A^*\rangle$ is a special pair.*
(2) *If $\langle G, A\rangle$ is not a strongly special pair, then $\langle G \times H, A^*\rangle$ is not a strongly special pair.*

PROOF. (1) Assume that $\mathrm{Z}(H) = \{1_H\}$ holds, $\langle g_*, h_*\rangle \in G \times H$ and

$$\varphi : \langle A^* \cup \{\langle g_*, h_*\rangle\}\rangle_{G\times H} \longrightarrow G \times H$$

is a monomorphism with $\varphi \restriction A^* = \mathrm{id}_{A^*}$ and $\varphi(\langle g_*, h_*\rangle) \neq \langle g_*, h_*\rangle$. Then $\langle k, 1_H\rangle \in \mathrm{dom}(\varphi)$ for every $k \in \langle A \cup \{g_*\}\rangle_G$ and $\varphi(\langle g_*, 1_H\rangle) \neq \langle g_*, 1_H\rangle$. Let $p_H : G \times H \longrightarrow H$ denote the canonical projection and define

$$\xi : \langle A \cup \{g_*\}\rangle_G \longrightarrow H;\ k \longmapsto (p_H \circ \varphi)(\langle k, 1_H\rangle).$$

Given $k \in \langle A \cup \{g_*\}\rangle_G$ and $h \in H$, we have

$$\begin{aligned}\xi(k) \cdot h &= (p_H \circ \varphi)(\langle k, 1_H\rangle) \cdot (p_H \circ \varphi)(\langle 1_G, h\rangle) = (p_H \circ \varphi)(\langle k, h\rangle) \\ &= (p_H \circ \varphi)(\langle 1_G, h\rangle) \cdot (p_H \circ \varphi)(\langle k, 1_H\rangle) = h \cdot \xi(k)\end{aligned}$$

and this shows $\operatorname{ran}(\xi) \subseteq \mathrm{Z}(H) = \{1_H\}$. We get a function

$$\bar{\varphi} : \langle A \cup \{g_*\}\rangle_G \longrightarrow G$$

with $\varphi(\langle k, 1_H\rangle) = \langle \bar{\varphi}(k), 1_H\rangle$ for all $k \in \langle A \cup \{g_*\}\rangle_G$. By our assumptions, $\bar{\varphi}$ is a monomorphism, $\bar{\varphi} \restriction A = \mathrm{id}_A$ and $\bar{\varphi}(g_*) \neq g_*$. This shows that $\langle G, A\rangle$ is not a special pair.

(2) Assume $g_* \in G$ and $\bar{\varphi} : \langle A \cup \{g\}\rangle_G \longrightarrow G$ is a homomorphism with $\bar{\varphi} \restriction A = \mathrm{id}_A$ and $\bar{\varphi}(g_*) \neq g_*$. If $\langle k, h\rangle \in \langle A^* \cup \{\langle g_*, 1_H\rangle\}\rangle_{G\times H}$, then $k \in \langle A \cup \{g_*\}\rangle_G$ and we can define

$$\varphi : \langle A^* \cup \{\langle g_*, 1_H\rangle\}\rangle_{G\times H} \longrightarrow G \times H;\ \langle k, h\rangle \longmapsto \langle \bar{\varphi}(k), h\rangle.$$

Then $\langle G\times H, A^*\rangle$ is not a strongly special pair, because φ is a homomorphism with $\varphi \restriction A^* = \mathrm{id}_{A^*}$ and $\varphi(\langle g_*, 1_H\rangle) \neq \langle g_*, 1_H\rangle$. □

For the remainder of this section, we **fix simple non-abelian groups** H **and** S **and a homomorphism** $\mathfrak{c} : \operatorname{Aut}(S) \longrightarrow \operatorname{Aut}(H)$ **with** $\operatorname{Inn}(H) \subseteq \operatorname{ran}(\mathfrak{c})$. Define

$$G = H \rtimes_{\mathfrak{c}} \operatorname{Aut}(S)$$

and $A = \{1_H\} \times \operatorname{Aut}(S)$.

Lemma 5.3.4. *The following statements are equivalent.*

1. *There is an isomorphism* $\Psi : H \longrightarrow S$ *with* $\mathfrak{c}(\pi) = \Psi^{-1} \circ \pi \circ \Psi$ *for all* $\pi \in \operatorname{Aut}(S)$.
2. $\langle G, A\rangle$ *is not a special pair.*

Proof. Assume (1) holds. Define

$$\phi : G \longrightarrow G;\ \langle h, \pi\rangle \longmapsto \langle h^{-1}, \iota_{\Psi(h)} \circ \pi\rangle.$$

Clearly, ϕ is injective and $\phi \restriction A = \mathrm{id}_A$. If $\langle h^{-1}, \iota_{\Psi(h)} \circ \pi\rangle = \langle h, \pi\rangle$ holds with $h \in H$ and $\pi \in \operatorname{Aut}(S)$, then $\iota_{\Psi(h)} = \mathrm{id}_S$ and this means $h = 1_H$. This shows $\phi \neq \mathrm{id}_G$. Given $\langle h_0, \pi_0\rangle, \langle h_1, \pi_1\rangle \in G$, we have

$$\begin{aligned}
\phi(\langle h_0, \pi_0\rangle \cdot \langle h_1, \pi_1\rangle) &= \phi(\langle h_0 \cdot \mathfrak{c}(\pi_0)(h_1), \pi_0 \circ \pi_1\rangle) \\
&= \langle \mathfrak{c}(\pi_0)(h_1^{-1}) \cdot h_0^{-1}, \iota_{\Psi(h_0 \cdot \mathfrak{c}(\pi_0)(h_1))} \circ \pi_0 \circ \pi_1\rangle \\
&= \langle h_0^{-1} \cdot \mathfrak{c}(\pi_0)(h_1^{-1})^{h_0}, \iota_{\Psi(h_0)} \circ \iota_{(\pi_0 \circ \Psi)(h_1)} \circ \pi_0 \circ \pi_1\rangle \\
&= \langle h_0^{-1} \cdot (\iota_{h_0} \circ \mathfrak{c}(\pi_0))(h_1^{-1}), \iota_{\Psi(h_0)} \circ \iota^{\pi_0}_{\Psi(h_1)} \circ \pi_0 \circ \pi_1\rangle \\
&= \langle h_0^{-1} \cdot \mathfrak{c}(\iota_{\Psi(h_0)} \circ \pi_0)(h_1^{-1}), \iota_{\Psi(h_0)} \circ \pi_0 \circ \iota_{\Psi(h_1)} \circ \pi_1\rangle \\
&= \langle h_0^{-1}, \iota_{\Psi(h_0)} \circ \pi_0\rangle \cdot \langle h_1^{-1}, \iota_{\Psi(h_1)} \circ \pi_1\rangle \\
&= \phi(\langle h_0, \pi_0\rangle) \cdot \phi(\langle h_1, \pi_1\rangle),
\end{aligned}$$

because our assumption implies that $\mathfrak{c}(\iota_{\Psi(h)}) = \iota_h$ holds for all $h \in H$. This computation shows that ϕ is a group monomorphism and $\langle G, A\rangle$ is not a special pair by Lemma 2.2.3.

In the other direction, assume that $\langle G, A\rangle$ is not a special pair. By Lemma 2.2.3, there is a $g_* = \langle h_*, \pi_*\rangle \in G$ and a monomorphism

$$\phi : \langle A \cup \{g_*\}\rangle_G \longrightarrow G$$

with $\phi \restriction A = \mathrm{id}_A$ and $\phi(g_*) \neq g_*$. This implies $h_* \neq 1_H$, $\langle h_*, \mathrm{id}_S\rangle \in \mathrm{dom}(\phi)$ and $\phi(\langle h_*, \mathrm{id}_S\rangle) \neq \langle h_*, \mathrm{id}_S\rangle$.

Let $N = \{h \in H \mid \langle h, \mathrm{id}_S\rangle \in \mathrm{dom}(\phi)\}$. If $h \in N$ and $k \in H$, then $\iota_k = \mathfrak{c}(\pi)$ for some $\pi \in \mathrm{Aut}(S)$,

$$\begin{aligned}\langle h^k, \mathrm{id}_S\rangle &= \langle \mathfrak{c}(\pi)(h), \mathrm{id}_S\rangle = \langle 1_H, \pi\rangle \cdot \langle h, \mathrm{id}_S\rangle \cdot \langle 1_H, \pi^{-1}\rangle \\ &= \langle h, \mathrm{id}_S\rangle^{\langle 1_H, \pi\rangle} \in \mathrm{dom}(\phi)\end{aligned}$$

and $h^k \in N$. This shows that N is a normal subgroup of H and therefore $N = H$, because $1_H \neq h_* \in N$.

Let $p_{\mathrm{Aut}(S)} : G \longrightarrow \mathrm{Aut}(S)$ denote the canonical projection map and define

$$\bar{\Psi} : H \longrightarrow \mathrm{Aut}(S);\ h \longmapsto (p_{\mathrm{Aut}(S)} \circ \phi)(\langle h, \mathrm{id}_S\rangle).$$

Assume, toward a contradiction, that $\ker(\bar{\Psi}) = H$. This assumption gives us a map $\xi : H \longrightarrow H$ with $\phi(\langle h, \mathrm{id}_S\rangle) = \langle \xi(h), \mathrm{id}_S\rangle$ for all $h \in H$. By our assumptions, ξ is a monomorphism. If $h, k \in H$ and $\pi \in \mathrm{Aut}(S)$ with $\mathfrak{c}(\pi) = \iota_k$, then

$$\phi(\langle h^k, \mathrm{id}_S\rangle) = \phi(\langle h, \mathrm{id}_S\rangle^{\langle 1_H, \pi\rangle}) = \phi(\langle h, \mathrm{id}_S\rangle)^{\langle 1_H, \pi\rangle} = \langle \xi(h)^k, \mathrm{id}_S\rangle,$$

and $\xi(h)^k = \xi(h^k) \in \mathrm{ran}(\xi)$. This shows that $\mathrm{ran}(\xi)$ is a normal subgroup of H. Since ϕ is injective and H is nontrivial, we can conclude that $H = \mathrm{ran}(\xi)$ and ξ is a nontrivial automorphism of H. Pick $h \in H$ and $\pi \in \mathrm{Aut}(S)$ with $\mathfrak{c}(\pi) = \iota_h$. If $k \in H$, then

$$\begin{aligned}\langle k^{\xi(h)}, \pi\rangle &= \langle k^{\xi(h)}, \mathrm{id}_S\rangle \cdot \langle 1_H, \pi\rangle = \phi(\langle \xi^{-1}(k)^h, \mathrm{id}_S\rangle) \cdot \phi(\langle 1_H, \pi\rangle) \\ &= \phi(\langle \mathfrak{c}(\pi)(\xi^{-1}(k)), \pi\rangle) \\ &= \phi(\langle 1_H, \pi\rangle) \cdot \phi(\langle \xi^{-1}(k), \mathrm{id}_S\rangle) = \langle 1_H, \pi\rangle \cdot \langle k, \mathrm{id}_S\rangle \\ &= \langle \mathfrak{c}(\pi)(k), \pi\rangle = \langle k^h, \pi\rangle\end{aligned}$$

and therefore $h^{-1} \cdot \xi(h) \in \mathrm{Z}(H) = \{1_H\}$. This shows $\xi = \mathrm{id}_H$, a contradiction.

By the above computations, $\bar{\Psi} : H \longrightarrow \mathrm{Aut}(S)$ is a monomorphism. If $\pi \in \mathrm{Aut}(S)$ and $h, k \in H$ with $\phi(\langle h, \mathrm{id}_S\rangle) = \langle k, \bar{\Psi}(h)\rangle$, then

$$\begin{aligned}\langle \mathfrak{c}(\pi)(k), \bar{\Psi}(h)^\pi\rangle &= \langle k, \bar{\Psi}(h)\rangle^{\langle 1_H, \pi\rangle} \\ &= \phi(\langle h, \mathrm{id}_s\rangle^{\langle 1_H, \pi\rangle}) = \phi(\langle \mathfrak{c}(\pi)(h), \mathrm{id}_S\rangle)\end{aligned} \tag{5.1}$$

and therefore $\bar{\Psi}(h)^\pi = \bar{\Psi}(\mathfrak{c}(\pi)(h)) \in \mathrm{ran}(\bar{\Psi})$. This shows that $\mathrm{ran}(\bar{\Psi})$ is a nontrivial normal subgroup of $\mathrm{Aut}(S)$. By Proposition 5.3.2, we have $\mathrm{Inn}(S) \cap \mathrm{ran}(\bar{\Psi}) \neq \{\mathrm{id}_S\}$ and this implies

$$\mathrm{Inn}(S) = \mathrm{Inn}(S) \cap \mathrm{ran}(\bar{\Psi}) = \mathrm{ran}(\bar{\Psi}),$$

because both $\mathrm{Inn}(S)$ and $\mathrm{ran}(\bar{\Psi})$ are simple groups. We have shown that $\bar{\Psi} : H \longrightarrow \mathrm{Inn}(S)$ is an isomorphism.

Define $\Psi : H \longrightarrow S$ to be the isomorphism $\iota_S^{-1} \circ \bar{\Psi}$. Given $\pi \in \mathrm{Aut}(S)$ and $h \in H$, the equalities in (5.1) show $\bar{\Psi}(\mathfrak{c}(\pi)(h)) = \bar{\Psi}(h)^\pi$ and this implies

$$\begin{aligned}\mathfrak{c}(\pi)(h) &= \bar{\Psi}^{-1}(\bar{\Psi}(h)^\pi) = \bar{\Psi}^{-1}\left(\iota^\pi_{\Psi(h)}\right) \\ &= (\Psi^{-1} \circ \iota_S^{-1})\left(\iota_{(\pi\circ\Psi)(h)}\right) = (\Psi^{-1} \circ \pi \circ \Psi)(h).\end{aligned}$$

This equality shows that Ψ is an isomorphism with the desired properties. □

COROLLARY 5.3.5. *If $\langle G, A\rangle$ is not a special pair, then $\mathfrak{c}$ is injective.* □

PROPOSITION 5.3.6. *$\langle G, A\rangle$ is not a strongly special pair.*

PROOF. Define

$$\varphi : G \longrightarrow G;\ \langle h, \pi\rangle \longmapsto \langle 1_H, \pi\rangle.$$

Then φ is a group homomorphism with $\varphi \restriction A = \mathrm{id}_A$ and $\varphi(\langle h, \mathrm{id}_S\rangle) \neq \langle h, \mathrm{id}_S\rangle$ for all $h \in H\setminus\{1_H\} \neq \emptyset$. By Lemma 5.1.2, this implies the statement of the proposition. □

We finish this chapter by stating the coding result mentioned above and proving Theorem 5.3.1.

THEOREM 5.3.7 ([**DGG01**, Corollary 4.7]). *Let κ be an uncountable regular cardinal and G be a group of cardinality at most κ. Then there exists a simple group S of cardinality κ such that G is isomorphic to* $\mathrm{Aut}(S)/\mathrm{Inn}(S)$.

PROOF OF THEOREM 5.3.1. Let κ be a regular uncountable cardinal. It is well-known that the group $\mathrm{Alt}(\kappa)$ is a simple, non-abelian group of cardinality κ. By Theorem 5.3.7, there is a simple group S of cardinality κ such that there is an isomorphism $\xi : \mathrm{Aut}(S)/\mathrm{Inn}(S) \longrightarrow \mathrm{Alt}(\kappa)$. If we define

$$\mathfrak{c} : \mathrm{Aut}(S) \longrightarrow \mathrm{Aut}(\mathrm{Alt}(\kappa));\ \pi \longmapsto \iota_{\xi(\pi\mathrm{Inn}(S))},$$

then $\mathfrak{c}$ is a non-injective group homomorphism with $\mathrm{Inn}(\mathrm{Alt}(\kappa)) \subseteq \mathrm{ran}(\mathfrak{c})$. We set $\bar{G} = \mathrm{Alt}(\kappa) \rtimes_{\mathfrak{c}} \mathrm{Aut}(S)$ and $\bar{A} = \{\mathrm{id}_\kappa\} \times \mathrm{Aut}(S)$. Since both S and $\mathrm{Aut}(S)/\mathrm{Inn}(S)$ have cardinality κ, $\mathrm{Aut}(S)$ has the same cardinality and $\bar{G}$ is a group of cardinality κ. Corollary 5.3.5 implies that $\langle \bar{G}, \bar{A}\rangle$ is a special pair and Proposition 5.3.6 shows that it is not strongly special.

Pick $\langle h, \pi\rangle \in \mathrm{C}_{\bar{G}}(\bar{A})$. Given $\sigma \in \mathrm{Aut}(S)$, we have

$$\langle h, \pi\rangle = \langle h, \pi\rangle^{\langle \mathrm{id}_\kappa, \sigma\rangle} = \langle \mathfrak{c}(\sigma)(h), \pi^\sigma\rangle$$

and this implies $\pi \in \mathrm{Z}(\mathrm{Aut}(S)) = \{\mathrm{id}_S\}$. If $k \in \mathrm{Alt}(\kappa)$ and $\sigma \in \mathrm{Aut}(S)$ with $\mathfrak{c}(\sigma) = \iota_k$, then

$$\langle h, \mathrm{id}_S\rangle = \langle h, \mathrm{id}_S\rangle^{\langle \mathrm{id}_\kappa, \sigma\rangle} = \langle \mathfrak{c}(\sigma)(h), \mathrm{id}_S\rangle = \langle h^k, \mathrm{id}_S\rangle$$

and hence $h \in \mathrm{Z}(\mathrm{Alt}(\kappa)) = \{\mathrm{id}_\kappa\}$.

Define $G = \bar{G} \times \mathrm{Alt}(\kappa)$ and $A = \bar{A} \times \{\mathrm{id}_\kappa\} \cup \{1_{\bar{G}}\} \times \mathrm{Alt}(\kappa)$. By Proposition 5.3.3, $\langle G, A\rangle$ is a special pair that is not strongly special. Moreover, it is easy to see that both G and A have cardinality κ and

$$\mathrm{C}_G(A) = \mathrm{C}_{\bar{G}}(\bar{A}) \times \mathrm{Z}(\mathrm{Alt}(\kappa)) = \{\langle 1_{\bar{G}}, \mathrm{id}_\kappa\rangle\}.$$

Let $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be an automorphism tower of G. Then G_1 has cardinality 2^κ, because the automorphism group of $\mathrm{Alt}(\kappa)$ is isomorphic to the group $\mathrm{Sym}(\kappa)$ of all permutations of κ and every automorphism of $\mathrm{Alt}(\kappa)$ induces a unique automorphism of G. By Theorem 2.2.5, $\langle G_1, A\rangle$ is a special pair with $\mathrm{C}_{G_1}(A) = \{1_G\}$. Finally, $\langle G_1, A\rangle$ is not a strongly special pair, because otherwise $\langle G, A\rangle$ would be a strongly special pair. $\square$

Part 2

Definability in generalized Baire spaces

CHAPTER 6

Generalized Baire spaces

In this chapter, we establish basic definitions to talk about definable subsets of generalized Baire spaces and their structural properties. We introduce two regularity properties that generalize classical notions from descriptive set theory. Then we investigate their structural implications and prove that, if κ is a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then it is consistent that all simply definable subsets of ${}^{\kappa}\kappa$ possess these properties.

This analysis allows us to show that the existence of simply definable well-orderings of subsets of ${}^{\kappa}\kappa$ of order-type 2^{κ} does not follow from the axioms of set theory for such cardinals κ. In combination with the results of the next chapter, this implies that the existence of such well-orderings is actually independent from the standard axioms of set theory.

The work presented in this chapter forms a part of [**Lücb**].

6.1. Introduction

If κ is an infinite cardinal and n is a natural number, then we equip the space $({}^{\kappa}\kappa)^n$ with the usual topological structure induced by basic open sets of the form

$$U_{s_0,\dots,s_{n-1}} = \{\langle x_0,\dots,x_{n-1}\rangle \in ({}^{\kappa}\kappa)^n \mid s_0 \subseteq x_0,\dots,s_{n-1} \subseteq x_{n-1}\}$$

with $s_0,\dots,s_{n-1} \in {}^{<\kappa}\kappa$. The resulting topological space is called *generalized Baire space for κ.*

It is easy to see that a subset of ${}^{\kappa}\kappa$ is closed with respect to this topology if and only if it is equal to the set $[T]$ of all cofinal branches through some tree T on κ^n of height κ.

Definition 6.1.1. Let κ be an infinite cardinal. A subset A of $({}^{\kappa}\kappa)^n$ is a *κ-Borel subset* if it is contained in the smallest algebra of sets on $({}^{\kappa}\kappa)^n$ that contains all open subsets and is closed under unions of size κ.

The following definition directly generalizes the notion of a projective subset of Baire Space to our setting.

Definition 6.1.2. Let κ be an infinite cardinal.

(1) A subset A of $({}^{\kappa}\kappa)^n$ is a $\mathbf{\Sigma}^1_1$*-subset* if there is a tree T on κ^{n+1} with $A = p[T]$.
(2) A subset A of $({}^{\kappa}\kappa)^n$ is a $\mathbf{\Pi}^1_k$*-subset* if $({}^{\kappa}\kappa)^n \setminus A$ is a $\mathbf{\Sigma}^1_k$-subset.
(3) A subset A of $({}^{\kappa}\kappa)^n$ is a $\mathbf{\Sigma}^1_{k+1}$*-subset* if there is a $\mathbf{\Pi}^1_k$-subset B of $({}^{\kappa}\kappa)^{n+1}$ with $A = \exists^x B$.

(4) A subset A of $({}^\kappa\kappa)^n$ is a $\mathbf{\Delta}^1_k$-*subset* if it is both a $\mathbf{\Sigma}^1_k$-subset and a $\mathbf{\Pi}^1_k$-subset.

Fix an uncountable regular κ with $\kappa = \kappa^{<\kappa}$. In Section 6.2 we will present a folklore result showing that the $\mathbf{\Sigma}^1_1$-subsets are exactly the subsets of $({}^\kappa\kappa)^n$ that are definable in the structure $\langle \mathrm{H}_{\kappa^+}, \in\rangle$ by a Σ_1-formula with parameters. This shows that the $\mathbf{\Sigma}^1_1$-subsets form an interesting and rich class of subsets. Moreover, this result can be used to show that the κ-Borel subsets of ${}^\kappa\kappa$ form a proper subclass of the class of $\mathbf{\Delta}^1_1$-subsets (see [**FHK**, Theorem 18]).

We will now specify what we mean by *simply definable well-order of a subset of* ${}^\kappa\kappa$.

Definition 6.1.3. A $\mathbf{\Sigma}^1_1$-*well-ordering of a subset of* ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$-subset R of ${}^\kappa\kappa \times {}^\kappa\kappa$ with the property that $\langle \mathrm{dom}(R), R\rangle$ is a well-ordering, where

$$\mathrm{dom}(R) = \{x \in {}^\kappa\kappa \mid (\exists y)\ [R(x,y) \vee R(y,x)]\}.$$

In the following sections, we will generalize the *perfect subset property* and the notion of $\mathbf{\Sigma}^1_2$-absoluteness to generalized Baire spaces. Then we will prove that certain fragments of these principles are consistent and derive some structural implications from them. We will use these results to prove the following statements about the possible non-existence of certain $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$.

Theorem 6.1.4. *Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$ and $\nu > \kappa$ be a cardinal. If G is* $\mathrm{Add}(\kappa,\nu)$*-generic over* V *and R is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in* $\mathrm{V}[G]$*, then* $\mathrm{dom}(R) \neq ({}^\kappa\kappa)^{\mathrm{V}[G]}$ *and the order-type of* $\langle \mathrm{dom}(R), R\rangle$ *has cardinality at most* $(2^\kappa)^{\mathrm{V}}$ *in* $\mathrm{V}[G]$.

To state the second theorem, we need to introduce a large cardinal property.

Definition 6.1.5. A cardinal ν is Σ_2-reflecting if it is inaccessible and the structure $\langle \mathrm{V}_\nu, \in\rangle$ is a Σ_2-elementary submodel of $\langle \mathrm{V}, \in\rangle$.

Note that the consistency strength of the existence of a Σ_2-reflecting cardinal is bounded by the consistency strength of the existence of a Mahlo cardinal.

Theorem 6.1.6. *Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$, $\nu > \kappa$ be an inaccessible cardinal and γ be a cardinal. Assume that either $\gamma > \nu$ or ν is a Σ_2-reflecting cardinal. If $G \times H$ is* $(\mathrm{Col}(\kappa,{<}\nu) \times \mathrm{Add}(\kappa,\gamma))$*-generic over* V *and R is $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in* $\mathrm{V}[G][H]$*, then the order-type of* $\langle \mathrm{dom}(R), R\rangle$ *has cardinality at most κ in* $\mathrm{V}[G][H]$.

6.2. $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ and H_{κ^+}

Given an uncountable regular cardinal κ, it is a well-known that a subset of ${}^\kappa\kappa$ is $\mathbf{\Sigma}^1_1$ if and only if it is definable in the structure $\langle \mathrm{H}_{\kappa^+}, \in\rangle$ by a Σ_1-formula with parameters. In this section, we will give a proof of this folklore result that emphasises the absoluteness properties of this correspondence.

Before we start, we fix some more notation. Given an ordinal λ closed under Gödel-Pairing, $f \in {}^\lambda X$ for some nonempty set X and $\alpha < \lambda$, we define $(f)_\alpha$ to be the unique function $g \in {}^\lambda X$ with $g(\beta) = f(\prec\alpha,\beta\succ)$ for all $\beta < \lambda$.

By using Gödel-Pairing to code κ-many branches into one branch, it is easy to prove the following proposition.

Proposition 6.2.1. *Let κ be an infinite cardinal.*

(1) *If $\langle T_\alpha \mid \alpha < \kappa\rangle$ is a sequence of trees on κ^{n+1}, then there are trees T_U and T_I on κ^{n+1} such that*
$$p[T_U] = \bigcup_{\alpha<\kappa} p[T_\alpha] \text{ and } p[T_I] = \bigcap_{\alpha<\kappa} p[T_\alpha]$$
hold in every transitive ZFC*-model that contains* V*.*

(2) *If T is a tree on κ^{n+2}, then there is a tree T_* on κ^{n+1} such that $p[T_*] = \exists^x p[T]$ holds in every transitive* ZFC*-model that contains* V*.* □

Given a limit ordinal λ closed under Gödel-Pairing and $x \in {}^\lambda 2$, we define $\in_x$ to be the unique binary relation on λ such that
$$\alpha \in_x \beta \iff x(\prec\alpha,\beta\succ) = 1$$
holds for all $\alpha, \beta < \lambda$.

Proposition 6.2.2. *Let κ be an uncountable regular cardinal. There is a tree T on $\kappa\times\kappa$ such that*
$$p[T] = \{x \in {}^\kappa 2 \mid \langle\kappa,\in_x\rangle \textit{ is well-founded and extensional }\} \tag{6.1}$$
holds in every transitive ZFC*-model that contains* V *and has the same ${}^{<\kappa}\kappa$ as* V*.*

Proof. Given $\lambda < \kappa$ closed under Gödel-Pairing, we define T^λ to be the set of all pairs $\langle s,t\rangle \in {}^\lambda 2 \times {}^\lambda\kappa$ such that $\langle\lambda,\in_s\rangle$ is well-founded and, if $\alpha,\beta,\gamma < \lambda$ with $\alpha \neq \beta$ and $t(\prec\alpha,\beta\succ) = \gamma$, then $s(\prec\gamma,\alpha\succ) \neq s(\prec\gamma,\beta\succ)$. We define T to be the tree on $\kappa\times\kappa$ consisting of all $\langle s,t\rangle$ with $\mathrm{lh}(s) = \mathrm{lh}(t)$ and $\langle s\restriction\lambda, t\restriction\lambda\rangle \in T^\lambda$ for all $\lambda \leq \mathrm{lh}(s)$ closed under Gödel-Pairing. □

Proposition 6.2.3. *Let κ be an infinite cardinal, $\varphi(v_0,\dots,v_{n-1})$ be a formula in the language of set theory and $\alpha_0,\dots,\alpha_{n-1} < \kappa$. There is a tree T on $\kappa\times\kappa$ such that*
$$p[T] = \{x \in {}^\kappa 2 \mid \langle\kappa,\in_x\rangle \models \varphi(\alpha_0,\dots,\alpha_{n-1})\} \tag{6.2}$$
holds in every transitive ZFC*-model that contains* V *and has the same ${}^{<\kappa}\kappa$ as* V*.*

Proof. We can assume that $\varphi(v_0,\dots,v_{n-1})$ is in prenex normal form. We construct the corresponding trees inductively. If φ is atomic (or the negation of an atomic formula), then T is simply the tree of all $\langle s,t\rangle \in {}^{<\kappa}\kappa\times{}^{<\kappa}\kappa$ with $\mathrm{lh}(s) = \mathrm{lh}(t)$ and either $\mathrm{lh}(s) \leq \prec\alpha_0,\alpha_1\succ$ or $s(\prec\alpha_0,\alpha_1\succ) = 1$ (or $s(\prec\alpha_0,\alpha_1\succ) = 0$ in the case of a negated atomic formula).

If $\varphi(v_0, \ldots, v_{n-1}) \equiv (\exists x)\ \varphi_0(v_0, \ldots, v_{n-1}, x)$ and $\alpha < \kappa$, then we can use the induction hypothesis to find a tree T_α on $\kappa \times \kappa$ such that

$$p[T_\alpha] = \{x \in {}^\kappa 2 \mid \langle \kappa, \in_x \rangle \models \varphi_0(\alpha_0, \ldots, \alpha_{n-1}, \alpha)\}$$

holds in every transitive ZFC-model that contains V and has the same ${}^{<\kappa}\kappa$ as V. By Proposition 6.2.1, there is a tree T on $\kappa \times \kappa$ with the property that $p[T] = \bigcup_{\alpha<\kappa} p[T_\alpha]$ holds upwards-absolutely. This implies that T satisfies (6.2) in every transitive ZFC-model that contains V and has the same ${}^{<\kappa}\kappa$ as V.

The trees in the universal quantifier case, the disjunction case and the conjunction case are constructed in the same fashion using Proposition 6.2.1. $\square$

Note that the sets mentioned in (6.1) and (6.2) are actually κ-Borel subsets of ${}^\kappa\kappa$. In particular, if κ has uncountable cofinality, then the set of codes for well-founded relations on κ is closed in ${}^\kappa\kappa$.

Lemma 6.2.4. *Let $\varphi(u_0, \ldots, v_{n+m-1})$ be a Σ_1-formula, κ be an uncountable regular cardinal and $x_n, \ldots, x_{n+m-1} \in \mathrm{H}_{\kappa^+}$. Then there is a tree T on κ^{n+1} such that*

$$p[T] = \{\langle x_0, \ldots, x_{n-1}\rangle \in ({}^\kappa\kappa)^n \mid \langle \mathrm{H}_{\kappa^+}, \in \rangle \models \varphi(x_0, \ldots, x_{n+m-1})\} \tag{6.3}$$

holds in every transitive ZFC*-model that contains* V *and has the same* ${}^{<\kappa}\kappa$ *as* V.

Proof. Fix bijections $b_j : \kappa \longrightarrow \mathrm{tc}(\{x_{n+j}\} \cup \kappa)$ for all $j < m$. Let M be a transitive ZFC-model containing V with the same ${}^{<\kappa}\kappa$ as V and $x_0, \ldots, x_{n-1} \in ({}^\kappa\kappa)^M$. Now, $\langle \mathrm{H}^M_{\kappa^+}, \in \rangle$ is a model of $\varphi(x_0, \ldots, x_{n+m-1})$ if and only if there is a transitive $N \in \mathrm{H}^M_{\kappa^+}$ with $\kappa, x_0, \ldots, x_{n+m-1} \in N$ and $\langle N, \in \rangle$ is a model of this statement.

If $\varphi(\vec{v}) \equiv (\exists x)\varphi_0(\vec{v}, x)$ for some Δ_0-formula φ_0, then the above statement is equivalent to the existence of $x \in ({}^\kappa 2)^M$ and $y, z_0, \ldots, z_m \in ({}^\kappa\kappa)^M$ with the following properties.

(1) $\langle \kappa, \in_x \rangle$ is well-founded and extensional and
$$\langle \kappa, \in_x \rangle \models \varphi_0(0, \ldots, n+m).$$
(2) $\langle \kappa, \in_x \rangle \models$ "$\omega \in \mathrm{On}$".
(3) $\langle \kappa, \in_x \rangle \models$ "$\omega + j + 1 = \mathrm{tc}(\{n+j\} \cup \omega)$" for all $j < m$.
(4) $\langle \kappa, \in_x \rangle \models$ "$i : \omega \longrightarrow \omega$" for all $i < n$.
(5) For all $\alpha, \beta < \kappa$, we have
$$\langle \kappa, \in_x \rangle \models \text{“}\alpha \dot{\in} \beta \wedge \beta \dot{\in} \omega\text{”}$$
if and only if $\alpha = y(\gamma)$ and $\beta = y(\delta)$ for some $\gamma < \delta < \kappa$.
(6) For all $\alpha, \beta < \kappa$ and $i < n$, we have
$$\langle \kappa, \in_x \rangle \models \text{“}i(\alpha) = \beta\text{”}$$
if and only if $\alpha = y(\gamma)$ and $\beta = (y \circ x_i)(\gamma)$ for some $\gamma < \kappa$.

(7) For all $\alpha, \beta < \kappa$ and $j < m$, we have

$$\langle \kappa, \in_x \rangle \models \text{“}\alpha \dot{\in} \beta \wedge \beta \dot{\in} (\omega + 1 + j)\text{”}$$

if and only if $\alpha = (z_j \circ b_j)(\gamma)$ and $\beta = (z_j \circ b_j)(\delta)$ for some $\gamma, \delta < \kappa$ with $b_j(\gamma) \in b_j(\delta)$.

Using Proposition 6.2.2 and 6.2.3, there is a tree T_0 on κ^{m+n+3} with the property that, for all M as above, $\langle x_0, \ldots, x_{n-1}, y, z_0, \ldots, z_m \rangle \in [T]^M$ if and only if $y, \vec{z}$ witness that $\langle \mathrm{H}^M_{\kappa^+}, \in \rangle \models \varphi(x_0, \ldots, x_{n+m-1})$ holds. By Proposition 6.2.1, this completes the proof of the lemma. □

Let κ be an infinite cardinal with $\kappa = \kappa^{<\kappa}$. Given $n < \omega$, there is a Σ_1-formula $\varphi(u_0, \ldots, u_{n-1}, v_0, v_1)$ such that for every tree T on κ^{n+1} the equality (6.3) holds with $m = 2$, $p_0 = \kappa$ and $p_1 = T$ in every transitive ZFC-model that contains V. This shows that $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ correspond to $\mathbf{\Sigma}_1(\mathrm{H}_{\kappa^+})$-subsets in a way that is upwards-absolute between transitive ZFC-models with the same ${}^{<\kappa}\kappa$. We will often use this folklore fact to keep constructions in our proofs simple.

There is a similar correspondence for κ-Borel subsets: a subset A of ${}^\kappa\kappa$ is κ-Borel if and only if there is a transitive set M of cardinality κ, a formula $\varphi \equiv \varphi(v_0, \ldots, v_{n-1})$ in the language of set theory expanded by an unary relation symbol and parameters $z_0, \ldots, z_{m-1} \in M$ such that $\kappa \in M$, $\langle M, \in \rangle \models \mathrm{ZF}^-$ and

$$x \in A \iff \langle M, \in, x \rangle \models \varphi(z_0, \ldots, z_{n-1})$$

holds for all $x \in {}^\kappa\kappa$.

6.3. The perfect subset property

We generalize the perfect subset property of subsets of Baire Space to subsets of arbitrary function spaces ${}^\kappa\kappa$ and establish a connection between this property and generic absoluteness.

In the remainder of this chapter, **we fix a regular uncountable cardinal κ that satisfies $\kappa = \kappa^{<\kappa}$.**

Definition 6.3.1. Let λ be a limit ordinal.

We say that a map $\iota : {}^{<\lambda}2 \longrightarrow ({}^{<\lambda}\lambda)^n$ is a *continuous order-embedding* if the following statements hold for all $s_0, s_1 \in {}^{<\lambda}2$ with $\iota(s_i) = \langle t^i_0, \ldots, t^i_{n-1} \rangle$.

(1) If $s_0 \subsetneq s_1$, then $t^0_k \subsetneq t^1_k$ for all $k < n$.
(2) If s_0 and s_1 are incompatible in ${}^{<\lambda}2$, then there is a $k < n$ such that t^0_k and t^1_k are incompatible in ${}^{<\lambda}\lambda$.
(3) If $\mathrm{lh}(s_0) \in \mathrm{Lim} \cap \lambda$ and $k < n$, then

$$t^0_k = \bigcup \{u^\alpha_k \mid (\exists \alpha < \mathrm{lh}(s_0))\ \iota(s_0 \upharpoonright \alpha) = \langle u^\alpha_0, \ldots, u^\alpha_{n-1} \rangle\}.$$

Definition 6.3.2. Let λ be a limit ordinal and A be a subset of ${}^\lambda\lambda$. We say that A *contains a perfect subset* if there is a continuous order-embedding

$\iota : {}^{<\lambda}2 \longrightarrow {}^{<\lambda}\lambda$ such that $[T_\iota] \subseteq A$, where T_ι is the tree

$$T_\iota = \{t \in {}^{<\lambda}\lambda \mid (\exists s \in {}^{<\lambda}2)\ t \subseteq \iota(s)\}.$$

on λ.

