The Derived Category of a Dynkin Quiver

Tim Seynnaeve

1 Derived Category of a Hereditary Abelian Category

Lemma 1.1. *Let $\mathcal{A}$ be an abelian category. Suppose $\mathcal{A}$ has enough projectives. Let A and B be objects in $\mathcal{A}$; $n \in \mathbb{Z}$. Then there is a canonical isomorphism*

$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A[0],B[n])$$

where we define $\mathrm{Ext}^n_{\mathcal{A}}(A,B) := 0$ for $n < 0$.

Proof. Let P^* be a projective resolution for A. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A[0],B[n]) &\cong \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(P^*,B[n])\\ &\cong \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(P^*,B[n])\\ &\cong H^{-n}(\mathrm{Hom}_{\mathcal{A}}(P^*,B))\\ &= \mathrm{Ext}^n_{\mathcal{A}}(A,B)\end{aligned}$$

where the second isomorphism follows from proposition 2.7 in talk 5.
The third isomorphism follows from the observation that giving a chain map $P^* \to B[n]$ amounts to giving a homomorphism $f : P^{-n} \to B$ s.t. $f \circ d^{-n-1} = 0$ (i.e. f is a cocycle), and that such a map is homotopic to zero if and only if there is a $g : P^{-n+1} \to B$ s.t. $f = g \circ d^{-n}$ (i.e. f is a coboundary). □

Remark 1.2. It follows immediately that for $n, m \in \mathbb{Z}$:

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A[n],B[m]) \cong \mathrm{Ext}^{m-n}_{\mathcal{A}}(A,B)$$

Remark 1.3. Recall from homological algebra: Yoneda's theory of extensions:
There is a bijection between $\mathrm{Ext}^1_{\mathcal{A}}(A,B)$ and the set of equivalence classes of extensions; which are short exact sequences of the form

$$0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0$$

(+ analogue for higher Ext-groups).
We can use this to define the Yoneda composition

$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) \times \mathrm{Ext}^m_{\mathcal{A}}(B,C) \to \mathrm{Ext}^{n+m}_{\mathcal{A}}(A,C)$$

For example, for $A \to B$ in $\mathrm{Ext}^0_{\mathcal{A}}(A,B) = \mathrm{Hom}_{\mathcal{A}}(A,B)$ and $C \to X \to B$ in $\mathrm{Ext}^1_{\mathcal{A}}(B,C)$, the Yoneda composition is given by

$$C \longrightarrow X \times_B A \longrightarrow A$$

One can check that under the identification $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A[n], B[m]) \cong \mathrm{Ext}^{m-n}_{\mathcal{A}}(A,B)$, the composition of morphisms $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A[l], B[m]) \times \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(B[m], C[n]) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A[l], C[n])$ corresponds to the Yoneda composition $\mathrm{Ext}^{m-l}_{\mathcal{A}}(A,B) \times \mathrm{Ext}^{n-m}_{\mathcal{A}}(B,C) \to \mathrm{Ext}^{n-l}_{\mathcal{A}}(A,C)$.

Recall that an abelian category is called semisimple, if every short exact sequence splits, or equivalentely, if $\mathrm{Ext}^1(\_,\_) = 0$.
In a previous talk, we proved that for $\mathcal{A}$ semisimple, the derived category $\mathrm{D}(\mathcal{A})$ is equivalent to $\mathcal{A}^{\mathbb{Z}}$. We will now slightly generalize this result.

Definition 1.4. An abelian category is called *hereditary* if $\mathrm{Ext}^2(\_,\_) = 0$ (or equivalentely, if $\mathrm{Ext}^n(\_,\_) = 0$ for all $n \geq 2$).

Examples 1.5.

- Any semisimple abelian category is hereditary.
- The category $\mathrm{Rep}_k Q$ of k-linear representations of a quiver Q is hereditary. (See later in this talk.)

