CLOSED MAXIMALITY PRINCIPLES AND GENERALIZED Baire SPACES

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Abstract. Given an uncountable regular cardinal \( \kappa \), we study the structural properties of the class of all sets of functions from \( \kappa \) to \( \kappa \) that are definable over the structure \( \langle H(\kappa^+), \in \rangle \) by a \( \Sigma_1 \)-formula with parameters. It is well-known that many important statements about these classes are not decided by the axioms of ZFC together with large cardinal axioms. In this paper, we present other canonical extensions of ZFC that provide a strong structure theory for these classes. These axioms are variations of the Maximality Principle introduced by Stavi and Väänänen and later rediscovered by Hamkins.

1. Introduction

Given an infinite regular cardinal \( \kappa \), the generalized Baire space of \( \kappa \) is the set \( {}^\kappa \kappa \) of all functions from \( \kappa \) to \( \kappa \) equipped with the topology whose basic open sets are of the form \( N_s = \{ x \in {}^\kappa \kappa \mid s \subseteq x \} \) for some \( s \) contained in the set \( {}^{<\kappa} \kappa \) of all functions \( t : \alpha \to \kappa \) with \( \alpha < \kappa \). For uncountable regular cardinals \( \kappa \), these spaces are a natural generalization of the classical Baire space \( {}^\omega \omega \) to these cardinalities. Moreover, many objects studied in set theory can be identified with definable subsets of these spaces. For example, there is a direct correspondence between the closed subsets of \( {}^\kappa \kappa \) and the sets of cofinal branches through set theoretic trees of height \( \kappa \). The investigation of generalized Baire spaces of uncountable cardinals and their definable subsets was initiated by Mekler and Väänänen (see [28] and [34]) and has recently gained increasing attention (see, for example, [6], [20] and [29]). Results of this analysis have been used in model theory and infinitary logic (see, for example, [6], [32] and [35]). In this paper, we are interested in the structural properties of classes of definable subsets of \( {}^\kappa \kappa \) of low complexity for uncountable regular cardinals \( \kappa \).

In the remainder of this paper, we let \( \kappa \) denote an uncountable regular cardinal. We generalize the classical definition of analytic sets of reals to the generalized Baire space of \( \kappa \) by calling a subset of \( ({}^\kappa \kappa)^n \) a \( \Sigma_1 \)-subset if it is equal to the projection of a closed subset of \( ({}^\kappa \kappa)^{n+1} \). A folklore results (see, for example, [24], Section 2) shows that for such cardinals \( \kappa \), every subset of \( ({}^\kappa \kappa)^n \) that is definable over the structure \( \langle H(\kappa^+), \in \rangle \) by a \( \Sigma_1 \)-formula with parameters is a \( \Sigma_1 \)-subset. If in addition \( \kappa = {}^{<\kappa} \kappa \) holds, then it is easy to see that the converse implication is also true (see Section 5). This shows that the class of \( \Sigma_1 \)-subsets of \( {}^\kappa \kappa \) contains a great variety

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of set theoretically interesting objects. Moreover, it shows that, in contrast to the classical Baire space, many basic statements about the class of $\Sigma_1^1$-subset are not settled by the axioms of ZFC together with large cardinal axioms, because it is possible to manipulated these objects by set theoretic methods. In Section 2, we will present important examples of such statements and present the corresponding independence results. This leads to the question whether there are other canonical extensions of ZFC that decide these statements by providing a suitable structure theory for the class of $\Sigma_1^1$-subsets of $^\kappa \kappa$. The aim of this paper is to show that certain forcing axioms that are motivated by modal logic are examples of such extensions of ZFC. These axioms are variations of the *Maximality Principle* introduced by Stavi and Vääänänen in [33] and later rediscovered by Hamkins (see [12]).

Given a sentence $\varphi$ in the language $L_\in$ of set theory, we say that $\varphi$ is *forceably necessary* if there is a partial order $P$ such that $1_{P*} \Vdash \varphi$ holds for every $P$-name $\dot{Q}$ for a partial order. The *Maximality Principle* (MP) is the scheme of axioms stating that every forceably necessary $L_\in$-sentence is true. This formulation can be motivated by the *Maximality Principle* $\Diamond \Box \varphi \rightarrow \varphi$ of modal logic by interpreting the modal statement $\Diamond \varphi$ ("$\varphi$ is possible") as "$\varphi$ holds in some forcing extension of the ground model" and the statement $\Box \varphi$ ("$\varphi$ is necessary") as "$\varphi$ holds in every forcing extension of the ground model". The consistency of the theory ZFC + MP follows from the consistency of ZFC (see [12, Theorem 2.1]). Similar to standard forcing axioms, this principle states that the universe $V$ is, in a certain sense, maximal in the collection of its forcing extensions.

Following [8] and [22], we will consider variations of this principle that arise from the following modifications of the above principle:

- A restriction of the complexity of the considered formulas.
- A restriction of the class of forcings that can be used to witness that a given statement is possible.
- A restriction of the class of forcings that need to be considered to show that a given statement is necessary.
- An extension of the principle to formulas containing parameters.

Note that the first two modifications weaken the principle, while the last two strengthen it. The axioms discussed in this paper are restricted to certain classes of $<\kappa$-closed forcings and allow statements with parameters of bounded hereditary cardinality. We will refer to these principle as *boldface closed maximality principles*. Such principles were already intensively studied by Fuchs in [8] and [9].

We outline the content of this paper. In Section 2, we present independence results that show that a number of important basic questions about the structural properties of the class of $\Sigma_1^1$-subsets of generalized Baire spaces are not decided by the axioms of ZFC together with large cardinal axioms. In the following, these questions will serve as test questions to evaluate the influence of other extension of ZFC on the structure theory of $\Sigma_1^1$-subsets of $^\kappa \kappa$. Section 3 contains the formulation of the closed maximality principle for the class of all $<\kappa$-closed forcing and statements with parameters in $H(\kappa^+)$. We present results that show that these principles answer most of the questions posed in the previous section. In contrast, we also present a question about the values of certain cardinal characteristics of the space $^\kappa \kappa$ that is not settled by such principles. Motivated by this observation, we consider maximality principles for formulas containing parameters of higher cardinalities in Section 4. A natural candidate for such a principle is the maximality
principle for \(<\kappa\)-closed forcing that satisfy the \(\kappa^+\)-chain condition and statements with parameters in \(H(2^\kappa)\). We will prove that this principle always fails for \(\aleph_1\). Motivated by this result, we will show that a restriction of this principle to a smaller class of forcings can lead to a consistent maximality principle that provides a strong structure for the the class of \(\Sigma^1_1\)-subsets of generalized Baire spaces. In particular, we will present an example of such a principle that answers all questions posed in the second section. Section 5 contains a number of results that demonstrate the influence of closed maximality principles on generalized Baire spaces and show how these principles answer our test questions. In the next section, we establish the consistency of the principles defined in Section 4 using iterated forcing. We close this paper with a discussion of extensions of ZFC that provide a strong structure theory for the classes of \(\Sigma^1_1\)-subsets of the generalized Baire spaces of all regular uncountable cardinals. Moreover, we present some open questions motivated by the results of this paper.

2. Independence results

In this section, we review several independence results showing that many basic question about the class of \(\Sigma^1_1\)-definable subsets of \(^\kappa\kappa\) for uncountable regular cardinals \(\kappa\) are not decided by the axioms of ZFC together with large cardinal axioms. In particular, these axioms do not provide a good structure theory for these classes of subsets. This observation suggests two ways to proceed:

(i) Consider subclasses of the class of all \(\Sigma^1_1\)-subsets of \(^\kappa\kappa\) that contain many interesting objects and have the property that ZFC together with large cardinal axioms provides a strong structure theory for them.

(ii) Consider other natural extensions of ZFC that provide a strong structure for the class of all \(\Sigma^1_1\)-subsets of \(^\kappa\kappa\).

The first approach is pursued in [25], where it is shown that large cardinals axioms resolve most of the independence issues discussed below for the class of subsets of \(^{\omega_1}\omega_1\) that are definable over \((H(\omega_2), \in)\) by a \(\Sigma^1_1\)-formula with parameter \(\omega_1\). In this paper, we will show that closed maximality principles are examples of extensions of ZFC that are suitable for the second approach by showing that these principles are in some sense natural and proving that the questions listed below are answered by these principles.

In the remainder of this paper, we let \(\kappa\) denote an uncountable regular cardinal.

2.1. Sets separating the club filter from the non-stationary ideal. As usual, we say that a subset \(A\) of \(^\kappa\kappa\) is a \(\Pi^1_1\)-subset if \(^\kappa\kappa\setminus A\) is a \(\Sigma^1_1\)-subset. Moreover, a subset of \(^\kappa\kappa\) is a \(\Delta^1_1\)-subset if it is both a \(\Sigma^1_1\)- and a \(\Pi^1_1\)-subset.

Given \(S \subseteq \kappa\), we define

\[
\text{Club}(S) = \{ x \in ^\kappa\kappa \mid \exists C \subseteq \kappa \text{ club } \forall \alpha \in C \cap S \ x(\alpha) > 0 \}
\]

and

\[
\text{NS}(S) = \{ x \in ^\kappa\kappa \mid \exists C \subseteq \kappa \text{ club } \forall \alpha \in C \cap S \ x(\alpha) = 0 \}.
\]

Then \(\text{Club}(S)\) and \(\text{NS}(S)\) are disjoint \(\Sigma^1_1\)-subsets of \(^\kappa\kappa\) for every \(S \subseteq \kappa\). We can identify \(\text{Club}(\kappa)\) with the club filter on \(\kappa\) and \(\text{NS}(\kappa)\) with the non-stationary ideal on \(\kappa\). In the light of the Lusin Separation Theorem theorem (see [19] Theorem 14.7)) the following question appears naturally.
Question 1. Is there a $\Delta^1_1$-subset $A$ of $^{\kappa}\kappa$ that separates $\text{Club}(\kappa)$ from $\text{NS}(\kappa)$, in
the sense that $\text{Club}(\kappa) \subseteq A \subseteq ^\kappa\kappa \setminus \text{NS}(\kappa)$ holds?

It is possible to combine results from [6] and [11] to show that forcing with the
partial order $\text{Add}(\kappa, \kappa^+)$ that adds $\kappa^*$-many Cohen subsets of $\kappa$ produces a negative
answer to this question. In Section 5, we will present a proof of the following result.

Theorem 2.1. If $\kappa = \kappa^{<\kappa}$ and $G$ is $\text{Add}(\kappa, \kappa^+)$-generic over $V$, then there is no
$\Delta^1_1$-subset $A$ of $^{\kappa}\kappa$ that separates $\text{Club}(\kappa)$ from $\text{NS}(\kappa)$ in $V[G]$.

In contrast, positive answers to Question 1 are also consistent. It was observed
in [7] that, if the non-stationary ideal on $\omega_1$ is $\omega_1$-dense, then $\text{Club}(\kappa)$ is a $\Delta^1_1$-subset
of $^{\omega_1}\omega_1$. Results of Woodin show that this assumption is consistent (see [4] Section
7.14]). Moreover, Friedman, Kulikov and Hytinnen used results on the existence
of $\kappa$-$\text{Canary tree}$ from [13] and [27] to establish the consistency of positive answers
to the above question for larger classes of cardinals. Note that, if $S$ is a stationary
subset of $\kappa$, then $\text{Club}(S)$ separates $\text{Club}(\kappa)$ from $\text{NS}(\kappa)$. Given an infinite regular
 cardinal $\mu < \kappa$, we use $S^\kappa_\mu$ to denote the set of all limit ordinals less than $\kappa$ of cofinality $\mu$.

Theorem 2.2 ([6 Theorem 49.5]). Assume that the GCH holds and $\kappa$ is not the
successor of a singular cardinal. Let $\mu < \kappa$ be an infinite regular cardinal. Then
there is a partial order $\mathbb{P}$ satisfying the $\kappa^+$-chain condition with the property that
forcing with $\mathbb{P}$ adds no bounded subsets of $\kappa$ and $\text{Club}(S^\kappa_\mu)$ is a $\Delta^1_1$-subset of $^{\kappa}\kappa$ in
every $\mathbb{P}$-generic extension of the ground model.

We will later show that certain closed maximality principles imply that all $\Delta^1_1$-
subsets possess a regularity property that generalizes the Baire property to higher
cardinalities. Since the results of [11] show that sets with this property cannot
separate $\text{Club}(\kappa)$ from $\text{NS}(\kappa)$, this implications shows that these principles answer
Question 1.

2.2. The Bernstein and the perfect set property. We say that a subset $A$ of
$^{\kappa}\kappa$ contains a perfect subset if there is a continuous injection $\iota : ^\omega 2 \rightarrow ^\kappa\kappa$ with
ran($\iota$) $\subseteq A$. A subset of $^{\kappa}\kappa$ has the Bernstein property if either $A$ or $^{\kappa}\kappa \setminus A$ contains
a perfect subset. Finally, we say that a subset $A$ of $^{\kappa}\kappa$ has the perfect set property
if either $A$ has cardinality at most $\kappa$ or $A$ contains a perfect subset. It is easy to see
that this definition of the perfect set property is equivalent to the definition using
transfinite games given in [28 Section 5].

We say that a subset of $^{\kappa}\kappa$ is $\kappa$-Borel if it is contained in the smallest algebra
of sets on $^{\kappa}\kappa$ that contains all open subsets of $^{\kappa}\kappa$ is closed under $\kappa$-unions. An easy argument (see Corollary 5.21) shows that, if $\kappa = \kappa^{<\kappa}$ holds, then all $\kappa$-Borel
subsets of $^{\kappa}\kappa$ have the Bernstein property. Since all $\kappa$-Borel subsets are $\Delta^1_1$-subsets
and there are always $\Delta^1_1$-subsets of $^{\kappa}\kappa$ that are not $\kappa$-Borel (see [6 Theorem 18]),
one naturally arrives at the following question.

Question 2. Does every $\Delta^1_1$-subset of $^{\kappa}\kappa$ have the Bernstein property?

Using forcing, it is possible to establish the consistency of a negative answer to
this question.

Theorem 2.3 ([13 Corollary 1.5]). Assume that $\kappa = \kappa^{<\kappa}$ and $2^\kappa$ is regular. Then
there is a $<\kappa$-closed partial order $\mathbb{P}$ with the property that forcing with $\mathbb{P}$ preserves
cardinals less than or equal to $2^\kappa$ and adds a $\Delta^1_1$-subset of $^\kappa\kappa$ without the Bernstein property.

In Section 5, we will show that it is consistent that all $\Delta^1_1$-subsets of $^\kappa\kappa$ have the Bernstein property.

**Theorem 2.4.** Assume that $\kappa = \kappa^\lt \kappa$ and let $G$ be $\text{Add}(\kappa, \kappa^+)$-generic over $V$. In $V[G]$, all $\Delta^1_1$-subsets of $^\kappa\kappa$ have the Bernstein property.

Since classical results in descriptive set theory show that all $\Sigma^1_1$-subsets of $\omega$ have the perfect set property, it is also natural to consider the following question.

**Question 3.** Does every $\Sigma^1_1$-subset of $^\kappa\kappa$ have the perfect set property?

Theorem 2.3 already established the consistency of a negative answer to this question. In addition, a small modification of a classical argument of Solovay (see [17, Section 4]) shows that the non-existence of inaccessible cardinals in $L$ implies the existence of closed counterexamples to the perfect set property.

In the other direction, results of Schlicht show that a positive answer to the above question can be established by collapsing inaccessible cardinals.

