CLOSED MAXIMALITY PRINCIPLES AND GENERALIZED BAIRE SPACES

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Abstract. Given an uncountable regular cardinal κ, we study the structural properties of the class of all sets of functions from κ to κ that are definable over the structure $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_1-formula with parameters. It is well-known that many important statements about these classes are not decided by the axioms of ZFC together with large cardinal axioms. In this paper, we present other canonical extensions of ZFC that provide a strong structure theory for these classes. These axioms are variations of the *Maximality Principle* introduced by Stavi and Väänänen and later rediscovered by Hamkins.

1. Introduction

Given an infinite regular cardinal κ, the *generalized Baire space of* κ is the set ${}^{\kappa}\kappa$ of all functions from κ to κ equipped with the topology whose basic open sets are of the form $N_s = \{x \in {}^{\kappa}\kappa \mid s \subseteq x\}$ for some s contained in the set ${}^{<\kappa}\kappa$ of all functions $t : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$. For uncountable regular cardinals κ, these spaces are a natural generalization of the classical Baire space ${}^{\omega}\omega$ to these cardinalities. Moreover, many objects studied in set theory can be identified with definable subsets of these spaces. For example, there is a direct correspondence between the closed subsets of ${}^{\kappa}\kappa$ and the sets of cofinal branches through set theoretic trees of height κ. The investigation of generalized Baire spaces of uncountable cardinals and their definable subsets was initiated by Mekler and Väänänen (see [28] and [34]) and has recently gained increasing attention (see, for example, [5], [6], [20] and [29]). Results of this analysis have been used in model theory and infinitary logic (see, for example, [6], [32] and [35]). In this paper, we are interested in the structural properties of classes of definable subsets of ${}^{\kappa}\kappa$ of low complexity for uncountable regular cardinals κ.

In the remainder of this paper, we let κ denote an uncountable regular cardinal. We generalize the classical definition of analytic sets of reals to the generalized Baire space of κ by calling a subset of $({}^{\kappa}\kappa)^n$ a $\mathbf{\Sigma}^1_1$-*subset* if it is equal to the projection of a closed subset of $({}^{\kappa}\kappa)^{n+1}$. A folklore results (see, for example, [24, Section 2]) shows that for such cardinals κ, every subset of $({}^{\kappa}\kappa)^n$ that is definable over the structure $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_1-formula with parameters is a $\mathbf{\Sigma}^1_1$-subset. If in addition $\kappa = \kappa^{<\kappa}$ holds, then it is easy to see that the converse implication is also true (see Section 5). This shows that the class of $\mathbf{\Sigma}^1_1$-subsets of ${}^{\kappa}\kappa$ contains a great variety

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of set theoretically interesting objects. Moreover, it shows that, in contrast to the classical Baire space, many basic statements about the class of $\mathbf{\Sigma}^1_1$-subset are not settled by the axioms of ZFC together with large cardinal axioms, because it is possible to manipulated these objects by set theoretic methods. In Section 2, we will present important examples of such statements and present the corresponding independence results. This leads to the question whether there are other canonical extensions of ZFC that decide these statements by providing a suitable structure theory for the class of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$. The aim of this paper is to show that certain forcing axioms that are motivated by modal logic are examples of such extensions of ZFC. These axioms are variations of the *Maximality Principle* introduced by Stavi and Väänänen in [33] and later rediscovered by Hamkins (see [12]).

Given a sentence φ in the language $\mathcal{L}_\in$ of set theory, we say that φ is *forceably necessary* if there is a partial order $\mathbb{P}$ such that $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \varphi$ holds for every $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a partial order. The *Maximality Principle* (MP) is the scheme of axioms stating that every forceably necessary $\mathcal{L}_\in$-sentence is true. This formulation can be motivated by the *Maximality Principle* $\Diamond\Box\varphi \longrightarrow \varphi$ of modal logic by interpreting the modal statement $\Diamond\varphi$ ("*φ is possible*") as "*φ holds in some forcing extension of the ground model*" and the statement $\Box\varphi$ ("*φ is necessary*") as "*φ holds in every forcing extension of the ground model*". The consistency of the theory $\mathrm{ZFC} + \mathrm{MP}$ follows from the consistency of ZFC (see [12, Theorem 2.1]). Similar to standard forcing axioms, this principle states that the universe V is, in a certain sense, maximal in the collection of its forcing extensions.

Following [8] and [22], we will consider variations of this principle that arise from the following modifications of the above principle:

- A restriction of the complexity of the considered formulas.
- A restriction of the class of forcings that can be used to witness that a given statement is possible.
- A restriction of the class of forcings that need to be considered to show that a given statement is necessary.
- An extension of the principle to formulas containing parameters.

Note that the first two modifications weaken the principle, while the last two strengthen it. The axioms discussed in this paper are restricted to certain classes of $<\kappa$-closed forcings and allow statements with parameters of bounded hereditary cardinality. We will refer to these principle as *boldface closed maximality principles*. Such principles were already intensively studied by Fuchs in [8] and [9].

We outline the content of this paper. In Section 2, we a present independence results that show that a number of important basic questions about the structural properties of the class of $\mathbf{\Sigma}^1_1$-subsets of generalized Baire spaces are not decided by the axioms of ZFC together with large cardinal axioms. In the following, these questions will serve as test questions to evaluate the influence of other extension of ZFC on the structure theory of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$. Section 3 contains the formulation of the closed maximality principle for the class of all $<\kappa$-closed forcing and statements with parameters in $\mathrm{H}(\kappa^+)$. We present results that show that these principles answer most of the questions posed in the previous section. In contrast, we also present a question about the values of certain cardinal characteristics of the space ${}^\kappa\kappa$ that is not settled by such principles. Motivated by this observation, we consider maximality principles for formulas containing parameters of higher cardinalities in Section 4. A natural candidate for such a principle is the maximality principle for $<\kappa$-closed forcing that satisfy the κ^+-chain condition and statements with parameters in $\mathrm{H}(2^\kappa)$. We will prove that this principle always fails for $\aleph_1$. Motivated by this result, we will show that a restriction of this principle to a smaller class of forcings can lead to a consistent maximality principle that provides a strong structure for the the class of $\mathbf{\Sigma}^1_1$-subsets of generalized Baire spaces. In particular, we will present an example of such a principle that answers all questions posed in the second section. Section 5 contains a number of results that demonstrate the influence of closed maximality principles on generalized Baire spaces and show how these principles answer our test questions. In the next section, we establish the consistency of the principles defined in Section 4 using iterated forcing. We close this paper with a discussion of extensions of ZFC that provide a strong structure theory for the classes of $\mathbf{\Sigma}^1_1$-subsets of the generalized Baire spaces of all regular uncountable cardinals. Moreover, we present some open questions motivated by the results of this paper.

2. Independence results

In this section, we review several independence results showing that many basic question about the class of $\mathbf{\Sigma}^1_1$-definable subsets of ${}^\kappa\kappa$ for uncountable regular cardinals κ are not decided by the axioms of ZFC together with large cardinal axioms. In particular, these axioms do not provide a good structure theory for these classes of subsets. This observation suggests two ways to proceed:

(i) Consider subclasses of the class of all $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ that contain many interesting objects and have the property that ZFC together with large cardinal axioms provides a strong structure theory for them.
(ii) Consider other natural extensions of ZFC that provide a strong structure for the class of all $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$.

The first approach is pursued in [25], where it is shown that large cardinals axioms resolve most of the independence issues discussed below for the class of subsets of ${}^{\omega_1}\omega_1$ that are definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a Σ_1-formula with parameter ω_1. In this paper, we will show that closed maximality principles are examples of extensions of ZFC that are suitable for the second approach by showing that these principles are in some sense natural and proving that the questions listed below are answered by these principles.

In the remainder of this paper, we let κ denote an uncountable regular cardinal.

2.1. **Sets separating the club filter from the non-stationary ideal.** As usual, we say that a subset A of ${}^\kappa\kappa$ is a $\mathbf{\Pi}^1_1$-subset if ${}^\kappa\kappa \setminus A$ is a $\mathbf{\Sigma}^1_1$-subset. Moreover, a subset of ${}^\kappa\kappa$ is a $\mathbf{\Delta}^1_1$-subset if it is both a $\mathbf{\Sigma}^1_1$- and a $\mathbf{\Pi}^1_1$-subset.

Given $S \subseteq \kappa$, we define

$$\mathsf{Club}(S) \ = \ \{x \in {}^\kappa\kappa \mid \exists C \subseteq \kappa \textit{ club } \forall \alpha \in C \cap S \ x(\alpha) > 0\}$$

and

$$\mathsf{NS}(S) \ = \ \{x \in {}^\kappa\kappa \mid \exists C \subseteq \kappa \textit{ club } \forall \alpha \in C \cap S \ x(\alpha) = 0\}.$$

Then $\mathsf{Club}(S)$ and $\mathsf{NS}(S)$ are disjoint $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ for every $S \subseteq \kappa$. We can identify $\mathsf{Club}(\kappa)$ with the *club filter* on κ and $\mathsf{NS}(\kappa)$ with the *non-stationary ideal* on κ. In the light of the *Lusin Separation Theorem theorem* (see [19, Theorem 14.7]) the following question appears naturally.

Question 1. Is there a $\mathbf{\Delta}^1_1$-subset A of ${}^\kappa\kappa$ that separates $\mathsf{Club}(\kappa)$ from $\mathsf{NS}(\kappa)$, in the sense that $\mathsf{Club}(\kappa) \subseteq A \subseteq {}^\kappa\kappa \setminus \mathsf{NS}(\kappa)$ holds?

It is possible to combine results from [6] and [11] to show that forcing with the partial order $\mathrm{Add}(\kappa,\kappa^+)$ that adds κ^+-many Cohen subsets of κ produces a negative answer to this question. In Section 5, we will present a proof of the following result.

Theorem 2.1. *If $\kappa = \kappa^{<\kappa}$ and G is $\mathrm{Add}(\kappa,\kappa^+)$-generic over V, then there is no $\mathbf{\Delta}^1_1$-subset A of ${}^\kappa\kappa$ that separates $\mathsf{Club}(\kappa)$ from $\mathsf{NS}(\kappa)$ in $\mathrm{V}[G]$.*

In contrast, positive answers to Question 1 are also consistent. It was observed in [7] that, if the non-stationary ideal on ω_1 is ω_1-dense, then $\mathsf{Club}(\kappa)$ is a $\mathbf{\Delta}^1_1$-subset of ${}^{\omega_1}\omega_1$. Results of Woodin show that this assumption is consistent (see [4, Section 7.14]). Moreover, Friedman, Kulikov and Hytinnen used results on the existence of κ-*Canary tree* from [15] and [27] to establish the consistency of positive answers to the above question for larger classes of cardinals. Note that, if S is a stationary subset of κ, then $\mathsf{Club}(S)$ separates $\mathsf{Club}(\kappa)$ from $\mathsf{NS}(\kappa)$. Given an infinite regular cardinal $\mu < \kappa$, we use S^κ_μ to denote the set of all limit ordinals less than κ of cofinality μ.

Theorem 2.2 ([6, Theorem 49.5])**.** *Assume that the* GCH *holds and κ is not the successor of a singular cardinal. Let $\mu < \kappa$ be an infinite regular cardinal. Then there is a partial order $\mathbb{P}$ satisfying the κ^+-chain condition with the property that forcing with $\mathbb{P}$ adds no bounded subsets of κ and $\mathsf{Club}(S^\kappa_\lambda)$ is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ in every $\mathbb{P}$-generic extension of the ground model.*

We will later show that certain closed maximality principles imply that all $\mathbf{\Delta}^1_1$-subsets posses a regularity property that generalizes the *Baire property* to higher cardinalities. Since the results of [11] show that sets with this property cannot separate $\mathsf{Club}(\kappa)$ from $\mathsf{NS}(\kappa)$, this implications shows that these principles answer Question 1.

2.2. **The Bernstein and the perfect set property.** We say that a subset A of ${}^\kappa\kappa$ *contains a perfect subset* if there is a continuous injection $\iota : {}^\kappa 2 \longrightarrow {}^\kappa\kappa$ with $\mathrm{ran}(\iota) \subseteq A$. A subset of ${}^\kappa\kappa$ has the *Bernstein property* if either A or ${}^\kappa\kappa \setminus A$ contains a perfect subset. Finally, we say that a subset A of ${}^\kappa\kappa$ has the *perfect set property* if either A has cardinality at most κ or A contains a perfect subset. It is easy to see that this definition of the perfect set property is equivalent to the definition using transfinite games given in [28, Section 5].

We say that a subset of ${}^\kappa\kappa$ is κ-Borel if it is contained in the smallest algebra of sets on ${}^\kappa\kappa$ that contains all open subsets of ${}^\kappa\kappa$ is closed under κ-unions. An easy argument (see Corollary 5.21) shows that, if $\kappa = \kappa^{<\kappa}$ holds, then all κ-Borel subsets of ${}^\kappa\kappa$ have the Bernstein property. Since all κ-Borel subsets are $\mathbf{\Delta}^1_1$-subsets and there are always $\mathbf{\Delta}^1_1$-subsets of ${}^\kappa\kappa$ that are not κ-Borel (see [6, Theorem 18]), one naturally arrives at the following question.

Question 2. Does every $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ have the Bernstein property?

Using forcing, it is possible to establish the consistency of a negative answer to this question.

Theorem 2.3 ([13, Corollary 1.5])**.** *Assume that $\kappa = \kappa^{<\kappa}$ and 2^κ is regular. Then there is a $<\kappa$-closed partial order $\mathbb{P}$ with the property that forcing with $\mathbb{P}$ preserves* *cardinals less than or equal to* 2^κ *and adds a* $\mathbf{\Delta}^1_1$*-subset of* ${}^\kappa\kappa$ *without the Bernstein property.*

In Section 5, we will show that it is consistent that all $\mathbf{\Delta}^1_1$-subsets of ${}^\kappa\kappa$ have the Bernstein property.

Theorem 2.4. *Assume that* $\kappa = \kappa^{<\kappa}$ *and let* G *be* $\mathrm{Add}(\kappa,\kappa^+)$*-generic over* V*. In* $\mathrm{V}[G]$*, all* $\mathbf{\Delta}^1_1$*-subsets of* ${}^\kappa\kappa$ *have the Bernstein property.*

Since classical results in descriptive set theory show that all $\mathbf{\Sigma}^1_1$-subsets of ${}^\omega\omega$ have the perfect set property, it is also natural to consider the following question.

Question 3. Does every $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ have the perfect set property?

Theorem 2.3 already established the consistency of a negative answer to this question. In addition, a small modification of a classical argument of Solovay (see [17, Section 4]) shows that the non-existence of inaccessible cardinals in L implies the existence of closed counterexamples to the perfect set property.

In the other direction, results of Schlicht show that a positive answer to the above question can be established by collapsing inaccessible cardinals.

Theorem 2.5 ([29, Theorem 1.2])**.** *Let* $\theta > \kappa$ *be an inaccessible cardinal and let* G *be* $\mathrm{Col}(\kappa,{<}\theta)$*-generic over* V*. In* $\mathrm{V}[G]$*, every subset of* ${}^\kappa\kappa$ *contained in* $\mathrm{HOD}({}^\kappa\mathrm{Ord})$ *has the perfect set property.*

We will later show that closed maximality principles imply that all $\mathbf{\Delta}^1_1$-subsets of ${}^\kappa\kappa$ have the Bernstein property. Moreover, we will show that certain principles also yield an affirmative answer to Question 3.

2.3. **The lengths of $\mathbf{\Sigma}^1_1$-definable wellorders.** We call a wellorder $\langle A,\lhd\rangle$ a $\mathbf{\Sigma}^1_1$-*wellordering of a subset of* ${}^\kappa\kappa$ if $\lhd$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa\times{}^\kappa\kappa$. Note that this implies that the domain A of $\lhd$ is also a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$. It is easy to see that, for every $\alpha<\kappa^+$, there is a $\mathbf{\Sigma}^1_1$-wellordering of a subset of ${}^\kappa\kappa$ of order-type α. Moreover, if there is an $x\subseteq\kappa$ such that κ^+ is not inaccessible in $\mathrm{L}[x]$, then there is a $\mathbf{\Sigma}^1_1$-wellordering of a subset of ${}^\kappa\kappa$ of order-type κ^+. The following question is motivated by the classical *Kunen-Martin Theorem* (see [19, Theorem 31.5]).

Question 4. What is the least upper bound for the order-types of $\mathbf{\Sigma}^1_1$-wellorderings of subsets of ${}^\kappa\kappa$?