Proposition 1.6. *If $\mathcal{A}$ is a hereditary abelian category, then every object in $\mathrm{D}(\mathcal{A})$ is isomorphic to a chain complex with all differentials 0.*

Proof. Let X^* be a chain complex. We want to show that in $\mathrm{D}(\mathcal{A})$, X^* is isomorphic to

$$\cdots \xrightarrow{0} H^{n-1}X \xrightarrow{0} H^nX \xrightarrow{0} H^{n+1}X \xrightarrow{0} \cdots$$

The short exact sequence

$$0 \longrightarrow \ker d^{n-1} \longrightarrow X^{n-1} \longrightarrow \operatorname{im} d^{n-1} \longrightarrow 0$$

yields a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(H^nX, \ker d^{n-1}) \to \quad & \mathrm{Hom}(H^nX, X^{n-1}) \to \mathrm{Hom}(H^nX, \operatorname{im} d^{n-1}) \\ \to \mathrm{Ext}^1(H^nX, \ker d^{n-1}) \to \quad & \mathrm{Ext}^1(H^nX, X^{n-1}) \to \mathrm{Ext}^1(H^nX, \operatorname{im} d^{n-1}) \\ \to \mathrm{Ext}^2(H^nX, \ker d^{n-1}) \to \quad & \cdots \end{aligned}$$

Since $\mathrm{Ext}^2(H^nX, \ker d^{n-1}) = 0$, the map $\mathrm{Ext}^1(H^nX, X^{n-1}) \to \mathrm{Ext}^1(H^nX, \operatorname{im} d^{n-1})$ is surjective. In particular, the extension

$$0 \to \operatorname{im} d^{n-1} \to \ker d^n \to H^nX \to 0$$

has an inverse image, i.e. there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^{n-1} & \longrightarrow & E^n & \longrightarrow & H^nX & \longrightarrow & 0 \\ & & \downarrow & \ulcorner & \downarrow & & \| & & \\ 0 & \longrightarrow & \operatorname{im} d^{n-1} & \longrightarrow & \ker d^n & \longrightarrow & H^nX & \longrightarrow & 0 \end{array}$$

with exact rows.
We obtain a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^nX & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & X^{n-1} & \longrightarrow & E^n & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots
\end{array}$$

where the vertical maps induce cohomology isomorphisms in degree n.
Combining these diagrams for all n:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^{n-2}X & \xrightarrow{0} & H^{n-1}X & \xrightarrow{0} & H^nX & \xrightarrow{0} & H^{n+1}X & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & X^{n-2} \oplus E^{n-2} & \longrightarrow & X^{n-1} \oplus E^{n-1} & \longrightarrow & X^n \oplus E^n & \longrightarrow & X^{n+1} \oplus E^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots
\end{array}$$

where the vertical maps induce cohomology isomorphisms in every degree. $\square$

Corollary 1.7. *Let $\mathcal{A}$ be a hereditary abelian category and $D^b(\mathcal{A})$ its bounded derived category. Consider the full subcategory $\overline{D^b(\mathcal{A})}$ of $D^b(\mathcal{A})$ whose objects are the finite direct sums of complexes of the form $A[n]$, for A an object of $\mathcal{A}$ and $n \in \mathbb{Z}$. Then the inclusion functor $\overline{D^b(\mathcal{A})} \to D^b(\mathcal{A})$ is an equivalence of categories.*
Homomorphisms in $\overline{D^b(\mathcal{A})}$ are given by

$$\mathrm{Hom}_{\overline{D^b(\mathcal{A})}}(A[n], B[m]) \cong \begin{cases} \mathrm{Hom}_{\mathcal{A}}(A, B) & \text{if } m = n \\ \mathrm{Ext}^1_{\mathcal{A}}(A, B) & \text{if } m = n+1 \\ 0 & \text{else} \end{cases}$$

and composition of morphisms is given by the Yoneda composition.
In particular, $A[n] \cong B[m]$ in $\overline{D^b(\mathcal{A})}$ if and only if $n = m$ and $A \cong B$.