**Theorem 2.5** ([29, Theorem 1.2]). Let $\theta > \kappa$ be an inaccessible cardinal and let $G$ be $\text{Col}(\kappa, <\theta)$-generic over $V$. In $V[G]$, every subset of $^\kappa\kappa$ contained in $\text{HOD}(^\kappa\text{Ord})$ has the perfect set property.

We will later show that closed maximality principles imply that all $\Delta^1_1$-subsets of $^\kappa\kappa$ have the Bernstein property. Moreover, we will show that certain principles also yield an affirmative answer to Question 3.

2.3. **The lengths of $\Sigma^1_1$-definable wellorders.** We call a wellorder $\langle A, <\rangle$ a $\Sigma^1_1$-wellordering of a subset of $^\kappa\kappa$ if $<\$ is a $\Sigma^1_1$-subset of $^\kappa\kappa \times ^\kappa\kappa$. Note that this implies that the domain $A$ of $<\$ is also a $\Sigma^1_1$-subset of $^\kappa\kappa$. It is easy to see that, for every $\alpha < \kappa^+$, there is a $\Sigma^1_1$-wellordering of a subset of $^\kappa\kappa$ of order-type $\alpha$. Moreover, if there is an $x \subseteq \kappa$ such that $\kappa^+$ is not inaccessible in $L[x]$, then there is a $\Sigma^1_1$-wellordering of a subset of $^\kappa\kappa$ of order-type $\kappa^+$. The following question is motivated by the classical Kunen-Martin Theorem (see [19, Theorem 31.5]).

**Question 4.** What is the least upper bound for the order-types of $\Sigma^1_1$-wellorderings of subsets of $^\kappa\kappa$?

With the help of generic coding techniques (see, for example, [14], [24] and Section 5 of this paper), it is possible to make arbitrary subsets of $^\kappa\kappa$ of the ground model $\Sigma^1_1$-definable in a cofinality preserving forcing extension. In particular, these techniques allow us to force the existence of a $\Sigma^1_1$-wellordering of a given length $\alpha$. The following theorem is a direct consequence of the main result of [24].

**Theorem 2.6** ([24, Theorem 1.5]). Assume that $\kappa = \kappa^\lt \kappa$. Given $\alpha < (2^\kappa)^+$, there is a partial order $P$ with the property that forcing with $P$ preserves all cofinalities and the value of $2^\kappa$ and there is a $\Sigma^1_1$-wellordering of a subset of $^\kappa\kappa$ of order-type $\alpha$ in every $P$-generic extension of the ground model.

In the other direction, the results of [24, Section 9] also show that both $\kappa^+$ and $2^\kappa > \kappa^+$ can consistently be upper bounds for the length of these wellorders.
Theorem 2.7 ([24 Corollary 9.5]). Assume that $\kappa = \kappa^{<\kappa}$. Let $\nu > \kappa$ be a cardinal, let $G$ be $\text{Add}(\kappa, \nu)$-generic over $V$ and let $\langle A, \triangleleft \rangle$ be a $\Sigma^1_1$-wellordering of a subset of $^{\kappa}\kappa$ in $V[G]$. Then $A \neq (^{\kappa}\kappa)^{V[G]}$ and the order-type of $\langle A, \triangleleft \rangle$ has cardinality at most $(2^{\kappa})^V$ in $V[G]$.

Remember that a cardinal $\theta$ is $\Sigma_n$-reflecting if $\theta$ is inaccessible and $\langle H(\theta), \in \rangle$ is an $\Sigma_n$-elementary submodel of $\langle V, \in \rangle$.

Theorem 2.8 ([24 Corollary 9.10]). Let $\nu > \kappa$ be an inaccessible cardinal and let $\mu > 0$ be a cardinal. Assume that either $\theta$ is $\Sigma_2$-reflecting or $\mu > \theta$. If $G \times H$ is $\langle \text{Col}(\kappa, (\kappa, \mu)) \times \text{Add}(\kappa, \mu) \rangle$-generic over $V$ and $\langle A, \triangleleft \rangle$ is a $\Sigma^1_1$-wellordering of a subset of $^{\kappa}\kappa$ in $V[G, H]$, then $A$ has cardinality $\kappa$ in $V[G, H]$.

Note that the above remarks show that the conclusion of the last theorem implies that $\kappa^+$ is inaccessible in $L[x]$ for every $x \subseteq \kappa$.

We will show that certain closed maximality principle exactly determine the least upper bound of the order-types of $\Sigma^1_1$-wellorderings of subsets of $^{\kappa}\kappa$. Depending on the specific principle, this upper bound will either be equal to $\kappa^+$ or $2^{\kappa} > \kappa^+$.

2.4. Cardinal characteristics of tree orderings. We close this section with a question of a somewhat different nature. Let $\kappa$ be an uncountable cardinal satisfying $\kappa = \kappa^{<\kappa}$, let $T_\kappa$ denote the class of all trees of cardinality and height $\kappa$ and let $TO_\kappa$ denote the class of all trees in $T_\kappa$ without a branch of length $\kappa$. Given $T_0, T_1 \in T_\kappa$, we write $T_0 \preceq T_1$ to denote that there is a function $f : T_0 \rightarrow T_1$ such that $f(s) <_{T_0} f(t)$ for all $s, t \in T_0$ with $s <_{T_0} t$. The resulting partial order $\langle TO_\kappa, \preceq \rangle$ arises naturally in infinitary model theory (see, for example, [16] and [30]) and it can be viewed as a substitute for the ordering of all countable ordinals in the uncountable setting (see [28], Section 1).

In the following, we use $\langle -, \rangle : \text{Ord} \times \text{Ord} \rightarrow \text{Ord}$ to denote the Gödel pairing function. Given $x \in ^{\kappa}\kappa$, define $\varepsilon_x$ to be the set of all pairs $\langle \alpha, \beta \rangle$ in $\kappa \times \kappa$ with $x(\alpha, \beta) = 1$. Let $T_\kappa$ denote the set of all $x \in ^{\kappa}\kappa$ such that $T_x = \langle \kappa, \varepsilon_x \rangle$ is an element of $T_\kappa$ and let $TO_\kappa$ denote the set of all $x \in T_\kappa$ such that $T_x$ is an element of $TO_\kappa$. It is easy to check that $T_\kappa$ is a $\kappa$-Borel subset of $^{\kappa}\kappa$ and $TO_\kappa$ is a $\Pi^1_1$-subset of $^{\kappa}\kappa$. Moreover the relation induced by $\preceq$ on $T_\kappa$ is easily seen to be $\Sigma^1_1$-definable. Therefore we may view the partial order $\langle TO_\kappa, \preceq \rangle$ as a $\Sigma^1_1$-definable ordering of a $\Pi^1_1$-subset of $^{\kappa}\kappa$.

We are interested in the order-theoretic properties of $\langle TO_\kappa, \preceq \rangle$. More specifically, we want to study the values of the following cardinal characteristics of this ordering of trees:

(i) The bounding number of $\langle TO_\kappa, \preceq \rangle$ is the smallest cardinal $b_{TO_\kappa}$ with the property that there is a $B \subseteq TO_\kappa$ of this cardinality such that there is no $T \in TO_\kappa$ with $\subseteq T$ for all $\subseteq T \in B$.

(ii) The dominating number of $\langle TO_\kappa, \preceq \rangle$ is the smallest cardinal $d_{TO_\kappa}$ with the property that there is a $D \subseteq TO_\kappa$ of this cardinality such that for every $\subseteq D \in TO_\kappa$ there is a $\subseteq T \in D$ with $\subseteq T \subseteq T$.

It is easy to see that

\[
\kappa^+ \leq b_{TO_\kappa} \leq d_{TO_\kappa} \leq 2^\kappa
\]

holds. In particular, if the GCH holds at $\kappa$, then these cardinal characteristics are equal. We may therefore ask if this is always the case.
Question 5. Is $b_{TO_0}$ equal to $d_{TO_0}$?

Assuming CH, [28 Theorem 15] shows that for every regular cardinal $\nu$ in the interval $[\omega_2, 2^{\omega_1}]$ there is a forcing that preserves all cofinalities and the value of $2^{\omega_1}$ and forces $\nu = b_{TO_{\omega_1}} = d_{TO_{\omega_1}}$. In unpublished work, Schlicht and Thompson strengthened this result by showing that these values can be forced to be equal to any pair of regular cardinals that satisfies the inequalities in \([1]\). In Section 5 we will prove the following result that shows that it is also possible to produce a negative answer to Question 5 by adding many Cohen subsets of $\kappa$.

**Theorem 2.9.** If $\kappa = \kappa^{<\kappa}$ holds and $G$ is $\text{Add}(\kappa, (2^\kappa)^+)$-generic over $V$, then

$$b_{TO_0} \leq V[G] \leq (2^\kappa)^V = d_{TO_0}.$$

We will later show that certain closed maximality principles imply that both of the above cardinal characteristics are equal to $2^\kappa$.

3. Closed maximality principles

We introduce the boldface maximality principles for the class of all $<\kappa$-closed partial orders first formulated by Fuchs in [8] and present results showing how this principle answers most questions stated in the previous section.

**Definition 3.1.** Let $\Phi(v_0, v_1)$ be an $\mathcal{L}_c$-formula.

(i) We say that $\Phi$ defines a class of partial orders if

$$\text{ZFC} \vdash \forall P \forall z \ [\Phi(P, z) \rightarrow "P is a partial order"].$$

(ii) If $\Phi$ defines a class of partial order, then $\Phi$ is suitable if

$$\text{ZFC} \vdash \forall z \forall P ["P is a trivial partial order" \rightarrow \Phi(P, z)].$$

The following definition contains our general formulation of boldface maximality principles.

**Definition 3.2.** Let $\Phi_0(v_0, v_1)$ and $\Phi_1(v_0, v_1)$ be $\mathcal{L}_c$-formulas defining classes of partial orders and let $z$ be a set.

(i) Given an $\mathcal{L}_c$-formula $\varphi(v_0, \ldots, v_{n-1})$ and parameters $x_0, \ldots, x_{n-1}$, we say that the statement $\varphi(x_0, \ldots, x_{n-1})$ is $(\Phi_0, \Phi_1, z)$-forceably necessary if there is a partial order $P$ with $\Phi_0(P, z)$ and $1_{\mathbb{P}} \Vdash \varphi(x_0, \ldots, x_{n-1})$ for every $\mathbb{P}$-name $\dot{Q}$ with $1_{\mathbb{P}} \Vdash \Phi_1(\dot{Q}, \dot{z})$.

(ii) Given an infinite cardinal $\theta$ and $n < \omega$, we let $(\Phi_0, \Phi_1, z)\neg \text{-MP}^n_\theta(\theta)$ denote the statement that every $(\Phi_0, \Phi_1, z)$-forceably necessary $\Sigma_n$-statement with parameters in $H(\theta)$ is true.

With the help of universal $\Sigma_n$-formulas, we can formulate principles of the form $(\Phi_0, \Phi_1, z)\neg \text{-MP}^n_\theta(\theta)$ as single statements in the language of set theory that only use the parameters $\theta$ and $z$. Moreover, if $(\Phi_0, \Phi_1, z)\neg \text{-MP}^1_\theta(\theta)$ holds and $P$ is a partial order with $\Phi_0(P, z)$, then forcing with $P$ preserves all cardinals less than $\theta$.

Let $\Phi_{cl}(v_0, v_1)$ be the canonical suitable formula defining a class of partial orders such that $\Phi_{cl}(\cdot, \kappa)$ defines the class of all $<\kappa$-closed partial orders if $\kappa$ is an infinite regular cardinal and $\Phi_{cl}(\cdot, z)$ defines the class of all partial orders if $z$ is not an infinite regular cardinal. In the following, we abbreviate $(\Phi_{cl}, \Phi_{cl}, \kappa)\neg \text{-MP}^n_\theta(\kappa^+)$ by CMP$_n^\kappa$. Principles of this form and their consequences were studied in depth by Fuchs in [8] and [9]. These principles may be viewed as a natural extension of ZFC,
because a classical argument of Silver shows that the principle CMP$_1$(κ) is always true. The short proof of the next proposition can be found in [24, Section 7].

**Proposition 3.3 (Silver).** Let ℙ be a $<κ$-closed partial order and let G be ℙ-generic over V. Then $(H(κ^+)^V, ∈)$ is a $Σ_1$-elementary substructure of $(H(κ^+)^G, ∈)$.

**Corollary 3.4.** The principle CMP$_1$(κ) holds.

**Proof.** Assume that $φ(v_0, . . . , v_{n−1})$ is a $Σ_1$-formula and $x_0, . . . , x_{n−1} ∈ H(κ^+)$. Let $P$ be a $<κ$-closed partial order witnessing this. Since the trivial partial order is $<κ$-closed, we have $P ⊨ φ(⃗x_0, . . . , ⃗x_{n−1})$. Let G be $ℙ$-generic over V. By the $Σ_1$-Reflection Principle, we know that $φ(⃗x_0, . . . , ⃗x_{n−1})$ holds in $H(κ^+)^V$. Since the trivial partial order is $<κ$-closed, we have $P ⊨ φ(⃗x_0, . . . , ⃗x_{n−1})$. Let G be $ℙ$-generic over V. By the $Σ_1$-Reflection Principle, we know that $φ(⃗x_0, . . . , ⃗x_{n−1})$ holds in $H(κ^+)^V$. Then $G$ is $<κ$-generic over V. By [24, Section 7], Proposition 3.3 implies that $φ(⃗x_0, . . . , ⃗x_{n−1})$ holds in $H(κ^+)^V$ and therefore it holds in V. □

We present a typical application of the principle CMP$_2$(κ). Note that, by the above remarks, the conclusion $κ = κ^{<κ}$ implies that the class of $Σ_1$-subsets of $κ^*$ coincides with the class of subsets of $κ^*$ that are definable over $(H(κ^+), ∈)$ by a $Σ_1$-formula with parameters. Moreover, this assumption allows us to generalize basic results on the structure of $κ$-Borel subsets to the uncountable setting (see [6, Section 1.2.1] for a discussion).

**Proposition 3.5.** If CMP$_2$(κ) holds, then $κ = κ^{<κ}$.

**Proof.** The statement that there is a surjection of $κ$ onto the collection $<κκ$ can be expressed by a $Σ_2$-formula with parameter $κ$ and the partial order $Add(κ, 1)$ witnesses that this statement is $(Φ_κ, Φ_κ, κ)$-forceably necessary. □

We now discuss the consistency of the above principles. In [8] and [9], Fuchs derived the following bounds for their consistency strength.

**Theorem 3.6 ([8 Theorem 3.8], [9 Lemma 3.9]).** Let $1 < n < ω$.

(i) If CMP$_n$(κ) holds and $θ = κ^+$, then $θ$ is $Σ_n$-reflecting in L.

(ii) If $θ > κ$ is a $Σ_{n+2}$-reflecting cardinal and $G$ is $Col(κ, <θ)$-generic over V, then CMP$_n$(κ) holds in $V[G]$.

The above result also allows us to establish the consistency of closed maximality principles for statements of arbitrary complexity. Let $L_κ$ denote the first-order language that extends $L_κ$ by an additional constant symbols $κ$ and $θ$. We let REF denote the $L_κ$-theory consisting of the axioms of ZFC together with the scheme of $L_κ$-sentences stating that $θ$ is a $Σ_n$-reflecting cardinal for all $0 < n < ω$ and the scheme stating that $κ$ is an uncountable regular cardinal smaller than $θ$. Note that, given a Mahlo cardinal $μ$ and a regular uncountable cardinal $κ < μ$, the set of $θ < μ$ with the property that $(H(μ), ∈, κ, θ)$ is a model of REF is stationary in $μ$. Next, let CMP denote the $L_κ$-theory consisting of the axioms of ZFC together with the scheme of $L_κ$-sentences stating that CMP$_n$(κ) holds for all $0 < n < ω$. The following meta-result shows that these theories are equiconsistent. It is a direct consequence of the above theorem.

