TAMENESS BEYOND O-MINIMALITY

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Abstract. While o-minimality covers much of what is now considered tame geometry, there is growing interest and success in studying well-behaved structures outside this important yet limited framework. In this survey, we provide an overview of the current state of research on tameness in expansions of the real field.

1. Introduction

"A lot of model theory is concerned with discovering and charting the 'tame' regions of mathematics, where wild phenomena like space filling curves and Gödel incompleteness are absent, or at least under control. As Hrushovski put it recently:

Model Theory = Geography of Tame Mathematics."

Lou van den Dries [18]

"What might it mean for a first-order expansion of the field of real numbers to be tame or well behaved? In recent years, much attention has been paid by model theorists and real-analytic geometers to the o-minimal setting[...]. But there are expansions of the real field that define sets with infinitely many connected components, yet are tame in some well-defined sense [...]. The analysis of such structures often requires a mixture of model-theoretic, analytic-geometric and descriptive set-theoretic techniques. An underlying idea is that first-order definability, in combination with the field structure, can be used as a tool for determining how complicated is a given set of real numbers."

Chris Miller [55]

At the core of logic and model theory is the observation that within mathematics, some objects (for model theorists, structures and their theories) must be considered tame, while others are considered wild. Many foundational results in logic from the first half of the 20th century focused on the existence of wild objects. For example, Gödel's proof of the undecidability of the theory of $(\mathbb{N}, +, \cdot)$ showed that this structure is wild from a logical perspective. These results are negative in nature, highlighting the limitations of mathematical reasoning. However, in the second half of the 20th century, the focus shifted. Model theorists discovered a vast number of mathematical structures that do not exhibit wildness and can be viewed

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as tame for various reasons. General frameworks for handling such tame structures were developed, and their properties have been studied extensively. The program of identifying and analyzing tame classes of structures whose model theory can be understood, came to be known as the geography of tame mathematics, and it has dominated model theory for the past fifty years.

In this survey, we pursue this program for expansions of the real field. While it is similar in spirit to Shelah's classification theory, the goal here is to classify structures over a fixed universe based on the geometric properties of their definable sets, rather than classifying theories by their combinatorial properties or the number of models. While classification theory is motivated by Morley's theorem, our tameness program originates from the remarkable success of o-minimality.

We now set the stage. Let $\overline{\mathbb{R}}$ denote the real field $(\mathbb{R}, +, \cdot)$. Consider a collection $\mathcal{X}$ of subsets of various $\mathbb{R}^n$. We are studying the expansion of $\overline{\mathbb{R}}$ by predicates for each $X \in \mathcal{X}$, that is, $(\overline{\mathbb{R}}, (X)_{X\in\mathcal{X}})$.

First, observe that the set of expansions of $\overline{\mathbb{R}}$ can be partially ordered. We say that $\mathcal{R}_1$ is a **reduct** of $\mathcal{R}_2$, written $\mathcal{R}_1 \leq \mathcal{R}_2$, if every set definable in $\mathcal{R}_1$ is also definable in $\mathcal{R}_2$. In this case, we also say that $\mathcal{R}_2$ is an **expansion** of $\mathcal{R}_1$. We say that $\mathcal{R}_1$ and $\mathcal{R}_2$ are **interdefinable** if $\mathcal{R}_1 \leq \mathcal{R}_2$ and $\mathcal{R}_2 \leq \mathcal{R}_1$. This partial order clearly has both a maximum (obtained by taking $\mathcal{X}$ to be $\bigcup_{n\in\mathbb{N}} \mathcal{P}(\mathbb{R}^n)$) and a minimum (obtained by taking $\mathcal{X} = \emptyset$).

With this notation in place, our overall research program can be formulated as follows.

Goal 1.1. *Classify expansions of $\overline{\mathbb{R}}$ up to interdefinability.*

As Miller put it in [41], our Goal 1.1 is "*too vague at best and intractable at worst.*" Instead of classifying expansions up to interdefinability, we aim to categorize them into classes based on the geometric tameness of their definable sets. We will replace interdefinability with a coarser and less precisely defined equivalence relation that captures this type of tameness. This distinction will become clearer later: our focus is on the geometric tameness of the definable sets, rather than on the model-theoretic tameness of the structure and its theory. Therefore, our approach is more definability-theoretic than model-theoretic.

We begin by reviewing some tameness notions and examples of expansions that satisfy them. Fix an expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$. When we say a set is *definable*, we mean definable in $\mathcal{R}$, possibly with parameters. The core tameness notion that gave rise to the systematic study of tameness in expansions of $\overline{\mathbb{R}}$ is o-minimality. We say $\mathcal{R}$ is **o-minimal** if every definable subset of $\mathbb{R}$ either has interior or is finite. O-minimality was isolated by van den Dries [14] in order to prove important results from semi-algebraic geometry in this generality, and developed by Pillay and Steinhorn [65] as a tameness notion in the setting of dense linear orders. Examples of o-minimal expansions of $\overline{\mathbb{R}}$ include $\overline{\mathbb{R}}$ itself, $(\overline{\mathbb{R}}, \exp)$, where exp is the graph of exponentiation, $\mathbb{R}_{an}$, the real field adjoined with all restricted analytic functions, and $(\mathbb{R}_{an}, \exp)$, the expansion that includes both all restricted analytic functions and the graph of exponentiation. While these structures are not interdefinable and there are model-theoretic differences between them, they are all o-minimal in terms of geometric tameness. This is the similarity in which we are interested.

Among the many results that transfer from semi-algebraic geometry to the setting of o-minimal structures are the monotonicity theorem for definable functions and the cell decomposition theorem. It is not the goal of this paper to discuss o-minimal structures or theorems about them; rather, we focus on surveying the larger framework of geometric tameness on $\mathbb{R}$. For readers interested in o-minimality, we refer to the excellent book by van den Dries [17].

It is worth noting that o-minimality implies model-theoretic tameness, a statement that we will see repeatedly fail for other tameness notions considered here. First, by the cell decomposition theorem, if a structure is o-minimal, then every elementarily equivalent structure is also o-minimal. Furthermore, an o-minimal structure is NIP, dp-minimal, and distal. In the model-theoretic universe, o-minimality plays a role in the ordered setting similar to that of strong minimality in the stable setting. Since the combinatorial tameness notions from model theory are only of secondary importance here, we refer the reader to Simon [69] for precise definitions.

Most of the tameness conditions we want to consider for $\mathcal{R}$ are of the following form:

every definable subset of $\mathbb{R}$ has interior or is *small*.

In the case of o-minimality, *small* means finite. A subset of $\mathbb{R}$ is finite if and only if it is closed, bounded, and discrete. Thus, the most obvious first step towards weaker notions of smallness is to drop one of the three conditions, say boundedness. We will later see that an expansion $\mathcal{R}$ defines an infinite discrete subset of $\mathbb{R}$ if and only if it defines an unbounded closed and discrete set.

We say that $\mathcal{R}$ is **weakly d-minimal** if every definable subset of $\mathbb{R}$ either has interior or is a finite union of discrete sets. We say that $\mathcal{R}$ is **d-minimal** if for every $n \in \mathbb{N}$ and every definable subset $A \subseteq \mathbb{R}^{n+1}$ there is $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^n$ either A_x has interior or is the union of N discrete sets.

In the case of weak d-minimality, being *small* means being the finite union of discrete sets. If *small* is defined as being discrete, then $\mathcal{R}$ is actually o-minimal. Indeed, it is an easy exercise to see that if $\mathcal{R}$ defines an infinite discrete subset of $\mathbb{R}$, then it defines a set that is the union of N discrete set, but which is not the union of $N-1$ discrete sets. Thus if every definable subset of $\mathbb{R}$ without interior is discrete, then every subset of $\mathbb{R}$ without interior has to be finite. Hence $\mathcal{R}$ is o-minimal in such a situation.

It is still an open question whether weak d-minimality implies d-minimalilty. When Miller introduced d-minimality in [55], he did not define weak d-minimality. Perhaps he expected the uniformity condition in the definition of d-minimality to be neccesary. It would be more consistent with other model-theoretic notation to use *strongly d-minimal* instead of *d-minimal*, and to use *d-minimal* instead *weakly d-minimal*. However, we stick here with Miller's original definition of d-minimality.

Let $a \in \mathbb{R}_{>0}$, and set $a^{\mathbb{Z}} := \{a^n : n \in \mathbb{Z}\}$. As pointed out in [55], it follows from van den Dries [15, Theorem III] that $(\overline{\mathbb{R}}, a^{\mathbb{Z}})$ is d-minimal. Indeed, more is true: if $\mathcal{R}$ is an o-minimal expansion of $\overline{\mathbb{R}}$ with field of exponents $\mathbb{Q}$, then $(\mathcal{R}, a^{\mathbb{Z}})$ is d-minimal by [55, Theorem 3.4.2]. So, in particular, the expansion $(\mathbb{R}_{an}, a^{\mathbb{Z}})$ is d-minimal. Let $\omega \in \mathbb{R}_{>0}$, and set $a := e^{2\pi/\omega}$. Miller and Speisseger observed (see [55, Corollary after Theorem 3.4.2]) that $(\mathbb{R}_{an}, a^{\mathbb{Z}})$ defines the logarithmic spiral

$$S_\omega := \{(e^t \cos(\omega t), e^t \sin(\omega t) \ : \ t \in \mathbb{R}\}.$$

Since d-minimality is preserved under reducts, we conclude that $(\overline{\mathbb{R}}, S_\omega)$ is d-minimal.

The logarithmic spiral S_ω is an example of a locally closed trajectory for a linear vector field. Miller [57] classifies all expansions of $\overline{\mathbb{R}}$ by collections of such trajectories, showing that S_ω is the only trajectory of this kind that, over $\overline{\mathbb{R}}$, generates a tame but not o-minimal expansion. For more examples of d-minimal expansions of $\overline{\mathbb{R}}$, see Friedman and Miller [34] and Miller and Tyne [60]. A countable cell decomposition theorem is known for d-minimal structures (see Miller [55] and Thamrongthanyalak [72, Theorem B]), but otherwise, the consequences of d-minimality have not been studied as extensively as those of o-minimality. Some examples include a d-minimal Whitney theorem by Miller and Thamrongthanyalak [59], a d-minimal Michael's selection theorem by Thamrongthanyalak [72], and a countable Gromov-Yomdin parametrization theorem by Jäger [47].

In contrast to o-minimality, d-minimality only implies geometric tameness, but not model-theoretic tameness. Let $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})^{\#}$ be the expansion of $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$ by every subset of every cartesian power of $2^{\mathbb{Z}}$. By Friedman and Miller [33], the expansion $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})^{\#}$ is d-minimal, yet defines an isomorphic copy of $(\mathbb{Z}, +, \cdot)$. Thus, its theory is undecidable, and it fails Shelah-style combinatorial tameness conditions like NIP and NTP2.

Of course, we can further expand the class of sets we want to consider as *small*. We say that $\mathcal{R}$ is **noiseless** if every definable subset of $\mathbb{R}$ either has interior or is nowhere dense. As a tameness condition, noiselessness is studied in [55], although the name *noiseless* was only suggested later by Chris Miller. The rationale behind the name is that the condition is equivalent to the statement that $\mathcal{R}$ does not define a set $X \subseteq \mathbb{R}$ such that $X \cap I$ is dense and co-dense in an open interval I. More in line with the aforementioned conditions, Fornasiero [28] used the term *i-minimal* (for interior-minimal) instead.

It is not easy to see that there are expansions of $\overline{\mathbb{R}}$ that are noiseless, but not d-minimal. A good example of a set without interior that is not a finite union of discrete sets is a **Cantor set**; that is, a compact, nonempty subset of $\mathbb{R}$ that has neither isolated nor interior points. The classical middle-thirds Cantor set is one example of such a set. Indeed, Friedman et al. [32] produce a Cantor set $K \subseteq [0, 1]$ such that every definable subset of $\mathbb{R}$ either has interior or is Hausdorff null[1]. So, in particular, $(\overline{\mathbb{R}}, K)$ is noiseless. A priori, this could be a stronger condition than noiselessness. However, we will later see that these two tameness conditions are

[1] In particular, K itself is Hausdorff null.

equivalent.

So, what about wildness? Consider $(\overline{\mathbb{R}}, \mathbb{Z})$, the expansion of the real field by a predicate for the set of integers. By Gödel's incompleteness results, its theory is undecidable. But much more is true: the structure $(\overline{\mathbb{R}}, \mathbb{Z})$ defines every Borel subset of $\mathbb{R}^n$ (see [49, (37.6)]), and hence also every projective subset of $\mathbb{R}^n$ in the sense of descriptive set theory (see [49, Chapter V]). In particular, all continuous functions are definable in this expansion. Thus, geometrically wild phenomena like space-filling curves and nowhere differentiable continuous functions appear in this structure. Even set-theoretic independence can arise from seemingly innocent questions, such as whether every set definable in $(\overline{\mathbb{R}}, \mathbb{Z})$ is Lebesgue measurable (Solovay [70]).

Because of such complications, the nondefinability of $\mathbb{Z}$ must be regarded as necessary for an expansion of $\overline{\mathbb{R}}$ to be tame.[2] Miller championed a research program to determine whether this nondefinability is also sufficient to enforce some form of well-defined tameness in such expansions.

Goal 1.2 (Miller's program)**.** *What kind of geometric tameness in expansions of $\overline{\mathbb{R}}$ can be deduced from the non-definability of $\mathbb{Z}$?*

This is an ambitious program. At first glance, the nondefinability of $\mathbb{Z}$ only guarantees that there is *some* Borel set that is not definable. Yet, it is far from obvious that any geometric pathologies can be ruled out. This research program arguably began in earnest with Miller's paper [56], but truly took off after [37].

Theorem 1.3 ([37, Theorem 1.1])**.** *Suppose $\mathcal{R}$ that does not define $\mathbb{Z}$. Let $D \subseteq \mathbb{R}$ be closed, discrete and definable. Then $\mathcal{R}$ does not define a function $f : D^n \to \mathbb{R}$ such that $f(D^n)$ is somewhere dense.*

By [38, Theorem A], the assumption that D is closed can be dropped. If D is closed and discrete, and $f(D)$ is somewhere dense, then the linear orders $(D, <)$ and $(f(D), <)$ are not order-isomorphic. One way of thinking about Theorem 1.3 is as a limitation on how a definable function can change the order type of a definable linear order. This restricts our ability to define dense and co-dense sets from closed definable sets and enforces rudimentary tameness on definable sets in such structures. We will make this observation explicit as a strong version of the Baire category theorem in Section 2 and deduce various geometric consequences from it. These results provide evidence for the viability of the following program:

Goal 1.4 (Tameness program)**.** *Study tameness in expansions of $\overline{\mathbb{R}}$ that do not define $\mathbb{Z}$, and classify these up to some common notion of tameness.*

Figure 1 gives a graphical representation of the tameness classes we have discussed so far.

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[2] Already in [14] van den Dries discusses o-minimality and the consequences of the nondefinability of $\mathbb{Z}$:

> "Postulating the finite type condition [i.e. o-minimality] is of course a rather drastic way of avoiding the Gödel phenomena that would appear if $\mathbb{N}$ were definable."