Let $\mathcal{C}$ be a class of subsets of ${}^{\kappa}\kappa$. We say that subsets in $\mathcal{C}$ have the *perfect subset property* if every subset in $\mathcal{C}$ of cardinality bigger than κ contains a perfect subset. We present existing results related to the above definitions following [**FHK**, Chapter IV].

- We call a tree T on κ a *weak κ-Kurepa tree* if $\mathrm{ht}(T) = \kappa$, $[T]$ has cardinality at least κ^+ and there are stationary many $\alpha < \kappa$ such that the cardinality of $T \cap {}^{\alpha}\kappa$ is at most the cardinality of α. The idea of using Kurepa trees to construct closed subsets without the perfect subset property goes back to [**MV93**, Section 5].

  Let $\iota : {}^{<\kappa}2 \longrightarrow {}^{<\kappa}\kappa$ be a continuous order-embedding and T be a tree on κ of height κ with $[T_\iota] \subseteq [T]$. First, assume that there is an $\alpha < \kappa$ such that $\iota"{}^{\alpha}2 \nsubseteq {}^{<\beta}\kappa$ for all $\beta < \kappa$. Let α be minimal with this property. By the regularity of κ, there is a $\beta < \kappa$ with $\iota"{}^{<\alpha}2 \subseteq {}^{<\beta}\kappa$. The set $C = \{s \in {}^{\alpha}2 \mid \mathrm{lh}(\iota(s)) \geq \beta\}$ has cardinality κ and $\iota(s) \restriction \beta \in T$ for all $s \in C$. We can conclude that $T \cap {}^{\beta}\kappa$ has cardinality at least κ in this case. Now, assume that for every $\alpha < \kappa$ there is a $\beta < \kappa$ with $\iota"{}^{\alpha}2 \subseteq T \cap {}^{<\beta}\kappa$. Then the set $\{\alpha < \kappa \mid \iota"{}^{\alpha}2 \subseteq T \cap {}^{\alpha}\kappa\}$ is closed and unbounded in κ. In both cases, T is not a weak κ-Kurepa tree.

  The existence of weak κ-Kurepa trees therefore provides examples for the failure of the perfect subset property for closed subsets of ${}^{\kappa}\kappa$. In particular, if "V = L" holds, then the perfect subset property for closed sets fails for all uncountable regular cardinals (see [**FHK**, Section IV.2]).
- If κ is successor cardinal, then we call a tree T on κ a *κ-Kurepa tree* if $\mathrm{ht}(T) = \kappa$, $[T]$ has cardinality at least κ^+ and $T \cap {}^{\alpha}\kappa$ has cardinality less than κ for every $\alpha < \kappa$. Given a limit cardinal κ, we call a tree T on κ a *κ-Kurepa tree* if $\mathrm{ht}(T) = \kappa$, $[T]$ has cardinality at least κ^+ and the cardinality of $T \cap {}^{\alpha}\kappa$ is at most the cardinality of α.

  If all closed subsets of ${}^{\kappa}\kappa$ have the perfect subset property, then there are no κ-Kurepa trees and κ^+ is inaccessible in L by an argument of Robert Solovay (see [**Jec71**, Section 4]).
- Let $\nu > \kappa$ be an inaccessible cardinal and G be $\mathrm{Col}(\kappa, {<}\nu)$-generic over V. An argument of Philipp Schlicht shows that $\mathbf{\Sigma}^1_1$-subsets of ${}^{\kappa}\kappa$ in V$[G]$ have the perfect subset property. We will provide a proof of this statement in Section 6.5 (Proposition 6.5.8).
- Large cardinal properties of κ do not imply the perfect subset property for closed subsets of ${}^{\kappa}\kappa$. If κ is a supercompact cardinal, then

there is a partial order that preserves the supercompactness of κ and adds a weak κ-Kurepa tree (see [**FHK**, Section IV.2]).

To further investigate the perfect subset property for $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$, we need a well-known result saying that ZFC proves *generic absoluteness for $\mathbf{\Sigma}^1_1({}^\kappa\kappa)$-formulae (i.e., formulae with parameters which define $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$) under $<\kappa$-closed forcings*.

PROPOSITION 6.3.3. *Let T be a tree on κ^n of height κ and $\mathbb{P}$ be a $<\kappa$-closed partial order. If there is a $p \in \mathbb{P}$ with $p \Vdash$ "$[\check{T}] \neq \emptyset$", then $[T] \neq \emptyset$.*

PROOF. Let $p \Vdash$ "$\langle \tau_0, \dots, \tau_{n-1}\rangle \in [\check{T}]$" for some names $\tau_0, \dots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$. Given $\alpha < \kappa$, the set of conditions $q \in \mathbb{P}$ with

$$(\exists \langle t_0, \dots, t_{n-1}\rangle \in T)\ \left[\mathrm{lh}(t_0) \geq \alpha \wedge q \Vdash \text{“}\check{t}_0 \subseteq \tau_0 \wedge \dots \wedge \check{t}_{n-1} \subseteq \tau_{n-1}\text{”}\right]$$

is dense below p. Since $\mathbb{P}$ is $<\kappa$-closed, we can define a $\leq_{\mathbb{P}}$-descending sequence $\langle p_\alpha \in \mathbb{P} \mid \alpha < \kappa\rangle$ and an ascending sequence

$$\langle \langle t^\alpha_0, \dots, t^\alpha_{n-1}\rangle \in T \mid \alpha < \kappa\rangle$$

in V such that $p_0 = p$, $\mathrm{lh}(t^\alpha_0) \geq \alpha$ and $p_\alpha \Vdash$ "$\check{t}^\alpha_i \subseteq \tau_i$" holds for all $\alpha < \kappa$ and $i < n$. But this construction implies that the tuple $\langle \bigcup_{\alpha<\kappa} t^\alpha_0, \dots, \bigcup_{\alpha<\kappa} t^\alpha_{n-1}\rangle$ is an element of $[T]$ in V. □

We look at a stronger version of the perfect subset property for $\mathbf{\Sigma}^1_1$-subsets.

DEFINITION 6.3.4. Let T be a tree on κ^{n+1}. An *$\exists^x$-perfect embedding into T* is a continuous order-embedding $\iota : {}^{<\kappa}2 \longrightarrow ({}^{<\kappa}\kappa)^{n+1}$ with the following properties.

(1) $\mathrm{ran}(\iota) \subseteq T$.
(2) If $s_0, s_1 \in {}^{<\kappa}2$ are incompatible sequences with $\iota(s_i) = \langle t^i_0, \dots, t^i_n\rangle$, then there is a $k < n$ such that the sequences t^0_k and t^1_k are incompatible in ${}^{<\kappa}\kappa$.

The idea behind the above definition is that a $\exists^x$-perfect embedding into T witnesses that the projection $p[T]$ has a perfect subset.

PROPOSITION 6.3.5. *Let T be a tree on $\kappa \times \kappa$ and ι be a $\exists^x$-perfect embedding into T. If we define $\bar{\iota} : {}^{<\kappa}2 \longrightarrow {}^{<\kappa}\kappa$ to be the continuous order-embedding such that $\bar{\iota}(s) = t_0$ for all $s \in {}^{<\kappa}2$ with $\iota(s) = \langle t_0, t_1\rangle$, then $\bar{\iota}$ witnesses that $p[T]$ has a perfect subset in every transitive* ZFC*-model containing* V. □

The following lemma establishes a connection between the existence of $\exists^x$-perfect embeddings and absoluteness properties of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$.

LEMMA 6.3.6. *The following statements are equivalent for every tree T on $\kappa \times \kappa$ of height κ.*

(1) *There is a $\exists^x$-perfect embedding into T.*
(2) *If $\mathbb{P}$ is $<\kappa$-closed partial order, then $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\mathcal{P}(\check{\kappa}) \nsubseteq \check{\mathrm{V}} \rightarrow p[\check{T}] \nsubseteq \check{\mathrm{V}}$".*

(3) $\mathbb{1}_{\mathrm{Add}(\kappa,1)} \Vdash$ "*$p[\check{T}] \nsubseteq \check{\mathrm{V}}$*".
(4) *There is a $<\kappa$-closed partial order $\mathbb{P}$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$p[\check{T}] \nsubseteq \check{\mathrm{V}}$".*

Proof. Assume (i) holds, ι is a $\exists^x$-perfect embedding into T and $\mathbb{P}$ is a $<\kappa$-closed partial order that adds a new subset of κ. If we define

$$S = \{\langle t_0 \restriction \alpha, t_1 \restriction \alpha\rangle \in T \mid \langle t_0, t_1\rangle \in \mathrm{ran}(\iota),\ \alpha \leq \mathrm{lh}(t_0)\},$$

then S is a subtree of T of height κ.

Let G be $\mathbb{P}$-generic over V, $x_0 \in ({}^\kappa 2)^{\mathrm{V}[G]} \setminus \mathrm{V}$ and define

$$y = \bigcup\{t_0 \mid (\exists \alpha < \kappa)\ \iota(x_0 \restriction \alpha) = \langle t_0, t_1\rangle\},$$

Clearly, $y \in p[S]^{\mathrm{V}[G]} \subseteq p[T]^{\mathrm{V}[G]}$. Assume, toward a contradiction, that $y \in \mathrm{V}$ holds. Then the tree $S_y = \{t \in {}^{<\kappa}\kappa \mid \langle y \restriction \mathrm{lh}(t), t\rangle \in S\}$ is an element of V and $[S_y]^{\mathrm{V}[G]} \neq \emptyset$. By Proposition 6.3.3, there is a $z \in [S_y]^{\mathrm{V}}$ and this means $\langle y, z\rangle \in [S]^{\mathrm{V}}$. But this means that there is an $x_1 \in ({}^\kappa 2)^{\mathrm{V}}$ with $y = \bigcup\{t_0 \mid (\exists \alpha < \kappa)\ \iota(x_1 \restriction \alpha) = \langle t_0, t_1\rangle\}$. Given $\alpha < \kappa$ with $x_0(\alpha) \neq x_1(\alpha)$ and $\iota(x_i \restriction (\alpha+1)) = \langle t_0^i, t_1^i\rangle$, we have t_0^0 and t_0^1 incompatible and $t_0^0, t_0^1 \subseteq y$, a contradiction.

Now, assume (iv) holds. Fix $\tau_0, \tau_1 \in \mathrm{V}^{\mathbb{P}}$ with

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}\tau_0 \notin \mathrm{V} \wedge \langle \tau_0, \tau_1\rangle \in [\check{T}]\text{”}.$$

We inductively construct order-embeddings $i : {}^{<\kappa}2 \longrightarrow \mathbb{P}$ and $\iota : {}^{<\kappa}2 \longrightarrow T$ with the following properties.

(1) ι is continuous.
(2) If $s \in {}^{<\kappa}2$ and $\iota(s) = \langle t_0, t_1\rangle$, then $i(s) \Vdash$ "$\check{t}_0 \subseteq \tau_0 \wedge \check{t}_1 \subseteq \tau_1$".
(3) If $s_0, s_1 \in {}^{<\kappa}2$ are incompatible, then $\iota(s_0), \iota(s_1) \in T$ are incompatible.

Assume that $i \restriction {}^{<\alpha}2$ and $\iota \restriction {}^{<\alpha}2$ are already constructed for some $\alpha < \kappa$. If $\alpha \in \mathrm{Lim}$ and $s \in {}^\alpha 2$, then there is a condition $i(s) \in \mathbb{P}$ with $p \leq_{\mathbb{P}} i(s \restriction \bar{\alpha})$ for all $\bar{\alpha} < \alpha$. Define $\langle t_0, t_1\rangle \in {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ by setting

$$t_i = \bigcup\{\bar{t}_i \mid (\exists \bar{\alpha} < \alpha)\ \iota(s \restriction \alpha) = \langle \bar{t}_0, \bar{t}_1\rangle\}.$$

By construction, $i(s) \Vdash$ "$\check{t}_i \subseteq \tau_i$" and this means $\langle t_0, t_1\rangle \in T$. Moreover, given incompatible $s_0, s_1 \in {}^\alpha 2$, there is an $\bar{\alpha} < \alpha$ such that $s_0 \restriction \bar{\alpha}$ and $s_1 \restriction \bar{\alpha}$ are incompatible and our assumptions imply that $\iota(s_0)$ and $\iota(s_1)$ are also incompatible.

If $\alpha = \bar{\alpha} + 1$ and $s \in {}^{\bar{\alpha}}2$, then there are conditions $q_0, q_1 \leq_{\mathbb{P}} i(s)$ and $\beta, \gamma_0, \gamma_1 < \kappa$ with $\beta \geq \mathrm{lh}(\iota(s))$, $q_i \Vdash$ "$\tau_0(\check{\beta}) = \check{\gamma}_i$" and $\gamma_0 \neq \gamma_1$, because we have $i(s) \Vdash$ "$\tau_0 \notin \mathrm{V}$". Given $i < 2$, we can find $i(s^\frown\langle i\rangle) \in \mathbb{P}$ and $\iota(s^\frown\langle i\rangle) = \langle t_0^i, t_1^i\rangle \in T$ with $i(s^\frown\langle i\rangle) \leq q_i$, $\mathrm{lh}(t_0^i) = \beta + 1$ and $i(s^\frown\langle i\rangle) \Vdash$ "$\check{t}_0^i \subseteq \tau_0 \wedge \check{t}_1^i \subseteq \tau_1$". It is easy to check that this partial embedding also satisfies the above properties. $\square$

In the following, we investigate the correlation between the existence of a perfect subset of $\mathbf{\Delta}^1_1$-subsets of the form $p[T_0]$ and the existence of $\exists^x$-perfect embeddings into T_0. We need another notion of absoluteness.

Definition 6.3.7. Let Γ be a class of partial orders. We say that a subset A of ${}^\kappa\kappa$ is *weakly* Γ*-persistently* $\mathbf{\Delta}^1_1$ if there are trees T_0 and T_1 on $\kappa \times \kappa$ such that $p[T_0] = A$, $p[T_1] = {}^\kappa\kappa \setminus A$ and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$p[\check{T}_1] = {}^{\check{\kappa}}\check{\kappa} \setminus p[\check{T}_0]$" holds for all partial orders $\mathbb{P}$ in Γ.

Proposition 6.3.8. *Let $\mathbb{P}$ be a $<\kappa$-closed partial order that adds a new subset of κ, A be a subset of ${}^\kappa\kappa$ and T_0, T_1 be trees on $\kappa \times \kappa$ witnessing that A is weakly $\mathbb{P}$-persistently $\mathbf{\Delta}^1_1$. Then A has a perfect subset if and only if $\mathbb{1}_{\mathbb{P}} \nVdash$ "$\check{A} = p[\check{T}_0]$".*

Proof. Pick $p \in \mathbb{P}$ with $p \Vdash$ "$\check{A} \neq p[\check{T}_0]$". Assume, towards a contradiction, that there is a $q \leq_{\mathbb{P}} p$ with $q \Vdash$ "$p[\check{T}_0] \subseteq \check{\mathrm{V}}$". Let G be $\mathbb{P}$-generic over V with $q \in G$ and pick $y \in p[T_0]^{\mathrm{V}[G]} \setminus A \subseteq \mathrm{V}$. Define $T_y = \{t \in {}^{<\kappa}\kappa \mid \langle y \restriction \mathrm{lh}(t), t\rangle \in T_0\} \in \mathrm{V}$. Then $[T_y]^{\mathrm{V}[G]} \neq \emptyset$ and this means $[T_y]^{\mathrm{V}} \neq \emptyset$ by Proposition 6.3.3. But this implies $y \in p[T_0]^{\mathrm{V}} = A$, a contradiction. Therefore $p \Vdash$ "$p[\check{T}_0] \nsubseteq \check{\mathrm{V}}$" and A has a perfect subset by Lemma 6.3.6.

In the other direction, let $\iota : {}^{<\kappa}2 \longrightarrow {}^{<\kappa}\kappa$ witnesses that A has a perfect subset and assume, toward a contradiction, that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{A} = p[\check{T}_0]$" holds. Let G be $\mathbb{P}$-generic over V. By construction, $[T_\iota]^{\mathrm{V}[G]} \nsubseteq \mathrm{V}$, $p[T_0]^{\mathrm{V}[G]} = A \subseteq \mathrm{V}$ and $p[T_1]^{\mathrm{V}[G]} = ({}^\kappa\kappa)^{\mathrm{V}[G]} \setminus p[T_0]^{\mathrm{V}[G]} = ({}^\kappa\kappa)^{\mathrm{V}[G]} \setminus A$. If we define $T = \{\langle t_0, t_1\rangle \in T_1 \mid t_0 \in T_\iota\} \in \mathrm{V}$, then $[T]^{\mathrm{V}[G]} \neq \emptyset$ and therefore $[T]^{\mathrm{V}} \neq \emptyset$. But this shows that $\emptyset \neq [T_\iota]^{\mathrm{V}} \cap p[T_1]^{\mathrm{V}} \subseteq p[T_0]^{\mathrm{V}} \cap p[T_1]^{\mathrm{V}} = \emptyset$, a contradiction. □

6.4. $\mathbf{\Sigma}^1_2$-absoluteness

In this section, we generalize the notion of $\mathbf{\Sigma}^1_2$-absoluteness to our uncountable context and investigate its structural implications.

Definition 6.4.1. Let Γ be a class of partial orders. We say that *generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under forcings in* Γ *holds* if the implication

$$p \Vdash \text{“}(\exists x_0, \ldots, x_n \in {}^{\check{\kappa}}\check{\kappa})(\forall y_1, \ldots, y_m \in {}^{\check{\kappa}}\check{\kappa})\langle x_0, \ldots, x_n, y_0, \ldots, y_m\rangle \notin [\check{T}]\text{”}$$
$$\longrightarrow (\exists x_0, \ldots, x_n \in {}^{\kappa}\kappa)(\forall y_0, \ldots, y_m \in {}^{\kappa}\kappa)\langle x_0, \ldots, x_n, y_0, \ldots, y_m\rangle \notin [T]$$

holds true for every partial order $\mathbb{P}$ in Γ, every condition $p \in \mathbb{P}$ and every tree T on κ^{m+n+2}.

In Section 6.5, we will show that the consistency of generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under forcing with $<\kappa$-closed partial orders can be established from a relatively mild large cardinal assumption (Lemma 6.5.6). We will also show that such generic absoluteness for Cohen forcing $\mathrm{Add}(\kappa, 1)$ holds in every $\mathrm{Add}(\kappa, \kappa^+)$-generic extension of the ground model (Corollary 6.5.3).

The referee of [**Lücb**] pointed out that it is possible to establish the consistency of $\mathbf{\Sigma}^1_2$-absoluteness under certain classes of $<\kappa$-closed partial orders without the use of large cardinals. Let Γ be a class of $<\kappa$-closed partial orders such that elements of Γ satisfy the κ^+-chain condition and Γ is closed under forcing iterations with $<\kappa$-support in the ground model and every generic extension by a forcing in Γ. If "$2^\kappa = \kappa^+$" holds in the ground model, then there is a forcing iteration $\langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \kappa^+\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \kappa^+\rangle\rangle$ of partial orders in Γ with $<\kappa$-support and a sequence $\langle\dot{t}_\alpha \in \mathrm{V}^{\vec{\mathbb{P}}_{<\alpha}} \mid \alpha < \kappa^+\rangle$ of names such that the following statements hold whenever $\alpha < \kappa^+$ with $\alpha = \prec\prec\beta, \gamma\succ, \delta\succ$, G is $\vec{\mathbb{P}}_{<\alpha}$-generic over V and $\bar{G}$ is the corresponding filter in $\vec{\mathbb{P}}_{<\beta}$.

(1) $\dot{t}^{\bar{G}}_\beta$ is an enumeration of all subtrees of ${}^{<\kappa}\kappa$ in $\mathrm{V}[\bar{G}]$ of length κ^+.

(2) If $\dot{t}^{\bar{G}}_\beta(\gamma) = T$ and $(\exists\mathbb{Q} \in \Gamma)\ \mathbb{1}_{\mathbb{Q}} \Vdash$ "$p[\check{T}] \neq {}^{\check{\kappa}}\check{\kappa}$" holds in $\mathrm{V}[G]$, then $\mathbb{1}_{\dot{\mathbb{P}}^G_\alpha} \Vdash$ "$p[\check{T}] \neq {}^{\check{\kappa}}\check{\kappa}$" holds in $\mathrm{V}[G]$.

If G is $\vec{\mathbb{P}}_{<\kappa^+}$-generic over V, then generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under forcings in Γ holds in $\mathrm{V}[G]$.

Proposition 6.4.2. *Let Γ be a class of $<\kappa$-closed partial order that contains the trivial partial order and assume that generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under forcings in Γ holds. Then every $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ is weakly Γ-persistently $\mathbf{\Delta}^1_1$.*

Proof. Let T_0 and T_1 witness that $p[T_0]$ is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$. By Proposition 6.2.1, there is a tree T such that "$p[T] = p[T_0] \cup p[T_1]$" holds in V and every generic extension of V by a forcing in Γ.

Assume, toward a contradiction, that $\mathbb{1}_{\mathbb{P}} \nVdash$ "${}^{\check{\kappa}}\check{\kappa} = p[\check{T}_0] \cup p[\check{T}_1]$" holds for some $\mathbb{P} \in \Gamma$. Then there is a $p \in \mathbb{P}$ with

$$p \Vdash \text{"}(\exists x \in {}^{\check{\kappa}}\check{\kappa})(\forall y \in {}^{\check{\kappa}}\check{\kappa})\ \langle x, y\rangle \notin [\check{T}]\text{"}.$$

By $\mathbf{\Sigma}^1_2$-absoluteness, there is an $x \in {}^\kappa\kappa$ with $x \notin p[T] = {}^\kappa\kappa$, a contradiction.

In the same way, we can use Proposition 6.3.3 to see that

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{"}p[\check{T}_0] \cap p[\check{T}_1] = \emptyset\text{"}$$

holds for every partial order $\mathbb{P}$ in Γ. □

Proposition 6.4.3. *If generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under* $\mathrm{Add}(\kappa, 1)$ *holds and T is a tree on $\kappa \times \kappa$ of height κ, then $p[T]$ contains a perfect subset if and only if $\mathbb{1}_{\mathrm{Add}(\kappa,1)} \Vdash$ "$p[\check{T}] \nsubseteq \check{\mathrm{V}}$".*

Proof. Let $\iota : {}^{<\kappa}2 \longrightarrow {}^{<\kappa}\kappa$ witness that $p[T]$ has a perfect subset and assume, toward a contradiction, that there is a $p \in \mathrm{Add}(\kappa, 1)$ with $p \Vdash$ "$p[\check{T}] \subseteq \check{\mathrm{V}}$". By the results of Section 6.2, there is a tree T_* on $\kappa \times \kappa$ such that $p[T_*] = p[T] \cup ({}^\kappa\kappa \setminus [T_\iota])$ holds in V and every $\mathrm{Add}(\kappa, 1)$-generic extension of V. Since $p \Vdash$ "$[\check{T}_\iota] \nsubseteq \check{\mathrm{V}}$", we get

$$p \Vdash \text{"}(\exists x \in {}^{\check{\kappa}}\check{\kappa})(\forall y \in {}^{\check{\kappa}}\check{\kappa})\ \langle x, y\rangle \notin [\check{T}_*]\text{"}$$

and absoluteness gives us an $x \in {}^\kappa\kappa$ with $x \notin p[T_*] = {}^\kappa\kappa$, a contradiction. $\square$

We apply the above results to prove statements about the length of definable well-orders on subsets of ${}^\kappa\kappa$ in the presence of $\mathbf{\Sigma}^1_2$-absoluteness.

Clearly, every $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ has order type less than $(2^\kappa)^+$ and for every $\gamma < \kappa^+$ there is such a well-ordering with order type γ. Moreover, the results of Chapter 7 will show that it is consistent to have a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ of order-type greater than 2^κ.

PROPOSITION 6.4.4. *Let Γ be a class of $<\kappa$-closed partial orders and assume that generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under forcings in Γ holds. If T is a tree on κ^3 of such that $p[T]$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ and $\mathbb{P} \in \Gamma$, then*

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}p[\check{T}] \textit{ is a } \mathbf{\Sigma}^1_1\textit{-well-ordering of a subset of } {}^{\check\kappa}\check\kappa\text{ ”}.$$

PROOF. We prove that $p[T]$ is a linear and well-founded relation in every generic extension by a forcing in Γ; the other properties of a well-ordering can be deduced in the same manner.

By the results of Section 6.2, there is a tree T_w in $\kappa \times \kappa$ such that

$$p[T_w] = \{x \in {}^\kappa\kappa \mid (\forall n < \omega)\ \langle (x)_{n+1}, (x)_n\rangle \in p[T]\}$$

holds in V and every generic extension of V by a forcing in Γ. By our assumptions, $p[T_w] = \emptyset$ and Proposition 6.3.3 shows that $\mathbb{1}_{\mathbb{P}} \vDash$ “$p[\check{T}_w] = \emptyset$” holds for all $\mathbb{P}$ in Γ. This shows that $p[T]^{\mathrm{V}[G]}$ is a well-founded relation in every $\mathbb{P}$-generic extension $\mathrm{V}[G]$ of V with $\mathbb{P} \in \Gamma$.

As above, there is a tree T_l on κ^7 such that $p[T_l]$ is equal to the set

$$\begin{aligned}&\{\langle x, x_0, x_1, y, y_0, y_1\rangle \in ({}^\kappa\kappa)^6 \mid \langle x, y\rangle \in p[T] \vee \langle y, x\rangle \in p[T]]\\ &\vee [\langle x, x_0, x_1\rangle \notin [T] \wedge \langle x_0, x, x_1\rangle \notin [T]] \vee [\langle y, y_0, y_1\rangle \notin [T] \wedge \langle y_0, y, y_1\rangle \notin [T]]\}\end{aligned}$$

in V and every generic extension of V by a forcing in Γ. Assume, toward a contradiction, that there is a $\mathbb{P}$ in Γ and a $\mathbb{P}$-generic extension $\mathrm{V}[G]$ of V such that $p[T]^{\mathrm{V}[G]}$ is not a linear order on its domain. Then there is a $p \in \mathbb{P}$ with

$$p \vDash \text{“}(\exists x, x_0, x_1, y, y_0, y_1 \in {}^{\check\kappa}\check\kappa)(\forall z \in {}^{\check\kappa}\check\kappa)\ \langle x, x_0, x_1, y, y_0, y_1, z\rangle \notin [T_l]\text{”}$$

and, by $\mathbf{\Sigma}^1_2$-absoluteness, $p[T]$ is not linear on its domain in V, a contradiction. $\square$

The proof of the following lemma uses an idea of Philipp Schlicht to show that $\mathbf{\Sigma}^1_2$-absoluteness implies that $\mathbf{\Sigma}^1_1$-well-orders have *small* domains.

LEMMA 6.4.5. *Assume that generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under* $\mathrm{Add}(\kappa, 1)$ *holds. If T is a tree on κ^3 such that $p[T]$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$, then* $\mathrm{dom}(p[T])$ *contains no perfect subset.*

Proof. There is a tree T_* on $\kappa \times \kappa$ such that $p[T_*] = \mathrm{dom}(p[T])$ holds in V and every $\mathrm{Add}(\kappa,1)$-generic extension of V. Assume, toward a contradiction, that $p[T_*]$ contains a perfect subset and let G be $\mathrm{Add}(\kappa,1)$-generic over V. We will construct sequences $\langle G_n \in \mathrm{V}[G] \mid n < \omega \rangle$ and $\langle x_n \in ({}^\kappa\kappa)^{\mathrm{V}[G]} \mid n < \omega \rangle$ such that the following statements hold true for all $n < \omega$.

(1) There is a $\bar{G} \in \mathrm{V}[G]$ such that $(G_n \times \bar{G})$ is $(\mathrm{Add}(\kappa,1) \times \mathrm{Add}(\kappa,1))$-generic over $\mathrm{V}[G_0,\ldots,G_{n-1}]$ and $\mathrm{V}[G] = \mathrm{V}[G_0,\ldots,G_{n-1}][G_n][\bar{G}]$.
(2) We have $x_n \in \mathrm{V}[G_0,\ldots,G_n]$, $\langle x_{n+1}, x_n\rangle \in p[T]^{\mathrm{V}[G_0,\ldots,G_{n+1}]}$ and

$$\langle \mathrm{V}[G_0,\ldots,G_n], \in\rangle \models \left[\mathbb{1}_{\mathrm{Add}(\kappa,1)} \Vdash (\exists x) \ \left[x \notin \check{\mathrm{V}} \wedge \langle x, \check{x}_n\rangle \in p[\check{T}]\right]\right]. \tag{6.4}$$

There are $H_0, H_1 \in \mathrm{V}[G]$ such that $H_0 \times H_1$ is $(\mathrm{Add}(\kappa,1) \times \mathrm{Add}(\kappa,1))$-generic over V with $\mathrm{V}[G] = \mathrm{V}[H_0][H_1]$. By our assumptions and Proposition 6.4.3, there are $y_0, y_1 \in \mathrm{V}[G]$ with $y_i \in p[T_*]^{\mathrm{V}[H_i]} \setminus \mathrm{V}$. Since $\mathrm{V}[H_0] \cap \mathrm{V}[H_1] = \mathrm{V}$, we have $y_0 \neq y_1$ and there is an $i_* < 2$ with $\langle y_{1-i_*}, y_{i_*}\rangle \in p[T]^{\mathrm{V}[G]}$. Define $x_0 = y_{i_*}$ and $G_0 = H_{i_*}$. The homogeneity of $\mathrm{Add}(\kappa,1)$ in $\mathrm{V}[G_0]$ and $y_{1-i_*} \in \mathrm{V}[G] \setminus \mathrm{V}[G_0]$ imply (6.4).

Now assume $G_0,\ldots,G_n$ and $x_0,\ldots,x_n$ with the above properties are already constructed. Hence there are $H_0, H_1 \in \mathrm{V}[G]$ such that $(H_0 \times H_1)$ is $(\mathrm{Add}(\kappa,1) \times \mathrm{Add}(\kappa,1))$-generic over $\mathrm{V}[G_0,\ldots,G_n]$ and

$$\mathrm{V}[G] = \mathrm{V}[G_0,\ldots,G_n][H_0][H_1].$$

By (6.4), there are $y_0, y_1 \in \mathrm{V}[G]$ with $y_i \in \mathrm{V}[G_0,\ldots,G_n,H_i] \setminus \mathrm{V}[G_0,\ldots,G_n]$ and $\langle y_i, x_n\rangle \in p[T]^{\mathrm{V}[G_0,\ldots,G_n,H_i]}$. Again, there is an $i_* < 2$ with $\langle y_{1-i_*}, y_{i_*}\rangle \in p[T]^{\mathrm{V}[G]}$ and we can define $G_{n+1} = H_{i_*}$ and $x_{n+1} = y_{i_*}$. As above, (6.4) holds true.

Our construction shows $\langle x_{n+1}, x_n\rangle \in p[T]^{\mathrm{V}[G]}$ for all $n < \omega$. But $p[T]^{\mathrm{V}[G]}$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$ by Proposition 6.4.4, a contradiction. □

Corollary 6.4.6. *If generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under* $\mathrm{Add}(\kappa,1)$ *holds, then there is no well-ordering of ${}^\kappa\kappa$ whose graph is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$.* □

6.5. Two models with a nice structure theory for $\mathbf{\Sigma}^1_1$-subsets

We show that certain fragments of $\mathbf{\Sigma}^1_2$-absoluteness hold in two well-known classes of ZFC-models and derive some consequences about the possible length of $\mathbf{\Sigma}^1_1$-well-orders of subsets of ${}^\kappa\kappa$ in these models. We start with a standard result about Cohen-generic extensions of a ground model.

Lemma 6.5.1. *Let $\nu > \kappa$ be a cardinal and X be a subset of ν of cardinality κ^+. If G is $\mathrm{Add}(\kappa,\nu)$-generic over* V *and $\bar{G} = G \cap \mathrm{Add}(\kappa,X)$, then there is an elementary embedding*

$$j : \mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}]}) \longrightarrow \mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[G]})$$

with $j \restriction \mathrm{On} = \mathrm{id}_{\mathrm{On}}$ and $j \restriction \mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}]} = \mathrm{id}_{\mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}]}}$.

PROOF. We define $P = \mathcal{P}(\kappa)^{V[G]}$ and $\bar{P} = \mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}]}$. By the construction of $\mathrm{L}(\bar{P})$, there is a surjection

$$s : [\mathrm{On}]^{<\omega} \times \bar{P} \longrightarrow \mathrm{L}(\bar{P})$$

definable in $\mathrm{L}(\bar{P})$ by a formula $\varphi \equiv \varphi(u, v_0, v_1, w)$ and the parameter $\bar{P}$. Define

$$\begin{aligned} j(a) = b \iff & (\exists x \in \bar{P})(\exists A \in [\mathrm{On}]^{<\omega}) \\ & [\langle \mathrm{L}(\bar{P}), \in\rangle \models \varphi(a, x, A, \bar{P}) \wedge \langle \mathrm{L}(P), \in\rangle \models \varphi(b, x, A, P)]. \end{aligned}$$

In order to show that j is a well-defined function and an elementary embedding with the above properties, it suffices to show that for all $x_0, \ldots, x_{n-1} \in \bar{P}$, $A \in [\mathrm{On}]^{<\omega}$ and every $\mathcal{L}_\in$-formula $\psi \equiv \psi(u_0, \ldots, u_{n-1}, v_0, \ldots, v_{m-1}, w)$

$$\langle \mathrm{L}(\bar{P}), \in\rangle \models \psi(\vec{x}, A, \bar{P}) \iff \langle \mathrm{L}(P), \in\rangle \models \psi(\vec{x}, A, P).$$

holds. There exist $\bar{G}_0, \bar{G}_1 \in \mathrm{V}[\bar{G}]$ such that $\bar{G}_0$ is $\mathrm{Add}(\kappa, 1)$-generic over V and $\bar{G}_1$ is $\mathrm{Add}(\kappa, \kappa^+)$-generic over $\mathrm{V}[\bar{G}_0]$ with $\vec{x} \in \mathrm{V}[\bar{G}_0]$ and $\mathrm{V}[\bar{G}] = \mathrm{V}[\bar{G}_0][\bar{G}_1]$. Moreover, there is $G_1 \in \mathrm{V}[G]$ that is $\mathrm{Add}(\kappa, \nu)$-generic over $\mathrm{V}[\bar{G}_0]$ with $\mathrm{V}[G] = \mathrm{V}[\bar{G}_0][G_1]$.

Let F be $\mathrm{Col}(\omega, 2^\kappa)^{\mathrm{V}[G]}$-generic over $\mathrm{V}[G]$. We show that there is a $H \in \mathrm{V}[G][F]$ that is $\mathrm{Add}(\kappa, \kappa^+)^{\mathrm{V}}$-generic over $\mathrm{V}[\bar{G}_0]$ and satisfies $\mathcal{P}(\kappa)^{\mathrm{V}[G]} = \mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}_0][H]}$.

Work in $\mathrm{V}[G][F]$. Let $\langle x_n \mid n < \omega\rangle$ enumerate $\mathcal{P}(\kappa)^{\mathrm{V}[G]}$ and $\langle \alpha_n \mid n < \omega\rangle$ be strictly increasing and cofinal in $\kappa^{+\mathrm{V}}$. Define $\mathbb{P} = \mathrm{Add}(\kappa, 1)^{\mathrm{V}[\bar{G}_0]}$, $\mathbb{P}_n = \prod_{i<n} \mathbb{P}$, $\mathbb{Q} = \mathrm{Add}(\kappa, \kappa^+)^{\mathrm{V}[\bar{G}_0]}$ and $\mathbb{Q}_n = \mathrm{Add}(\kappa, \alpha_n)^{\mathrm{V}[\bar{G}_0]}$. Using the factor-property of Cohen-Forcing, it is easy to define a sequence $\langle H_n \mid n < \omega\rangle$ of filters in $\mathbb{P}$ that satisfy the following properties for all $n < \omega$.

(1) $H_n \in \mathrm{V}[G]$.
(2) H_n is $\mathbb{P}$-generic over $\mathrm{V}[\bar{G}_0][H_0, \ldots, H_{n-1}]$ and x_n is an element of $\mathrm{V}[\bar{G}_0][H_0, \ldots, H_n]$.
(3) There is a $G' \in \mathrm{V}[G]$ that $\mathrm{Add}(\kappa, \nu)$-generic over $\mathrm{V}[\bar{G}_0][H_0, \ldots, H_n]$ and satisfies $\mathrm{V}[G] = \mathrm{V}[\bar{G}_0][H_0, \ldots, H_n][G']$.