**Corollary 3.7.**

(i) Assume that $(V, ∈, κ, θ)$ is a model of CMP and $θ = κ^+$. Then $(L_κ, ∈, κ, θ)$ is a model of REF.

(ii) Assume that $(V, ∈, κ, θ)$ is a model of REF. If $G$ is $Col(κ, <θ)$-generic over V, then $(V[G], ∈, κ, θ)$ is a model of CMP. □
In Section 5, we will show that the principle CMP\(_2(\kappa)\) induces a strong structure theory for the class of \(\Sigma^1_1\)-subsets of \(\kappa\). These results will allow us to show that this axiom settles most of the questions posed in Section 2.

**Theorem 3.8.** Assume that CMP\(_2(\kappa)\) holds.

(i) No \(\Delta^1_1\)-subset of \(\kappa\) separates \(\text{Club}(\kappa)\) from \(\text{NS}(\kappa)\).

(ii) Every \(\Sigma^1_1\)-subset of \(\kappa\) has the perfect set property.

(iii) The least upper bound for the order-types of \(\Sigma^1_1\)-wellorderings of subsets of \(\kappa\) is equal to \(\kappa^+\).

In contrast, a combination of results from [8] and observations about the structure of \(\langle TO_\kappa, \leq \rangle\) in Add(\(\kappa, \nu\))-generic extensions will allow us to show that axioms of the form CMP\(_n(\kappa)\) do not answer Question 5. We will prove the following result in Section 5.

**Theorem 3.9.** If the theory CMP is consistent, then it does not decide the statement \(b_{TO_\kappa} = d_{TO_\kappa}\).

The proof of this result suggests that it is also interesting to consider closed maximality principles for statements containing parameters of higher cardinalities. We present an example of such a principle in the next section.

**4. Closed Maximality Principles with More Parameters**

Motivated by Theorem 3.9, we want to consider closed maximality principle for statements with parameters of larger cardinalities. In order to obtain a consistent principle, we have to restrict ourselves to classes of forcings that preserve more cardinalities. If we assume that \(\kappa = \kappa^{<\kappa}\) holds, then the class of all \(<\kappa\)-closed partial orders that satisfy the \(\kappa^+\)-chain condition is a canonical candidate for a rich class of partial orders that preserve all cardinals. Note that the cardinal arithmetic assumption \(\kappa = \kappa^{<\kappa}\) is needed to make this class of partial orders non-trivial. Moreover, the discussion preceding Proposition 3.5 shows that this assumption has many desirable implications on the basic structure of generalized Baire spaces.

In the following, we will argue that we have to restrict the class of partial orders even further. This will follow from a connection between boldface maximality principles and generalizations of classical forcing axioms to larger cardinalities.

Given a partial order \(P\) and an infinite cardinal \(\mu\), we let FA\(_\mu(P)\) denote the statement that, for every collection \(D\) of \(\mu\)-many dense subsets of \(P\), there is a filter \(G\) on \(P\) that meets all elements of \(D\). The following proposition reformulates results of Bagaria (see [1]) and Stavi-Väänänen (see [33, Theorem 25]) to connect maximality principles for \(\Sigma_1\)-statements with forcing axioms.

**Proposition 4.1.** Let \(\Phi_0(v_0, v_1)\) and \(\Phi_1(v_0, v_1)\) be formulas defining classes of partial orders, let \(z\) be set and let \(\mu \geq \kappa\) be a cardinal.

(i) If \((\Phi_0, \Phi_1, z) - \text{MP}_1(\mu^+)\) holds and \(P\) is a partial order of cardinality at most \(\mu\) with \(\Phi_0(P, z)\), then FA\(_\mu(P)\) holds.

(ii) Assume that every partial order \(P\) with \(\Phi_0(P, z)\) satisfies the \(\mu^+\)-chain condition and has the property that FA\(_\mu(P)\) holds. If \(\Phi_1\) is suitable, then \((\Phi_0, \Phi_1, z) - \text{MP}_1(\mu^+)\) holds.

**Proof.** (i) We may assume that the underlying set of \(P\) is an ordinal less than or equal to \(\mu\). Let \(D\) be a collection of \(\mu\)-many dense subsets of \(P\). Then \(D, P \in H(\mu^+)\)
and the statement that there is a $\mathcal{D}$-generic filter on $\mathbb{P}$ can be expressed by a $\Sigma_1$-formula with parameters $\mathbb{P}$ and $\mathcal{D}$. Moreover, $\mathbb{P}$ witnesses that this statement is $(\Phi_0, \Phi_1, z)$-forcibly necessary and therefore true in $V$.

(ii) Fix a $\Sigma_0$-formula $\varphi(v_0, \ldots, v_n)$ and $x_0, \ldots, x_{n-1} \in H(\mu^+)$. Assume that $\mathbb{P}$ is a partial order with $\Phi_0(\mathbb{P}, z)$ that witnesses that the corresponding $\Sigma_1$-statement $\exists x \varphi(x_0, \ldots, x_{n-1}, x)$ is $(\Phi_0, \Phi_1, z)$-forcibly necessary. Since $\Phi_1$ is suitable, this implies that $\mathbb{I}_\mathbb{P} \models "\exists x \varphi(x_0, \ldots, x_{n-1}, x)^\cap \mathcal{D}$.

Picking a sufficiently large regular cardinal $\theta$ and an elementary submodel $M$ of $H(\theta)$ of cardinality $\mu$ with $\mathbb{P}, x_0, \ldots, x_{n-1} \in N$ and $\mu + 1 \subseteq M$. Let $\pi : M \rightarrow N$ be the corresponding transitive collapse and let $\mathcal{D}$ be the collection of all dense open subsets of $\mathbb{P}$ contained in $M$. Then $\text{FA}_{\mu}(\mathbb{P})$ holds and there is a $\mathcal{D}$-generic filter $G$ on $\mathbb{P}$. Since $\mathbb{P}$ satisfies the $\mu^+$-chain condition and $\pi \upharpoonright \mu = \text{id}_\mu$, it follows that $G = \pi[G \cap M]$ is $\pi(\mathbb{P})$-generic over $N$. By elementarity, we have $\pi(x_i) = x_i$ for all $i < n$ and $\exists x \varphi(x_0, \ldots, x_{n-1}, x)$ holds in $N[G]$. Since $N[G]$ is transitive, we can conclude that this statement also holds in $V$. 

Let $\Phi_{\text{dec}}(v_0, v_1)$ be the canonical suitable formula defining a class of partial orders such that $\Phi_{\text{dec}}(\cdots, \kappa)$ defines the class of all $<\kappa$-closed partial orders satisfying the $\kappa^+$-chain condition if $\kappa$ is an infinite regular cardinal and $\Phi_{\text{dec}}(\cdots, z)$ defines the class of all partial orders if $z$ is not an infinite regular cardinal. We will use the following result of Shelah to show that the corresponding closed maximality principle $(\Phi_{\text{dec}}, \Phi_{\text{dec}}, \aleph_1)$–$\text{MP}_2(\aleph_3)$ is provably false.

**Theorem 4.2** ([31 Theorem 6]). Assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} > \aleph_2$. Then there is a $\sigma$-closed partial order $\mathbb{P}$ of cardinality $\aleph_2$ such that $\mathbb{P}$ satisfies the $\aleph_2$-chain condition and $\text{FA}_{\aleph_2}(\mathbb{P})$ fails.

**Corollary 4.3.** If $2^{\aleph_0} = \aleph_1$ holds, then $(\Phi_{\text{dec}}, \Phi_{\text{dec}}, \aleph_1)$–$\text{MP}_2(\aleph_3)$ fails.

**Proof.** Assume, towards a contradiction, that $(\Phi_{\text{dec}}, \Phi_{\text{dec}}, \aleph_1)$–$\text{MP}_2(\aleph_3)$ holds. Then the partial order $\text{Add}(\aleph_1, \aleph_2)$ has cardinality $\aleph_2$ and the first part of Proposition 4.1 shows that our assumption implies that $\text{FA}_{\aleph_2}(\text{Add}(\aleph_1, \aleph_2))$ holds. This implies that $2^{\aleph_1} > \aleph_2$ and we can use Theorem 4.2 to find a partial order $\mathbb{P}$ with the property that $\Phi_{\text{dec}}(\mathbb{P}, \aleph_1)$ holds and $\text{FA}_{\aleph_2}(\mathbb{P})$ fails. By the first part of Proposition 4.1, this contradicts our assumption.

The above result suggests that, in order to obtain consistent maximality principles for statements containing parameters of cardinality greater than $\kappa$, we should restrict the class of forcings that can be used to witness that a given statement is forceably necessary even further. In particular, it should be consistent that $\text{FA}_{\kappa^+}(\mathbb{P})$ holds for every partial order $\mathbb{P}$ in this class. An example of such a class is contained in Shelah’s work on generalization of Martin’s Axiom to higher cardinals (see [30]). Remember that a partial order $\mathbb{P}$ is well-met if all compatible conditions have a greatest lower bound in $\mathbb{P}$. Given a infinite regular cardinal $\mu$, we say that a partial order $\mathbb{P}$ is stationary $\mu^+$-linked if for every sequence $(p_\gamma : \gamma < \mu^+)$ of conditions in $\mathbb{P}$, there is a club $C$ in $\mu^+$ and a regressive function $r : \mu^+ \rightarrow \mu^+$ with the property that the conditions $p_\gamma$ and $p_\delta$ are compatible in $\mathbb{P}$ for all $\gamma, \delta \in C \cap S_\mu^+$ with $r(\gamma) = r(\delta)$. Let $\Phi_{\text{Sh}}(v_0, v_1)$ be the canonical suitable formula defining a class of partial orders such that $\Phi_{\text{Sh}}(\cdots, \kappa)$ defines the class of all stationary $\kappa^+$-linked, well-met partial orders that are $<\kappa$-closed with greatest lower bounds if $\kappa$ is an infinite regular cardinal and $\Phi_{\text{Sh}}(\cdots, z)$ defines the class of all partial orders if $z$ is not an infinite regular cardinal. Following [30], we let GMA$_\kappa$ denote the statement that
FA_\mu(\mathbb{P}) holds for every \mu < 2^\kappa and all partial orders \mathbb{P} with \Phi_{SH}(\mathbb{P}, \kappa). The results of [30] show that, if \kappa = \kappa^{<\kappa} holds, then GMA_\kappa holds in a cofinality-preserving forcing extension of the ground model.

In the following, we study the maximality principles (\Phi_{SH}, \Phi_{clc}, \kappa) – MP_n(2^\kappa) associated to this class. We use SMP_\kappa to denote the conjunction of this principle and the statement \kappa = \kappa^{<\kappa}. Note that similar principles were already studied in [33, Section 2.5]. The following proposition shows that these principles are natural in the sense that they directly generalize Shelah’s forcing axiom GMA. We will later show (see Theorem 4.7) that the assumption of this proposition is a consequence of SMP_2(\kappa).

**Proposition 4.4.** Assume that \mu^{<\kappa} < 2^\mu holds for all \mu < 2^\kappa. Then SMP_1(\kappa) holds if and only if GMA_\kappa and \kappa = \kappa^{<\kappa} hold.

**Proof.** By standard arguments, our cardinal arithmetic assumption implies that GMA_\kappa holds if and only if FA_\mu(\mathbb{P}) holds for all \mu < 2^\kappa and every partial order \mathbb{P} of cardinality less than 2^\kappa with \Phi_{SH}(\mathbb{P}, \kappa). Using this observation, the statement of the proposition follows directly from Proposition 4.1. \qed

The following result gives bounds for the consistency strength of the above principles that resemble the bounds given by Theorem 3.6. The proof of this theorem is contained in Section 3.

**Theorem 4.5.**

(i) If SMP_n(\kappa) holds for some 1 < n < \omega and \theta = 2^\kappa, then \theta is \Sigma_n-reflecting in L.

(ii) Let \theta > \kappa be an inaccessible cardinal and let \triangleleft be a wellordering of H(\theta) of order-type \theta. Assume that \kappa = \kappa^{<\kappa} holds. Then there is a \kappa-closed partial order \mathbb{B}(\kappa, \triangleleft) that is uniformly definable in parameters \kappa and \triangleleft with the property that, if \theta is \Sigma_{n+2}-reflecting for some 1 < n < \omega, then SMP_n(\kappa) holds in every \mathbb{B}(\kappa, \triangleleft)-generic extension of the ground model V.

As above, we also consider versions of this maximality principle for statements of unbounded complexity. We let SMP denote the \mathcal{L}_\kappa-theory consisting of the axioms of ZFC together with the scheme of \mathcal{L}_\kappa-sentences stating that SMP_n(\kappa) holds for all 0 < n < \omega.

**Corollary 4.6.**

(i) Assume that \langle V, \in, \kappa, \theta \rangle is a model of SMP and \theta = 2^\kappa. Then \langle L, \in, \kappa, \theta \rangle is a model of REFL.

(ii) Assume that \kappa = \kappa^{<\kappa} holds, \langle V, \in, \kappa, \theta \rangle is a model of REFL and \triangleleft is a wellordering of H(\theta) of order-type \theta. If G is \mathbb{B}(\kappa, \triangleleft)-generic over V, then \langle V[G], \in, \kappa, \theta \rangle is a model of SMP. \qed

The following theorem summarizes several results that will be proven in the next section. In combination, they show that the principle SMP_2(\kappa) answers all questions posed in Section 2. Moreover, we can combine these results with Proposition 4.4 to conclude that SMP_2(\kappa) implies GMA_\kappa.

**Theorem 4.7.** Assume that SMP_2(\kappa) holds.

(i) The cardinal 2^\kappa is weakly inaccessible and \mu^{<\kappa} < 2^\kappa holds for all \mu < 2^\kappa.

(ii) No \Delta_1^1-subset of \kappa^{<\kappa} separates Clb(\kappa) from NS(\kappa).

(iii) Every \Sigma_1^1-subset of \kappa of cardinality 2^\kappa contains a perfect subset. In particular, every \Delta_1^1-subset of \kappa has the Bernstein property.
Every subset of $\kappa$ of cardinality less than $2^\kappa$ is equal to the union of $\kappa$-many closed subsets of $\kappa$.

(v) The least upper bound for the order-types of $\Sigma_1^1$-wellorderings of subsets of $\kappa$ is equal to $2^\kappa$.

(vi) $\mathfrak{d}_{\mathcal{O}_n} = \mathfrak{d}_{\mathcal{O}_n} = 2^\kappa$.

5. Structural implications

In this section, we show that the principles CMP$_2(\kappa)$ and SMP$_2(\kappa)$ induce a strong structure theory for the class of $\Sigma_1^1$-subsets of $\kappa$. We start by reviewing some basic definitions concerning $\Sigma_1^1$-subsets of $\kappa$.

We call a subset $T$ of $(\prec\kappa)\kappa$ a subtree of $(\prec\kappa)\kappa$ if $\text{lh}(t_0) = \ldots = \text{lh}(t_{n-1})$ and $\langle t_0 \upharpoonright \alpha, \ldots, t_{n-1} \upharpoonright \alpha \rangle \in T$ holds for all $\langle t_0, \ldots, t_{n-1} \rangle \in T$ and $\alpha < \text{lh}(t_0)$. Given such a subtree, we define

$$[T] = \{ \langle x_0, \ldots, x_{n-1} \rangle \in (\kappa^\kappa)^n \mid \forall \alpha < \kappa \langle x_0 \upharpoonright \alpha, \ldots, x_{n-1} \upharpoonright \alpha \rangle \in T \}$$

to be the set of all cofinal branches through $T$. It is easy to see that a subset $A$ of $(\kappa^\kappa)^n$ is closed if and only if it is of the form $[T]$ for some subtree $T$ of $(\prec\kappa)\kappa$.