With the help of generic coding techniques (see, for example, [14], [24] and Section 5 of this paper), it is possible to make arbitrary subsets of ${}^\kappa\kappa$ of the ground model $\mathbf{\Sigma}^1_1$-definable in a cofinality preserving forcing extension. In particular, these techniques allow us to force the existence of a $\mathbf{\Sigma}^1_1$-wellordering of a given length α. The following theorem is a direct consequence of the main result of [24].

Theorem 2.6 ([24, Theorem 1.5])**.** *Assume that* $\kappa=\kappa^{<\kappa}$*. Given* $\alpha<(2^\kappa)^+$*, there is a partial order* $\mathbb{P}$ *with the property that forcing with* $\mathbb{P}$ *preserves all cofinalities and the value of* 2^κ *and there is a* $\mathbf{\Sigma}^1_1$*-wellordering of a subset of* ${}^\kappa\kappa$ *of order-type* α *in every* $\mathbb{P}$*-generic extension of the ground model.*

In the other direction, the results of [24, Section 9] also show that both κ^+ and $2^\kappa>\kappa^+$ can consistently be upper bounds for the length of these wellorders.

Theorem 2.7 ([24, Corollary 9.5])**.** *Assume that $\kappa = \kappa^{<\kappa}$. Let $\nu > \kappa$ be a cardinal, let G be $\mathrm{Add}(\kappa, \nu)$-generic over V and let $\langle A, \lhd\rangle$ be a $\mathbf{\Sigma}^1_1$-wellordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$. Then $A \neq ({}^\kappa\kappa)^{\mathrm{V}[G]}$ and the order-type of $\langle A, \lhd\rangle$ has cardinality at most $(2^\kappa)^{\mathrm{V}}$ in $\mathrm{V}[G]$.*

Remember that a cardinal θ is *Σ_n-reflecting* if θ is inaccessible and $\langle \mathrm{H}(\theta), \in\rangle$ is an Σ_n-elementary submodel of $\langle \mathrm{V}, \in\rangle$.

Theorem 2.8 ([24, Corollary 9.10])**.** *Let $\nu > \kappa$ is an inaccessible cardinal and let $\mu > 0$ be a cardinal. Assume that either θ is Σ_2-reflecting or $\mu > \theta$. If $G \times H$ is $(\mathrm{Col}(\kappa, {<}\theta) \times \mathrm{Add}(\kappa, \mu))$-generic over V and $\langle A, \lhd\rangle$ is a $\mathbf{\Sigma}^1_1$-wellordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G, H]$, then A has cardinality κ in $\mathrm{V}[G, H]$.*

Note that the above remarks show that the conclusion of the last theorem implies that κ^+ is inaccessible in $\mathrm{L}[x]$ for every $x \subseteq \kappa$.

We will show that certain closed maximality principle exactly determine the least upper bound of the order-types of $\mathbf{\Sigma}^1_1$-wellorderings of subsets of ${}^\kappa\kappa$. Depending on the specific principle, this upper bound will either be equal to κ^+ or $2^\kappa > \kappa^+$.

2.4. **Cardinal characteristics of tree orderings.** We close this section with a question of a somewhat different nature. Let κ be an uncountable cardinal satisfying $\kappa = \kappa^{<\kappa}$, let $\mathcal{T}_\kappa$ denote the class of all trees of cardinality and height κ and let $\mathcal{TO}_\kappa$ denote the class of all trees in $\mathcal{T}_\kappa$ without a branch of length κ. Given $\mathbb{T}_0, \mathbb{T}_1 \in \mathcal{T}_\kappa$, we write $\mathbb{T}_0 \preceq \mathbb{T}_1$ to denote that there is a function $f : \mathbb{T}_0 \longrightarrow \mathbb{T}_1$ with $f(s) <_{\mathbb{T}_0} f(t)$ for all $s, t \in \mathbb{T}_0$ with $s <_{\mathbb{T}_1} t$. The resulting partial order $\langle \mathcal{TO}_\kappa, \preceq\rangle$ arises naturally in infinitary model theory (see, for example, [16] and [36]) and it can be viewed as a substitute for the ordering of all countable ordinals in the uncountable setting (see [28, Section 1]).

In the following, we use $\prec\cdot,\cdot\succ : \mathrm{Ord} \times \mathrm{Ord} \longrightarrow \mathrm{Ord}$ to denote the Gödel pairing function. Given $x \in {}^\kappa\kappa$, define $\in_x$ to be the set of all pairs $\langle \alpha, \beta\rangle$ in $\kappa \times \kappa$ with $x(\prec\alpha, \beta\succ) = 1$. Let T_κ denote the set of all $x \in {}^\kappa\kappa$ such that $\mathbb{T}_x = \langle \kappa, \in_x\rangle$ is an element of $\mathcal{T}_\kappa$ and let TO_κ denote the set of all $x \in \mathsf{T}_\kappa$ such that $\mathbb{T}_x$ is an element of $\mathcal{TO}_\kappa$. It is easy to check that T_κ is a κ-Borel subset of ${}^\kappa\kappa$ and TO_κ is a $\mathbf{\Pi}^1_1$-subset of ${}^\kappa\kappa$. Moreover the relation induced by $\preceq$ on T_κ is easily seen to be $\mathbf{\Sigma}^1_1$-definable. Therefore we may view the partial order $\langle \mathcal{TO}_\kappa, \preceq\rangle$ as a $\mathbf{\Sigma}^1_1$-definable ordering of a $\mathbf{\Pi}^1_1$-subset of ${}^\kappa\kappa$.

We are interested in the order-theoretic properties of $\langle \mathcal{TO}_\kappa, \preceq\rangle$. More specifically, we want to study the values of the following cardinal characteristics of this ordering of trees:

(i) The *bounding number of* $\langle \mathcal{TO}_\kappa, \preceq\rangle$ is the smallest cardinal $\mathfrak{b}_{\mathcal{TO}_\kappa}$ with the property that there is a $B \subseteq \mathcal{TO}_\kappa$ of this cardinality such that there is no tree $\mathbb{T} \in \mathcal{TO}_\kappa$ with $\mathbb{S} \preceq \mathbb{T}$ for all $\mathbb{S} \in B$.

(ii) The *dominating number of* $\langle \mathcal{TO}_\kappa, \preceq\rangle$ is the smallest cardinal $\mathfrak{d}_{\mathcal{TO}_\kappa}$ with the property that there is a subset $D \subseteq \mathcal{TO}_\kappa$ of this cardinality such that for every $\mathbb{S} \in \mathcal{TO}_\kappa$ there is a $\mathbb{T} \in D$ with $\mathbb{S} \leq \mathbb{T}$.

It is easy to see that

$$\kappa^+ \ \leq \ \mathfrak{b}_{\mathcal{TO}_\kappa} \ \leq \ \mathfrak{d}_{\mathcal{TO}_\kappa} \ \leq \ 2^\kappa \tag{1}$$

holds. In particular, if the GCH holds at κ, then these cardinal characteristics are equal. We may therefore ask if this is always the case.

Question 5. Is $\mathfrak{b}_{\mathcal{TO}_\kappa}$ equal to $\mathfrak{d}_{\mathcal{TO}_\kappa}$?

Assuming CH, [28, Theorem 15] shows that for every regular cardinal ν in the interval $[\omega_2, 2^{\omega_1}]$ there is a forcing that preserves all cofinalities and the value of $2^{\aleph_1}$ and forces $\nu = \mathfrak{b}_{\mathcal{TO}_{\aleph_1}} = \mathfrak{d}_{\mathcal{TO}_{\aleph_1}}$. In unpublished work, Schlicht and Thompson strengthened this result by showing that these values can be forced to be equal to any pair of regular cardinals that satisfies the inequalities in (1). In Section 5, we will prove the following result that shows that it is also possible to produce a negative answer to Question 5 by adding many Cohen subsets of κ.

Theorem 2.9. *If $\kappa = \kappa^{<\kappa}$ holds and G is $\mathrm{Add}(\kappa, (2^\kappa)^+)$-generic over V, then*

$$\mathfrak{b}^{\mathrm{V}[G]}_{\mathcal{TO}_\kappa} \leq (2^\kappa)^{\mathrm{V}} < (2^\kappa)^{\mathrm{V}[G]} = \mathfrak{d}^{\mathrm{V}[G]}_{\mathcal{TO}_\kappa}.$$

We will later show that certain closed maximality principles imply that both of the above cardinal characteristics are equal to 2^κ.

3. Closed maximality principles

We introduce the boldface maximality principles for the class of all $<\kappa$-closed partial orders first formulated by Fuchs in [8] and present results showing how this principle answers most questions stated in the previous section.

Definition 3.1. Let $\Phi(v_0, v_1)$ be an $\mathcal{L}_\in$-formula.

(i) We say that Φ *defines a class of partial orders* if

$$\mathrm{ZFC} \vdash \forall \mathbb{P}\ \forall z\ [\Phi(\mathbb{P}, z) \longrightarrow \text{“}\mathbb{P} \textit{ is a partial order}\text{”}].$$

(ii) If Φ defines a class of partial order, then Φ is *suitable* if

$$\mathrm{ZFC} \vdash \forall z\ \forall \mathbb{P}\ [\text{“}\mathbb{P} \textit{ is a trivial partial order}\text{”} \longrightarrow \Phi(\mathbb{P}, z)].$$

The following definition contains our general formulation of boldface maximality principles.

Definition 3.2. Let $\Phi_0(v_0, v_1)$ and $\Phi_1(v_0, v_1)$ be $\mathcal{L}_\in$-formulas defining classes of partial orders and let z be a set.

(i) Given an $\mathcal{L}_\in$-formula $\varphi(v_0, \dots, v_{n-1})$ and parameters $x_0, \dots, x_{n-1}$, we say that the statement $\varphi(x_0, \dots, x_{n-1})$ is (Φ_0, Φ_1, z)-*forceably necessary* if there is a partial order $\mathbb{P}$ with $\Phi_0(\mathbb{P}, z)$ and $\mathbb{1}_{\mathbb{P} * \dot{\mathbb{Q}}} \Vdash \varphi(\check{x}_0, \dots, \check{x}_{n-1})$ for every $\mathbb{P}$-name $\dot{\mathbb{Q}}$ with $\mathbb{1}_{\mathbb{P}} \Vdash \Phi_1(\dot{\mathbb{Q}}, \check{z})$.

(ii) Given an infinite cardinal θ and $n < \omega$, we let $(\Phi_0, \Phi_1, z){-}\mathrm{MP}_n(\theta)$ denote the statement that every (Φ_0, Φ_1, z)-forceably necessary Σ_n-statement with parameters in $\mathrm{H}(\theta)$ is true.

With the help of universal Σ_n-formulas, we can formulate principles of the form $(\Phi_0, \Phi_1, z){-}\mathrm{MP}_n(\theta)$ as single statements in the language of set theory that only use the parameters θ and z. Moreover, if $(\Phi_0, \Phi_1, z){-}\mathrm{MP}_1(\theta)$ holds and $\mathbb{P}$ is a partial order with $\Phi_0(\mathbb{P}, z)$, then forcing with $\mathbb{P}$ preserves all cardinals less than θ.

Let $\Phi_{cl}(v_0, v_1)$ be the canonical suitable formula defining a class of partial orders such that $\Phi_{cl}(\,\cdot\,, \kappa)$ defines the class of all $<\kappa$-closed partial orders if κ is an infinite regular cardinal and $\Phi_{cl}(\,\cdot\,, z)$ defines the class of all partial orders if z is not an infinite regular cardinal. In the following, we abbreviate $(\Phi_{cl}, \Phi_{cl}, \kappa){-}\mathrm{MP}_n(\kappa^+)$ by $\mathrm{CMP}_n(\kappa)$. Principles of this form and their consequences were studied in depth by Fuchs in [8] and [9]. These principles may be viewed as a natural extension of ZFC, because a classical argument of Silver shows that the principle $\mathrm{CMP}_1(\kappa)$ is always true. The short proof of the next proposition can be found in [24, Section 7].

Proposition 3.3 (Silver)**.** *Let $\mathbb{P}$ be a $<\kappa$-closed partial order and let G be $\mathbb{P}$-generic over* V. *Then $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}}, \in\rangle$ is a Σ_1-elementary substructure of $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$.*

Corollary 3.4. *The principle $\mathrm{CMP}_1(\kappa)$ holds.*

Proof. Assume that $\varphi(v_0, \dots, v_{n-1})$ is a Σ_1-formula and $x_0, \dots, x_{n-1} \in \mathrm{H}(\kappa^+)$ with the property that the statement $\varphi(x_0, \dots, x_{n-1})$ is $(\Phi_{cl}, \Phi_{cl}, \kappa)$-forceably necessary. Pick a $<\kappa$-closed partial order $\mathbb{P}$ witnessing this. Since the trivial partial order is $<\kappa$-closed, we have $\mathbb{1}_{\mathbb{P}} \Vdash \varphi(\check{x}_0, \dots, \check{x}_{n-1})$. Let G be $\mathbb{P}$-generic over V. By the Σ_1-Reflection Principle, we know that $\varphi(x_0, \dots, x_{n-1})$ holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. In this situation, Proposition 3.3 implies that $\varphi(x_0, \dots, x_{n-1})$ holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ and therefore it holds in V. □

We present a typical application of the principle $\mathrm{CMP}_2(\kappa)$. Note that, by the above remarks, the conclusion $\kappa = \kappa^{<\kappa}$ implies that the class of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ coincides with the class of subsets of ${}^\kappa\kappa$ that are definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameters. Moreover, this assumption allows us to generalize basic results on the structure of κ-Borel subsets to the uncountable setting (see [6, Section 1.2.1] for a discussion).

Proposition 3.5. *If $\mathrm{CMP}_2(\kappa)$ holds, then $\kappa = \kappa^{<\kappa}$.*

Proof. The statement that there is a surjection of κ onto the collection ${}^{<\kappa}\kappa$ can be expressed by a Σ_2-formula with parameter κ and the partial order $\mathrm{Add}(\kappa, 1)$ witnesses that this statement is $(\Phi_{cl}, \Phi_{cl}, \kappa)$-forceably necessary. □

We now discuss the consistency of the above principles. In [8] and [9], Fuchs derived the following bounds for their consistency strength.

Theorem 3.6 ([8, Theorem 3.8], [9, Lemma 3.9])**.** *Let $1 < n < \omega$.*

(i) If $\mathrm{CMP}_n(\kappa)$ holds and $\theta = \kappa^+$, then θ is Σ_n-reflecting in L.
(ii) If $\theta > \kappa$ is a Σ_{n+2}-reflecting cardinal and G is $\mathrm{Col}(\kappa, <\theta)$-generic over V, *then $\mathrm{CMP}_n(\kappa)$ holds in* $\mathrm{V}[G]$.

The above result also allow us to establish the consistency of closed maximality principles for statements of arbitrary complexity. Let $\mathcal{L}_*$ denote the first-order language that extends $\mathcal{L}_\in$ by an additional constant symbols $\dot{\kappa}$ and $\dot{\theta}$. We let REFL denote the $\mathcal{L}_*$-theory consisting of the axioms of ZFC together with the scheme of $\mathcal{L}_*$-sentences stating that $\dot{\theta}$ is a Σ_n-reflecting cardinal for all $0 < n < \omega$ and the sentence stating that $\dot{\kappa}$ is an uncountable regular cardinal smaller than $\dot{\theta}$. Note that, given a Mahlo cardinal μ and a regular uncountable cardinal $\kappa < \mu$, the set of $\theta < \mu$ with the property that $\langle \mathrm{H}(\mu), \in, \kappa, \theta\rangle$ is a model of REFL is stationary in μ. Next, let CMP denote the $\mathcal{L}_*$-theory consisting of the axioms of ZFC together with the scheme of $\mathcal{L}_*$-sentences stating that $\mathrm{CMP}_n(\dot{\kappa})$ holds for all $0 < n < \omega$. The following meta-result shows that these theories are equiconsistent. It is a direct consequence of the above theorem.

Corollary 3.7. *(i) Assume that $\langle \mathrm{V}, \in, \kappa, \theta\rangle$ is a model of* CMP *and $\theta = \kappa^+$. Then $\langle \mathrm{L}, \in, \kappa, \theta\rangle$ is a model of* REFL.
(ii) Assume that $\langle \mathrm{V}, \in, \kappa, \theta\rangle$ is a model of REFL. *If G is $\mathrm{Col}(\kappa, <\theta)$-generic over* V, *then $\langle \mathrm{V}[G], \in, \kappa, \theta\rangle$ is a model of* CMP. □

In Section 5, we will show that the principle $\mathrm{CMP}_2(\kappa)$ induces a strong structure theory for the class of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$. These results will allow us to show that this axiom settles most of the questions posed in Section 2.