For all $n < \omega$, we let $e_n : \mathbb{P}_n \longrightarrow \mathbb{P}_{n+1}$ denote the natural inclusion. In $\mathrm{V}[\bar{G}_0]$, there are isomorphisms $i_n : \mathbb{P}_n \longrightarrow \mathbb{Q}_n$ with $i_n = i_{n+1} \circ e_n$ for all $n < \omega$. For all $n < \omega$, $H_0 \times \cdots \times H_{n-1}$ is $\mathbb{P}_n$-generic over $\mathrm{V}[\bar{G}_0]$ and we can define

$$H = \bigcup_{n<\omega} i_n``(H_0 \times \cdots \times H_{n-1}) \in \mathcal{P}(\mathbb{Q})^{\mathrm{V}[G][F]}.$$

We have

- $\mathbb{Q}_n \subseteq \mathbb{Q}_{n+1} \subseteq \bigcup_{n<\omega} \mathbb{Q}_n = \mathbb{Q}$,
- $e_n^{-1}``(H_0 \times \cdots \times H_n) = H_0 \times \cdots \times H_{n-1}$ and
- $i_n``H_0 \times \cdots \times H_{n-1} = H \cap \mathbb{Q}_n$ is a filter in $\mathbb{Q}_n$ for all $n < \omega$.

This allows us to conclude that H is a filter in $\mathbb{Q}$. We show that H is also $\mathbb{Q}$-generic over $\mathrm{V}[\bar{G}_0]$. If $\mathcal{A} \in \mathrm{V}[\bar{G}_0]$ is a maximal antichain in $\mathbb{Q}$, then $\mathcal{A} \subseteq \mathbb{Q}_n$ for some $n < \omega$, because the $\mathbb{Q}$ satisfies the κ^+-chain condition in $\mathrm{V}[\bar{G}_0]$. By the above remarks, $H \cap \mathbb{Q}_n = i_n``(H_0 \times \cdots \times H_{n-1})$ is $\mathbb{Q}_n$-generic over $\mathrm{V}[\bar{G}_0]$. Therefore, we get $\mathcal{A} \cap H \neq \emptyset$. Since $\mathbb{Q}$ satisfies the κ^+-chain condition, it is easy to see that $\mathcal{P}(\kappa)^{\mathrm{V}[G]} = \mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}_0][H]}$ holds.

The weak homogeneity of $\mathrm{Add}(\kappa, \kappa^+)$ in $\mathrm{V}[\bar{G}_0]$ yields the following equivalences.

$$
\begin{aligned}
& \langle \mathrm{L}(\bar{P}), \in \rangle \models \psi(\vec{x}, A, \bar{P}) \\
\Longleftrightarrow\ & \langle \mathrm{V}[\bar{G}_0][\bar{G}_1], \in \rangle \models (\exists p) \ \left[p = \mathcal{P}(\kappa) \wedge \psi(\vec{x}, A, p)^{\mathrm{L}(p)} \right] \\
\Longleftrightarrow\ & \langle \mathrm{V}[\bar{G}_0], \in \rangle \models \left[\mathbb{1}_{\mathrm{Add}(\kappa,\kappa^+)} \Vdash (\exists p) \ \left[p = \mathcal{P}(\check{\kappa}) \wedge \psi(\check{\vec{x}}, \check{A}, p)^{\mathrm{L}(p)} \right] \right] \\
\Longleftrightarrow\ & \langle \mathrm{V}[\bar{G}_0][H], \in \rangle \models (\exists p) \ \left[p = \mathcal{P}(\kappa) \wedge \psi(\vec{x}, A, p)^{\mathrm{L}(p)} \right] \\
\Longleftrightarrow\ & \langle \mathrm{L}(P), \in \rangle \models \psi(\vec{x}, A, P).
\end{aligned}
$$

□

This result has two useful corollaries in our context.

Corollary 6.5.2. *Let $\nu > \kappa$ be a cardinal and G be* $\mathrm{Add}(\kappa, \nu)$*-generic over* V. *Then the* axiom of choice *fails in* $\langle \mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[G]}), \in \rangle$. *In particular, the graph of a well-order of ${}^\kappa\kappa$ is not a $\mathbf{\Sigma}^1_n$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$ in* $\mathrm{V}[G]$.

Proof. This follows directly from Lemma 6.5.1 and [**Kan03**, Proposition 5.1(b)]. □

Corollary 6.5.3. *Let λ and ν be cardinals with $\nu > \kappa$. If G is* $\mathrm{Add}(\kappa, \nu)$*-generic over* V, *then generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under* $\mathrm{Add}(\kappa, \lambda)$ *holds in* $\mathrm{V}[G]$.

Proof. Let $T \in \mathrm{V}[G]$ be a tree on κ^{m+n+1} and assume

$$
\begin{aligned}
\mathbb{1}_{\mathrm{Add}(\kappa,\lambda)} \Vdash \text{``}(\exists x_0, \ldots, x_n \in {}^{\check{\kappa}}\check{\kappa})(\forall y_1, \ldots, y_m \in {}^{\check{\kappa}}\check{\kappa}) \\
\langle x_0, \ldots, x_n, y_0, \ldots, y_m \rangle \notin [\check{T}]\text{''}
\end{aligned}
$$

holds in $\mathrm{V}[G]$. We may assume that $T \in \mathrm{V}[\bar{G}]$ with $\bar{G} = G \cap \mathrm{Add}(\kappa, \kappa^+)$.

Let $\gamma = \max\{\lambda^+, \nu^+\}$ and F be $\mathrm{Add}(\kappa, \gamma)$-generic over V with $G = F \cap \mathrm{Add}(\kappa, \nu)$. There are $H_0, H_1 \in \mathrm{V}[F]$ such that H_0 is $\mathrm{Add}(\kappa, \lambda)$-generic over $\mathrm{V}[G]$, H_1 is $\mathrm{Add}(\kappa, \gamma)$-generic over $\mathrm{V}[G][H_0]$ and $\mathrm{V}[F] = \mathrm{V}[G][H_0][H_1]$. By the above assumption, the statement

$$
(\exists x_0, \ldots, x_n \in {}^\kappa\kappa)(\forall y_1, \ldots, y_m \in {}^\kappa\kappa)\langle x_0, \ldots, x_n, y_0, \ldots, y_m \rangle \notin [T]
$$

holds in $\mathrm{V}[G][H_0]$. An application of Proposition 6.3.3 shows that this statement also holds in $\mathrm{V}[F] = \mathrm{V}[G][H_0][H_1]$ and hence in $\mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[F]})$. By Lemma 6.5.1, it holds in $\mathrm{L}(\mathcal{P}(\kappa)^{\mathrm{V}[\bar{G}]})$ and in $\mathrm{V}[\bar{G}]$. Since $\mathrm{V}[G]$ is either equal to $\mathrm{V}[G]$ or an $\mathrm{Add}(\kappa, \nu)$-generic extension of $\mathrm{V}[\bar{G}]$, we can use Proposition 6.3.3 again to conclude that the statement holds in $\mathrm{V}[G]$. □

Proposition 6.5.4. *Let $\nu > \kappa$ be a cardinal. If G is* $\mathrm{Add}(\kappa, \nu)$*-generic over* V *and A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ of cardinality bigger than $(2^\kappa)^{\mathrm{V}}$ in* $\mathrm{V}[G]$, *then A has a perfect subset in* $\mathrm{V}[G]$.

PROOF. Fix a tree T on $\kappa \times \kappa$ with $A = p[T]^{\mathrm{V}[G]}$. There are $G_0, G_1 \in \mathrm{V}[G]$ such that G_0 is $\mathrm{Add}(\kappa, \kappa^+)$-generic over V, $T \in \mathrm{V}[G_0]$, G_1 is $\mathrm{Add}(\kappa, \nu)$-generic over $\mathrm{V}[G_0]$ and $\mathrm{V}[G] = \mathrm{V}[G_0][G_1]$. Since $(2^\kappa)^{\mathrm{V}[G_0]} = (2^\kappa)^{\mathrm{V}}$, we get $p[T]^{\mathrm{V}[G_0]} \subsetneq p[T]^{\mathrm{V}[G]}$ and there is a $\exists^x$-perfect embedding into T in $\mathrm{V}[G_0]$ by Lemma 6.3.6. Proposition 6.3.5 implies that $p[T]$ has a perfect subset in $\mathrm{V}[G]$. □

The combination of Lemma 6.4.5, Corollary 6.5.3 and Proposition 6.5.4 directly imply the statement of Theorem 6.1.4. The absoluteness properties of the coding forcing developed in Chapter 7 will show that it is consistent to have $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ of order-type $(2^\kappa)^{\mathrm{V}}$ in $\mathrm{Add}(\kappa, \nu)$-generic extensions of the ground model V. Therefore, the statement of Theorem 6.1.4 is optimal with respect to this assumption.

Another way to produce models of set theory with a nice structure theory for $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ is to use large cardinals and the *Levy-Collapse* and mimic classical constructions. We will repeatedly use the following folkloristic fact.

LEMMA 6.5.5. *Let ν be a cardinal with $\nu = \nu^{<\kappa}$ and $\mathbb{P}$ be a $<\kappa$-closed partial order.*

(1) *If $\mathbb{P}$ has cardinality at most ν, then $\mathrm{Col}(\kappa, \nu)$ and $\mathbb{P} \times \mathrm{Col}(\kappa, \nu)$ are forcing equivalent.*
(2) *If $\mathbb{P}$ has cardinality less than ν and $\lambda^{<\kappa} < \nu$ holds for all $\lambda < \nu$, then $\mathrm{Col}(\kappa, <\nu)$ and $\mathbb{P} \times \mathrm{Col}(\kappa, <\nu)$ are forcing equivalent.*

PROOF. See [**Fuc08**, Corollary 2.3] and [**Fuc08**, Corollary 2.4]. □

Generic absoluteness for $\mathbf{\Sigma}^1_3({}^\omega\omega)$-formulae is equiconsistent with the existence of a Σ_2-reflecting cardinal (see [**BF01**] and [**FMW92**]). The consistency of generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under forcing with $<\kappa$-closed partial orders follows from a direct generalization of the proof of this result.

LEMMA 6.5.6. *Let $\nu > \kappa$ be a Σ_2-reflecting cardinal and γ be a cardinal. If $G \times H$ is $(\mathrm{Col}(\kappa, <\nu) \times \mathrm{Add}(\kappa, \gamma))$-generic over* V*, then generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under $<\kappa$-closed forcings holds in* $\mathrm{V}[G][H]$.

PROOF. In $\mathrm{V}[G][H]$, let T be a tree on κ^{m+n+1} and $\mathbb{Q}$ be a $<\kappa$-closed partial order such that

$$p \Vdash \text{“}(\exists x_0, \dots, x_n \in {}^{\check\kappa}\check\kappa)(\forall y_1, \dots, y_m \in {}^{\check\kappa}\check\kappa)\langle x_0, \dots, x_n, y_0, \dots, y_m\rangle \notin [\check T]\text{”}$$

holds for some $p \in \mathbb{Q}$.

We can find $F \in \mathrm{V}[G][H]$, $\bar\nu < \nu$ and $i < 2$ such that F is $(\mathrm{Col}(\kappa, \bar\nu) \times \mathrm{Add}(\kappa, i))$-generic over V, $T \in \mathrm{V}[F]$ and $\mathrm{V}[G][H]$ is a $(\mathrm{Col}(\kappa, \nu) \times \mathrm{Add}(\kappa, \bar\gamma))$-generic extension of $\mathrm{V}[F]$ for some $\bar\gamma \in \{0, \gamma\}$. Then ν is a Σ_2-reflecting cardinal in $\mathrm{V}[F]$. Let $\dot{\mathbb{Q}} \in \mathrm{V}[F]$ be a $(\mathrm{Col}(\kappa, \nu) \times \mathrm{Add}(\kappa, \bar\gamma))$-name for $\mathbb{Q}$ such that $\mathbb{1}_{\mathrm{Col}(\kappa,\nu)\times\mathrm{Add}(\kappa,\bar\gamma)} \Vdash$ “$\dot{\mathbb{Q}}$ *is a $<\kappa$-closed partial order*”.

By Lemma 6.5.5, there is a cardinal $\lambda > \bar{\nu}$ such that the partial order

$$((\mathrm{Col}(\kappa, \nu) \times \mathrm{Add}(\kappa, \bar{\gamma})) * \dot{\mathbb{Q}}) \times \mathrm{Col}(\kappa, {<}\lambda)$$

is forcing equivalent to $\mathrm{Col}(\kappa, {<}\lambda)$ in $\mathrm{V}[F]$. Proposition 6.3.3 and the weak homogeneity of $\mathrm{Col}(\kappa, {<}\lambda)$ imply that

$$\begin{aligned} &(\exists \lambda > \bar{\nu})(\exists \mathbb{P}) \ \big[\lambda \textit{ is a cardinal, } \mathbb{P} = \mathrm{Col}(\kappa, {<}\lambda) \textit{ and} \\ &\mathbb{1}_{\mathbb{P}} \Vdash \text{“}(\exists x_0, \dots, x_n \in {}^{\check{\kappa}}\check{\kappa})(\forall y_1, \dots, y_m \in {}^{\check{\kappa}}\check{\kappa}) \\ &\qquad \langle x_0, \dots, x_n, y_0, \dots, y_m \rangle \notin [\check{T}]\text{”}\big] \end{aligned} \tag{6.5}$$

holds in $\mathrm{V}[F]$. We can apply Σ_2-elementarity to see that (6.5) holds in $\mathrm{V}_\nu[F]$ and hence there is a cardinal $\lambda_* < \nu$ that witnesses that (6.5) holds in $\mathrm{V}[F]$. There is $F_* \in \mathrm{V}[G][H]$ such that F_* is $\mathrm{Col}(\kappa, {<}\lambda_*)$-generic over $\mathrm{V}[F]$ and $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[F][F_*]$ by a ${<}\kappa$-closed partial order. A final application of Proposition 6.3.3 shows that

$$(\exists x_0, \dots, x_n \in {}^\kappa\kappa)(\forall y_0, \dots, y_m \in {}^\kappa\kappa) \ \langle x_0, \dots, x_n y_0, \dots, y_m \rangle \notin [T]$$

holds in $\mathrm{V}[G][H]$. □

Another generalization of the proof of the absoluteness result mentioned above shows that the consistency strength of $\mathbf{\Sigma}^1_2$-absoluteness under ${<}\kappa$-closed partial orders is exactly a Σ_2-reflecting cardinal.

LEMMA 6.5.7. *Assume that generic absoluteness for $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-formulae under ${<}\kappa$-closed forcings holds. Then κ^+ is a Σ_2-reflecting cardinal in* L.

PROOF. Let $\nu = \kappa^+$. First, assume, toward a contradiction, that there is an $\alpha < \kappa^+$ such that $\nu = (\alpha^+)^{\mathrm{L}}$. By the results of Section 6.2, there is a tree T on $\kappa \times \kappa$ such that $p[T]$ is equal to the set of all $x \in {}^\kappa 2$ with

$$\begin{aligned} \langle \mathrm{H}_{\kappa^+}, \in \rangle \models \text{“}\langle \kappa, \in_x \rangle &\textit{ is a well-order of order-type } \beta \textit{ and there is a} \\ &\textit{surjection } f : \alpha \longrightarrow \beta \textit{ that is an element of } \mathrm{L}\text{”} \end{aligned}$$

in V and every generic extension of V by a ${<}\kappa$-closed forcing.

Let G be $\mathrm{Col}(\kappa, \nu)$-generic over V and $x \in ({}^\kappa 2)^{\mathrm{V}[G]}$ such that $\langle \kappa, \in_x \rangle$ is a well-order of order-type ν. Then $x \notin p[T]^{\mathrm{V}[G]}$ and there is an $x_0 \in ({}^\kappa\kappa)^{\mathrm{V}}$ with $x_0 \notin p[T]^{\mathrm{V}}$, a contradiction. Hence ν is inaccessible in L.

Let $\varphi(v_0, v_1, v_2)$ be a Δ_0-formula, $z \in \mathrm{L}_\nu$ and $\mu > \nu$ such that

$$\langle \mathrm{L}, \in \rangle \models (\forall y) \ \varphi(x_0, y, z)$$

holds for some $x_0 \in \mathrm{L}_\mu$. Assume, toward a contradiction, that

$$\langle \mathrm{L}_\nu, \in \rangle \models (\forall x)(\exists y) \ \neg\varphi(x, y, z).$$

This implies

$$\langle \mathrm{H}_\nu, \in \rangle \models (\forall x \in \mathrm{L})(\exists y \in \mathrm{L}) \ \neg\varphi(x, y, z)$$

and this is a Π_2-statement with parameters in H_ν. Let G be $\mathrm{Col}(\kappa, \mu)$-generic over V. By $\mathbf{\Sigma}^1_2$-absoluteness and the results of Section 6.2, we have

$$\langle \mathrm{H}_{(\kappa^+)^{\mathrm{V}[G]}}, \in \rangle \models (\exists y \in \mathrm{L}) \ \neg\varphi(x_0, y, z)$$

and we can conclude that $\neg\varphi(x_0, y_0, z)$ holds for some $y_0 \in \mathrm{L}$, a contradiction. □

Next, we present an argument due to Philipp Schlicht showing that it is consistent that the class of all $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ has the perfect subset property.

PROPOSITION 6.5.8. *Let $\nu > \kappa$ be an inaccessible cardinal and γ be a cardinal. If $G \times H$ is $(\mathrm{Col}(\kappa, {<}\nu) \times \mathrm{Add}(\kappa, \gamma))$-generic over* V *and A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ of cardinality greater than κ in $\mathrm{V}[G][H]$, then A contains a perfect subset in $\mathrm{V}[G][H]$.*

PROOF. Let $A = p[T]$. As above, there is a $\bar{\nu} < \nu$, $i < 2$ and $F \in \mathrm{V}[G][H]$ such that F is $\mathrm{Col}(\kappa, {<}\bar{\nu}) \times \mathrm{Add}(\kappa, i)$-generic over V, $T \in \mathrm{V}[F]$ and $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[F]$ by a ${<}\kappa$-closed forcing. The set $p[T]^{\mathrm{V}[F]}$ has cardinality κ in $\mathrm{V}[G][H]$ and this means $p[T]^{\mathrm{V}[F]} \subsetneq p[T]^{\mathrm{V}[G][H]}$. By Lemma 6.3.6, there is a $\exists^x$-perfect embedding into T in $\mathrm{V}[F]$ and the set $p[T]^{\mathrm{V}[G][H]}$ contains a perfect subset in $\mathrm{V}[G][H]$ by Proposition 6.3.5. □

As above, we can combine Lemma 6.4.5, Proposition 6.5.4, Lemma 6.5.6 and Proposition 6.5.8 to derive the statement of Theorem 6.1.6.

CHAPTER 7

$\mathbf{\Sigma}^1_1$-definability at uncountable regular cardinals

In this chapter, we present and prove the main result of [**Lücb**]: *if κ is an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$ and A is an arbitrary subset of ${}^\kappa\kappa$, then there is $<\kappa$-closed partial order $\mathbb{P}$ of cardinality 2^κ that satisfies the κ^+-chain condition and forces A to be a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ in every $\mathbb{P}$-generic extension of the ground model* V. The proof of this theorem builds upon modifications of the well-known forcing that adds a κ-Kurepa trees and the *almost disjoint coding forcing* developed by Robert Solovay.

One of the key features of this coding forcing is its absoluteness with respect to further forcings: if we first force with the above coding forcing and then with another σ-closed forcing that preserves the regularity of κ, then the coded subset will still be a $\mathbf{\Sigma}^1_1$-subset in the final forcing extension. Moreover, there is a class of $<\kappa$-closed forcings that also preserves the $\mathbf{\Delta}^1_1$-definition produced by our coding forcing. This absoluteness has many interesting implications, e.g. it allows us to show that generic absoluteness for $\mathbf{\Sigma}^1_3({}^\kappa\kappa)$-formulae always fails. The first section of this chapter contains the statements of our coding theorems and a detailed outline of their applications. The following sections contain the proofs of these results.

7.1. Introduction

The initial motivation behind the work of [**Lücb**] was to find generalizations of the following coding result due to Leo Harrington to uncountable regular cardinals κ.

THEOREM 7.1.1 ([**Har77**, Theorem 1.7]). *Assume $\omega_1 = \omega_1^{\mathrm{L}}$. For every subset A of ${}^\omega\omega$, there is a partial order $\mathbb{P}$ with the following properties.*

(1) *$\mathbb{P}$ satisfies the countable chain condition.*
(2) *If G is $\mathbb{P}$-generic over* V*, then A is a $\mathbf{\Pi}^1_2$-subset of ${}^\omega\omega$ in* $\mathrm{V}[G]$.

We give a brief overview of related existing results. If κ is an uncountable regular cardinal and "$\mathrm{V} = \mathrm{L}[x]$" holds in the ground model for some $x \subseteq \kappa$, then we can apply Solovay's *almost disjoint coding* forcing (see [**JS70**]) to make an arbitrary subset of ${}^\kappa\kappa$ $\mathbf{\Sigma}^1_1$-definable in a forcing extension of $\mathrm{L}[x]$ and in any further forcing extension in which κ remains a cardinal. This follows from the absoluteness properties of this coding and the fact that $({}^\kappa\kappa)^{\mathrm{L}[x]}$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ in all such extensions. Section 7.3 of this chapter contains a detailed outline of the properties of this forcing.

Now, assume that the GCH holds at κ. If A is an arbitrary subset of ${}^\kappa\kappa$, then results of Sy-David Friedman show that there is a $<\kappa$-closed partial order that satisfies the κ^+-chain condition and adds a $\mathbf{\Sigma}^1_1$-definition of A. Moreover, this coding is absolute with respect to all further forcing extensions that preserve the regularity of κ and κ^+. This coding technique is called *Canonical Function coding.* A detailed discussion of this technique can be found in [**Fri10**], [**AF**] and [**FHa**].

We will present a coding result which only requires the assumption that the set of bounded subsets of κ has cardinality κ in the ground model. In particular, the hypothesis "$2^\kappa = \kappa^+$" is not needed. Before we state this result, we need to introduce some vocabulary.

DEFINITION 7.1.2. Let κ be an infinite regular cardinal and Γ be a class of partial orders that contains the trivial partial order. We say that a subset A of ${}^\kappa\kappa$ is Γ-*persistently* $\mathbf{\Sigma}^1_1$ if there is a tree T on $\kappa \times \kappa$ such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{A} = p[\check{T}]$" holds for every $\mathbb{P}$ in Γ.

We are now ready to state our first main result. See [**Cum10**, Definition 5.14] for the definition of α-strategically closed partial orders. As usual, we write σ-*strategically closed* instead of $(\omega + 1)$-strategically closed.

THEOREM 7.1.3 ([**Lücb**, Theorem 1.5]). *Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. For every subset A of ${}^\kappa\kappa$, there is a partial order $\mathbb{P}$ that satisfies the following statements.*

(1) *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
(2) *If G is $\mathbb{P}$-generic over V, then A is Γ-persistently $\mathbf{\Sigma}^1_1$ in $\mathrm{V}[G]$, where Γ is the class of all σ-strategically closed partial orders in $\mathrm{V}[G]$ that preserve the regularity of κ.*

By combining the above absoluteness properties with uncountable versions of results from the proof of Theorem 7.1.1 in [**Har77**], we are able to prove our second main result.

THEOREM 7.1.4 ([**Lücb**, Theorem 1.6]). *Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. For every subset A of ${}^\kappa\kappa$, there is a partial order $\mathbb{P}$ that satisfies the following statements.*

(1) *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
(2) *If G is $\mathbb{P}$-generic over V, then A is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$.*

This coding will also have certain absoluteness properties.

DEFINITION 7.1.5. Let κ be an infinite regular cardinal and Γ be a class of partial orders that contains the trivial partial order. We say that a subset A of ${}^\kappa\kappa$ is Γ-*persistently* $\mathbf{\Delta}^1_1$ if there are trees T_0 and T_1 on $\kappa\times\kappa$ such that T_0 witnesses that A is Γ-persistently $\mathbf{\Sigma}^1_1$ and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$p[\check{T}_1] = {}^{\check{\kappa}}\check{\kappa} \setminus p[\check{T}_0]$" holds for all $\mathbb{P}$ in Γ.

The proof of Theorem 7.1.4 will show that there is a nontrivial class Γ of $<\kappa$-closed partial orders that satisfy the κ^+-chain condition such that the set coded in Theorem 7.1.4 is actually Γ-persistently $\mathbf{\Delta}^1_1$ in the generic extension. Both the forcing $\mathrm{Add}(\kappa,1)$ that adds a Cohen-subset of κ and the *almost disjoint coding* forcings at κ are contained in this class Γ and this allows us to analyze certain structural properties of A in $\mathrm{V}[G]$.

Note that Proposition 6.3.8 implies that, given a class Γ is of $<\kappa$-closed partial orders that contains both the trivial partial order and a partial order that adds a new subset of κ and a Γ-persistently $\mathbf{\Delta}^1_1$-subset A of ${}^\kappa\kappa$, A contains no perfect subset if and only if it is Γ-persistently $\mathbf{\Delta}^1_1$.

In the following, we present some applications of the above results and the methods used in their proofs.

The *Anticoding Theorem*, proven by Itay Neeman and Jindřich Zapletal (see [**NZ98**]), says that in the presence of large cardinals proper forcings do not code any set of ordinals from the ground model into $\mathrm{L}(\mathbb{R})$ of the forcing extension unless that set is already an element of $\mathrm{L}(\mathbb{R})$ of the ground model. Given an uncountable regular cardinal κ with $\kappa=\kappa^{<\kappa}$, an easy application of the above results shows that it is possible to code new sets of ordinals into $\mathrm{L}(\mathcal{P}(\kappa))$ by forcing with a κ-proper partial order (see [**HR01**, Definition 3.4] for a definition of this class of partial orders).

Corollary 7.1.6. *If κ is a regular uncountable cardinal with $\kappa=\kappa^{<\kappa}$ and X is an arbitrary set, then there is a partial order $\mathbb{P}$ with the following properties.*

(1) *$\mathbb{P}$ is $<\kappa$-closed and satisfies the κ^+-chain condition.*
(2) *If G is $\mathbb{P}$-generic over V, then $\mathbb{1}_{\mathbb{Q}} \Vdash$ "$\check{X}\in\mathrm{L}(\mathcal{P}(\check{\kappa}))$" holds in $\mathrm{V}[G]$ for every σ-strategically closed partial order $\mathbb{Q}$ in $\mathrm{V}[G]$ that preserve the regularity of κ.*

Next, we consider definable well-orders of ${}^\kappa\kappa$. In [**FHa**], Sy-David Friedman and Peter Holy construct a class-sized partial order preserving ZFC and large cardinals that forces GCH and adds a well-order of ${}^\kappa\kappa$ whose graph is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa\times{}^\kappa\kappa$ for every uncountable regular cardinal κ. In another direction, David Asperó and Sy-David Friedman showed in [**AF09**] that there is a class-sized partial order with the above preservation properties that forces GCH and adds a well-order ${}^\kappa\kappa$ that is definable in the structure $\langle \mathrm{H}_{\kappa^+},\in\rangle$ by a formula without parameters for every uncountable regular cardinal κ. A detailed discussion of the above results and the related problem of obtaining lightface well-orders of low quantifier complexity can be found in the first part of [**Fri10**].

We apply Theorem 7.1.3 to add a definable well-order of ${}^\kappa\kappa$ with a forcing that preserves both cofinalities and the value of 2^κ.

Theorem 7.1.7 ([**Lücb**, Theorem 1.9]). *If κ is a regular uncountable cardinal with $\kappa=\kappa^{<\kappa}$, then there is a partial order $\mathbb{P}$ with the following properties.*

(1) $\mathbb{P}$ *is* $<\kappa$*-closed, satisfies the* κ^+*-chain condition and has cardinality* 2^κ.
(2) *If* G *is* $\mathbb{P}$*-generic over* V*, then there is a well-ordering of* $({}^\kappa\kappa)^{\mathrm{V}[G]}$ *whose graph is a* $\mathbf{\Delta}^1_2$*-subset of* ${}^\kappa\kappa \times {}^\kappa\kappa$ *in* $\mathrm{V}[G]$.

Our next application deals with a quasi-ordering of trees that arises naturally in infinitary model theory (see [**HV90**] and [**Vää95**]). Remember that a structure $\langle \mathbb{T}, \lhd_\mathbb{T}\rangle$ is a *tree* if $\lhd_\mathbb{T}$ is a well-founded strict ordering on $\mathbb{T}$ and the set $\mathrm{prec}_\mathbb{T}(t) = \{u \in \mathbb{T} \mid t \leq_\mathbb{T} u\}$ is well-ordered by $\lhd_\mathbb{T}$ for each $t \in \mathbb{T}$. As usual, we will just write $\mathbb{T}$ instead of $\langle \mathbb{T}, \lhd_\mathbb{T}\rangle$. As above, a branch through a tree $\mathbb{T}$ is a linearly ordered subset of $\mathbb{T}$. Given an infinite cardinal κ, we let $\mathcal{T}_\kappa$ denote the class of all trees $\mathbb{T}$ of cardinality at most κ such that every branch through $\mathbb{T}$ has length less than κ.

Let $\mathbb{T}_0$ and $\mathbb{T}_1$ be elements of $\mathcal{T}_\kappa$. We say that $\mathbb{T}_0$ *is order-preserving embeddable into* $\mathbb{T}_1$(abbreviated by $\mathbb{T}_0 \leq \mathbb{T}_1$) if there is a function $f : \mathbb{T}_0 \longrightarrow \mathbb{T}_1$ such that

$$t_0 \lhd_{\mathbb{T}_0} t_1 \longrightarrow f(t_0) \lhd_{\mathbb{T}_1} f(t_1)$$

holds for all $t_0, t_1 \in \mathbb{T}_0$. Note that f need not be injective.

There is a natural correspondence between elements of $\mathcal{T}_\omega$ and countable ordinals and the above ordering of trees is equal to the ordering of the ordinals under this correspondence. We may therefore think of elements of $\mathcal{T}_\kappa$ as analogs of ordinals. We can combine Theorem 7.1.3 with the *Boundedness Lemma for* ${}^\kappa\kappa$ to get an easy and short proof of the following statement that was proved in [**MV93**, Proof of Theorem 15] in the case "$\kappa = \omega_1$".

Theorem 7.1.8 ([**Lücb**, Theorem 1.10]). *If* κ *is a regular uncountable cardinal with* $\kappa = \kappa^{<\kappa}$*, then there is a partial order* $\mathbb{P}$ *with the following properties.*

(1) $\mathbb{P}$ *is* $<\kappa$*-closed, satisfies the* κ^+*-chain condition and has cardinality at most* 2^κ.
(2) *If* G *is* $\mathbb{P}$*-generic over* V*, then there is a* $\mathbb{T}_G \in \mathcal{T}_\kappa^{\mathrm{V}[G]}$ *such that* $\mathbb{T} \leq \mathbb{T}_G$ *holds for every tree* $\mathbb{T} \in \mathcal{T}_\kappa^{\mathrm{V}}$.

Next, we consider stronger notions of projective absoluteness. Given an uncountable regular cardinal κ with $\kappa = \kappa^{<\kappa}$, the constructions carried out in the proof of Theorem 7.1.7 show that we can define a $\mathbf{\Sigma}^1_3$-subset of ${}^\kappa\kappa$ that is empty in every $\mathrm{Add}(\kappa, \kappa^+)$-generic extension of the ground model and nonempty in a generic extension of the ground model by a certain $<\kappa$-closed forcing that satisfies the κ^+-chain condition. This shows that *generic absoluteness for* $\mathbf{\Sigma}^1_3({}^\kappa\kappa)$*-formulae under forcings with the above properties is inconsistent with the axioms of* ZFC for such cardinals κ.

Theorem 7.1.9 ([**Lücb**, Theorem 1.11]). *Let* κ *be an uncountable regular cardinal with* $\kappa = \kappa^{<\kappa}$ *and* $a \subseteq \kappa$ *such that* ${}^{<\kappa}\kappa \in \mathrm{L}[a]$*. Then there is a tree* T *on* κ^3 *contained in* $\mathrm{L}[a]$ *and a partial order* $\mathbb{P}$ *such that the following statements hold.*

(1) $\mathbb{P}$ *is* $<\kappa$*-closed and satisfies the* κ^+*-chain condition.*
(2) $\mathbb{1}_{\mathbb{P}} \Vdash$ "$(\exists x \in {}^{\check{\kappa}}\check{\kappa})(\forall y \in {}^{\check{\kappa}}\check{\kappa})\ \langle x, y\rangle \in p[\check{T}]$ ".
(3) $\mathbb{1}_{\mathrm{Add}(\kappa,\kappa^+)} \Vdash$ "$(\forall x \in {}^{\check{\kappa}}\check{\kappa})(\exists y \in {}^{\check{\kappa}}\check{\kappa})\ \langle x, y\rangle \notin p[\check{T}]$ ".

Again, the above result was known in the case where the GCH holds at κ. The *Canonical Function coding* mentioned above can be applied to construct such trees and extensions under this assumption.

7.2. Generic tree coding

This section contains the proof of Theorem 7.1.3. Our coding forcing will be a modification of the standard forcing that adds a Kurepa tree (see [**Jec71**, §3]). The main idea behind this modification is that it is possible to code information about the elements of a subset A of ${}^{\kappa}\kappa$ into the cofinal branches of the generic tree.

For the remainder of this chapter, we **fix an uncountable regular cardinal** κ that satisfies $\kappa = \kappa^{<\kappa}$ and **an enumeration** $\langle s_\alpha \mid \alpha < \kappa\rangle$ **of** ${}^{<\kappa}\kappa$ with $\mathrm{lh}(s_\alpha) \leq \alpha$ for all $\alpha < \kappa$ and $\{\alpha < \kappa \mid s = s_\alpha\}$ unbounded in κ for all $s \in {}^{<\kappa}\kappa$.

Definition 7.2.1. We call a pair $\langle A, s\rangle$ a κ*-coding basis* if the following statements hold.

(1) A is a nonempty subset of ${}^{\kappa}\kappa$ and $s : \kappa \longrightarrow {}^{<\kappa}\kappa$.
(2) $\mathrm{ran}(s)$ contains $\{x \restriction \alpha \mid x \in A,\ \alpha < \kappa\}$ and all constant functions in ${}^{<\kappa}\kappa$.
(3) For all $\alpha < \kappa$, $\mathrm{lh}(s(\alpha)) \leq \alpha$ and $\{\beta < \kappa \mid s(\alpha) = s(\beta)\}$ is unbounded in κ.

If $s : \kappa \longrightarrow {}^{<\kappa}\kappa$ is the function defined by $s(\beta) = s_\beta$ for all $\beta < \kappa$ and A is an arbitrary nonempty subset of A, then $\langle A, s\rangle$ is a κ-coding basis. In view of applications of this coding forcing in Chapter 8, we work without the assumption that s is surjective. In addition to the above objects, we also **fix a** κ**-coding basis** $\langle A, s\rangle$ for the remainder of this section.

Definition 7.2.2. We define $\mathbb{P}_s(A)$ to be the partial order consisting of conditions $p = \langle T_p, f_p, h_p\rangle$ with the following properties.

(1) T_p is a subtree of ${}^{<\kappa}2$ that satisfies the following statements.
  (a) T_p has cardinality less than κ.
  (b) If $t \in T_p$ with $\mathrm{lh}(t) + 1 < \mathrm{ht}(T_p)$, then t has two immediate successors in T_p.
(2) $f_p : A \xrightarrow{part} [T_p]$ is a partial function such that $\mathrm{dom}(f_p)$ is a nonempty set of cardinality less than κ.
(3) $h_p : A \xrightarrow{part} \kappa$ is a partial function with the following properties.
  (a) $\mathrm{dom}(h_p) = \mathrm{dom}(f_p)$.
  (b) For all $x \in \mathrm{dom}(h_p)$ and $\alpha, \beta < \mathrm{ht}(T_p)$ with $\alpha = \prec h_p(x), \beta\succ$, we have

$$s(\beta) \subseteq x \iff f_p(x)(\alpha) = 1.$$

We define $p \leq_{\mathbb{P}_s(A)} q$ to hold if the following statements are satisfied.

(a) T_p is an end-extension of T_q.
(b) For all $x \in \mathrm{dom}(f_q)$, $x \in \mathrm{dom}(f_p)$ and $f_q(x)$ is an initial segment of $f_p(x)$.
(c) $h_q = h_p \restriction \mathrm{dom}(h_q)$.