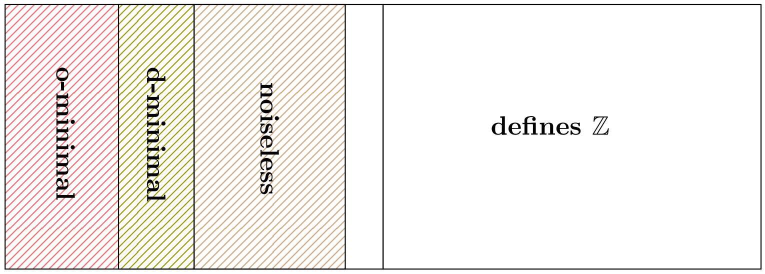

FIGURE 1. Tameness map for expansions of $\overline{\mathbb{R}}$

It is important to stress again that only o-minimality can be considered a model-theoretic tameness notion. We already discussed this for d-minimality, but it is even more poignant for noiselessness. Indeed, the aforementioned noiseless expansion by a Cantor set, constructed in [32], does not define $\mathbb{Z}$, yet it defines a Borel isomorph of the structure $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$. However, it follows from noiselessness that such complications must live on nowhere dense sets, and thus have limited impact on the geometry of all definable sets. For more well-behaved noiseless expansions, see [40].

The distinction between model-theoretic and geometric tameness should not be considered problematic. Rather, it highlights the fact that our investigation has a different goal. This is reminiscent of the distinction between model-theoretic and computational tameness addressed in the following quote:

> "What about decidability of the theory? Just as biological taxonomy does not tell us whether a species is tasty, the classification here does not deal with decidability."

Saharon Shelah [67]

We focus here on expansions of $\overline{\mathbb{R}}$, but similar questions arise and yield analogous answers for expansions of $(\mathbb{R}, <, +)$. Some of these are discussed in Section 3. It is also worth noting that a much older program, predating o-minimality by several decades, has studied expansions of $(\mathbb{N}, +)$. See Bès [4] for an excellent survey. This program on expansions of Presburger arithmetic is closely connected to the work presented here.

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Notation. Let $X \subseteq \mathbb{R}^{m+n}$. We denote by $\overline{X}$ the topological closure of X, by $\mathring{X}$ the interior of X, by $\operatorname{bd}(X)$ the boundary $\overline{X} \setminus \mathring{X}$ of X, and by $\operatorname{fr}(X)$ the frontier

$\overline{X} \setminus X$ of X. For $x \in \mathbb{R}^m$, we use X_x to denote the set $\{y \in \mathbb{R}^n \ : \ (x,y) \in X\}$. A **box** is a subset of $\mathbb{R}^n$ given as a product of n nonempty open intervals.

We always use i, j, k, m, n, N for natural numbers and $r, s, t, \varepsilon, \delta$ for real numbers. We let $|x| := \max\{|x_1|, \ldots, |x_n|\}$ be the l_∞ norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$, set

$$B_r(x) := \{y \in \mathbb{R}^n \ : \ |x - y| < r\}.$$

For $x \in \mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$, set

$$\operatorname{dist}(x, X) := \inf\{|x - y| \ : \ y \in X\}.$$

2. Miller's conjecture

The notion of noiselessness suggests that we aim to exclude the definability of dense and co-dense sets in tame structures. However, this is not the case. Instead, our goal is to control noise when it arises. In this section, we outline the main conjecture in this area, proposed by Chris Miller, on how to achieve this objective. Before stating the conjecture, it is helpful to introduce the concept of an open core.

Definition 2.1 (Miller-Speisseger [58])**.** Let $\mathcal{R}$ be an expansion of $(\mathbb{R}, <)$, we define the **open core** of $\mathcal{R}$, denoted $\mathcal{R}^\circ$, as the reduct of $\mathcal{R}$ generated by all open sets of all arities.

Since the complement of a closed set is an open set, every closed subset of $\mathbb{R}^n$ definable in $\mathcal{R}$ is also definable in $\mathcal{R}^\circ$. In particular, the topological closure of *every* subset of $\mathbb{R}^n$ definable in $\mathcal{R}$ is definable in $\mathcal{R}^\circ$. We say that $\mathcal{R}$ has an **o-minimal open core** (or d-minimal open core, noiseless open core) if $\mathcal{R}^\circ$ is o-minimal (d-minimal, noiseless).

We give a few natural examples of structures with o-minimal open cores that are not o-minimal. Let $\mathcal{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$, and let $\mathcal{S}$ be an elementary substructure of $\mathcal{R}$ whose universe is a proper dense subset of $\mathbb{R}$. The pair $(\mathcal{R}, \mathcal{S})$ is called a **dense pair**, and is studied by van den Dries [16]. By [16, Theorem 5], every open set definable in $(\mathcal{R}, \mathcal{S})$ is already definable in $\mathcal{R}$. Thus, the open core of $(\mathcal{R}, \mathcal{S})$ is $\mathcal{R}$, and hence o-minimal. Since the universe of $\mathcal{S}$ is dense and co-dense in $\mathbb{R}$, the structure $\mathcal{R}$ is not o-minimal itself. The prototypical example of such a pair is the pair of real closed fields $(\overline{\mathbb{R}}, \mathbb{Q}^{\mathrm{rc}})$, where $\mathbb{Q}^{\mathrm{rc}}$ is the subfield of real algebraic numbers. The study of this pair predates o-minimality and the tameness program, as Robinson [66], answering a question of Tarski, already showed that its theory is decidable.

For a related example, let Γ be a dense multiplicative subgroup of $\mathbb{R}^\times$ of finite rank. The structure $(\overline{\mathbb{R}}, \Gamma)$ is studied by van den Dries and Günaydın [19], and the results in that paper are used in Berenstein, Ealy, and Günaydın [3, Theorem 58] to show that its open core is o-minimal. Other examples include expansions by dense independent sets studied by Dolich, Miller, and Steinhorn [12], finite rank subgroups of the unit circle studied by Belegradek and Zilber [2], and rational points of elliptic curves studied by Günaydın and Hieronymi [35].

All the examples mentioned above work by the *Mordell-Lang principle*, which states that the structure induced on the predicate by the larger structures is simple. In the second example, this follows from the Mordell-Lang conjecture, whose usefulness in model theory was first observed by Pillay [63]. A precise framework for this observation has been developed by Block, Gorman, et al. [5]. For an in-depth study of the geometry of definable sets in these examples, see Eleftheriou et al. [22].

There are also natural examples of structures whose open core is d-minimal, but not o-minimal. Günaydın [36] studies the structure $(\overline{\mathbb{R}}, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$, providing a complete axiomatization of its theory. Boxall and Hieronymi [8, p. 116] use this work to show that the open core of this structure is $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$, and hence d-minimal. Currently, there is no documented example of an expansion that is not noiseless, yet whose open core is noiseless and not d-minimal.

While the open core plays a crucial role in studying geometric tameness, model-theoretic tameness of the open core $\mathcal{R}^\circ$ does not imply model-theoretic tameness of $\mathcal{R}$. Indeed, although all the above examples with o-minimal open core are NIP, we can generate arbitrarily complicated structures with o-minimal open cores.

Theorem 2.2 (H.-Nell-Walsberg [42])**.** *Let $\mathcal{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$. Then there is an expansion $\mathcal{S}$ of $\mathcal{R}$ that interprets $(\overline{\mathbb{R}}, \mathbb{Z})$, yet $\mathcal{S}^\circ = \mathcal{R}$.*

Thus having o-minimal open core is *not* a model-theoretic tameness condition. Now that we have introduced the notion of an open core and given examples of structures with tame open core, we are ready to state Miller's conjecture.

Conjecture 2.3 (Miller's conjecture)**.** *Let $\mathcal{R}$ be an expansion of $\overline{\mathbb{R}}$ that does not define $\mathbb{Z}$. Then $\mathcal{R}$ has noiseless open core.*

Loosely speaking, this conjecture states that:

If you throw away all the noise in a minimally tame[3] expansion of $\overline{\mathbb{R}}$, then the expansion generated by the remaining sets[4] does not contain any noise[5].

There is another way of looking at it. Miller's conjecture implies that for every set definable in an expansion of $\overline{\mathbb{R}}$ that does not define $\mathbb{Z}$, its topological closure is definable in a noiseless expansion. Accordingly, every set $X \subseteq \mathbb{R}^n$ such that $(\overline{\mathbb{R}}, X)$ does not define $\mathbb{Z}$, is dense in a set definable in a noiseless expansion. Thus, "from far away", such a set X looks like a set in a noiseless expansion.

If $\mathcal{R}$ is an expansion of $\overline{\mathbb{R}}$ by open and closed sets, then it is easy to see that the open core $\mathcal{R}^\circ$ is just $\mathcal{R}$. This stays true for expansions by boolean combinations of such sets. Indeed, by Dougherty and Miller [13, Corollary 1], definable boolean combinations of open sets are boolean combinations of definable open sets. Let us call boolean combinations of open sets **constructible**. Thus, if $\mathcal{R}$ is an expansion by constructible sets, then $\mathcal{R}^\circ = \mathcal{R}$. Therefore,

Studying open cores is the same as studying expansions by constructible sets.

We obtain immediately the following equivalent restatement of Miller's conjecture.

Conjecture 2.4 (Restated Miller's conjecture)**.** *Let $\mathcal{R}$ be an expansion of $\overline{\mathbb{R}}$ by constructible sets that does not define $\mathbb{Z}$. Then $\mathcal{R}$ is noiseless.*

[3] That is, you take the open (or closed) definable sets in an expansion of $\overline{\mathbb{R}}$ that doesn't define $\mathbb{Z}$.

[4] That is the open core.

[5] That is, it doesn't define any dense and codense set.

Nondefinability of $\mathbb{Z}$ is necessary: observe that $(\overline{\mathbb{R}}, \mathbb{Z})$ defines $\mathbb{Q}$, and hence is not noiseless. The truth of Miller's conjecture would give us an excellent understanding of the expansions by constructible sets, enabling us to arguably achieve the goal of classifying such structures up to common geometric tameness. Figure 2 visualizes this.

There is already non-trivial evidence for the truth of Miller's conjecture and there exist potential strategies for a proof. The first result in this direction is the following slightly restated version of a theorem of Miller and Speissegger.

Theorem 2.5 ([58, p. 194, Corollary (2)]). *Let $\mathcal{R}$ be an expansion of $\overline{\mathbb{R}}$ such that every closed definable subset of $\mathbb{R}$ either has interior or is finite. Then $\mathcal{R}^{\circ}$ is o-minimal.*

This result establishes that an assumption on definable closed sets can sometimes be extended to every set definable in the open core. Of course, every closed subset of $\mathbb{R}$ either has interior or is nowhere dense. Hence a natural strategy to approach Miller's conjecture is to follow the argument of Miller and Speissegger and extend this to all sets definable in the open core of an expansion that does not define $\mathbb{Z}$. A first step in this direction has actually been achieved, as we explain now.

Definition 2.6 ([58]). We say $E \subseteq \mathbb{R}^n$ is D_Σ if there is a definable set $X \subseteq \mathbb{R}_{>0} \times \mathbb{R}^n$ such that

(1) $E = \bigcup_{r>0} X_r$,
(2) X_r is compact for every $r \in \mathbb{R}_{>0}$,
(3) $X_r \subseteq X_s$ for $r \leq s$.

It is easy to see that we can replace *compact* by *closed* in (2) without changing the definition. Thus, we can think of D_Σ sets as a definable analogue of F_σ sets from descriptive set theory. Indeed, it is not hard to show that each D_Σ set is F_σ. By [58, Corollary p.202] a set $E \subseteq \mathbb{R}^m$ is D_Σ if and only if E is the image of some closed definable set $F \subseteq \mathbb{R}^{m+n}$ under the coordinate projection onto the first m coordinates. It follows immediately that all D_Σ sets of $\mathcal{R}$ are definable in $\mathcal{R}^{\circ}$.

Theorem 2.7 (Strong Baire Category Theorem (SBCT)). *Let $\mathcal{R}$ be an expansion of $\overline{\mathbb{R}}$ that does define $\mathbb{Z}$. Then each D_Σ set either has interior or is nowhere dense.*

The classical Baire category theorem states that every F_σ subset of $\mathbb{R}^n$ either has interior or is a countable union of nowhere dense sets. A countable union of nowhere

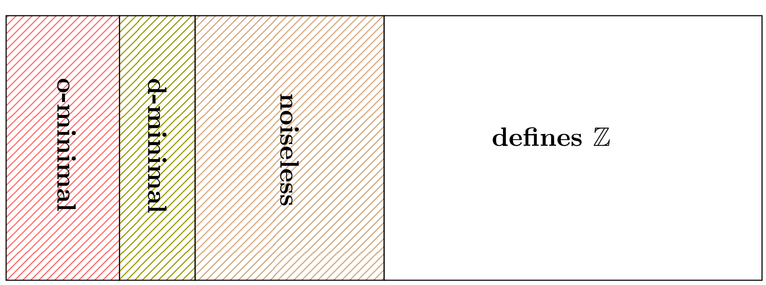

FIGURE 2. Tameness map for expansions of $\overline{\mathbb{R}}$ by constructible sets assuming Miller's conjecture

dense sets is called **meagre**, and can be somewhere dense. Meagre sets are treated by descriptive set theorists as small or negligible. However, their topological closure is in general not negligible in any sense of the word, and hence the classical notion of meagre is arguably not useful from a geometrical point of view. In contrast the topological closure of a nowhere dense is still nowhere dense.

This form of the SBCT was first proved by Fornasiero in the unpublished paper [28, Theorem 1.10]. More general statements for restrained definable complete fields have been published in Fornasiero and Hieronymi [29] and for type A expansions of the real ordered additive group in Fornasiero, Hieronymi and Walsberg [31].

It is natural to think of the open core as a definable analogue of the Borel or the projective hierarchy. In this metaphor, the D_Σ sets correspond to the F_σ sets, and constitute the first step in this hierarchy beyond closed sets. In this sense the SBCT is the first step in the outlined strategy to prove Miller's conjecture. However, while this trivially extends to complements of D_Σ sets, we are currently unable to complete this approach. Part of the problem is that under the assumptions of Theorem 2.5 all set definable in the open core are D_Σ. In general, this is known to be false.

3. An interlude: expansions of $(\mathbb{R}, <, +)$

In this section, we take a slight detour and discuss the tameness program in the more general setting of expansions of the ordered real additive group. In this generality, definability of $\mathbb{Z}$ does not necessarily imply wildness. For example, the structure $(\mathbb{R}, <, +, \mathbb{Z})$ obviously defines $\mathbb{Z}$, yet it is **locally o-minimal** (see, for example, Friedman and Miller [33]). That is, every definable subset of $\mathbb{R}$ either has interior or is closed and discrete. Similarly, $(\mathbb{R}, <, +, \sin)$ defines $\pi\mathbb{Z}$, but is locally o-minimal (see Toffalori and Vorzoris [73, Theorem 2.7]).

In the setting of local o-minimality, *small* means closed and discrete. In particular, every locally o-minimal expansion of $(\mathbb{R}, <, +)$ is d-minimal and, hence, should be considered tame. It is an easy exercise to see that an expansion of $\mathcal{R}$ or $\mathcal{OR}$ is locally o-minimal if and only if $\mathcal{R}$ is o-minimal. Hence, local o-minimality—at least when working over the real numbers—is only relevant in its own right when multiplication is not definable. For more on local o-minimality, we refer the reader to [73] and Kawakami et al. [48].

Fix an expansion $\mathcal{R}$ of $(\mathbb{R}, <, +)$. Definable throughout means definable in $\mathcal{R}$, possibly with parameters. Since the definability of $\mathbb{Z}$ is no longer our prototype of wildness, we instead consider a consequence of the definability of $\mathbb{Z}$ over $\overline{\mathbb{R}}$. We say that $\mathcal{R}$ is **type C** if it defines every compact set. In this situation, $\mathcal{R}$ also defines every bounded Borel set and even every bounded projective set, and hence must be considered wild for the same reasons we outlined for $(\overline{\mathbb{R}}, \mathbb{Z})$. However, it is not necessarily true that *all* projective sets are definable: by Pillay, Scowcroft, and Steinhorn [64, Theorem 2.1], the expansion of $(\mathbb{R}, <, +)$ by all compact sets does not even define multiplication on all of $\mathbb{R}$.