2 Reminder about quiver representations

Throughout the rest of the talk: let k be an algebraically closed field

Definition 2.1.
- A *quiver* Q consists of the following data:
  - a finite set Q_0, called *vertices*

  - a finite set Q_1, called *arrows*
  - two maps $s, t : Q_1 \to Q_0$, called *source map* resp. *target map*

- A *finite-dimensional (k-linear) representation* V of Q consists of:
  - For every vertex $i \in Q_0$ a finite-dimensional k-vectorspace V_i.
  - For every arrow $\rho \in Q_1$ a k-linear map $V_{s(\rho)} \to V_{t(\rho)}$.

- A *morphism of representations* $V \to W$ consists of for every $i \in Q_0$ a k-linear map $V_i \to W_i$ s.t. for any $\rho \in Q_1$ the following diagram commutes:

$$\begin{array}{ccc} V_{s(\rho)} & \longrightarrow & V_{t(\rho)} \\ \downarrow & & \downarrow \\ V_{s(\rho)} & \longrightarrow & W_{t(\rho)} \end{array}$$

  So we have an (abelian, k-linear) category of k-linear representations, denoted $\operatorname{Rep}_k Q$.

- A *nontrivial path* in Q is a sequence $p = \rho_m \rho_{m-1} \dots \rho_1$ with $s(\rho_i) = t(\rho_{i-1}) \forall 1 < i \leq m$. We denote $s(p) = s(\rho_1)$ and $t(p) = t(\rho_m)$.

- For every vertex i, we introduce a *trivial path* e_i. Denote $s(e_i) = t(e_i) = i$

- The *path algebra* kQ of Q is the k-vectorspace with basis given by the set of trivial and nontrivial paths. Multiplication is given by concatenation of paths:

  1. For $p = \rho_m \rho_{m-1} \dots \rho_1$ and $p' = \rho'_{m'} \rho'_{m'-1} \dots \rho'_1$, we have

  $$pp' = \begin{cases} \rho_m \rho_{m-1} \dots \rho_1 \rho'_{m'} \rho'_{m'-1} \dots \rho'_1 & \text{if } t(p') = s(p) \\ 0 & \text{else} \end{cases}$$

  2. $e_i p = \begin{cases} p & \text{if } t(p) = i \\ 0 & \text{else} \end{cases}$ and $pe_i = \begin{cases} p & \text{if } s(p) = i \\ 0 & \text{else} \end{cases}$

Example 2.2. Let $Q = \vec{A}_n^{\,op}$ be given by

$$1 \xleftarrow{\alpha_1} 2 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{n-1}} n$$

Then kQ is the k-vectorspace with basis given by the paths p_{ij} for $i \geq j$, where $p_{ii} := e_i$ and $p_{ij} = \alpha_j \dots \alpha_{i-1}$; with the multiplication defined before. It is not hard to check that kQ is isomorphic to the algebra of upper triangular $n \times n$-matrices, where the isomorphism is defined by $p_{ij} \mapsto E_{ij}$. (Here E_{ij} is the matrix with coefficient 1 in position (i, j), and all other coefficients 0.)

Proposition 2.3. *Let Q be a quiver. There is an equivalence of categories*

$$\mathrm{Rep}_k\, Q^{op} \simeq \mathrm{mod}\,-kQ$$

(Here $\mathrm{mod}\,-kQ$ *denotes the category of finite-dimensional right kQ-modules, and Q^{op} is obtained from Q by reversing all arrows.)*

Proposition 2.4. *Denote $A = kQ$. Every right A-module M has a projective resolution of length at most 2.*
This implies that $\mathrm{Ext}^2_{A-\mathrm{mod}}(\_,\_)$ *vanishes, i.e.* $A-\mathrm{mod}$ *is a hereditary abelian category.*

Theorem 2.5 (Gabriel)**.** 1. *A quiver Q has only finetely many isoclasses of indecomposable representations if and only if its underlying unoriented graph is a "simply laced Dynkin diagram", i.e. one of the graphs in the following figure. In this case Q is called a* Dynkin quiver.