Theorem 3.8. *Assume that* $\mathrm{CMP}_2(\kappa)$ *holds.*

(i) *No* $\mathbf{\Delta}^1_1$*-subset of* ${}^\kappa\kappa$ *separates* $\mathsf{Club}(\kappa)$ *from* $\mathsf{NS}(\kappa)$.
(ii) *Every* $\mathbf{\Sigma}^1_1$*-subset of* ${}^\kappa\kappa$ *has the perfect set property.*
(iii) *The least upper bound for the order-types of* $\mathbf{\Sigma}^1_1$*-wellorderings of subsets of* ${}^\kappa\kappa$ *is equal to* κ^+.

In contrast, a combination of results from [8] and observations about the structure of $\langle \mathcal{TO}_\kappa, \preceq\rangle$ in $\mathrm{Add}(\kappa,\nu)$-generic extensions will allow us to show that axioms of the form $\mathrm{CMP}_n(\kappa)$ do not answer Question 5. We will prove the following result in Section 5.

Theorem 3.9. *If the theory* CMP *is consistent, then it does not decide the statement* $\mathfrak{b}_{\mathcal{TO}_{\dot\kappa}} = \mathfrak{d}_{\mathcal{TO}_{\dot\kappa}}$.

The proof of this result suggests that it is also interesting to consider closed maximality principles for statements containing parameters of higher cardinalities. We present an example of such a principles in the next section.

4. Closed maximality principles with more parameters

Motivated by Theorem 3.9, we want to consider closed maximality principle for statements with parameters of larger cardinalities. In order to obtain a consistent principle, we have to restrict ourselves to classes of forcings that preserve more cardinalities. If we assume that $\kappa = \kappa^{<\kappa}$ holds, then the class of all $<\kappa$-closed partial orders that satisfy the κ^+-chain condition is a canonical candidate for a rich class of partial orders that preserve all cardinals. Note that the cardinal arithmetic assumption $\kappa = \kappa^{<\kappa}$ is needed to make this class of partial orders non-trivial. Moreover, the discussion preceding Proposition 3.5 shows that this assumption has many desirable implications on the basic structure of generalized Baire spaces.

In the following, we will argue that we have to restrict the class of partial orders even further. This will follow from a connection between boldface maximality principles and generalizations of classical forcing axioms to larger cardinalities.

Given a partial order $\mathbb{P}$ and an infinite cardinal μ, we let $\mathsf{FA}_\mu(\mathbb{P})$ denote the statement that, for every collection $\mathcal{D}$ of μ-many dense subsets of $\mathbb{P}$, there is a filter G on $\mathbb{P}$ that meets all elements of $\mathcal{D}$. The following proposition reformulates results of Bagaria (see [1]) and Stavi-Väänänen (see [33, Theorem 25]) to connect maximality principles for Σ_1-statements with forcing axioms.

Proposition 4.1. *Let* $\Phi_0(v_0,v_1)$ *and* $\Phi_1(v_0,v_1)$ *be formulas defining classes of partial orders, let* z *be set and let* $\mu \geq \kappa$ *be a cardinal.*

(i) *If* $(\Phi_0,\Phi_1,z)\text{--}\mathrm{MP}_1(\mu^+)$ *holds and* $\mathbb{P}$ *is a partial order of cardinality at most* μ *with* $\Phi_0(\mathbb{P},z)$, *then* $\mathsf{FA}_\mu(\mathbb{P})$ *holds.*
(ii) *Assume that every partial order* $\mathbb{P}$ *with* $\Phi_0(\mathbb{P},z)$ *satisfies the* μ^+*-chain condition and has the property that* $\mathsf{FA}_\mu(\mathbb{P})$ *holds. If* Φ_1 *is suitable, then* $(\Phi_0,\Phi_1,z)\text{--}\mathrm{MP}_1(\mu^+)$ *holds.*

Proof. (i) We may assume that the underlying set of $\mathbb{P}$ is an ordinal less than or equal to μ. Let $\mathcal{D}$ be a collection of μ-many dense subsets of $\mathbb{P}$. Then $\mathcal{D},\mathbb{P} \in \mathrm{H}(\mu^+)$

and the statement that there is a $\mathcal{D}$-generic filter on $\mathbb{P}$ can be expressed by a Σ_1-formula with parameters $\mathbb{P}$ and $\mathcal{D}$. Moreover, $\mathbb{P}$ witnesses that this statement is (Φ_0, Φ_1, z)-forceably necessary and therefore true in V.

(ii) Fix a Σ_0-formula $\varphi(v_0, \ldots, v_n)$ and $x_0, \ldots, x_{n-1} \in \mathrm{H}(\mu^+)$. Assume that $\mathbb{P}$ is a partial order with $\Phi_0(\mathbb{P}, z)$ that witnesses that the corresponding Σ_1-statement $\exists x \ \varphi(x_0, \ldots, x_{n-1}, x)$ is (Φ_0, Φ_1, z)-forceably necessary. Since Φ_1 is suitable, this implies that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\exists x \ \varphi(x_0, \ldots, x_{n-1}, x)$". Pick a sufficiently large regular cardinal θ and an elementary submodel M of $\mathrm{H}(\theta)$ of cardinality μ with $\mathbb{P}, x_0, \ldots, x_{n-1} \in N$ and $\mu+1 \subseteq M$. Let $\pi : M \longrightarrow N$ be the corresponding transitive collapse and let $\mathcal{D}$ be the collection of all dense open subsets of $\mathbb{P}$ contained in M. Then $\mathsf{FA}_\mu(\mathbb{P})$ holds and there is a $\mathcal{D}$-generic filter G on $\mathbb{P}$. Since $\mathbb{P}$ satisfies the μ^+-chain condition and $\pi \restriction \mu = \mathrm{id}_\mu$, it follows that $\bar{G} = \pi[G \cap M]$ is $\pi(\mathbb{P})$-generic over N. By elementarity, we have $\pi(x_i) = x_i$ for all $i < n$ and $\exists x \ \varphi(x_0, \ldots, x_{n-1}, x)$ holds in $N[\bar{G}]$. Since $N[\bar{G}]$ is transitive, we can conclude that this statement also holds in V. $\square$

Let $\Phi_{clc}(v_0, v_1)$ be the canonical suitable formula defining a class of partial orders such that $\Phi_{clc}(\ \cdot \ , \kappa)$ defines the class of all $<\kappa$-closed partial orders satisfying the κ^+-chain condition if κ is an infinite regular cardinal and $\Phi_{clc}(\ \cdot \ , z)$ defines the class of all partial orders if z is not an infinite regular cardinal. We will use the following result of Shelah to show that the corresponding closed maximality principle $(\Phi_{clc}, \Phi_{clc}, \aleph_1){-}\mathrm{MP}_2(\aleph_3)$ is provably false.

Theorem 4.2 ([31, Theorem 6])**.** *Assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} > \aleph_2$. Then there is a σ-closed partial order $\mathbb{P}$ of cardinality $\aleph_2$ such that $\mathbb{P}$ satisfies the $\aleph_2$-chain condition and $\mathsf{FA}_{\aleph_2}(\mathbb{P})$ fails.*

Corollary 4.3. *If $2^{\aleph_0} = \aleph_1$ holds, then $(\Phi_{clc}, \Phi_{clc}, \aleph_1){-}\mathrm{MP}_2(\aleph_3)$ fails.*

Proof. Assume, towards a contradiction, that $(\Phi_{clc}, \Phi_{clc}, \aleph_1){-}\mathrm{MP}_2(\aleph_3)$ holds. Then the partial order $\mathrm{Add}(\aleph_1, \aleph_2)$ has cardinality $\aleph_2$ and the first part of Proposition 4.1 shows that our assumption implies that $\mathsf{FA}_{\aleph_2}(\mathrm{Add}(\aleph_1, \aleph_2))$ holds. This implies that $2^{\aleph_1} > \aleph_2$ and we can use Theorem 4.2 to find a partial order $\mathbb{P}$ with the property that $\Phi_{clc}(\mathbb{P}, \aleph_1)$ holds and $\mathsf{FA}_{\aleph_2}(\mathbb{P})$ fails. By the first part of Proposition 4.1, this contradicts our assumption. $\square$

The above result suggests that, in order to obtain consistent maximality principles for statements containing parameters of cardinality greater than κ, we should restrict the class of forcings that can be used to witness that a given statement is forceably necessary even further. In particular, it should be consistent that $\mathsf{FA}_{\kappa^+}(\mathbb{P})$ holds for every partial order $\mathbb{P}$ in this class. An example of such a class is contained in Shelah's work on generalization of Martin's Axiom to higher cardinalities (see [30]). Remember that a partial order $\mathbb{P}$ is *well-met* if all compatible conditions have a greatest lower bound in $\mathbb{P}$. Given a infinite regular cardinal μ, we say that a partial order $\mathbb{P}$ is *stationary μ^+-linked* if for every sequence $\langle p_\gamma \mid \gamma < \mu^+ \rangle$ of conditions in $\mathbb{P}$, there is a club C in μ^+ and a regressive function $r : \mu^+ \longrightarrow \mu^+$ with the property that the conditions p_γ and p_δ are compatible in $\mathbb{P}$ for all $\gamma, \delta \in C \cap S^{\mu^+}_\mu$ with $r(\gamma) = r(\delta)$. Let $\Phi_{Sh}(v_0, v_1)$ be the canonical suitable formula defining a class of partial orders such that $\Phi_{Sh}(\ \cdot \ , \kappa)$ defines the class of all stationary κ^+-linked, well-met partial orders that are $<\kappa$-closed with greatest lower bounds if κ is an infinite regular cardinal and $\Phi_{Sh}(\ \cdot \ , z)$ defines the class of all partial orders if z is not an infinite regular cardinal. Following [30], we let GMA_κ denote the statement that

$\mathsf{FA}_\mu(\mathbb{P})$ holds for every $\mu < 2^\kappa$ and all partial orders $\mathbb{P}$ with $\Phi_{Sh}(\mathbb{P},\kappa)$. The results of [30] show that, if $\kappa = \kappa^{<\kappa}$ holds, then GMA_κ holds in a cofinality-preserving forcing extension of the ground model.

In the following, we study the maximality principles $(\Phi_{Sh},\Phi_{clc},\kappa)\text{−}\mathrm{MP}_n(2^\kappa)$ associated to this class. We use $\mathrm{SMP}_n(\kappa)$ to denote the conjunction of this principle and the statement $\kappa = \kappa^{<\kappa}$. Note that similar principles were already studied in [33, Section 2.5]. The following proposition shows that these principles are natural in the sense that they directly generalize Shelah's forcing axiom GMA_κ. We will later show (see Theorem 4.7) that the assumption of this proposition is a consequence of $\mathrm{SMP}_2(\kappa)$.

Proposition 4.4. *Assume that $\mu^{<\kappa} < 2^\kappa$ holds for all $\mu < 2^\kappa$. Then $\mathrm{SMP}_1(\kappa)$ holds if and only if GMA_κ and $\kappa = \kappa^{<\kappa}$ hold.*

Proof. By standard arguments, our cardinal arithmetic assumption implies that GMA_κ holds if and only if $\mathsf{FA}_\mu(\mathbb{P})$ holds for all $\mu < 2^\kappa$ and every partial order $\mathbb{P}$ of cardinality less than 2^κ with $\Phi_{Sh}(\mathbb{P},\kappa)$. Using this observation, the statement of the proposition follows directly from Proposition 4.1. □

The following result gives bounds for the consistency strength of the above principles that resemble the bounds given by Theorem 3.6. The proof of this theorem is contained in Section 6.

Theorem 4.5. *(i) If $\mathrm{SMP}_n(\kappa)$ holds for some $1 < n < \omega$ and $\theta = 2^\kappa$, then θ is Σ_n-reflecting in L.*
(ii) Let $\theta > \kappa$ be an inaccessible cardinal and let $\lhd$ be a wellordering of $\mathrm{H}(\theta)$ of order-type θ. Assume that $\kappa = \kappa^{<\kappa}$ holds. Then there is a $<\kappa$-closed partial order $\mathbb{B}(\kappa,\lhd)$ that is uniformly definable in parameters κ and $\lhd$ with the property that, if θ is Σ_{n+2}-reflecting for some $1 < n < \omega$, then $\mathrm{SMP}_n(\kappa)$ holds in every $\mathbb{B}(\kappa,\lhd)$-generic extension of the ground model V.

As above, we also consider versions of this maximality principle for statements of unbounded complexity. We let SMP denote the $\mathcal{L}_*$-theory consisting of the axioms of ZFC together with the scheme of $\mathcal{L}_*$-sentences stating that $\mathrm{SMP}_n(\dot\kappa)$ holds for all $0 < n < \omega$.

Corollary 4.6. *(i) Assume that $\langle \mathrm{V},\in,\kappa,\theta\rangle$ is a model of SMP and $\theta = 2^\kappa$. Then $\langle \mathrm{L},\in,\kappa,\theta\rangle$ is a model of REFL.*
(ii) Assume that $\kappa = \kappa^{<\kappa}$ holds, $\langle \mathrm{V},\in,\kappa,\theta\rangle$ is a model of REFL and $\lhd$ is a wellordering of $\mathrm{H}(\theta)$ of order-type θ. If G is $\mathbb{B}(\kappa,\lhd)$-generic over V, then $\langle \mathrm{V}[G],\in,\kappa,\theta\rangle$ is a model of SMP. □

The following theorem summarizes several results that will be proven in the next section. In combination, they show that the principle $\mathrm{SMP}_2(\kappa)$ answers all questions posed in Section 2. Moreover, we can combine these results with Proposition 4.4 to conclude that $\mathrm{SMP}_2(\kappa)$ implies GMA_κ.

Theorem 4.7. *Assume that $\mathrm{SMP}_2(\kappa)$ holds.*

(i) The cardinal 2^κ is weakly inaccessible and $\mu^{<\kappa} < 2^\kappa$ holds for all $\mu < 2^\kappa$.
(ii) No $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ separates $\mathsf{Club}(\kappa)$ from $\mathsf{NS}(\kappa)$.
(iii) Every $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ of cardinality 2^κ contains a perfect subset. In particular, every $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ has the Bernstein property.

(iv) *Every subset of* ${}^\kappa\kappa$ *of cardinality less than* 2^κ *is equal to the union of* κ*-many closed subsets of* ${}^\kappa\kappa$.

(v) *The least upper bound for the order-types of* $\mathbf{\Sigma}^1_1$*-wellorderings of subsets of* ${}^\kappa\kappa$ *is equal to* 2^κ.

(vi) $\mathfrak{b}_{\mathcal{TO}_\kappa} = \mathfrak{d}_{\mathcal{TO}_\kappa} = 2^\kappa$.

5. Structural implications

In this section, we show that the principles $\mathrm{CMP}_2(\kappa)$ and $\mathrm{SMP}_2(\kappa)$ induce a strong structure theory for the class of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$. We start by reviewing some basic definitions concerning $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$.

We call a subset T of $({}^{<\kappa}\kappa)^n$ a *subtree of* $({}^{<\kappa}\kappa)^n$ if $\mathrm{lh}(t_0) = \ldots = \mathrm{lh}(t_{n-1})$ and $\langle t_0 \restriction \alpha, \ldots, t_{n-1} \restriction \alpha \rangle \in T$ holds for all $\langle t_0, \ldots, t_{n-1} \rangle \in T$ and $\alpha < \mathrm{lh}(t_0)$. Given such a subtree, we define

$$[T] \ = \ \{\langle x_0, \ldots, x_{n-1} \rangle \in ({}^\kappa\kappa)^n \mid \forall \alpha < \kappa \ \langle x_0 \restriction \alpha, \ldots, x_{n-1} \restriction \alpha \rangle \in T\}$$

to be the set of all *cofinal branches though* T. It is easy to see that a subset A of $({}^\kappa\kappa)^n$ is closed if and only if it is of the form $[T]$ for some subtree T of $({}^{<\kappa}\kappa)^n$. This shows that the $\mathbf{\Sigma}^1_1$-subsets of $({}^\kappa\kappa)^n$ are exactly of the form

$$p[T] \ = \ \{\langle x_0, \ldots, x_n \rangle \in ({}^\kappa\kappa)^n \mid \exists x_n \ \langle x_0, \ldots, x_n \rangle \in [T]\}$$

for some subtree T of $({}^{<\kappa}\kappa)^{n+1}$. This also shows that the assumption $\kappa = \kappa^{<\kappa}$ implies that the class of all $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ coincides with the class of all subsets of ${}^\kappa\kappa$ that are definable over the structure $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_1-formula with parameters.

We now start to derive structural implications from closed maximality principles. It turns out that many of these implications factor through the following concept.

Definition 5.1. Given a partial order $\mathbb{P}$, we say that $\mathbf{\Sigma}_2(\mathrm{H}(\kappa^+))$*-absoluteness holds for* $\mathbb{P}$ if $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}}, \in \rangle$ is a Σ_2-elementary substructure of $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ whenever G is $\mathbb{P}$-generic over V.