We now want to analyze the case when $\mathcal{R}$ is not type C. Here, we distinguish whether a **dense ω-order** is definable or not.

Definition 3.1. Let $X \subseteq \mathbb{R}^n$. We say that $(X, \prec)$ is an **ω-order** if

(1) $(X, \prec)$ is a linear order,
(2) X and $\prec$ are definable,
(3) $(X, \prec)$ has order type ω.

We say such an ω-order is **dense** if X is a dense subset of an open subinterval of $\mathbb{R}$. An **ω-orderable set** X is a definable set such that there is a definable order $\prec$ on X with order type ω.

One can think of an ω-order as a definably countable set, although one should be careful with this analogy, as "definably countable" could be defined in many other ways. The canonical example of an ω-orderable set is the following.

Example 3.2. Let $D \subseteq \mathbb{R}_{>0}$ be closed in $\mathbb{R}$, infinite and discrete. Then the restriction of the usual order $<$ to D is a definable order on D with order type ω.

Lemma 3.3. *Let $D \subseteq \mathbb{R}^\ell$ be an ω-orderable set, and let $f : D \to \mathbb{R}^m$ be a definable function. Then*

(1) *D^n is ω-orderable.*
(2) *$f(D)$ is either finite or ω-orderable.*

Proof. Let $\prec$ be a definable order on D with order type ω. Let $\prec_{\text{lex}}$ be the lexicographic order on D^n obtained from $\prec$, and let $\max_\prec : D^n \to D$ map $(x_1, \dots, x_n)$ to the $\prec$-maximum of $\{x_1, \dots, x_n\}$. Both $\prec_{\text{lex}}$ and $\max_\prec$ are definable. For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in D^n$, we write $x \prec_n y$ if either

- $\max_\prec x \prec \max_\prec y$, or
- $\max_\prec x = \max_\prec y$ and $x \prec_{\text{lex}} y$.

Observe that $\prec_n$ is a definable linear order, and $(D^n, \prec_n)$ has order type ω, since for every $x \in D^n$ there are only finitely many $y \in D^n$ with $y \prec_n x$. Thus, (1) holds.

For (2), suppose that $f(D)$ is infinite. For $x, y \in f(D)$, we write $x \prec_f y$ if there is $d \in D$ such that $f(d) = x$ and $f(e) \neq y$ for all $e \preceq d$. It is easy to check that $(f(D), \prec_f)$ is an ω-order. □

Combining Example 3.2 and Lemma 3.3 we obtain the following corollary.

Corollary 3.4. *Let $D \subseteq \mathbb{R}$ be definable, closed and discrete and let $f : D^n \to \mathbb{R}$ be a definable function such that $f(D^n)$ is somewhere dense. Then $\mathcal{R}$ defines a dense ω-order.*

Example 3.5. We give several natural examples of dense ω-orders that arise via Corollary 3.4.

(1) Let $a, b \in \mathbb{R}_{>1}$ be such that $\log_a(b) \notin \mathbb{Q}$. Then $a^{\mathbb{N}} \cup b^{\mathbb{N}}$ is closed and discrete. Consider the map $f : (a^{\mathbb{N}} \cup b^{\mathbb{N}})^2 \to \mathbb{R}$ sending (x, y) to x/y. Since $\log_a(b) \notin \mathbb{Q}$, the image of f is somewhere dense. Thus, the structure $(\mathbb{R}, <, +, \cdot, a^{\mathbb{N}}, b^{\mathbb{N}})$ defines a dense ω-order.
(2) The image of $\mathbb{N}$ under the sine function is dense in $[-1, 1]$. Hence, the structure $(\mathbb{R}, <, +, \sin, \mathbb{N})$ defines a dense ω-order.

(3) Let α be an irrational number, and let $f : \alpha\mathbb{N} \to [0,1)$ be
$$\alpha n \mapsto \alpha n - \lfloor \alpha n \rfloor.$$
The image of f is dense by Kronecker's Approximation Theorem (see [1, Theorem 7.8]). It is easy to see that $(\mathbb{R},<,+,\mathbb{N},\alpha\mathbb{N})$ defines f, and hence a dense ω-order.

We say that $\mathcal{R}$ is **type A** if it does not define a dense ω-order, and **type B** if it define a dense ω-order, but is not type C. The expansions mentioned in Example 3.5(1)&(2) are type C by [37, Theorem 1.3] and Hieronymi and Tychonievich [43, Thorem D]. However, for quadratic α, the expansion $(\mathbb{R},<,+,\mathbb{Z},\alpha\mathbb{Z})$ is type B by [39, Theorem A]. Before discussing type A and B in more detail, we show that the classification of expansions of $(\mathbb{R},<,+)$ into type A, type B and type C expansions is indeed a trichotomy.

Proposition 3.6. *Suppose that $\mathcal{R}$ is type C. Then $\mathcal{R}$ defines a dense ω-order, and hence is not type A.*

Proof. Set
$$E := \left\{ \frac{1}{n} : n \in \mathbb{N}_{>0} \right\},$$
and note that $(E,>)$ has order type ω. Let $f : E \to \mathbb{Q} \cap [0,1]$ be a bijection. Let $G \subseteq \mathbb{R}^2$ be the graph of f and let C be its topological closure in $\mathbb{R}^2$. Then C is compact, as G is bounded, and hence definable. It is easy to see that
$$G = C \cap \{(x,y) \in \mathbb{R}^2 : x > 0\}.$$
Thus, G itself is definable, and hence f is a definable function. Since $\mathbb{Q} \cap [0,1]$ is the image of f, we also know that $\mathbb{Q} \cap [0,1]$ is definable. Since E is ω-orderable, the set $\mathbb{Q} \cap [0,1]$ is ω-orderable by Lemma 3.3. $\square$

Observe that all noiseless, and hence all o-minimal and all d-minimal expansions, are type A. This suggests that we should treat type A as the ultimate generalization of o-minimality. Non-definability of a dense ω-order is introduced as a tameness notion in [31], and systematically studied as type A in Hieronymi and Walsberg [45]. In these two papers, several foundational results from o-minimality are generalized to type A. In particular, it is shown that there is a good theory of dimension, definable selection (at least for D_Σ sets), and generic smoothness:

Theorem 3.7 (H.-Walsberg [45, Theorem B])**.** *Suppose $\mathcal{R}$ is type A. Let $k \in \mathbb{N}_{>0}$, and let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be definable such that f is continuous and U is open. Then there is an open definable set $V \subseteq U$ dense in U such that the restriction of f to V is C^k.*

We are going to prove this theorem in the next section, although only for expansions of $\overline{\mathbb{R}}$. In its stated generality, Theorem 3.7 allows us to turn our trichotomy into a proper and non-trivial tetrachotomy for expansions of $(\mathbb{R},<,+)$ (see Figure 3), in which we divide expansions not only by whether they are tame or not, but also by whether they define certain fields.

Definition 3.8. We say $\mathcal{R}$ is **field-type** if there is a field $(I,\oplus,\odot)$, where $I \subseteq \mathbb{R}$ is an interval and $\oplus$ and $\odot$ are definable (as subsets of $\mathbb{R}^3$) such that $(I,<,\oplus,\odot)$ and $(\mathbb{R},<,+,\cdot)$ are isomorphic.

It is a classical problem in model theory, dating back to Zilber's trichotomy conjecture [75], to analyze whether model-theoretically tame structures (in our case, type A expansions) that exhibit well-defined non-linear behavior actually define fields. In the case of o-minimal structures, Peterzil and Starchenko [61] indeed show that non-linearity yields a definable field. This result can be generalized to type A structures thanks to the following theorem.

Theorem 3.9 (Marker-Peterzil-Pillay [53])**.** *If $\mathcal{R}$ defines a C^2 non-affine function $I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval, then $\mathcal{R}$ is field-type.*

While the proof in [53] assumes that $\mathcal{R}$ is o-minimal, it can easily be adjusted to hold in the general case (see the proof of [45, Theorem F]). Combining Theorem 3.7 and Theorem 3.9, we obtain the following strong linearity result for type A structures that are not field-type.

Theorem 3.10 ([45, Theorem A])**.** *Suppose that $\mathcal{R}$ is type A and not field-type. Let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be definable such that f is continuous and U is open. Then there is an open definable set $V \subseteq U$ dense in U such that for every connected component C of V the restriction of f to C is affine.*

In the type A case, this establishes that structures with non-linear behavior are field-type. Outside the type A case, being field-type distinguishes between type B and type C structures. Indeed, it follows from the proof of Theorem 1.3 that expansions of field-type cannot be type B.

Theorem 3.11 (See [45, Fact 1.3])**.** *Suppose that $\mathcal{R}$ is field-type. Then $\mathcal{R}$ is type C if and only if $\mathcal{R}$ defines a dense ω-order.*

Even when moving from expansions of the real ordered additive groups to expansions of the ordered $\mathbb{R}$-vector space of real numbers, type B disappears.

Theorem 3.12 ([30, Theorem 3.9])**.** *Suppose that $\mathcal{R}$ expands $(\mathbb{R}, <, +, (x \to rx)_{r\in\mathbb{R}})$. Then $\mathcal{R}$ is type C if and only if $\mathcal{R}$ defines a dense ω-order.*

Although we already mentioned an example of a type B structure earlier, we now consider another instructive example in more detail. Let $k \in \mathbb{N}_{>0}$, and let $\mathcal{T}_k$ be the

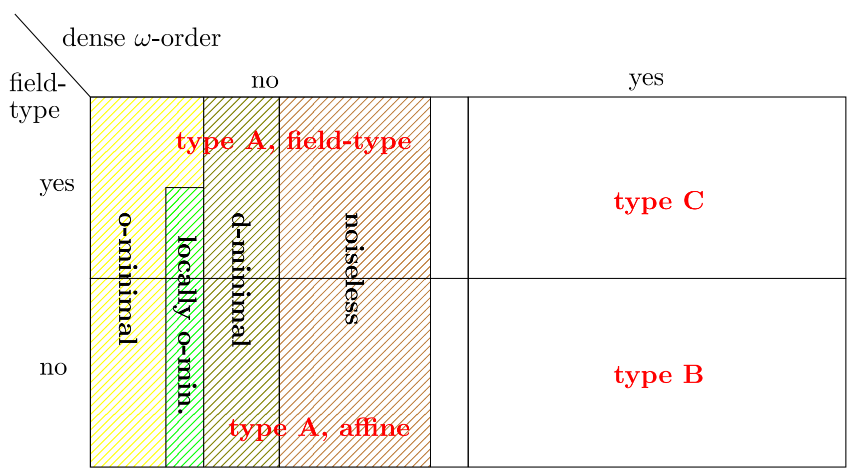

FIGURE 3. Tetrachotomy for expansions of $(\mathbb{R}, <, +)$

expansion of $(\mathbb{R},<,+)$ by the ternary predicate V_k defined as follows: $(a,b,i) \in V_k$ if and only if

- $b \in k^{\mathbb{Z}}$ and $i \in \{0,1,\dots,k-1\}$ and
- i is the digit corresponding to b in a k-ary expansion of a.

Boigelot, Rassart and Wolper [7] were the first to study this structure, although their motivation came from computer science and not from tame geometry. By [7, Theorem 6] the theory of $\mathcal{T}_k$ is decidable. This result itself is a corollary of Büchi's theorem [9] on the decidability of the monadic second-order theory of one successor. It follows that $\mathcal{T}_k$ can not be type C. Thus to show that $\mathcal{T}_k$ is type B, we just need to argue that $\mathcal{T}_k$ defines a dense ω-orderable set.

Let $W \subseteq (0,1)$ be the set of $a \in (0,1)$ such that a has a finite k-ary representation. It is easy to see that W is dense in $(0,1)$ and definable in $\mathcal{T}_k$. Let $\mu : W \to k^{-\mathbb{N}}$ map $a \in W$ to the smallest element $y \in k^{-\mathbb{N}}$ such that y appears with non-zero digit in the finite k-ary representation of a. For example, if $k=2$ and $a=\frac{3}{4}$, then $\mu(a)=2^{-2}$. This function μ is definable in $\mathcal{T}_k$. For $a_1,a_2 \in W$, we write $a_1 \prec_W a_2$ if and only if either

- $\mu(a_1) > \mu(a_2)$, or,
- $\mu(a_1) = \mu(a_2)$ and $a_1 < a_2$.

Thus if $k=2$, we get

$$\frac{1}{2} \prec_W \frac{1}{4} \prec_W \frac{3}{4} \prec_W \frac{1}{8} \prec_W \frac{3}{8} \prec_W \frac{5}{8} \prec_W \frac{7}{8} \prec \frac{1}{16} \dots$$

The reader may check that $(W,\prec_W)$ is a dense ω-order.

The type B expansions given in the above example define many well-known fractals. For example, $\mathcal{T}_3$ defines the usual "middle-thirds" Cantor set, which is obtained by iteratively removing the open middle third starting with $[0,1]$. Indeed, since it is the set of exactly those elements in $[0,1]$ that have a ternary representation in which 1 does not appear, it can easily be defined using V_3. Even though fractal objects are definable, we can recover some tameness of continuous functions. By Block Gorman et al. [6, Theorem 7.3], a continuous function $f:[0,1]\to\mathbb{R}$ definable in some $\mathcal{T}_k$ is locally affine outside a nowhere dense set. However, the proof relies heavily on automata theory rather than tame geometry, and is unlikely to generalize to other type B expansions.

Question 3.13 ([45, Question 1.4]). *Let $\mathcal{R}$ be type B and let $f:[0,1]\to\mathbb{R}$ be definable and continuous. Is there a nowhere dense definable subset of $[0,1]$ such that f is affine on each connected component of $[0,1]\setminus V$?*

The best result in the generality of type B expansions is the following.

Theorem 3.14 ([45, Theorem 8.1]). *Let $\mathcal{R}$ be type B and let $f:[0,1]\to\mathbb{R}$ be C^2 and definable. Then f is affine.*

3.1. **Connections to neostability.** We already argued that

geometric tameness $\not\Rightarrow$ model-theoretic tameness.

The converse is true to a certain extent.

Theorem 3.15 (H.-Walsberg [44, Theorem A]). *If $\mathcal{R}$ defines a dense ω-order, then $\mathcal{R}$ defines an isomorphic copy of the two-sorted structure $(\mathbb{N},\mathcal{P}(\mathbb{N}),\in,+1)$.*

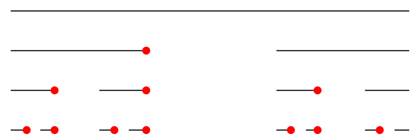

FIGURE 4. The highlighted points are the left endpoints of bounded complementary intervals of the Cantor ternary set after the first few steps of the construction.

The structure $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, +1)$ can be seen as the standard model of the monadic second-order theory of one successor. It is an easy exercise to see that $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, +1)$ does not satisfy any of the Shelah-style combinatorial tameness conditions, such as NIP or NTP2. Thus, by Theorem 3.15, an expansion that is not type A cannot satisfy any of these conditions either. Therefore, all model-theoretic tameness in the sense of Shelah is confined to type A expansions.

Theorem 3.16 (Miller's conjecture for NIP expansions, Walsberg [74]). *Let $\mathcal{R}$ be an NIP expansion. Then $\mathcal{R}^\circ$ is noiseless.*

By [44], an NIP expansion cannot define a Cantor set, and Walsberg [74] deduces from this that a noiseless NIP expansion is **strongly noiseless**; that is, if X, Y are definable subsets of $\mathbb{R}^n$ with $X \subseteq Y$, then X is either nowhere dense in Y or X has interior in Y. It is open whether this can be strengthened further.