LEMMA 7.2.3. *$\mathbb{P}_s(A)$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*

PROOF. If $\lambda \in \mathrm{Lim} \cap \kappa$ and $\langle p_\mu \mid \mu < \lambda \rangle$ is a strictly $\leq_{\mathbb{P}_s(A)}$-descending sequence in $\mathbb{P}_s(A)$, then we define $T = \bigcup_{\mu<\lambda} T_{p_\mu}$, $h = \bigcup_{\mu<\lambda} h_\mu$ and

$$f(x) = \bigcup\{f_{p_\mu}(x) \mid \mu < \lambda, x \in \mathrm{dom}(f_{p_\mu})\}$$

for all $x \in \mathrm{dom}(h)$. It is easy to see that $p = \langle T, f, h \rangle \in \mathbb{P}_s(A)$ and $p \leq_{\mathbb{P}_s(A)} p_\mu$ holds for all $\mu < \lambda$.

Next, assume that $\langle p_\mu \mid \mu < \kappa^+ \rangle$ enumerates an antichain in $\mathbb{P}_s(A)$. By our assumptions, we can assume $T_{p_\mu} = T_{p_\rho}$ for all $\mu, \rho < \kappa^+$. A Δ-system argument allows us to assume the existence of an $r \subseteq A$ with $r = \mathrm{dom}(f_{p_\mu}) \cap \mathrm{dom}(f_{p_\rho})$, $f_{p_\mu} \restriction r = f_{p_\rho} \restriction r$ and $h_{p_\mu} \restriction r = h_{p_\rho} \restriction r$ for all $\mu < \rho < \kappa^+$. But this shows that $\langle T_{p_0}, f_{p_0} \cup f_{p_1}, h_{p_0} \cup h_{p_1} \rangle$ is a common extension of p_0 and p_1, a contradiction.

Finally, the assumption $\kappa = \kappa^{<\kappa}$ implies that there are only κ-many subtrees of ${}^{<\kappa}2$ of height less than κ and 2^κ-many partial functions with the above properties. □

The next lemma will allow us to show that various subsets of $\mathbb{P}_s(A)$ are dense.

LEMMA 7.2.4. *Let p be a condition in $\mathbb{P}_s(A)$ and $\langle c_x \in {}^\kappa 2 \mid x \in \mathrm{dom}(f_p) \rangle$ be a sequence of functions. There exists a $\leq_{\mathbb{P}_s(A)}$-descending sequence*

$$\langle p_\mu \in \mathbb{P}_s(A) \mid \mathrm{ht}(T_p) \leq \mu < \kappa \rangle$$

such that $p = p_{\mathrm{ht}(T_p)}$ and the following statements hold for all ordinals in the interval $[\mathrm{ht}(T_p), \kappa)$.

(1) $\mathrm{dom}(f_{p_\mu}) = \mathrm{dom}(f_p)$ *and* $\mathrm{ht}(T_{p_\mu}) = \mu$.
(2) *If $x \in \mathrm{dom}(f_p)$ and $\mu \neq \prec h_p(x), \beta \succ$ for all $\beta < \kappa$, then*

$$f_{p_{\mu+1}}(x)(\mu) = c_x(\mu).$$

(3) *If $\mu \in \mathrm{Lim}$, then* $\mathrm{ran}(f_{p_\mu}) = T_{p_{\mu+1}} \cap {}^\mu 2$.

PROOF. We construct the sequences inductively. If $\mu \in \mathrm{Lim}$, then we define $T_{p_\mu} = \bigcup\{T_{p_{\bar\mu}} \mid \mathrm{ht}(T_p) \leq \bar\mu < \mu\}$. Given $x \in \mathrm{dom}(f_p)$, we define

$$f_{p_\mu}(x) = \bigcup\{f_{p_{\bar\mu}}(x) \mid \mathrm{ht}(T_p) \leq \bar\mu < \mu\}.$$

If $\mu = \bar\mu + 1$ with $\bar\mu \notin \mathrm{Lim}$, then $T_{p_{\bar\mu}}$ has a maximal level and there is only one suitable tree T_{p_μ} of height μ end-extending it. In particular,

$f_{p_{\bar{\mu}}}(x) \in T_{p_\mu}$ for all $x \in \mathrm{dom}(f_p)$. For all $x \in \mathrm{dom}(f_p)$, we define $f_{p_\mu}(x)$ to be the unique element t of ${}^{\mu}2$ with $f_{p_{\bar{\mu}}}(x) \subseteq t$ and

$$t(\bar{\mu}) = \begin{cases} 1, & \text{if } \bar{\mu} = \prec h_p(x), \beta \succ \text{ and } s(\beta) \subseteq x, \\ 0, & \text{if } \bar{\mu} = \prec h_p(x), \beta \succ \text{ and } s(\beta) \nsubseteq x, \\ c_x(\bar{\mu}), & \text{otherwise.} \end{cases}$$

Finally, if $\mu = \bar{\mu} + 1$ with $\bar{\mu} \in \mathrm{Lim}$, then we set $T_{p_\mu} = T_{p_{\bar{\mu}}} \cup \mathrm{ran}(f_{p_{\bar{\mu}}})$ and define f_{p_μ} as in the first successor case. □

COROLLARY 7.2.5. *The following sets are dense subsets of* $\mathbb{P}_s(A)$.

(1) $C_\mu = \{p \in \mathbb{P}_s(A) \mid \mathrm{ht}(T_p) > \mu\}$ *for all* $\mu < \kappa$.
(2) $D_x = \{p \in \mathbb{P}_s(A) \mid x \in \mathrm{dom}(f_p)\}$ *for all* $x \in A$.
(3) $E_{x,y} = \{p \in \mathbb{P}_s(A) \mid x, y \in \mathrm{dom}(f_p), f_p(x) \neq f_p(y)\}$ *for all* $x, y \in A$.
(4) $F_z = \{p \in \mathbb{P}_s(A) \mid \mathrm{ht}(T_p) = \mu + 1,\ z \restriction \mu \notin T_p\}$ *for all* $z \in {}^{\kappa}2$.

PROOF. (i) This statement follows directly from Lemma 7.2.4.

(ii) Given $p \in \mathbb{P}_s(A)$ with $x \notin \mathrm{dom}(f_p)$ and $b \in [T_p] \neq \emptyset$, we define

$$q = \langle T_p, f_p \cup \{\langle x, b\rangle\}, h_p \cup \{\langle x, \mathrm{ht}(T_p)\rangle\}\rangle.$$

Then $q \in D_x$ and $q \leq_{\mathbb{P}_s(A)} p$.

(iii) Given $p \in \mathbb{P}_s(A)$, we can apply the above result to find $q \leq_{\mathbb{P}_s(A)} p$ with $x, y \in \mathrm{dom}(f_q)$. There is $\mathrm{ht}(T_q) \leq \mu < \kappa$ with

$$\prec h_q(x), \beta_0 \succ \neq \mu \neq \prec h_q(y), \beta_1 \succ$$

for all $\beta_0, \beta_1 < \kappa$ and we can use Lemma 7.2.4 to find $q^* \leq_{\mathbb{P}_s(A)} q$ with $\mathrm{ht}(T_{q^*}) = \mu + 1$ and $f_{q^*}(x)(\mu) \neq f_{q^*}(y)(\mu)$.

(iv) Fix $p \in \mathbb{P}_s(A)$ and $\mathrm{ht}(T_p) \leq \mu < \kappa$ with $\mu \neq \prec h_p(x), \beta \succ$ for all $x \in \mathrm{dom}(f_p)$ and $\beta < \kappa$. Using Lemma 7.2.4, we can find $q \leq_{\mathbb{P}_s(A)} p$ with $\mathrm{ht}(T_q) = \mu + 1$, $\mathrm{dom}(f_q) = \mathrm{dom}(f_p)$ and $f_q(x)(\mu) = 1 - z(\mu)$ for all $x \in \mathrm{dom}(f_p)$.

In particular, $z \restriction (\mu + 1) \notin \mathrm{ran}(f_q)$. Another application of the above lemma gives us conditions $s \leq_{\mathbb{P}_s(A)} r \leq_{\mathbb{P}_s(A)} q$ with

$$\mathrm{ht}(T_s) = \mathrm{ht}(T_r) + 1 = \mathrm{ht}(T_q) + \omega + 1,$$

$\mathrm{dom}(f_s) = \mathrm{dom}(f_p)$ and $T_s \cap {}^{\mathrm{ht}(T_r)}2 = \mathrm{ran}(f_r)$. Since $z \restriction \mathrm{ht}(T_r) \neq f_r(x)$ for all $x \in \mathrm{dom}(f_p)$, we have $z \restriction \mathrm{ht}(T_r) \notin T_s$. □

COROLLARY 7.2.6. *Let G be $\mathbb{P}_s(A)$-generic over* V. *The following statements hold true in* V$[G]$.

(1) $T_G = \bigcup_{p \in G} T_p$ *is subtree of* ${}^{<\kappa}2$ *of height* κ *with* $[T_G] \cap \mathrm{V} = \emptyset$.
(2) *If we define* $F_G(x) = \bigcup\{f_p(x) \mid p \in G, x \in \mathrm{dom}(f_p)\}$ *for all* $x \in A$, *then* $F_G : A \longrightarrow [T_G]$ *is an injection.*
(3) *Let* $H_G = \bigcup_{p \in G} h_g$. *Then* $H_G : A \longrightarrow \kappa$ *and*

$$s(\beta) \subseteq x \iff F_G(x)(\prec H_G(x), \beta \succ) = 1 \tag{7.1}$$

for all $x \in A$ *and* $\beta < \kappa$. □

We now show how the branches of T_G correspond to elements of A in an absolute and bijective way.

LEMMA 7.2.7. *Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}_s(A)$-name such that*

$$\mathbb{1}_{\mathbb{P}_s(A)} \Vdash \text{"}\dot{\mathbb{Q}} \textit{ is a } \sigma\textit{-strategically closed partial order and forcing with } \dot{\mathbb{Q}} \textit{ preserves the regularity of } \check{\kappa}\text{"}. \tag{7.2}$$

If $G_0 * G_1$ is $(\mathbb{P}_s(A) * \dot{\mathbb{Q}})$-generic over V, then $F_{G_0} : A \longrightarrow [T_{G_0}]^{\mathrm{V}[G_0][G_1]}$ is surjective.

PROOF. Fix names $\dot{F}, \dot{T} \in \mathrm{V}^{\mathbb{P}_s(A)*\dot{\mathbb{Q}}}$ such that $\dot{F}^{H_0 * H_1} = F_{H_0}$ and $\dot{T}^{H_0*H_1} = T_{H_0}$ holds whenever $H_0 * H_1$ is $(\mathbb{P}_s(A)*\dot{\mathbb{Q}})$-generic over V. Assume, toward a contradiction, that there is a name $\tau \in \mathrm{V}^{\mathbb{P}_s(A)*\dot{\mathbb{Q}}}$ and a condition $\langle p, \dot{q}\rangle$ in $\mathbb{P}_s(A) * \dot{\mathbb{Q}}$ with

$$\langle p, \dot{q}\rangle \Vdash \text{"}\tau \in [\dot{T}] \wedge \tau \notin \check{\mathrm{V}} \wedge \tau \notin \operatorname{ran}(\dot{F})\text{"}.$$

For each $r \leq_{\mathbb{P}_s(A)*\dot{\mathbb{Q}}} \langle p, \dot{q}\rangle$, we define a partial function $t_r : \kappa \xrightarrow{part} 2$ in V by setting

$$t_r = \bigcup\{s \in {}^{<\kappa}2 \mid r \Vdash \text{"}\check{s} \subseteq \tau\text{"}\}.$$

We have $t_r \in {}^{<\kappa}2$ for all $r \leq_{\mathbb{P}_s(A)*\dot{\mathbb{Q}}} \langle p, \dot{q}\rangle$, because $r \Vdash \text{"}\tau \notin \check{V}\text{"}$. Moreover, since $\langle p, \dot{q}\rangle \Vdash \text{"}(\forall \alpha < \check{\kappa})\ \tau \restriction \alpha \in \check{\mathrm{V}}\text{"}$, the set

$$\{r \leq_{\mathbb{P}_s(A)*\dot{\mathbb{Q}}} \langle p, \dot{q}\rangle \mid \alpha \subseteq \operatorname{dom}(t_r)\}$$

is dense below $\langle p, \dot{q}\rangle$ for all $\alpha < \kappa$.

Let $\langle p', \dot{q}'\rangle \leq_{\mathbb{P}_s(A)*\dot{\mathbb{Q}}} \langle p, \dot{q}\rangle$ and $d = \operatorname{dom}(f_{p'})$. Since

$$\langle p, \dot{q}\rangle \Vdash \text{"} \textit{The cardinality of } \check{d} \textit{ is less than } \operatorname{cof}(\check{\kappa}) \textit{ and } (\forall x \in \check{d})\ \tau \neq \dot{F}(x)\text{"},$$

there is an $r \leq_{\mathbb{P}_s(A)*\dot{\mathbb{Q}}} \langle p', \dot{q}'\rangle$ and an $\alpha < \kappa$ such that

$$r \Vdash \text{"}(\forall x \in \check{d})(\exists \beta < \check{\alpha})\ \tau(\beta) \neq \dot{F}(x)(\beta)\text{"}.$$

Then there is a condition $r_* = \langle p'', \dot{q}''\rangle \leq_{\mathbb{P}_s(A)*\dot{\mathbb{Q}}} r$ such that $\alpha \subseteq \operatorname{dom}(t_{r_*})$ and $\operatorname{ht}(T_{p''}) \geq \alpha$. This implies that for all $x \in \operatorname{dom}(f_{p'})$ there is a $\beta \in \operatorname{dom}(t_{r_*})$ such that $f_{p''}(x)(\beta) \neq t_{r_*}(\beta)$.

Let $\dot{\sigma}$ be a $\mathbb{P}_s(A)$-name with

$$\mathbb{1}_{\mathbb{P}_s(A)} \Vdash \text{"}\dot{\sigma} \textit{ is a winning strategy for Player } \text{Even in } \mathcal{G}_{\omega+1}(\dot{\mathbb{Q}})\text{"}.$$

Given $\langle p_0, \dot{q}_1\rangle \leq_{\mathbb{P}_s(A)*\dot{\mathbb{Q}}} \langle p, \dot{q}\rangle$, the above remarks allow us to construct a strictly $\leq_{\mathbb{P}_s(A)*\dot{\mathbb{Q}}}$-descending sequence $\langle\langle p_n, \dot{q}_{2n+1}\rangle \mid n < \omega\rangle$ of conditions in $\mathbb{P}_s(A) * \dot{\mathbb{Q}}$ and a sequence $\langle \dot{q}_{2n} \in \mathrm{V}^{\mathbb{P}_s(A)} \mid n < \omega\rangle$ of names such that the following statements hold for all $n < \omega$.

(1) $\dot{q}_0 = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}}$, $\mathbb{1}_{\mathbb{P}_s(A)} \Vdash \text{"}\dot{q}_{2n} \in \dot{\mathbb{Q}}\text{"}$, $p_n \Vdash \text{"}\dot{q}_{2n+1} \leq_{\dot{\mathbb{Q}}} \dot{q}_{2n}\text{"}$ and

$$p_n \Vdash \text{"}\dot{q}_{2n+2} = \dot{\sigma}(\dot{q}_0, \ldots, \dot{q}_{2n+1})\text{"}.$$

(2) $\operatorname{ht}(T_{p_n}) \subseteq \operatorname{dom}(t_{\langle p_{n+1}, \dot{q}_{2n+3}\rangle})$ and $\operatorname{dom}(t_{\langle p_n, \dot{q}_{2n+1}\rangle}) \subsetneq \operatorname{ht}(T_{p_{n+1}})$.

(3) If $x \in \mathrm{dom}(f_{p_n})$, then there is an $\alpha \in \mathrm{dom}(t_{\langle p_{n+1}, \dot{q}_{2n+3}\rangle})$ with

$$f_{p_{n+1}}(x)(\alpha) \neq t_{\langle p_{n+1}, \dot{q}_{2n+3}\rangle}(\alpha).$$

By the proof of Lemma 7.2.3, the sequence $\langle p_n \mid n < \omega\rangle$ has a greatest lower bound p_ω in $\mathbb{P}_s(A)$. Note that $T_{p_\omega} = \bigcup_{n<\omega} T_{p_n}$ and $\mathrm{dom}(f_{p_\omega}) = \bigcup_{n<\omega} \mathrm{dom}(f_{p_n})$ hold. If $\dot{R} \in \mathrm{V}^{\mathbb{P}_s(A)}$ is the canonical name for the sequence $\langle \dot{q}_n \mid n < \omega\rangle$, then

$$p_\omega \Vdash \text{“}\dot{R} \textit{ is a run of } \mathcal{G}_\omega(\dot{\mathbb{Q}}) \textit{ in which } \text{Even played according to } \dot{\sigma}\text{”}.$$

Hence we can find a name $\dot{q}_\omega \in \mathrm{V}^{\mathbb{P}_s(A)}$ with $\mathbb{1}_{\mathbb{P}_s(A)} \Vdash$ “$\dot{q}_\omega \in \dot{\mathbb{Q}}$” and $p_\omega \Vdash$ “$\dot{q}_\omega \leq_{\dot{\mathbb{Q}}} \dot{q}_n$” for all $n < \omega$. This implies $\langle p_\omega, \dot{q}_\omega\rangle \leq_{\mathbb{P}_s(A)*\dot{Q}} \langle p_n, \dot{q}_{2n+1}\rangle$ for all $n < \omega$. We define $t = t_{\langle p_\omega, \dot{q}_\omega\rangle} \restriction \mathrm{ht}(T_{p_\omega}) \in [T_{p_\omega}]$. Since we have $\langle p_\omega, \dot{q}_\omega\rangle \Vdash$ “$\check{t} \subseteq \tau \wedge \tau \in [\dot{T}]$”, we can conclude $\langle p_\omega, \dot{q}_\omega\rangle \Vdash$ “$\check{t} \in \dot{T}$”.

By our construction, we have $\mathrm{ht}(T_{p_\omega}) \in \mathrm{Lim}$ and $t \notin \mathrm{ran}(f_{p_\omega})$. We can apply Lemma 7.2.4 to find a condition $p^* \leq_{\mathbb{P}_s(A)} p_\omega$ with $\mathrm{ht}(T_{p^*}) = \mathrm{ht}(T_{p_\omega}) + 1$ and $t \notin T_{p^*}$. This obviously implies $\langle p^*, \dot{q}_\omega\rangle \Vdash$ “$\check{t} \notin \dot{T}$”, a contradiction. □

Corollary 7.2.8. *Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}_s(A)$-name such that* (7.2) *holds. If $G_0 * G_1$ is $(\mathbb{P}_s(A) * \dot{\mathbb{Q}})$-generic over* V*, then the following statements are equivalent for all $y \in ({}^\kappa\kappa)^{\mathrm{V}[G_0][G_1]}$.*

(1) *$y \in A$.*

(2) *There is $z \in [T_{G_0}]^{\mathrm{V}[G_0][G_1]}$ and $\gamma < \kappa$ such that*

$$s(\beta) \subseteq y \iff z(\prec\gamma, \beta\succ) = 1 \tag{7.3}$$

holds for all $\beta < \kappa$

Proof. If $y \in A$, then the equivalence (7.3) holds with $z = F_{G_0}(y)$ and $\gamma = H_{G_0}(y)$ by Corollary 7.2.6.

In the other direction, let $z \in [T_{G_0}]^{\mathrm{V}[G_0][G_1]}$ and $\gamma < \kappa$ witness that (7.3) holds for $y \in ({}^\kappa\kappa)^{\mathrm{V}[G_0][G_1]}$. By Lemma 7.2.7, we have $z = F_{G_0}(x) \in \mathrm{V}[G_0]$ for some $x \in A$. Pick $p \in G_0$ with $x \in \mathrm{dom}(f_p)$.

Assume, toward a contradiction, that $\gamma \neq h_p(x) = H_{G_0}(x)$. By Lemma 7.2.4, this implies that the set

$$D_s = \{q \leq_{\mathbb{P}_s(A)} p \mid \mathrm{ht}(T_q) = \mu + 1,\ \mu = \prec\gamma, \beta\succ,\ f_q(x)(\mu) = 0,\ s(\beta) = s\}$$

is dense below p for all $s \in \mathrm{ran}(s)$ and there is a $q \in G_0 \cap D_{y\restriction 1}$ with $q \leq_{\mathbb{P}_s(A)} p$. Then there is a $\beta < \kappa$ with $\mathrm{ht}(T_q) = \prec\gamma, \beta\succ + 1$, $z(\prec\gamma, \beta\succ) = 0$ and $s(\beta) = y \restriction 1 \subseteq y$, contradicting (7.3). This shows $\gamma = H_{G_0}(x)$ and we can conclude that

$$\begin{aligned} s(\beta) \subseteq y &\iff z(\prec\gamma, \beta\succ) = 1 \\ &\iff F_{G_0}(x)(\prec H_{G_0}(x), \beta\succ) = 1 \iff s(\beta) \subseteq x \end{aligned}$$

holds for all $\beta < \kappa$. Since every initial segment of x is of the form $s(\beta)$ for some $\beta < \kappa$, we can conclude $y = x \in A$. □

We are now ready to prove our first main result.

Proof of Theorem 7.1.3. Let G_0 be $\mathbb{P}_s(A)$-generic over V. In $\mathrm{V}[G_0]$, define T to be the set consisting of pairs $\langle t, u\rangle$ such that $t \in {}^{<\kappa}\kappa$, $u \in {}^{<\kappa}\kappa$ and there is an ordinal $\gamma < \kappa$ and a $v \in T_{G_0}$ with $\mathrm{lh}(z) = \mathrm{lh}(u) = \mathrm{lh}(v)$, $u(\alpha) = \prec\gamma, v(\alpha)\succ$ for all $\alpha < \mathrm{lh}(s)$ and

$$s(\beta) \subseteq t \iff v(\prec\gamma, \beta\succ) = 1$$

for all $\beta < \mathrm{lh}(s)$ with $\prec\gamma, \beta\succ < \mathrm{lh}(s)$. It is easy to check that T is a tree.

Let $\mathbb{Q}$ be a σ-strategically closed partial order in $\mathrm{V}[G_0]$ and G_1 be $\mathbb{Q}$-generic over $\mathrm{V}[G_0]$. There is a name $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}_s(A)}$ such that $\mathbb{Q} = \dot{\mathbb{Q}}^{G_0}$ and (7.2) holds in V.

If $\langle x, y\rangle \in [T]^{\mathrm{V}[G_0][G_1]}$, then there is $z \in [T_{G_0}]^{\mathrm{V}[G_0][G_1]}$ and $\gamma < \kappa$ with $y(\beta) = \prec\gamma, z(\beta)\succ$ and

$$s(\beta) \subseteq x \iff z(\prec\gamma, \beta\succ) = 1$$

for all $\beta < \kappa$. By Corollary 7.2.8, this implies $x \in A$.

Conversely, if $x \in A$ and $y \in ({}^{\kappa}\kappa)^{\mathrm{V}[G_0]}$ with

$$y(\alpha) = \prec H_{G_0}(x), F_{G_0}(x)(\alpha)\succ,$$

then $\langle x, y\rangle \in [T]^{\mathrm{V}[G_0][G_1]}$ by our assumptions on s and Corollary 7.2.8. □

We close this section by showing that Theorem 7.1.3 directly implies the statement of Corollary 7.1.6. Given functions $x, y \in {}^{\kappa}\kappa$, we let $\prec x, y\succ$ denote the unique function $z \in {}^{\kappa}\kappa$ with $x = (z)_0$, $y = (z)_1$ and $(z)_\alpha = \mathrm{id}_\kappa$ for all $1 < \alpha < \kappa$.

Proof of Corollary 7.1.6. Let ν be the cardinality of $\mathrm{tc}(\{X\})$ and let $\dot{e} \in \mathrm{V}^{\mathrm{Add}(\kappa,\nu^+)}$ be a name for an injection of $\mathrm{tc}(\{X\})$ into ${}^{\kappa}\kappa \setminus \{\mathrm{id}_\kappa\}$. Let $\dot{A}, \dot{s} \in \mathrm{V}^{\mathrm{Add}(\kappa,\nu^+)}$ be names with the property that, whenever G is $\mathrm{Add}(\kappa, \nu^+)$-generic over V, then $\langle \dot{A}^G, \dot{s}^G\rangle$ is a κ-coding basis in $\mathrm{V}[G]$ and

$$\dot{A}^G = \{\prec\dot{e}^G(b), \dot{e}^G(c)\succ \mid b, c \in \mathrm{tc}(\{X\}),\ b \in c\} \ \cup\ \{\prec\mathrm{id}_\kappa, \dot{e}^G(b)\succ \mid b \in X\}.$$

Pick a name $\dot{\mathbb{P}} \in \mathrm{V}^{\mathrm{Add}(\kappa,\nu^+)}$ with $\mathbb{1}_{\mathrm{Add}(\kappa,\nu^+)} \Vdash$ "$\dot{\mathbb{P}} = \mathbb{P}_{\dot{s}}(\dot{A})$". The partial order $\mathrm{Add}(\kappa, \nu^+) * \dot{\mathbb{P}}$ is $<\kappa$-closed and satisfies the κ^+-chain condition.

Let $G_0 * G_1$ be $(\mathrm{Add}(\kappa, \nu^+) * \dot{\mathbb{P}})$-generic over V, $\mathbb{Q}$ be a σ-strategically closed partial order in $\mathrm{V}[G_0][G_1]$ that preserve the regularity of κ and H be $\mathbb{Q}$-generic over $\mathrm{V}[G_0][G_1]$. By Theorem 7.1.3, $\dot{A}^{G_0}$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^{\kappa}\kappa$ in $\mathrm{V}[G_0][G_1][H]$. This shows that both $\mathrm{ran}(\dot{e}^{G_0})$ and the relation

$$E = \{\langle \dot{e}^{G_0}(b), \dot{e}^{G_1}(c)\rangle \mid b, c \in \mathrm{tc}(\{X\}),\ b \in c\}$$

are elements of $\mathrm{L}(\mathcal{P}(\kappa))$ in $\mathrm{V}[G_0][G_1][H]$. Since this model can compute the transitive collapse of the well-founded and extensional relation $\langle \mathrm{ran}(\dot{b}^{G_0}), E\rangle$ and this function is equal to the inverse of $\dot{e}^{G_0}$, we can conclude that $\mathrm{tc}(\{X\})$ is an element of $\mathrm{L}(\mathcal{P}(\kappa))$ in $\mathrm{V}[G_0][G_1][H]$. Finally, we have

$$X = \{b \in \mathrm{tc}(\{X\}) \mid \prec\mathrm{id}_\kappa, \dot{e}^{G_0}(b)\succ \in \dot{A}^{G_0}\}$$

and we can conclude that X is also an element of $\mathrm{L}(\mathcal{P}(\kappa))$ in $\mathrm{V}[G_0][G_1][H]$. $\square$

7.3. Almost disjoint coding

In [**Har77**, Section 1], Leo Harrington uses the method of *almost disjoint coding* forcing invented by Robert Solovay (see [**JS70**]) to prove Theorem 7.1.1. Working towards a proof of Theorem 7.1.4, we generalize this approach to uncountable cardinalities. Note that all results of this section are also true if κ is countable.

Definition 7.3.1. Given $A \subseteq {}^{\kappa}\kappa$, we define $\mathbb{Q}(A)$ to be the partial order consisting of conditions $p = \langle t_p, a_p \rangle$ with $t_p \in {}^{<\kappa}2$ and $a_p \in [A]^{<\kappa}$. The ordering $p \leq_{\mathbb{Q}(A)} q$ is defined by the following clauses.

(1) $t_q \subseteq t_p$ and $a_q \subseteq a_p$.
(2) $(\forall x \in a_q)(\forall \alpha \in \mathrm{dom}(t_p) \setminus \mathrm{dom}(t_q))\ [s_\alpha \subseteq x \longrightarrow t_p(\alpha) = 0]$.

It is easy to check that this is in fact a partial order. In addition, it is easy to see that two conditions p and q are compatible if and only if t_p and t_q are compatible as elements of ${}^{<\kappa}2$ and $\langle t_p \cup t_q, a_p \cup a_q \rangle \leq_{\mathbb{Q}(A)} p, q$.

Lemma 7.3.2. *$\mathbb{Q}(A)$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*

Proof. If $\mu < \kappa$, $\langle p_\alpha \mid \alpha < \mu \rangle$ is a $\leq_{\mathbb{Q}(A)}$-descending sequence, $t = \bigcup_{\alpha<\mu} t_{p_\alpha}$ and $a = \bigcup_{\alpha<\mu} a_{p_\alpha}$, then $\langle t, a \rangle \in \mathbb{Q}(A)$ and $\langle t, a \rangle \leq_{\mathbb{Q}(A)} p_\alpha$ for all $\alpha < \mu$. It is easy to see that any two conditions in $\mathbb{Q}(A)$ with the same first coordinate are compatible and this shows that any antichain in $\mathbb{Q}(A)$ has cardinality at most $\kappa^{<\kappa} = \kappa$. The cardinality statement follows directly from our assumptions on κ. $\square$

Proposition 7.3.3. *The following sets are dense subsets of $\mathbb{Q}(A)$.*

(1) $C_\mu = \{p \in \mathbb{Q}(A) \mid \mu \in \mathrm{dom}(t_p)\}$ *for all* $\mu < \kappa$.
(2) $D_x = \{p \in \mathbb{Q}(A) \mid x \in a_p\}$ *for all* $x \in A$.
(3) $E_{\alpha,y} = \{p \in \mathbb{Q}(A) \mid (\exists \beta \in \mathrm{dom}(t_p) \setminus \alpha)\ [t_p(\beta) = 1 \wedge s_\beta \subseteq y]\}$ *for all* $\alpha < \kappa$ *and* $y \in {}^{\kappa}\kappa \setminus A$.

Proof. (i) Given $\alpha < \kappa$ and $p \in \mathbb{Q}(A)$ with $\alpha \notin \mathrm{dom}(t_p)$, we define

$$t(\beta) = \begin{cases} t_p(\beta), & \text{if } \beta \in \mathrm{dom}(t_p), \\ 0, & \text{if } \beta \in (\alpha+1) \setminus \mathrm{dom}(s_p). \end{cases}$$

Obviously, $\langle t, a_p \rangle \leq_{\mathbb{Q}(A)} p$ and $\langle t, a_p \rangle \in C_\alpha$.

(ii) For all $p \in \mathbb{Q}(A)$, $p^* = \langle t_p, a_p \cup \{x\} \rangle \leq_{\mathbb{Q}(A)} p$ and $p^* \in D_x$.

(iii) Given $p \in \mathbb{Q}(A)$, there is an $\alpha < \beta \in \kappa \setminus \mathrm{dom}(s_p)$ with $x \restriction \beta \neq y \restriction \beta$ for all $x \in a_p$. We can find $\beta \leq \gamma < \kappa$ with $s_\gamma = y \restriction \beta$ and define $t : \gamma + 1 \longrightarrow 2$ by

$$t(\delta) = \begin{cases} t_p(\delta), & \text{if } \delta \in \mathrm{dom}(s_p), \\ 0, & \text{if } \delta \in \gamma \setminus \mathrm{dom}(t_p). \\ 1, & \text{if } \delta = \gamma. \end{cases}$$

Then $\langle t, a_p\rangle \leq_{\mathbb{Q}(A)} p$ and $\langle t, a_p\rangle \in E_{\alpha,y}$. $\square$

The following theorem summarizes the properties of $\mathbb{Q}(A)$.

THEOREM 7.3.4. *Let G be $\mathbb{Q}(A)$-generic over* V *and define*

$$t_G = \bigcup\{t_p \mid p \in G\}.$$

Then $t_G \in {}^\kappa 2$ and

$$x \in A \Longleftrightarrow (\exists\beta < \kappa)(\forall\beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \rightarrow t_G(\alpha) = 0] \tag{7.4}$$

for all $x \in ({}^\kappa\kappa)^{\mathrm{V}}$. Moreover,

$$\begin{aligned} G = \{p \in \mathbb{Q}(A) \mid t_p \subseteq t_G \wedge (\forall\alpha \in \kappa)[\alpha \notin \operatorname{dom}(t_p) \\ \vee (\forall x \in a_p)\ \ [s_\alpha \subseteq x \rightarrow t_G(\alpha) = 0]]\}. \end{aligned} \tag{7.5}$$

PROOF. By Proposition 7.3.3, t_G is a function with domain κ and for every $x \in A$ there is a $p \in G$ with $x \in a_p$.

Assume, toward a contradiction, that $t_G(\alpha) = 1$ and $s_\alpha \subseteq x$ holds for some $\alpha \in \kappa \setminus \operatorname{dom}(t_p)$. There is a $q \in G$ with $q \leq_{\mathbb{Q}(A)} p$ and $\alpha \in \operatorname{dom}(t_q)$. But this means $0 = t_q(\alpha) = t_G(\alpha)$, a contradiction. Given $<\in {}^\kappa\kappa \setminus A$ and $\beta < \kappa$, there is $p \in G \cap E_{\beta,y}$ and this shows that there is an $\beta < \alpha < \kappa$ with $t_G(\alpha) = 1$ and $s_\alpha \subseteq y$.

Given $p \in G$, the above argument shows that p is also an element of the right set. Next, assume $p \in \mathbb{Q}(A)$ is a member of the set on the right. There is a $q \in G$ with $a_p \subseteq a_q$ and $t_p \subseteq t_q$. If $\alpha \in \operatorname{dom}(t_q) \setminus \operatorname{dom}(t_p)$ and $x \in a_p$ with $s_\alpha \subseteq x$, then $t_q(\alpha) = t_G(\alpha) = 0$. This shows $q \leq_{\mathbb{Q}(A)} p$ and $p \in G$. $\square$

We close this section by introducing two forcing-theoretical properties and investigating their relevance to $\mathbb{Q}(A)$.

DEFINITION 7.3.5. We call a partial order $\mathbb{P}$ a *q-lattice ("quasi-lower-semi-lattice")* if the $\mathbb{P}$-minimum $p_0 \wedge_{\mathbb{P}} p_1$ exists for all compatible conditions $p_0, p_1 \in \mathbb{P}$. Let $\mathbb{Q}$ be a suborder of $\mathbb{P}$ and a q-lattice itself. We call $\mathbb{Q}$ a *sublattice of* $\mathbb{P}$ if $q_0 \wedge_{\mathbb{P}} q_1 = q_0 \wedge_{\mathbb{Q}} q_1$ holds for all $q_0, q_1 \in \mathbb{Q}$, which are compatible in $\mathbb{Q}$.

The partial order $\mathrm{Add}(\kappa, 1)$ is clearly a q-lattice and the remark following the definition of $\mathbb{Q}(A)$ directly implies that $\mathbb{Q}(A)$ is also a q-lattice with

$$p \wedge_{\mathbb{Q}(A)} q = \langle t_p \cup t_q, a_p \cup a_q\rangle$$

for all compatible $p, q \in \mathbb{Q}(A)$. Moreover, if $B \subseteq A$, then $\mathbb{Q}(B)$ is a sublattice of $\mathbb{Q}(A)$, every antichain in $\mathbb{Q}(B)$ is an antichain in $\mathbb{Q}(A)$ and every $\mathbb{Q}(B)$-nice name is a $\mathbb{Q}(A)$-nice name.

DEFINITION 7.3.6. Let $\mathbb{P}$ be a partial order and $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name. We call $\dot{\mathbb{Q}}$ a $\mathbb{P}$-*innocuous forcing* if there is a q-lattice $\mathbb{Q}_0$ with

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}\dot{\mathbb{Q}} \textit{ is a sublattice of } \check{\mathbb{Q}}_0\text{”}.$$

We give a simple example of $\mathbb{P}$-innocuous forcings that will be important later.

PROPOSITION 7.3.7. *If $\mathbb{P}$ is a $<\kappa$-closed forcing and $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ with*

$$\mathbb{1}_{\mathbb{P}} \Vdash (\exists B) \ \left[B \subseteq \check{A} \wedge \dot{\mathbb{Q}} = \mathbb{Q}(B)\right],$$

then $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-innocuous forcing.

PROOF. Set $\mathbb{Q}_0 = \mathbb{Q}(A)$. We show $\mathbb{1}_{\mathbb{P}} \Vdash$ "*$\dot{\mathbb{Q}}$ is a sublattice of $\check{\mathbb{Q}}_0$*". Let G be $\mathbb{P}$-generic over V. We have $\mathbb{Q}_0 = \mathbb{Q}(A)^{\mathrm{V}[G]}$, because $\mathbb{P}$ is $<\kappa$-closed. An application of the above remarks in V[G] shows that $\dot{\mathbb{Q}}^G$ is a sublattice of $\mathbb{Q}_0$ in V[G]. □

7.4. Innocent forcings

In this section, we complete the proof of Theorem 7.1.4. As mentioned in the Introduction, the $\mathbf{\Delta}^1_1$-coding we construct will have certain absoluteness properties. We are now ready to introduce the corresponding class of partial orders.