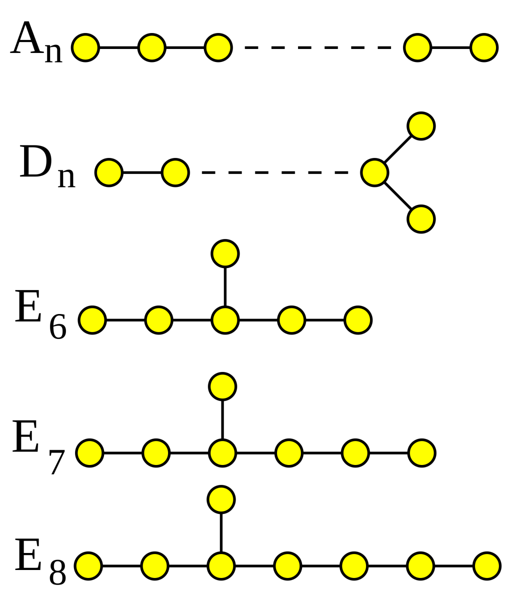


Figure 1: The simply laced Dynkin diagrams. The number of beads is equal to n.

2. *The indecomposable representations are in a one-to-one correspondence with the positive roots of the root system of the Dynkin diagram.*

Our goal is to give a discription of the derived category $\mathrm{D}^b(\mathrm{mod}\,-kQ)$ for Q a Dynkin quiver. By Gabriel's theorem, $\mathrm{mod}\,-kQ$ has only finitely many indecomposable objects up to isomorphism. We also know that every object of $\mathrm{mod}\,-kQ$ decomposes into a finite sum of indecomposable objects, unique up to isomorphism and permutation. Since $\mathrm{mod}\,-kQ$ is hereditary, we can use corollary 1.7 to see that the indecomposable objects of $\mathrm{D}^b(\mathrm{mod}\,-kQ)$ are given by the chain complexes $M[n]$ (where $M \in \mathrm{ind}(\mathrm{mod}\,-kQ)$, $n \in \mathbb{Z}$), the map

$$\mathrm{ind}(\mathrm{mod}\,-kQ)/\!\cong \times\, \mathbb{Z} \to \mathrm{ind}(\mathrm{D}^b(\mathrm{mod}\,-kQ))/\!\cong\, : (M, n) \mapsto M[n]$$

is a bijection, and that any object $\mathrm{D}^b(\mathrm{mod}-kQ)$ can be written in a unique way as a finite direct sum of indecomposables. So the understand the category $\mathrm{D}^b(\mathrm{mod}-kQ)$, it suffices to understand the subcategory $\mathrm{ind}(\mathrm{D}^b(\mathrm{mod}-kQ))$ of indecomposable objects. It will turn out that the category $\mathrm{ind}(\mathrm{D}^b(\mathrm{mod}-kQ))$ is equivalent to the so-called *mesh category* of Q, which we will define in the following section.

3 The mesh category of a quiver

Remark 3.1.
- One can define infinite quivers in the same way as quivers, but without the assumption that Q_0 and Q_1 are finite.
- An isomorphism $Q \xrightarrow{\sim} Q'$ of (infinite) quivers consists of bijections $Q_0 \xrightarrow{\sim} Q'_0$ and $Q_1 \xrightarrow{\sim} Q'_1$ commuting with the source and target maps.

Definition 3.2. Let Q be a quiver. The *repetition quiver* $\mathbb{Z}Q$ of Q is the infinite quiver with:

- Vertices $(\mathbb{Z}Q)_0 = \mathbb{Z} \times Q_0$.
- For every arrow $\alpha : i \to j$ in Q and every $p \in \mathbb{Z}$, we have an arrow $(p, \alpha) : (p, i) \to (p, j)$ and an arrow $\sigma(p, \alpha) : (p-1, j) \to (p, i)$.