Question 3.17 ([42]). *Is every noiseless NIP expansion d-minimal?*

There are several known results using strengthenings of NIP to deduce geometric tameness in expansions of $(\mathbb{R}, <, +)$. By Simon [68, Theorem 3.6], every dp-minimal expansion of $(\mathbb{R}, <, +)$ is o-minimal. If $\mathcal{R}$ is a strong (not necessarily NIP) expansion of $(\mathbb{R}, <, +, \cdot)$, then $\mathcal{R}$ has o-minimal open core by Dolich and Goodrick [11, Corollary 2.4].

3.2. **Connections to metric dimensions.** Dense ω-orderable sets appear naturally when studying expansions of $(\mathbb{R}, <, +)$ by fractal objects. Indeed, let $C \subseteq \mathbb{R}$ be a Cantor set, that is, C is a compact set without interior or isolated points. We say an open interval in $\mathbb{R}$ is a **complementary interval of** C if it is a connected component of the complement of C. Let L be the set of all left endpoints of bounded complementary intervals of C. Observe that L is dense in C and definable in $(\mathbb{R}, <, +, C)$. See Figure 4 for a visualization when C is the classical Cantor ternary set. We now construct an ω-order on L, also definable in $(\mathbb{R}, <, +, C)$.

Let $\tau : L \to \mathbb{R}_{>0}$ map $z \in L$ to the length of the complementary interval whose endpoint is z. It is easy to check that τ is definable in $(\mathbb{R}, <, +, C)$. Now, for $a_1, a_2 \in L$, we write $a_1 \prec_C a_2$ if and only if either

- $\tau(a_1) > \tau(a_2)$, or
- $\tau(a_1) = \tau(a_2)$ and $a_1 < a_2$.

Since C is compact, we know that for every $b \in \mathbb{R}$ there are only finitely many $a \in L$ with $\tau(a) > b$. Hence, $(L, \prec_C)$ is an ω-order.

We now explain the connection to metric dimensions. For $A \subseteq \mathbb{R}^n$, let $\dim_{\mathcal{H}} A$ denote the **Hausdorff dimension** of A. We refer the reader to Falconer [24] for a definition of Hausdorff dimension, but we collect the following two properties that we need here.

Fact 3.18. *Let $A, B \subseteq \mathbb{R}^n$. Then*

(1) $\dim_{\mathcal{H}} A + \dim_{\mathcal{H}} B \leq \dim_{\mathcal{H}}(A \times B)$,
(2) *if A is Borel and $\dim_{\mathcal{H}} A > m$, then there is an orthogonal projection $\pi : \mathbb{R}^n \to \mathbb{R}^m$ such that $\pi(A)$ has positive m-dimensional Lebesgue measure.*

The second statement is known as Marstrand's projection theorem, a fundamental result in geometric measure theory.

Lemma 3.19 (Edgar-Miller [20, Lemma 1])**.** *Let $E \subseteq \mathbb{R}$ be compact such that $\dim_{\mathcal{H}} E > 0$. Then there exists $n \in \mathbb{N}$ and an $\mathbb{R}$-linear function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(E^n)$ has interior.*

Proof. Let $k \in \mathbb{N}$ be such that $k \dim_{\mathcal{H}}(E) > 1$. By Fact 3.18(1),

$$\dim_{\mathcal{H}}(E^k) \geq k \dim_{\mathcal{H}}(E) > 1.$$

By Fact 3.18(2), there exists an orthogonal projection $\pi : \mathbb{R}^k \to \mathbb{R}$ such that $\pi(E^k)$ has positive Lebesgue measure. Thus, the difference set $\{a - b : a, b \in \pi(E^k)\}$ has interior by Steinhaus [71]. Now, let $f : \mathbb{R}^{2k} \to \mathbb{R}$ map $(a_1, \dots, a_{2k})$ to $\pi(a_1, \dots, a_k) - \pi(a_{k+1}, \dots, a_{2k})$. $\square$

Corollary 3.20 ([30, Theorem B])**.** *Let $C \subseteq \mathbb{R}$ be a Cantor set such that $\dim_{\mathcal{H}}(C) > 0$. Then $(\mathbb{R}, <, +, (x \mapsto rx)_{r\in\mathbb{R}}, E)$ is type C.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $\mathbb{R}$-linear map such that $f(C^n)$ has interior. Since L is dense in C and f is continuous, we have that $f(L^n)$ is dense in $f(C^n)$ and hence somewhere dense. Thus, $f(L^n)$ is a dense ω-order in $(\mathbb{R}, <, +, C, f)$ by Lemma 3.3. With Theorem 3.12, we conclude that $(\mathbb{R}, <, +, (x \mapsto rx)_{r\in\mathbb{R}}, E)$ is type C. $\square$

Therefore, Cantor sets with positive Hausdorff dimension cannot be defined in expansions of the real ordered vector space on $\mathbb{R}$ that can be considered tame in any logical or model-theoretic sense. We really need the vector space structure here: recall that $\mathcal{T}_3$ is type B, yet defines the middle-thirds Cantor set, whose Hausdorff dimension is $\log_3(2)$.

4. Strong Baire Category Theorem

In this section, we prove the SBCT for expansions of $\overline{\mathbb{R}}$, roughly following the argument in the proof of [41, 2.14]. Before doing so, we collect some preliminary results about D_Σ sets. Let $\mathcal{R}$ be an expansion of $\overline{\mathbb{R}}$.

Lemma 4.1 (Properties of D_Σ sets)**.**

(1) *The image of a D_Σ set under a continuous definable map is D_Σ.*
(2) *Finite unions and finite intersections of D_Σ sets are D_Σ.*
(3) *If $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is D_Σ, then A_x is D_Σ for every $x \in \mathbb{R}^m$.*
(4) *If $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is D_Σ, then $\{x \in \mathbb{R}^m : A_x \text{ has interior}\}$ is D_Σ.*

Proof. The first three items follow easily from well-known properties of compact subsets of $\mathbb{R}^n$.

For (4), let $X \subseteq \mathbb{R}_{>0} \times \mathbb{R}^{m+n}$ be a definable set witnessing that A is D_Σ. In particular, $A = \bigcup_{r>0} X_r$. It follows from the classical Baire category theorem that

$$\{x \in \mathbb{R}^m \ : \ A_x \text{ has interior}\} = \bigcup_{r>0} \{x \in \mathbb{R}^m : \exists y \in \mathbb{R}^n \ \prod_{i=1}^{n} [y_i, y_i + \frac{1}{r}] \subseteq X_{(r,x)}\}$$

Since X_r is compact for each $r \in \mathbb{R}_{>0}$, it is easy to check that the same is true for

$$\{x \in \mathbb{R}^m : \exists y \in \mathbb{R}^n \prod_{i=1}^{n} [y_i, y_i + \frac{1}{r}] \subseteq X_{(r,x)}\}.$$

□

Lemma 4.2. *The following are equivalent:*
(1) $\mathcal{R}^\circ$ *is o-minimal,*
(2) $\mathcal{R}$ *defines no infinite closed discrete set,*
(3) $\mathcal{R}$ *defines no infinite discrete set.*

Proof. (3) ⇒ (1): Suppose every definable discrete set is finite. Let $X \subseteq \mathbb{R}$ be open and definable. Thus, X is a countable union of disjoint intervals, and it is easy to check that the set of midpoints of those intervals is definable in $\mathcal{R}^\circ$. Let Y denote this set. Clearly, Y is discrete, and hence finite. Therefore, X has finitely many connected components. By Theorem 2.5, the open core of $\mathcal{R}$ is o-minimal.

(1) ⇒ (2): Since $\mathcal{R}^\circ$ is o-minimal, every closed subset of $\mathbb{R}$ definable in $\mathcal{R}$ either has interior or is finite. Since discrete sets do not have interior, every closed discrete set definable in $\mathcal{R}$ has to be finite.

(2) ⇒ (3): Let $D \subseteq \mathbb{R}$ be infinite, discrete, and definable in $\mathcal{R}$. We now construct a definable set $E \subseteq \mathbb{R}$ such that E is infinite, closed, and discrete. For $t \in \mathbb{R}_{>0}$, set

$$X_t := \{d \in D : (d - t, d + t) \cap D = \{d\}\}.$$

This definable family is decreasing; that is, $X_s \subseteq X_t$ whenever $s, t \in \mathbb{R}_{>0}$ and $t < s$. Further observe that each X_t is closed and discrete. Thus, if X_t is infinite for some $t \in \mathbb{R}_{>0}$, we are done. Hence we can assume that X_t is finite for every $t \in \mathbb{R}_{>0}$.

Since D is infinite, there is a $u \in \mathbb{R}_{>0}$ such that X_u has cardinality at least 2. Let $g : (0, u) \to D$ be defined by

$$t \mapsto \max\left(\left\{\max\left(\{(d - e)^{-1}, d - e\}\right) : d, e \in X_t, d > e\right\}\right).$$

Note that $g((0, u))$ is infinite and unbounded, since D is infinite. However, $g((t, u))$ is finite for every $t \in (0, u)$. Since g is strictly decreasing, we get that $(1, N) \cap g((0, u))$ is finite for every $N \in \mathbb{N}$. Thus, $g((0, u))$ is closed and discrete. □

The implication (2)⇒(3) first appeared in Fornasiero [27, Remark 4.16], but the proof presented here is from [38, Lemma 3].

4.1. **Proof of SBCT.** Let $\mathcal{R}$ be an expansion of $\overline{\mathbb{R}}$ that does not define $\mathbb{Z}$. We now prove the SBCT. First, suppose that $\mathcal{R}$ does not define an infinite, closed, and discrete set. Then, by Lemma 4.2, the open core $\mathcal{R}^\circ$ of $\mathcal{R}$ is o-minimal. Since every D_Σ set in $\mathcal{R}$ is definable in $\mathcal{R}^\circ$, the SBCT follows. Thus, we can assume that $\mathcal{R}$ defines an infinite, closed, and discrete set $D \subseteq \mathbb{R}$. Note that D must be unbounded. By possibly replacing the elements of D by their negations, we can assume that $D \subseteq \mathbb{R}_{>0}$. The order $(D,<)$ has order-type ω. The successor function $\sigma : D \to D$ mapping $d \in D$ to $\min\big(D \cap (d,\infty)\big)$ is clearly definable in $\mathcal{R}$.

Lemma 4.3. *Let $E \subseteq [0,1]^{n+1}$ be D_Σ such that E_a has empty interior for every $a \in [0,1]^n$. Then there is $\varepsilon > 0$ such that for all $a \in [0,1]^n$ there is $x \in [0,1]$ with*

$$\big(x-\varepsilon, x+\varepsilon\big) \cap E_a = \emptyset.$$

Proof. Suppose otherwise. Since E is D_Σ, there is an increasing definable family $(X_y)_{y>0}$ such that $E = \bigcup_{y>0} X_y$ and X_y is closed for each $y \in \mathbb{R}_{>0}$. Since D is unbounded, we have that $E = \bigcup_{d\in D} X_d$. Because $X_{(d,a)} \subseteq E_a$ for every $a \in [0,1]^n$ and E_a has empty interior, the set $X_{(d,a)}$ has empty interior as well. Since X_d is closed, so is $X_{(d,a)}$ and hence $X_{(d,a)}$ is nowhere dense. Let $(Y_{(d,a)})_{d\in D, a\in[0,1]^n}$ be the definable family such that $Y_{(d,a)}$ is the set of midpoints of bounded complimentary interval of $X_{(d,a)}$. Let $(g_{(d,a)} : Y_{(d,a)} \to X_{(d,a)})_{d\in D,a\in[0,1]^n}$ be the definable family of maps given by

$$g_{(d,a)}(x) := \sup\big(X_{(d,a)} \cap (-\infty, x]\big).$$

Note that $Y_{(d,a)}$ is a discrete subset of $(0,1)$ and its image is dense in $X_{(d,a)}$.

We will now define a discrete set $D' \subseteq \mathbb{R}$. For $d \in D$, let $\delta(d) \in D$ be the smallest element $d' \in D$ larger than d such that

$$Z_{(d,d')} := \{a \in [0,1]^n \ : \ \forall x \in [0,1] \exists y \in X_{(d',a)} \ |x-y| \leq d^{-1}\}$$

is non-empty. Since $X_{\delta(d)}$ is closed, the set $Z_{d,\delta(d)}$ is closed as well. Let $b : D \to [0,1]^n$ map $d \in D$ to $\operatorname{lexmin}(Z_{(d,\delta(d))})$. The function b is definable. Observe that

$$\bigcup_{d\in D} X_{(\delta(d),b(d))} \text{ is dense in } [0,1].$$

Set

$$D' := \{d + (\sigma(d)-d)c \ : \ d \in D, c \in Y_{(\delta(d),b(d))}\}.$$

Let $f : D' \to [0,1]$ be the function mapping $d + (\sigma(d)-d)c$ to $g_{\delta(d),b(d)}(c)$. It is easy to see that f is well-defined and definable, that D' is discrete and, by our assumption on E, that $f(D')$ is dense in $[0,1]$. By Theorem 1.3 and the remark following it, we conclude that $\mathcal{R}$ defines $\mathbb{Z}$. This is a contradiction. ☐

Corollary 4.4. *Let $E \subseteq [0,1]^{n+1}$ be D_Σ such that E_a has empty interior for every $a \in [0,1]^n$. Then there are intervals $I_1,\dots,I_p \subseteq [0,1]$ such that for all $a \in [0,1]^n$ there is $i \in \{1,\dots,p\}$ with $I_i \subseteq [0,1] \setminus E_a$.*

Proof. By Lemma 4.3 there is $\varepsilon > 0$ such that for all $a \in [0,1]^n$ there is $x \in [0,1]$ with $|x-y| > \varepsilon$ for all $y \in E_a$. Now let $p \in \mathbb{N}_{>0}$ be such that $p > 2\varepsilon^{-1}$. For $i \in \{1,\dots,p\}$, set

$$I_i := [\frac{(i-1)}{p}, \frac{i}{p}].$$

For each i the length of I_i is $1/p$, and hence less than $\frac{\varepsilon}{2}$. Thus for each $a \in [0,1]$ there is $i \in \{1,\dots,p\}$ such that $I_i \subseteq [0,1] \setminus E_a$. ☐

We need the following easy fact from descriptive set theory.

Fact 4.5 (see [58, 1.5(3)]). *Let $A \subseteq \mathbb{R}^{m+n}$ be F_σ with empty interior. Then*

$$\{x \in \mathbb{R}^m \ : \ A_x \text{ has interior}\}$$

has empty interior.

Proof. Let $(A_k)_{k\in\mathbb{N}}$ be a family of closed subsets of $\mathbb{R}^{m+n}$ such that $\bigcup_{k\in\mathbb{N}} A_k = A$. Suppose that $\{x \in \mathbb{R}^m \ : \ A_x \text{ has interior}\}$ has non-empty interior. Let $\mathcal{V}$ be the set of all open boxes in $\mathbb{R}^n$ with rational endpoints. Note that $\mathcal{V}$ is countable. Since $A_x = \bigcup_{k\in\mathbb{N}}(A_k)_x$, we obtain from the the classical Baire category theorem that

$$\begin{aligned}\{x \in \mathbb{R}^m \ : \ A_x \text{ has interior}\} &= \bigcup_{k\in\mathbb{N}} \{x \in \mathbb{R}^m \ : \ (A_k)_x \text{ has interior}\} \\ &= \bigcup_{k\in\mathbb{N}} \bigcup_{V\in\mathcal{V}} \{x \in \mathbb{R}^m \ : \ V \subseteq (A_k)_x\}.\end{aligned}$$

Again by the classical Baire category theorem, we conclude that there is $k \in \mathbb{N}$, $V \in \mathcal{V}$ and an open box $U \subseteq \mathbb{R}^m$ such that

$$U \subseteq \{x \in \mathbb{R}^m \ : \ V \subseteq (A_k)_x\}.$$

Hence $U \times V \subseteq A_k \subseteq A$. This contradicts our assumption on A. □

Proof of SBCT. We proceed by induction on n. We first consider the case that $n = 1$. Let $A \subseteq \mathbb{R}$ be D_Σ with empty interior. Suppose towards a contradiction that there is an open interval $I \subseteq \mathbb{R}$ such that $A \cap I$ is dense in I. Since D_Σ sets are preserved under affine maps by Lemma 4.1(1), we can assume that $I = (0, 1)$. Note that $A \cap (0, 1)$ is D_Σ by Lemma 4.1(2). By Lemma 4.3 there are $\varepsilon > 0$ and $x \in (0, 1)$ such that

$$\big(x - \varepsilon, x + \varepsilon\big) \cap A \cap (0, 1) = \emptyset.$$

This contradicts the density of $A \cap (0, 1)$ in $(0, 1)$.