This shows that the $\Sigma_1^1$-subsets of $(\kappa^\kappa)^n$ are exactly of the form

$$p[T] = \{ \langle x_0, \ldots, x_n \rangle \in (\kappa^\kappa)^n \mid \exists x_n \langle x_0, \ldots, x_n \rangle \in [T] \}$$

for some subtree $T$ of $(\prec\kappa)\kappa + 1$. This also shows that the assumption $\kappa = \kappa^{<\kappa}$ implies that the class of all $\Sigma_1^1$-subsets of $\kappa$ coincides with the class of all subsets of $\kappa$ that are definable over the structure $\langle H(\kappa^+), \in \rangle$ by a $\Sigma_1^1$-formula with parameters.

We now start to derive structural implications from closed maximality principles. It turns out that many of these implications factor through the following concept.

Definition 5.1. Given a partial order $\mathbb{P}$, we say that $\Sigma_2(H(\kappa^+))-absoluteness holds for $\mathbb{P}$ if $\langle H(\kappa^+)^V, \in \rangle$ is a $\Sigma_2$-elementary substructure of $\langle H(\kappa^+)^{\mathbb{P}}, \in \rangle$ whenever $\mathbb{P}$ is $\mathbb{P}$-generic over $V$.

Proposition 5.2. If $\Sigma_2(H(\kappa^+))-absoluteness holds for $\text{Add}(\kappa, 1)$, then $\kappa = \kappa^{<\kappa}$.

Proof. If $G$ is $\text{Add}(\kappa, 1)$-generic over $V$, then $\kappa = \kappa^{<\kappa}$ holds in $V[G]$ and this implies that the collection $\kappa^{<\kappa}$ is a set in $H(\kappa^+)^{V[G]}$. Since this statement can be formulated by a $\Sigma_2$-formula with parameter $\kappa$, it also holds in $H(\kappa^+)^V$ and hence $\kappa = \kappa^{<\kappa}$ holds in $V$. □

The following lemma is essentially a reformulation of [9] Theorem 3.6.

Lemma 5.3. If $\Phi$ is an $\mathcal{L}_m$-formula that is either equal to $\Phi_{cl}$ or to $\Phi_{Sh}$ and $(\Phi, \Phi_{cl}, \kappa)\text{-MP}_2(\kappa^+)$ holds, then $\Sigma_2(H(\kappa^+))-absoluteness holds all $\mathbb{P}$ with $\Phi(\mathbb{P}, \kappa)$.

Proof. Let $\mathbb{P}$ be a partial order with $\Phi(\mathbb{P}, \kappa)$ and let $G$ be $\mathbb{P}$-generic over $V$. Fix a $\Sigma_2$-formula $\varphi(v_0, \ldots, x_{n-1})$ and $v_0, \ldots, x_{n-1} \in H(\kappa^+)$. If $\varphi(x_0, \ldots, x_{n-1})$ holds in $H(\kappa^+)^V$, then Proposition 3.3 directly implies that this statement also holds in $H(\kappa^+)^{\mathbb{P}}$. In the other direction, assume that $\varphi(x_0, \ldots, x_{n-1})$ holds in $H(\kappa^+)^{\mathbb{P}}$. Pick a condition $p$ in $\mathbb{P}$ that forces this and let $\bar{\mathbb{P}}$ denote the partial order consisting of all extensions of $p$ in $\mathbb{P}$. Since $\Phi$ is either equal to $\Phi_{cl}$ or to $\Phi_{Sh}$, we know that $\Phi(\bar{\mathbb{P}}, \kappa)$ holds. If $\bar{Q}$ is a $\bar{\mathbb{P}}$-name for a $\kappa$-closed partial order and $G \ast H$ is $(\bar{\mathbb{P}} \ast \bar{Q})$-generic over $V$, then Proposition 3.3 implies that $\varphi(x_0, \ldots, x_{n-1})$ also holds in $H(\kappa^+)^{\mathbb{P}}$. This argument shows that $\bar{\mathbb{P}}$ witnesses that the statement...
that $\varphi(x_0, \ldots, x_{n-1})$ holds in $H(\kappa^+)$ is $(\Phi, \Phi_{cl}, \kappa)$-forceably necessary. Since we can formulate this statement by a $\Sigma_2$-formula with parameters $\kappa, x_0, \ldots, x_{n-1} \in H(\kappa^+)$, our assumptions imply that this statement holds in $V$ and therefore $\varphi(x_0, \ldots, x_{n-1})$ holds in $H(\kappa^+)^V$.

Next, we show that $\Sigma_2(H(\kappa^+))$-absoluteness implies that all $\Delta^1_1$-subsets of $\kappa^\kappa$ possess a regularity property that generalizes the Baire property to subsets of the generalized Baire space of $\kappa$. A detailed discussion of this regularity property can be found in [6, Section IV.3].

**Definition 5.4.** We say that a subset $X$ of $\kappa^\kappa$ has the $\kappa$-Baire property if there is an open subset $U$ of $\kappa^\kappa$ and a sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ of closed nowhere dense subsets of $\kappa^\kappa$ with the property that $X \Delta U \subseteq \bigcup \{ A_\alpha \mid \alpha < \kappa \}$.

A standard proof shows that the assumption $\kappa = \kappa^{<\kappa}$ implies that every $\kappa$-Borel subset of $\kappa^\kappa$ has the $\kappa$-Baire property (see [10] and Corollary 5.10 below). Moreover, it is also known that the $\Sigma^1_1$-subsets $\text{Club}(\kappa)$ and $\text{NS}(\kappa)$ do not have the $\kappa$-Baire property (see [11, Theorem 4.2]). Finally, it is consistent that every $\Delta^1_3$-subset has the $\kappa$-Baire property (see [6, Theorem 49]). In the following, we will present an alternative proof of this result that makes use of the following notion.

**Definition 5.5.** Given a partial order $\mathbb{P}$, we say that a subset $X$ of $\kappa^\kappa$ is a $\mathbb{P}$-absolutely $\Delta^1_1$-definable if there are subtrees $T_0$ and $T_1$ of $<\kappa^\kappa \times <\kappa^\kappa$ with the property that $p[T_0] = X$, $p[T_1] = \kappa^\kappa \setminus X$ and $1_\mathbb{P} \models " \kappa^\mathbb{P} = p[T_0] \cup p[T_1]"$.

The following observation is a direct consequence of Definition 5.1.

**Proposition 5.6.** If $\Sigma_2(H(\kappa^+))$-absoluteness holds for a partial order $\mathbb{P}$, then every $\Delta^1_1$-subset of $\kappa^\kappa$ is $\mathbb{P}$-absolutely $\Delta^1_1$-definable.

Moreover, the proof of [20, Lemma 1.11] yields the following statement.

**Proposition 5.7.** If $\mathbb{P}$ is a $<\kappa$-closed partial order, then every $\kappa$-Borel subset of $\kappa^\kappa$ is $\mathbb{P}$-absolutely $\Delta^1_1$-definable.

A careful review of the definition of the forcing relation (see, for example, [21, Section VII.3]) yields the following observation that is used throughout this paper.

**Proposition 5.8.** Given $0 < n < \omega$ and a $\Sigma_n$-formula $\varphi(v_0, \ldots, v_{k-1})$, there are $\Sigma_n$-formulas $\psi_0(v_0, \ldots, v_{k+1})$ and $\psi_1(v_0, \ldots, v_{k+1})$ such that

$\text{ZFC}^- \vdash \forall x_0, \ldots, x_{k-1} \forall \mathbb{P} \text{ partial order } \forall p \in \mathbb{P}$

$\left[ (\psi_0(x_0, \ldots, x_{k-1}, \mathbb{P}, p) \iff p \Vdash p \varphi(\tilde{x}_0, \ldots, \tilde{x}_{k-1})) \right. \\
\left. \wedge \left( \psi_1(x_0, \ldots, x_{k-1}, \mathbb{P}, p) \iff p \Vdash \neg \varphi(\tilde{x}_0, \ldots, \tilde{x}_{k-1}) \right) \right]$.}

**Lemma 5.9.** If $\kappa = \kappa^{<\kappa}$, then every $\text{Add}(\kappa, 1)$-absolutely $\Delta^1_1$-definable subset of $\kappa^\kappa$ has the $\kappa$-Baire property.

**Proof.** Let $X$ be such a subset of $\kappa^\kappa$ and let $T_0$ and $T_1$ be subtrees of $<\kappa^\kappa \times <\kappa^\kappa$ with $p[T_0] = X$, $p[T_1] = \kappa^\kappa \setminus X$ and

$1_{\text{Add}(\kappa, 1)} \models " \kappa^\mathbb{P} = p[T_0] \cup p[T_1]"$.

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1 By $\text{ZFC}^-$, we mean the usual axioms of $\text{ZFC}$ without the power set axiom, however including the Collection scheme instead of the Replacement scheme. Note that $H(\nu)$ is a model of this theory for every uncountable regular cardinal $\nu$. 
Since the assumption $\kappa = \kappa^{<\kappa}$ implies that \cite{2} can be formulated by a $\Sigma_1$-formula with parameters in $\text{H}(\kappa^+)$, Proposition \ref{5.8} and the $\Sigma_1$-Reflection Principle imply that \cite{2} also holds in $\text{H}(\kappa^+)$. Let $M$ be an elementary submodel of $\text{H}(\kappa^+)$ of cardinality $\kappa$ with $\kappa + 1 \subseteq M$ and $T_0, T_1 \in M$. Then $M$ is transitive, the assumption $\kappa = \kappa^{<\kappa}$ implies that $\kappa^{<\kappa} \subseteq M$ and elementarity implies that \cite{2} holds in $M$. Let $\langle A_\alpha \mid \alpha < \kappa \rangle$ enumerate all closed nowhere dense subsets of $\kappa$ contained in $M$. We let $\dot{x}$ denote the canonical $\text{Add}(\kappa, 1)$-name for the generic function from $\kappa$ to $\kappa$ and define

$$U = \bigcup \{ N_s \mid s \models \text{Add}(\kappa, 1)^M \text{“} \dot{x} \in p[T_0] \text{”}\} \subseteq \kappa.$$  

**Claim.** $X \setminus U \subseteq \bigcup \{ A_\alpha \mid \alpha < \kappa \}.$

**Proof of the Claim.** Pick $x \in X \setminus \bigcup \{ A_\alpha \mid \alpha < \kappa \}$. Since $x$ is an element of every dense open subset of $\kappa$ contained in $M$, the filter $G_x = \{ x \mid \alpha < \kappa \}$ is $\text{Add}(\kappa, 1)$-generic over $M$. Moreover, we have $p[T_1]^{|M[G_x]} \models p[T_1] = \kappa \setminus X$. Since \cite{2} holds in $M$, we know that

$$\kappa \cap M[G_x] = p[T_0]^{|M[G_x]} \cup p[T_1]^{|M[G_x]}$$

and therefore $x \in p[T_0]^{|M[G_x]}$. Then there is $\alpha < \kappa$ with $x \models \text{Add}(\kappa, 1)^M \text{“} \dot{x} \in p[T_0] \text{”}.\Box$

This allows us to conclude that $x \in N_{x} \alpha \subseteq U$.\hfill\Box

**Claim.** $U \setminus X \subseteq \bigcup \{ A_\alpha \mid \alpha < \kappa \}$.

**Proof of the Claim.** Pick $x \in U \setminus \bigcup \{ A_\alpha \mid \alpha < \kappa \}$. As above, the function $x$ is an element of every dense open subset of $\kappa$ contained in $M$ and therefore the filter $G_x = \{ x \mid \alpha < \kappa \}$ is $\text{Add}(\kappa, 1)$-generic over $M$. Since $x \in U$, we can conclude that $x \in p[T_0]^{|M[G_x]} \subseteq p[T_0] = X$.\hfill\Box

In combination, the above claims show that the open subset $U$ and the sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ witness that $X$ has the $\kappa$-Baire property.\hfill\Box

**Corollary 5.10.** If $\kappa = \kappa^{<\kappa}$ holds, then every $\kappa$-Borel subset of $\kappa$ has the $\kappa$-Baire property.\hfill\Box

**Corollary 5.11.** If $\Sigma_2(\text{H}(\kappa^+))$-absoluteness holds for $\text{Add}(\kappa, 1)$, then every $\Delta^1_1$-subset of $\kappa$ has the $\kappa$-Baire property.\hfill\Box

**Corollary 5.12.** If $\kappa = \kappa^{<\kappa}$ and $G$ is $\text{Add}(\kappa, \kappa^+)$-generic over $V$, then all $\Delta^1_1$-subsets of $\kappa$ have the $\kappa$-Baire property in $V[G]$.\hfill\Box

**Proof.** By \cite{23} Lemma 9.1], $\Sigma_2(\text{H}(\kappa^+))$-absoluteness holds for $\text{Add}(\kappa, 1)$ in $V[G]$. Therefore the statement of the corollary follows directly from Lemma 5.9.\hfill\Box

We now show that the first statement of Theorem \ref{5.8} and the second statement of Theorem \ref{4.7} are direct consequences of the above lemma and the results of \cite{11}.

**Definition 5.13.** A subset $X$ of $\kappa$ super-dense if $\bigcap \{ U_\alpha \cap X \mid \alpha < \kappa \} \neq \emptyset$ for all sequences $\langle U_\alpha \mid \alpha < \kappa \rangle$ of dense open subsets of non-empty open subsets $U$ of $\kappa$.

**Proposition 5.14** (\cite{26} Proposition 3.7]). Assume that $A$ and $B$ are disjoint super-dense subsets of $\kappa$. If $A \subseteq X \subseteq \kappa \setminus B$, then $X$ does not have the $\kappa$-Baire property.

The following statement is essentially shown in the proof of \cite{11} Theorem 4.2.

**Theorem 5.15.** The subsets $\text{Club}(\kappa)$ and $\text{NS}(\kappa)$ of $\kappa$ are super-dense.
Other interesting examples of disjoint super-dense $\Sigma^1_1$-subsets of $^\omega \kappa$ are constructed in [26, Section 3].

**Corollary 5.16.** If all $\Delta^1_1$-subsets of $^\omega \kappa$ have the $\kappa$-Baire property, then no $\Delta^1_1$-subset of $^\omega \kappa$ separates $\text{Club}(\kappa)$ from $\text{NS}(\kappa)$. □

**Proof of Theorem 2.1.** Assume that $\kappa = \kappa^{\lt \kappa}$ and let $G$ be $\text{Add}(\kappa, \kappa^+)$-generic over $V$. Then Corollary 5.12 implies that all $\Delta^1_1$-subsets of $^\omega \kappa$ have the $\kappa$-Baire property in $V[G]$. By Corollary 5.16 there is no $\Delta^1_1$-subset of $^\omega \kappa$ in $V[G]$ that separates $\text{Club}(\kappa)$ from $\text{NS}(\kappa)$. □

**Proof of Clause (i) of Theorem 3.8 and Clause (ii) of Theorem 4.7.** If we assume that either CMP$_2(\kappa)$ or SMP$_2(\kappa)$ holds, then Lemma 5.3 implies that $\Sigma_2(\text{H}(\kappa^+))$-absoluteness holds every $\text{Add}(\kappa, 1)$ and Lemma 5.9 shows that all $\Delta^1_1$-subsets of $^\omega \kappa$ have the $\kappa$-Baire property. This allows us to use Corollary 5.16 to conclude that there is no $\Delta^1_1$-subset of $^\omega \kappa$ separates $\text{Club}(\kappa)$ from $\text{NS}(\kappa)$. □

Next, we discuss the Bernstein property and the perfect set property. We will make use of the following notion defined in [24].