DEFINITION 7.4.1. Let M be an inner model, ν be a cardinal, $\mathbb{P}$ be a partial order contained in M and G be $\mathbb{P}$-generic over M. We define $\Gamma_M(\mathbb{P}, G, \nu)$ to be the class of all $<\nu$-closed partial orders $\mathbb{Q}$ that satisfy the ν^+-chain condition and have the property that there is a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ in M with $\mathbb{Q} = \dot{\mathbb{Q}}^G$ and

$$\langle M, \in\rangle \models \text{"}\dot{\mathbb{Q}} \textit{ is a } \mathbb{P}\textit{-innocuous forcing"}.$$

If $\mathbb{P}$ is a partial order and G is $\mathbb{P}$-generic over V, then results of Richard Laver (see [**Lav07**, Theorem 3]) show that V is a class in V[G]. In particular, $\Gamma_{\mathrm{V}}(\mathbb{P}, G, \nu)$ is a class in V[G] for every cardinal ν.

In the following, we continue to modify coding results from [**Har77**] to our context to prove the following *absoluteness version* of Theorem 7.1.4.

THEOREM 7.4.2 ([**Lücb**, Theorem 5.2]). *Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. For every subset A of ${}^\kappa\kappa$, there is a partial order $\mathbb{P}$ with the following properties.*

1. *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
2. *If G is $\mathbb{P}$-generic over V, then A is a $\Gamma_{\mathrm{V}}(\mathbb{P}, G, \kappa)$-persistently $\mathbf{\Delta}^1_1$ in V[G].*

Following [**Har77**], we start by introducing another notion of forcing.

DEFINITION 7.4.3. Given $A \subseteq {}^\kappa\kappa$, we define $\mathbb{Q}^+(A) = \bigoplus_{\gamma<\kappa^+} \mathbb{Q}(A)$ to be the κ^+-product forcing of $\mathbb{Q}(A)$ with $<\kappa$-support.

LEMMA 7.4.4. *$\mathbb{Q}^+(A)$ is a $<\kappa$-closed q-lattice that satisfies the κ^+-chain condition and has cardinality at most 2^κ.*

PROOF. Since $\mathbb{Q}^+(A)$ is the product with $<\kappa$-support and $\mathbb{Q}(A)$ is $<\kappa$-closed, it follows directly that $\mathbb{Q}^+(A)$ is also $<\kappa$-closed.

Given two compatible conditions $\vec{q}_0 = \left(q^0_\gamma\right)_{\gamma<\kappa^+}$ and $\vec{q}_1 = \left(q^1_\gamma\right)_{\gamma<\kappa^+}$, it is easy to check that q^0_γ and q^1_γ are compatible for all $\gamma < \kappa^+$ and $\left(q^0_\gamma \wedge_{\mathbb{Q}(A)} q^1_\gamma\right)_{\gamma<\kappa^+}$ is the $\mathbb{Q}^+(A)$-minimum of $\vec{q}_0$ and $\vec{q}_1$.

Assume, toward a contradiction, that $\langle \vec{q}_\delta \mid \delta < \kappa^+\rangle$ enumerates an antichain in $\mathbb{Q}^+(A)$ with $\vec{q}_\delta = \left(q^\delta_\gamma\right)_{\gamma<\kappa^+}$ for each $\delta < \kappa^+$. By the Δ-System Lemma, we may assume that there is an $r \subseteq \kappa^+$ of cardinality less than κ such that $\operatorname{supp}(\vec{q}_\delta) \cap \operatorname{supp}(\vec{q}_{\bar{\delta}}) = r$ holds for all $\delta < \bar{\delta} < \kappa^+$. The set $\{\langle t_{q^\delta_\gamma} \in {}^{<\kappa}\kappa \mid \gamma \in r\rangle \mid \delta < \kappa^+\}$ is a subset of ${}^r({}^{<\kappa}\kappa)$ and this set has cardinality κ by our assumptions. Hence there are $\delta < \bar{\delta} < \kappa^+$ with $t_{q^\delta_\gamma} = t_{q^{\bar{\delta}}_\gamma}$ for all $\gamma \in r$. But this shows that $\vec{q}_\delta$ and $\vec{q}_{\bar{\delta}}$ are compatible in $\mathbb{Q}^+(A)$, a contradiction.

By our assumptions, the set $S = \{\operatorname{supp}(\vec{q}) \mid \vec{q} \in \mathbb{Q}^+(A)\}$ has cardinality κ^+ and for each $s \in S$ there are at most 2^κ-many conditions $\vec{q} \in \mathbb{Q}^+(A)$ with $\operatorname{supp}(\vec{q}) = s$. □

Let $G = \bigoplus_{\gamma<\kappa^+} G_\alpha$ be $\mathbb{Q}^+(A)$-generic over V and $x \in ({}^\kappa\kappa)^{\mathrm{V}[G]}$. Since $\mathbb{Q}^+(A)$ satisfies the κ^+-chain condition in V, there is an $\delta < \kappa^+$ with $x \in \mathrm{V}[\langle G_\beta \mid \gamma < \delta\rangle]$. Now, Theorem 7.3.4 shows that

(7.6)

$$x \in A \iff (\forall \gamma < \kappa^+)(\exists \beta < \kappa)(\forall \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \longrightarrow t_{G_\gamma}(\alpha) = 0]$$

holds in $\mathrm{V}[G]$. This shows that a $\mathbf{\Sigma}^1_1$-definition of the set $\{t_{G_\gamma} \in {}^\kappa\kappa \mid \gamma < \kappa^+\}$ would yield a $\mathbf{\Pi}^1_1$-definition of A in $\mathrm{V}[G]$. In order to make the set of all t_{G_γ}'s $\mathbf{\Sigma}^1_1$-definable, we need to show that the equivalence (7.6) also holds in certain forcing extensions of $\mathrm{V}[G]$. We introduce a class of forcings with this property.

Definition 7.4.5. Let $\mathbb{P}$ be a q-lattice. We call $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ a *$\mathbb{P}$-innocent forcing* if $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is a partial order" and there is a dense subset $D \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ such that

$$p_0 \wedge_{\mathbb{P}} p_1 \Vdash \text{“}\dot{q}_0 \textit{ and } \dot{q}_1 \textit{ are compatible in } \dot{\mathbb{Q}}\text{”}$$

holds for all compatible $\langle p_0, \dot{q}_0\rangle, \langle p_1, \dot{q}_1\rangle \in D$.

Lemma 7.4.6. *Let $\dot{\mathbb{P}}$ be an $\mathbb{Q}^+(A)$-innocent forcing with*

$$\mathbb{1}_{\mathbb{Q}^+(A)} \Vdash \text{“}\dot{\mathbb{P}} \textit{ is } {<}\check{\kappa}\textit{-closed and satisfies the } \check{\kappa}^+\textit{-chain condition}\text{”}.$$

If $G_0 * G_1$ is $(\mathbb{Q}^+(A) * \dot{\mathbb{P}})$-generic over V *with $G_0 = \bigoplus_{\gamma<\kappa^+} \bar{G}_\gamma$, then*

$$x \in A \iff (\forall \gamma < \kappa^+)(\exists \beta < \kappa)(\forall \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \longrightarrow t_{\bar{G}_\gamma}(\alpha) = 0]$$

holds in $\mathrm{V}[G_0][G_1]$ *for all $x \in ({}^\kappa\kappa)^{\mathrm{V}[G_0][G_1]}$.*

Proof. Let D be a dense subset of $\mathbb{Q}^+(A) * \dot{\mathbb{P}}$ witnessing that $\dot{\mathbb{P}}$ is a $\mathbb{Q}^+(A)$-innocent forcing. Let $\dot{\eta}_0$ be a $\mathbb{Q}^+(A)$-name with with the property that, whenever G is $\mathbb{Q}^+(A)$-generic over V and $G = \bigoplus_{\gamma<\kappa^*} \bar{G}_\gamma$, then

$$\dot{\eta}_0^G : \kappa^+ \longrightarrow ({}^\kappa 2)^{\mathrm{V}[G]};\ \gamma \longmapsto t_{\bar{G}_\gamma}$$

Let $\dot{\eta}$ denote the canonical $(\mathbb{Q}^+(A) * \dot{\mathbb{P}})$-name corresponding to $\dot{\eta}_0$.

Assume, toward a contradiction, that there is a name $\dot{x} \in \mathrm{V}^{\mathbb{Q}^+(A)*\dot{\mathbb{P}}}$ and a condition $r_0 \in \mathbb{Q}^+(A) * \dot{\mathbb{P}}$ such that

$$r_0 \Vdash \text{“}\dot{x} \in ({}^{\check{\kappa}}\check{\kappa} \setminus \check{A}) \ \wedge \ (\forall \gamma < \check{\kappa}^+)(\exists \beta < \check{\kappa})(\forall \beta \leq \alpha < \check{\kappa}) \qquad [\check{s}_\alpha \subseteq \dot{x} \longrightarrow \dot{\eta}(\gamma)(\alpha) = 0]\text{”}$$

holds. Given $\alpha < \kappa$, we pick a maximal antichain $\mathcal{A}_\alpha$ in the set

$$\{r \in D \mid r \Vdash \text{“}\check{s}_{\check{\alpha}} \subseteq \dot{x}\text{”}\}$$

and define $\mathcal{A} = \bigcup\{\mathcal{A}_\alpha \mid \alpha < \kappa\}$. Our assumptions imply that $\mathbb{Q}^+(A) * \dot{\mathbb{P}}$ satisfies the κ^+-chain condition and therefore $\mathcal{A}$ has cardinality κ. This shows that there is a $\gamma_* < \kappa^+$ with the property that, whenever $\langle \vec{q}, \dot{p}\rangle \in \mathcal{A}$ and $\vec{q} = (q_\gamma)_{\gamma<\kappa^+}$, then $q_{\gamma_*} = \mathbb{1}_{\mathbb{Q}(A)}$.

We can find an $r_1 \in D$ and $\beta_* < \kappa$ with $r_1 \leq_{\mathbb{Q}^+(A)*\dot{\mathbb{P}}} r_0$ and

$$r_1 \Vdash \text{“}(\forall \check{\beta}_* \leq \alpha < \check{\kappa}) \ [\check{s}_\alpha \subseteq \dot{x} \longrightarrow \eta(\check{\gamma}_*)(\alpha) = 0]\text{”}.$$

Let $r_1 = \langle \vec{q}_1, \dot{p}_1\rangle$, $\vec{q}_1 = \left(q^1_\gamma\right)_{\gamma<\kappa^+}$ and $q^1_{\gamma_*} = \langle t_1, a_1\rangle$.

Now, let $G_0 * G_1$ be $(\mathbb{Q}^+(A) * \dot{\mathbb{P}})$-generic over V with $r_1 \in G_0 * G_1$ and set $x = \dot{x}^{G_0*G_1}$. Since this partial order is $<\kappa$-closed and every initial segment of x is an element of V, we can find an $\alpha_* < \kappa$ with $\beta_* < \alpha_*$, $\mathrm{dom}(t_1) < \alpha_*$, $s_{\alpha_*} \subseteq x$ and $s_{\alpha_*} \nsubseteq y$ for all $y \in a_1$.

Our construction ensures that there is an $r_2 \in \mathcal{A}_{\alpha_*} \cap G$. Let $r_2 = \langle \vec{q}_2, \dot{p}_2\rangle$ and $\vec{q}_2 = \left(q^2_\gamma\right)_{\gamma<\kappa^+}$. The conditions r_1 and r_2 are compatible and elements of D. Hence, we can find an $r = \langle \vec{q}, \dot{p}\rangle \leq_{\mathbb{Q}^+(A)*\dot{\mathbb{P}}} r_1, r_2$ with $\vec{q} = (q_\gamma)_{\gamma<\kappa}$ and $q_\gamma = q^1_\gamma \wedge_{\mathbb{Q}(A)} q^2_\gamma$ for all $\gamma < \kappa^+$. In particular, $q_{\gamma_*} = q^1_{\gamma_*} = \langle t_1, a_1\rangle$.

We define $t_* \in {}^{<\kappa}\kappa$ by setting

$$t_*(\delta) = \begin{cases} t_1(\delta), & \text{if } \delta \in \mathrm{dom}(t_1), \\ 0, & \text{if } \delta \in \alpha_* \setminus \mathrm{dom}(t_1), \\ 1, & \text{if } \delta = \alpha_*. \end{cases}$$

By the choice of α_*, we have $\langle t_*, a_1\rangle \leq_{\mathbb{Q}(A)} \langle t_1, a_1\rangle = q_{\gamma^*}$. If we define

$$q^*_\gamma = \begin{cases} q_\gamma, & \text{if } \gamma \neq \gamma_*, \\ \langle t_*, a_1\rangle, & \text{if } \gamma = \gamma^*, \end{cases}$$

then $r_* = \langle \left(q^*_\gamma\right)_{\gamma<\kappa^+}, \dot{p}\rangle \leq r$. Let H be $(\mathbb{Q}^+(A) * \dot{\mathbb{P}})$-generic over V with $H = H_0 * H_1$, $H_0 = \bigoplus_{\gamma<\kappa^+} \bar{H}_\gamma$ and $r_* \in H$. The above construction yields $r_1 \in H$, $s_{\alpha_*} \subseteq \dot{x}^H$ and $t_{\bar{H}_{\gamma_*}}(\alpha_*) = 1$, a contradiction. $\square$

Let $\mathbb{P}$ be a partial order, $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ “$\dot{\mathbb{Q}}$ *is a partial order* ” and G be $\mathbb{P}$-generic over V. Using $\mathbb{P}$, $\dot{\mathbb{Q}}$ and G as parameters, we can recursively define a class function

$$\mathrm{t}_G : \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}} \longrightarrow \mathrm{V}[G]^{\dot{\mathbb{Q}}^G}$$

in $\mathrm{V}[G]$ that satisfies

$$\mathrm{t}_G(\sigma) = \{\langle \mathrm{t}_G(\tau), \dot{q}^G\rangle \mid \langle \tau, \langle p, \dot{q}\rangle\rangle \in \sigma, p \in G\}$$

for all $\sigma \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$. If H is $\dot{\mathbb{Q}}^G$-generic over $\mathrm{V}[G]$, then an easy induction shows that $\sigma^{G*H} = \mathrm{t}_G(\sigma)^H$ holds for all $\sigma \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$. Given $\sigma \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$, we let $\mathrm{T}(\sigma)$ denote the class of all $\tau \in \mathrm{V}^{\mathbb{P}}$ such that $\tau^G = \mathrm{t}_G(\sigma)$ whenever G is $\mathbb{P}$-generic over V.

Next, suppose $\dot{\mathbb{R}} \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ with $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash$ "$\dot{\mathbb{R}}$ *is a partial order* " and $\dot{\mathbb{S}} \in \mathrm{V}^{\mathbb{P}}$. We write $\dot{\mathbb{S}} = \dot{\mathbb{Q}} *_{\mathbb{P}} \dot{\mathbb{R}}$ if there is a $\sigma \in \mathrm{T}(\dot{\mathbb{R}})$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{S}} = \dot{\mathbb{Q}} * \sigma$".

By the above remarks, there is a map

$$\iota : (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}} \longrightarrow \mathbb{P} * \dot{\mathbb{S}}$$

such that for every $\langle\langle p, \dot{q}\rangle, \dot{r}\rangle \in (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$ there is an $\dot{s} \in \mathrm{V}^{\mathbb{P}}$ and $\rho \in \mathrm{T}(\dot{r})$ with $\iota(\langle p, \dot{q}\rangle, \dot{r}) = \langle p, \dot{s}\rangle$ and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{s} = \langle \dot{q}, \rho\rangle$".

LEMMA 7.4.7. *The map ι is a dense embedding.* □

LEMMA 7.4.8. *Let $\mathbb{P}$ be a q-lattice, $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ be a $\mathbb{P}$-innocuous forcing and $\dot{\mathbb{R}} \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ be a $(\mathbb{P} * \dot{\mathbb{Q}})$-innocuous forcing. If $\dot{\mathbb{S}} \in \mathrm{V}^{\mathbb{P}}$ satisfies $\dot{\mathbb{S}} = \dot{\mathbb{Q}} *_{\mathbb{P}} \dot{\mathbb{R}}$, then $\dot{\mathbb{S}}$ is a $\mathbb{P}$-innocent forcing.*

PROOF. Let $\mathbb{Q}_0$ witness that $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-innocuous forcing and $\mathbb{R}_0$ witness that $\dot{\mathbb{R}}$ is a $(\mathbb{P} * \dot{\mathbb{Q}})$-innocuous forcing. We define D_0 to be the set

$$\{\langle\langle p, \dot{q}\rangle, \dot{r}\rangle \in (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}} \mid (\exists q \in \mathbb{Q}_0)(\exists r \in \mathbb{R}_0) \\ [p \Vdash \text{“}\dot{q} = \check{q}\text{”} \wedge \langle p, \dot{q}\rangle \Vdash \text{“}\dot{r} = \check{r}\text{”}]\}.$$

Pick $\langle\langle p_0, \dot{q}_0\rangle, \dot{r}\rangle \in (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$. There is a $\langle p_1, \dot{q}\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_0, \dot{q}_0\rangle$ and $r \in \mathbb{R}_0$ with $\langle p_1, \dot{q}\rangle \Vdash$ "$\dot{r} = \check{r}$". In addition, there is a $p \leq_{\mathbb{P}} p_1$ and a $q \in \mathbb{Q}_0$ with $p \Vdash$ "$\dot{q} = \check{q}$". This means $\langle\langle p, \dot{q}\rangle, \dot{r}\rangle \in D_0$ and $\langle\langle p, \dot{q}\rangle, \dot{r}\rangle \leq_{(\mathbb{P}*\dot{\mathbb{Q}})*\dot{\mathbb{R}}} \langle\langle p_0, \dot{q}_0\rangle, \dot{r}\rangle$.

Let $\langle\langle p_0, \dot{q}_0\rangle, \dot{r}_0\rangle, \langle\langle p_1, \dot{q}_1\rangle, \dot{r}_1\rangle, \langle\langle p_2, \dot{q}_2\rangle, \dot{r}_2\rangle \in (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$ with

$$\langle\langle p_0, \dot{q}_0\rangle, \dot{r}_0\rangle \leq_{(\mathbb{P}*\dot{\mathbb{Q}})*\dot{\mathbb{R}}} \langle\langle p_1, \dot{q}_1\rangle, \dot{r}_1\rangle, \langle\langle p_2, \dot{q}_2\rangle, \dot{r}_2\rangle$$

and $\langle\langle p_1, \dot{q}_1\rangle, \dot{r}_1\rangle, \langle\langle p_2, \dot{q}_2\rangle, \dot{r}_2\rangle \in D_0$. Fix conditions $q_1, q_2 \in \mathbb{Q}_0$ and $r_1, r_2 \in \mathbb{R}_0$ with $p_i \Vdash$ "$\dot{q}_i = \check{q}_i$" and $\langle p_i, \dot{q}_i\rangle \Vdash$ "$\dot{r}_i = \check{r}_i$". Clearly, $p_1 \wedge_{\mathbb{P}} p_2$ exists and there is a $p \in \mathbb{P}$ and $q \in \mathbb{Q}_0$ such that $p \leq_{\mathbb{P}} p_0$ and $p \Vdash$ "$\check{q} = \dot{q}_0 \leq_{\dot{\mathbb{Q}}} \check{q}_1, \check{q}_2$". But this shows that $q \leq_{\mathbb{Q}_0} q_1, q_2$ and $q_1 \wedge_{\mathbb{Q}_0} q_2$ exists. If we pick $\dot{q} \in \mathrm{V}^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{q} \in \dot{\mathbb{Q}}$" and $(p_1 \wedge_{\mathbb{P}} p_2) \Vdash$ "$\dot{q} = \check{q}_1 \wedge_{\check{\mathbb{Q}}_0} \check{q}_2$", then

$$\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{q}\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_1, \dot{q}_1\rangle, \langle p_2, \dot{q}_2\rangle.$$

In the same way, we can show that $r_1 \wedge_{\mathbb{R}_0} r_2$ exists and there is an $\dot{r} \in \mathrm{V}^{\mathbb{R}}$ with $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash$ "$\dot{r} \in \dot{\mathbb{R}}$" and $\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{q}\rangle \Vdash$ "$\dot{r} = \check{r}_1 \wedge_{\check{\mathbb{R}}_0} \check{r}_2$". This means

$$\langle\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{q}\rangle, \dot{r}\rangle \leq_{(\mathbb{P}*\dot{\mathbb{Q}})*\dot{\mathbb{R}}} \langle\langle p_1, \dot{q}_1\rangle, \dot{r}_1\rangle, \langle\langle p_2, \dot{q}_2\rangle, \dot{r}_2\rangle.$$

Define $D \subseteq \mathbb{P}*\dot{\mathbb{S}}$ to be the image of D_0 under ι. By the above Lemma, D is a dense subset of $D \subseteq \mathbb{P} * \dot{\mathbb{S}}$. Given two compatible conditions $d_0, d_1 \in D$

with $d_i = \iota(\langle\langle p_i, \dot{q}_i\rangle, \dot{r}_i\rangle) = \langle p_i, \dot{s}_i\rangle$, we have shown that there are $\dot{q} \in \mathrm{V}^{\mathbb{P}}$ and $\dot{r} \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ with

$$\langle\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{q}\rangle, \dot{r}\rangle \leq_{(\mathbb{P}*\dot{\mathbb{Q}})*\dot{\mathbb{R}}} \langle\langle p_1, \dot{q}_1\rangle, \dot{r}_1\rangle, \langle\langle p_2, \dot{q}_2\rangle, \dot{r}_2\rangle.$$

This gives us an $\dot{s} \in \mathrm{V}^{\mathbb{P}}$ with

$$\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{s}\rangle = \iota(\langle\langle p_1 \wedge_{\mathbb{P}} p_2, \dot{q}\rangle, \dot{r}\rangle) \leq_{\mathbb{P}*\dot{\mathbb{S}}} d_0, d_1.$$

□

The techniques developed above allow us to prove the *absoluteness version* of our second main result.

Proof of Theorem 7.4.2. For the remainder of the proof, we fix a subset B of ${}^{\kappa}\kappa$ of cardinality κ^+ such that

$$(x)_\alpha = (y)_\beta \iff [x = y \wedge \alpha = \beta]$$

holds for all $x, y \in B$ and $\alpha, \beta < \kappa$. In addition, we fix an injective enumeration $\langle b_\gamma \mid \gamma < \kappa^+\rangle$ of B.

By Theorem 7.1.3, there is a $<\kappa$-closed forcing $\mathbb{P}_0$ of cardinality at most 2^κ satisfying the κ^+-chain condition with the property that, whenever G_0 is $\mathbb{P}_0$-generic over V and $\mathbb{Q} \in \mathrm{V}[G_0]$ is a $<\kappa$-closed partial order, then both A and B are $\mathbf{\Sigma}^1_1$-subsets of $({}^{\kappa}\kappa)^{\mathrm{V}[G_0][G_1]}$ in every $\mathbb{Q}$-generic extension $\mathrm{V}[G_0][G_1]$ of $\mathrm{V}[G_0]$.

If G_0 is $\mathbb{P}_0$-generic over V, then $\mathbb{Q}(A)^{\mathrm{V}} = \mathbb{Q}(A)^{\mathrm{V}[G_0]}$ and $\mathbb{Q}^+(A)^{\mathrm{V}} = \mathbb{Q}^+(A)^{\mathrm{V}[G_0]}$. This shows that $\mathbb{P}_0 \times \mathbb{Q}^+(A)$ is a $<\kappa$-closed forcing that satisfies the κ^+-chain condition in V. In addition, there are names $\dot{C}, \dot{\mathbb{R}} \in \mathrm{V}^{\mathbb{P}_0 \times \mathbb{Q}^+(A)}$ with the property that, whenever $G_0 \times G_1$ is $(\mathbb{P}_0 \times \mathbb{Q}^+(A))$-generic over V with $G_1 = \bigoplus_{\gamma<\kappa^+} \bar{G}_\gamma$, then

$$\dot{C}^{G_0 \times G_1} = \{(b_\gamma)_{\prec\bar{\alpha},\alpha\succ} \in {}^{\kappa}\kappa \mid \alpha, \bar{\alpha} < \kappa, \gamma < \kappa^+, \bar{\alpha} = t_{\bar{G}_\gamma}(\alpha)\}$$

and $\dot{\mathbb{R}}^{G_0 \times G_1} = \mathbb{Q}(\dot{C}^{G_0 \times G_1})$ in $\mathrm{V}[G_0][G_1]$. Notice $\dot{C}^{G_0 \times G_1} \subseteq ({}^{\kappa}\kappa)^{\mathrm{V}}$ and $\dot{\mathbb{R}}^{G_0 \times G_1}$ is a sublattice of $\mathbb{Q}({}^{\kappa}\kappa)^{\mathrm{V}}$ in $\mathrm{V}[G_0][G_1]$. We define

$$\mathbb{P} = (\mathbb{P}_0 \times \mathbb{Q}^+(A)) * \dot{\mathbb{R}}.$$

This partial order is $<\kappa$-closed and satisfies the κ^+-chain condition.

In V, we define

$$D_0 = \{\langle p, \vec{q}, r\rangle \in \mathbb{P}_0 \times \mathbb{Q}^+(A) \times \mathbb{Q}({}^{\kappa}\kappa) \mid \langle p, \vec{q}\,\rangle \Vdash \text{“}\check{r} \in \dot{\mathbb{R}}\text{”}\}.$$

For each $\vec{d} = \langle p, \vec{q}, r\rangle \in D_0$, there is an $s_{\vec{d}} \in (\mathbb{P}_0 \times \mathbb{Q}^+(A)) * \dot{\mathbb{R}}$ with $s_{\vec{d}} = \langle\langle p, \vec{q}\rangle, \dot{r}\rangle$ and $\langle p, \vec{q}\rangle \Vdash$ “$\dot{r} = \check{r}$”. Clearly, there is a subset D of $\mathbb{P}$ that is closed under descending $\leq_{\mathbb{P}}$-sequences of length less than κ, has cardinality at most 2^κ and contains the dense subset $\{s_{\vec{d}} \mid \vec{d} \in D_0\}$. The partial order $\langle D, \leq_{\mathbb{P}}\restriction (D \times D)\rangle$ satisfies the κ^+-chain condition and is forcing-equivalent to $\mathbb{P}$. We continue to work with $\mathbb{P}$.

Let $G = (G_0 \times G_1) * G_2$ be $\mathbb{P}$-generic over V. There are trees $T_0, T_B \in \mathrm{V}[G_0]$ on $\kappa \times \kappa$ such that $A = p[T_0]^{\mathrm{V}[G_0][\bar{G}]}$ and $B = p[T_B]^{\mathrm{V}[G_0][\bar{G}]}$ hold in every generic extension $\mathrm{V}[G_0][\bar{G}]$ of $\mathrm{V}[G_0]$ by a $<\kappa$-closed forcing.

The results of Section 6.2 show that there is a tree $T_S \in \mathrm{V}[G]$ on $\kappa \times \kappa$ such that $p[T_S]$ is the set of all $x \in {}^\kappa\kappa$ with

$$\begin{aligned}(\exists y \in p[T_B])(\forall \alpha, \bar{\alpha} < \kappa)\big[&x(\alpha) = \bar{\alpha} \\ &\longleftrightarrow (\exists \bar{\beta} < \kappa)(\forall \beta \leq \bar{\beta} < \kappa)[s_{\bar{\beta}} \subseteq (y)_{\prec \bar{\alpha}, \alpha \succ} \longrightarrow t_{G_2}(\bar{\beta}) = 0]\big]\end{aligned} \tag{7.7}$$

in every transitive ZFC-model that contains $\mathrm{V}[G]$ and has the same ${}^{<\kappa}\kappa$ as $\mathrm{V}[G]$.

Fix a $\gamma < \kappa^+$. The definition of $\dot{C}$ and the equivalence (7.4) imply that the function $b_\gamma \in B = p[T_B]^{\mathrm{V}[G]}$ witnesses that $t_{\bar{G}_\gamma} \in p[T_S]^{\mathrm{V}[G]}$ holds.

By the results of Section 6.2, there is a tree $T_1 \in \mathrm{V}[G]$ on $\kappa \times \kappa$ such that $p[T_1]$ is equal to the set

$$\{x \in {}^\kappa\kappa \mid (\exists y \in p[T_S])(\forall \beta < \kappa)(\exists \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \wedge y(\alpha) = 1]\} \tag{7.8}$$

in every transitive ZFC-model that contains $\mathrm{V}[G]$ and has the same ${}^{<\kappa}\kappa$ as $\mathrm{V}[G]$.

Let $\mathbb{S}$ be an element of the class $\Gamma_{\mathrm{V}}(\mathbb{P}, G, \kappa)$. We work in $\mathrm{V}[G_0]$. Since $\dot{\mathbb{R}}^{G_0 \times G_1}$ is a sublattice of $\mathbb{Q}({}^\kappa\kappa)^{\mathrm{V}[G_0]}$, we can find a $\mathbb{Q}^+(A)$-innocuous forcing $\dot{\mathbb{R}}_0 \in \mathrm{V}[G_0]^{\mathbb{Q}^+(A)}$ with $\dot{\mathbb{R}}_0^{G_1} = \dot{\mathbb{R}}^{G_0 \times G_1}$. By our assumptions, there is a $(\mathbb{Q}^+(A) * \dot{\mathbb{R}}_0)$-innocuous forcing $\dot{\mathbb{S}} \in \mathrm{V}[G_0]^{\mathbb{Q}^+(A) * \dot{\mathbb{R}}_0}$ with $\mathbb{S} = \dot{\mathbb{S}}^{G_1 * G_2}$. Pick $\dot{\mathbb{T}} \in \mathrm{V}[G_0]^{\mathbb{Q}^+(A)}$ with $\dot{\mathbb{T}} = \dot{\mathbb{R}}_0 *_{\mathbb{Q}^+(A)} \dot{\mathbb{S}}$. This means

$$\mathbb{1}_{\mathbb{Q}^+(A)} \Vdash \text{“}\dot{\mathbb{T}} \textit{ is } <\check{\kappa}\textit{-closed and satisfies the } \check{\kappa}^+\textit{-chain condition”}$$

and $\dot{\mathbb{T}}$ is a $\mathbb{Q}^+(A)$-innocent forcing by Lemma 7.4.8.

Let H be $\mathbb{S}$-generic over $\mathrm{V}[G]$. Then $A = p[T_0]^{\mathrm{V}[G][H]}$, $B = p[T_B]^{\mathrm{V}[G][H]}$ and $\{t_{\bar{G}_\gamma} \mid \gamma < \kappa^+\} \subseteq p[T_S]^{\mathrm{V}[G][H]}$.

Suppose $x \in p[T_S]^{\mathrm{V}[G][H]}$. Then x satisfies (7.7) in $\mathrm{V}[G][H]$ and there is a $\gamma < \kappa^+$ with

$$\begin{aligned} x(\alpha) = \bar{\alpha} &\iff (\exists \bar{\beta} < \kappa)(\forall \bar{\beta} \leq \delta < \kappa)\ [s_\delta \subseteq (b_\gamma)_{\prec \bar{\alpha}, \alpha \succ} \rightarrow t_{G_2}(\delta) = 0] \\ &\iff (b_\gamma)_{\prec \bar{\alpha}, \alpha \succ} \in \dot{C}^{G_0 \times G_1} \end{aligned}$$

for all $\alpha, \bar{\alpha} < \kappa$. We can conclude $x = t_{\bar{G}_\gamma}$ and $p[T_S]^{\mathrm{V}[G][H]} = \{t_{\bar{G}_\gamma} \mid \gamma < \kappa^+\}$.

We can find a $\bar{H} \in \mathrm{V}[G][H]$ that is $\dot{\mathbb{T}}^{G_1}$-generic over $\mathrm{V}[G_0][G_1]$ with $\mathrm{V}[G][H] = \mathrm{V}[G_0][G_1][\bar{H}]$. The above remarks and Lemma 7.4.6 show that $x \in ({}^\kappa\kappa)^{\mathrm{V}[G][H]}$ is an element of A if and only if

$$(\forall \gamma < \kappa^+)(\exists \beta < \kappa)(\forall \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \longrightarrow t_{\bar{G}_\gamma}(\alpha) = 0].$$

By the above computations, $x \in ({}^\kappa\kappa)^{\mathrm{V}[G][H]}$ is *not* an element of A if and only if

$$(\exists y \in p[T_S])(\forall \beta < \kappa)(\exists \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \wedge y(\alpha) = 1]$$

holds in $\mathrm{V}[G][H]$. Since the equality (7.8) still holds in $\mathrm{V}[G][H]$, we can conclude

$$p[T_1]^{\mathrm{V}[G][H]} = ({}^{\kappa}\kappa)^{\mathrm{V}[G][H]} \setminus A.$$

□

7.5. Definable well-orders of ${}^{\kappa}\kappa$

This section is devoted to the proof of the following result that directly implies the statement of Theorem 7.1.7. Moreover, we will combine this theorem with the results of Chapter 6 to prove Theorem 7.1.9.

THEOREM 7.5.1. *Let $\lhd$ be a well-ordering of ${}^{\kappa}\kappa$, $\langle A, s\rangle$ be a κ-coding basis with*

$$A = \{\prec x, y \succ \mid x, y \in {}^{\kappa}\kappa \textit{ with either } x = y \textit{ or } x \lhd y\}$$

and G be $\mathbb{P}_s(A)$-generic over V.

(1) *There is a well-ordering of $({}^{\kappa}\kappa)^{\mathrm{V}[G]}$ whose graph is a $\boldsymbol{\Delta}^1_2$-subset of ${}^{\kappa}\kappa$ in $\mathrm{V}[G]$.*
(2) *The set $({}^{\kappa}\kappa)^{\mathrm{V}[G]}$ is Γ_κ-persistently $\boldsymbol{\Sigma}^1_1$ in $\mathrm{V}[G]$, where Γ_κ is the class of all $<\kappa$-closed partial orders in $\mathrm{V}[G]$.*

The idea behind the proof of this statement is to use $\lhd$ in the $\mathbb{P}_s(A)$-generic extension to define a well-ordering $\lhd^*$ of $\mathrm{H}^{\mathrm{V}}_{\kappa^+}$ in $\mathrm{H}^{\mathrm{V}[G]}_{\kappa^+}$ and well-order $({}^{\kappa}\kappa)^{\mathrm{V}[G]}$ by identifying functions in ${}^{\kappa}\kappa$ with the $\lhd^*$-least nice name in $\mathrm{H}^{\mathrm{V}}_{\kappa^+}$ representing this function. We introduce some vocabulary needed in the following arguments.

DEFINITION 7.5.2. Let Γ be a class of partial orders that contains the trivial partial order. We say that a set X is Γ*-persistently* $\boldsymbol{\Sigma}_1(\mathrm{H}_{\kappa^+})$ if there is a Σ_1-formula $\varphi \equiv \varphi(u, v_0, \ldots, v_{n-1})$ and parameters $y_0, \ldots, y_{n-1} \in \mathrm{H}_{\kappa^+}$ such that

$$X = \{x \in \mathrm{H}^{\mathrm{V}[G]}_{\kappa^+} \mid \langle \mathrm{H}^{\mathrm{V}[G]}_{\kappa^+}, \in\rangle \models \varphi(x, y_0, \ldots, y_{n-1})\}$$

holds whenever $\mathbb{Q}$ is a partial order in Γ and G is $\mathbb{Q}$-generic over V.

PROPOSITION 7.5.3. *Let A be a subset of ${}^{\kappa}\kappa$ and G be $\mathbb{P}_s(A)$-generic over* V. *If Γ_κ denotes the class of all $<\kappa$-closed partial orders in $\mathrm{V}[G]$, then the sets A, $\mathbb{P}_s(A)^{\mathrm{V}}$, G, $\mathbb{P}_s(A)^{\mathrm{V}} \setminus G$ and the relation*

$$I_{\mathbb{P}_s(A)^{\mathrm{V}}} = \{\langle p, q\rangle \in \mathbb{P}_s(A)^{\mathrm{V}} \times \mathbb{P}_s(A)^{\mathrm{V}} \mid p \textit{ and } q \textit{ are incompatible in } \mathbb{P}_s(A)^{\mathrm{V}}\}$$

are Γ_κ-persistently $\boldsymbol{\Sigma}_1(\mathrm{H}^{\mathrm{V}[G]}_{\kappa^+})$ in $\mathrm{V}[G]$.

PROOF. We work in $\mathrm{V}[G]$. Theorem 7.1.3 directly implies that A is Γ_κ-persistently $\boldsymbol{\Sigma}_1(\mathrm{H}^{\mathrm{V}[G]}_{\kappa^+})$. If $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[G]$ by a forcing in Γ_κ, then $\langle A, s\rangle$ is a κ-coding basis in $\mathrm{V}[G][H]$,

$$\mathbb{P}_s(A)^{\mathrm{V}} = \mathbb{P}_s(A) = \mathbb{P}_s(A)^{\mathrm{V}[G][H]} \subseteq \mathrm{H}_{\kappa^+}$$

$\mathbb{P}_s(A)$ is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}_{\kappa^+}^{\mathrm{V}[G]})$ by the absoluteness of the $\mathbf{\Sigma}^1_1$-definition of A.