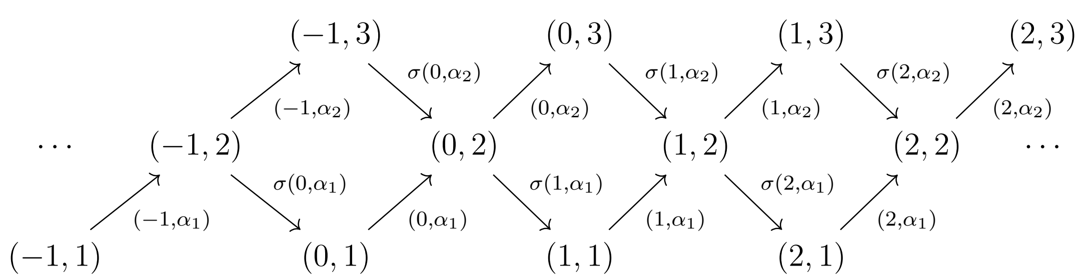
$\mathbb{Z}Q$ comes equipped with a map

$$\begin{aligned} \sigma : (\mathbb{Z}Q)_1 \to (\mathbb{Z}Q)_1 :& (p,\alpha) \mapsto \sigma(p,\alpha) \\ & \sigma(p,\alpha) \mapsto (p-1,\alpha) \end{aligned}$$

and an automorphism τ given by

$$\begin{aligned} \tau : (\mathbb{Z}Q)_0 \to (\mathbb{Z}Q)_0 :& (p,i) \mapsto (p-1,i) \\ \tau : (\mathbb{Z}Q)_1 \to (\mathbb{Z}Q)_1 :& (p,\alpha) \mapsto (p-1,\alpha) \\ & \sigma(p,\alpha) \mapsto \sigma(p-1,\alpha) \end{aligned}$$

Example 3.3. For $Q = \vec{A}_3$, the repetition quiver is given by:



Observation 3.4. If Q is a quiver s.t. the underlying graph has no loops, and Q' is obtained from Q by reversing an arrow, then their repetition quivers $\mathbb{Z}Q$ and $\mathbb{Z}Q'$ are isomorphic (in a way that is compatible with σ and τ).

Definition 3.5. The *path category* $k(\Gamma)$ of an (infinite) quiver Γ is the k-linear category with:

- Objects: vertices of Γ.
- $\mathrm{Hom}(i, j)$ is the k-vectorspace with basis the set of all paths from i to j.
- The composition is induced by concatenation of paths.

Remark 3.6.

- An *ideal* $\mathcal{I}$ of a k-linear category $\mathcal{C}$ consists of for any $\mathrm{Hom}(x, y)$ a subspace $\mathcal{I}(x, y)$ s.t. for any $f : x' \to x$, $h : y \to y'$ and $g \in \mathcal{I}(x, y)$ we have $h \circ g \circ f \in \mathcal{I}(x', y')$.
- The *quotient category* $\mathcal{C}/\mathcal{I}$ has the same objects as $\mathcal{C}$, and

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(x, y) := \mathrm{Hom}_{\mathcal{C}}(x, y)/\mathcal{I}(x, y)$$

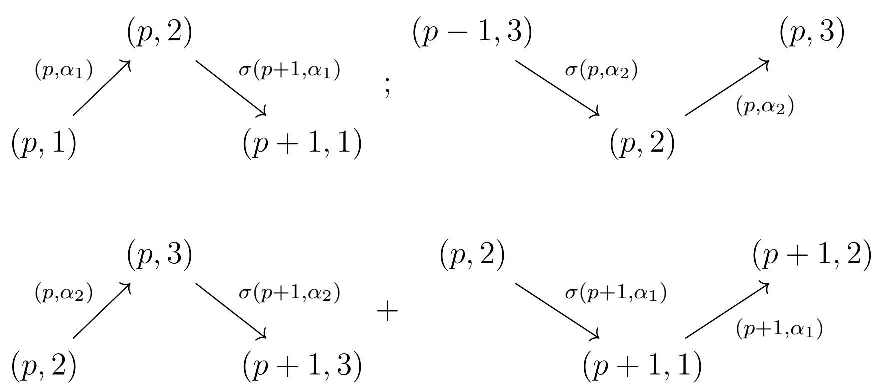
Definition 3.7.