Now suppose the statement of the SBCT holds for n. Let $A \subseteq \mathbb{R}^{n+1}$ be D_Σ with empty interior. We need to show that A is nowhere dense. Let $U \subseteq \mathbb{R}^{n+1}$ be open. We will find an open set $V \subseteq U$ such that $A \cap V = \emptyset$.

First consider

$$Z := \{x \in \mathbb{R}^n \ : \ A_x \text{ has interior}\}.$$

This set is D_Σ by Lemma 4.1(4). By induction Z either has interior or is nowhere dense. Thus Z has empty interior by Fact 4.5, and hence is nowhere dense. Hence there is an open set $W \subseteq \mathbb{R}^n$ such that $W \cap Z = \emptyset$ and $(W \times \mathbb{R}) \cap U \neq \emptyset$. Now replace U by $(W \times \mathbb{R}) \cap U$. Observe that $(A \cap U)_a$ is nowhere dense for all $a \in \mathbb{R}^n$.

Since $A \cap U$ is D_Σ, we can assume that $A = A \cap U$. Since the collection of D_Σ is closed under definable continuous function by Lemma 4.1(1), we can reduce to the case that $U = (0, 1)^{n+1}$. Thus it is left to find an open set $V \subseteq (0, 1)^{n+1}$ with $A \cap V = \emptyset$.

Since A is D_Σ, there are open intervals $I_1, \dots, I_p \subseteq [0, 1]$ such that for all $a \in [0, 1]^n$ there is $i \in \{1, \dots, p\}$ with $I_i \subseteq [0, 1] \setminus A_a$. For $i = 1, \dots, p$, set

$$J_i := \{a \in [0, 1]^n \ : \ A_a \cap I_i = \emptyset\}.$$

Since J_i is a complement of a D_Σ set, each $[0, 1]^n \setminus J_i$ either has interior or is nowhere dense by induction. Since $\bigcup_{i=1}^p J_i = [0, 1]^n$, there is $j \in \{1, \dots, p\}$ such that J_j has interior. Fix this j, and let V_0 be an open subset of J_j. Then set $V := V_0 \times I_j$ and observe that $A \cap V = \emptyset$. □

Corollary 4.6. *Let $(A_t)_{t\in\mathbb{R}_{>0}}$ be an increasing definable family of nowhere dense subsets of $\mathbb{R}^n$. Then $\bigcup_{t\in\mathbb{R}_{>0}} A_t$ is nowhere dense.*

Proof. Note that $\overline{A_t}$ is nowhere dense for each $t \in \mathbb{R}_{>0}$. Thus, by the Baire category theorem, $\bigcup_{t\in\mathbb{R}_{>0}} \overline{A_t}$ does not have interior. Since it is D_Σ, the SBCT then implies that it is nowhere dense. □

Corollary 4.7. *Let $A \subseteq \mathbb{R}^n$ be D_Σ. Then $\mathbb{R}^n \setminus A$ either has interior or is nowhere dense, and* $\operatorname{fr}(A)$ *is nowhere dense.*

Proof. Suppose that $\mathbb{R}^n \setminus A$ does not have interior. Let $U \subseteq \mathbb{R}^n$ be an open box. In order to show that $\mathbb{R}^n \setminus A$ is nowhere dense, it suffices to find an open subset $V \subseteq U$ such that $(\mathbb{R}^n \setminus A) \cap V = \emptyset$. Note that $A \cap U$ is D_Σ, and hence either has interior or is nowhere dense. Since $\mathbb{R}^n \setminus A$ does not have interior, the intersection $A \cap U$ is dense in U. By SBCT we get that $A \cap U$ has nonempty interior. Let V be this interior.

Note that the boundary of A is the union of two D_Σ sets:

$$\mathbb{R}^n \setminus \operatorname{bd}(A) = (\mathbb{R}^n \setminus \overline{A}) \cup \mathring{A}.$$

Thus $\operatorname{bd}(A)$ has interior or is nowhere dense. If $\operatorname{bd}(A)$ has interior, there is an open box $U \subseteq \mathbb{R}^n$ such that $A \cap U$ is dense and codense in U. This contradict the fact that A is D_Σ. Thus $\operatorname{bd}(A)$ is nowhere dense. Since the frontier $\operatorname{fr}(A)$ is a subset of the boundary $\operatorname{bd}(A)$, it is nowhere dense, too. □

5. Consequences of SBCT

> "I would like to say a few words now about some topological considerations which have made me understand the necessity of new foundations for 'geometric' topology. [...] 'General topology' was developed (during the thirties and forties) by analysts and in order to meet the needs of analysis. [...] Even now, just as in the heroic times when one anxiously witnessed the first time curves cheerfully filling squares and cubes, when one tries to do topological geometry in the technical context of topological spaces, one is confronted at each step with spurious difficulties related to wild phenomena."
>
> Alexander Grothendieck in "Esquisse d'un Programme"

Let $\mathcal{R}$ be an expansion of $(\mathbb{R}, <, +, \cdot)$ that does not define $\mathbb{Z}$. As noted in Goal 1.2, part of the tameness program is to study geometric tameness of definable sets in $\mathcal{R}$. Here, we understand geometric tameness in the sense of Grothendieck's *topologie modérée*, and in this section aim to prove weak analogues of results known for o-minimal structures. Two of the most fundamental results in o-minimality are the monotonicity theorem and the smoothness theorem, and we will establish analogues of these theorems for D_Σ sets in $\mathcal{R}$, basically as corollaries of the SBCT.

Before doing so, we want to collect one important, yet simple fact about D_Σ sets. Let $X \subseteq \mathbb{R}_{>0} \times \mathbb{R}^n$ be a definable family of compact subsets such that $X_r \subseteq X_s$ for all $r, s \in \mathbb{R}_{>0}$ with $r \geq s$. Then $\bigcup_{r>0} X_r$ is D_Σ. Indeed, the family $(X_{\frac{1}{r}})_{r\in\mathbb{R}_{>0}}$ is an increasing definable family of compact set witnessing that $\bigcup_{r>0} X_r$ is D_Σ. We will use this observation freely throughout the remainder.

Let $X \subseteq \mathbb{R}^m$, let $f : X \to \mathbb{R}^n$ and let $\varepsilon > 0$. For $x \in X$, we say that f has **ε-oscillation at** x if for all $\delta > 0$, there are $y, z \in X$ such that $|x - y| < \delta$ and $|x - z| < \delta$, yet $|f(y) - f(z)| \geq \varepsilon$. Let $\mathcal{D}_\varepsilon$ denote the set of all $x \in X$ at which f has ε-oscillation. Set $\mathcal{D}(f) := \bigcup_{\varepsilon>0} \mathcal{D}_\varepsilon(f)$. It is an easy exercise to check that $\mathcal{D}_\varepsilon(f)$ is closed in X, and f is discontinuous at $x \in X$ if and only if $x \in \mathcal{D}(f)$.

Lemma 5.1. *Let $U \subseteq \mathbb{R}^m$ be open and definable, let $f : U \to \mathbb{R}^n$ be definable. Then $\mathcal{D}(f)$ is D_Σ, and one of the following holds:*

(1) *There is an open dense definable $V \subseteq U$ such that the restriction of f to V is continuous.*
(2) *There is a definable open $V \subseteq U$ and $\varepsilon > 0$ such that $V \subseteq D_\varepsilon(f)$.*

Proof. Each $\mathcal{D}_\varepsilon(f)$ is closed in U and hence D_Σ. Thus

$$\mathcal{D}(f) = U \cap \bigcup_{\varepsilon \in \mathbb{R}_{>0}} \overline{\mathcal{D}_\varepsilon(f)}.$$

Then $\mathcal{D}(f)$ is D_Σ by Lemma 4.1(1). Applying the SBCT, we obtain that $\mathcal{D}(f)$ either has interior or is nowhere dense. If $\mathcal{D}(f)$ has no interior, then (1) holds. Suppose now that $\mathcal{D}(f)$ has interior. By the classical Baire category theorem there is $\varepsilon > 0$ such that $D_\varepsilon(f)$ has interior. Hence (2) holds. □

To illustrate that (2) in Lemma 5.1 can fail even for tame structures we give the following example.

Example 5.2. Let $\mathcal{R}$ be $(\mathbb{R}, <, +, \cdot, \mathbb{Q}^{rc})$, the expansion of the real field by the real algebraic numbers. Recall that $\mathcal{R}^o$ is o-minimal. Consider the characteristic function $\chi_{\mathbb{Q}^{rc}}$ of the real algebraic numbers. Clearly, this function has ε-oscillation at every point for all $\varepsilon < 1$.

Let $U \subseteq \mathbb{R}^m$ be open and $f : U \to \mathbb{R}^n$. We say f is **Baire class one** if it is a pointwise limit of a sequence of continuous functions $(f_i : U \to \mathbb{R}^n)_{i\in\mathbb{N}}$.

Proposition 5.3. *Let $U \subseteq \mathbb{R}^m$ be open and let $f : U \to \mathbb{R}^n$ be definable and Baire class one. Then f is continuous on an open dense definable subset of U.*

Proof. By a theorem of Baire (see [49, (24.14)]), the set $\mathcal{D}(f)$ is meager whenever f is Baire class one. Thus condition (2) of Lemma 5.1 fails. Hence there is an open definable $V \subseteq U$ that is dense in U, such that the restriction of f to V is continuous. □

Let $f : X \subseteq \mathbb{R}^m \to \mathbb{R}$ and $x \in X$. We say f is **lower semicontinuous at** x if $\liminf_{y\to x} f(y) \geq f(x)$, and f is **lower semicontinuous** if it is lower semicontinuous at every $x \in X$. A lower semicontinuous function is Baire class one (see [49, (24.16)]).

Proposition 5.4. *Let $A \subseteq \mathbb{R}^{m+n}$ be definable and compact, and let $U \subseteq \mathbb{R}^m$ be open and definable such that $A_x \neq \emptyset$ for all $x \in U$, and let $f : U \to \mathbb{R}^n$ map $x \in U$ to the lexicographically minimal element of A_x. Then $\mathcal{D}(f)$ is nowhere dense and thus there is a definable dense open subset V of U such that f is continuous on V.*

Proof. Since A is compact, the image $f(U)$ is bounded. Thus for all $x \in U$, we have that $\liminf_{y\to x} f(y) \in A_x$ and

$$f(x) \leq \liminf_{y\to x} f(y).$$

Hence f is lower semicontinuous. Now apply Proposition 5.3.

Let $n \in \mathbb{N}_{>1}$ and suppose the statement holds for $n-1$. Let $\varepsilon > 0$. By the SBCT it is enough to show that $\mathcal{D}_\varepsilon(f)$ is nowhere dense. Let C be the projection of A onto the first $m+n-1$ coordinates. Let $g : C \to \mathbb{R}$ map $(x,y) \in C \cap \mathbb{R}^m \times \mathbb{R}^{n-1}$ to $\min A_{(x,y)}$. By the base case $n = 1$, the set $\mathcal{D}_\varepsilon(g)$ is nowhere dense in $\mathbb{R}^m \times \mathbb{R}^{n-1}$. Let $h : U \to \mathbb{R}^{n-1}$ map $x \in U$ to $\operatorname{lexmin} C_x$. By induction, $\mathcal{D}_\varepsilon(h)$ is nowhere dense. It follows that $\mathcal{D}_\varepsilon(f)$ is nowhere dense. □

We are going to prove the fiber lemma for D_Σ sets. It is known to hold for all definable sets in noiseless structures by [55, Main Lemma]. Here we follow the argument in the proof of [29, Lemma 49(2)].

Corollary 5.5 (Fiber Lemma)**.** *Let $A \subseteq \mathbb{R}^{m+n}$ be D_Σ. Then*

$$\{x \in \mathbb{R}^m \ : \ \overline{(A_x)} \neq \overline{A}_x\}$$

is nowhere dense.

Proof. Let $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ be the coordinate projection onto the first m coordinates. Set

$$E := \{x \in \mathbb{R}^m \ : \ \overline{A}_x \setminus \overline{(A_x)} \neq \emptyset\}.$$

For each open box $B \subseteq \mathbb{R}^n$, set

$$E_B := \{x \in \mathbb{R}^m \ : \ \overline{A}_x \cap B \neq \emptyset, \overline{(A_x)} \cap B = \emptyset\}.$$

Observe that

$$E_B \subseteq \operatorname{fr}(\pi(\mathbb{R}^m \times B) \cap A)).$$

Since $\pi(\mathbb{R}^m \times B) \cap A)$ is D_Σ, its frontier is nowhere dense by Corollary 4.7 that $\operatorname{fr}(\pi(\mathbb{R}^m \times B) \cap A))$. Thus E_B is nowhere dense. Define

$$C := \{(r,x,y) \in \mathbb{R}_{>0} \times \mathbb{R}^m \times \mathbb{R}^n \ : \ |y| \leq r^{-1}, (x,y) \in \overline{A}, \operatorname{dist}(y, A_x) \geq r\}$$

Observe that

$$E = \bigcup_{r>0} \pi(C_r).$$

By Corollary 4.6 it is left to show that $\pi(C_r)$ is nowhere dense for each $r > 0$. Suppose not. Let $r \in \mathbb{R}_{>0}$ be such that $\pi(C_r)$ has interior, and let $U \subseteq \mathbb{R}^m$ be an open subset of $\pi(C_r)$. Set $Z := \overline{(C_r)}$, and let $f : U \to \mathbb{R}^m$ map $x \in U$ to $\operatorname{lexmin} Z_x$. By Proposition 5.4 there is an open dense subset V of U such that $f|_V$ is continuous. Note that the graph of f is a subset of Z. By continuity of f, we can find an open box $B \subseteq \mathbb{R}^n$ with side lengths less than r and open subset $W \subseteq V$ such that $f(W) \subseteq B$. Let $(x,y) \in C_r \cap (\mathbb{R}^m \times B)$. Since B has side lengths less than r, we have that $A_x \cap B = \emptyset$. It follows that

$$\pi(C_r \cap (\mathbb{R}^m \times B)) \subseteq E_B.$$

Then

$$W \subseteq \pi(Z \cap (\mathbb{R}^m \times B)) \subseteq \overline{\pi(C_r \cap (\mathbb{R}^m \times B))} \subseteq \overline{E_B}.$$

This contradicts nowhere denseness of E_B. □

Corollary 5.6. *Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be D_Σ. Then $\mathcal{D}(f)$ is nowhere dense.*

Proof. Let $V \subseteq U$ be an open box. It is enough to find an open subset of V that does not intersect $\mathcal{D}(f)$. For each $r \in \mathbb{R}_{>0}$, let

$$V_r := \{x \in V \ : \ |f(x)| \leq r\}.$$

By Corollary 4.6 there is $r \in \mathbb{R}_{>0}$ such that V_r is somewhere dense. Since V_r is D_Σ, it has interior. Then by Corollary 5.5 there is an open subset $W \subseteq A_r$ such that for all $x \in W$

$$\overline{\operatorname{graph}(f)}_x = \{f(x)\}.$$

Thus f is continuous on W. Thus $\mathcal{D}(f) \cap W = \emptyset$. □

Theorem 5.7 (Monotonicity)**.** *Let $f : U \to \mathbb{R}$ be definable and continuous, and let $U \subseteq \mathbb{R}$. Then there is a definable open set $V \subseteq U$ dense in U such that on each connected component of V the restriction of f to V is either strictly increasing, strictly decreasing or constant.*

Proof. Let

$$V_1 = \{x \in U \ : \ f \text{ is constant on an open interval around } x\}$$
$$V_2 = \{x \in U \ : \ f \text{ is strictly increasing on an open interval around } x\}$$
$$V_3 = \{x \in U \ : \ f \text{ is strictly decreasing on an open interval around } x\}.$$

Each V_1, V_2, V_3 is open. Set $V := V_1 \cup V_2 \cup V_3$. Let C be a connected component of V. Observe that there is $i \in \{1, 2, 3\}$ such that C is a connected component of V_i. Furthermore, it is easy to check that if $i = 1$, then f is constant on C, if $i = 2$, then f is strictly increasing on C and, otherwise, f is strictly decreasing on C. It is left to show that $U \setminus (V_1 \cup V_2 \cup V_3)$ has empty interior.