A pair $\langle p, q\rangle$ of conditions in $\mathbb{P}_s(A)$ is an incompatible in $\mathbb{P}_s(A)$ if and only if one of the following statements holds true.

(1) T_p is not an end-extension of T_q or T_q is not an end-extension of T_p.
(2) T_p is an end-extension of T_q and there is an $x \in \mathrm{dom}(f_p) \cap \mathrm{dom}(f_q)$ with either $f_q(x) \nsubseteq f_p(x)$ or $h_p(x) \neq h_q(x)$.
(3) Same as (2), but with the roles of p and q exchanged.
(4) T_p is an end-extension of T_q and there is an $x \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_p)$ such that for all $z \in [T_p]$ with $f_q(x) \subseteq z$ there is $\beta < \kappa$ with $\prec h_q(x), \beta\succ < \mathrm{ht}(T_p)$ and either $s(\beta) \subseteq x$ and $z(\prec h_q(x), \beta\succ) = 0$ or $s(\beta) \nsubseteq x$ and $z(\prec h_q(x), \beta\succ) = 1$.
(5) Same as (4), but with the roles of p and q exchanged.

Since all of those statements are absolute between $\mathrm{V}[G]$ and generic extensions of $\mathrm{V}[G]$ by forcings in Γ_κ, we can conclude that $I_{\mathbb{P}_s(A)^\mathrm{V}}$ is Γ_κ-persistent $\mathbf{\Sigma}_1(\mathrm{H}_{\kappa^+})$.

Given $y \in A$ and $\gamma < \kappa$, the proof of Corollary 7.2.8 shows that $H_G(x) = \gamma$ holds if and only if there is a $z \in [T_G]$ such that (7.3) holds for all $\beta < \kappa$. By Lemma 7.2.7, $[T_G] = [T_G]^{\mathrm{V}[G][H]}$ holds whenever $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[G]$ by a forcing in Γ_κ. This shows that the graph of H_G is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}_{\kappa^+})$. In combination with (7.1), this implies that the graph of F_G is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}_{\kappa^+})$.

The filter G consists of all conditions p in $\mathbb{P}_s(A)$ such that T_G is an end extension of T_p and, if $x \in \mathrm{dom}(f_p)$, then $f_p(x) = F_G(x) \upharpoonright \mathrm{ht}(T_p)$ and $h_p(x) = H_G(x)$. In combination with the above computations, this allows us to conclude that G is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}_{\kappa^+})$.

Finally, a condition p in $\mathbb{P}_s(A)$ is not an element of G if there is a q in G that is incompatible with p. Using the above computations, $\mathbb{P}_s(A)^\mathrm{V} \setminus G$ is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}_{\kappa^+})$. □

Proof of Theorem 7.5.1. (i) Work in $\mathrm{V}[G]$ and let Γ_κ denote the class of all $<\kappa$-closed partial orders in $\mathrm{V}[G]$. We have

$$x \in \mathrm{V} \iff \langle \mathrm{H}_{\kappa^+}, \in\rangle \models (\exists z \in A)(\forall \alpha < \kappa)\ x(\alpha) = z(\prec 0, \alpha\succ)$$

for all $z \in {}^\kappa\kappa$. This shows that $({}^\kappa\kappa)^\mathrm{V}$ is Γ_κ-persistently $\mathbf{\Sigma}^1_1(\mathrm{H}_{\kappa^+})$.

Define $\psi \equiv \psi(u, v, w)$ to be the Σ_1-formula

$$\begin{aligned}(\exists f : w \longrightarrow \mathrm{tc}(\{u\} \cup w)\ \textit{bijection})(\forall \alpha, \beta < w)\big[(v(\prec 0, \prec\alpha, \beta\succ\succ) = 1 \\ \longleftrightarrow f(\alpha) \in f(\beta)) \wedge (v(\prec 1, \alpha\succ) = 1 \leftrightarrow f(\alpha) \in u)\big].\end{aligned} \tag{7.9}$$

Let $\mathrm{V}[G][H]$ be a generic extension of $\mathrm{V}[G]$ by a forcing in Γ_κ. Given a function $x \in ({}^\kappa 2)^{\mathrm{V}[G][H]}$, we let e_x denote the relation on κ defined by

$$\alpha\ e_x\ \beta \iff x(\prec 0, \prec\alpha, \beta\succ\succ) = 1.$$

If $\langle\kappa, e_x\rangle$ is well-founded and extensional, then we let t_x denote image of the corresponding collapsing map c_x and $a_x = \{c_x(\alpha) \mid x(\prec 1, \alpha\succ) = 1\}$.

Given $a \in \mathrm{H}_{\kappa^+}^{\mathrm{V}[G][H]}$, there is an $x \in ({}^\kappa 2)^{\mathrm{V}[G][H]}$ such that $\psi(a, x, \kappa)$ holds in $\mathrm{H}_{\kappa^+}^{\mathrm{V}[G][H]}$. Moreover, if $\psi(a, x, \kappa)$ holds in $\mathrm{H}_{\kappa^+}^{\mathrm{V}[G][H]}$, then $\langle\kappa, e_x\rangle$ is well-founded and extensional, $a = a_x$, $\mathrm{tc}(\{a\} \cup \kappa) = t_x$ and c_x is the unique bijection witnessing that $\psi(a, x, \kappa)$ holds. In particular, if $a, b \in \mathrm{H}_{\kappa^+}^{\mathrm{V}[G][H]}$ and $x \in ({}^\kappa 2)^{\mathrm{V}[G][H]}$ such that both $\psi(a, x, \kappa)$ and $\psi(b, x, \kappa)$ hold in $\mathrm{H}_{\kappa^+}^{\mathrm{V}[G][H]}$, then $a = b$. Finally, these computations show that a is an element of $\mathrm{H}_{\kappa^+}^{\mathrm{V}}$ if and only if $\psi(a, x, \kappa)$ holds in $\mathrm{H}_{\kappa^+}^{\mathrm{V}[G][H]}$ for some $x \in ({}^\kappa 2)^{\mathrm{V}}$. We can conclude that $\mathrm{H}_{\kappa^+}^{\mathrm{V}}$ is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}_{\kappa^+})$.

Let N denote the set of all functions $n : \kappa \times \kappa \longrightarrow \mathbb{P}_s(A)$ in V with the property that the set $A^n_\alpha = \{n(\alpha, \beta) \in \mathbb{P}_s(A) \mid \beta < \kappa\}$ is an anti-chain in $\mathbb{P}_s(A)$ for all $\alpha < \kappa$. By Proposition 7.5.3 and the above computations, N is Γ_κ-persistently $\mathbf{\Sigma}_1(\mathrm{H}_{\kappa^+})$.

By the results of Section 6.2, there is a tree T on κ^3 with the property that, whenever $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[G]$ by a forcing in Γ_κ, then $p[T]^{\mathrm{V}[G][H]}$ is equal to the set of all $\langle x, y\rangle \in ({}^\kappa\kappa)^{\mathrm{V}[G][H]} \times ({}^\kappa 2)^{\mathrm{V}}$ such that

$$\begin{aligned} &\psi(n, y, \kappa) \wedge (\forall \alpha, \beta < \kappa) \\ &\qquad \big[(x(\alpha) = \beta \rightarrow (\exists \gamma < \kappa)\ n(\prec\alpha, \beta\succ, \gamma) \in G) \\ &\qquad \wedge (x(\alpha) \neq \beta \rightarrow (\forall \gamma < \kappa)\ n(\prec\alpha, \beta\succ, \gamma) \notin G)\big]. \end{aligned} \tag{7.10}$$

holds in $\langle \mathrm{H}_{\kappa^+}^{\mathrm{V}[G][H]}, \in\rangle$ for some $n \in N$. For every $x \in {}^\kappa\kappa$ there is a $y \in ({}^\kappa 2)^{\mathrm{V}}$ with $\langle x, y\rangle \in p[T]$, because there is an $n \in N$ such that

$$\tau_n^G = \{\prec\alpha, \beta\succ \mid \alpha, \beta < \kappa,\ x(\alpha) = \beta\},$$

where τ_n is the $\mathbb{P}_s(A)$-nice name $\bigcup_{\alpha<\kappa}\{\check{\alpha}\} \times A^n_\alpha$. Moreover, if $\langle x_0, y\rangle, \langle x_1, y\rangle \in p[T]$, then $x_0 = x_1$.

Now, define $x_0 \lhd^* x_1$ by

$$\begin{aligned} &(\exists z_0, z_1 \in ({}^\kappa 2)^{\mathrm{V}})\ [\langle x_0, z_0\rangle, \langle x_1, z_1\rangle \in p[T] \wedge z_0 \lhd z_1 \wedge (\forall \bar{z}_0, \bar{z}_1 \in ({}^\kappa 2)^{\mathrm{V}}) \\ &\qquad [(\bar{z}_0 \lhd z_0 \wedge \bar{z}_1 \lhd z_1) \rightarrow (\langle x_0, \bar{z}_0\rangle \notin p[T] \vee \langle x_1, \bar{z}_1\rangle \notin p[T])]]. \end{aligned}$$

for all $x_0, x_1 \in {}^\kappa\kappa$. By the above constructions and the results of Section 6.2, the graph of this relation is a $\mathbf{\Sigma}^1_2$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$. It is easy to check that this relation is linear, strict and total. In particular, its graph is a $\mathbf{\Delta}^1_2$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$. Assume, toward a contradiction, that there is a strictly $\lhd^*$-descending sequence of elements in ${}^\kappa\kappa$ of length ω. The definition gives us a strictly $\lhd$-descending sequence of elements in $({}^\kappa 2)^{\mathrm{V}}$ of the same length. Since $\mathbb{P}_s(A)$ is σ-closed, this sequence is an element of V, a contradiction.

(ii) By Proposition 6.2.1, there is a tree T_* on $\kappa \times \kappa$ such that

$$p[T_*]^{\mathrm{V}[G][H]} = \exists^x (p[T]^{\mathrm{V}[G][H]})$$

holds whenever $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[G]$ by a forcing in Γ_κ. Let $\mathrm{V}[G][H]$ be such an extension and x be an element of $p[T_*]^{\mathrm{V}[G][H]}$. There is a $y \in ({}^\kappa\kappa)^{\mathrm{V}[G][H]}$ with $\langle x, y\rangle \in p[T]^{\mathrm{V}[G][H]}$. By the construction of T, y is an element of $({}^\kappa 2)^{\mathrm{V}}$ and there is an $n \in N$ witnessing that (7.10) holds in $\mathrm{H}_{\kappa^+}^{\mathrm{V}[G][H]}$. In V, we can construct the $\mathbb{P}_s(A)$-nice name τ_n and, since $\tau_n^G \in \mathrm{V}[G]$, we can conclude $x \in \mathrm{V}[G]$. This shows that $p[T_*]^{\mathrm{V}[G][H]} \subseteq ({}^\kappa\kappa)^{\mathrm{V}[G]}$ and the above computations already show

$$({}^\kappa\kappa)^{\mathrm{V}[G]} = \exists^x(p[T]^{\mathrm{V}[G]}) = p[T_*]^{\mathrm{V}[G]} \subseteq p[T_*]^{\mathrm{V}[G][H]}.$$

□

In combination with the results of Chapter 6, the above result allows us to show that generic absoluteness for $\mathbf{\Sigma}^1_3({}^\kappa\kappa)$-formulae under $<\kappa$-closed forcings that satisfy the κ^+-chain condition always fails.

Proof of Theorem 7.1.9. We fix an $a \in \mathcal{P}(\kappa)$ with ${}^{<\kappa}\kappa \in \mathrm{L}[a]$ and bijections

$$f : \kappa \longrightarrow \{\langle t_0, t_1\rangle \in {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa \mid \mathrm{lh}(t_0) = \mathrm{lh}(t_1)\}$$

and $g : \kappa \longrightarrow {}^{<\kappa}2$ contained in $\mathrm{L}[a]$. Given $x \in {}^\kappa\kappa$, we define $\iota_x = f \circ x \circ g^{-1}$ and $T_x = \{f(\alpha) \mid x(\alpha) = 1\}$. By the results of Section 6.2, there is a tree $T \in \mathrm{L}[a]$ on κ^3 such that

$$p[T] = \{\langle x, y\rangle \in {}^\kappa\kappa \times {}^\kappa\kappa \mid \text{“}T_x \textit{ is a tree on } \kappa \times \kappa\text{”} \ \wedge \ y \in p[T_x] \\ \wedge \ \text{“}\iota_y \textit{ is not a } \exists^x\textit{-perfect embedding into } T_x\text{”}\}$$

holds in every transitive ZFC-model that contains $\mathrm{L}[a]$ and has the same ${}^{<\kappa}\kappa$ as $\mathrm{L}[a]$. This implies that in any ZFC-model with the above properties

$$(\exists x \in {}^\kappa\kappa)(\forall y \in {}^\kappa\kappa)\ \langle x, y\rangle \in p[T] \tag{7.11}$$

is equivalent to the existence of a tree T_* on $\kappa \times \kappa$ such that “$p[T_*] = {}^\kappa\kappa$” holds and there is no $\exists^x$-perfect embedding into T_*.

We show that

$$\mathbb{1}_{\mathrm{Add}(\kappa,\kappa^+)} \Vdash \text{“}(\forall x \in {}^{\check\kappa}\check\kappa)(\exists y \in {}^{\check\kappa}\check\kappa)\ \langle x, y\rangle \notin p[\check T]\text{”}$$

holds in V. Assume, toward a contradiction, that G is $\mathrm{Add}(\kappa, \kappa^+)$-generic over V and $T_* \in \mathrm{V}[G]$ witnesses that (7.11) holds in $\mathrm{V}[G]$. Since $T_* = T_x$ for some $x \in ({}^\kappa 2)^{\mathrm{V}[G]}$, there is an $\alpha < \kappa^+$ with $T_* \in \mathrm{V}[G \cap \mathrm{Add}(\kappa, \alpha)]$ and $\mathrm{V}[G]$ is an $\mathrm{Add}(\kappa, \kappa)^+$-generic extension of $\mathrm{V}[G \cap \mathrm{Add}(\kappa, \alpha)]$ with

$$({}^\kappa\kappa)^{\mathrm{V}[G\cap\mathrm{Add}(\kappa,\alpha)]} \subsetneq ({}^\kappa\kappa)^{\mathrm{V}[G]} = p[T_*]^{\mathrm{V}[G]}.$$

This means

$$\langle \mathrm{V}[G \cap \mathrm{Add}(\kappa, \alpha)], \in\rangle \models \big[(\exists p \in \mathrm{Add}(\kappa, \kappa^+))\ p \Vdash \text{“}p[\check T_*] \nsubseteq \check{\mathrm{V}}\text{”}\big]$$

and there is a $\exists^x$-perfect embedding into T_* in $\mathrm{V}[G \cap \mathrm{Add}(\kappa, \alpha)]$ by Lemma 6.3.6. But this map is also a $\exists^x$-perfect embedding into T_* in $\mathrm{V}[G]$, a contradiction.

In the other direction, define $A \subseteq {}^{\kappa}\kappa$ as in Theorem 7.5.1 and let G be $\mathbb{P}(A)$-generic over V. By the second part of the Theorem, there is a tree T_* on $\kappa \times \kappa$ in $\mathrm{V}[G]$ such that $p[T_*]^{\mathrm{V}[G][H]} = ({}^{\kappa}\kappa)^{\mathrm{V}[G]}$ holds whenever $\mathrm{V}[G][H]$ is a generic extension of $\mathrm{V}[G]$ by a $<\kappa$-closed forcing in $\mathrm{V}[G]$. This obviously implies that "$p[T_*] = {}^{\kappa}\kappa$" holds in $\mathrm{V}[G]$ and we can apply Lemma 6.3.6 to show that there are no $\exists^x$-perfect embeddings into T_* in $\mathrm{V}[G]$. We can conclude that $\mathbb{1}_{\mathbb{P}(A)} \Vdash$ "$(\exists x \in {}^{\check{\kappa}}\check{\kappa})(\forall y \in {}^{\check{\kappa}}\check{\kappa})\ \langle x, y\rangle \in p[\check{T}]$" holds in V. □

7.6. Embeddings of trees

In this short section, we present an easy proof of Theorem 7.1.8 with the help of our first main result. Let $\mathcal{TO}_\kappa$ denote the class of all $x \in {}^{\kappa}\kappa$ such that $\mathbb{T}_x = \langle \kappa, \in_x\rangle$ is a tree that is an element of $\mathcal{T}_\kappa$.

Let $\bar{T}$ be the set of all pairs $\langle s, t\rangle$ in ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ such that $\mathrm{lh}(s) = \mathrm{lh}(t) = \gamma + 1$ for some $\gamma < \kappa$ and either $\langle \lambda, \in_{s \restriction \lambda}\rangle$ is not a tree for some $\lambda \leq \gamma$ closed under Gödel-Pairing or t is injective and

$$(\forall \alpha < \beta \leq \gamma)\ \ [\prec t(\alpha), t(\beta)\succ \leq \gamma \rightarrow s(\prec t(\alpha), t(\beta)\succ) = 1]\,.$$

We define T to be the tree $\{\langle s \restriction \alpha, t \restriction \beta\rangle \mid \langle s, t\rangle \in \bar{T},\ \alpha \leq \mathrm{lh}(s)\}$ on $\kappa \times \kappa$. It is easy to check that $\mathcal{TO}_\kappa = {}^{\kappa}\kappa \setminus p[T]$ holds in V and every generic extension of V by a $<\kappa$-closed forcing.

Given $y \in {}^{\kappa}\kappa$, we define $T(y)$ to be the tree $\{t \in {}^{<\kappa}\kappa \mid \langle y \restriction \mathrm{lh}(t), t\rangle \in T\}$ on κ. If $y \in \mathcal{TO}_\kappa$ and $\alpha < \kappa$, then $\langle \{\alpha\} \cup \mathrm{prec}_{\mathbb{T}_y}(\alpha), \in_y\rangle$ is a well-order of successor length and we let $t_y(\alpha) \in {}^{<\kappa}\kappa$ denote the corresponding uncollapsing map. Our construction yields $t_y(\alpha) \in T(y)$ and the map $[\alpha \mapsto t_y(\alpha)]$ shows that $\mathbb{T}_y$ is order-preserving embeddable into $\langle T(y), \subsetneq\rangle$.

The following result was proved in [**MV93**] in the case "$\kappa = \omega_1$", but the proof given there directly generalizes to higher cardinalities. It is the uncountable version of the classical *Boundedness Lemma*.

Lemma 7.6.1 (Boundedness Lemma for ${}^{\kappa}\kappa$, [**MV93**, Corollary 13]). *If A is a $\mathbf{\Sigma}^1_1$-subset of ${}^{\kappa}\kappa$ with $A \subseteq \mathcal{TO}_\kappa$, then there is a tree $\mathbb{T}$ in $\mathcal{T}_\kappa$ such that $\mathbb{T}_y \leq \mathbb{T}$ holds for every $y \in A$.*

Proof. Let S be a tree on $\kappa \times \kappa$ with $A = p[S]$ and T be the tree on $\kappa \times \kappa$ defined above. Define S_* to be the tree on κ^3 consisting of triples $\langle s, t, u\rangle$ with $\langle s, t\rangle \in T$ and $\langle s, u\rangle \in S$. Assume towards a contradiction, that there is a $\langle x, y, z\rangle \in [S_*]$. Then $x \in p[S] \cap p[T] = A \cap ({}^{\kappa}\kappa \setminus \mathcal{TO}_\kappa) = \emptyset$, a contradiction. If $y \in A$ with $\langle y, z\rangle \in [S]$ and $t \in T(y)$, then $\langle y \restriction \mathrm{lh}(t), t, z \restriction \mathrm{lh}(t)\rangle \in S_*$ and the map $[t \mapsto \langle y \restriction \mathrm{lh}(t), t, z \restriction \mathrm{lh}(t)\rangle]$ shows that $\langle T(y), \subsetneq\rangle$ is order-preserving embeddable into $\mathbb{T} = \langle S_*, \lhd_*\rangle$, where $\lhd_*$ is the natural order on S_*. By the above remarks, this shows that $\mathbb{T}_y \leq \mathbb{T}$ holds for every $y \in A$. □

Proof of Theorem 7.1.8. Let $\mathbb{P}$ be the forcing produced an application of Theorem 7.1.3 that codes the subset $\mathcal{TO}_\kappa$ of ${}^{\kappa}\kappa$ and G be $\mathbb{P}$-generic over V. By the above remarks and Proposition 6.3.3, we have $\mathcal{TO}_\kappa^{\mathrm{V}} \subseteq \mathcal{TO}_\kappa^{\mathrm{V}[G]}$ and $\mathcal{TO}_\kappa^{\mathrm{V}}$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^{\kappa}\kappa$ in $\mathrm{V}[G]$. Lemma 7.6.1

shows that there is a $\mathbb{T}_G \in \mathcal{T}_\kappa^{\mathrm{V}[G]}$ with $\mathbb{T}_x \leq \mathbb{T}_G$ for all $x \in \mathcal{TO}_\kappa^{\mathrm{V}}$. For every $\mathbb{T} \in \mathcal{T}_\kappa^{\mathrm{V}}$ there is an $x \in \mathcal{TO}_\kappa^{\mathrm{V}}$ with $\mathbb{T}$ isomorphic to $\mathbb{T}_x$ and this completes the proof of the theorem. □

7.7. Open problems

We close this chapter with some questions motivated by the work of [**Lücb**].

It is natural to ask whether Theorem 7.1.4 is optimal with respect to the complexity of the coded subset in the generic extension of the ground model.

QUESTION 7.7.1. *Is there a partial order* $\mathbb{P}$ *with the following properties?*

(1) $\mathbb{P}$ *preserves cofinalities and cardinalities.*
(2) *If* G *is* $\mathbb{P}$*-generic over* V*, then* $({}^\kappa\kappa)^{\mathrm{V}}$ *is a* κ*-Borel subset of* ${}^\kappa\kappa$ *without a perfect subset in* V$[G]$.

A positive answer to this question would imply that every subset of ${}^\kappa\kappa$ is κ-Borel in a cofinality-preserving generic extension of the ground model, because such a forcing could be combined with *almost disjoint coding*. In the other direction, an answer to the following question might provide a negative answer to Question 7.7.1.

QUESTION 7.7.2. *Does* ZFC *(plus large cardinal axioms) prove nontrivial statements about the possible lengths of well-orders of subsets of* ${}^\kappa\kappa$ *whose graph is a* κ*-Borel subset of* ${}^\kappa\kappa \times {}^\kappa\kappa$ *?*

A positive answer to Question 7.7.1 would also show that the absoluteness statement of Theorem 7.4.2 holds for other classes of partial orders.

QUESTION 7.7.3. *Does the statement of Theorem 7.4.2 hold if we replace* $\Gamma_{\mathrm{V}}(\mathbb{P}, G, \kappa)$ *by the class of all* $<\kappa$*-closed partial orders?*

If we restrict the canonical well-order of L to ${}^\kappa\kappa$, then we get a well-order whose graph is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$. Results of Sy-David Friedman and Peter Holy in [**FHa**] show that there is a partial order that forces "$2^\kappa = \kappa^+$ " and the existence of a $\mathbf{\Delta}^1_1$-well-order of ${}^\kappa\kappa$. We may therefore ask whether the existence of a $\mathbf{\Delta}^1_1$-well-order of ${}^\kappa\kappa$ is compatible with a failure of the GCH at κ.

QUESTION 7.7.4. *Does the existence of a well-order of* ${}^\kappa\kappa$ *whose graph is a* $\mathbf{\Delta}^1_1$*-subset of* ${}^\kappa\kappa \times {}^\kappa\kappa$ *imply that* $2^\kappa = \kappa^+$ *holds?*

There are many open questions concerning the perfect subset property and weakenings of it. We present two interesting examples.

QUESTION 7.7.5. *Is it consistent that all* $\mathbf{\Pi}^1_1$*-subsets of* ${}^\kappa\kappa$ *have the perfect subset property?*

QUESTION 7.7.6. *Is it consistent that every subset of* ${}^\kappa\kappa$ *in* $\mathrm{L}(\mathcal{P}(\kappa))$ *either has cardinality less than* 2^κ *or contains a perfect subset?*

CHAPTER 8

$\mathbf{\Sigma}^1_1$-definability at supercompact cardinals

Given an inaccessible cardinal κ and a subset A of the corresponding generalized Baire space ${}^\kappa\kappa$, the results of the last chapter show that there is a forcing that adds a $\mathbf{\Sigma}^1_1$-definition of A and preserves the inaccessibility of κ and the value of 2^κ. If we consider stronger large cardinal properties and ask whether the coding forcing constructed in the proof of Theorem 7.1.3 preserves these properties, then it is possible to construct scenarios in which this forcing destroys large cardinal properties like measurability. It is therefore natural to ask whether it is possible to have a coding forcing that preserves large cardinal properties of κ.

This chapter contains joint work with Sy-David Friedman that provides a positive answer to this question in the case of supercompact cardinals using class forcing. This forcing will be a class-sized iteration of a variation of the coding forcing $\mathbb{P}_s(A)$ developed in Chapter 7. By carefully defining the support of this iteration and using structural properties of the partial order $\mathbb{P}_s(A)$, we will be able to lift certain supercompact embeddings to our forcing extension of the ground model (see [**Cum10**, Section 9] for more details on the idea of *extending elementary embeddings to generic extensions*). We then use this result to construct a class-sized partial order that preserves the inaccessibility of inaccessible cardinals and the supercompactness of supercompact cardinals and forces the existence of well-orders of H_{κ^+} definable in the structure $\langle \mathrm{H}_{\kappa^+}, \in\rangle$ for every inaccessible cardinal κ.

The results of this chapter are contained in [**FLa**].

8.1. Introduction

Remember that an uncountable cardinal κ is called *γ-supercompact* with $\gamma \geq \kappa$ if there is an elementary embedding $j : \mathrm{V} \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$, $\gamma < j(\kappa)$ and ${}^\gamma M \subseteq M$. The existence of such an embedding is equivalent to the existence of a normal ultrafilter on the set $\mathcal{P}_\kappa(\gamma)$ of all subsets of γ of cardinality less than κ (see [**Kan03**, Theorem 22.7]). Given such an ultrafilter U, we let M_U denote the transitive collapse of the corresponding ultrapower $\mathrm{Ult}_U(\mathrm{V})$ and $j_U : \mathrm{V} \longrightarrow M_U$ denote the corresponding elementary embedding. Finally, we call a cardinal κ *supercompact* if κ is γ-supercompact for all $\gamma \geq \kappa$.

Let κ be a supercompact cardinal and A be an arbitrary subset of ${}^\kappa\kappa$. We want to construct an outer model W of the ground model V such that κ is still supercompact in W, $(2^\kappa)^{\mathrm{V}} = (2^\kappa)^{\mathrm{W}}$ and A is definable in the structure

$\langle \mathrm{H}^{\mathrm{W}}_{\kappa^+}, \in\rangle$. By extending coding methods developed in Chapter 7, this aim is achieved in the following theorem.

THEOREM 8.1.1 ([**FLa**, Theorem 1.1]). *There is a* ZFC*-preserving class forcing* $\mathbb{P}$ *definable without parameters that satisfies the following statements.*

(1) *Let* κ *be a cardinal with the property that there is no singular limit of inaccessible cardinals* ν *with* $\nu^+ < \kappa \leq 2^\nu$*. Then forcing with* $\mathbb{P}$ *does not collapse* κ *and, if* κ *is regular, then* $\mathbb{P}$ *preserves the regularity of* κ*.*
(2) $\mathbb{P}$ *preserves the inaccessibility of inaccessible cardinals and the supercompactness of supercompact cardinals.*
(3) *If* α *is an inaccessible cardinal and* G *is* $\mathbb{P}$ *generic over* V*, then* $(2^\alpha)^{\mathrm{V}} = (2^\alpha)^{\mathrm{V[G]}}$.
(4) *If* κ *is an inaccessible cardinal and* A *is a subset of* ${}^\kappa\kappa$*, then there is a condition* p *in* $\mathbb{P}$ *with the property that* A *is a* $\mathbf{\Sigma}^1_1$*-subset of* ${}^\kappa\kappa$ *in* V[G] *whenever* G *is* $\mathbb{P}$*-generic over* V *with* $p \in \mathrm{G}$*.*

In addition, if the class of inaccessible cardinals is bounded in On*, then* $\mathbb{P}$ *is forcing equivalent to a set-sized forcing.*

In particular, if the *Singular Cardinal Hypothesis* holds in the ground model, then forcing with $\mathbb{P}$ preserves cofinalities and cardinalities.

The proof of this result will actually show that certain degrees of supercompactness are preserved after forcing with $\mathbb{P}$. Let κ be γ-supercompact such that γ is a cardinal with $\gamma = \gamma^{<\kappa}$, $2^\gamma = \gamma^+$ and $2^\nu \leq \gamma$, where ν is the supremum of all inaccessible cardinals smaller or equal to γ. Then κ will still be γ-supercompact after forcing with $\mathbb{P}$. Given a supercompact cardinal κ, we will use a classical result due to Robert Solovay to show that there is a proper class of cardinals γ that satisfy the above properties with respect to κ.

We want to use the above coding result to produce ZFC-models with definable well-orders of H_{κ^+} for every supercompact cardinal κ. We give a brief overview of related existing results. A detailed discussion of this topic can be found in the first part of [**Fri10**]. In [**FHa**], Sy-David Friedman and Peter Holy constructed a class forcing that adds such definable well-orders of low quantifier complexity and preserves various large cardinals.

THEOREM 8.1.2 ([**Fri10**, Theorem 9]). *There is a class forcing which forces* GCH*, preserves all supercompact cardinals (as well as a proper class of* n*-huge cardinals for each* $n < \omega$*) and adds a well-order of* H_{κ^+} *that is definable in* $\langle \mathrm{H}_{\kappa^+}, \in\rangle$ *by a* Σ_1*-formula with parameters for every uncountable regular cardinal* κ*.*

If the GCH holds in the ground model, then results due to David Asperó and Sy-David Friedman show that it possible to produce *lightface* definable well-orders of H_{κ^+} for every uncountable regular cardinal κ.

THEOREM 8.1.3 ([**AF09**, Theorem 1.1] and [**AF**, Theorem 1.1]). *Assume* GCH*. There is a formula* $\varphi(x, y)$ *without parameters and a definable* *class-sized partial order* $\mathbb{P}$ *preserving* ZFC, GCH *and cofinalities that satisfy the following statements.*

(1) $\mathbb{P}$ *forces that there is a well-order* $\leq$ *of the universe such that*

$$\{\langle a,b\rangle \in \mathrm{H}^2_{\kappa^+} \mid \langle \mathrm{H}_{\kappa^+}, \in\rangle \models \varphi(a,b)\}$$

is the restriction $\leq\restriction \mathrm{H}_{\kappa^+}$ *and is a well-order of* H_{κ^+} *whenever* κ *is a regular uncountable cardinal.*

(2) *For all regular cardinals* $\kappa \leq \lambda$*, if* κ *is a* λ*-supercompact cardinal in* V*, then* κ *remains* λ*-supercompact after forcing with* $\mathbb{P}$*.*

The second result of this chapter shows that it is possible to add definable well-orders of H_{κ^+} for every inaccessible cardinal κ without assuming GCH with a class forcing that preserves supercompact cardinals and failures of the GCH at inaccessible cardinals.

THEOREM 8.1.4 ([**FLa**, Theorem 1.4]). *There is a* ZFC*-preserving class forcing* $\mathbb{P}$ *definable without parameters that satisfies the following statements.*

(1) *Let* κ *be a cardinal with the property that there is no singular limit of inaccessible cardinals* ν *with* $\nu^+ < \kappa \leq 2^\nu$*. Then forcing with* $\mathbb{P}$ *does not collapse* κ *and, if* κ *is regular, then* $\mathbb{P}$ *preserves the regularity of* κ*.*

(2) $\mathbb{P}$ *preserves the inaccessibility of inaccessible cardinals and the supercompactness of supercompact cardinals.*

(3) *If* α *is an inaccessible cardinal and* G *is* $\mathbb{P}$ *generic over* V*, then* $(2^\alpha)^{\mathrm{V}} = (2^\alpha)^{\mathrm{V[G]}}$ *and there is a well-order of* $\mathrm{H}^{\mathrm{V[G]}}_{\kappa^+}$ *that is definable in the structure* $\langle \mathrm{H}^{\mathrm{V[G]}}_{\kappa^+}, \in\rangle$ *by a formula with parameters.*

In fact, the partial order $\mathbb{P}$ constructed in the proof of this result satisfies the statements listed in Theorem 8.1.1.

8.2. Coding well-orders of H_{κ^+}

Given an uncountable regular cardinal κ with $\kappa = \kappa^{<\kappa}$, we show in this section how the results of Chapter 7 can be applied to force the existence of a definable well-order of H_{κ^+} with a partial order that is uniformly definable in parameter κ.

DEFINITION 8.2.1. Let κ be an infinite cardinal and $\langle A, s\rangle$ be a κ-coding basis. We say that $\langle A, s\rangle$ *codes a well-order of* ${}^\kappa\kappa$ if there is a well-order $\lhd$ of ${}^\kappa\kappa$ such that

$$A = \{\prec x, y\succ \mid x, y \in {}^\kappa\kappa \textit{ with either } x = y \textit{ or } x \lhd y\}.$$

The following result is a direct consequence of Theorem 7.5.1.

COROLLARY 8.2.2. *Let* κ *be an uncountable regular cardinal that satisfies* $\kappa = \kappa^{<\kappa}$*. If* $\langle A, s\rangle$ *is a* κ*-coding basis that codes a well-order of* ${}^\kappa\kappa$ *and* G *is* $\mathbb{P}_s(A)$*-generic over* V*, then there is a well-order of* $\mathrm{H}^{\mathrm{V}[G]}_{\kappa^+}$ *that is definable in* $\langle \mathrm{H}^{\mathrm{V}[G]}_{\kappa^+}, \in\rangle$ *by a formula with parameters.*

PROOF. We work in $\mathrm{V}[G]$. By Theorem 7.5.1, there is a well-ordering $\lhd^*$ of ${}^\kappa\kappa$ that is definable in $\langle \mathrm{H}_{\kappa^+}, \in\rangle$ by a formula with parameters.

Define R to be the set of all pairs $\langle a, x\rangle$ in $\mathrm{H}_{\kappa^+} \times {}^\kappa 2$ such that there is a bijection $b : \kappa \longrightarrow \mathrm{tc}(\{a\} \cup \kappa)$ with the following properties.

(1) For all $\alpha, \beta < \kappa$, $x(\prec 0, \prec\alpha, \beta\succ\succ) = 1$ if and only if $b(\alpha) \in b(\beta)$.
(2) For all $\alpha < \kappa$, $x(\prec 1, \alpha\succ) = 1$ if and only if $b(\alpha) \in a$.

This relation is definable in $\langle \mathrm{H}_{\kappa^+}, \in\rangle$. If $\langle a_0, x\rangle, \langle a_1, x\rangle \in R$, then it is easy to see that $a_0 = a_1$ holds.

Define $r : \mathrm{H}_{\kappa^+} \longrightarrow {}^\kappa 2$ to be the function that sends $a \in \mathrm{H}_{\kappa^+}$ to the $\lhd^*$-least $x \in {}^\kappa 2$ with $R(a, x)$. This function is definable in $\langle \mathrm{H}_{\kappa^+}, \in\rangle$ and injective. It therefore yields a well-order of H_{κ^+} that is definable in the structure $\langle \mathrm{H}_{\kappa^+}, \in\rangle$ by a formula with parameters. □

Next, we introduce partial orders $\mathbb{C}_\alpha$ that *randomly* well-order ${}^\alpha\alpha$ if α is a regular uncountable cardinal with $\alpha = \alpha^{<\alpha}$. This coding is *random* in the sense that the generic filter chooses the well-order of ${}^\alpha\alpha$ that is coded using a partial order of the form $\mathbb{P}_s(A)$.

DEFINITION 8.2.3. If α is not a regular uncountable cardinal with $\alpha = \alpha^{<\alpha}$, then we define $\mathbb{C}_\alpha$ to be the trivial partial order. Otherwise, we define the domain of $\mathbb{C}_\alpha$ to consist of conditions $\langle A, s, p\rangle$ such that either $A = s = p = \emptyset$ or $\langle A, s\rangle$ is an α-coding basis that codes a well-ordering of ${}^\alpha\alpha$ and $p \in \mathbb{P}_s(A)$. We set $\langle A, s, p\rangle \leq_{\mathbb{C}_\alpha} \langle B, t, q\rangle$ if either $B = \emptyset$ or $A = B \neq \emptyset$, $s = t$ and $p \leq_{\mathbb{P}_s(A)} q$.