- The *mesh ideal* $\mathrm{mesh}(\mathbb{Z}Q)$ is the ideal of $k(\mathbb{Z}Q)$, generated by the *mesh relators*

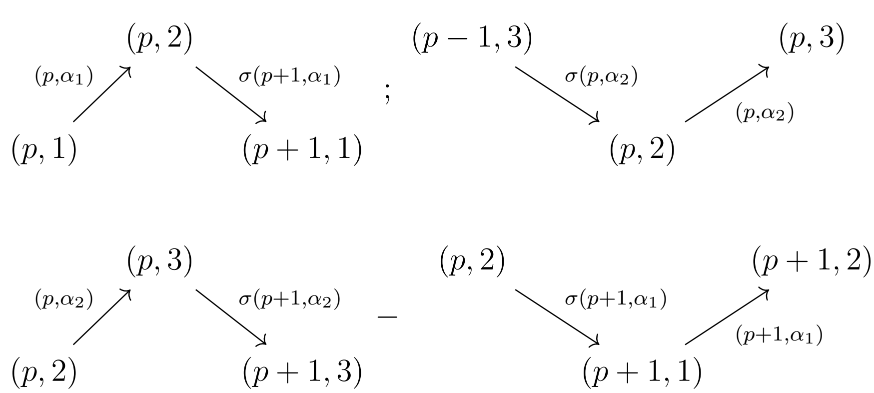
$$r_v = \sum_{\alpha : t(\alpha) = v} \alpha\sigma(\alpha) \in \mathrm{Hom}_{k(\mathbb{Z}Q)}(\tau(v), v)$$

  where v runs through the vertices of $\mathbb{Z}Q$.
- The *mesh category* $k\langle \mathbb{Z}Q \rangle$ of $\mathbb{Z}Q$ is the quotient category $k(\mathbb{Z}Q)/\,\mathrm{mesh}(\mathbb{Z}Q)$.

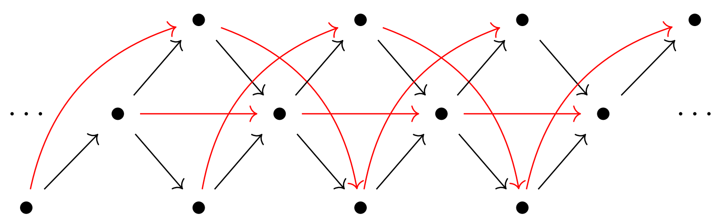
Example 3.8. For $Q = \vec{A}_3$, the mesh ideal $\mathrm{mesh}(\mathbb{Z}Q)$ is generated by

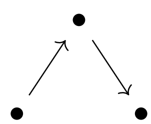
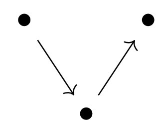


In order to avoid trouble with signs, we will instead work with the ideal generated by



(we changed the plus in a minus). It's easy to see that the obtained quotient categories are equivalent (for example by replacing (p, α_1) by $-(p, \alpha_1)$).
We can see that the only Hom-spaces in $k\langle \mathbb{Z}Q \rangle$ that are nonzero are $\mathrm{Hom}_{k\langle \mathbb{Z}Q \rangle}((p,1),(p,2))$, $\mathrm{Hom}_{k\langle \mathbb{Z}Q \rangle}((p,1),(p,3))$, $\mathrm{Hom}_{k\langle \mathbb{Z}Q \rangle}((p,2),(p,3))$, $\mathrm{Hom}_{k\langle \mathbb{Z}Q \rangle}((p,2),(p+1,1))$, $\mathrm{Hom}_{k\langle \mathbb{Z}Q \rangle}((p,2),(p+1,2))$, $\mathrm{Hom}_{k\langle \mathbb{Z}Q \rangle}((p,3),(p+1,2))$, $\mathrm{Hom}_{k\langle \mathbb{Z}Q \rangle}((p,3),(p+2,1))$, and these are 1-dimensional. So $k\langle \mathbb{Z}Q \rangle$ is isomorphic to the category given by the following figure.