Suppose not. Let I be a compact interval contained in $U \setminus (V_1 \cup V_2 \cup V_3)$. Since $I \cap V_1 = \emptyset$, we have that f is non-constant on I. Since f is continuous, the image $f(I)$ contains an open interval by the intermediate value theorem. Let J be such an open subinterval of $f(I)$. Define the function $g : J \to \mathbb{R}$ such that $g(r) = \min(I \cap f^{-1}(r))$. Observe that g is injective by construction. By Proposition 5.4 this function g is lower-semicontinuous, and thus there is an open subinterval $J' \subseteq J$ such that the restriction of g to J' is continuous. Thus g maps J' homeomorphically onto an subinterval $I' \subseteq I$, and hence the restriction of f to I' is strictly monotone. □

Another fundamental result for o-minimal structures is that every definable function from $\mathbb{R}$ to $\mathbb{R}$ is differentiable outside finitely many points. A similar result holds for continuous functions definable in $\mathcal{R}$.

Theorem 5.8 (Differentiability)**.** *Let $k \in \mathbb{N}_{>0}$, let $f : U \to \mathbb{R}$ be definable and continuous, and let $U \subseteq \mathbb{R}^n$ be open. Then there is an open definable set $V \subseteq U$ dense in U such that the restriction of f to V is C^k.*

Instead of the technical proof in full generality, we will give a quick proof under the assumption that $n = 1$. The full proof can be founded in [29, Lemma 51].

Proof of Theorem 5.8, $n = 1$. Since the derivative of a definable differentiable functions is definable, it is enough to consider the case that $k = 1$. By Theorem 5.7, there is an open dense set $V_0 \subseteq U$ such that f is either strictly monotone or constant on each connected component of V_0. Replacing U by V_0, we can assume that $U = V_0$.

Consider the definable set

$$Y := \{x \in U \ : \ f \text{ is not differentiable at } x\}.$$

We will show that Y is nowhere dense. Suppose towards a contradiction that there is an interval $I \subseteq U$ such that $Y \cap I$ is dense in I. By Lebesgue's differentiability theorem, the set Y is Lebesgue null and hence $Y \cap I$ is also co-dense in I. Set

$$Z := \{(x,h,y) \in I \times \mathbb{R}_{\neq 0} \times \mathbb{R} \ : \ h \neq 0, y = \frac{f(x+h) - f(x)}{h}\}.$$

Note that Z is D_Σ. By Corollary 5.5 there is a dense open definable subset V_1 of I such that $\overline{Z_x} = \overline{Z}_x$ for all $x \in V_1$. Thus for all $x \in V_1 \setminus Y$,

$$(\overline{Z}_x)_0 = (\overline{Z_x})_0 = \{f'(x)\}.$$

Set

$$G := \{(x,y) \in \mathbb{R}^2 \ : \ (x,0,y) \in \overline{Z}\}, \quad H := \{(r,x) \in \mathbb{R}_{>0} \times V_1 \ : \ G_x \subseteq [-r,r]\}.$$

Note that the $\bigcup_{r>0} H_r$ contains $V_1 \setminus Y$, and hence is dense in I. Thus by Corollary 4.6 there is $r \in \mathbb{R}_{>0}$ such that $\overline{H_r}$ has interior. Let J be an open interval contained in this interior, let $g_{\min} : J \to \mathbb{R}$ be the function mapping x to $\min G_x \cap [-r,r]$, and let $g_{\max} : J \to \mathbb{R}$ map x to $\max G_x \cap [-r,r]$. Observe that for all $x \in J \setminus Y$

$$g_{\min}(x) = f'(x) = g_{\max}(x).$$

Using Proposition 5.4, we can find a dense open definable set $V_2 \subseteq J$ such that $g_{\min}$ and $g_{\max}$ are continuous on V_2. Since $g_{\min}$ and $g_{\max}$ agree on $V_2 \setminus Y$ and this set is dense in V_2, we get that $g_{\min} = g_{\max}$ on V_2. It follows that for all $x \in V_2$

$$g_{\min}(x) = g_{\max}(x) = f'(x).$$

This contradicts density of $Y \cap I$ in I.

Thus there is an open dense definable subset $V \subseteq U$ such that f is differentiable on V. The above argument can now be used to find an open dense definable subset of V on which f is C^1. $\square$

6. More on metric geometry

> "A set on which all usual dimensions coincide is called dimensionally concordant. Otherwise, it is a fractal."
> "This definition of a fractal took care of the frontier against Euclid."
>
> Benoît Mandelbrot (slightly rephrased)

In our attempt to make progress toward Miller's conjecture, we now return to metric dimensions. Our goal in this section is to show that all usual dimensions coincide for D_Σ sets definable in expansions of $\overline{\mathbb{R}}$ that do not define $\mathbb{Z}$. Thus, in the above terminology, all such sets are dimensionally concordant, and hence exhibit tameness in the Mandelbrotian sense.

6.1. **Topological dimensions.** We begin by introducing two notions of topological dimension. The first is a classical dimension for topological spaces. Let X be a topological space. We define the **small inductive dimension**, denoted $\operatorname{ind} X$, inductively: if X is empty, then $\operatorname{ind} X = -\infty$, and for $X \neq \emptyset$, we set $\operatorname{ind} X$ to be the infimum over all $k \in \mathbb{N}$ such that for every $x \in X$ and open neighborhood V of x, there exists an open set $U \subseteq X$ such that $x \in \overline{U} \subseteq V$ and the boundary of U (regarded as a topological space via the subspace topology) has small inductive dimension strictly less than k. Here, a subset E of $\mathbb{R}^n$ is equipped with the topology induced by the Euclidean topology on $\mathbb{R}^n$. Thus, $\operatorname{ind} E$ refers to the small inductive dimension with respect to this topology. On such spaces, the small inductive dimension coincides with the large inductive dimension and the Lebesgue covering dimension. We refer to Engelking [23] for more details about these topological dimensions and the proofs of the following facts.

Fact 6.1. *Let $A, B \subseteq \mathbb{R}^n$. Then*

(1) $\operatorname{ind} A = n$ *if and only if A has nonempty interior in $\mathbb{R}^n$,*
(2) $\operatorname{ind} A \leq \operatorname{ind} B$, *if $A \subseteq B$,*
(3) $\operatorname{ind} A = \max_{i\in\mathbb{N}} \operatorname{ind} A_i$ *for $A = \bigcup_{i\in\mathbb{N}} A_i$ with each A_i closed.*
(4) *if A is F_σ and* $\operatorname{ind} A \geq m$, *then there is a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^m$ such that $\pi(A)$ has interior.*

The compact case of (4) is due to Nöbeling, see [23, 1.10.23]. In o-minimality, another notion of topological dimension is often used. Let $E \subseteq \mathbb{R}^n$ be nonempty. The **naive dimension** of E, denoted $\dim E$, is the maximum $m \in \mathbb{N}$ for which there exists a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^m$ such that $\pi(E)$ has interior. We define $\dim \emptyset := -\infty$. This notion of dimension is much more convenient with regard to definability. For example, it is easy to check that whenever $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is definable and $k \in \mathbb{N}$, then

$$\{x \in \mathbb{R}^m : \dim A_x \geq k\}$$

is definable. We now collect the following facts, leaving the easy proofs to the reader.

Fact 6.2. *Let $A, B \subseteq \mathbb{R}^n$. Then*

(1) $\dim A = n$ *if and only if A has nonempty interior.*
(2) $\dim A \leq \dim B$, *if $A \subseteq B$,*
(3) $\dim A \times B = \dim A + \dim B$.

In o-minimal structures these dimensions coincide. However, in general they do not. For example, consider a space filling curve, that is a continuous surjection $f : [0, 1] \to [0, 1]^2$. Its graph has naive dimension 2, as its projection onto the last two coordinates has interior, however its small inductive dimension is 1. Towards our goal to show that all usual dimensions coincide, we first establish that these two topological dimensions agree on D_Σ sets.

Proposition 6.3. *Let $E \subseteq \mathbb{R}^n$ be D_Σ. Then* $\dim E = \operatorname{ind} E$.

Before we give the proof of Proposition 6.3, we need a lemma about the small inductive dimension.

Lemma 6.4. *Let $A \subseteq \mathbb{R}^m$ be D_Σ and let $f : A \to \mathbb{R}^n$ be continuous and definable. Then* $\operatorname{ind} f(A) \leq \operatorname{ind} A$.

Proof. Since A is D_Σ, it can be written as a countable union of compact definable sets. Thus, by Fact 6.1(3) we can assume that A is compact. By Fact 6.1(4) there is a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^{\operatorname{ind} f(A)}$ such that the image of $f(A)$ under π has interior. After replacing f with $\pi \circ f$, we may assume that $n = \operatorname{ind} f(A)$ and that $f(A)$ has nonempty interior. Let $U \subseteq \mathbb{R}^n$ be an open box in this interior. Define $g : U \to \mathbb{R}^m$ by sending $u \in U$ to the lexicographic minimum of $f^{-1}(\{u\}) \cap A$. By Proposition 5.4, there is an open subset $V \subseteq U$ such that $g|_V$ is continuous. Note that $f(g(x)) = x$ for all $x \in V$, and $g(f(y)) = y$ for all $y \in g(V)$. Since g is continuous, it is an homeomorphism. Hence

$$\operatorname{ind} f(A) = \operatorname{ind} V = \operatorname{ind} g(V) \leq \operatorname{ind} A. \qquad \square$$

Proof of Proposition 6.3. By Fact 6.1(4), we already know that $\dim E \geq \operatorname{ind} E$. It is left to show that $\operatorname{ind} E \geq \dim E$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{\dim E}$ be a coordinate projection such that $\pi(E)$ has interior. By Lemma 6.4 and Facts 6.1 and 6.2

$$\operatorname{ind} E \geq \operatorname{ind} \pi(E) = \dim E. \qquad \square$$

Corollary 6.5. *Let $E \subseteq \mathbb{R}$ be D_Σ and nowhere dense, and let $f : E^n \to \mathbb{R}$ be definable and continuous. Then $f(E^n)$ is nowhere dense.*

Proof. By Lemma 6.4 and Proposition 6.3

$$\dim f(E^n) = \operatorname{ind} f(E^n) \leq \operatorname{ind} E^n = \dim E^n = n \dim E = 0.$$

Since $f(E^n)$ is D_Σ, the SBCT implies that $f(E^n)$ is nowhere dense. $\square$

Since the graphs of continuous functions are closed, we can deduce as a corollary that no space-filling curve is definable in $\mathcal{R}$, one of the wild phenomena described by Grothendieck in the quote above. However, there are definable sets in tame expansions for which the naive and small inductive dimensions differ. Returning to Example 5.2, the graph of the characteristic function $\chi_{\mathbb{Q}^{rc}}$ has small inductive dimension 0, since it is totally disconnected. However, the projection onto the first coordinate is $\mathbb{R}$, and hence the naive dimension of the graph is 1.

6.2. **Metric dimensions.** We now turn our attention to metric dimensions. Let $E \subseteq \mathbb{R}^n$. For $r \in \mathbb{R}_{>0}$, let $N(E, r)$ be the minimum number of boxes of side length r needed to cover E. For $E \neq \emptyset$, the **upper Minkowski dimension** of E is defined as

$$\overline{\dim}_M E := \limsup_{r \to 0} \frac{\log(N(E,r))}{\log(\frac{1}{r})}.$$

We set $\overline{\dim}_M \emptyset = -\infty$. We say E is **M-null** if $\overline{\dim}_M E = 0$. The upper Minkowski dimension also appears as upper Minkowski–Bouligand dimension and upper box-counting dimension in the literature.

Fact 6.6. *Let $A, B \subseteq \mathbb{R}^n$ be bounded and let $f : A \to \mathbb{R}^m$ be Lipschitz. Then*

1. $\overline{\dim}_M A = n$ *if A has nonempty interior.*
2. $\overline{\dim}_M A \leq \overline{\dim}_M B$, *if $A \subseteq B$,*
3. $\overline{\dim}_M A \cup B = \max\{\overline{\dim}_M A, \overline{\dim}_M B\}$,
4. $\overline{\dim}_M A = \overline{\dim}_M \overline{A}$,
5. $\overline{\dim}_M f(A) \leq \overline{\dim}_M A$,
6. $\overline{\dim}_M A^k = k \overline{\dim}_M A$,
7. $\overline{\dim}_M (A \times B) \leq \overline{\dim}_M A + \overline{\dim}_M B$.

In contrast to the Hausdorff dimension the upper Minkowski dimension can assign positive values to countable sets. In particular,

$$\overline{\dim}_M\{\frac{1}{n} \ : \ n \in \mathbb{N}_{>0}\} = \frac{1}{2}, \text{ yet } \operatorname{ind}\{\frac{1}{n} \ : \ n \in \mathbb{N}_{>0}\} = 0$$

See [24, Example 3.5] for a proof. Thus the non-definability of $\mathbb{Z}$ is necessary to have equality of upper Minkowski dimension and the topological dimensions even for closed definable subsets of $\mathbb{R}^n$. The following theorem states that this non-definability is also a sufficient condition.

Theorem 6.7 (H.-Miller [41])**.** *Let $E \subseteq \mathbb{R}^n$ be D_Σ and bounded. Then*

$$\operatorname{ind} E = \dim E = \overline{\dim}_M E.$$

If Theorem 6.7 holds for every set definable in $\mathcal{R}^\circ$ and not just for Σ-definable sets, then Miller's conjecture is true. Indeed, suppose that Theorem 6.7 is true in this broader context, and let $E \subseteq \mathbb{R}^n$ be definable in $\mathcal{R}^\circ$. Suppose that E does not have interior, yet is somewhere dense. Let $U \subseteq \mathbb{R}^n$ be an open box such that $U \cap E$ is dense in U. Then

$$\dim(U \cap E) = \overline{\dim}_M(U \cap E) = \overline{\dim}_M \overline{U \cap E} \geq \overline{\dim}_M U = n.$$

Thus, $U \cap E$ has interior, which contradicts our assumption that E does not have interior.

In the statement of Theorem 6.7, we can replace $\overline{\dim}_M E$ with the **Assouad dimension** of E, a "uniform" version of the Minkowski dimension. To be precise, the Assouad dimension of E is defined as the infimum over all α such that there exist constants C and ρ with the property that, for $0 < r < R < \rho$, we have

$$\sup_{x \in E} N(B_R(x) \cup E, r) \leq C \left(\frac{R}{r}\right)^\alpha.$$

Here, we will only provide the proof of Theorem 6.7 for $n \in \{1, 2\}$ and the upper Minkowski dimension. When $n = 2$, we will also assume that E is compact. This simplifies the argument by allowing us to ignore the bookkeeping necessary to make the argument uniform for $n > 1$, while still explaining the key trick in the proof from [41].