PROPOSITION 8.2.4. *Let α be a regular uncountable cardinal with $\alpha = \alpha^{<\alpha}$.*

(1) *$\mathbb{C}_\alpha$ is $<\alpha$-closed.*
(2) *A filter G is $\mathbb{C}_\alpha$-generic over V if and only if there is an α-coding basis $\langle A, s\rangle$ coding a well-order of ${}^\alpha\alpha$ in V and a filter H in $\mathbb{P}_s(A)$ that is generic over V and satisfies*

$$G = \{\langle \emptyset, \emptyset, \emptyset\rangle\} \cup \{\langle A, s, p\rangle \in \mathbb{C}_\alpha \mid p \in H\}. \tag{8.1}$$

In particular, $\mathrm{V}[G] = \mathrm{V}[H]$ holds in the above situation, forcing with $\mathbb{C}_\alpha$ preserves cofinalities, cardinalities and 2^α and every set of ordinals of cardinality at most α in a $\mathbb{C}_\alpha$-generic extension of the ground model V is covered by a set that is an element of V and has cardinality α in V.
(3) *If G is $\mathbb{C}_\alpha$-generic over V, then there is a well-order of $\mathrm{H}_{\alpha^+}^{\mathrm{V}[G]}$ that is definable in $\langle \mathrm{H}_{\alpha^+}^{\mathrm{V}[G]}, \in\rangle$ by a formula with parameters.* □

Note that $\mathbb{C}_\alpha$ is uniformly definable in parameter α.

8.3. Iterated coding forcing

In this section, we use the coding forcing developed above in an iterated forcing construction. Our account of iterated forcing follows [**Bau83**] and [**Cum10**] and we will repeatedly use results proved there.

By the results of the last section, there is a unique forcing iteration

$$\langle\langle\vec{\mathbb{C}}_{<\alpha} \mid \alpha \in \mathrm{On}\rangle, \langle\dot{\mathbb{C}}_\alpha \mid \alpha \in \mathrm{On}\rangle\rangle$$

with *Easton support* (see [**Cum10**, Definition 7.5]) satisfying the following properties.

(1) If $\beta < \alpha$ and α is inaccessible, then $\vec{\mathbb{C}}_{<\beta}, \dot{\mathbb{C}}_\beta \in \mathrm{V}_\alpha$.
(2) If α is not an inaccessible cardinal, then $\mathbb{1}_{\vec{\mathbb{C}}_{<\alpha}} \Vdash$ "$\dot{\mathbb{C}}_\alpha$ *is trivial*".
(3) If α is an inaccessible cardinal, then $\mathbb{1}_{\vec{\mathbb{C}}_{<\alpha}} \Vdash$ "$\dot{\mathbb{C}}_\alpha = \mathbb{C}_{\check{\alpha}}$".

For all $\nu \leq \mu$, we let $\dot{\mathbb{C}}_{[\nu,\mu)}$ denote the canonical $\vec{\mathbb{C}}_{<\nu}$-name with

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\nu}} \Vdash \text{“}\dot{\mathbb{C}}_{[\nu,\beta)} \textit{ is a partial order with domain } \{\vec{p} \restriction [\check{\nu}, \check{\mu}) \mid \vec{p} \in \check{\vec{\mathbb{C}}}_{<\mu}\}\text{”}$$

such that there is a dense embedding $e_{[\nu,\mu)} : \vec{\mathbb{C}}_{<\mu} \longrightarrow \vec{\mathbb{C}}_{<\nu} * \dot{\mathbb{C}}_{[\nu,\mu)}$ with $e_{[\nu,\mu)}(\vec{p}) = \langle \vec{p} \restriction \nu, \dot{q}\rangle$ and $\mathbb{1}_{\vec{\mathbb{C}}_{<\nu}} \Vdash$ "$\dot{q} = \check{\vec{p}} \restriction [\check{\nu}, \check{\mu})$" (see [**Bau83**, Section 5]).

Proposition 8.3.1. *Let $\alpha < \mu$ and μ be a regular cardinal. Assume that there are no inaccessible cardinals in (α, μ) and $\vec{\mathbb{C}}_{<\alpha+1}$ has the property that every set of ordinals of cardinality less than μ in a $\vec{\mathbb{C}}_{<\alpha+1}$-generic extension of the ground model is covered by a set of cardinality less than μ in the ground model. Then*

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\alpha+1}} \Vdash \text{“}\dot{\mathbb{C}}_{[\alpha+1,\nu)} \textit{ is } {<}\check{\mu}\textit{-closed”}$$

for all $\nu > \alpha$.

Proof. For all $\alpha < \beta < \mu$, we have $\mathbb{1}_{\vec{\mathbb{C}}_{<\beta}} \Vdash$ "$\dot{\mathbb{C}}_\beta$ *is trivial*" by the definition of $\vec{\mathbb{C}}_{<\nu}$ and our assumptions on μ. This shows that

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\beta}} \Vdash \text{“}\dot{\mathbb{C}}_\beta \textit{ is } {<}\check{\mu}\textit{-closed”}$$

holds for all $\beta > \alpha$. Moreover, $\vec{\mathbb{C}}_{<\beta}$ is an inverse limit for every limit ordinal $\beta > \alpha$ with $\mathrm{cof}(\beta) < \mu$. We can apply [**Cum10**, Proposition 7.12] to deduce the statement of the claim. □

Proposition 8.3.2. *If α is an inaccessible cardinal, then $\vec{\mathbb{C}}_{<\alpha}$ preserves the inaccessibility of α.*

Proof. Let G be $\vec{\mathbb{C}}_{<\alpha}$-generic over V. Fix $\beta < \alpha$ and let $G_{\beta+1}$ denote the corresponding filter in $\vec{\mathbb{C}}_{<\beta+1}$. If $\mu = (|\vec{\mathbb{C}}_{<\beta}|^+ + |\beta|)^+ < \alpha$, then there are no inaccessible cardinals in (β, μ) and $\dot{\mathbb{C}}^{\mathrm{V}[G_{\beta+1}]}_{[\beta+1,\alpha)}$ is ${<}\beta^+$-closed by Proposition 8.3.1. This shows $({}^\beta\alpha)^{\mathrm{V}[G]} \subseteq \mathrm{V}[G_{\beta+1}]$. Since $\vec{\mathbb{C}}_{<\beta+1} \in \mathrm{V}_\alpha$ and α is inaccessible in $\mathrm{V}[G_{\beta+1}]$, the statement of the claim follows directly. □

PROPOSITION 8.3.3. *$\vec{\mathbb{C}}_{<\nu}$ preserves the inaccessibility of all inaccessible cardinals.*

PROOF. By Proposition 8.3.2 and our assumptions, $\vec{\mathbb{C}}_{<\nu}$ preserves the cofinality, cardinality and inaccessibility of all inaccessible cardinals greater or equal to ν.

Let $\alpha < \nu$ be an inaccessible cardinal. By Proposition 8.3.2, $\vec{\mathbb{C}}_{<\alpha}$ preserves the inaccessibility of α and $\mathbb{1}_{\vec{\mathbb{C}}_{<\alpha}} \Vdash$ "$\dot{\mathbb{C}}_\alpha$ *is not trivial*". Proposition 8.2.4 shows that $\vec{\mathbb{C}}_{<\alpha+1}$ preserves the inaccessibility of α. If $\mu = (|\vec{\mathbb{C}}_{<\alpha+1}|^+ + \alpha)^+$, then there are no inaccessible cardinals in (α, μ) and

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\alpha+1}} \Vdash \text{“}\dot{\mathbb{C}}_{[\alpha+1,\nu)} \textit{ is } {<}\check{\mu}\textit{-closed”}.$$

In particular, $\vec{\mathbb{C}}_{<\nu}$ preserves the inaccessibility of α. □

LEMMA 8.3.4. *Let $\alpha < \nu$ and α be an inaccessible cardinal. If G is $\vec{\mathbb{C}}_{<\nu}$-generic over V, $\bar{G}$ is the corresponding filter in $\vec{\mathbb{C}}_{<\alpha}$ and G_α is the corresponding filter in $\dot{\mathbb{C}}_\alpha^{\bar{G}}$, then the following statements hold.*

1. $(2^\alpha)^{\mathrm{V}[G]} = (2^\alpha)^{\mathrm{V}}$.
2. $\dot{\mathbb{C}}_\alpha^{\bar{G}} = \mathbb{C}_\alpha^{\mathrm{V}[\bar{G}]}$ *is not the trivial partial order.*
3. *If $\langle A, s\rangle$ is an α-coding basis coding a well-order of ${}^\alpha\alpha$ in $\mathrm{V}[\bar{G}]$ with $\langle A, s, \mathbb{1}_{\mathbb{P}_s(A)}\rangle \in G_\alpha$, then A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\alpha\alpha$ in $\mathrm{V}[G]$ and there is a well-order of $\mathrm{H}_{\alpha^+}^{\mathrm{V}[G]}$ that is definable in $\langle \mathrm{H}_{\alpha^+}^{\mathrm{V}[G]}, \in\rangle$ by a formula with parameters.*

PROOF. It follows directly from the definition of the forcing iteration that the partial order $\vec{\mathbb{C}}_{<\alpha}$ has cardinality α. This implies $(2^\alpha)^{\mathrm{V}[\bar{G}]} = (2^\alpha)^{\mathrm{V}}$ and we can apply Lemma 7.2.3 and Proposition 8.2.4 to conclude $(2^\alpha)^{\mathrm{V}[\bar{G}][G_\alpha]} = (2^\alpha)^{\mathrm{V}}$. By Proposition 8.3.2, α is an inaccessible cardinal in $\mathrm{V}[\bar{G}]$ and there is an α-coding basis $\langle A, s\rangle$ in $\mathrm{V}[\bar{G}]$ such that $\langle A, s, \mathbb{1}_{\mathbb{P}_s(A)}\rangle \in G_\alpha$. The proof of Theorem 7.1.3 shows that A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\alpha\alpha$ in $\mathrm{V}[\bar{G}][G_\alpha]$ and, by Corollary 8.2.2, there is a well-order of $\mathrm{H}_{\alpha^+}^{\mathrm{V}[\bar{G}][G_\alpha]}$ that is definable in the structure $\langle \mathrm{H}_{\alpha^+}^{\mathrm{V}[\bar{G}][G_\alpha]}, \in\rangle$ by a formula with parameters. As above, it is easy to show that $\dot{\mathbb{C}}_{[\alpha+1,\nu)}^{\bar{G}*G_\alpha}$ adds no new α-sequences of ordinals. We can conclude $(2^\alpha)^{\mathrm{V}[G]} = (2^\alpha)^{\mathrm{V}}$, $({}^\alpha\alpha)^{\mathrm{V}[G]} = ({}^\alpha\alpha)^{\mathrm{V}[\bar{G}*G_\alpha]}$ and $\mathrm{H}_{\alpha^+}^{\mathrm{V}[G]} = \mathrm{H}_{\alpha^+}^{\mathrm{V}[\bar{G}][G_\alpha]}$. □

PROPOSITION 8.3.5. *Let κ be an infinite cardinal with the property that $\kappa \notin (\nu^+, 2^\nu]$ holds whenever ν is a singular limit of inaccessible cardinals. Given $\mu > \kappa$, $\vec{\mathbb{C}}_{<\mu}$ preserves the cardinality of κ and, if κ is regular, then $\vec{\mathbb{C}}_{<\mu}$ preserves the regularity of κ.*

PROOF. By Proposition 8.3.3, we may assume that κ is not inaccessible. Let

$$\nu = \sup\{\alpha < \kappa \mid \alpha \textit{ is an inaccessible cardinal}\}.$$

If $\nu = 0$ or ν is inaccessible, then $\nu < \kappa$, $\vec{\mathbb{C}}_{<\nu+1}$ satisfies the κ-chain condition and

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\nu+1}} \Vdash \text{“}\dot{\mathbb{C}}_{[\nu+1,\mu)} \textit{ is } {<}\check{\kappa}^{+}\textit{-closed”}$$

holds by Proposition 8.3.1.

If ν is singular and $\kappa = \nu$, then κ is a limit of inaccessible cardinals and $\vec{\mathbb{C}}_{<\mu}$ preserves the cardinality of κ by Proposition 8.3.3.

Let ν be singular and $\kappa = \nu^{+}$. Assume, toward a contradiction, that κ has cardinality less or equal to ν in some $\vec{\mathbb{C}}_{<\mu}$-generic extension $\mathrm{V}[G]$ of the ground model. Then there is an inaccessible cardinal α such with $\mathrm{cof}(\kappa)^{\mathrm{V}[G]} < \alpha < \nu$. If $\bar{G}$ is the filter in $\vec{\mathbb{C}}_{<\alpha+1}$ induced by G, then $\mathrm{cof}(\kappa)^{\mathrm{V}[\bar{G}]} < \kappa$, because $\dot{\mathbb{C}}^{\bar{G}}_{[\alpha+1,\mu)}$ is ${<}\alpha$-closed by Proposition 8.3.1. But $\vec{\mathbb{C}}_{<\alpha+1}$ satisfies the κ-chain condition, a contradiction. This shows that $\vec{\mathbb{C}}_{<\mu}$ preserves the cardinality and cofinality of ν^{+}.

If ν is singular and $\kappa > 2^{\nu}$, then $\vec{\mathbb{C}}_{<\nu+1}$ satisfies the κ-chain condition and $\mathbb{1}_{\vec{\mathbb{C}}_{<\nu+1}} \Vdash$ “$\dot{\mathbb{C}}_{[\nu+1,\mu)}$ *is* ${<}\check{\kappa}^{+}$*-closed*” holds by Proposition 8.3.1. □

8.4. Preserving supercompactness

This section is devoted to the proof of the following theorem.

THEOREM 8.4.1. *Let γ be a cardinal with $2^{\gamma} = \gamma^{+}$ and $2^{\nu} \leq \gamma$, where*

$$\nu = \sup\{\alpha \leq \gamma \mid \alpha \textit{ is an inaccessible cardinal}\}.$$

If κ is γ-supercompact with $\gamma = \gamma^{<\kappa}$, then

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\lambda}} \Vdash \textit{“}\check{\kappa} \textit{ is } \check{\gamma}\textit{-supercompact”}$$

holds for all $\lambda > \nu$.

The following structural property of the coding forcing $\mathbb{P}_s(A)$ will be essential in our proof of supercompactness preservation.

LEMMA 8.4.2. *Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$ and $\langle A, s\rangle$ be a κ-coding basis. Assume $P \subseteq \mathbb{P}_s(A)$ satisfies the following properties.*

(1) $\eta = \mathrm{lub}\{\mathrm{ht}(T_p) \mid p \in P\} \in \mathrm{Lim} \cap \kappa$.
(2) $D = \bigcup\{\mathrm{dom}(f_p) \mid p \in P\}$ *has cardinality less than* κ.
(3) *If $p_0, p_1 \in P$, then there is $q \in P$ with $q \leq_{\mathbb{P}_s(A)} p_0, p_1$.*

Then there is a unique condition $p_P \in \mathbb{P}_s(A)$ with $\mathrm{ht}(T_{p_P}) = \eta$, $\mathrm{dom}(f_{p_P}) = D$ and $p_P \leq_{\mathbb{P}_s(A)} p$ for all $p \in P$.

PROOF. Set $T = \bigcup\{T_p \mid p \in P\}$. Then T is a tree of height η and an end-extension of T_p for all $p \in P$. If we define

$$F : D \longrightarrow [T];\ x \longmapsto \bigcup\{f_p(x) \mid p \in P,\ x \in \mathrm{dom}(f_p)\} \in [T],$$

then this is a well-defined function. Moreover, for all $x \in D$ there is a unique $H(x) < \kappa$ with $h_p(x) = H(x)$ for all $p \in P$ with $x \in \mathrm{dom}(f_p)$ and we can define $H : D \longrightarrow \kappa$ in this way.

If $x \in D$ and $\alpha, \beta < \eta$ with $\alpha = \prec H(x), \beta \succ$, then there is $p \in P$ with $x \in \mathrm{dom}(f_p)$ and $\alpha, \beta < \mathrm{ht}(T_p)$. We can conclude

$$s(\beta) \subseteq x \iff f_p(x)(\alpha) = 1 \iff F(x)(\alpha) = 1.$$

This shows that $p_P = \langle T, F, H \rangle$ is a condition in $\mathbb{P}$ with $p_P \leq p$ for all $p \in P$.

Let $q \in \mathbb{P}_s(A)$ be a condition with $\mathrm{ht}(T_q) = \eta$, $\mathrm{dom}(f_q) = D$ and $q \leq_{\mathbb{P}_S(A)} p$ for all $p \in P$. Since $\eta \in \mathrm{Lim}$, for every $t \in T_q$ there is a $p \in P$ with $\mathrm{lh}(t) < \mathrm{ht}(T_p)$ and therefore $t \in T_p$. This shows $T_q = \bigcup\{T_p \mid p \in P\} = T$. In the same way, we can show $f_q(x) = \bigcup\{f_p(x) \mid p \in P,\ x \in \mathrm{dom}(f_p)\} = F(x)$ and $h_q(x) = H(x)$ for all $x \in D$. This means $q = p_P$. □

Proof of Theorem 8.4.1. By our assumptions, $\mathrm{cof}(\gamma) \geq \kappa$ and $\nu \in [\kappa, \gamma)$ is a strong limit cardinal.

Let U be a normal ultrafilter on $\mathcal{P}_\kappa(\gamma)$. We will prove a number of claims that will allow us to show that κ is γ-supercompact in every $\vec{\mathbb{C}}_{<\nu+1}$-generic extension of the ground model. Given $\alpha \leq \beta \in \mathrm{On}$, we define $\vec{\mathbb{Q}}_{<\alpha} = \vec{\mathbb{C}}^{M_U}_{<\alpha}$, $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{C}}^{M_U}_\beta$ and $\dot{\mathbb{Q}}_{[\alpha,\beta)} = \dot{\mathbb{C}}^{M_U}_{[\alpha,\beta)}$.

Since ν is either an inaccessible cardinal or a limit of inaccessible cardinals, we have $\vec{\mathbb{C}}_{<\alpha} \in \mathrm{V}_\nu \subseteq M_U$ for all $\alpha < \nu$ and this shows $\vec{\mathbb{C}}_{<\nu} \in M_U$, because ${}^\gamma M_U \subseteq M_U$ holds. The definition of $\vec{\mathbb{C}}_{<\alpha}$ is absolute between V and M_U for every $\alpha \leq \nu$. Hence elementarity implies $\vec{\mathbb{C}}_{<\nu} = \vec{\mathbb{Q}}_{<\nu}$. In particular, if $\bar{G}$ is $\vec{\mathbb{C}}_{<\nu}$-generic over V, then $\bar{G}$ is $\vec{\mathbb{Q}}_{<\nu}$-generic over M_U.

Claim 1. *If $\bar{G}$ is $\vec{\mathbb{C}}_{<\nu}$-generic over* V*, then* $({}^\gamma M_U[\bar{G}])^{\mathrm{V}[\bar{G}]} \subseteq M_U[\bar{G}]$.

Proof of the claim. Let $x \in \mathrm{V}[\bar{G}]$ with $x \subseteq \gamma$. We can find a $\vec{\mathbb{C}}_{<\nu}$-nice name $\tau = \bigcup_{\alpha<\gamma}\{\check{\alpha}\} \times A_\alpha$ with $x = \tau^{\bar{G}}$. By the above remarks, we have $\vec{\mathbb{C}}_{<\nu} \subseteq {}^\nu\mathrm{V}_\nu$ and every A_α has cardinality at most $2^\nu \leq \gamma$. This shows that every A_α is an element of M_U and we also get $\langle A_\alpha \mid \alpha < \gamma \rangle \in M_U$. Hence $\tau \in M_U$ and $x = \tau^{\bar{G}} \in M_U[\bar{G}]$. We can conclude $({}^\gamma 2)^{\mathrm{V}[\bar{G}]} \subseteq M_U[\bar{G}]$.

Let $X \in \mathrm{V}[\bar{G}]$ with $X \subseteq \mathrm{On}$ and $|X|^{\mathrm{V}[\bar{G}]} \leq \gamma$. Since $\vec{\mathbb{C}}_{<\nu}$ satisfies the γ-chain condition in V, there is an $X_0 \in \mathrm{V}$ with $X \subseteq X_0$ and $|X_0|^{\mathrm{V}} \leq \gamma$. By our assumptions, $X_0 \in M_U$ and $|X_0|^{M_U} \leq \gamma$. Let $\langle \eta_\alpha \mid \alpha < \gamma \rangle$ be an enumeration of X_0 in M_U and $x = \{\alpha < \gamma \mid \eta_\alpha \in X\} \in \mathrm{V}[\bar{G}]$. By the above argument, $x \in M_U[\bar{G}]$ and this shows $X \in M_U[\bar{G}]$.

The argument shows $({}^\gamma\mathrm{On})^{\mathrm{V}[\bar{G}]} \subseteq M_U[\bar{G}]$ and this implies the statement of the claim, because $M_U[\bar{G}]$ is a transitive ZFC-model with $\mathrm{On} \subseteq M_U[\bar{G}] \subseteq \mathrm{V}[\bar{G}]$. □

Claim 2. *If $\bar{G}$ is $\vec{\mathbb{C}}_{<\nu}$-generic over* V*, then* $\dot{\mathbb{C}}^{\bar{G}}_\nu = \dot{\mathbb{Q}}^{\bar{G}}_\nu$.

Proof of the claim. If ν is not an inaccessible cardinal in V, then ν is not inaccessible in M_U and both partial orders are trivial.

Now, assume that ν is inaccessible in V and M_U. By Lemma 8.3.4, $(2^\nu)^{\mathrm{V}[\bar{G}]} = (2^\nu)^{\mathrm{V}} \leq \gamma$ and Claim 1 implies $\mathcal{P}({}^\nu\nu)^{\mathrm{V}[\bar{G}]} = \mathcal{P}({}^\nu\nu)^{M_U[\bar{G}]}$. This allows us to conclude $\dot{\mathbb{C}}^{\bar{G}}_\nu = \mathbb{C}^{\mathrm{V}[\bar{G}]}_\nu = \mathbb{C}^{M_U[\bar{G}]}_\nu = \dot{\mathbb{Q}}^{\bar{G}}_\nu$. □

In particular, if G is $\vec{\mathbb{C}}_{<\nu+1}$-generic over V, then G is $\vec{\mathbb{Q}}_{<\nu+1}$-generic over M_U.

Claim 3. *If G is $\vec{\mathbb{C}}_{<\nu+1}$-generic over* V*, then* $({}^{\gamma}M_U[G])^{\mathrm{V}[G]} \subseteq M_U[G]$.

Proof of the claim. Let $\bar{G}$ be the filter in $\vec{\mathbb{C}}_{<\nu}$ corresponding to G and G_ν be the filter in $\dot{\mathbb{C}}_\nu^{\bar{G}}$ corresponding to G. By Proposition 8.2.4 and the above claims, there is a partial order $\mathbb{P}$ in $M_U[\bar{G}]$ and $H \in M_U[G]$ such that $\mathbb{P}$ satisfies the ν^+-chain condition in $\mathrm{V}[\bar{G}]$, H is $\mathbb{P}$-generic over $\mathrm{V}[\bar{G}]$ and H induces G_ν as in (8.1). Every anti-chain in $\mathbb{P}$ in $\mathrm{V}[\bar{G}]$ has cardinality at most γ in $\mathrm{V}[\bar{G}]$ and $({}^{\gamma}M_U[\bar{G}])^{\mathrm{V}[\bar{G}]} \subseteq M_U[\bar{G}]$, we can repeat the proof of Claim 1 and deduce the statement of the claim. □

The proofs of the above claims show that every set of ordinals of cardinality at most γ in a $\vec{\mathbb{C}}_{<\nu+1}$-generic extension of V is covered by a set of cardinality γ in V. By our assumptions, this implies that every set of ordinals of cardinality at most γ in a $\vec{\mathbb{Q}}_{<\nu+1}$-generic extension of M_U is covered by a set of cardinality γ in M_U. In particular, forcing with $\vec{\mathbb{Q}}_{<\nu+1}$ preserves $(\gamma^+)^{M_U} = (\gamma^+)^{\mathrm{V}}$.

Claim 4. *If G is $\vec{\mathbb{C}}_{<\nu+1}$-generic over* V*, then $\dot{\mathbb{Q}}^G_{[\nu+1,\mu)}$ is $<\gamma^+$-closed in $M_U[G]$ for all $\mu > \nu$ and the power set of $\dot{\mathbb{Q}}^G_{[\nu+1,j_U(\nu))}$ in $M_U[G]$ has cardinality at most γ^+ in* $\mathrm{V}[G]$.

Proof of the claim. In M_U, the interval (ν,γ^+) contains no inaccessible cardinals, because ${}^{\gamma}M_U \subseteq M_U$ holds and no ordinal in this interval is inaccessible in V. By the above remark and an application of Proposition 8.3.1 in M_U, we can conclude that $\dot{\mathbb{Q}}^G_{[\nu+1,\mu)}$ is $<\gamma^+$-closed in $M_U[G]$ for all $\mu > \nu$.

By the definition of the partial order $\dot{\mathbb{C}}_{[\alpha,\beta)}$ and elementarity, the cardinality of $\dot{\mathbb{Q}}^G_{[\nu+1,j_U(\nu))}$ in $M_U[G]$ is less or equal to the cardinality of $\vec{\mathbb{Q}}_{<j_U(\nu)}$ in M_U. The above computations and elementarity show that the cardinality of $\vec{\mathbb{Q}}_{<j_U(\nu)}$ in M_U is at most $j_U(2^\nu)$ and this ordinal is smaller or equal to $j_U(\gamma)$. If $\alpha < j_U(\gamma)$, then α is represented in M_U by a function $f : \mathcal{P}_\kappa(\gamma) \longrightarrow \gamma$ contained in V. By our assumptions, $\mathcal{P}_\kappa(\gamma)$ has cardinality γ in V and there are at most 2^γ-many such functions in V. Since $2^\gamma = \gamma^+$ holds in V and $(\gamma^+)^{\mathrm{V}[G]} = (\gamma^+)^{\mathrm{V}}$, this shows that $j_U(\gamma)$ has cardinality at most γ^+ in $\mathrm{V}[G]$. □

Since $\vec{\mathbb{C}}_{<\nu} \in M_U$ has cardinality at most γ in V, we have $j_U \restriction \vec{\mathbb{C}}_{<\nu} \in M_U$ and there is a sequence

$$\langle \dot{G}_\alpha \in (\mathrm{V}^{\vec{\mathbb{Q}}_{<\nu}})^{M_U} \mid j_U(\kappa) \leq \alpha < j_U(\nu)\rangle$$

of names in M_U with the property that $\dot{G}_\alpha^{\bar{G}} = \{j_U(\vec{p}) \restriction \alpha \mid \vec{p} \in \bar{G}\}$ for all $\alpha \in [j_U(\kappa), j_U(\nu))$ whenever $\bar{G}$ is $\vec{\mathbb{Q}}_{<\nu}$-generic over M_U.

CLAIM 5. *Let $\alpha \in [j_U(\kappa), j_U(\nu))$ be an inaccessible cardinal in M_U, H be $\vec{\mathbb{Q}}_{<\alpha}$-generic over M_U and $\bar{G}$ be the filter in $\vec{\mathbb{Q}}_{<\nu}$ induced by H. If $\dot{G}^{\bar{G}}_\alpha \subseteq H$ and $j_U(\vec{p})(\alpha)^H \neq \mathbb{1}_{\mathbb{C}_\alpha^{M_U[H]}}$ for some $\vec{p} \in \bar{G}$, then the following statements hold.*

(1) *There is a unique α-coding basis $\langle A_\alpha, s_\alpha\rangle$ coding a well-order of ${}^\alpha\alpha$ in $M_U[H]$ such that for all $\vec{p} \in \bar{G}$ with $j_U(\vec{p})(\alpha)^H \neq \mathbb{1}_{\mathbb{C}_\alpha^{M_U[H]}}$ there is a $q \in \mathbb{P}_{s_\alpha}(A_\alpha)^{M_U[H]}$ with $j_U(\vec{p})(\alpha)^H = \langle A_\alpha, s_\alpha, q\rangle$.*

(2) *The set*

$$P_\alpha = \{q \in \mathbb{P}_{s_\alpha}(A_\alpha)^{M_U[H]} \mid (\exists \vec{p} \in \bar{G})\ j_U(\vec{p})(\alpha)^H = \langle A_\alpha, s_\alpha, q\rangle\}$$

satisfies the statements (i)-(iii) *of Lemma 8.4.2 in $M_U[H]$.*

PROOF OF THE CLAIM. If $\vec{p} \in \bar{G}$ and $\beta < \nu$, then $\mathbb{1}_{\vec{\mathbb{C}}_{<\beta}} \Vdash$ "$\vec{p}(\beta) \in \dot{\mathbb{C}}_\beta$". By elementarity, we have $\mathbb{1}_{\vec{\mathbb{Q}}_{<\alpha}} \Vdash$ "$j_U(\vec{p})(\alpha) \in \dot{\mathbb{Q}}_\alpha$" and, by Proposition 8.3.2, this implies

$$Q_\alpha = \{j_U(\vec{p})(\alpha)^H \mid \vec{p} \in \bar{G}\} \subseteq \dot{\mathbb{Q}}_\alpha^H = \mathbb{C}_\alpha^{M_U[H]}.$$

Given $\vec{p}_0, \vec{p}_1 \in \bar{G}$, there is a $\vec{p} \in \bar{G}$ with $\vec{p} \leq_{\vec{\mathbb{C}}_{<\nu}} \vec{p}_0, \vec{p}_1$ and hence

$$\vec{p} \restriction \beta \Vdash \text{“}\vec{p}(\beta) \leq_{\dot{\mathbb{P}}_\beta} \vec{p}_0(\beta), \vec{p}_1(\beta)\text{”}$$

for all $\beta < \nu$. Since $j_U(\vec{p}) \restriction \alpha \in \dot{G}^{\bar{G}}_\alpha \subseteq H$, this argument shows that the elements of Q_α are pairwise compatible.

Pick $\vec{p}_* \in \bar{G}$ with $j_U(\vec{p}_*)(\alpha)^H \neq \mathbb{1}_{\mathbb{C}_\alpha^{M_U[H]}}$ and define $\langle A_\alpha, s_\alpha\rangle \in M_U[H]$ to be the unique α-coding basis coding a well-order of ${}^\alpha\alpha$ with $j_U(\vec{p}_*)(\alpha)^H = \langle A_\alpha, s_\alpha, q\rangle$ for some condition $q \in \mathbb{P}_{s_\alpha}(A_\alpha)^{M_U[H]}$. By the above computations, every element of Q_α is either of the form $\mathbb{1}_{\mathbb{C}_\alpha^{M_U[H]}}$ or $\langle A_\alpha, s_\alpha, q\rangle$ for some $q \in \mathbb{P}_{s_\alpha}(A_\alpha)^{M_U[H]}$.

Since $\bar{G}$ has cardinality at most γ in $M_U[H]$, $\gamma < j_U(\kappa) \leq \alpha$ and α is regular in $M_U[H]$, we know that $\eta = \operatorname{lub}\{\operatorname{ht}(T_q) \mid q \in P_\alpha\} < \alpha$ and $\bigcup\{\operatorname{dom}(f_q) \mid q \in P_\alpha\}$ has cardinality less than α in $M_U[H]$.

We show that $\eta \in \operatorname{Lim} \cap \alpha$. Let $\vec{p} \in \bar{G}$ and $p \in \mathbb{P}_{s_\alpha}(A_\alpha)^{M_U[H]}$ with $\langle A_\alpha, s_\alpha, p\rangle = j_U(\vec{p})(\alpha)^H \neq \mathbb{1}_{\mathbb{C}_\alpha^{M_U[H]}}$. Let D be the set consisting of all conditions $\vec{q} \in \vec{\mathbb{C}}_{<\nu}$ with $\vec{q} \leq_{\vec{\mathbb{C}}_{<\nu}} \vec{p}$ and

$$\begin{aligned} \vec{q} \restriction \beta \Vdash \text{“}(\forall A, s, p)\ \big[\big(\dot{\mathbb{C}}_\beta = \mathbb{C}_{\check{\beta}} \wedge \vec{p}(\beta) = \langle A, s, p\rangle \neq \mathbb{1}_{\mathbb{C}_{\check{\beta}}}\big) \\ \longrightarrow (\exists \bar{p})[\vec{q}(\beta) = \langle A, s, \bar{p}\rangle \wedge \operatorname{ht}(T_p) < \operatorname{ht}(T_{\bar{p}})]\big]\text{”} \end{aligned}$$

for all $\beta < \nu$. An easy inductive construction using Lemma 7.2.4 shows that D is dense below $\vec{p}$ in V. If $\vec{q} \in D \cap \bar{G}$ with $j_U(\vec{q})(\alpha)^H = \langle A_\alpha, s_\alpha, q\rangle$, then $\operatorname{ht}(T_q) > \operatorname{ht}(T_p)$ holds in $M_U[H]$ by elementarity. This shows that η is a limit ordinal.

Finally, the conditions in P_α are pairwise compatible, because the conditions in Q_α are pairwise compatible and the first part of the claim shows that every condition in P_α belongs to a condition in Q_α. □

In M_U, we define a sequence $\vec{q}_* = \langle \dot{q}_\alpha \in (\mathrm{V}^{\vec{\mathbb{Q}}_{<\alpha}})^{M_U} \mid \alpha < j_U(\nu)\rangle$ such that the following statements hold in M_U for all $\alpha < j_U(\nu)$.

(1) If $\alpha < j_U(\kappa)$ or α is not an inaccessible cardinal, then
$$\mathbb{1}_{\vec{\mathbb{Q}}_{<\alpha}} \Vdash \text{“}\dot{q}_\alpha = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\alpha}\text{”}.$$

(2) If α is an inaccessible cardinal in $[j_U(\kappa), j_U(\nu))$, then $\dot{q}_\alpha$ is a canonical $\vec{\mathbb{Q}}_{<\alpha}$-name τ such that the following statements hold whenever H is $\vec{\mathbb{Q}}_{<\alpha}$-generic over M_U and $\bar{G}$ is the filter in $\vec{\mathbb{Q}}_{<\nu}$ induced by H.
   (a) If $\dot{G}_\alpha^{\bar{G}} \subseteq H$ and $j_U(\vec{p})(\alpha)^H \neq \mathbb{1}_{\dot{\mathbb{Q}}_\alpha^H}$ for some $\vec{p} \in \bar{G}$, then $\tau^H = \langle A_\alpha, s_\alpha, p_{P_\alpha}\rangle$, where A_α, s_α and P_α are defined as in Claim 5 and p_{P_α} is defined as in Lemma 8.4.2.
   (b) Otherwise, $\tau^H = \mathbb{1}_{\dot{\mathbb{Q}}_\alpha^H}$.

Claim 6. $\vec{q}_* \in \vec{\mathbb{Q}}_{<j_U(\nu)}$.

Proof of the claim. Let $\alpha \in [j_U(\kappa), j_U(\nu))$ be a regular cardinal in M_U. For all $\vec{p} \in \vec{\mathbb{C}}_{<\nu}$ there is an $\bar{\alpha}_{\vec{p}} < \alpha$ with $j_U(\vec{p})(\beta) = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta}$ for all $\bar{\alpha}_{\vec{p}} \leq \beta < \alpha$. Since j_U"$\vec{\mathbb{C}}_{<\nu}$ is an element of M_U and has cardinality less than α in M_U, we can find an $\bar{\alpha} \in (j_U(\kappa), \alpha)$ with $j_U(\vec{p})(\beta) = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta}$ for all $\vec{p} \in \vec{\mathbb{C}}_{<\nu}$ and $\bar{\alpha} \leq \beta < \alpha$. If $\beta \in (\bar{\alpha}, \alpha)$ is an inaccessible cardinal, H is $\vec{\mathbb{Q}}_{<\alpha}$-generic over M_U and $\bar{G}$ is the filter in $\vec{\mathbb{Q}}_{<\nu}$ induced by H, then $j_U(\vec{p})(\beta)^H = \mathbb{1}_{\dot{\mathbb{Q}}_\beta^H}$ for all $\vec{p} \in \bar{G}$ and $\dot{q}_\beta^H = p_{P_\beta} = \mathbb{1}_{\dot{\mathbb{Q}}_\beta^H}$ by the uniqueness of p_{P_β}. By the definition of $\dot{q}_\beta$, this shows $\dot{q}_\beta = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta}$. Therefore $\vec{q}_*$ is a sequence with Easton support. □

Claim 7. *If H is $\vec{\mathbb{Q}}_{<j_U(\nu)}$-generic over M_U with $\vec{q}_* \in H$ and $\bar{G}$ is the corresponding filter in $\vec{\mathbb{Q}}_{<\nu}$, then j_U"$\bar{G} \subseteq H$.*

Proof of the claim. Let $\alpha \in [\nu, j_U(\nu))$ and F be $\vec{\mathbb{Q}}_{<\alpha}$-generic over M_U with $\vec{q}_* \restriction \alpha \in F$. Assume that F induces $\bar{G}$ in $\vec{\mathbb{Q}}_{<\nu}$ and
$$\vec{q}_* \restriction [\nu, \alpha) \leq_{\dot{\mathbb{Q}}^{\bar{G}}_{[\nu,\alpha)}} j_U(\vec{p}) \restriction [\nu, \alpha) \tag{8.2}$$
holds for all $\vec{p} \in \bar{G}$. Pick $\vec{p} \in \bar{G}$. There is a $\bar{\kappa} < \kappa$ such that $\vec{p}(\beta) = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta}$ for all $\beta \in [\bar{\kappa}, \kappa)$ and
$$j_U(\vec{p})(\beta) = \begin{cases} \vec{p}(\beta), & \text{if } \beta < \bar{\kappa}, \\ \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta}, & \text{if } \bar{\kappa} \leq \beta < \nu. \end{cases}$$
by the definition of $\vec{\mathbb{C}}_{<\nu}$. In particular, $\vec{p} \leq_{\vec{\mathbb{Q}}_{<\nu}} j_U(\vec{p}) \restriction \nu$. By our assumption, there is a $\vec{p}_* \in \bar{G}$ with $\vec{p}_* \leq_{\vec{\mathbb{Q}}_{<\nu}} \vec{p}$ and
$$\vec{p}_* * (\vec{q}_* \restriction [\nu, \alpha)) \leq_{\vec{\mathbb{Q}}_{<\nu} * \dot{\mathbb{Q}}_{[\nu,\alpha)}} i_{[\nu,\alpha)}(j_U(\vec{p}) \restriction \alpha).$$
This implies $j_U(\vec{p}) \restriction \alpha \in F$ and hence $\dot{G}_\alpha^{\bar{G}} \subseteq F$.