Proposition 5.2. *If* $\mathbf{\Sigma}_2(\mathrm{H}(\kappa^+))$*-absoluteness holds for* $\mathrm{Add}(\kappa, 1)$*, then* $\kappa = \kappa^{<\kappa}$.

Proof. If G is $\mathrm{Add}(\kappa, 1)$-generic over V, then $\kappa = \kappa^{<\kappa}$ holds in $\mathrm{V}[G]$ and this implies that the collection $\kappa^{<\kappa}$ is a set in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. Since this statement can be formulated by a Σ_2-formula with parameter κ, it also holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ and hence $\kappa = \kappa^{<\kappa}$ holds in V. □

The following lemma is essentially a reformulation of [9, Theorem 3.6].

Lemma 5.3. *If* Φ *is an* $\mathcal{L}_\in$*-formula that is either equal to* Φ_{cl} *or to* Φ_{Sh} *and* $(\Phi, \Phi_{cl}, \kappa){-}\mathrm{MP}_2(\kappa^+)$ *holds, then* $\mathbf{\Sigma}_2(\mathrm{H}(\kappa^+))$*-absoluteness holds all* $\mathbb{P}$ *with* $\Phi(\mathbb{P}, \kappa)$.

Proof. Let $\mathbb{P}$ be a partial order with $\Phi(\mathbb{P}, \kappa)$ and let G be $\mathbb{P}$-generic over V. Fix a Σ_2-formula $\varphi(v_0, \ldots, x_{n-1})$ and $x_0, \ldots, x_{n-1} \in \mathrm{H}(\kappa^+)$. If $\varphi(x_0, \ldots, x_{n-1})$ holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}}$, then Proposition 3.3 directly implies that this statement also holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. In the other direction, assume that $\varphi(x_0, \ldots, x_{n-1})$ holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$. Pick a condition p in $\mathbb{P}$ that forces this and let $\bar{\mathbb{P}}$ denote the partial order consisting of all extensions of p in $\mathbb{P}$. Since Φ is either equal to Φ_{cl} or to Φ_{Sh}, we know that $\Phi(\bar{\mathbb{P}}, \kappa)$ holds. If $\dot{\mathbb{Q}}$ is a $\bar{\mathbb{P}}$-name for a $<\kappa$-closed partial order and $\bar{G} * H$ is $(\bar{\mathbb{P}} * \dot{\mathbb{Q}})$-generic over V, then Proposition 3.3 implies that $\varphi(x_0, \ldots, x_{n-1})$ also holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}[\bar{G}, H]}$. This argument shows that $\bar{\mathbb{P}}$ witnesses that the statement that $\varphi(x_0,\ldots,x_{n-1})$ holds in $\mathrm{H}(\kappa^+)$ is (Φ,Φ_{cl},κ)-forceably necessary. Since we can formulate this statement by a Σ_2-formula with parameters $\kappa,x_0,\ldots,x_{n-1}\in\mathrm{H}(\kappa^+)$, our assumptions imply that this statement holds in V and therefore $\varphi(x_0,\ldots,x_{n-1})$ holds in $\mathrm{H}(\kappa^+)^{\mathrm{V}}$. $\square$

Next, we show that $\boldsymbol{\Sigma}_2(\mathrm{H}(\kappa^+))$-absoluteness implies that all $\boldsymbol{\Delta}^1_1$-subsets of ${}^\kappa\kappa$ possess a regularity property that generalizes the Baire property to subsets of the generalized Baire space of κ. A detailed discussion of this regularity property can be found in [6, Section IV.3.].

Definition 5.4. We say that a subset X of ${}^\kappa\kappa$ has the *κ-Baire property* if there is an open subset U of ${}^\kappa\kappa$ and a sequence $\langle A_\alpha \mid \alpha<\kappa\rangle$ of closed nowhere dense subsets of ${}^\kappa\kappa$ with the property that $X_\Delta U\subseteq\bigcup\{A_\alpha\mid\alpha<\kappa\}$.

A standard proof shows that the assumption $\kappa=\kappa^{<\kappa}$ implies that every κ-Borel subset of ${}^\kappa\kappa$ has the κ-Baire property (see [10] and Corollary 5.10 below). Moreover, it is also known that the $\boldsymbol{\Sigma}^1_1$-subsets $\mathsf{Club}(\kappa)$ and $\mathsf{NS}(\kappa)$ do not have the κ-Baire property (see [11, Theorem 4.2]). Finally, it is consistent that every $\boldsymbol{\Delta}^1_1$-subset has the κ-Baire property (see [6, Theorem 49]). In the following, we will present an alternative proof of this result that makes use of the following notion.

Definition 5.5. Given a partial order $\mathbb{P}$, we say that a subset X of ${}^\kappa\kappa$ is a *$\mathbb{P}$-absolutely $\boldsymbol{\Delta}^1_1$-definable* if there are subtrees T_0 and T_1 of ${}^{<\kappa}\kappa\times{}^{<\kappa}\kappa$ with the property that $p[T_0]=X$, $p[T_1]={}^\kappa\kappa\setminus X$ and $\mathbb{1}_{\mathbb{P}}\Vdash$ "${}^{\check\kappa}\check\kappa=p[\check T_0]\cup p[\check T_1]$".

The following observation is a direct consequence of Definition 5.1.

Proposition 5.6. *If $\boldsymbol{\Sigma}_2(\mathrm{H}(\kappa^+))$-absoluteness holds for a partial order $\mathbb{P}$, then every $\boldsymbol{\Delta}^1_1$-subset of ${}^\kappa\kappa$ is $\mathbb{P}$-absolutely $\boldsymbol{\Delta}^1_1$-definable.* $\square$

Moreover, the proof of [26, Lemma 1.11] yields the following statement.

Proposition 5.7. *If $\mathbb{P}$ is a $<\kappa$-closed partial order, then every κ-Borel subset of ${}^\kappa\kappa$ is $\mathbb{P}$-absolutely $\boldsymbol{\Delta}^1_1$-definable.*

A careful review of the definition of the forcing relation (see, for example, [21, Section VII.3]) yields the following observation that is used throughout this paper.

Proposition 5.8. *Given $0<n<\omega$ and a Σ_n-formula $\varphi(v_0,\ldots,v_{k-1})$, there are Σ_n-formulas $\psi_0(v_0,\ldots,v_{k+1})$ and $\psi_1(v_0,\ldots,v_{k+1})$ such that*[1]

$$\begin{aligned}&\mathrm{ZFC}^-\vdash\forall x_0,\ldots,x_{k-1}\ \forall\mathbb{P}\ \textit{partial order}\ \forall p\in\mathbb{P}\\ &\qquad\big[(\psi_0(x_0,\ldots,x_{k-1},\mathbb{P},p)\ \longleftrightarrow\ p\Vdash_{\mathbb{P}}\varphi(\check x_0,\ldots,\check x_{k-1}))\\ &\qquad\ \wedge\ (\neg\psi_1(x_0,\ldots,x_{k-1},\mathbb{P},p)\ \longleftrightarrow\ p\Vdash_{\mathbb{P}}\neg\varphi(\check x_0,\ldots,\check x_{k-1}))\big].\end{aligned}$$

Lemma 5.9. *If $\kappa=\kappa^{<\kappa}$, then every $\mathrm{Add}(\kappa,1)$-absolutely $\boldsymbol{\Delta}^1_1$-definable subset of ${}^\kappa\kappa$ has the κ-Baire property.*

Proof. Let X be such a subset of ${}^\kappa\kappa$ and let T_0 and T_1 be subtrees of ${}^{<\kappa}\kappa\times{}^{<\kappa}\kappa$ with $p[T_0]=X$, $p[T_1]={}^\kappa\kappa\setminus X$ and

$$\mathbb{1}_{\mathrm{Add}(\kappa,1)}\Vdash\text{“}{}^{\check\kappa}\check\kappa=p[\check T_0]\cup p[\check T_1]\text{”}.\tag{2}$$

[1] By ZFC^-, we mean the usual axioms of ZFC without the power set axiom, however including the Collection scheme instead of the Replacement scheme. Note that $\mathrm{H}(\nu)$ is a model of this theory for every uncountable regular cardinal ν.

Since the assumption $\kappa = \kappa^{<\kappa}$ implies that (2) can be formulated by a Σ_1-formula with parameters in $\mathrm{H}(\kappa^+)$, Proposition 5.8 and the Σ_1-Reflection Principle imply that (2) also holds in $\mathrm{H}(\kappa^+)$. Let M be an elementary submodel of $\mathrm{H}(\kappa^+)$ of cardinality κ with $\kappa+1 \subseteq M$ and $T_0, T_1 \in M$. Then M is transitive, the assumption $\kappa = \kappa^{<\kappa}$ implies that ${}^{<\kappa}\kappa \subseteq M$ and elementarity implies that (2) holds in M. Let $\langle A_\alpha \mid \alpha < \kappa \rangle$ enumerate all closed nowhere dense subsets of ${}^\kappa\kappa$ contained in M. We let $\dot{x}$ denote the canonical $\mathrm{Add}(\kappa,1)$-name for the generic function from κ to κ and define

$$U \;=\; \bigcup\{N_s \mid s \Vdash^M_{\mathrm{Add}(\kappa,1)} \text{“}\dot{x} \in p[\check{T}_0]\text{”}\} \;\subseteq\; {}^\kappa\kappa.$$

Claim. $X \setminus U \subseteq \bigcup\{A_\alpha \mid \alpha < \kappa\}$.

Proof of the Claim. Pick $x \in X \setminus \bigcup\{A_\alpha \mid \alpha < \kappa\}$. Since x is an element of every dense open subset of ${}^\kappa\kappa$ contained in M, the filter $G_x = \{x \restriction \alpha \mid \alpha < \kappa\}$ is $\mathrm{Add}(\kappa,1)$-generic over M. Moreover, we have $p[T_1]^{M[G_x]} \subseteq p[T_1] = {}^\kappa\kappa \setminus X$. Since (2) holds in M, we know that

$${}^\kappa\kappa \cap M[G_x] \;=\; p[T_0]^{M[G_x]} \cup p[T_1]^{M[G_x]}$$

and therefore $x \in p[T_0]^{M[G_x]}$. Then there is $\alpha < \kappa$ with $x{\restriction}\alpha \Vdash^M_{\mathrm{Add}(\kappa,1)}$ “$\dot{x} \in p[\check{T}_0]$”. This allows us to conclude that $x \in N_{x\restriction\alpha} \subseteq U$. □

Claim. $U \setminus X \subseteq \bigcup\{A_\alpha \mid \alpha < \kappa\}$.

Proof of the Claim. Pick $x \in U \setminus \bigcup\{A_\alpha \mid \alpha < \kappa\}$. As above, the function x is an element of every dense open subset of ${}^\kappa\kappa$ contained in M and therefore the filter $G_x = \{x \restriction \alpha \mid \alpha < \kappa\}$ is $\mathrm{Add}(\kappa,1)$-generic over M. Since $x \in U$, we can conclude that $x \in p[T_0]^{M[G_x]} \subseteq p[T_0] = X$. □

In combination, the above claims show that the open subset U and the sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ witness that X has the κ-Baire property. □

Corollary 5.10. *If $\kappa = \kappa^{<\kappa}$ holds, then every κ-Borel subset of ${}^\kappa\kappa$ has the κ-Baire property.* □

Corollary 5.11. *If $\boldsymbol{\Sigma}_2(\mathrm{H}(\kappa^+))$-absoluteness holds for* $\mathrm{Add}(\kappa,1)$*, then every $\boldsymbol{\Delta}^1_1$-subset of ${}^\kappa\kappa$ has the κ-Baire property.* □

Corollary 5.12. *If $\kappa = \kappa^{<\kappa}$ and G is $\mathrm{Add}(\kappa,\kappa^+)$-generic over* V*, then all $\boldsymbol{\Delta}^1_1$-subsets of ${}^\kappa\kappa$ have the κ-Baire property in* $\mathrm{V}[G]$.

Proof. By [24, Lemma 9.1], $\boldsymbol{\Sigma}_2(\mathrm{H}(\kappa^+))$-absoluteness holds for $\mathrm{Add}(\kappa,1)$ in $\mathrm{V}[G]$. Therefore the statement of the corollary follows directly from Lemma 5.9. □

We now show that the first statement of Theorem 3.8 and the second statement of Theorem 4.7 are direct consequences of the above lemma and the results of [11].

Definition 5.13. A subset X of ${}^\kappa\kappa$ *super-dense* if $\bigcap\{U_\alpha \cap X \mid \alpha < \kappa\}) \neq \emptyset$ for all sequences $\langle U_\alpha \mid \alpha < \kappa\rangle$ of dense open subsets of non-empty open subsets U of ${}^\kappa\kappa$.

Proposition 5.14 ([26, Proposition 3.7])**.** *Assume that A and B are disjoint super-dense subsets of ${}^\kappa\kappa$. If $A \subseteq X \subseteq {}^\kappa\kappa \setminus B$, then X does not have the κ-Baire property.*

The following statement is essentially shown in the proof of [11, Theorem 4.2].

Theorem 5.15. *The subsets* $\mathsf{Club}(\kappa)$ *and* $\mathsf{NS}(\kappa)$ *of ${}^\kappa\kappa$ are super-dense.*

Other interesting examples of disjoint super-dense $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ are constructed in [26, Section 3].

Corollary 5.16. *If all $\mathbf{\Delta}^1_1$-subsets of ${}^\kappa\kappa$ have the κ-Baire property, then no $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ separates $\mathsf{Club}(\kappa)$ from $\mathsf{NS}(\kappa)$.* □

Proof of Theorem 2.1. Assume that $\kappa = \kappa^{<\kappa}$ and let G be $\mathrm{Add}(\kappa, \kappa^+)$-generic over V. Then Corollary 5.12 implies that all $\mathbf{\Delta}^1_1$-subsets of ${}^\kappa\kappa$ have the κ-Baire property in V$[G]$. By Corollary 5.16, there is no $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ in V$[G]$ that separates $\mathsf{Club}(\kappa)$ from $\mathsf{NS}(\kappa)$. □

Proof of Clause (i) of Theorem 3.8 and Clause (ii) of Theorem 4.7. If we assume that either $\mathrm{CMP}_2(\kappa)$ or $\mathrm{SMP}_2(\kappa)$ holds, then Lemma 5.3 implies that $\mathbf{\Sigma}_2(\mathrm{H}(\kappa^+))$-absoluteness holds every $\mathrm{Add}(\kappa, 1)$ and Lemma 5.9 shows that all $\mathbf{\Delta}^1_1$-subsets of ${}^\kappa\kappa$ have the κ-Baire property. This allows us to use Corollary 5.16 to conclude that there is no $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ separates $\mathsf{Club}(\kappa)$ from $\mathsf{NS}(\kappa)$. □

Next, we discuss the Bernstein property and the perfect set property. We will make use of the following notion defined in [24].

Definition 5.17. Given a subtree T of $({}^{<\kappa}\kappa)^{n+1}$, a map $\iota : {}^{<\kappa}2 \longrightarrow T$ is a $\exists^x$-*perfect embedding into* T if the following statements hold for all $s_0, s_1 \in {}^{<\kappa}2$ with $\iota(s_i) = \langle t^i_0, \ldots, t^i_n \rangle$.