Luukainen [52] writes, using $\overline{\dim}_B$ instead of $\overline{\dim}_M$,

> "A *fractal* might be defined - adapting Mandelbrot - as non-empty compact metric space X for which at least one of the always valid inequalties
>
> $$\dim X \leq \dim_H X \leq \underline{\dim}_B X \leq \overline{\dim}_B X \leq \dim_A X$$
>
> for the topological, Hausdorff, lower box-counting, upper box-counting, and Assouad, respectively, of X is strict. [...] On the other hand, if the inequalities above are all equalities, that is, if $\dim_A X = \dim X$, then the compact metric space X and its metric might be called *antifractal* or *flat*."

Using this notation, Theorem 6.7 gives that all definable compact subsets of $\mathbb{R}^n$ are antifractal or dimensionally corcordant in Mandelbrot's terminology.

Proof of Theorem 6.7, assuming $\mathcal{R}^\circ$ is o-minimal. Since E is D_Σ, it is definable in $\mathcal{R}^\circ$. By Kurdyka and Parusiński [51], there are $N \in \mathbb{N}$ and for each $i = 1, \dots, N$ a bi-Lipshitz map $f_i : U_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^n$ such that

(1) $n_1 \leq \cdots \leq n_N = \dim E$,
(2) U_i is open for $i = 1, \dots, N$, and
(3) $E = \bigcup_{i=1}^N f_i(U_i)$.

By Fact 6.6(5), we have that for every $i = 1, \dots, N$

$$\overline{\dim}_M f_i(U_i) = \overline{\dim}_M U_i = n_i.$$

Thus by Fact 6.6(3)

$$\overline{\dim}_M E = \max\{n_i \ : \ i = 1, \dots, N\} = n_N = \dim E.$$

□

6.3. **The unary case.** We begin by proving the case $n = 1$ of Theorem 6.7. As an input we use the following variant of Marstrand's projection theorem for upper Minkowski dimension.

Theorem 6.8 (Projection lemma, Falconer-Howroyd [25])**.** *Let $m, n \in \mathbb{N}$ be such that $m \leq n$, and let $E \subseteq \mathbb{R}^n$ be analytic. Then there is a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\overline{\dim}_M T(E) \geq \frac{\overline{\dim}_M E}{1 + (\frac{1}{m} - \frac{1}{n})\overline{\dim}_M E}.$$

Corollary 6.9. *Let $E \subseteq \mathbb{R}$ be analytic such that $\overline{\dim}_M E > 0$. Then there is $n \in \mathbb{N}$ and a linear map $T : \mathbb{R}^n \to \mathbb{R}$ such that $\overline{\dim}_M T(E^n) > \frac{1}{2}$.*

Proof. By Theorem 6.8, for every $n \in \mathbb{N}$ there is a linear map $T : \mathbb{R}^n \to \mathbb{R}$ such that

$$\overline{\dim}_M T(E^n) \geq \frac{\overline{\dim}_M E^n}{1 + (1 - \frac{1}{n}) \dim E^n} = \frac{n\overline{\dim}_M E}{1 + (n-1)\overline{\dim}_M E} = \frac{\overline{\dim}_M E}{\frac{1}{n} + (1 - \frac{1}{n})\overline{\dim}_M E}.$$

The right hand side goes to 1 as $n \to \infty$. □

For a subset $X \subseteq \mathbb{R}$, the **set of difference quotients** of X is defined as

$$Q(X) := \{\frac{x_1 - x_2}{x_3 - x_4} \ : \ x_1, x_2, x_3, x_4 \in X, x_3 \neq x_4\}.$$

Note that $Q(X)$ is the set of slopes of non-vertical lines connecting distinct points in X^2.

Lemma 6.10 (Falconer's trick)**.** *Let $I \subseteq \mathbb{R}$ be an open interval. Then there is $c \in \mathbb{R}_{>0}$ and a rotation $Z : \mathbb{R}^2 \to \mathbb{R}^2$ such that for every $E \subseteq \mathbb{R}$ with $Q(E) \cap I = \emptyset$ there is a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant c such that $Z(E^2)$ is contained in the graph of f.*

Proof. Let $a, b \in \mathbb{R}$ be such that $I = (a, b)$. Set

$$C := \{(x, y) \in \mathbb{R}_{>0} \times \mathbb{R} : ax < y < bx\} \cup \{(x, y) \in \mathbb{R}_{<0} \times \mathbb{R} : bx < y < ax\}.$$

Observe that C is an open double cone centered at the origin. Let ℓ be the axis of C, and $\ell^\perp$ be the line through the origin perpendicular to ℓ. Let Z be the counter-clockwise rotation of $\mathbb{R}^2$ mapping ℓ to the y-axis and $\ell^\perp$ to the x-axis. Note that $Z(C)$ is again an open double cone centered at the origin. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the coordinate projection onto the first coordinate. See Figure 5 for a visualization.

Let $E \subseteq \mathbb{R}$ be such that $Q(E) \cap I = \emptyset$. We are going to show that $Z(E^2)$ is contained in the graph of a Lipschitz function with the Lipschitz constant just depending on C, but not on E. First, we observe that $\{u - v \ : \ u, v \in E^2\}$ is disjoint from C. Hence

$$\{Zu - Zv \ : \ u, v \in E^2\} = Z(\{u - v \ : \ u, v \in E^2\})$$

is disjoint for $Z(C)$. Thus, the restriction of π to $Z(E)^2$ is injective. Indeed, if $\pi(Zu) = \pi(Zv)$ for $u, v \in E^2$ with $u \neq v$, then $Zu - Zv \in Z(\ell)$, and in particular, in $Z(C)$. This is a contradiction. Let $g : \pi(Z(E^2)) \to \mathbb{R}$ map x to the unique $y \in \mathbb{R}$ such that $(x, y) \in Z(E^2)$. Note that the graph of g contains $Z(E^2)$. Since $\{Zu - Zv \ : \ u, v \in E^2\} \cap Z(C) = \emptyset$, it is easy to see that g is Lipschitz with a Lipschitz constant depending only on the cone $Z(C)$. By Kirszbraun's theorem [50], we can extend g to a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ without changing the Lipschitz constant. ☐

Falconer's trick is implicit in the proof of [30, Lemma]. It is due to Kenneth Falconer and replaces, in [30], an explicit box-counting argument that relied on additive combinatorics.

Lemma 6.11 (Fornasiero-H.-Miller [30])**.** *Let $E \subseteq \mathbb{R}$ be bounded such that $\overline{\dim}_M E > 0$. Then there exist $n \in \mathbb{N}$ and a linear $T : \mathbb{R}^n \to \mathbb{R}$ such that $Q(T(E^n))$ is dense in $\mathbb{R}$.*

Proof. By Corollary 6.9 there are $n \in \mathbb{N}$ and $T : \mathbb{R}^n \to \mathbb{R}$ linear such that $\overline{\dim}_M T(E^n) > \frac{1}{2}$. Replacing E by $T(E^n)$, we may assume that $\overline{\dim}_M(E) > \frac{1}{2}$ and show that $Q(E)$ is dense. Towards a contradiction, suppose that $Q(E)$ is not dense in $\mathbb{R}$. By Lemma 6.10, there is a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ and a rotation $Z : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Z(E^2)$ is contained in the graph of f. The graph of a Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$ has upper Minkowski dimension 1 by Fact 6.6(5). Since Z is bi-Lipschitz, we have that

$$\overline{\dim}_M E^2 = \overline{\dim}_M Z(E^2) \leq 1.$$

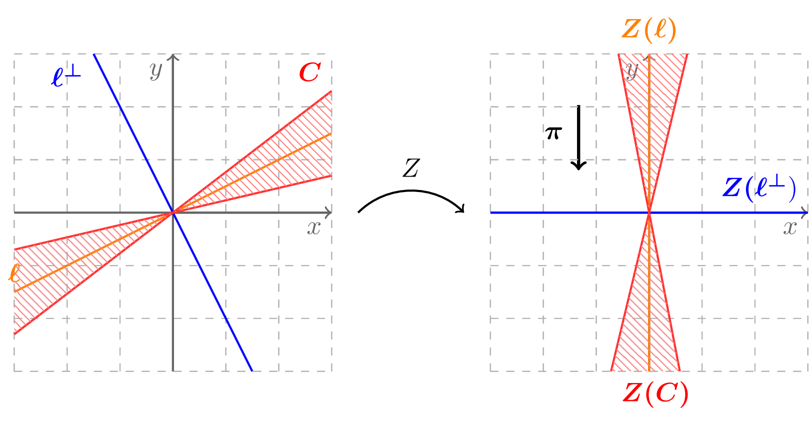

FIGURE 5. Visualization of Falconer's trick in Lemma 6.10

Thus
$$\overline{\dim}_M E = \frac{\overline{\dim}_M E^2}{2} \leq \frac{1}{2}.$$
This contradicts our assumption on E. $\square$

Proof of Theorem 6.7, $n = 1$. Let $E \subseteq \mathbb{R}$ be D_Σ. Then by SBCT, we know that E either has interior or is nowhere dense. If E has interior,
$$\dim E = 1 = \overline{\dim}_M E.$$
Now suppose that E is nowhere dense. Then $\dim E = 0$, and it is left to show that $\overline{\dim}_M E = 0$. Suppose towards a contradiction that $\overline{\dim}_M E > 0$. By Lemma 6.11 there exist $n \in \mathbb{N}$ and a linear $T : \mathbb{R}^n \to \mathbb{R}$ such that $Q(T(E^n))$ is dense in $\mathbb{R}$. This contradicts Corollary 6.5. $\square$

6.4. **A stronger Projection Lemma and uniform M-nullness.** For the proof of Theorem 6.7 when $n = 2$, we need a uniform version of the result in the unary case. To achieve this, we require a minor strengthening of Theorem 6.8. As in [25], we denote by $G_{n,m}$ the Grassmannian of all m-dimensional subspaces of $\mathbb{R}^n$, and by $\gamma_{n,m}$ the natural invariant measure on $G_{n,m}$, which we assume to be normalized. For $V \in G_{n,m}$, we write $\pi_V : \mathbb{R}^n \to V$ for the orthogonal projection onto V.

Lemma 6.12. *(*[25, Corollary 2]*) Let $E \subseteq \mathbb{R}^n$, $r \in \mathbb{R}_{>0}$ and $s \in \mathbb{R}$ such that $N(E, r) \geq r^{-s}$. Then*
$$\int_{G_{n,m}} \frac{1}{N(\pi_V(E), h)} \mathrm{d}\gamma_{n,m}(V) \leq (c+1) r^s,$$
where c only depends on n and m, and $h = r^{1+s(1/m-1/n)}$.

We will now use this lemma to strengthen [25, Theorem 3(a)]. The proof of this strengthening is essentially the same as that of the original result.

Lemma 6.13 (Uniform projection lemma)**.** *Let $X \subseteq \mathbb{R}^m \times \mathbb{R}^n$, $s \in \mathbb{R}$, $\varepsilon \in \mathbb{R}_{>0}$ and let $(a_k)_{k\in\mathbb{N}}$ be a sequence of elements of $\mathbb{R}^m$ and $(r_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers such that $\lim_{k\to\infty} r_k = 0$ and for all $k \in \mathbb{N}$*
$$N(X_{a_k}, r_k) \geq r_k^{-s}.$$
Then there is a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ and a sequence $(k_\ell)_{\ell\in\mathbb{N}}$ of natural numbers such that for all $\ell \in \mathbb{N}$
$$N(T(X_{a_{k_\ell}}), h_{k_\ell}) \geq h_{k_\ell}^{-t_\varepsilon},$$
where $t_\varepsilon = \frac{s-\varepsilon}{1+s(1/m-1/n)}$ and $h_k := r_k^{1+s(1/m-1/n)}$.

Proof. By Lemma 6.12,
$$\int_{G_{n,m}} \frac{1}{N(\pi_V(E_{a_k}), h_k)} \mathrm{d}\gamma_{n,m}(V) \leq (c+1) r_k^s.$$
Hence
$$\gamma_{n,m}\big(\{V \in G_{n,m} \ : \ \frac{1}{N(\pi_V(E_{a_k}), h_k)}\big) \geq r_k^{s-\varepsilon}\}\big) \leq (c+1) r_k^{\varepsilon}.$$

Let $t_\varepsilon := \frac{s-\varepsilon}{1+s(1/m-1/n)}$. Then

$$\bigcup_{j=1}^{\infty}\bigcap_{k\geq j}\{V \in G_{n,m} \ : N(\pi_V(E_{a_k}), h_k) \leq h_k^{t_\varepsilon}\}$$
$$\subseteq \bigcup_{j=1}^{\infty}\bigcap_{k\geq j}\{V \in G_{n,m} \ : \ \frac{1}{N(\pi_V(E_{a_k}), h_k)} \geq r_k^{s-\varepsilon}\},$$

has $\gamma_{n,m}$-measure 0. Pick V in the complement of this set, and let T be π_V. □

We say a family $(X_a)_{a\in Y}$ of subsets of $\mathbb{R}^n$ is **uniformly M-null** if

$$\limsup_{r\to 0} \frac{\log(N(X_a, r))}{\log \frac{1}{r}} = 0 \text{ uniformly in } a.$$

Proposition 6.14. *Let $E \subseteq [0,1]^{n+1}$ be D_Σ such that E_a is nowhere dense for all $a \in [0,1]^n$. Then $(E_a)_{a\in[0,1]^n}$ is uniformly M-null.*

Proof. Suppose not. Then there is $s \in (0,1)$ such that for arbitrarily small $r \in \mathbb{R}_{>0}$ there is $a \in [0,1]^m$ with

$$N(E_a, r) \geq r^{-s}.$$

Pick a sequence $(r_k)_{k\in\mathbb{N}}$ of real numbers with $\lim_{k\to\infty} r_k = 0$ and a sequence $(a_k)_{k\in\mathbb{N}}$ of elements of $[0,1]^m$ such that for every $k \in \mathbb{N}$

$$N(E_{a_k}, r_k) \geq r_k^{-s}.$$

Note that every $d \in \mathbb{N}$ and $k \in \mathbb{N}$

$$N(E_{a_k}^d, r_k) > r_k^{-sd}.$$

We now apply Lemma 6.13. Taking $d \in \mathbb{N}$ large enough, we can find by an $t \in \mathbb{R}$ with $t > 1/2$ and a linear map $T : \mathbb{R}^n \to \mathbb{R}$ such that (after replacing $(a_k)_{k\in\mathbb{N}}$ by a subsequence and r_k by h_k) for all $k \in \mathbb{N}$

$$N(T(E_{a_k}^d), r_k) \geq r_k^{-t}.$$

Consider now the set

$$A := \{(a,x) \in [0,1]^n \times [0,1] : x \in Q(T(E_a^n))\}.$$

By Corollary 6.5, the fiber A_a is nowhere dense for every $a \in [0,1]^n$. Since the graph of $Q \circ T$ is D_Σ, we have that A is D_Σ as well. Thus by Corollary 4.4 there are intervals $J_1, \dots, J_p \subseteq \mathbb{R}$ such that for each $a \in [0,1]^m$ there is $i \in \{1, \dots, p\}$ with $J_i \subseteq [0,1] \setminus A_a$. From Lemma 6.10 we obtain Lipschitz constants $c_1, \dots, c_p$ and rotations $Z_1, \dots, Z_p$ such that for each $a \in [0,1]^n$ there is $i \in \{1, \dots, p\}$ and a Lipschitz function $f_a : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant c_i such that

$$T(E_a^d)^2 \subseteq Z_i\big(\operatorname{graph}(f_a)\big).$$

Observe that there is $r_0 > 0$ such that for all $r < r_0$

$$N(T(E_a^d)^2, r) \leq N(Z_i \operatorname{graph}(f_a), r) < r^{-2t}.$$

However for all $k \in \mathbb{N}$,

$$N(T(E_a^d)^2, r_k) \geq r_k^{-2t}.$$

□

6.5. **The planar case.** We consider the case $n = 2$ and E compact of Theorem 6.7. Let $E \subseteq [0,1]^2$ be compact. For $x, y \in [0,1]$, we will use the following notation:

$$E_x := \{z \in [0,1] \ : \ (x,z) \in E\}, \ E^y := \{z \in [0,1] \ : \ (z,y) \in E\}.$$

Let $f_E : [0,1]^2 \to [0,1]$ be the function mapping (x,y) to

$$\min\big((E_x \cup \{2\}) \cap [y,\infty)\big).$$

For $y \in [0,1]$, we write $f_E(\_,y)$ for the function mapping $x \in [0,1]$ to $f_E(x,y)$. We need to analyze points of discontinuity of these functions. For $\varepsilon \in \mathbb{R}_{>0}$, set

$$\mathcal{D}_\varepsilon(E) := E \cup \{(x,y) \in [0,1]^2 \ : \ x \in \mathcal{D}_\varepsilon(f_E(\_,y))\}.$$

Set $\mathcal{D}(E) := \bigcup_{\varepsilon>0} \mathcal{D}_\varepsilon(E)$.