**Definition 5.17.** Given a subtree $T$ of $(^{<\kappa} \kappa)^{n+1}$, a map $\iota : ^{<\kappa} 2 \rightarrow T$ is a $\exists^\kappa$-perfect embedding into $T$ if the following statements hold for all $s_0, s_1 \in ^{<\kappa} 2$ with $\iota(s_i) = \langle t^i_0, \ldots, t^i_n \rangle$.

(i) If $s_0 \subseteq s_1$, then $t^0_k \subseteq t^1_k$ for all $k \leq n$.

(ii) If the sequences $s_0$ and $s_1$ are incompatible, then there is a $k < n$ such that the sequences $t^0_k$ and $t^1_k$ are incompatible.

(iii) If $\text{lh}(s_0) \in \text{Lim}$ and $k \leq n$, then

$$t^0_k = \bigcup \{ u^0_\alpha \mid \exists \alpha < \text{lh}(s_0) \ \iota(s_0) \upharpoonright \alpha = \langle u^0_\alpha, \ldots, u^n_\alpha \rangle \}. $$

**Proposition 5.18 ([24, Proposition 7.5]).** If $T$ is a subtree of $(^{<\kappa} \kappa)^{n+1}$ such that there is a $\exists^\kappa$-perfect embedding into $T$, then $p[T]$ contains a perfect subset.

**Lemma 5.19 ([24, Lemma 7.6]).** Assume $\kappa = \kappa^{<\kappa}$ and let $T$ be a subtree of $(^{<\kappa} \kappa)^{n+1}$. Then there exists a $\exists^\kappa$-perfect embedding into $T$ if and only if there is a $<\kappa$-closed partial order $\mathbb{P}$ with $\mathbb{P} \models "p[T] \not\subseteq V \"$. 

**Lemma 5.20.** If $\kappa = \kappa^{<\kappa}$, then every $\text{Add}(\kappa, 1)$-absolutely $\Delta^1_1$-definable subset of $^\omega \kappa$ has the Bernstein property.

Proof. Let $X$ be such a subset of $^\omega \kappa$, let $T_0$ and $T_1$ be subtrees of $^{<\kappa} \kappa$ witnessing this, let $G$ be $\text{Add}(\kappa, 1)$-generic over $V$ and let $x$ denote the generic function in $V[G]$. By our assumptions, there is $i < 2$ with $x \in p[T_i]^{V[G]}$ and hence $p[T_i]^{V[G]} \not\subseteq V$. By Lemma 5.19 this shows that there is a $\exists^\kappa$-perfect embedding into $T_i$ in $V$ and Proposition 5.18 shows that either $X$ or $^\omega \kappa \setminus X$ contains a perfect subset in $V$. □

**Corollary 5.21.** If $\kappa = \kappa^{<\kappa}$ holds, then every $\kappa$-Borel subset of $^\omega \kappa$ has the Bernstein property. □

**Corollary 5.22.** If $\Sigma_2(\text{H}(\kappa^+))$-absoluteness holds for $\text{Add}(\kappa, 1)$, then every $\Delta^1_1$-subset of $^\omega \kappa$ has the Bernstein property. □

**Proof of Theorem 2.4.** Assume that $\kappa = \kappa^{<\kappa}$ and let $G$ be $\text{Add}(\kappa, \kappa^+)$-generic over $V$. By [24, Lemma 9.1], $\Sigma_2(\text{H}(\kappa^+))$-absoluteness holds for $\text{Add}(\kappa, 1)$ holds in $V[G]$. In this situation, Corollary 5.22 implies the statement of the theorem. □
Proof of Clause (ii) of Theorem 3.8 Assume that CMP\(_2(\kappa)\) holds and let \(T\) be a subtree of \(\preceq_\kappa \times \preceq_\kappa\). If \(\mathcal{Q}\) is a Col(\(\kappa, 2^\kappa\))\(^+\)-name for a \(\preceq_\kappa\)-closed partial order and \(G \ast H\) is (Col(\(\kappa, 2^\kappa\)) \ast \mathcal{Q})\)-generic over \(V\), then either \(p[T]^V \subseteq p[T]^V[G, H]\) or \(p[T]^V[H, G]\) has cardinality \(\kappa\) in \(V[G, H]\). By Lemma 5.19, the first alternative implies that there is a \(\exists^x\)-perfect embedding into \(T\) in \(V\) and this function is still a \(\exists^x\)-perfect embedding into \(T\) in \(V[G, H]\). This shows that Col(\(\kappa, 2^\kappa\))\(^+\) witnesses that the statement that either \(p[T]\) has cardinality \(\kappa\) or there is a \(\exists^x\)-perfect embedding into \(T\) in \(V\) and this function is still a \(\exists^x\)-perfect embedding into \(T\) in \(V[G, H]\). This shows that Add(\(\kappa, 2^\kappa\))\(^+\)-forcing with \(\langle \Phi_{cl}, \Phi_{cl}, \kappa \rangle\)-forceably necessary. By Proposition 3.5, this statement can be formulated by a \(\Sigma_2\)-formula with parameters in \(H(\kappa^+)\). Hence CMP\(_2(\kappa)\) implies that the statement is true in \(V\) and Proposition 5.18 shows that \(p[T]\) has the perfect subset property in \(V\).

Proof of Clause (iii) of Theorem 4.7 Assume that SMP\(_2(\kappa)\) holds and let \(T\) be a subtree of \(\preceq_\kappa \times \preceq_\kappa\). Let \(\mathcal{Q}\) be an Add(\(\kappa, 2^\kappa\))\(^+\)-name for a \(\preceq_\kappa\)-closed partial order satisfying the \(\kappa^+\)-chain condition and let \(G \ast H\) is (Add(\(\kappa, 2^\kappa\)) \ast \mathcal{Q})\)-generic over \(V\). Since forcing with \(\langle \text{Add}(\kappa, 2^\kappa)^+ \ast \mathcal{Q} \rangle\) preserves all cardinals, we know that either \(p[T]^V \subseteq p[T]^V[G, H]\) holds or there is no surjection from \(p[T]^V[G, H]\) onto \(\mathcal{P}(\kappa)^V[G, H]\). As above, the first alternative implies that there is a \(\exists^x\)-perfect embedding into \(T\) in \(V[G, H]\). This shows that Add(\(\kappa, 2^\kappa\))\(^+\) witnesses that the statement that either there is a \(\exists^x\)-perfect embedding into \(T\) or there is no surjection from \(p[T]\) onto \(\mathcal{P}(\kappa)\) is \(\langle \Phi_{Sh}, \Phi_{cl}, \kappa \rangle\)-forceably necessary. Since SMP\(_2(\kappa)\) entails \(\kappa = \kappa^{<\kappa}\), we can formulate this statement by a \(\Sigma_2\)-formula with parameters in \(H(\kappa^+)\) and therefore SMP\(_2(\kappa)\) implies that the statement holds in \(V\). We can conclude that \(p[T]\) either contains a perfect subset or has cardinality less than \(2^\kappa\).

In the following, we will use the above results to prove statements about the length of \(\Sigma_1^1\)-wellorderings in the presence of closed maximality principles. The starting point is the following result from [24] that is used in the proof of Theorem 2.7 and Theorem 2.8.

Lemma 5.23 ([24] Lemma 7.15). Assume that \(\Sigma_2(H(\kappa^+))\)-absoluteness holds for Add(\(\kappa, 1\)). If \(\langle A, \triangleright \rangle\) is a \(\Sigma_1^1\)-wellordering of a subset of \(\kappa^+\), then \(A\) does not contain a perfect subset.

Proof of Clause (iii) of Theorem 3.8 Assume that CMP\(_2(\kappa)\) holds. Then Lemma 5.3 shows that the assumption of Lemma 5.23 is satisfied. Since the domains of \(\Sigma_1^1\)-wellorderings of subsets of \(\kappa^+\) are \(\Sigma_1^1\)-subsets of \(\kappa^+\), a combination of Lemma 5.23 with the second part of Theorem 3.8 shows that these domains have cardinality at most \(\kappa\). In the other direction, it is easy to see that every subset of \((\kappa^+)^\kappa\) of cardinality at most \(\kappa\) is a \(\Sigma_1^1\)-subset. In particular, for every \(\gamma < \kappa^+\), there is a \(\Sigma_1^1\)-wellordering of a subset of \(\kappa^+\) of order-type \(\gamma\).

In order to prove the corresponding result for the principle SMP\(_2(\kappa)\), we first need to derive the fourth statement of Theorem 2.7 to show that this principle implies that for every \(\gamma < 2^\kappa\) there is a \(\Sigma_1^1\)-wellordering of a subset of \(\kappa^+\) of order-type \(\gamma\). The proof of this result relies on the generic coding techniques studied in [14], Section 2].

Definition 5.24. Assume that \(A\) is a subset of \(\kappa^+\) and \(\vec{s} = \langle s_\beta \mid \beta < \kappa \rangle\) is an enumeration of \(\preceq_\kappa \times \preceq_\kappa\) with the property that every element of \(\preceq_\kappa \times \preceq_\kappa\) is enumerated
unboundedly often. We let \( C_\vec{s}(A) \) denote the unique partial order defined by the following clauses:

(i) A condition in \( C_\vec{s}(A) \) is a pair \( p = \langle t_p, a_p \rangle \) such that \( t_p : \alpha_p \to 2 \) for some \( \alpha_p < \kappa \) and \( a_p \) is a subset of \( A \) of cardinality less than \( \kappa \).

(ii) Given conditions \( p \) and \( q \) in \( C_\vec{s}(A) \), we have \( p \leq_{C_\vec{s}(A)} q \) if and only if \( t_q \subseteq t_p \), \( \alpha_q \leq \alpha_p \) and

\[
s_\beta \subseteq x \implies t_p(\beta) = 1
\]

for all \( x \in a_q \) and \( \alpha_q \leq \beta < \alpha_p \).

**Proposition 5.25.** In the situation of Definition 5.24, the partial order \( C_\vec{s}(A) \) is stationary \( \kappa^+ \)-linked, well-met and \( <\kappa \)-closed with greatest lower bounds.

**Proof.** By [14 Proposition 2.2], \( C_\vec{s}(A) \) is \( \kappa \)-linked and \( <\kappa \)-closed with greatest lower bounds. In particular, this implies that \( C_\vec{s}(A) \) is stationary \( \kappa^+ \)-linked. Fix compatible conditions \( p \) and \( q \) in \( C_\vec{s}(A) \) with \( \alpha_p \geq \alpha_q \). Then we have \( t_q \subseteq t_p \) and \( r = \langle t_p, a_p \cup a_q \rangle \) is a condition in \( C_\vec{s}(A) \) that extends \( p \). Pick \( x \in a_q \) and \( \alpha_q \leq \beta < \alpha_p \) with \( s_\beta \subseteq x \). Pick a condition \( s \) in \( C_\vec{s}(A) \) with \( s \leq_{C_\vec{s}(A)} p, q \). Then \( t_p \subseteq t_s \), \( \beta < \alpha_s \) and \( t_p(\beta) = t_s(\beta) = 1 \). This shows that \( r \) is also an extension of \( q \). Finally, it is easy to check that \( u \leq_{C_\vec{s}(A)} r \) holds for all conditions \( u \) in \( C_\vec{s}(A) \) with \( u \leq_{C_\vec{s}(A)} p, q \).

Under the assumptions listed in Definition 5.24 there is a canonical sequence \( \langle \dot{T}_\alpha \mid \alpha < \kappa \rangle \) of \( C_\vec{s}(A) \)-names for subtrees of \( \langle \kappa \rangle \subset \kappa \) with the property that, whenever \( G \) is \( C_\vec{s}(A) \)-generic over \( V \) and \( t_G = \bigcup \{ t_p \mid p \in G \} : \kappa \to 2 \), then

\[
\dot{T}_\alpha^G = \{ t \in \langle \kappa \rangle \mid \forall \beta < \kappa [ t_G(\beta) = 1 \implies s_\beta \not\subseteq t] \}
\]

for all \( \alpha < \kappa \).

**Theorem 5.26 (14 Corollary 4.3).** In the situation of Definition 5.24, if \( \dot{\mathcal{Q}} \) is a \( C_\vec{s}(A) \)-name for a \( <\kappa \)-closed partial order and \( G \ast H \) is \( (C_\vec{s}(A) \ast \dot{\mathcal{Q}}) \)-generic over \( V \), then \( A = \bigcup \{ T_\alpha^G \mid \alpha < \kappa \} \) is \( \langle \kappa \rangle \)-closed in \( V[G, H] \).

**Proof of Clause (iv) of Theorem 4.7.** Assume that \( \text{SMP}_2(\kappa) \) holds and \( A \) is a subset of \( \langle \kappa \rangle \subset \kappa \) of cardinality less than \( 2^\kappa \). Since \( \kappa = \kappa^{<\kappa} \) holds, there is an enumeration \( \vec{s} \) of \( \langle \kappa \rangle \subset \kappa \) that satisfies the requirements of Definition 5.24. If \( \dot{\mathcal{Q}} \) is a \( C_\vec{s}(A) \)-name for a \( <\kappa \)-closed partial order and \( G \ast H \) is \( (C_\vec{s}(A) \ast \dot{\mathcal{Q}}) \)-generic over \( V \), then Theorem 5.26 implies that \( A = \bigcup \{ T_\alpha^G \mid \alpha < \kappa \} \) is \( \langle \kappa \rangle \)-closed in \( V[G, H] \). By Proposition 5.25, this shows that the partial order \( C_\vec{s}(A) \) witnesses that the statement that there is a sequence \( \langle T_\alpha \mid \alpha < \kappa \rangle \) of sub-trees of \( \langle \kappa \rangle \subset \kappa \) with the property that \( A = \bigcup \{ T_\alpha \mid \alpha < \kappa \} \) is \( \langle \Phi_{\text{Sth}}, \Phi_{\text{ad}}, \kappa \rangle \)-forcibly necessary. Since this statement can be expressed by a \( \Sigma_2 \)-formula with parameters \( \kappa, A \in H(2^\kappa) \), our assumptions imply that it holds in \( V \).

**Proof of Clause (v) of Theorem 4.7.** Assume that \( \text{SMP}_2(\kappa) \) holds. Then Lemma 5.3 shows that the assumption of Lemma 5.23 is satisfied. Hence we can apply the third part of Theorem 4.7 to conclude that the domains of \( \Sigma_1^1 \)-wellorderings of subsets of \( \langle \kappa \rangle \subset \kappa \) have cardinality less than \( 2^\kappa \) and therefore these orderings have order-type less than \( 2^\kappa \). In the other direction, the fourth part of Theorem 4.7 implies that every subset of \( \langle \kappa \rangle \times \langle \kappa \rangle \subset \kappa \) of cardinality less than \( 2^\kappa \) is a \( \Sigma_1^1 \)-subset. In particular, for every \( \gamma < 2^\kappa \), there is a \( \Sigma_1^1 \)-wellordering of a subset of \( \langle \kappa \rangle \subset \kappa \) of order-type \( \gamma \).
In the remainder of this section, we will study the influence of closed maximality principles on the cardinal characteristics of the partial order \(<\mathcal{TO}_\kappa, \preceq>\). These results will also allow us to prove Theorem 2.9. The starting point of this investigation is the following **Boundedness Lemma**. This result was first proven for \(\kappa = \omega_1\) in [28] and the proof given there directly generalizes to higher regular cardinalities (see [24] Lemma 8.1). 