Next, we show that (8.2) holds in $M_U[G]$ for all $\vec{p} \in \bar{G}$ and $\alpha \in [\nu, j_U(\nu)]$ by induction. The case "$\alpha = \nu$" is trivial and the case "$\alpha \in \mathrm{Lim}$" follows directly from the induction hypothesis.

Assume $\alpha = \bar{\alpha} + 1$ with $\bar{\alpha} \geq \nu$. We may assume that $\bar{\alpha}$ is an inaccessible cardinal in M_U. It suffices to show that

$$\vec{q}_*(\bar{\alpha})^F \leq_{\dot{\mathbb{Q}}^F_{\bar{\alpha}}} j_U(\vec{p})(\bar{\alpha})^F$$

holds in $M_U[F]$ whenever $\vec{p} \in \bar{G}$ and F is $\vec{\mathbb{Q}}_{<\bar{\alpha}}$-generic over M_U such that $\vec{q}_* \restriction \bar{\alpha} \in F$ and F induces $\bar{G}$ in $\vec{\mathbb{Q}}_{<\nu}$. We may assume that there is a $\vec{p} \in \bar{G}$ with $j_U(\vec{p})(\bar{\alpha})^F \neq \mathbb{1}_{\dot{\mathbb{Q}}^F_{\bar{\alpha}}}$. By the induction hypothesis and the above computations, we directly get $\dot{G}^{\bar{G}}_{\bar{\alpha}} \subseteq F$. The definition of $\vec{q}_*(\bar{\alpha})$ and Claim 5 imply

$$\vec{q}_*(\bar{\alpha})^F = \langle A_\alpha, s_\alpha, p_{P_{\bar{\alpha}}} \rangle \leq_{\dot{\mathbb{Q}}^F_{\bar{\alpha}}} j_U(\vec{p})(\bar{\alpha})^F$$

for all $\vec{p} \in \bar{G}$.

This induction shows that (8.2) holds if $\alpha = j_U(\nu)$ and $\vec{p} \in G$. This allows us to repeat the above computation and conclude j_U"$G \subseteq H$. □

Claim 8. $\mathbb{1}_{\vec{\mathbb{C}}_{<\nu+1}} \Vdash$ *"$\check{\kappa}$ is $\check{\gamma}$-supercompact".*

Proof of the claim. Let G be $\vec{\mathbb{C}}_{<\nu+1}$-generic over V, $\bar{G}$ be the corresponding filter in $\vec{\mathbb{C}}_{<\nu}$ and G_ν be the corresponding filter in $\dot{\mathbb{C}}^{\bar{G}}_\nu$. Claim 4 combined with Claim 3 shows that there is a $\bar{H} \in \mathrm{V}[G]$ such that $\vec{q}_* \in \bar{H}$, $\bar{H}$ is $\vec{\mathbb{Q}}_{<j_U(\nu)}$-generic over M_U and $\bar{H}$ induces G in $\vec{\mathbb{Q}}_{<\nu+1}$. By Claim 7, we have j_U"$\bar{G} \subseteq \bar{H}$ and we can apply [**Cum10**, Proposition 9.1] to define an elementary embedding $j : \mathrm{V}[\bar{G}] \longrightarrow M_U[\bar{H}]$ extending j_U in $\mathrm{V}[G]$ that by setting $j(\tau^{\bar{G}}) = j_U(\tau)^{\bar{H}}$ for all $\tau \in \mathrm{V}^{\vec{\mathbb{C}}_{<\nu}}$.

We show that there is a $H_* \in \mathrm{V}[G]$ such that H_* is $\dot{\mathbb{Q}}^{\bar{H}}_{j_U(\nu)}$-generic over M_U and j"$G_\nu \subseteq H_*$. We may assume that ν is an inaccessible cardinal. This implies $(2^\nu)^{\mathrm{V}[\bar{G}]} = (2^\nu)^{\mathrm{V}} \leq \gamma$. By Proposition 8.2.4, there is a ν-coding basis $\langle A, s \rangle \in \mathrm{V}[\bar{G}]$ coding a well-order of ${}^\nu\nu$ and a filter $F_\nu \in \mathrm{V}[G]$ such that F_ν is $\mathbb{P}_s(A)^{\mathrm{V}[\bar{G}]}$-generic over $\mathrm{V}[\bar{G}]$ and F_ν induces G_ν as in (8.1).

By Claim 3, we have $({}^\gamma\mathrm{On})^{\mathrm{V}[G]} \subseteq M_U[G] \subseteq M_U[\bar{H}] \subseteq \mathrm{V}[G]$ and this implies that $({}^\gamma M_U[\bar{H}])^{\mathrm{V}[G]} \subseteq M_U[\bar{H}]$ holds. In particular, both $\mathbb{P}_s(A)^{\mathrm{V}[\bar{G}]}$ and $j \restriction \mathbb{P}_s(A)^{\mathrm{V}[\bar{G}]}$ are elements of $M_U[\bar{H}]$, because $\mathbb{P}_s(A)^{\mathrm{V}[\bar{G}]}$ has cardinality at most γ in $\mathrm{V}[\bar{G}]$. If $j(\langle A, s \rangle) = \langle \bar{A}, \bar{s} \rangle$ and $P = j$"F_ν, then $\langle \bar{A}, \bar{s} \rangle$ is a $j_U(\nu)$-coding basis that codes a well-order of ${}^{j_U(\nu)}j_U(\nu)$ in $M_U[\bar{H}]$, $P \subseteq \mathbb{P}_{\bar{s}}(\bar{A})^{M_U[\bar{H}]}$ and $P \in M_U[\bar{H}]$, because F_ν is an element of $M_U[\bar{H}]$. As in the proof of Claim 5, the set P satisfies the statements (i)-(iii) of Lemma 8.4.2 in $M_U[\bar{H}]$ and we can find a condition $p_P \in \mathbb{P}_{\bar{s}}(\bar{A})^{M_U[\bar{H}]}$ as in the statement of the Lemma.

In $M_U[\bar{H}]$, $\mathbb{P}_{\bar{s}}(\bar{A})^{M_U[\bar{H}]}$ is $<\gamma^+$-closed and has cardinality at most $j_U(\gamma)$. By the proof of Claim 4, $j_U(\gamma)$ has cardinality at most γ^+ in $\mathrm{V}[G]$ and there is a $F_* \in \mathrm{V}[G]$ such that $p_P \in F_*$ and F_* is $\mathbb{P}_{\bar{s}}(\bar{A})^{M_U[\bar{H}]}$-generic over $M_U[\bar{H}]$. If $H_* \in \mathrm{V}[G]$ is the filter in $\mathbb{C}_{j_U(\nu)}^{M_U[\bar{H}]}$ corresponding to F_*, then H_* is $\dot{\mathbb{Q}}_{j_U(\nu)}^{\bar{H}}$-generic over $M_U[\bar{H}]$ and our construction ensures $j"G_\nu \subseteq H_*$. Another application of [**Cum10**, Proposition 9.1] to define an elementary embedding $j_* : \mathrm{V}[G] \longrightarrow M_U[\bar{H}][H_*]$ in $\mathrm{V}[G]$ that extends j. Since $({}^\gamma\mathrm{On})^{\mathrm{V}[G]} \subseteq M_U[\bar{H}][H_*] \subseteq \mathrm{V}[G]$, this argument shows that κ is γ-supercompact in $\mathrm{V}[G]$. □

CLAIM 9. *If $\lambda > \nu$, then $\mathbb{1}_{\vec{\mathbb{C}}_{<\lambda}} \Vdash$ "$\check{\kappa}$ is $\check{\gamma}$-supercompact".*

PROOF OF THE CLAIM. Let H be $\vec{\mathbb{C}}_{<\lambda}$-generic over V and G be the corresponding filter in $\vec{\mathbb{C}}_{<\nu+1}$. There are no inaccessible cardinals in (ν, γ^+) and the above computations show that $\vec{\mathbb{C}}_{<\nu+1}$ has the property that every set of ordinals of cardinality at most γ in a $\vec{\mathbb{C}}_{<\nu+1}$-generic extension of the ground model is covered by a set of cardinality γ in V. By Proposition 8.3.1, $\dot{\mathbb{C}}_{[\nu+1,\lambda)}^G$ is $<\gamma^+$-closed in $\mathrm{V}[G]$.

By Claim 8, there is a normal filter U^* on $\mathcal{P}_\kappa(\gamma)$ in $\mathrm{V}[G]$ and U^* is also a normal filter on $\mathcal{P}_\kappa(\gamma)$ in $\mathrm{V}[H]$, because $\mathrm{V}[H]$ is a $\dot{\mathbb{C}}_{[\nu+1,\lambda)}^G$-generic extension of $\mathrm{V}[G]$ and $<\gamma^+$-closed forcing preserve normal filters on $\mathcal{P}_\kappa(\gamma)$. □

This completes the proof of the theorem. □

The following result due to Robert Solovay shows that, given a supercompact cardinal κ, there is a proper class of cardinals γ satisfying the assumptions of Theorem 8.4.1 with respect to κ. Remember that an uncountable cardinal is *strongly compact* if for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S. Every supercompact cardinal is strongly compact (see [**Kan03**, Corollary 22.18]).

THEOREM 8.4.3 ([**Sol74**, Theorem 1]). *If κ is a strongly compact cardinal and γ is a singular strong limit cardinal greater than κ, then $2^\gamma = \gamma^+$.*

Let κ be a cardinal and $\gamma_0 \geq \kappa$. There is a singular strong limit cardinal $\gamma > \gamma_0$ such that $\mathrm{cof}(\gamma) \geq \kappa$ and there are no inaccessible cardinals in $(\gamma_0, \gamma]$. If κ is supercompact, then $2^\gamma = \gamma^+$ by Theorem 8.4.3 and γ satisfies the assumptions of Theorem 8.4.1. This proves the following statement.

COROLLARY 8.4.4. *If κ is supercompact and $\gamma \in \mathrm{On}$, then there is a $\nu \in \mathrm{On}$ with*

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\lambda}} \Vdash \text{“}\check{\kappa} \textit{ is } \check{\gamma}\textit{-supercompact”}$$

for all $\lambda > \nu$. □

8.5. Proofs of the main results

Given $\alpha \leq \beta \in \mathrm{On}$, let $\epsilon_{\alpha,\beta} : \vec{\mathbb{C}}_{<\alpha} \longrightarrow \vec{\mathbb{C}}_{<\beta}$ denote the canonical embedding of partial orders. Let D be the class of all $\vec{p}$ such that there is a $\beta \in \mathrm{On}$ with $\vec{p} \in \vec{\mathbb{C}}_{<\beta}$ and $\vec{p} \neq \epsilon_{\alpha,\beta}(\vec{q})$ for all $\alpha < \beta$ and $\vec{q} \in \vec{\mathbb{C}}_{<\alpha}$. Define $\mathbb{P}$ to be the class forcing with domain D ordered by $\vec{p} \leq_{\mathbb{P}} \vec{q}$ if there are $\alpha, \beta, \gamma \in \mathrm{On}$

with $\alpha, \beta \leq \gamma$, $\vec{p} \in \vec{\mathbb{C}}_{<\alpha}$, $\vec{q} \in \vec{\mathbb{C}}_{<\beta}$ and $\epsilon_{\alpha,\gamma}(\vec{p}) \leq_{\vec{\mathbb{C}}_{<\gamma}} \epsilon_{\beta,\gamma}(\vec{q})$. This means that $\mathbb{P}$ is a direct limit of the directed system

$$\langle\langle\vec{\mathbb{C}}_{<\alpha} \mid \alpha \in \mathrm{On}\rangle, \langle\epsilon_{\alpha,\beta} \mid \alpha \leq \beta \in \mathrm{On}\rangle\rangle.$$

Since $\vec{\mathbb{C}}_{<\alpha}$ is uniformly definable in parameter α, $\mathbb{P}$ is definable without parameters.

Proof of Theorem 8.1.4. First, assume that the inaccessible cardinals are bounded in On and define

$$\nu = \sup\{\alpha \in \mathrm{On} \mid \alpha \textit{ is an inaccessible cardinal}\}.$$

We have $\mathbb{1}_{\vec{\mathbb{C}}_{<\nu+1}} \Vdash$ "$\dot{\mathbb{C}}_{[\nu+1,\lambda)}$ *is trivial*" for all $\lambda > \nu$ and this shows that $\mathbb{P}$ is forcing equivalent to $\vec{\mathbb{C}}_{<\nu+1}$. Since ν is definable without parameters and each $\vec{\mathbb{C}}_\alpha$ is definable in parameter α, the partial order $\vec{\mathbb{C}}_{<\nu+1}$ is definable without parameters. Proposition 8.3.3, Lemma 8.3.4 and Corollary 8.4.4 show that $\vec{\mathbb{C}}_{<\nu+1}$ satisfies the statements listed in Theorem 8.1.4 under this assumption.

Now, assume that there are unboundedly many inaccessible cardinals in On. Let G be $\mathbb{P}$-generic over V.

For each $\beta \in \mathrm{On}$, we define

$$G_\beta = \{\epsilon_{\alpha,\beta}(\vec{p}) \mid \alpha \leq \beta,\ \vec{p} \in \mathrm{G} \cap \vec{\mathbb{C}}_{<\alpha}\}.$$

Then G_β is $\vec{\mathbb{C}}_{<\beta}$-generic over V, V[G] is the union of all V[G_β] and G_α is the filter induced by G_β in $\vec{\mathbb{C}}_{<\alpha}$ whenever $\alpha \leq \beta \in \mathrm{On}$.

Claim 1. *If α is an inaccessible cardinal in* V *and $x \in$* V[G] *is a subset of α, then $x \in$* V[$G_{\alpha+1}$].

Proof of the claim. There is a $\beta > \alpha$ with $x \in$ V[G_β]. Since $\vec{\mathbb{C}}_{<\alpha+1}$ satisfies the α^+-chain condition in V, we can apply Proposition 8.3.1 to show that $\dot{\mathbb{C}}_{[\alpha+1,\beta)}$ is $<\alpha^+$-closed in V[$G_{\alpha+1}$] and this implies $x \in$ V[$G_{\alpha+1}$]. □

Claim 2. *Let x be an element of* V[G]. *There is an inaccessible cardinal α such that $y \in$* V[$G_{\alpha+1}$] *for all $y \in$* V[G] *with $y \subseteq x$. In particular,* V[G] *satisfies the* Power Set Axiom.

Proof of the claim. By our assumption, we can find an inaccessible cardinal α in V such that $x \in$ V[$G_{\alpha+1}$] and $|x|^{\mathrm{V}[G_{\alpha+1}]} \leq \alpha$. Let $i : x \longrightarrow \alpha$ be an injection in V[$G_{\alpha+1}$]. If $y \in$ V[G] is a subset of x, then there is $\beta > \alpha$ with $y \in$ V[G_β]. By Claim 1, we have $f"y \in$ V[$G_{\alpha+1}$] and therefore $y \in$ V[$G_{\alpha+1}$]. This argument shows that $\mathcal{P}(x)^{\mathrm{V}[G_{\alpha+1}]}$ is the power set of x in V[G]. □

Claim 3. V[G] *is a model of* ZFC.

Proof of the claim. Let $\vec{p}$ be a condition in $\mathbb{P}$, A be an element of V and $\langle D_a \mid a \in A\rangle$ be a V-definable sequence of dense subclasses of $\mathbb{P}$. Then there is an $\alpha \in \mathrm{On}$ with $\vec{p} \in \vec{\mathbb{C}}_{<\alpha}$. Given $a \in A$, define

$$d_a = \{\vec{q} \restriction \alpha \mid (\exists \beta \geq \alpha)\ \vec{q} \in D_a \cap \vec{\mathbb{C}}_{<\beta}\} \in \mathrm{V}.$$

Then $\langle d_a \mid a \in A\rangle \in \mathrm{V}$ and each d_a is predense in $\mathbb{P}$. This shows that $\mathbb{P}$ is *pretame* with respect to V (see [**Fri00**, page 33]). By [**Fri00**, Lemma 2.19], this implies that V[G] is a model of ZFC^-. □

Claim 4. *Let κ be a cardinal in* V *with the property that there is no singular limit of inaccessible cardinals ν with $\nu^+ < \kappa \leq 2^\nu$ in* V*. Then κ is a cardinal in* V[G] *and, if κ is regular in* V*, then κ is regular in* V[G].

Proof of the claim. By Proposition 8.3.5, κ is a cardinal in $\mathrm{V}[G_\mu]$ for every $\mu \in \mathrm{On}$ and, if κ is regular in V, then κ is regular in every $\mathrm{V}[G_\mu]$. In combination with the above remarks, this directly implies the statement of the claim. □

Claim 5. *If κ is a supercompact cardinal in* V*, then κ is supercompact in* V[G].

Proof of the claim. Given $\gamma \in \mathrm{On}$, Corollary 8.4.4 shows that there is a $\nu \in \mathrm{On}$ such that κ is γ-supercompact in $\mathrm{V}[G_\beta]$ for all $\beta > \nu$. By Claim 2, there is an inaccessible cardinal α such that $\mathcal{P}(\mathcal{P}_\kappa(\gamma))^{\mathrm{V[G]}} = \mathcal{P}(\mathcal{P}_\kappa(\gamma))^{\mathrm{V}[G_\alpha]}$ and therefore $\mathcal{P}(\mathcal{P}_\kappa(\gamma))^{\mathrm{V}[G_\alpha]} = \mathcal{P}(\mathcal{P}_\kappa(\gamma))^{\mathrm{V}[G_\beta]}$ for all $\beta > \nu$. We can conclude that κ is γ-supercompact in V[G]. □

Claim 6. *If α is an inaccessible cardinal in* V*, then α is an inaccessible cardinal in* V[G] *and* $(2^\alpha)^{\mathrm{V[G]}} = (2^\alpha)^{\mathrm{V}}$.

Proof of the claim. By Proposition 8.3.3, α is an inaccessible cardinal in $\mathrm{V}[G_{\alpha+1}]$ and Lemma 8.3.4 shows that $(2^\alpha)^{\mathrm{V}[G_\alpha+1]} = (2^\alpha)^{\mathrm{V}}$ holds. The statement of the claim follows directly from Claim 1. □

Claim 7. *Let α be an inaccessible cardinal in* V*. There is a well-order of $\mathrm{H}_{\alpha^+}^{\mathrm{V[G]}}$ that is definable in $\langle \mathrm{H}_{\alpha^+}^{\mathrm{V[G]}}, \in\rangle$ by a formula with parameters.*

Proof of the claim. By Claim 2, there is a $\nu > \alpha$ with $\mathrm{H}_{\alpha^+}^{\mathrm{V[G]}} = \mathrm{H}_{\alpha^+}^{\mathrm{V}[G_\nu]}$. The statements of the Claim follows directly from Lemma 8.3.4. □

This completes the proof of the theorem. □

Proof of Theorem 8.1.1. Let α be an inaccessible cardinal and A be a subset of ${}^\alpha\alpha$. There is a $\vec{\mathbb{C}}_{<\alpha}$-name $\dot{p}$ with the property that, whenever G is $\vec{\mathbb{C}}_{<\alpha}$-generic over V, then there is a α-coding basis $\langle \bar{A}, \bar{s}\rangle$ coding a well-order of ${}^\alpha\alpha$ in $\mathrm{V}[G]$ that satisfies the following statements in $\mathrm{V}[G]$.

(1) $\dot{p}^G = \langle \bar{A}, \bar{s}, \mathbb{1}_{\mathbb{P}_{\bar{s}}(\bar{A})^{\mathrm{V}[G]}}\rangle \in \dot{\mathbb{C}}_\alpha^G$.
(2) There is a well-order $\lhd$ of ${}^\alpha\alpha$ witnessing that $\langle \bar{A}, \bar{s}\rangle$ codes a well-order of ${}^\alpha\alpha$ such that A is an initial segment of this order and the order-type of this initial segment is equal to the cardinality of A.

Pick $\vec{p} \in \vec{\mathbb{C}}_{<\alpha+1}$ with $\vec{p}(\alpha) = \dot{p}$. Then p is a condition in $\mathbb{P}$.

Let G be $\mathbb{P}$-generic over V with $p \in$ G. For each $\beta \in$ On, define G_β as in the proof of Theorem 8.1.4 and let

$$\dot{p}^{G_\alpha} = \langle \bar{A}, \bar{s}, \mathbb{1}_{\mathbb{P}_{\bar{s}}(\bar{A})^{\mathrm{V}[G_\alpha]}} \rangle \in \mathrm{V}[G_\alpha].$$

By Claim 2 in the above proof, there is a $\nu > \alpha$ with $\mathrm{H}_{\alpha^+}^{\mathrm{V[G]}} = \mathrm{H}_{\alpha^+}^{\mathrm{V}[G_\nu]}$. Lemma 8.3.4 implies that $\bar{A}$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\alpha\alpha$ in $\mathrm{V}[G_\nu]$ and therefore also in V[G]. Let $\lhd$ denote the well-order of $({}^\alpha\alpha)^{\mathrm{V}[G_\alpha]}$ produced by the above construction. Then $\lhd$ is definable in $\langle \mathrm{H}_{\alpha^+}^{\mathrm{V[G]}}, \in \rangle$ and A is either equal to the domain of $\lhd$ or to the set of all $\lhd$-predecessors of an element of this domain. This shows that A is definable in $\langle \mathrm{H}_{\alpha^+}^{\mathrm{V[G]}}, \in \rangle$ by a Σ_1-formula with parameters. By the results of Section 6.2, A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\alpha\alpha$. $\square$

8.6. Open problems

We close this chapter with some open problems related to the above results.

If the *Singular Cardinal Hypothesis* holds, then forcing with the class-sized partial order constructed in Theorem 8.1.4 does not collapse cardinals. It is not obvious if the converse of this implication also holds.

QUESTION 8.6.1. *Is it consistent that the partial order constructed in the proof of Theorem 8.1.4 collapses cardinals?*

Given a κ-coding basis $\langle A, s \rangle$, an easy argument shows that forcing with $\mathbb{P}_s(A)$ adds a Cohen-subset of κ. Therefore, a positive answer to the above question would follow from the existence of certain *scales* (see [**Jec03**, Definition 24.6]). The proof of [**Hon10**, Observation 4.3] contains the idea behind this approach.

As mentioned in the abstract, Theorem 8.1.4 can be viewed as a *boldface* version of Theorem 8.1.3 in the absence of the GCH. We may therefore ask whether a *lightface* version of Theorem 8.1.4 is possible.

QUESTION 8.6.2. *Let κ be a regular uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$ and $2^\kappa > \kappa^+$. Is there a cardinal preserving partial order $\mathbb{P}$ with the property that, whenever $\mathrm{V}[G]$ is a $\mathbb{P}$-generic extension of the ground model, then there is a well-order of $\mathrm{H}_{\kappa^+}^{\mathrm{V}[G]}$ that is definable in $\langle \mathrm{H}_{\kappa^+}^{\mathrm{V}[G]}, \in \rangle$ by a formula* without *parameters?*

In [**FHb**], Sy-David Friedman and Radek Honzik use a κ^{++}-strong cardinal to produce a model with a measurable κ with $2^\kappa = \kappa^{++}$ and the property that there is a well-order of H_{κ^+} that is definable in $\langle \mathrm{H}_{\kappa^+}, \in \rangle$ by a formula without parameters. It is natural to ask whether this statement is optimal.

QUESTION 8.6.3. *Is it consistent that there is a measurable cardinal κ such that $2^\kappa > \kappa^{++}$ and there is a well-order of H_{κ^+} that is definable in $\langle \mathrm{H}_{\kappa^+}, \in \rangle$ by a formula without parameters?*

The result mentioned above is used in [**FHb**] to establish the consistency of a *definable failure of the Singular Cardinal Hypothesis*, i.e. if the existence of a κ^{++}-strong cardinal is consistent, then it is consistent that $\aleph_\omega$ is a strong limit cardinal, $2^{\aleph_\omega} = \aleph_{\omega+2}$ and there is a well-order of $\mathrm{H}_{\aleph_{\omega+1}}$ that is definable in $\langle \mathrm{H}^{\mathrm{V}[G]}_{\aleph_{\omega+1}}, \in \rangle$ by a formula without parameters.

Starting from a supercompact cardinal, we can apply the *Laver preparation* (see [**Lav78**]) and Theorem 8.1.4 to produce a positive answer to the *boldface* version of Question 8.6.3. We may therefore ask whether the existence of stronger definable failure of the *Singular Cardinal Hypothesis* is consistent.

QUESTION 8.6.4. *Is it consistent that there is a singular strong limit cardinal ν such that $2^\nu > \nu^{++}$ and there is a well-order of H_{ν^+} that is definable in $\langle \mathrm{H}_{\nu^+}, \in \rangle$ by a formula with parameters?*

Finally, we ask whether the existence of a definable well-order of $\mathrm{H}_{\aleph_{\omega+1}}$ can be forced without applying some variation of Prikry-Forcing.

QUESTION 8.6.5. *Is there a partial order $\mathbb{P}$ with cardinality less than the least inaccessible cardinal and the property that, whenever $\mathrm{V}[G]$ is a $\mathbb{P}$-generic extension of the ground model, then there is a well-order of $\mathrm{H}^{\mathrm{V}[G]}_{\aleph_{\omega+1}}$ that is definable in $\langle \mathrm{H}^{\mathrm{V}[G]}_{\aleph_{\omega+1}}, \in \rangle$ by a formula with parameters?*

Bibliography

[AF] David Asperó and Sy-David Friedman. Definable wellorderings of H_{ω_2} and CH. To appear.

[AF09] David Asperó and Sy-David Friedman. Large cardinals and locally defined wellorders of the universe. *Ann. Pure Appl. Logic*, 157(1):1–15, 2009.

[Bar75] Jon Barwise. *Admissible sets and structures*. Springer-Verlag, Berlin, 1975. An approach to definability theory, Perspectives in Mathematical Logic.

[Bau83] James E. Baumgartner. Iterated forcing. In *Surveys in set theory*, volume 87 of *London Math. Soc. Lecture Note Ser.*, pages 1–59. Cambridge Univ. Press, Cambridge, 1983.

[Bec11] Howard Becker. A note on the automorphism tower of a countable group. Handwritten notes, 2011.

[BF01] Joan Bagaria and Sy-David Friedman. Generic absoluteness. In *Proceedings of the XIth Latin American Symposium on Mathematical Logic (Mérida, 1998)*, volume 108, pages 3–13, 2001.

[BGM71] Jon Barwise, Robin O. Gandy, and Yiannis N. Moschovakis. The next admissible set. *J. Symbolic Logic*, 36:108–120, 1971.

[BK96] Howard Becker and Alexander S. Kechris. *The descriptive set theory of Polish group actions*, volume 232 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.

[Cum10] James Cummings. Iterated forcing and elementary embeddings. In *The Handbook of Set Theory (M. Foreman, A. Kanamori, and M. Magidor, eds.)*, volume 2, pages 775–884. Springer, Berlin, 2010.

[DGG01] Manfred Droste, Michèle Giraudet, and Rüdiger Göbel. All groups are outer automorphism groups of simple groups. *J. London Math. Soc. (2)*, 64(3):565–575, 2001.

[Far11] Ilijas Farah. All automorphisms of the Calkin algebra are inner. *Ann. of Math. (2)*, 173(2):619–661, 2011.

[FHa] Sy-David Friedman and Peter Holy. Condensation and large cardinals. To appear in *Fundamenta Mathematicae*.

[FHb] Sy-David Friedman and Radek Honzik. A definable failure of the singular cardinal hypothesis. To appear in the *Israel Journal of Mathematics*.

[FH08] Gunter Fuchs and Joel David Hamkins. Changing the heights of automorphism towers by forcing with Souslin trees over L. *J. Symbolic Logic*, 73(2):614–633, 2008.

[FHK] Sy-David Friedman, Tapani Hyttinen, and Vadim Kulikov. Generalised descriptive set theory and classification theory. To appear.

[FLa] Sy-David Friedman and Philipp Lücke. Large cardinals and definable wellorders, without the GCH. Submitted.

[FLb] Gunter Fuchs and Philipp Lücke. Iteratively changing the heights of automorphism towers. To appear in the *Notre Dame Journal of Formal Logic*.

[FMW92] Qi Feng, Menachem Magidor, and Hugh Woodin. Universally Baire sets of reals. In *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of *Math. Sci. Res. Inst. Publ.*, pages 203–242. Springer, New York, 1992.

[Fri00] Sy-David Friedman. *Fine structure and class forcing*, volume 3 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2000.

[Fri10] Sy-David Friedman. Forcing, combinatorics and definability. In *Proceedings of the 2009 RIMS Workshop on Combinatorical Set Theory and Forcing Theory in Kyoto, Japan, RIMS Kokyuroku No. 1686*, pages 24–40. 2010.

[Fuc08] Gunter Fuchs. Closed maximality principles: implications, separations and combinations. *J. Symbolic Logic*, 73(1):276–308, 2008.

[GP08] Paul Gartside and Bojana Pejić. Uniqueness of Polish group topology. *Topology Appl.*, 155(9):992–999, 2008.

[Ham02] Joel David Hamkins. How tall is the automorphism tower of a group? In *Logic and algebra*, volume 302 of *Contemp. Math.*, pages 49–57. Amer. Math. Soc., Providence, RI, 2002.

[Har77] Leo Harrington. Long projective wellorderings. *Ann. Math. Logic*, 12(1):1–24, 1977.

[Hod93] Wilfrid Hodges. *Model theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[Hon10] Radek Honzik. Global singularization and the failure of SCH. *Ann. Pure Appl. Logic*, 161(7):895–915, 2010.

[HR01] Tapani Hyttinen and Mika Rautila. The canary tree revisited. *J. Symbolic Logic*, 66(4):1677–1694, 2001.

[HT00] Joel David Hamkins and Simon Thomas. Changing the heights of automorphism towers. *Ann. Pure Appl. Logic*, 102(1-2):139–157, 2000.

[Hul70] J. A. Hulse. Automorphism towers of polycyclic groups. *J. Algebra*, 16:347–398, 1970.

[HV90] Tapani Hyttinen and Jouko Väänänen. On Scott and Karp trees of uncountable models. *J. Symbolic Logic*, 55(3):897–908, 1990.

[Jec71] Thomas Jech. Trees. *J. Symbolic Logic*, 36:1–14, 1971.

[Jec03] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[Jen72] R. Björn Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308; erratum, ibid. 4 (1972), 443, 1972. With a section by Jack Silver.

[JS70] R. Björn Jensen and Robert M. Solovay. Some applications of almost disjoint sets. In *Mathematical Logic and Foundations of Set Theory (Proc. Internat. Colloq., Jerusalem, 1968)*, pages 84–104. North-Holland, Amsterdam, 1970.

[JST99] Winfried Just, Saharon Shelah, and Simon Thomas. The automorphism tower problem revisited. *Adv. Math.*, 148(2):243–265, 1999.

[Kal86] Robert R. Kallman. Uniqueness results for homeomorphism groups. *Trans. Amer. Math. Soc.*, 295(1):389–396, 1986.

[Kan03] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kri64] Saul A. Kripke. Transfinite recursion on admissible ordinals I, II (abstracts). *J. Symb. Logic*, 29:161–162, 1964.

[KS09] Itay Kaplan and Saharon Shelah. The automorphism tower of a centerless group without choice. *Arch. Math. Logic*, 48(8):799–815, 2009.

[Kun80] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1980. An introduction to independence proofs.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[Lav78] Richard Laver. Making the supercompactness of κ indestructible under κ-directed closed forcing. *Israel J. Math.*, 29(4):385–388, 1978.

[Lav07] Richard Laver. Certain very large cardinals are not created in small forcing extensions. *Ann. Pure Appl. Logic*, 149(1-3):1–6, 2007.

[LT11] Philipp Lücke and Simon Thomas. Automorphism groups of ultraproducts of finite symmetric groups. *Communications in Algebra*, 39(10):3625–3630, 2011.

[Lüca] Philipp Lücke. Groups with unbounded potential automorphism tower heights. To appear in the proceedings of the *PhDs in Logic III* conference in Brussels 2011.

[Lücb] Philipp Lücke. $\mathbf{\Sigma}^1_1$-definability at uncountable regular cardinals. Submitted.

[Lücc] Philipp Lücke. Special pairs and automorphisms of centreless groups. To appear in the proceedings of the Groups and Model Theory conference in Mülheim an der Ruhr.

[Mac57] George W. Mackey. Borel structure in groups and their duals. *Trans. Amer. Math. Soc.*, 85:134–165, 1957.

[Mos74] Yiannis N. Moschovakis. *Elementary induction on abstract structures*. North-Holland Publishing Co., Amsterdam, 1974. Studies in Logic and the Foundations of Mathematics, Vol. 77.

[Mos75] Yiannis N. Moschovakis. On nonmonotone inductive definability. *Fund. Math.*, 82:39–83, 1974/75. Collection of articles dedicated to Andrzej Mostowski on the occasion of his sixtieth birthday, V.

[MV93] Alan Mekler and Jouko Väänänen. Trees and Π^1_1-subsets of ${}^{\omega_1}\omega_1$. *J. Symbolic Logic*, 58(3):1052–1070, 1993.

[NS78] Mark Nadel and Jonathan Stavi. $L_{\infty\lambda}$-equivalence, isomorphism and potential isomorphism. *Trans. Amer. Math. Soc.*, 236:51–74, 1978.

[NZ98] Itay Neeman and Jindřich Zapletal. Proper forcings and absoluteness in $L(\mathbf{R})$. *Comment. Math. Univ. Carolin.*, 39(2):281–301, 1998.

[Pla66] Richard Alan Platek. *Foundations of recursion theory*. ProQuest LLC, Ann Arbor, MI, 1966. Thesis (Ph.D.)–Stanford University.

[Rub89] Matatyahu Rubin. On the reconstruction of topological spaces from their groups of homeomorphisms. *Trans. Amer. Math. Soc.*, 312(2):487–538, 1989.

[Sol74] Robert M. Solovay. Strongly compact cardinals and the GCH. In *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)*, pages 365–372, Providence, R.I., 1974. Amer. Math. Soc.

[SS88] Saharon Shelah and Juris Steprāns. PFA implies all automorphisms are trivial. *Proc. Amer. Math. Soc.*, 104(4):1220–1225, 1988.

[Ste83] John R. Steel. Scales in $L(\mathbf{R})$. In *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Math.*, pages 107–156. Springer, Berlin, 1983.

[SV02] Saharon Shelah and Pauli Väisänen. The number of $L_{\infty\kappa}$-equivalent nonisomorphic models for κ weakly compact. *Fund. Math.*, 174(2):97–126, 2002.

[Tho] Simon Thomas. *The Automorphism Tower Problem.* To appear.

[Tho85] Simon Thomas. The automorphism tower problem. *Proc. Amer. Math. Soc.*, 95(2):166–168, 1985.

[Tho98] Simon Thomas. The automorphism tower problem. II. *Israel J. Math.*, 103:93–109, 1998.

[TV99] Stevo Todorčević and Jouko Väänänen. Trees and Ehrenfeucht-Fraïssé games. *Ann. Pure Appl. Logic*, 100(1-3):69–97, 1999.

[Vää91] Jouko Väänänen. A Cantor-Bendixson theorem for the space $\omega_1^{\omega_1}$. *Fund. Math.*, 137(3):187–199, 1991.

[Vää95] Jouko Väänänen. Games and trees in infinitary logic: A survey. In *Quantifiers (M. Krynicki, M. Mostowski and L. Szczerba, eds.)*, pages 105–138. Kluwer Academic Publishers, 1995.

[Wie39] Helmut Wielandt. Eine Verallgemeinerung der invarianten Untergruppen. *Math. Z.*, 45(1):209–244, 1939.

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Symbols and Abbreviations

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