(i) If $s_0 \subsetneq s_1$, then $t^0_k \subsetneq t^1_k$ for all $k \leq n$.
(ii) If the sequences s_0 and s_1 are incompatible, then there is a $k < n$ such that the sequences t^0_k and t^1_k are incompatible.
(iii) If $\mathrm{lh}(s_0) \in \mathrm{Lim}$ and $k \leq n$, then
$$t^0_k = \bigcup \{ u^\alpha_k \mid \exists \alpha < \mathrm{lh}(s_0) \ \iota(s_0 \restriction \alpha = \langle u^\alpha_0, \ldots, u^\alpha_n \rangle) \}.$$

Proposition 5.18 ([24, Proposition 7.5])**.** *If T is a subtree of $({}^{<\kappa}\kappa)^{n+1}$ such that there is a $\exists^x$-perfect embedding into T, then $p[T]$ contains a perfect subset.*

Lemma 5.19 ([24, Lemma 7.6])**.** *Assume $\kappa = \kappa^{<\kappa}$ and let T be a subtree of $({}^{<\kappa}\kappa)^{n+1}$. Then there exists a $\exists^x$-perfect embedding into T if and only if there is a $<\kappa$-closed partial order $\mathbb{P}$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$p[\check{T}] \nsubseteq \check{\mathrm{V}}$".*

Lemma 5.20. *If $\kappa = \kappa^{<\kappa}$, then every $\mathrm{Add}(\kappa, 1)$-absolutely $\mathbf{\Delta}^1_1$-definable subset of ${}^\kappa\kappa$ has the Bernstein property.*

Proof. Let X be such a subset of ${}^\kappa\kappa$, let T_0 and T_1 be subtrees of ${}^{<\kappa}\kappa$ witnessing this, let G be $\mathrm{Add}(\kappa, 1)$-generic over V and let x denote the generic function in V$[G]$. By our assumptions, there is $i < 2$ with $x \in p[T_i]^{\mathrm{V}[G]}$ and hence $p[T_i]^{\mathrm{V}[G]} \nsubseteq \mathrm{V}$. By Lemma 5.19, this shows that there is a $\exists^x$-perfect embedding into T_i in V and Proposition 5.18 shows that either X or ${}^\kappa\kappa \setminus X$ contains a perfect subset in V. □

Corollary 5.21. *If $\kappa = \kappa^{<\kappa}$ holds, then every κ-Borel subset of ${}^\kappa\kappa$ has the Bernstein property.* □

Corollary 5.22. *If $\mathbf{\Sigma}_2(\mathrm{H}(\kappa^+))$-absoluteness holds for $\mathrm{Add}(\kappa, 1)$, then every $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ has the Bernstein property.* □

Proof of Theorem 2.4. Assume that $\kappa = \kappa^{<\kappa}$ and let G be $\mathrm{Add}(\kappa, \kappa^+)$-generic over V. By [24, Lemma 9.1], $\mathbf{\Sigma}_2(\mathrm{H}(\kappa^+))$-absoluteness holds for $\mathrm{Add}(\kappa, 1)$ holds in V$[G]$. In this situation, Corollary 5.22 implies the statement of the theorem. □

Proof of Clause (ii) of Theorem 3.8. Assume that $\mathrm{CMP}_2(\kappa)$ holds and let T be a subtree of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$. If $\dot{\mathbb{Q}}$ is a $\mathrm{Col}(\kappa, 2^\kappa)$-name for a $<\kappa$-closed partial order and $G * H$ is $(\mathrm{Col}(\kappa, 2^\kappa) * \dot{\mathbb{Q}})$-generic over V, then either $p[T]^{\mathrm{V}} \subsetneq p[T]^{\mathrm{V}[G,H]}$ or $p[T]^{\mathrm{V}[G,H]}$ has cardinality κ in $\mathrm{V}[G, H]$. By Lemma 5.19, the first alternative implies that there is a $\exists^x$-perfect embedding into T in V and this function is still a $\exists^x$-perfect embedding into T in $\mathrm{V}[G, H]$. This shows that $\mathrm{Col}(\kappa, 2^\kappa)$ witnesses that the statement that either $p[T]$ has cardinality κ or there is a $\exists^x$-perfect embedding into T is $(\Phi_{cl}, \Phi_{cl}, \kappa)$-forceably necessary. By Proposition 3.5, this statement can be formulated by a Σ_2-formula with parameters in $\mathrm{H}(\kappa^+)$. Hence $\mathrm{CMP}_2(\kappa)$ implies that the statement is true in V and Proposition 5.18 shows that $p[T]$ has the perfect subset property in V. □

Proof of Clause (iii) of Theorem 4.7. Assume that $\mathrm{SMP}_2(\kappa)$ holds and let T be a subtree of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$. Let $\dot{\mathbb{Q}}$ be an $\mathrm{Add}(\kappa, (2^\kappa)^+)$-name for a $<\kappa$-closed partial order satisfying the κ^+-chain condition and let $G * H$ is $(\mathrm{Add}(\kappa, (2^\kappa)^+) * \dot{\mathbb{Q}})$-generic over V. Since forcing with $(\mathrm{Add}(\kappa, (2^\kappa)^+) * \dot{\mathbb{Q}})$ preserves all cardinals, we know that either $p[T]^{\mathrm{V}} \subsetneq p[T]^{\mathrm{V}[G,H]}$ holds or there is no surjection from $p[T]^{\mathrm{V}[G,H]}$ onto $\mathcal{P}(\kappa)^{\mathrm{V}[G,H]}$ in $\mathrm{V}[G, H]$. As above, the first alternative implies that there is a $\exists^x$-perfect embedding into T in $\mathrm{V}[G, H]$. This shows that $\mathrm{Add}(\kappa, (2^\kappa)^+)$ witnesses that the statement that either there is a $\exists^x$-perfect embedding into T or there is no surjection from $p[T]$ onto $\mathcal{P}(\kappa)$ is $(\Phi_{Sh}, \Phi_{clc}, \kappa)$-forceably necessary. Since $\mathrm{SMP}_2(\kappa)$ entails $\kappa = \kappa^{<\kappa}$, we can formulate this statement by a Σ_2-formula with parameters in $\mathrm{H}(\kappa^+)$ and therefore $\mathrm{SMP}_2(\kappa)$ implies that the statement holds in V. We can conclude that $p[T]$ either contains a perfect subset or has cardinality less than 2^κ. □

In the following, we will use the above results to prove statements about the length of $\mathbf{\Sigma}^1_1$-wellorderings in the presence of closed maximality principles. The starting point is the following result from [24] that is used in the proof of Theorem 2.7 and Theorem 2.8.

Lemma 5.23 ([24, Lemma 7.15])**.** *Assume that $\mathbf{\Sigma}_2(\mathrm{H}(\kappa^+))$-absoluteness holds for* $\mathrm{Add}(\kappa, 1)$. *If $\langle A, \lhd\rangle$ is a $\mathbf{\Sigma}^1_1$-wellordering of a subset of ${}^\kappa\kappa$, then A does not contain a perfect subset.*

Proof of Clause (iii) of Theorem 3.8. Assume that $\mathrm{CMP}_2(\kappa)$ holds. Then Lemma 5.3 shows that the assumption of Lemma 5.23 is satisfied. Since the domains of $\mathbf{\Sigma}^1_1$-wellorderings of subsets of ${}^\kappa\kappa$ are $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$, a combination of Lemma 5.23 with the second part of Theorem 3.8 shows that these domains have cardinality at most κ. In the other direction, it is easy to see that every subset of $({}^\kappa\kappa)^n$ of cardinality at most κ is a $\mathbf{\Sigma}^1_1$-subset. In particular, for every $\gamma < \kappa^+$, there is a $\mathbf{\Sigma}^1_1$-wellordering of a subset of ${}^\kappa\kappa$ of order-type γ. □

In order to prove the corresponding result for the principle $\mathrm{SMP}_2(\kappa)$, we first need to derive the fourth statement of Theorem 4.7 to show that this principle implies that for every $\gamma < 2^\kappa$ there is a $\mathbf{\Sigma}^1_1$-wellordering of a subset of ${}^\kappa\kappa$ of order-type γ. The proof of this result relies on the generic coding techniques studied in [14, Section 2].

Definition 5.24. Assume that A is a subset of ${}^\kappa\kappa$ and $\vec{s} = \langle s_\beta \mid \beta < \kappa\rangle$ is an enumeration of ${}^{<\kappa}\kappa$ with the property that every element of ${}^{<\kappa}\kappa$ is enumerated unboundedly often. We let $\mathbb{C}_{\vec{s}}(A)$ denote the unique partial order defined by the following clauses:

(i) A condition in $\mathbb{C}_{\vec{s}}(A)$ is a pair $p = \langle t_p, a_p\rangle$ such that $t_p : \alpha_p \longrightarrow 2$ for some $\alpha_p < \kappa$ and a_p is a subset of A of cardinality less than κ.

(ii) Given conditions p and q in $\mathbb{C}_{\vec{s}}(A)$, we have $p \leq_{\mathbb{C}_{\vec{s}}(A)} q$ if and only if $t_q \subseteq t_p$, $\alpha_q \leq \alpha_p$ and
$$s_\beta \subseteq x \ \longrightarrow \ t_p(\beta) = 1$$
for all $x \in a_q$ and $\alpha_q \leq \beta < \alpha_p$.

Proposition 5.25. *In the situation of Definition 5.24, the partial order $\mathbb{C}_{\vec{s}}(A)$ is stationary κ^+-linked, well-met and $<\kappa$-closed with greatest lower bounds.*

Proof. By [14, Proposition 2.2], $\mathbb{C}_{\vec{s}}(A)$ is κ-linked and $<\kappa$-closed with greatest lower bounds. In particular, this implies that $\mathbb{C}_{\vec{s}}(A)$ is stationary κ^+-linked. Fix compatible conditions p and q in $\mathbb{C}_{\vec{s}}(A)$ with $\alpha_p \geq \alpha_q$. Then we have $t_q \subseteq t_p$ and $r = \langle t_p, a_p \cup a_q\rangle$ is a condition in $\mathbb{C}_{\vec{s}}(A)$ that extends p. Pick $x \in a_q$ and $\alpha_q \leq \beta < \alpha_p$ with $s_\beta \subseteq x$. Pick a condition s in $\mathbb{C}_{\vec{s}}(A)$ with $s \leq_{\mathbb{C}_{\vec{s}}(A)} p, q$. Then $t_p \subseteq t_s$, $\beta < \alpha_s$ and $t_p(\beta) = t_s(\beta) = 1$. This shows that r is also an extension of q. Finally, it is easy to check that $u \leq_{\mathbb{C}_{\vec{s}}(A)} r$ holds for all conditions u in $\mathbb{C}_{\vec{s}}(A)$ with $u \leq_{\mathbb{C}_{\vec{s}}(A)} p, q$. □

Under the assumptions listed in Definition 5.24, there is a canonical sequence $\langle \dot{T}_\alpha \mid \alpha < \kappa\rangle$ of $\mathbb{C}_{\vec{s}}(A)$-names for subtrees of ${}^{<\kappa}\kappa$ with the property that, whenever G is $\mathbb{C}_{\vec{s}}(A)$-generic over V and $t_G = \bigcup\{t_p \mid p \in G\} : \kappa \longrightarrow 2$, then
$$\dot{T}^G_\alpha \ = \ \{t \in {}^{<\kappa}\kappa \mid \forall \alpha < \beta < \kappa \ [t_G(\beta) = 0 \longrightarrow s_\beta \nsubseteq t]\}$$
for all $\alpha < \kappa$.

Theorem 5.26 ([14, Corollary 4.3])**.** *In the situation of Definition 5.24, if $\dot{\mathbb{Q}}$ is a $\mathbb{C}_{\vec{s}}(A)$-name for a $<\kappa$-closed partial order and $G * H$ is $(\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}})$-generic over* V*, then $A = \bigcup\{[\dot{T}^G_\alpha]^{\mathrm{V}[G,H]} \mid \alpha < \kappa\}$.*

Proof of Clause (iv) of Theorem 4.7. Assume that $\mathrm{SMP}_2(\kappa)$ holds and A is a subset of ${}^\kappa\kappa$ of cardinality less than 2^κ. Since $\kappa = \kappa^{<\kappa}$ holds, there is an enumeration $\vec{s}$ of ${}^{<\kappa}\kappa$ that satisfies the requirements of Definition 5.24. If $\dot{\mathbb{Q}}$ is a $\mathbb{C}_{\vec{s}}(A)$-name for a $<\kappa$-closed partial order and $G * H$ is $(\mathbb{C}_{\vec{s}}(A) * \dot{\mathbb{Q}})$-generic over V, then Theorem 5.26 implies that $A = \bigcup\{[\dot{T}^G_\alpha]^{\mathrm{V}[G,H]} \mid \alpha < \kappa\}$ is equal to the union of κ-many closed subsets of ${}^\kappa\kappa$ in $\mathrm{V}[G, H]$. By Proposition 5.25, this shows that the partial order $\mathbb{C}_{\vec{s}}(A)$ witnesses that the statement that there is a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ of subtrees of ${}^{<\kappa}\kappa$ with the property that $A = \bigcup\{[T_\alpha] \mid \alpha < \kappa\}$ is $(\Phi_{Sh}, \Phi_{cl}, \kappa)$-forceably necessary. Since this statement can be expressed by a Σ_2-formula with parameters $\kappa, A \in \mathrm{H}(2^\kappa)$, our assumptions imply that it holds in V. □

Proof of Clause (v) of Theorem 4.7. Assume that $\mathrm{SMP}_2(\kappa)$ holds. Then Lemma 5.3 shows that the assumption of Lemma 5.23 is satisfied. Hence we can apply the third part of Theorem 4.7 to conclude that the domains of $\mathbf{\Sigma}^1_1$-wellorderings of subsets of ${}^\kappa\kappa$ have cardinality less than 2^κ and therefore these orderings have order-type less than 2^κ. In the other direction, the fourth part of Theorem 4.7 implies that every subset of ${}^\kappa\kappa \times {}^\kappa\kappa$ of cardinality less than 2^κ is a $\mathbf{\Sigma}^1_1$-subset. In particular, for every $\gamma < 2^\kappa$, there is a $\mathbf{\Sigma}^1_1$-wellordering of a subset of ${}^\kappa\kappa$ of order-type γ. □

In the remainder of this section, we will study the influence of closed maximality principles on the cardinal characteristics of the partial order $\langle\mathcal{TO}_\kappa,\preceq\rangle$. These results will also allow us to prove Theorem 2.9. The starting point of this investigation is the following *Boundedness Lemma.* This result was first proven for $\kappa=\omega_1$ in [28] and the proof given there directly generalizes to higher regular cardinalities (see [24, Lemma 8.1]).

Lemma 5.27 ([28, Corollary 13])**.** *If $\kappa=\kappa^{<\kappa}$ holds and A is a $\mathbf{\Sigma}^1_1$-subset of TO_κ, then there is a tree $\mathbb{T}$ in $\mathcal{TO}_\kappa$ with $\mathbb{T}_x\preceq\mathbb{T}$ for all $x\in A$.*

Proof of Clause (vi) of Theorem 4.7. Assume that $\mathrm{SMP}_2(\kappa)$ holds. Let B be a subset of $\mathcal{TO}_\kappa$ of cardinality less than 2^κ. Pick a subset A of TO_κ of the same cardinality such that for every $\mathbb{S}\in B$, there is an $x\in A$ such that the trees $\mathbb{S}$ and $\mathbb{T}_x$ are isomorphic. In this situation, the fourth part of Theorem 4.7 implies that A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ and Lemma 5.27 shows that there is a tree $\mathbb{T}$ in $\mathcal{TO}_\kappa$ with $\mathbb{S}\preceq\mathbb{T}$ for all $\mathbb{S}\in B$. In combination with (1), these computations show that $\mathrm{SMP}_2(\kappa)$ implies that $\mathfrak{b}_{\mathcal{TO}_\kappa}=\mathfrak{d}_{\mathcal{TO}_\kappa}=2^\kappa$. □

In order to prove Theorem 2.9 and Theorem 3.9, we study the behaviour of the partial order $\langle\mathcal{TO}_\kappa,\preceq\rangle$ in $\mathrm{Add}(\kappa,\nu)$-generic extensions of the ground model.

Lemma 5.28. *If $\kappa=\kappa^{<\kappa}$ holds, G is $\mathrm{Add}(\kappa,1)$-generic over V and $\mathbb{T}\in\mathcal{TO}_\kappa^{\mathrm{V}[G]}$, then there is $\mathbb{S}\in\mathcal{TO}_\kappa^{\mathrm{V}}$ such that $\mathbb{S}\not\preceq\mathbb{T}$ in $\mathrm{V}[G]$.*

Proof. Work in V. Let $\dot{x}$ be an $\mathrm{Add}(\kappa,1)$-nice name for an element of TO_κ. Then $\dot{x}\in\mathrm{H}(\kappa^+)$. Define

$$A(\dot{x}) \ = \ \{x\in\mathsf{T}_\kappa \mid \exists p\in\mathrm{Add}(\kappa,1)\ p\Vdash_{\mathrm{Add}(\kappa,1)} \text{“}\mathbb{T}_{\check{x}}\preceq\mathbb{T}_{\dot{x}}\text{”}\}.$$

Since the ordering $\preceq$ is defined by a Σ_1-formula with parameter κ, we can use Proposition 5.8 and the Σ_1-Reflection Principle to show that $A(\dot{x})$ is definable over $\langle\mathrm{H}(\kappa^+),\in\rangle$ by a Σ_1-formula with parameters.