**Lemma 5.27** ([28] Corollary 13). If \(\kappa = \kappa^{<\kappa}\) holds and \(A\) is a \(\Sigma^1_1\)-subset of \(\mathcal{TO}_\kappa\), then there is a tree \(T \in \mathcal{TO}_\kappa\) with \(T_x \preceq T\) for all \(x \in A\).

**Proof of Clause (vi) of Theorem 4.7.** Assume that SMP\(_2\)(\(\kappa\)) holds. Let \(B\) be a subset of \(\mathcal{TO}_\kappa\) of cardinality less than \(2^\kappa\). Pick a subset \(A\) of \(\mathcal{TO}_\kappa\) of the same cardinality such that for every \(S \in B\), there is an \(x \in A\) such that the trees \(S\) and \(T_x\) are isomorphic. In this situation, the fourth part of Theorem 4.7 implies that \(A\) is a \(\Sigma^1_1\)-subset of \(\kappa\) and Lemma 5.27 shows that there is a tree \(T \in \mathcal{TO}_\kappa\) with \(S \preceq T\) for all \(S \in B\). In combination with [1], these computations show that SMP\(_2\)(\(\kappa\)) implies that \(b_{\mathcal{TO}_\kappa} = b_{\mathcal{TO}_\kappa} = 2^\kappa\).

In order to prove Theorem 2.9 and Theorem 3.9, we study the behaviour of the partial order \(<\mathcal{TO}_\kappa, \preceq>\) in \(\text{Add}(\kappa, \nu)\)-generic extensions of the ground model.

**Lemma 5.28.** If \(\kappa = \kappa^{<\kappa}\) holds, \(G\) is Add(\(\kappa, 1\))-generic over \(V\) and \(T \in \mathcal{TO}_{\kappa}[G]\), then there is \(S \in \mathcal{TO}_{\kappa}[V]\) such that \(S \not\preceq T\) in \(V[G]\).

**Proof.** Work in \(V\). Let \(x\) be an \(\text{Add}(\kappa, 1)\)-nice name for an element of \(\mathcal{TO}_\kappa\). Then \(\dot{x} \in H(\kappa^+)\). Define

\[
A(\dot{x}) = \{x \in \mathcal{TO}_\kappa \mid \exists p \in \text{Add}(\kappa, 1) \ p \models_{\text{Add}(\kappa, 1)} \text{“}T_x \preceq T_x\text{”}\}.
\]

Since the ordering \(\preceq\) is defined by a \(\Sigma_1\)-formula with parameter \(\kappa\), we can use Proposition 5.8 and the \(\Sigma_1\)-Reflection Principle to show that \(A(\dot{x})\) is definable over \(<H(\kappa^+), \in>\) by a \(\Sigma_1\)-formula with parameters.

Next, pick \(x \in A(\dot{x})\) and let \(G\) be \(\text{Add}(\kappa, 1)\)-generic over \(V\) such that \(T_x \preceq T_x\) holds in \(V[G]\). Since \(T_x \in \mathcal{TO}_{\kappa}[G]\), we have \(x \in \mathcal{TO}_{\kappa}[G]\) and Proposition 3.3 implies that \(x \in \mathcal{TO}_{\kappa}^V\). These computations show that \(A(\dot{x})\) is a \(\Sigma^1_1\)-subset of \(\mathcal{TO}_\kappa\) in \(V\). Since results of Kurepa (see [16] Lemma 2.1) show that the assumption \(\kappa = \kappa^{<\kappa}\) implies that \(<\mathcal{TO}_\kappa, \preceq>\) contains no maximal elements, we can use Lemma 5.27 to find a tree \(T(\dot{x}) \in \mathcal{TO}_\kappa^V\) with \(T_x \preceq T(\dot{x})\) and \(T(\dot{x}) \not\preceq T_x\) for all \(x \in A(\dot{x})\).

Now, let \(G\) be \(\text{Add}(\kappa, 1)\)-generic over \(V\) and let \(T\) be an element of \(\mathcal{TO}_{\kappa}[G]\). Pick an \(\text{Add}(\kappa, 1)\)-nice name \(\dot{x}\) for an element of \(\mathcal{TO}_\kappa\) such that the tree \(T_{\dot{x}}\) is isomorphic to \(T\) in \(V[G]\). Then \(T(\dot{x}) \in \mathcal{TO}_\kappa^V\). Assume, towards a contradiction, that \(T(\dot{x}) \preceq T\) holds in \(V[G]\). Then there is an \(x \in A(\dot{x})\) such that the tree \(T_x\) is isomorphic to \(T(\dot{x})\) in \(V\). But this is a contradiction, because \(x \in A(\dot{x})\) implies that \(T(\dot{x}) \not\preceq T_x\) holds in \(V\).

**Lemma 5.29.** Assume that \(\kappa = \kappa^{<\kappa}\) holds. If \(\nu > \kappa\) and \(G\) is \(\text{Add}(\kappa, \nu)\)-generic over \(V\), then \(b_{\mathcal{TO}_\kappa}^{V[G]} \leq (2^\kappa)^V\).

**Proof.** Assume, towards a contradiction, that there is \(T \in \mathcal{TO}_{\kappa}[G]\) with the property that \(T_x \preceq T\) holds in \(V[G]\) for all \(x \in \mathcal{TO}_\kappa^V\). Pick \(y \in \mathcal{TO}_{\kappa}[G]\) with the property that the trees \(T\) and \(T_y\) are isomorphic in \(V[G]\). Then we can find \(H \in V[G]\) such that \(H\) is \(\text{Add}(\kappa, 1)\)-generic over \(V\) with \(y \in V[H]\) and \(V[G]\) is an \(\text{Add}(\kappa, \nu)\)-generic
extension of $V[H]$. In this situation, Lemma 5.28 yields an $x \in \text{TO}_\kappa^V$ with the property that $\mathbb{T}_x \not\preceq \mathbb{T}_y$ holds in $V[H]$. By Proposition 3.3 this implies that $\mathbb{T}_x \not\preceq \mathbb{T}_y$ also holds in $V[G]$, a contradiction. □

In the other direction, the following lemma will enable us to find lower bounds for $\delta_{\mathcal{T}_\alpha}$ in Add($\kappa, \nu$)-generic extensions of the ground model.

**Lemma 5.30.** If $\kappa = \kappa^{<\kappa}$ holds, then there is a Add($\kappa, 1$)-generic $\mathbb{T}$ for an element of $\mathcal{T}_\alpha$ such that $1^\text{Add}(\kappa, 1) \models "\mathbb{T} \not\preceq \mathbb{T}_x"$ holds for all $x \in \text{TO}_\kappa$.

Proof. Let $\dot{x}$ be the canonical Add($\kappa, 1$)-name for the generic function from $\kappa$ to 2 and let $\mathbb{T}$ be a canonical Add($\kappa, 1$)-name for the subtree of $^{<\kappa}\kappa$ with the property that, whenever $G$ is Add($\kappa, 1$)-generic over $V$, then $T^G$ consists of all strictly increasing, continuous functions $s$ in $^{<\kappa}\kappa$ with $\dot{x}^G(s(\alpha)) = 1$ for all $\alpha \in \text{dom}(s)$. If $G$ is Add($\kappa, 1$)-generic over $V$, then the set $\{\alpha < \kappa \mid \dot{x}^G(\alpha) = 1\}$ is both a stationary and costationary subset of $\kappa$ in $V[G]$ and this implies that $\mathbb{T}^G \in \mathcal{T}_\alpha^{V[G]}$.

Assume, towards a contradiction, that there is an $x \in \text{TO}_\kappa$, an Add($\kappa, 1$)-name $\dot{f}$ for a function with domain $\mathbb{T}$ and a condition $p_0$ in Add($\kappa, 1$) with

$$p_0 \forces_{\text{Add}(\kappa, 1)} "f : (\mathbb{T}, \subseteq) \rightarrow \mathbb{T}_x \text{ witnesses that } (\mathbb{T}, \subseteq) \preceq \mathbb{T}_x."$$

Then we can inductively construct sequences $(s_\alpha \in {^{<\kappa}\kappa} \mid \alpha < \kappa), (\gamma_\alpha < \kappa \mid \alpha < \kappa)$ and $(p_\alpha \in \text{Add}(\kappa, 1) \mid \alpha < \kappa)$ such that the following statements hold for all $\alpha < \kappa$.

(i) If $\bar{\alpha} < \alpha$, then $s_{\bar{\alpha}} \subseteq s_\alpha$ and $p_\alpha \leq \text{Add}(\kappa, 1) \cdot p_{\bar{\alpha}}$.

(ii) $s_\alpha$ is a strictly increasing, continuous function and $p_\alpha(s(\gamma)) = 1$ for all $\gamma \in \text{dom}(s_\alpha)$.

(iii) $\text{dom}(s_\alpha)$ has a maximal element and $s_\alpha(\text{max}(\text{dom}(s_\alpha)))$ is the maximum of $\text{dom}(p_\alpha)$.

(iv) $p_{\alpha+1} \forces_{\text{Add}(\kappa, 1)} "\dot{f}(s_\alpha) = \gamma_\alpha."$

But then $\gamma_\alpha \in_x \gamma_\alpha$ holds for all $\bar{\alpha} < \alpha < \kappa$ and the set $\{\gamma < \kappa \mid \exists \alpha < \kappa \gamma \in_x \gamma_\alpha\}$ is a cofinal branch through $\mathbb{T}_x$ in $V$, a contradiction. □

**Lemma 5.31.** Assume that $\kappa = \kappa^{<\kappa}$ and $\nu \geq 2^\kappa$ is a cardinal with $\text{cof}(\nu) > \kappa$. If $G$ is Add($\kappa, \nu$) over $V$, then $\delta_{\mathcal{T}_\nu}^{V[G]} = \nu = (2^\kappa)^{V[G]}$.

Proof. Assume, towards a contradiction, that, in $V[G]$, there is a subset $D$ of $\text{TO}_\kappa$ of cardinality less than $\nu$ with the property that, for every $T \in \mathcal{T}_\nu$, there is $x \in D$ with $T \preceq T_x$. Since Add($\kappa, \nu$) satisfies the $\kappa^+$-chain condition in $V$, we can find $H \in V[G]$ such that $H$ is Add($\kappa, \nu$)-generic over $V$ with $D \in V[H]$ and $V[G]$ is an Add($\kappa, 1$)-generic extension of $V[H]$. In this situation, Lemma 5.30 shows that there is a tree $T \in \mathcal{T}_\alpha^{V[G]}$ such that $T \not\preceq T_x$ holds in $V[G]$ for all $x \in \text{TO}_\kappa^{V[H]}$. But our assumptions imply that there is an $x \in D \subseteq \text{TO}_\kappa^{V[H]}$ with $T \preceq T_x$, a contradiction. □

**Proof of Theorem 2.9.** Assume that $\kappa = \kappa^{<\kappa}$ holds and $G$ is Add($\kappa, (2^\kappa)^+$)-generic over $V$. Then Lemma 5.29 implies that

$$b_{\text{TO}_\kappa}^{V[G]} \leq (2^\kappa)^V < ((2^\kappa)^+)^V = (2^\kappa)^{V[G]}.$$

In addition, we can apply Lemma 5.31 to conclude that $\delta_{\text{TO}_\kappa}^{V[G]} = (2^\kappa)^{V[G]}$. □

The following absoluteness result of Fuchs is the last ingredient needed for the proof of Theorem 3.9.
**Theorem 5.32** ([8] Lemma 4.10). Assume that \(\langle V, \varepsilon, \kappa, \kappa^+ \rangle\) is a model of CMP. If \(G\) is Add(\(\kappa, \nu\))-generic over \(V\), then \(\langle V[G], \varepsilon, \kappa, \kappa^+ \rangle\) is also a model of CMP.

**Proof of Theorem 5.32**. Assume that the theory CMP is consistent. By Corollary 3.7, this assumption implies that the theory CMP + GCH is also consistent and this theory proves that “\(\forall \tau \in \omega \times \kappa^+ \) \(\tau\) is \(\kappa\)-compact.” In the other direction, we can combine our assumption with Theorem 2.9 and Theorem 5.32 to conclude that the theory CMP + “\(\forall \tau \in \omega \times \kappa^+ \) \(\tau\) is \(\kappa\)-compact” is also consistent.

6. Equiconsistency results

This section contains results that provide upper and lower bounds for the consistency strength of closed maximality principles of the form SMP\(_n\)(\(\kappa\)). We start by completing the proof of Theorem 4.7.

**Proof of Clause (i) of Theorem 4.7**. Assume that SMP\(_2\)(\(\kappa\)) holds and let \(\mu\) denote a cardinal smaller than \(2^\kappa\).

First, let \(\hat{Q}\) be an Add(\(\kappa, (\mu^{<\kappa})^+\))-name for a \(<\kappa\)-closed partial order that satisfies the \(\kappa^+\)-chain condition. If \(G \ast H\) is Add(\(\kappa, (\mu^{<\kappa})^+\) * \(\hat{Q}\))-generic over \(V\), then \((\kappa^+ \ast \mu^{<\kappa})\) V\([G, H]\) \(\subseteq V\) and hence \(\mu^{<\kappa} < 2^\kappa\) holds in \(V[G, H]\). This shows that the partial order Add(\(\kappa, (\mu^{<\kappa})^+\)) witnesses that the statement there is no surjection from \(\mu^{<\kappa}\) onto \(P(\kappa)\) is \(\Phi_{Sh}, \Phi_{clc}, \kappa\)-forceably necessary. Since this statement can be expressed by a \(\Sigma_2\)-formula with parameters contained in \(H(2^\kappa)\), our assumptions imply that it holds in \(V\) and we can conclude that \(\mu^{<\kappa} < 2^\kappa\).

Next, pick an injection \(\iota: \mu \to \kappa^\kappa\). Given \(X \subseteq \mu\), we can use the fourth part of Theorem 4.7 to find a subtree \(T_X\) of \(\kappa^\kappa\) with \(\iota[X] = p[T_X]\). Since \(\kappa = \kappa^{<\kappa}\) holds, there is an \(x \in T_{\kappa}\) such that \(T_X\) is isomorphic to the tree \(T_x\). This allows us to construct an injection from \(2^\mu\) into \(2^\kappa\) and we can conclude that \(2^\kappa = 2^\mu\) holds for all \(\mu < 2^\kappa\).

Finally, let \(\hat{Q}\) be an Add(\(\kappa, (2^\kappa)^+\))-name for a partial order that satisfies the \(\kappa^+\)-chain condition. If \(G \ast H\) is Add(\(\kappa, (2^\kappa)^+\) * \(\hat{Q}\))-generic over \(V\), then we know that, in \(V[G, H]\), there is a set \(X\) with the property that there is no surjection from \(\mu\) onto \(X\) and no surjection from \(X\) onto \(P(\kappa)\). In particular, the partial order Add(\(\kappa, (2^\kappa)^+\)) witnesses that this statement is \(\Phi_{Sh}, \Phi_{clc}, \kappa\)-forceably necessary. Since this statement can be expressed by a \(\Sigma_2\)-formula with parameters contained in \(H(2^\kappa)\), it holds in \(V\) and hence we have \(\mu^{<\kappa} < 2^\kappa\).

The above computations show that \(2^\kappa\) is a limit cardinal with \(2^\kappa = 2^\mu\) for all \(\kappa \leq \mu < 2^\kappa\). By [13] Corollary 5.17, we can conclude that \(2^\kappa\) is regular.