Here every arrow represents a 1-dimensional Hom-space, and we can choose generators for these Hom-spaces s.t. the generator of a horizontal red arrow is given by the composition of the generators  which equals the composition of the generators , and the generator of another red arrow is the composition of the generators of the 2 corresponding black arrows.

Theorem 3.9. *Let Q be a Dynkin quiver.*

- *There is a canonical bijection from the vertices of $\mathbb{Z}Q$ to the set of indecompoasable objects of $\mathrm{D}^b(\mathrm{mod}-kQ)$, taking (0,i) to the indecomposable projective module e_iA.*
- *The above bijection extends to an equivalence of categories*

$$k\langle \mathbb{Z}Q \rangle \simeq \mathrm{ind}(\mathrm{D}^b(\mathrm{mod}-kQ))$$

A full proof of this theorem would take too much time. We will only give a proof for the case $Q = \vec{A}_3$.

Corollary 3.10. *By observation 3.4, it follows that if Q and Q' are Dynkin quivers with the same underlying Dynkin diagram, then their (bounded) derived categories are equivalent.*

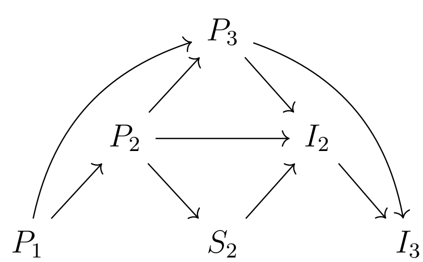
4 The category $\mathrm{ind}(\mathrm{D}^b(\mathrm{mod}-k\vec{A}_3))$

By Gabriel's theorem, $\vec{A}_3^{\,op}$ has only finitely many indecomposable representations. It's an easy exercise to list all of them (we'll denote the path algebra $k\vec{A}_3$ by A :

- $(k \leftarrow 0 \leftarrow 0) \cong e_1A =: P_1 = S_1$ (projective, simple)

- $(0 \leftarrow k \leftarrow 0) =: S_2$ (simple)
- $(0 \leftarrow 0 \leftarrow k) =: I_3 = S_3$ (injective, simple)
- $(k \overset{\sim}{\leftarrow} k \leftarrow 0) \cong e_2 A =: P_2$ (projective)
- $(0 \leftarrow k \overset{\sim}{\leftarrow} k) =: I_2$ (injective)
- $(k \overset{\sim}{\leftarrow} k \overset{\sim}{\leftarrow} k) \cong e_3 A =: P_3 = I_1$ (projective, injective)

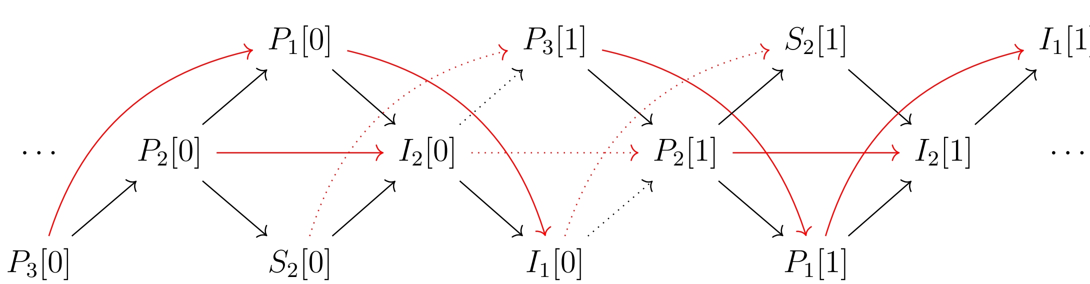
All of these are obviously non-isomorphic indecomposable representations. To see that there are no others, either use part 2 of Gabriel's theorem, or just check explicitely.
It's easy to compute $\mathrm{Hom}_{\mathrm{mod}\,-k\vec{A_3}}(M, N)$ for M and N indecomposable modules. We get the following picture of $\mathrm{ind}(\mathrm{mod}\,-k\vec{A_3})$ (here $M \to N$ means $\mathrm{Hom}_{\mathrm{mod}\,-k\vec{A_3}}(M, N) \cong k$, else $\mathrm{Hom}_{\mathrm{mod}\,-k\vec{A_3}}(M, N) = 0$; furthermore if we replace every arrow by a well-chosen generator of the corresponding vectorspace, the figure becomes a commutative diagram).