Next, pick $x\in A(\dot{x})$ and let G be $\mathrm{Add}(\kappa,1)$-generic over V such that $\mathbb{T}_x\preceq\mathbb{T}_{\dot{x}^G}$ holds in $\mathrm{V}[G]$. Since $\mathbb{T}_{\dot{x}^G}\in\mathcal{TO}_\kappa^{\mathrm{V}[G]}$, we have $x\in\mathsf{TO}_\kappa^{\mathrm{V}[G]}$ and Proposition 3.3 implies that $x\in\mathsf{TO}_\kappa^{\mathrm{V}}$. These computations show that $A(\dot{x})$ is a $\mathbf{\Sigma}^1_1$-subset of TO_κ in V. Since results of Kurepa (see [16, Lemma 2.1]) show that the assumption $\kappa=\kappa^{<\kappa}$ implies that $\langle\mathcal{TO}_\kappa,\preceq\rangle$ contains no maximal elements, we can use Lemma 5.27 to find a tree $\mathbb{T}(\dot{x})$ in $\mathcal{TO}_\kappa^{\mathrm{V}}$ with $\mathbb{T}_x\preceq\mathbb{T}(\dot{x})$ and $\mathbb{T}(\dot{x})\not\preceq\mathbb{T}_x$ for all $x\in A(\dot{x})$.

Now, let G be $\mathrm{Add}(\kappa,1)$-generic over V and let $\mathbb{T}$ be an element of $\mathcal{TO}_\kappa^{\mathrm{V}[G]}$. Pick an $\mathrm{Add}(\kappa,1)$-nice name $\dot{x}$ for an element of TO_κ such that the tree $\mathbb{T}_{\dot{x}^G}$ is isomorphic to $\mathbb{T}$ in $\mathrm{V}[G]$. Then $\mathbb{T}(\dot{x})\in\mathcal{TO}_\kappa^{\mathrm{V}}$. Assume, towards a contradiction, that $\mathbb{T}(\dot{x})\preceq\mathbb{T}$ holds in $\mathrm{V}[G]$. Then there is an $x\in A(\dot{x})$ such that the tree $\mathbb{T}_x$ is isomorphic to $\mathbb{T}(\dot{x})$ in V. But this is a contradiction, because $x\in A(\dot{x})$ implies that $\mathbb{T}(\dot{x})\not\preceq\mathbb{T}_x$ holds in V. □

Lemma 5.29. *Assume that $\kappa=\kappa^{<\kappa}$ holds. If $\nu>\kappa$ and G is $\mathrm{Add}(\kappa,\nu)$-generic over V, then $\mathfrak{b}^{\mathrm{V}[G]}_{\mathcal{TO}_\kappa}\leq(2^\kappa)^{\mathrm{V}}$.*

Proof. Assume, towards a contradiction, that there is $\mathbb{T}\in\mathcal{TO}_\kappa^{\mathrm{V}[G]}$ with the property that $\mathbb{T}_x\preceq\mathbb{T}$ holds in $\mathrm{V}[G]$ for all $x\in\mathsf{TO}_\kappa^{\mathrm{V}}$. Pick $y\in\mathsf{TO}_\kappa^{\mathrm{V}[G]}$ with the property that the trees $\mathbb{T}$ and $\mathbb{T}_y$ are isomorphic in $\mathrm{V}[G]$. Then we can find $H\in\mathrm{V}[G]$ such that H is $\mathrm{Add}(\kappa,1)$-generic over V with $y\in\mathrm{V}[H]$ and $\mathrm{V}[G]$ is an $\mathrm{Add}(\kappa,\nu)$-generic extension of $\mathrm{V}[H]$. In this situation, Lemma 5.28 yields an $x \in \mathsf{TO}_\kappa^{\mathrm{V}}$ with the property that $\mathbb{T}_x \not\preceq \mathbb{T}_y$ holds in $\mathrm{V}[H]$. By Proposition 3.3, this implies that $\mathbb{T}_x \not\preceq \mathbb{T}_y$ also holds in $\mathrm{V}[G]$, a contradiction. $\square$

In the other direction, the following lemma will enable us to find lower bounds for $\mathfrak{d}_{\mathcal{TO}_\kappa}$ in $\mathrm{Add}(\kappa,\nu)$-generic extensions of the ground model.

Lemma 5.30. *If $\kappa = \kappa^{<\kappa}$ holds, then there is a $\mathrm{Add}(\kappa,1)$-name $\dot{\mathbb{T}}$ for an element of $\mathcal{TO}_\kappa$ such that $\mathbb{1}_{\mathrm{Add}(\kappa,1)} \Vdash$ "$\dot{\mathbb{T}} \not\preceq \mathbb{T}_{\check{x}}$" holds for all $x \in \mathsf{TO}_\kappa$.*

Proof. Let $\dot{x}$ be the canonical $\mathrm{Add}(\kappa,1)$-name for the generic function from κ to 2 and let $\dot{T}$ be a canonical $\mathrm{Add}(\kappa,1)$-name for the subtree of ${}^{<\kappa}\kappa$ with the property that, whenever G is $\mathrm{Add}(\kappa,1)$-generic over V, then $\dot{T}^G$ consists of all strictly increasing, continuous functions s in ${}^{<\kappa}\kappa$ with $\dot{x}^G(s(\alpha)) = 1$ for all $\alpha \in \mathrm{dom}(s)$. If G is $\mathrm{Add}(\kappa,1)$-generic over V, then the set $\{\alpha < \kappa \mid \dot{x}^G(\alpha) = 1\}$ is both a stationary and costationary subset of κ in $\mathrm{V}[G]$ and this implies that $\dot{T}^G \in \mathcal{TO}_\kappa^{\mathrm{V}[G]}$.

Assume, towards a contradiction, that there is an $x \in \mathrm{TO}_\kappa^{\mathrm{V}}$, an $\mathrm{Add}(\kappa,1)$-name $\dot{f}$ for a function with domain $\dot{T}$ and a condition p_0 in $\mathrm{Add}(\kappa,1)$ with

$$p_0 \Vdash_{\mathrm{Add}(\kappa,1)} \text{“}\dot{f} : \langle \dot{T}, \subseteq\rangle \longrightarrow \mathbb{T}_{\check{x}} \textit{ witnesses that } \langle \dot{T}, \subseteq\rangle \preceq \mathbb{T}_{\check{x}}\text{”}.$$

Then we can inductively construct sequences $\langle s_\alpha \in {}^{<\kappa}\kappa \mid \alpha < \kappa\rangle$, $\langle \gamma_\alpha < \kappa \mid \alpha < \kappa\rangle$ and $\langle p_\alpha \in \mathrm{Add}(\kappa,1) \mid \alpha < \kappa\rangle$ such that the following statements hold for all $\alpha < \kappa$.

(i) If $\bar{\alpha} < \alpha$, then $s_{\bar{\alpha}} \subsetneq s_\alpha$ and $p_\alpha \leq_{\mathrm{Add}(\kappa,1)} p_{\bar{\alpha}}$.
(ii) s_α is a strictly increasing, continuous function and $p_\alpha(s(\gamma)) = 1$ for all $\gamma \in \mathrm{dom}(s_\alpha)$.
(iii) $\mathrm{dom}(s_\alpha)$ has a maximal element and $s_\alpha(\max(\mathrm{dom}(s_\alpha)))$ is the maximum of $\mathrm{dom}(p_\alpha)$.
(iv) $p_{\alpha+1} \Vdash_{\mathrm{Add}(\kappa,1)}$ "$\dot{f}(\dot{s}_\alpha) = \check{\gamma}_\alpha$".

But then $\gamma_{\bar{\alpha}} \in_x \gamma_\alpha$ holds for all $\bar{\alpha} < \alpha < \kappa$ and the set $\{\gamma < \kappa \mid \exists \alpha < \kappa\ \gamma \in_x \gamma_\alpha\}$ is a cofinal branch through $\mathbb{T}_x$ in V, a contradiction. $\square$

Lemma 5.31. *Assume that $\kappa = \kappa^{<\kappa}$ and $\nu \geq 2^\kappa$ is a cardinal with $\mathrm{cof}(\nu) > \kappa$. If G is $\mathrm{Add}(\kappa,\nu)$ over V, then $\mathfrak{d}_{\mathcal{TO}_\kappa}^{\mathrm{V}[G]} = \nu = (2^\kappa)^{\mathrm{V}[G]}$.*

Proof. Assume, towards a contradiction, that, in $\mathrm{V}[G]$, there is a subset D of TO_κ of cardinality less than ν with the property that, for every $\mathbb{T} \in \mathcal{TO}_\kappa$, there is $x \in D$ with $\mathbb{T} \preceq \mathbb{T}_x$. Since $\mathrm{Add}(\kappa,\nu)$ satisfies the κ^+-chain condition in V, we can fnd $H \in \mathrm{V}[G]$ such that H is $\mathrm{Add}(\kappa,\nu)$-generic over V with $D \in \mathrm{V}[H]$ and $\mathrm{V}[G]$ is an $\mathrm{Add}(\kappa,1)$-generic extension of $\mathrm{V}[H]$. In this situation, Lemma 5.30 shows that there is a tree $\mathbb{T} \in \mathcal{TO}_\kappa^{\mathrm{V}[G]}$ such that $\mathbb{T} \not\preceq \mathbb{T}_x$ holds in $\mathrm{V}[G]$ for all $x \in \mathsf{TO}_\kappa^{\mathrm{V}[H]}$. But our assumptions imply that there is an $x \in D \subseteq \mathsf{TO}_\kappa^{\mathrm{V}[H]}$ with $\mathbb{T} \preceq \mathbb{T}_x$, a contradiction. $\square$

Proof of Theorem 2.9. Assume that $\kappa = \kappa^{<\kappa}$ holds and G is $\mathrm{Add}(\kappa,(2^\kappa)^+)$-generic over V. Then Lemma 5.29 implies that

$$\mathfrak{b}_{\mathcal{TO}_\kappa}^{\mathrm{V}[G]} \ \leq\ (2^\kappa)^{\mathrm{V}} \ <\ ((2^\kappa)^+)^{\mathrm{V}} \ =\ (2^\kappa)^{\mathrm{V}[G]}.$$

In addition, we can apply Lemma 5.31 to conclude that $\mathfrak{d}_{\mathcal{TO}_\kappa}^{\mathrm{V}[G]} = (2^\kappa)^{\mathrm{V}[G]}$. $\square$

The following absoluteness result of Fuchs is the last ingredient needed for the proof of Theorem 3.9.

Theorem 5.32 ([8, Lemma 4.10]). *Assume that* $\langle \mathrm{V}, \in, \kappa, \kappa^+\rangle$ *is a model of* CMP. *If* G *is* $\mathrm{Add}(\kappa,\nu)$*-generic over* V*, then* $\langle \mathrm{V}[G], \in, \kappa, \kappa^+\rangle$ *is also a model of* CMP.

Proof of Theorem 3.9. Assume that the theory CMP is consistent. By Corollary 3.7, this assumption implies that the theory CMP + GCH is also consistent and this theory proves that "$\mathfrak{b}_{\mathcal{TO}_\kappa} = \mathfrak{d}_{\mathcal{TO}_\kappa}$". In the other direction, we can combine our assumption with Theorem 2.9 and Theorem 5.32 to conclude that the theory CMP + "$\mathfrak{b}_{\mathcal{TO}_\kappa} < \mathfrak{d}_{\mathcal{TO}_\kappa}$" is also consistent. □

6. Equiconsistency results

This section contains results that provide upper and lower bounds for the consistency strength of closed maximality principles of the form $\mathrm{SMP}_n(\kappa)$. We start by completing the proof of Theorem 4.7.

Proof of Clause (i) of Theorem 4.7. Assume that $\mathrm{SMP}_2(\kappa)$ holds and let μ denote a cardinal smaller than 2^κ.

First, let $\dot{\mathbb{Q}}$ be an $\mathrm{Add}(\kappa,(\mu^{<\kappa})^+)$-name for a $<\kappa$-closed partial order that satisfies the κ^+-chain condition. If $G*H$ is $(\mathrm{Add}(\kappa,(\mu^{<\kappa})^+)*\dot{\mathbb{Q}})$-generic over V, then $({}^{<\kappa}\mu)^{\mathrm{V}[G,H]} \subseteq \mathrm{V}$ and hence $\mu^{<\kappa} < 2^\kappa$ holds in $\mathrm{V}[G,H]$. This shows that the partial order $\mathrm{Add}(\kappa,(\mu^{<\kappa})^+)$ witnesses that the statement that there is no surjection from ${}^{<\kappa}\mu$ onto $\mathcal{P}(\kappa)$ is $(\Phi_{Sh},\Phi_{clc},\kappa)$-forceably necessary. Since this statement can be expressed by a Σ_2-formula with parameters contained in $\mathrm{H}(2^\kappa)$, our assumptions imply that it holds in V and we can conclude that $\mu^{<\kappa} < 2^\kappa$.

Next, pick an injection $\iota : \mu \longrightarrow {}^\kappa\kappa$. Given $X \subseteq \mu$, we can use the fourth part of Theorem 4.7 to find a subtree T_X of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ with $\iota[X] = p[T_X]$. Since $\kappa = \kappa^{<\kappa}$ holds, there is an $x \in \mathsf{T}_\kappa$ such that T_X is isomorphic to the tree $\mathbb{T}_x$. This allows us to construct an injection from 2^μ into 2^κ and we can conclude that $2^\kappa = 2^\mu$ holds for all $\mu < 2^\kappa$.

Finally, let $\dot{\mathbb{Q}}$ be an $\mathrm{Add}(\kappa,(2^\kappa)^+)$-name for a partial order that satisfies the κ^+-chain condition. If $G*H$ is $(\mathrm{Add}(\kappa,(2^\kappa)^+)*\dot{\mathbb{Q}})$-generic over V, then we know that, in $\mathrm{V}[G,H]$, there is a set X with the property that there is no surjection from μ onto X and no surjection from X onto $\mathcal{P}(\kappa)$. In particular, the partial order $\mathrm{Add}(\kappa,(2^\kappa)^+)$ witnesses that this statement is $(\Phi_{Sh},\Phi_{clc},\kappa)$-forceably necessary. Since this statement can be expressed by a Σ_2-formula with parameters contained in $\mathrm{H}(2^\kappa)$, it holds in V and hence we have $\mu^+ < 2^\kappa$.

The above computations show that 2^κ is a limit cardinal with $2^\kappa = 2^\mu$ for all $\kappa \leq \mu < 2^\kappa$. By [18, Corollary 5.17], we can conclude that 2^κ is regular. □

A similar argument also allows us to prove the first part of Theorem 4.5.

Proof of Clause (i) of Theorem 4.5. Assume that $1 < n < \omega$ and $\mathrm{SMP}_n(\kappa)$ holds. Set $\theta = 2^\kappa$. By the first part of Theorem 4.7, θ is weakly inaccessible and therefore θ is an inaccessible cardinal in L. By induction, we show that $\langle \mathrm{L}_\theta, \in\rangle$ is a Σ_n-elementary substructure of $\langle \mathrm{L}, \in\rangle$.

Assume that $\langle \mathrm{L}_\theta, \in\rangle$ is a Σ_m-elementary substructure of $\langle \mathrm{L}, \in\rangle$ for some $m < n$. Fix a Π_m-formula $\varphi(v_0,\ldots,v_k)$ and $z_0,\ldots,z_{k-1} \in \mathrm{L}_\theta$ with the property that the statement $\exists x\ \varphi(x,z_0,\ldots,z_{k-1})$ holds in L. Pick a cardinal μ with the property that there is some $y \in \mathrm{L}_\mu$ such that $\varphi(y,z_0,\ldots,z_{k-1})$ holds in L. Let $\dot{\mathbb{Q}}$ be an $\mathrm{Add}(\kappa,\mu)$-name for a $<\kappa$-closed partial order satisfying the κ^+-chain condition and let $G*H$ be $(\mathrm{Add}(\kappa,\nu)*\dot{\mathbb{Q}})$-generic over V. In $\mathrm{V}[G,H]$, there is an ordinal λ with the property that there is no surjection from λ onto $\mathcal{P}(\kappa)$ and $\varphi(y, z_0, \ldots, z_{k-1})$ holds in L for some $y \in \mathrm{L}_\lambda$. This shows that the partial order $\mathrm{Add}(\kappa, \nu)$ witnesses that this statement is $(\Phi_{Sh}, \Phi_{clc}, \kappa)$-forceably necessary. By our assumptions, this statement can be expressed by a Σ_n-formula with parameters in $\mathrm{H}(2^\kappa)$ and therefore it holds in V. This shows that there is $y \in \mathrm{L}_\theta$ with the property that $\varphi(y, z_0, \ldots, z_{k-1})$ holds in L. Using our induction hypothesis, we can conclude that $\varphi(y, z_0, \ldots, z_{k-1})$ holds in L_θ. In combination with our induction hypothesis, these computations show that $\langle \mathrm{L}_\theta, \in \rangle$ is a Σ_{m+1}-elementary substructure of $\langle \mathrm{L}, \in \rangle$. □

In the remainder of this section, we use iterated forcing to prove the second part of Theorem 4.5. The main difficulty in this argument arises from the fact that the class of partial orders $\mathbb{P}$ satisfying $\Phi_{Sh}(\mathbb{P}, \kappa)$ is not closed under $<\kappa$-support iterations. We deal with this complication by proving the consistency of a maximality principle for the class of $<\kappa$-support iterations of such partial orders.