A similar argument also allows us to prove the first part of Theorem 4.5.

**Proof of Clause (i) of Theorem 4.5**. Assume that \(1 < n < \omega\) and SMP\(_n\)(\(\kappa\)) holds. Set \(\theta = 2^\kappa\). By the first part of Theorem 4.7, \(\theta\) is weakly inaccessible and therefore \(\theta\) is an inaccessible cardinal in \(L\). By induction, we show that \(\langle L_\theta, \varepsilon \rangle\) is a \(\Sigma_n\)-elementary substructure of \(\langle L, \varepsilon \rangle\).

Assume that \(\langle L_\theta, \varepsilon \rangle\) is a \(\Sigma_m\)-elementary substructure of \(\langle L, \varepsilon \rangle\) for some \(m < n\). Fix a \(\Pi_m\)-formula \(\varphi(v_0, \ldots, v_k)\) and \(z_0, \ldots, z_{k-1} \in L_\theta\) with the property that the statement \(\exists x \varphi(x, z_0, \ldots, z_{k-1})\) holds in \(L\). Pick a cardinal \(\mu\) with the property that there is some \(y \in L_\mu\) such that \(\varphi(y, z_0, \ldots, z_{k-1})\) holds in \(L\). Let \(\hat{Q}\) be an Add(\(\kappa, \mu\))-name for a \(<\kappa\)-closed partial order satisfying the \(\kappa^+\)-chain condition and let \(G \ast H\) be Add(\(\kappa, \nu\) * \(\hat{Q}\))-generic over \(V\). In \(V[G, H]\), there is an ordinal \(\lambda\) with the
Definition 6.1. Let $\Phi$ be an $\mathcal{L}_{\infty}$-formula defining a class of partial orders.

(i) A forcing iteration $\overrightarrow{\mathbb{P}} = \langle \langle \overrightarrow{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \overrightarrow{\mathbb{P}}_{\alpha} \mid \alpha < \lambda \rangle \rangle$ is a $(\Phi, \kappa)$-iteration if it has $<\kappa$-support and $\mathbb{1}_{\overrightarrow{\mathbb{P}}_{<\alpha}} \models \Phi(\overrightarrow{\mathbb{P}}_{\alpha}, \kappa)$ holds for all $\alpha < \lambda$.

(ii) A partial order $\mathbb{Q}$ is equivalent to a $(\Phi, \kappa)$-iteration if there exists a $(\Phi, \kappa)$-iteration $\langle \langle \overrightarrow{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \overrightarrow{\mathbb{P}}_{\alpha} \mid \alpha < \lambda \rangle \rangle$ with the property that the partial order $\overrightarrow{\mathbb{P}}_{<\lambda}$ is isomorphic to $\mathbb{Q}$.

The following iteration result is contained in the proof of [30, Theorem 1.1] and motivates the above definition of stationary $\kappa^+$-linked partial orders.

Theorem 6.2 (30). Assume that $\kappa = \kappa^{<\kappa}$ and let $\langle \langle \overrightarrow{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \overrightarrow{\mathbb{P}}_{\alpha} \mid \alpha < \lambda \rangle \rangle$ be a $(\Phi_{\text{Sh}}, \kappa)$-iteration. Then the partial order $\overrightarrow{\mathbb{P}}_{<\lambda}$ is $<\kappa$-closed and satisfies the $\kappa^+$-chain condition.

Our proof of Theorem 1.5 relies on two technical lemmas that are consequences of Baumgartner’s detailed analysis of the tails of forcing iterations in [2]. Given such an iteration $\langle \langle \overrightarrow{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \overrightarrow{\mathbb{P}}_{\alpha} \mid \alpha < \lambda \rangle \rangle$ and $\alpha \leq \lambda$, we let $\overrightarrow{\mathbb{P}}_{[\alpha, \lambda]}$ denote the canonical $\overrightarrow{\mathbb{P}}_{<\alpha}$-name for the corresponding tail forcing constructed in [2, Section 5].

Lemma 6.3. If $\Phi$ defines a class of partial orders and $\langle \langle \overrightarrow{\mathbb{P}}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \overrightarrow{\mathbb{P}}_{\alpha} \mid \alpha < \lambda \rangle \rangle$ is a $(\Phi, \kappa)$-iteration, then

$$\mathbb{1}_{\overrightarrow{\mathbb{P}}_{<\alpha}} \models \text{“} \overrightarrow{\mathbb{P}}_{[\alpha, \lambda]} \text{ is equivalent to a $(\Phi, \kappa)$-iteration”}$$

for all $\alpha < \lambda$ with the property that $\overrightarrow{\mathbb{P}}_{<\alpha}$ is $<\kappa$-distributive.

We will show that this lemma is a direct consequence of the results of [2, Section 5]. Following Baumgartner, we say that a set $X$ of ordinals is $\kappa$-thin if $X \cap \delta$ is bounded in $\delta$ for every limit ordinal $\delta$ of cofinality greater than or equal to $\kappa$.

Proposition 6.4. Let $\mathbb{P}$ be a $<\kappa$-distributive partial order and let $G$ be $\mathbb{P}$-generic over $V$. If $X$ is a $\kappa$-thin set of ordinals in $V[G]$, then $X \in V$ and $X$ is $\kappa$-thin in $V$.

Proof. Since $X$ is $\kappa$-thin, we know that the club of limit points of $X$ contains no ordinal of cofinality $\kappa$ in $V[G]$ and therefore $X$ has cardinality less than $\kappa$ in $V[G]$. In particular, $X$ is contained in $V$. Given a limit ordinal $\delta$ with $\text{cof}(\delta)^V \geq \kappa$, we have $\text{cof}(\delta)^V \geq \kappa$ and $X \cap \delta$ is bounded in $\delta$. Hence $X$ is $\kappa$-thin in $V$. $\square$
Proof of Lemma 6.3. Fix $\alpha < \lambda$ such that $\overrightarrow{P}_{<\alpha}$ is $\langle \kappa \rangle$-distributive and pick $\bar{\lambda} \leq \lambda$ with $\lambda = \alpha + \bar{\lambda}$. Let $G$ be $\overrightarrow{P}_{<\alpha}$-generic over $V$ and work in $V[G]$. By [2, Theorem 5.2], there is a canonical forcing iteration $\overrightarrow{Q} = \langle (\overrightarrow{Q}_{<\beta} \mid \beta \leq \lambda), (\overrightarrow{Q}_{\beta} \mid \beta < \bar{\lambda}) \rangle$ and a sequence $(i_{\beta} : \overrightarrow{P}^{G}_{(\alpha, \alpha+1)} \rightarrow \overrightarrow{Q}_{\beta} \mid \beta \leq \lambda)$ of isomorphisms of partial orders such that the following statements hold for all $\beta$: for a partial order. If $\overrightarrow{Q}$ is then there is a $\overrightarrow{P}$-name $\bar{p}$ with the property that, whenever $\bar{q}$ satisfies and hence we can use this result together with [2, Theorem 5.3] to conclude $\overrightarrow{Q}$ has $\langle \kappa \rangle$-support. This shows that $\overrightarrow{Q}$ is a $(\Phi, \kappa)$-iteration and the partial order $\overrightarrow{P}^{G}_{(\alpha, \lambda)}$ is equivalent to such an iteration in $V[G]$. 

The next lemma shows that the converse of the above implication is also true.

Lemma 6.5. Let $\Phi$ be a suitable formula defining a class of partial order, let $\overrightarrow{P} = \langle \overrightarrow{P}_{<\alpha} \mid \alpha \leq \lambda \rangle, \langle \overrightarrow{P}_{\alpha} \mid \alpha < \lambda \rangle$ be a $(\Phi, \kappa)$-iteration and let $\overrightarrow{Q}$ be a $\overrightarrow{P}_{<\lambda}$-name for a partial order. If $\overrightarrow{P}_{<\alpha}$ is $\langle \kappa \rangle$-distributive and

$$1_{\overrightarrow{P}_{<\beta}} \models "\overrightarrow{Q} is equivalent to a $(\Phi, \kappa)$-iteration",$$

then there is a $(\Phi, \kappa)$-iteration $\langle (\overrightarrow{P}^{G}_{<\alpha} \mid \alpha \leq \lambda + \eta), (\overrightarrow{P}^{G}_{\alpha} \mid \alpha < \lambda + \eta) \rangle$ extending $\overrightarrow{P}$ with the property that, whenever $G$ is $\overrightarrow{P}_{<\lambda}$-generic over $V$, then the partial orders $\overrightarrow{P}^{G}_{(\lambda, \lambda+\eta)}$ and $\overrightarrow{Q}^{G}$ are isomorphic in $V[G]$.

Proof. With the help of a maximal antichain in $\overrightarrow{P}_{<\lambda}$, we find a set of ordinals $A$ with the property that, whenever $G$ is $\overrightarrow{P}_{<\lambda}$-generic over $V$, then there is a $\lambda \in A$ and a $(\Phi, \kappa)$-iteration $\langle (\overrightarrow{Q}_{<\alpha} \mid \alpha \leq \lambda), (\overrightarrow{Q}_{\alpha} \mid \alpha < \bar{\lambda}) \rangle$ in $V[G]$ such that the partial orders $\overrightarrow{Q}^{G}$ and $\overrightarrow{Q}_{\bar{\lambda}}$ are isomorphic in $V[G]$. Set $\eta = \sup(A)$. Since $\Phi$ is suitable, we can find sequences $\langle \bar{I}_{\alpha} \mid \alpha \leq \eta \rangle$ and $\langle \bar{P}_{\alpha} \mid \alpha < \eta \rangle$ of $\overrightarrow{P}_{<\alpha}$-names with the property that, whenever $G$ is $\overrightarrow{P}_{<\lambda}$-generic over $V$, then $\bar{I}_{G} = \langle \langle \bar{I}^{G}_{\alpha} \mid \alpha \leq \eta \rangle, \langle \bar{P}^{G}_{\alpha} \mid \alpha < \eta \rangle \rangle$ is a $(\Phi, \kappa)$-iteration in $V[G]$ and the partial orders $\overrightarrow{Q}^{G}$ and $\overrightarrow{I}^{G}_{\eta}$ are isomorphic in $V[G]$. In the following, we will derive the statement of the lemma by constructing a $(\Phi, \kappa)$-iteration $\langle \langle \bar{I}^{G}_{\alpha} \mid \alpha \leq \lambda + \eta \rangle, \langle \bar{P}^{G}_{\alpha} \mid \alpha < \lambda + \eta \rangle \rangle$ that extends $\overrightarrow{P}$ and has the property that, whenever $G$ is $\overrightarrow{P}_{<\lambda}$-generic over $V$ and $\alpha \leq \eta$, then $\langle \langle \bar{I}^{G}_{\beta} \mid \beta \leq \alpha \rangle, \langle \bar{P}^{G}_{\beta} \mid \beta < \alpha \rangle \rangle$ is the canonical forcing iteration in $V[G]$ such that there is a sequence $\langle i_{\beta} : \overrightarrow{P}^{G}_{(\lambda, \lambda+\alpha)} \rightarrow \overrightarrow{I}^{G}_{\beta} \mid \beta \leq \alpha \rangle$ of isomorphisms satisfying the properties (i) and (ii) listed in the proof of Lemma 6.3.

First, pick $\alpha < \eta$ and assume that we have constructed such an extension of $\overrightarrow{P}$ of length $\eta + \alpha$. Then there is an extension $\langle (\overrightarrow{P}_{<\beta} \mid \beta \leq \lambda + \alpha + 1), (\overrightarrow{P}_{\beta} \mid \beta \leq \lambda + \alpha) \rangle$ of this iteration with the property that, whenever $F$ is $\overrightarrow{P}_{<\lambda+\alpha}$-generic over $V$, $G \ast H$ is the filter on $\overrightarrow{P}_{<\lambda+\alpha}$ induced by $F$ and $\langle i_{\beta} : \overrightarrow{P}^{F}_{(\lambda, \lambda+\alpha)} \rightarrow \overrightarrow{I}^{F}_{\beta} \mid \beta \leq \alpha \rangle$ is the sequence of isomorphisms with the above properties, then $\overrightarrow{P}^{F}_{\lambda+\alpha} = (\bar{P}^{G}_{\alpha})_{\bar{\alpha}[H]}$. This
definition ensures that we find an isomorphism \( i_{\alpha+1} : \tilde{P}_{\langle \lambda, \lambda+\alpha+1 \rangle} \rightarrow \tilde{I}_{\alpha+1} \) with the desired properties whenever \( G \) is \( \tilde{P}_{\lambda} \)-generic over \( V \).

Next, assume that \( \alpha \leq \eta \) is a limit ordinal and we constructed a \((\Phi, \kappa)\)-iteration \( \langle <\tilde{P}_{\beta} | \beta \leq \lambda + \alpha \rangle, \langle \tilde{P}_{\beta} | \beta < \lambda + \alpha \rangle \rangle \) such that every initial segment satisfies the above properties. Let \( G \) be \( \tilde{P}_{\lambda} \)-generic over \( V \). Then our assumptions imply that there is a sequence \( \langle i_\beta : \tilde{P}_{\langle \lambda, \lambda+\beta \rangle} \rightarrow \tilde{I}_{\beta} | \beta < \lambda + \alpha \rangle \) of isomorphisms in \( V[G] \) that satisfy the above properties. Let \( i_\alpha : \tilde{P}_{\langle \lambda, \lambda+\alpha \rangle} \rightarrow \tilde{I}_{\alpha} \) denote the function defined by \( i_\alpha(\bar{p}) = \bigcup \{ i_\beta(\bar{p} \upharpoonright [\lambda, \lambda + \beta]) | \beta < \alpha \} \). Since Proposition 4.4 shows that we can use [2, Theorem 5.4] to show that the canonical iteration corresponding to \( \tilde{P}_{\langle \lambda, \lambda+\alpha \rangle} \) also has \( \kappa \)-support in \( V[G] \), we can conclude that \( i_\alpha \) is also an isomorphism. \( \square \)

**Proof of Clause (ii) of Theorem 4.5.** Assume that \( \kappa = \kappa \), \( \theta > \kappa \) is an inaccessible cardinal and \( \langle \rangle \) is a wellordering of \( H(\theta) \) of order-type \( \theta \).

Let \( \Phi(v_0, v_1) \) be the canonical suitable formula defining a class of partial orders such that \( \Phi(\cdot, \kappa) \) defines the class of all partial orders that are equivalent to a \((\Phi_{\langle \lambda, \kappa \rangle}, \kappa)\)-iteration if \( \kappa \) is an infinite regular cardinal and \( \Phi(\cdot, z) \) defines the class of all partial orders if \( z \) is not an infinite regular cardinal.