Lemma 6.15. *Let $\varepsilon > 0$. Then $\mathcal{D}_\varepsilon(E)$ is closed, and hence $\mathcal{D}(E)$ is D_Σ.*

Proof. Let $(x_0,y_0) \in [0,1]^2 \setminus \mathcal{D}_\varepsilon(E)$. Since $(x_0,y_0) \notin E$, there is $\delta_0 \in \mathbb{R}$ that

$$B_{\delta_0}((x,y)) \subseteq [0,1]^2 \setminus E.$$

Thus for every $(x,y) \in B_{\delta_0}((x_0,y_0))$,

$$f_E(x,y) = f_E(x,y_0). \tag{1}$$

Since $x_0 \notin \mathcal{D}_\varepsilon(f_E(\_,y_0))$, there is a $\delta_1 \in \mathbb{R}_{>0}$ such that for all $z \in B_{\delta_1}(x_0)$,

$$f_E(B_{\delta_1}(x_0),y_0) \subseteq B_\varepsilon(f_E(z,y_0)). \tag{2}$$

Let $\delta \in \mathbb{R}_{>0}$ be such that $\delta + \delta_1 < \delta_0$, and let $(x,y) \in B_\delta((x_0,y_0))$. Then for each $z \in B_{\delta_1}(x)$, we have that $(z,y) \in B_{\delta_0}((x_0,y_0))$ and by (1)

$$f_E(z,y) = f_E(z,y_0).$$

Thus by (2) for every $z \in B_{\delta_1}(x)$,

$$f_E(B_{\delta_1}(x),y) = f_E(B_{\delta_1}(x),y_0) \subseteq B_\varepsilon(f_E(z,y_0)) = B_\varepsilon(f_E(z,y)).$$

Hence $x \notin \mathcal{D}_\varepsilon(f_E(\_,y))$. Thus $B_\delta((x_0,y_0)) \subseteq [0,1]^2 \setminus \mathcal{D}_\varepsilon(E)$. □

Proof of Theorem 6.7, $n = 2$, E compact. If E has interior, then

$$\dim E = \overline{\dim}_M E = 2.$$

Hence, we can assume that E does not have interior. If $\dim E = 0$, then the image of E under each coordinate projection has empty interior, and since E is closed, it is nowhere dense. By case $n = 1$ of Theorem 6.7, these two sets are M-null. Thus, E is a subset of the product of two M-null sets, and therefore, $\overline{\dim}_M E = 0$.

We have reduced to the case where $\dim E = 1$. Consider the definable sets

$$A_1 := \{x \in [0,1] : E_x \text{ has interior}\}, \quad A_2 := \{y \in [0,1] : E^y \text{ has interior}\}.$$

Then, A_1 and A_2 are both D_Σ by Lemma 4.1(4), and each either has interior or is nowhere dense by the SBCT. Since E does not have interior, both A_1 and A_2 do not have interior by Fact 4.5. The unary case of Theorem 6.7 then implies that A_1 and A_2 are M-null. Set $A := \overline{A_1 \cup A_2}$. Since A_1 and A_2 are M-null, so is A. By Fact 6.6(7),

$$\overline{\dim}_M(A \times [0,1]) \leq 1.$$

Towards a contradiction, suppose that $\overline{\dim}_M E > d > 1$. Set

$$E' := E \setminus (A \times [0,1]).$$

Then, $\overline{\dim}_M E' > d$ by Fact 6.6(3). Since E is compact and A is closed, it follows that E' is D_Σ. Since E'_x has empty interior for every $x \in [0,1]$, we know that E'_x is nowhere dense for every $x \in [0,1]$. Hence, by Proposition 6.14, the families $(E'_x)_{x\in[0,1]}$ and $(E'^y)_{y\in[0,1]}$ are uniformly M-null. By Lemma 6.15, the set $\mathcal{D}(E)$ is D_Σ. By Proposition 5.4, we know that each $\mathcal{D}(E)^y$ is nowhere dense whenever $y \notin A$. Observe that $\mathcal{D}(E) \cap E'$ is D_Σ. Hence, the family $(\mathcal{D}(E)^y \cap E'^y)_{y\in[0,1]}$ is uniformly M-null by Proposition 6.14.

We finally reach the moment when we have to do actual box counting. To do this efficiently, it is convenient to introduce further notation. For $X \subseteq \mathbb{R}^n$ and $m \in \mathbb{N}$, set

$$N_m(X) := \operatorname{card}\{(i,j) \in \{0,\dots,m-1\}^2 \ : \ m^{-1}([i,i+1]\times[j,j+1]) \cap X\}.$$

Observe that

$$2^{-n}N_m(X) \le N(X,m^{-1}) \le N_m(X).$$

Since these three families are uniformly M-null and $\overline{\dim}_M E' > d$, we can pick an $m \in \mathbb{N}$ and $p \in \mathbb{R}_{>0}$ such that

(1) $d > 1 + p$,
(2) $N_m(E') > 4m^d$,
(3) $N_m(A \times [0,1]) \le m^{1+p}$,
(4) for $x \in [0,1]$ and $y \in [0,1]$,

$$\max\{N_m(E'_x), N_m(E'^y), N_m(\mathcal{D}(E)^y \cap E'^y)\} < m^p.$$

For convenience, set

$$\begin{aligned} F := \{(i,j) \in \{0,\dots,m-1\}^2 \ : \ & m^{-1}([i,i+1]\times[j,j+1]) \cap E' \neq \emptyset, \\ & m^{-1}([i,i+1]\times[j,j+1]) \cap (A\times[0,1]) = \emptyset\}. \end{aligned}$$

Set

$$G := \{(i,j) \in F : m^{-1}([i,i+1]\times\{j\}) \cap \mathcal{D}(E) = \emptyset\}.$$

Here, F corresponds to the set of boxes that intersect E', but not E, and G represents the boxes which, in addition, do not contain any of the discontinuities of E. For ease of notation, we will write f_j for $f_E(\_, \frac{j}{m})$.

We now proceed with some elementary counting arguments: first, note that by (2) and (3),

$$\operatorname{card}(F) \ge N_m(E') - N_m(A\times[0,1]) \ge 4m^d - m^{1+p} \ge 3m^d.$$

By (4),

$$\operatorname{card}(F \setminus G) \le m \cdot m^p = m^{1+p}.$$

Thus, by (1) and (2),

$$\operatorname{card}(G) = \operatorname{card}(F) - \operatorname{card}(F\setminus G) \ge 3m^d - m^{1+p} \ge 2m^d.$$

Set

$$H := \{(i,j) \in F : f_j(m^{-1}[i,i+1]) \subseteq m^{-1}[j,j+1]\}.$$

Consider $(i,j) \in G \setminus H$. Then

- f_j is continuous on $m^{-1}[i,i+1]$
- there is $x_0 \in m^{-1}[i,i+1]$ such that $f_j(x_0) \in m^{-1}([j,j+1])$
- there is $x_1 \in m^{-1}[i,i+1]$ such that $f_j(x_1) \notin m^{-1}([j,j+1])$

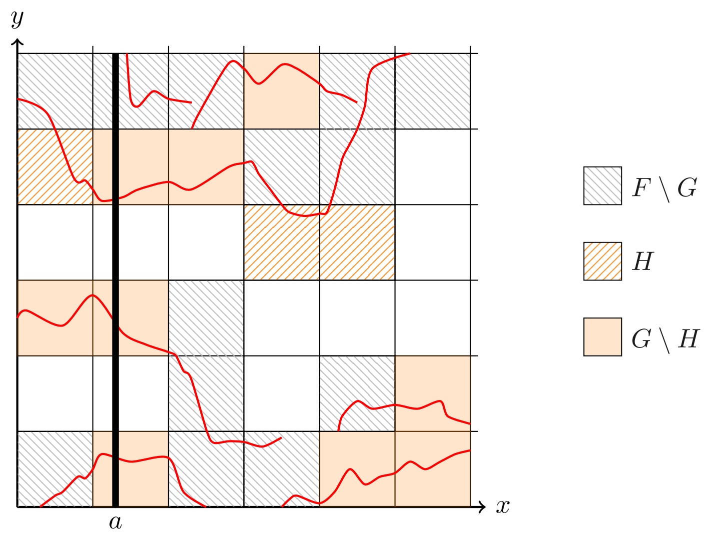

FIGURE 6. A visualization of the counting argument in the proof of the planar case of Theorem 6.7. The ways lines indicate the graphs of the various f_j.

Thus, by the intermediate value theorem, there exists $x_2 \in m^{-1}([i, i+1])$ such that $f_j(x_2) = m^{-1}(j+1)$. So

$$[i, i+1] \cap E'^{m^{-1}(j+1)} \neq \emptyset.$$

Hence by (4)

$$\operatorname{card}(G \setminus H) \leq \sum_{j=0}^{m-1} N_m(E'^{m^{-1}(j+1)}) < m \cdot m^p = m^{1+p} < m^d.$$

Thus

$$\operatorname{card}(H) = \operatorname{card}(G) - \operatorname{card}(G \setminus H) > 2m^d - m^d = m^d.$$

Since $1 + p < d$, there is $i_0 \in \mathbb{N}$ such that

$$\operatorname{card}\{j \in \mathbb{N}_{>0} \ : \ (i_0, j) \in H\} > m^p.$$

Fix this i_0. Let $a \in m^{-1}[i_0, i_0 + 1]$ and let $j \in \mathbb{N}_{>0}$ be such that $(i_0, j) \in H$. Since $(i_0, j) \in H$,

$$f_j(a) \in m^{-1}[j, j+1].$$

Hence, $[j/m, (j+1)/m] \cap E'_a \neq \emptyset$. Thus

$$N_m(E'_a) \geq \operatorname{card}\{j \in \mathbb{N}_{>0} \ : \ (i_0, j) \in H\} > m^p,$$

contradicting (4).

□

Figure 6 visualizes the counting argument from the preceding proof. The wavy lines represent the graphs of the various f_j. The striped boxes, with stripes running from the top-left to the bottom-right, correspond to boxes whose bottom-left corners lie in $F \setminus G$; that is, boxes in F where the corresponding f_j has points of discontinuity. The striped boxes with stripes running from the bottom-left to the top-right correspond to boxes whose bottom-left corners are in H; that is, boxes in F where f_j is continuous but leaves the box. The shaded boxes correspond to boxes with bottom-left corners in $G \setminus H$.

In the above proof, we show that if E' has a large upper Minkowski dimension, then many boxes have bottom-left corners in $G \setminus H$. This leads to a contradiction with the fact that the fibers of E' are uniformly M-null. Indeed, while each fiber E'_a does not necessarily intersect each box containing an element of E', it must intersect every striped box, as highlighted by the thick vertical line. This is the key observation in the proof: if there are many shaded boxes, there must be $a \in [0, 1]$ such that E'_a intersects too many boxes.

7. Outlook and open questions

While Miller's conjecture has been the main focus of this survey, there are other active directions in this research program. We list some of these here.

7.1. **Generalizations of Pila-Wilkie.** The Pila-Wilkie theorem [62], which counts rational points on definable sets in o-minimal structures, has been a driving force behind the recent surge of applications of o-minimality, particularly in arithmetic geometry. Surprisingly, it remains an open question whether this theorem generalizes to noiseless expansions of $\overline{\mathbb{R}}$. This question is explicitly raised in Comte and Miller [10], where the authors establish Pila-Wilkie results over the real numbers, but outside the o-minimal context. For certain non-o-minimal expansions of $\overline{\mathbb{R}}$ that have an o-minimal open core, Eleftheriou [21] proves an appropriate version of the Pila-Wilkie theorem, accounting for the additional noise in these structures.

7.2. **Definable completeness.** In this survey, we focus on expansions of the real field. A similar analysis can be conducted for expansions of arbitrary real closed fields that satisfy a definable analogue of topological completeness. We say an expansion of a dense linear order $(R, <)$ is **definably complete** if every bounded definable subset of R has both a supremum and an infimum in R. Such structures are introduced by Miller [54] under the name *structures with the intermediate value property.* Noiselessness in this generality was already studied in [28], and a dichotomy corresponding to Theorem 1.3 is established by Fornasiero and Hieronymi [29, Theorem A]. The results presented in Section 5 are already proven in [29, Section 6.3] for definably complete expansions that do not define a discrete subring. Feller [26] extends some of the results from Section 3 to definably complete expansions of ordered groups. The advantage of definable completeness lies in its first-order expressibility. Since it is preserved under elementary equivalence, the compactness theorem can be applied to establish uniformity in parameters.

7.3. **A more detailed analysis of noiselessness.** Noiseless structures have been studied in [55], and the results from Section 5 hold for all sets definable in such structures, simply because the conclusion of the SBCT holds in this generality. While this provides analogues of important theorems from o-minimality, there is no known analogue of the cell decomposition theorem for these structures. The methods used in the current proofs of the cell decomposition theorem for d-minimal structures are unlikely to extend, as they rely on the existence of isolated points in subsets of $\mathbb{R}$ that are definable in a d-minimal structure and have empty interior.

7.4. **Classification.** Due to the intractability of Goal 1.1, we replaced interdefinability with a less well-defined, coarser equivalence relation. There is another way to address the apparently insoluble Goal 1.1: replace the set of all expansions of $\overline{\mathbb{R}}$ with some interesting subset. We already mentioned the classification of all expansions of $\overline{\mathbb{R}}$ by collections of locally closed trajectories of a linear vector field in [57]. See Miller's chapter in this volume for more on classifications of expansions by trajectories of more general vector fields. It is also natural to investigate expansions by algebraic objects. Hieronymi, Walsberg, and Xu [46] establish a classification of expansions of $\overline{\mathbb{R}}$ by infinite discrete subgroups of $GL_n(\mathbb{C})$, the general linear group of degree n over the field of complex numbers. There, the question is raised whether a similar classification holds for expansions by finite-rank subgroups of $GL_n(\mathbb{C})$. Another very interesting question is whether a classification can be obtained for expansions of $\overline{\mathbb{R}}$ by subfields of $\mathbb{R}$. Perhaps here, interdefinability is again too fine of an equivalence relation, and a more realistic goal is a coarser classification up to some form of tameness. Indeed, Miller asks in [55] whether every expansion of $\overline{\mathbb{R}}$ by a proper subfield of $\mathbb{R}$ either defines $\mathbb{Z}$ or has o-minimal open core. This question remains open.

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