Definition 6.1. Let Φ be an $\mathcal{L}_\in$-formula defining a class of partial orders.

(i) A forcing iteration $\vec{\mathbb{P}} = \langle \langle \vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \dot{\mathbb{P}}_\alpha \mid \alpha < \lambda \rangle \rangle$ is a (Φ, κ)-*iteration* if it has $<\kappa$-support and $\mathbb{1}_{\vec{\mathbb{P}}_{<\alpha}} \Vdash \Phi(\dot{\mathbb{P}}_\alpha, \check{\kappa})$ holds for all $\alpha < \lambda$.

(ii) A partial order $\mathbb{Q}$ is *equivalent to a* (Φ, κ)-*iteration* if there exists a (Φ, κ)-iteration $\langle \langle \vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \dot{\mathbb{P}}_\alpha \mid \alpha < \lambda \rangle \rangle$ with the property that the partial order $\vec{\mathbb{P}}_{<\lambda}$ isomorphic to $\mathbb{Q}$.

The following iteration result is contained in the proof of [30, Theorem 1.1] and motivates the above definition of stationary κ^+-linked partial orders.

Theorem 6.2 ([30]). *Assume that $\kappa = \kappa^{<\kappa}$ and let $\langle \langle \vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \dot{\mathbb{P}}_\alpha \mid \alpha < \lambda \rangle \rangle$ be a (Φ_{Sh}, κ)-iteration. Then the partial order $\vec{\mathbb{P}}_{<\lambda}$ is $<\kappa$-closed and satisfies the κ^+-chain condition.*

Our proof of Theorem 4.5 relies on two technical lemmas that are consequences of Baumgartner's detailed analysis of the tails of forcing iterations in [2]. Given such an iteration $\langle \langle \vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \dot{\mathbb{P}}_\alpha \mid \alpha < \lambda \rangle \rangle$ and $\alpha \leq \lambda$, we let $\dot{\mathbb{P}}_{[\alpha,\lambda)}$ denote the canonical $\vec{\mathbb{P}}_{<\alpha}$-name for the corresponding tail forcing constructed in [2, Section 5].

Lemma 6.3. *If Φ defines a class of partial orders and $\langle \langle \vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \dot{\mathbb{P}}_\alpha \mid \alpha < \lambda \rangle \rangle$ is a (Φ, κ)-iteration, then*

$$\mathbb{1}_{\vec{\mathbb{P}}_{<\alpha}} \Vdash \text{"}\dot{\mathbb{P}}_{[\alpha,\lambda)} \textit{ is equivalent to a } (\Phi, \check{\kappa})\textit{-iteration"}$$

for all $\alpha < \lambda$ with the property that $\vec{\mathbb{P}}_{<\alpha}$ is $<\kappa$-distributive.

We will show that this lemma is a direct consequence of the results of [2, Section 5]. Following Baumgartner, we say that a set X of ordinals is κ-*thin* if $X \cap \delta$ is bounded in δ for every limit ordinal δ of cofinality greater than or equal to κ.

Proposition 6.4. *Let $\mathbb{P}$ be a $<\kappa$-distributive partial order and let G be $\mathbb{P}$-generic over* V. *If X is a κ-thin set of ordinals in* V$[G]$, *then $X \in$* V *and X is κ-thin in* V.

Proof. Since X is κ-thin, we know that the club of limit points of X contains no ordinal of cofinality κ in V$[G]$ and therefore X has cardinality less than κ in V$[G]$. In particular, X is contained in V. Given a limit ordinal δ with $\mathrm{cof}(\delta)^{\mathrm{V}} \geq \kappa$, we have $\mathrm{cof}(\delta)^{\mathrm{V}[G]} \geq \kappa$ and $X \cap \delta$ is bounded in δ. Hence X is κ-thin in V. □

Proof of Lemma 6.3. Fix $\alpha < \lambda$ such that $\vec{\mathbb{P}}_{<\alpha}$ is $<\kappa$-distributive and pick $\bar{\lambda} \leq \lambda$ with $\lambda = \alpha + \bar{\lambda}$. Let G be $\vec{\mathbb{P}}_{<\alpha}$-generic over V and work in V$[G]$. By [2, Theorem 5.2], there is a canonical forcing iteration $\vec{\mathbb{Q}} = \langle\langle\vec{\mathbb{Q}}_{<\beta} \mid \beta \leq \bar{\lambda}\rangle, \langle\dot{\mathbb{Q}}_\beta \mid \beta < \bar{\lambda}\rangle\rangle$ and a sequence $\langle i_\beta : \dot{\mathbb{P}}^G_{[\alpha,\alpha+\beta)} \longrightarrow \vec{\mathbb{Q}}_{<\beta} \mid \beta \leq \bar{\lambda}\rangle$ of isomorphisms of partial orders such that the following statements hold for all $\beta \leq \bar{\lambda}$:

(i) If $\vec{p} \in \dot{\mathbb{P}}^G_{[\alpha,\alpha+\beta)}$ and $\gamma \leq \beta$, then $i_\gamma(\vec{p} \restriction [\alpha, \alpha+\gamma)) = i_\beta(\vec{p}) \restriction \gamma$.

(ii) If $\beta < \bar{\lambda}$, H is $\vec{\mathbb{Q}}_{<\beta}$-generic over V$[G]$ and F is the filter on $\vec{\mathbb{P}}_{<\alpha+\beta}$ induced by $G * (i_\beta^{-1}[H])$, then $\dot{\mathbb{Q}}^H_\beta = \dot{\mathbb{P}}^F_{\alpha+\beta}$ and $i_\beta(\vec{p})(\beta)^H = \vec{p}(\alpha+\beta)^F$ for all $\vec{p} \in \dot{\mathbb{P}}^G_{[\alpha,\lambda)}$.

These properties imply that $\mathbb{1}_{\vec{\mathbb{Q}}_{<\beta}} \Vdash \Phi(\dot{\mathbb{Q}}_\beta, \check{\kappa})$ holds in V$[G]$ for all $\beta < \bar{\lambda}$. Moreover, Proposition 6.4 implies that the assumptions of [2, Theorem 5.4] are satisfied and hence we can use this result together with [2, Theorem 5.3] to conclude that $\vec{\mathbb{Q}}$ has $<\kappa$-support. This shows that $\vec{\mathbb{Q}}$ is a (Φ, κ)-iteration and the partial order $\dot{\mathbb{P}}^G_{[\alpha,\lambda)}$ is equivalent to such an iteration in V$[G]$. □

The next lemma shows that the converse of the above implication is also true.

Lemma 6.5. *Let Φ be a suitable formula defining a class of partial order, let $\vec{\mathbb{P}} = \langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \lambda\rangle\rangle$ be a (Φ, κ)-iteration and let $\dot{\mathbb{Q}}$ be a $\vec{\mathbb{P}}_{<\lambda}$-name for a partial order. If $\vec{\mathbb{P}}_{<\lambda}$ is $<\kappa$-distributive and*

$$\mathbb{1}_{\vec{\mathbb{P}}_{<\lambda}} \Vdash \text{“}\dot{\mathbb{Q}} \textit{ is equivalent to a } (\Phi, \check{\kappa})\textit{-iteration”},$$

then there is a (Φ, κ)-iteration $\langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda+\eta\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \lambda+\eta\rangle\rangle$ extending $\vec{\mathbb{P}}$ with the property that, whenever G is $\vec{\mathbb{P}}_{<\lambda}$-generic over V*, then the partial orders $\dot{\mathbb{P}}^G_{[\lambda,\lambda+\eta)}$ and $\dot{\mathbb{Q}}^G$ are isomorphic in* V$[G]$.

Proof. With the help of a maximal antichain in $\vec{\mathbb{P}}_{<\lambda}$, we find a set of ordinals A with the property that, whenever G is $\vec{\mathbb{P}}_{<\lambda}$-generic over V, then there is a $\bar{\lambda} \in A$ and a (Φ, κ)-iteration $\langle\langle\vec{\mathbb{Q}}_{<\alpha} \mid \alpha \leq \bar{\lambda}\rangle, \langle\dot{\mathbb{Q}}_\alpha \mid \alpha < \bar{\lambda}\rangle\rangle$ in V$[G]$ such that the partial orders $\dot{\mathbb{Q}}^G$ and $\vec{\mathbb{Q}}_{<\bar{\lambda}}$ are isomorphic in V$[G]$. Set $\eta = \sup(A)$. Since Φ is suitable, we can find sequences $\langle\dot{I}_\alpha \mid \alpha \leq \eta\rangle$ and $\langle\dot{P}_\alpha \mid \alpha < \eta\rangle$ of $\vec{\mathbb{P}}_{<\lambda}$-names with the property that, whenever G is $\vec{\mathbb{P}}_{<\lambda}$-generic over V, then $\vec{I}_G = \langle\langle\dot{I}^G_\alpha \mid \alpha \leq \eta\rangle, \langle\dot{P}^G_\alpha \mid \alpha < \eta\rangle\rangle$ is a (Φ, κ)-iteration in V$[G]$ and the partial orders $\dot{\mathbb{Q}}^G$ and $\dot{I}^G_\eta$ are isomorphic in V$[G]$. In the following, we will derive the statement of the lemma by constructing a (Φ, κ)-iteration $\langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda+\eta\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \lambda+\eta\rangle\rangle$ that extends $\vec{\mathbb{P}}$ and has the property that, whenever G is $\vec{\mathbb{P}}_{<\lambda}$-generic over V and $\alpha \leq \eta$, then $\langle\langle\dot{I}^G_\beta \mid \beta \leq \alpha\rangle, \langle\dot{P}^G_\beta \mid \beta < \alpha\rangle\rangle$ is the canonical forcing iteration in V$[G]$ such that there is a sequence $\langle i_\beta : \dot{\mathbb{P}}^G_{[\lambda,\lambda+\beta)} \longrightarrow \dot{I}^G_\beta \mid \beta \leq \alpha\rangle$ of isomorphisms satisfying the properties (i) and (ii) listed in the proof of Lemma 6.3.

First, pick $\alpha < \eta$ and assume that we have constructed such an extension of $\vec{\mathbb{P}}$ of length $\eta + \alpha$. Then there is an extension $\langle\langle\vec{\mathbb{P}}_{<\beta} \mid \beta \leq \lambda+\alpha+1\rangle, \langle\dot{\mathbb{P}}_\beta \mid \beta \leq \lambda+\alpha\rangle\rangle$ of this iteration with the property that, whenever F is $\vec{\mathbb{P}}_{<\lambda+\alpha}$-generic over V, $G*H$ is the filter on $\vec{\mathbb{P}}_{<\lambda} * \dot{\mathbb{P}}_{[\lambda,\lambda+\alpha)}$ induced by F and $\langle i_\beta : \dot{\mathbb{P}}^G_{[\lambda,\lambda+\beta)} \longrightarrow \dot{I}^G_\beta \mid \beta \leq \alpha\rangle$ is the sequence of isomorphisms with the above properties, then $\dot{\mathbb{P}}^F_{\lambda+\alpha} = (\dot{P}^G_\alpha)^{e_\alpha[H]}$. This definition ensures that we find an isomorphism $i_{\alpha+1} : \dot{\mathbb{P}}^G_{[\lambda,\lambda+\alpha+1)} \longrightarrow \dot{I}^G_{\alpha+1}$ with the desired properties whenever G is $\vec{\mathbb{P}}_{<\lambda}$-generic over V.

Next, assume that $\alpha \leq \eta$ is a limit ordinal and we constructed a (Φ,κ)-iteration $\langle\langle\vec{\mathbb{P}}_{<\beta} \mid \beta \leq \lambda+\alpha\rangle, \langle\dot{\mathbb{P}}_\beta \mid \beta < \lambda+\alpha\rangle\rangle$ such that every initial segment satisfies the above properties. Let G be $\vec{\mathbb{P}}_{<\lambda}$-generic over V. Then our assumptions imply that there is a sequence $\langle i_\beta : \dot{\mathbb{P}}^G_{[\lambda,\lambda+\beta)} \longrightarrow I^G_\beta \mid \beta < \lambda+\alpha\rangle$ of isomorphisms in $\mathrm{V}[G]$ that satisfy the above properties. Let $i_\alpha : \dot{\mathbb{P}}^G_{[\lambda,\lambda+\alpha)} \longrightarrow \dot{I}^G_\alpha$ denote the function defined by $i_\alpha(\vec{p}) = \bigcup\{i_\beta(\vec{p} \upharpoonright [\lambda,\lambda+\beta)) \mid \beta < \alpha\}$. Since Proposition 6.4 shows that we can use [2, Theorem 5.4] to show that the canonical iteration corresponding to $\dot{\mathbb{P}}^G_{[\lambda,\lambda+\alpha)}$ also has $<\kappa$-support in $\mathrm{V}[G]$, we can conclude that i_α is also an isomorphism. □

Proof of Clause (ii) of Theorem 4.5. Assume that $\kappa = \kappa^{<\kappa}$, $\theta > \kappa$ is an inaccessible cardinal and $\lhd$ is a wellordering of $\mathrm{H}(\theta)$ of order-type θ.

Let $\Phi(v_0,v_1)$ be the canonical suitable formula defining a class of partial orders such that $\Phi(\,\cdot\,,\kappa)$ defines the class of all partial orders that are equivalent to a (Φ_{Sh},κ)-iteration if κ is an infinite regular cardinal and $\Phi(\,\cdot\,,z)$ defines the class of all partial orders if z is not an infinite regular cardinal.

Following [3, Section I.9], let $\mathsf{Fml} \subseteq \mathrm{V}_\omega$ denote the set of all formalized $\mathcal{L}_\in$-formulas, let $\# : \mathsf{Fml} \longrightarrow \omega$ denote the corresponding arity function and let Sat denote the corresponding formalized satisfaction relation. Given an $\mathcal{L}_\in$-formula φ, let $\ulcorner\varphi\urcorner \in \mathsf{Fml}$ denote the canonical code for this formula. Let $f : \mathsf{Fml} \longrightarrow \mathsf{Fml}$ be the canonical recursive function with the property that $\#(f(a)) = \#(a)+2$ for all $a \in \mathsf{Fml}$ and $f(\ulcorner\varphi\urcorner) = \ulcorner\psi\urcorner$ whenever $\varphi(v_0,\ldots,v_{n-1})$ is an $\mathcal{L}_\in$-formula and $\psi(v_0,\ldots,v_{n+1})$ is the induced formula stating that

"*v_{n+1} witnesses that $\varphi(v_0,\ldots,v_{n-1})$ is (Φ,Φ_{clc},v_n)-forceably necessary*".

In the following, we inductively construct a continuous, strictly increasing sequence $\langle c_\gamma < \theta \mid \gamma < \theta\rangle$, a (Φ_{Sh},κ)-iteration $\vec{\mathbb{P}}^\theta = \langle\langle\vec{\mathbb{P}}_{<\delta} \mid \delta \leq \theta\rangle, \langle\dot{\mathbb{P}}_\delta \mid \delta < \theta\rangle\rangle$ and a sequence $\langle\dot{b}_\delta \mid \delta < \theta\rangle$ such that $\vec{\mathbb{P}}^\gamma = \langle\langle\vec{\mathbb{P}}_{<\delta} \mid \delta \leq c_\gamma\rangle, \langle\dot{\mathbb{P}}_\delta \mid \delta < c_\gamma\rangle\rangle$ is an element of $\mathrm{H}(\theta)$ for every $\gamma < \theta$ and $\dot{b}_\delta$ is a $\vec{\mathbb{P}}_{<\delta}$-name for a bijection between θ and $\mathrm{H}(\theta)$ for every $\delta < \theta$.