Following [3, Section 1.9], let \( \text{Fml} \subseteq \langle V_\omega \rangle \) denote the set of all formalized \( \mathcal{L}_\varepsilon \)-formulas, let \( \# : \text{Fml} \rightarrow \omega \) denote the corresponding arity function and let \( \text{Sat} \) denote the corresponding formalized satisfaction relation. Given an \( \mathcal{L}_\varepsilon \)-formula \( \psi \), let \( \text{Sat} \langle \langle v_0, \ldots, v_{n+1} \rangle \rangle \) denote the canonical code for this formula. Let \( f : \text{Fml} \rightarrow \text{Fml} \) be the canonical recursive function with the property that \( \#(f(a)) = \#(a) + 2 \) for all \( a \in \text{Fml} \) and \( f(\psi) = \rho \psi \) whenever \( \psi(v_0, \ldots, v_{n+1}) \) is an \( \mathcal{L}_\varepsilon \)-formula and \( \psi(v_0, \ldots, v_{n+1}) \) is the induced formula stating that

\[ \text{"witnesses that } \psi(v_0, \ldots, v_{n+1}) \text{ is } (\Phi, \Phi_{\text{dc}}, v_n)\text{-forceably necessary".} \]

In the following, we inductively construct a continuous, strictly increasing sequence \( \langle c_\gamma \rangle < \theta \rangle | \gamma < \theta \rangle \), a \((\Phi_{\langle \lambda, \kappa \rangle}, \kappa)\)-iteration \( \tilde{P}^\varphi = \langle <\tilde{P}_\delta | \delta \leq \theta \rangle, \langle \tilde{P}_\delta | \delta < \theta \rangle \rangle \) and a sequence \( \langle \tilde{b}_\delta | \delta < \theta \rangle \) such that \( \tilde{P}^\varphi = \langle <\tilde{P}_\delta | \delta \leq c_\gamma \rangle, \langle \tilde{P}_\delta | \delta < c_\gamma \rangle \rangle \) is an element of \( H(\theta) \) for every \( \gamma < \theta \) and \( \tilde{b}_\delta \) is a \( \tilde{P}_{\langle \delta, \kappa \rangle} \)-name for a bijection between \( \theta \) and \( H(\theta) \) for every \( \delta < \theta \).

Assume that \( \gamma = <\delta, \varepsilon > < \theta \) and we already constructed \( \langle c_\delta | \delta \leq \gamma \rangle, \tilde{P}^\varphi \) and \( \langle \tilde{b}_\delta | \delta < c_\gamma \rangle \) with the above properties. Then there is a \( \tilde{P}_{\langle \delta, \kappa \rangle} \)-name \( \tilde{Q} \) with the property that the following statements hold whenever \( G \) is \( \tilde{P}_{\langle \delta, \varepsilon \rangle} \)-generic over \( V \) and \( \bar{G} \) is the filter on \( \tilde{P}_{\delta} \) induced by \( G \):

(a) If \( \tilde{b}_\delta^G(\varepsilon) = (a, x_0, \ldots, x_{n-1}) \) with \( a \in \text{Fml} \) and \( \#a = n \) and there is a partial order \( \tilde{Q} \in H(\theta)^{V[G]} \) such that

\[ \text{Sat}(H(\theta)^{V[G]}, f(a), (x_0, \ldots, x_{n-1}, \kappa, \tilde{Q})) \]

holds in \( V[G] \), then \( \tilde{Q}^G \in H(\theta)^{V[G]} \) and

\[ \text{Sat}(H(\theta)^{V[G]}, f(a), (x_0, \ldots, x_{n-1}, \kappa, \tilde{Q}^G)) \]

holds in \( V[G] \).

(b) If the above assumptions are not satisfied, then \( \tilde{Q}^G \) is the trivial partial order in \( H(\theta)^{V[G]} \).
Since $\vec{P}_{c_\gamma}$ is contained in $H(\theta)$, there is a $\vec{P}_{c_\gamma}$-name with these properties in $H(\theta)$ and we let $\bar{Q}$ denote the $c$-least such name. Then the statement

\[ \bar{Q} \text{ is equivalent to a } (\Phi_{Sh}, \bar{\kappa})\text{-iteration} \]

holds in $H(\theta)$ and an application of Lemma 6.3 in $H(\theta)$ yields $c_\gamma < c_{\gamma+1} < \theta$ and a $(\Phi_{Sh}, \kappa)$-iteration \(<\vec{P}_{c_\gamma}^\gamma, \vec{P}_\delta|\delta < c_{\gamma+1}\>\) extending $\vec{P}_\gamma$ with the property that, whenever $G$ is $\vec{P}_{c_\gamma}$-generic over $V$, then the partial orders $\vec{P}_\gamma^G$ and $\bar{Q}^G$ are isomorphic in $H(\theta)^{V[G]}$. Let $\vec{P}_\gamma^{\gamma+1}$ denote the $c$-least iteration with these properties in $H(\theta)$. In $V$, $\vec{P}_\gamma^{\gamma+1}$ is also a $(\Phi_{Sh}, \kappa)$-iteration that extends $\vec{P}_\gamma$. Given $c_\gamma \le \delta < c_{\gamma+1}$, let $\bar{b}_\delta$ be the canonical $\vec{P}_{c_\gamma}$-name with the property that, whenever $G$ is $\vec{P}_{c_\gamma}$-generic over $V$ and $\bar{Q}$ is the canonical wellordering of $H(\theta)^{V[G]}$ of order-type $\theta$ induced by $\triangleleft$ (i.e. given $x, y \in H(\theta)^{V[G]}$, we have $y \triangleleft G y$ if and only if there is a $\vec{P}_{c_\gamma}$-name $\bar{x} \in H(\theta)^V$ with $x = \bar{x}^G$ and $\bar{y} \triangleleft \bar{y}$ for all $\vec{P}_{c_\gamma}$-names $\bar{y} \in H(\theta)^V$ with $y = \bar{y}^G$), then $\bar{b}_\delta : \theta \rightarrow H(\theta)^{V[G]}$ is the monotone enumeration of the wellorder $(H(\theta)^{V[G]}, \bar{Q})$.

Since our assumptions on the support of the iteration determine the definition of $c_\gamma$ and $\vec{P}_\gamma$ for all $\gamma \in \text{Lim} \cap \theta$, this completes the definition of our forcing iteration.

Set $B(\kappa, <\kappa) = \vec{P}_{c_\gamma}$. Note that the above definition ensures that this partial order is uniformly definable in the parameters $\kappa$ and $<\kappa$. By Theorem 6.2, the partial order $B(\kappa, <\kappa)$ is $<\kappa$-closed and satisfies the $\kappa^+$-chain condition.

From now on, assume that $\theta$ is a $\Sigma_{n+2}$-reflecting cardinal for some $1 < n < \omega$ and let $G$ be $B(\kappa, <\kappa)$-generic over $V$. Given $\delta < \theta$, let $G_\delta$ denote the filter on $\vec{P}_\delta$ induced by $G$.

**Claim.** If $\varphi(v_0, \ldots, v_{k-1})$ is a $\Sigma_n$-formula and $x_0, \ldots, x_{k-1} \in H(\theta)^{V[G]}$ such that the statement $\varphi(x_0, \ldots, x_{k-1})$ is $(\Phi_{Sh}, \Phi_{dc}, \kappa)$-forceably necessary in $V[G]$, then $\varphi(x_0, \ldots, x_{k-1})$ holds in $V[G]$.

**Proof of the Claim.** Since $\vec{P}_\delta$ is a $<\kappa$-support iteration of forcings of size less than $\theta$, we can find $\delta < \theta$ with $x_0, \ldots, x_{k-1} \in H(\theta)^{V[G_\delta]}$. Then there is an ordinal $\gamma < \delta$ with $\vec{P}_\delta(\gamma) = \langle \vec{P}_{\gamma+1}^G, \vec{P}_\delta|\delta < \gamma+1\>$. Set $\gamma < \delta$, $\varphi^\gamma$. Since the formula $\Phi_{Sh}$ is suitable, there is a $(\Phi_{Sh}, \kappa)$-iteration $\langle \vec{P}_{<\gamma}^\gamma|\delta < \gamma+1\rangle, \vec{P}_\delta|\delta \le \theta\rangle$ in $V$ extending $\vec{P}_\delta$ such that $\vec{P}_\delta^G$ witnesses that the statement $\varphi(x_0, \ldots, x_{k-1})$ is $(\Phi_{Sh}, \Phi_{dc}, \kappa)$-forceably necessary in $V[G]$. In this situation, an application of Lemma 6.3 shows that $\vec{P}_{\gamma+1}^G$ witnesses that the statement $\varphi(x_0, \ldots, x_{k-1})$ is $(\Phi_{Sh}, \Phi_{dc}, \kappa)$-forceably necessary in $V[G]$. In combination with Theorem 6.2, these observations show that $\Phi(\vec{P}, \kappa)$ holds for a partial order $\vec{P}$ if and only if there is an ordinal $\alpha$ such that this
statement holds in \( H(\nu) \) for all regular cardinals \( \nu > \alpha \). In particular, the class of these forcings can be uniformly defined by a \( \Sigma_3 \)-formula with parameter \( \kappa \).

Together with Proposition 5.8, these remarks show that we can formulate the statement that \( \varphi(x_0, \ldots, x_{k-1}) \) is \( (\Phi, \Phi_{}\text{clc}, \kappa) \)-forceably necessary by a \( \Sigma_{n+2} \)-formula using parameters contained in \( H(\theta)^V \). Using the fact that \( \mathbb{P}^c_{\kappa} \) is contained in \( H(\theta)^V \), a routine application of Proposition 5.8 shows that \( \theta \) is still \( \Sigma_{n+2} \)-reflecting in \( V[G, c] \) and we can conclude that the \( \varphi(x_0, \ldots, x_{k-1}) \) is also \( (\Phi, \Phi_{\text{clc}, \kappa}) \)-forceably necessary in \( H(\theta)^{V[G, c]} \). This shows that there is a partial order \( Q \) in \( H(\theta)^{V[G, c]} \) such that

\[
\text{Sat}(H(\theta)^{V[G, c]}, f(\varphi^\gamma), (x_0, \ldots, x_{k-1}, \kappa, Q))
\]

holds in \( V[G, c] \). By our definition of \( \mathbb{P}^{c_{\kappa}+1} \), we know that

\[
\text{Sat}(H(\theta)^{V[G, c]}, f(\varphi^\gamma), (x_0, \ldots, x_{k-1}, \kappa, \mathbb{P}^{G, c_{\kappa}+1}_{\kappa, c_{\kappa}+1}))
\]

holds in \( V[G, c] \) and this means that \( \mathbb{P}^{G, c_{\kappa}+1}_{\kappa, c_{\kappa}+1} \) witnesses that \( \varphi(x_0, \ldots, x_{k-1}) \) is \( (\Phi, \Phi_{\text{clc}, \kappa}) \)-forceably necessary in \( H(\theta)^{V[G, c]} \). As above, our assumptions imply that this statement also holds in \( V[G,c] \). By Theorem 6.2 and Lemma 6.3, we know that \( \Phi_{\text{clc}}(\mathbb{P}^{G, c_{\kappa}+1}_{\kappa, c_{\kappa}+1}, \theta) \) holds in \( V[G,c_{\kappa}+1] \). This allows us to conclude that the statement \( \varphi(x_0, \ldots, x_{k-1}) \) holds in \( V[G] \). \( \square \)

Claim. \( \theta = (2^\kappa)^{V[G]} \).

Proof of the Claim. In \( V \), \( \theta \) is inaccessible and \( B(\kappa, \prec) \) is a partial order of cardinality \( \theta \) that satisfies the \( \kappa^+ \)-chain condition. This implies that there are at most \( \theta \)-many \( B(\kappa, \prec) \)-nice names for subset of \( \kappa \) in \( V \) and hence \( (2^\kappa)^{V[G]} \leq \theta \).

Given a cardinal \( \mu < \theta \), the partial order \( \text{Add}(\kappa, \mu^+) \) witnesses that the statement that there is no surjection from \( \mu \) onto \( P(\kappa) \) is \( (\Phi_{\text{Sh}}, \Phi_{\text{clc}, \kappa}) \)-forceably necessary in \( V[G] \). Since this statement can be formulated by a \( \Sigma_2 \)-formula with parameters in \( H(\theta)^{V[G]} \), the above claim shows that it holds in \( V[G] \) and hence \( \mu < (2^\kappa)^{V[G]} \).

This shows that we also have \( (2^\kappa)^{V[G]} \geq \theta \). \( \square \)

The combination of the above claims shows that \( \text{SMP}_{\kappa}(\kappa) \) holds in \( V[G] \). \( \square \)

7. Concluding remarks and open questions

We close this paper with some open questions motivated by the above results and a discussion of possible directions of further research.

The above results show that closed maximality principles exactly determine the possible lengths of \( \Sigma^1_1 \)-definable wellorders. Since definable prewellorders play a more important role in classical descriptive set theory, one naturally arrives at the following question.

**Question 7.1.** Do axioms of the form \( \text{CMP}_{\kappa}(\kappa) \) or \( \text{SMP}_{\kappa}(\kappa) \) determine the least upper bound of the lengths of \( \Delta^1_1 \)-definable prewellorders on \( ^\kappa \kappa \)?

The results of this paper show that the principles \( \text{CMP}_2(\kappa) \) and \( \text{SMP}_2(\kappa) \) induce a strong structure theory for the class of \( \Sigma^1_1 \)-subsets of \( ^\kappa \kappa \). Therefore it is natural to ask whether similar implications hold for stronger maximality principles and larger classes of definable subsets of \( ^\kappa \kappa \).

**Question 7.2.** Assume that \( \langle V, \in, \kappa, \kappa^+ \rangle \) is a model of CMP.
(i) Does every subset of \(^{\kappa}\kappa\) that is an element of \(L(\mathcal{P}(\kappa))\) have the perfect set property?
(ii) Does the axiom of choice fail in \(L(\mathcal{P}(\kappa))\)?

**Question 7.3.** Assume that \((V, \in, \kappa, 2^{\kappa})\) is a model of SMP.

(i) Does every subset of \(^{\kappa}\kappa\) that is an element of \(L(\mathcal{P}(\kappa))\) have the Bernstein property?
(ii) Does the axiom of choice fail in \(L(\mathcal{P}(\kappa))\)?

Note that in both cases, an affirmative answer to the first part of the question yields an affirmative answer to the second part of the question, because a wellordering of \(^{\kappa}\kappa\) allows us to construct a subset of \(^{\kappa}\kappa\) without the Bernstein property.

Since the principles CMP\(_2(\kappa)\) and SMP\(_2(\kappa)\) settle many questions about the class of \(\Sigma^1_2\)-subsets of the generalized Baire space of \(\kappa\), it is natural to ask whether these closed maximality principles can hold for all uncountable regular cardinals \(\kappa\).

In [8, Section 6], Fuchs shows that the full maximality principles for the class of all closed partial orders cannot hold globally. A closer look at the proof of [8, Theorem 6.9] shows that the class of uncountable regular cardinals \(\kappa\) with the property that CMP\(_3(\kappa)\) holds is bounded in the ordinals. A similar argument shows that the same is true for the class of all uncountable regular cardinals \(\kappa\) such that SMP\(_3(\kappa)\) holds. In a subsequent paper [23], we will prove that this is also true for the class of cardinals such that SMP\(_2(\kappa)\) holds. This leaves open the following question.

**Question 7.4.** Is it consistent that CMP\(_2(\kappa)\) holds for all uncountable regular cardinals \(\kappa\)?)

In [8, Section 5], Fuchs introduces a weakening of CMP that restricts this principle to local statements that are provably forceably necessary and shows that it is consistent that this principle holds for every uncountable regular cardinal. Moreover, he argues that many interesting consequences of CMP derived in [8] are also consequences of this weaker principles. In [23], we will show that the consequences listed in Theorem 3.8 can also be derived from this restricted principle. Therefore we may view it as a natural example of an extension of ZFC that can answers the questions posed in Section 2 in a global way.

In [23], we will also formulate a similar restriction of the principle SMP to local statements that are provably forceably and construct a model in which this principle holds for all uncountable regular cardinal \(\kappa\) satisfying \(\kappa = \kappa^{<\kappa}\) and unboundedly many cardinals with this property exist. Finally, we will also show that the statements listed in Theorem 4.7 are also consequences of this weaker principle.

**References**


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