It's also easy to find projective resolutions for the indecomposable objects:

$$0 \to P_1 \to P_2 \to S_2 \to 0$$
$$0 \to P_1 \to P_3 \to I_2 \to 0$$
$$0 \to P_2 \to P_3 \to I_3 \to 0$$

We can use these to compute $\mathrm{Ext}^1_{\mathrm{mod}\,-k\vec{A_3}}(M, N)$ for M and N indecomposable modules. We find that all Ext-spaces are 0, except for the following 5:
$\mathrm{Ext}^1(S_2, P_1)$, $\mathrm{Ext}^1(I_2, P_1)$, $\mathrm{Ext}^1(I_2, P_2)$, $\mathrm{Ext}^1(I_3, P_2)$, $\mathrm{Ext}^1(I_3, S_2)$
which are 1-dimensional.
So, using corollary 1.7, we get the following picture of $\mathrm{ind}(\mathrm{D}^b(\mathrm{mod}\,-k\vec{A_3}))$:



The only thing left to check to see that this category is equivalent to the category of example 3.8, is that (after choosing generators) the red dotted arrows are the compositions of 2 black ones. I.e. we need the maps $\mathrm{Ext}^1(I_2, P_1) \to \mathrm{Ext}^1(S_2, P_1)$, $\mathrm{Ext}^1(I_3, P_2) \to \mathrm{Ext}^1(I_2, P_2)$, $\mathrm{Ext}^1(I_2, P_1) \to \mathrm{Ext}^1(I_2, P_2)$, $\mathrm{Ext}^1(I_3, P_2) \to \mathrm{Ext}^1(I_3, S_2)$ to be bijective. This can be easily checked using the long exact sequences associated to the short exact sequences $0 \to S_2 \to I_2 \to I_3 \to 0$ and $0 \to P_1 \to P_2 \to S_2 \to 0$.
This concludes the proof of theorem 3.9 in the case $Q = \vec{A}_3$.

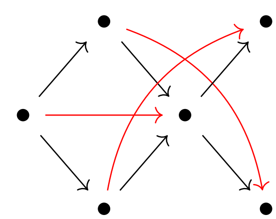
Remark 4.1. Instead of computing the Ext-spaces via projective resolutions, we could also have used the formula

$$\dim \mathrm{Hom}_{\mathrm{mod}\,-k\vec{A}_3}(M, N) - \dim \mathrm{Ext}^1_{\mathrm{mod}\,-k\vec{A}_3}(M, N) = \langle \underline{\dim}\, M, \underline{\dim}\, N \rangle$$

where $\langle \_, \_ \rangle : \mathbb{N}^{Q_0} \times \mathbb{N}^{Q_0} \to \mathbb{Z}$ is the *Euler form*, defined by

$$\langle \underline{x}, \underline{y} \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\rho \in Q_1} x_{s(\rho)} y_{t(\rho)}$$

Example 4.2. For the quiver $1 \to 2 \leftarrow 3$, the category $\mathrm{ind}(\mathrm{mod}\,-kQ)$ looks like



which is not equivalent to $\mathrm{ind}(\mathrm{mod}\,-k\vec{A}_3)$. However, if we construct from this the derived category $\mathrm{ind}(\mathrm{D}^b(\mathrm{mod}\,-kQ))$ as we did above, we get the same picture as before, as predicted by corollary 3.10.