Assume that $\gamma = \prec\delta,\epsilon\succ < \theta$ and we already constructed $\langle c_\delta \mid \delta \leq \gamma\rangle$, $\vec{\mathbb{P}}^\gamma$ and $\langle\dot{b}_\delta \mid \delta < c_\gamma\rangle$ with the above properties. Then there is a $\vec{\mathbb{P}}_{<c_\gamma}$-name $\dot{\mathbb{Q}}$ with the property that the following statements hold whenever G is $\vec{\mathbb{P}}_{<c_\gamma}$-generic over V and $\bar{G}$ is the filter on $\vec{\mathbb{P}}_{<\delta}$ induced by G:

(a) If $\dot{b}^{\bar{G}}_\delta(\epsilon) = \langle a,x_0,\ldots,x_{n-1}\rangle$ with $a \in \mathsf{Fml}$ and $\#a = n$ and there is a partial order $\mathbb{Q}$ in $\mathrm{H}(\theta)^{\mathrm{V}[G]}$ such that

$$\mathsf{Sat}(\mathrm{H}(\theta)^{\mathrm{V}[G]}, f(a), \langle x_0,\ldots,x_{n-1},\kappa,\mathbb{Q}\rangle)$$

holds in $\mathrm{V}[G]$, then $\dot{\mathbb{Q}}^G \in \mathrm{H}(\theta)^{\mathrm{V}[G]}$ and

$$\mathsf{Sat}(\mathrm{H}(\theta)^{\mathrm{V}[G]}, f(a), \langle x_0,\ldots,x_{n-1},\kappa,\dot{\mathbb{Q}}^G\rangle)$$

holds in $\mathrm{V}[G]$.

(b) If the above assumptions are not satisfied, then $\dot{\mathbb{Q}}^G$ is the trivial partial order in $\mathrm{H}(\theta)^{\mathrm{V}[G]}$.

Since $\vec{\mathbb{P}}_{<c_\gamma}$ is contained in $\mathrm{H}(\theta)$, there is a $\vec{\mathbb{P}}_{<c_\gamma}$-name with these properties in $\mathrm{H}(\theta)$ and we let $\dot{\mathbb{Q}}$ denote the $\lhd$-least such name. Then the statement

$$\mathbb{1}_{\vec{\mathbb{P}}_{<c_\gamma}} \Vdash \text{“}\dot{\mathbb{Q}} \textit{ is equivalent to a } (\Phi_{Sh}, \check{\kappa})\textit{-iteration}\text{”}$$

holds in $\mathrm{H}(\theta)$ and an application of Lemma 6.5 in $\mathrm{H}(\theta)$ yields $c_\gamma < c_{\gamma+1} < \theta$ and a (Φ_{Sh}, κ)-iteration $\langle\langle \vec{\mathbb{P}}_{<\delta} \mid \delta \leq c_{\gamma+1}\rangle, \langle \dot{\mathbb{P}}_\delta \mid \delta < c_{\gamma+1}\rangle\rangle$ extending $\vec{\mathbb{P}}^\gamma$ with the property that, whenever G is $\vec{\mathbb{P}}_{<c_\gamma}$-generic over V, then the partial orders $\dot{\mathbb{P}}^G_{[c_\gamma, c_{\gamma+1})}$ and $\dot{\mathbb{Q}}^G$ are isomorphic in $\mathrm{H}(\theta)^{\mathrm{V}[G]}$. Let $\vec{\mathbb{P}}^{\gamma+1}$ denote the $\lhd$-least iteration with these properties in $\mathrm{H}(\theta)$. In V, $\vec{\mathbb{P}}^{\gamma+1}$ is also a (Φ_{Sh}, κ)-iteration that extends $\vec{\mathbb{P}}^\gamma$. Given $c_\gamma \leq \delta < c_{\gamma+1}$, let $\dot{b}_\delta$ be the canonical $\vec{\mathbb{P}}_{<\delta}$-name with the property that, whenever G is $\vec{\mathbb{P}}_{<\delta}$-generic over V and $\lhd_G$ is the canonical wellordering of $\mathrm{H}(\theta)^{\mathrm{V}[G]}$ of order-type θ induced by $\lhd$ (i.e. given $x, y \in \mathrm{H}(\theta)^{\mathrm{V}[G]}$, we have $y \lhd_G y$ if and only if there is a $\vec{\mathbb{P}}_{<\gamma}$-name[2] $\dot{x} \in \mathrm{H}(\theta)^{\mathrm{V}}$ with $x = \dot{x}^G$ and $\dot{x} \lhd \dot{y}$ for all $\vec{\mathbb{P}}_{<\gamma}$-names $\dot{y} \in \mathrm{H}(\theta)^{\mathrm{V}}$ with $y = \dot{y}^G$), then $\dot{b}^G_\gamma : \theta \longrightarrow \mathrm{H}(\theta)^{\mathrm{V}[G]}$ is the monotone enumeration of the wellorder $\langle \mathrm{H}(\theta)^{\mathrm{V}[G]}, \lhd_G\rangle$.

Since our assumptions on the support of the iteration determine the definition of c_γ and $\vec{\mathbb{P}}^\gamma$ for all $\gamma \in \mathrm{Lim} \cap \theta$, this completes the definition of our forcing iteration.

Set $\mathbb{B}(\kappa, \lhd) = \vec{\mathbb{P}}_{<\theta}$. Note that the above definition ensures that this partial order is uniformly definable in the parameters κ and $\lhd$. By Theorem 6.2, the partial order $\mathbb{B}(\kappa, \lhd)$ is $<\kappa$-closed and satisfies the κ^+-chain condition.

From now on, assume that θ is a Σ_{n+2}-reflecting cardinal for some $1 < n < \omega$ and let G is $\mathbb{B}(\kappa, \lhd)$-generic over V. Given $\delta < \theta$, let G_δ denote the filter on $\vec{\mathbb{P}}_{<\delta}$ induced by G.

Claim. *If $\varphi(v_0, \ldots, v_{k-1})$ is a Σ_n-formula and $x_0, \ldots, x_{k-1} \in \mathrm{H}(\theta)^{\mathrm{V}[G]}$ such that the statement $\varphi(x_0, \ldots, x_{k-1})$ is $(\Phi_{Sh}, \Phi_{clc}, \kappa)$-forceably necessary in $\mathrm{V}[G]$, then $\varphi(x_0, \ldots, x_{k-1})$ holds in $\mathrm{V}[G]$.*

Proof of the Claim. Since $\vec{\mathbb{P}}^\theta$ is a $<\kappa$-support iteration of forcings of size less than θ, we can find $\delta < \theta$ with $x_0, \ldots, x_{k-1} \in \mathrm{H}(\theta)^{\mathrm{V}[G_\delta]}$. Then there is an ordinal $\epsilon < \theta$ with $\dot{b}^{G_\delta}_\delta(\epsilon) = \langle \ulcorner\varphi\urcorner, x_0, \ldots, x_{k-1}\rangle$. Set $\gamma = \prec\delta, \epsilon\succ$. Since the formula Φ_{Sh} is suitable, there is a (Φ_{Sh}, κ)-iteration $\langle\langle \vec{\mathbb{P}}_{<\delta} \mid \delta \leq \theta+1\rangle, \langle \dot{\mathbb{P}}_\delta \mid \delta \leq \theta\rangle\rangle$ in V extending $\vec{\mathbb{P}}^\theta$ such that $\dot{\mathbb{P}}^G_\theta$ witnesses that the statement $\varphi(x_0, \ldots, x_{k-1})$ is $(\Phi_{Sh}, \Phi_{clc}, \kappa)$-forceably necessary in $\mathrm{V}[G]$. In this situation, an application of Lemma 6.3 shows that $\dot{\mathbb{P}}^{G_{c_\gamma}}_{[c_\gamma, \theta+1)}$ witnesses that the statement $\varphi(x_0, \ldots, x_{k-1})$ is $(\Phi, \Phi_{clc}, \kappa)$-forceably necessary in $\mathrm{V}[G_{c_\gamma}]$.

Note that, given a partial order $\mathbb{P}$, the statement $\Phi_{Sh}(\mathbb{P}, \kappa)$ holds if and only if there is an ordinal α such that $\Phi_{Sh}(\mathbb{P}, \kappa)$ holds in $\mathrm{H}(\nu)$ for all regular cardinals $\nu > \alpha$. This shows that, given a partial order $\mathbb{P}$ with $\Phi_{clc}(\mathbb{P}, \kappa)$, a condition p in $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a partial order, we have $p \Vdash_{\mathbb{P}} \Phi_{Sh}(\dot{\mathbb{Q}}, \check{\kappa})$ if and only if there is an ordinal α such that $p \Vdash^{\mathrm{H}(\nu)}_{\mathbb{P}} \Phi_{Sh}(\dot{\mathbb{Q}}, \check{\kappa})$ holds for all regular cardinals $\nu > \alpha$, because $\mathrm{H}(\nu)^{\mathrm{V}[G]} = \mathrm{H}(\nu)[G]$ holds for all sufficiently large regular cardinals ν whenever G is $\mathbb{P}$-generic over V. In combination with Theorem 6.2, these observations show that $\Phi(\mathbb{P}, \kappa)$ holds for a partial order $\mathbb{P}$ if and only if there is an ordinal α such that this

[2]Note that the first statement implies that for every $x \in \mathrm{H}(\theta)^{\mathrm{V}[G]}$, there is a $\vec{\mathbb{P}}_{<\gamma}$-name $\dot{x} \in \mathrm{H}(\theta)^{\mathrm{V}}$ with $x = \dot{x}^G$.

statement holds in $\mathrm{H}(\nu)$ for all regular cardinals $\nu > \alpha$. In particular, the class of these forcings can be uniformly defined by a Σ_3-formula with parameter κ.

Together with Proposition 5.8, these remarks show that we can formulate the statement that $\varphi(x_0,\ldots,x_{k-1})$ is (Φ,Φ_{clc},κ)-forceably necessary by a Σ_{n+2}-formula using parameters contained in $\mathrm{H}(\theta)^{\mathrm{V}[G_{c_\gamma}]}$. Using the fact that $\vec{\mathbb{P}}_{<c_\gamma}$ is contained in $\mathrm{H}(\theta)^{\mathrm{V}}$, a routine application of Proposition 5.8 shows that θ is still Σ_{n+2}-reflecting in $\mathrm{V}[G_{c_\gamma}]$ and we can conclude that the $\varphi(x_0,\ldots,x_{k-1})$ is also (Φ,Φ_{clc},κ)-forceably necessary in $\mathrm{H}(\theta)^{\mathrm{V}[G_{c_\gamma}]}$. This shows that there is a partial order $\mathbb{Q}$ in $\mathrm{H}(\theta)^{\mathrm{V}[G_{c_\gamma}]}$ such that

$$\mathrm{Sat}(\mathrm{H}(\theta)^{\mathrm{V}[G_{c_\gamma}]}, f(\ulcorner\varphi\urcorner), \langle x_0,\ldots,x_{k-1},\kappa,\mathbb{Q}\rangle)$$

holds in $\mathrm{V}[G_{c_\gamma}]$. By our definition of $\vec{\mathbb{P}}^{c_{\gamma+1}}$, we know that

$$\mathrm{Sat}(\mathrm{H}(\theta)^{\mathrm{V}[G_{c_\gamma}]}, f(\ulcorner\varphi\urcorner), \langle x_0,\ldots,x_{k-1},\kappa,\dot{\mathbb{P}}^{G_{c_\gamma}}_{[c_\gamma,c_{\gamma+1})}\rangle)$$

holds in $\mathrm{V}[G_{c_\gamma}]$ and this means that $\dot{\mathbb{P}}^{G}_{[c_\gamma,c_{\gamma+1})}$ witnesses that $\varphi(x_0,\ldots,x_{k-1})$ is (Φ,Φ_{clc},κ)-forceably necessary in $\mathrm{H}(\theta)^{\mathrm{V}[G_{c_\gamma}]}$. As above, our assumptions imply that this statement also holds in $\mathrm{V}[G_{c_\gamma}]$. By Theorem 6.2 and Lemma 6.3, we know that $\Phi_{clc}(\dot{\mathbb{P}}^{G_{c_{\gamma+1}}}_{[c_{\gamma+1},\theta)})$ holds in $\mathrm{V}[G_{c_{\gamma+1}}]$. This allows us to conclude that the statement $\varphi(x_0,\ldots,x_{k-1})$ holds in $\mathrm{V}[G]$. □

Claim. $\theta = (2^\kappa)^{\mathrm{V}[G]}$.

Proof of the Claim. In V, θ is inaccessible and $\mathbb{B}(\kappa,\lhd)$ is a partial order of cardinality θ that satisfies the κ^+-chain condition. This implies that there are at most θ-many $\mathbb{B}(\kappa,\lhd)$-nice names for subset of κ in V and hence $(2^\kappa)^{\mathrm{V}[G]} \leq \theta$.

Given a cardinal $\mu < \theta$, the partial order $\mathrm{Add}(\kappa,\mu^+)$ witnesses that the statement that there is no surjection from μ onto $\mathcal{P}(\kappa)$ is $(\Phi_{Sh},\Phi_{clc},\kappa)$-forceably necessary in $\mathrm{V}[G]$. Since this statement can be formulated by a Σ_2-formula with parameters in $\mathrm{H}(\theta)^{\mathrm{V}[G]}$, the above claim shows that it holds in $\mathrm{V}[G]$ and hence $\mu < (2^\kappa)^{\mathrm{V}[G]}$. This shows that we also have $(2^\kappa)^{\mathrm{V}[G]} \geq \theta$. □

The combination of the above claims shows that $\mathrm{SMP}_n(\kappa)$ holds in $\mathrm{V}[G]$. □

7. Concluding remarks and open questions

We close this paper with some open questions motivated by the above results and a discussion of possible directions of further research.

The above results show that closed maximality principles exactly determine the possible lengths of $\mathbf{\Sigma}^1_1$-definable wellorders. Since definable prewellorders play a more important role in classical descriptive set theory, one naturally arrives at the following question.

Question 7.1. Do axioms of the form $\mathrm{CMP}_n(\kappa)$ or $\mathrm{SMP}_n(\kappa)$ determine the least upper bound of the lengths of $\mathbf{\Delta}^1_1$-definable prewellorders on ${}^\kappa\kappa$?

The results of this paper show that the principles $\mathrm{CMP}_2(\kappa)$ and $\mathrm{SMP}_2(\kappa)$ induce a strong structure theory for the class of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$. Therefore it is natural to ask whether similar implications hold for stronger maximality principles and larger classes of definable subsets of ${}^\kappa\kappa$.

Question 7.2. Assume that $\langle \mathrm{V},\in,\kappa,\kappa^+\rangle$ is a model of CMP.

(i) Does every subset of ${}^{\kappa}\kappa$ that is an element of $\mathrm{L}(\mathcal{P}(\kappa))$ have the perfect set property?
(ii) Does the axiom of choice fail in $\mathrm{L}(\mathcal{P}(\kappa))$?

Question 7.3. Assume that $\langle \mathrm{V}, \in, \kappa, 2^{\kappa}\rangle$ is a model of SMP.

(i) Does every subset of ${}^{\kappa}\kappa$ that is an element of $\mathrm{L}(\mathcal{P}(\kappa))$ have the Bernstein property?
(ii) Does the axiom of choice fail in $\mathrm{L}(\mathcal{P}(\kappa))$?

Note that in both cases, an affirmative answer to the first part of the question yields an affirmative answer to the second part of the question, because a wellordering of ${}^{\kappa}\kappa$ allows us to construct a subset of ${}^{\kappa}\kappa$ without the Bernstein property.

Since the principles $\mathrm{CMP}_2(\kappa)$ and $\mathrm{SMP}_2(\kappa)$ settle many questions about the class of $\mathbf{\Sigma}^1_1$-subsets of the generalized Baire space of κ, it is natural to ask whether these closed maximality principles can hold for all uncountable regular cardinals κ. In [8, Section 6], Fuchs shows that the full maximality principles for the class of all closed partial orders cannot hold globally. A closer look at the proof of [8, Theorem 6.9] shows that the class of uncountable regular cardinals κ with the property that $\mathrm{CMP}_3(\kappa)$ holds is bounded in the ordinals. A similar argument shows that the same is true for the class of all uncountable regular cardinals κ such that $\mathrm{SMP}_3(\kappa)$ holds. In a subsequent paper [23], we will prove that this is also true for the class of cardinals such that $\mathrm{SMP}_2(\kappa)$ holds. This leaves open the following question.

Question 7.4. Is it consistent that $\mathrm{CMP}_2(\kappa)$ holds for all uncountable regular cardinals κ?

In [8, Section 5], Fuchs introduces a weakening of CMP that restricts this principle to *local* statements that are *provably forceably necessary* and shows that it is consistent that this principle holds for every uncountable regular cardinal. Moreover, he argues that many interesting consequences of CMP derived in [8] are also consequences of this weaker principles. In [23], we will show that the consequences listed in Theorem 3.8 can also be derived from this restricted principle. Therefore we may view it as a natural example of an extension of ZFC that can answers the questions posed in Section 2 in a global way.

In [23], we will also formulate a similar restriction of the principle SMP to local statements that are provably forceably and construct a model in which this principles holds for all uncountable regular cardinals κ satisfying $\kappa = \kappa^{<\kappa}$ and unboundedly many cardinals with this property exist. Finally, we will also show that the statements listed in Theorem 4.7 are also consequences of this weaker principle.

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