Verdier Localisation and the Derived Category

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In this seminar talk, the concept of the so called *Verdier localisation* of a triangulated category and the derived category of an abelian category will be introduced. The main goal of this talk is to establish the following theorem:

Theorem 0.1 (Verdier). *For a* triangulated subcategory *$\mathcal{C}$ of a triangulated category $\mathcal{D}$, there exists a triangulated category denoted "$\mathcal{D}/\mathcal{C}$" and a* triangulated functor *$F : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ such that $\mathcal{C}$ is contained in the* kernel *of F and F is universal with this property, i. e. all triangulated functors $T : \mathcal{D} \to \mathcal{E}$ whose kernel contains $\mathcal{C}$ factor uniquely through F.*

After having explained the relevant vocabulary of this theorem, we will construct the category $\mathcal{D}/\mathcal{C}$ in order to prove the theorem. Eventually we will de derived category of an abelian category by taking the Verdier localisation of its homotopy category.

If not stated otherwise, this talk follows the argumentation from [Nee01, pp. 73-99] and [Mur07]. The proofs of most technical lemmas are ommited and can be found at the same place. Also, any set theoretical considerations are waived.

1. Necessary Definitions

The terms in Verdier's theorem undefined yet are to be made precise in this section. One might expect a *triangulated functor* between triangulated categories to map triangles to triangles; however, such a functor has to be compatible with the suspension functor Σ in an appropriate sense which motivates the following definition:

Definition 1.1. *A* triangulated functor *between two triangulated categories $\mathcal{D}$ and $\mathcal{E}$ consists of an additive functor $F : \mathcal{D} \to \mathcal{D}'$ together with a natural isomorphism $F\Sigma \cong \Sigma F$, such that it maps triangles to triangles.*

Using this notion, we can reformulate the definition of a triangulated subcategory as follows:

Definition 1.2. *A strictly full*[1] *additive subcategory $\mathcal{C}$ of a triangulated category $\mathcal{D}$ is called a* triangulated subcategory *if it satisfies either of the following equivalent properties:*

1. *if viewed with the triangulated structure of $\mathcal{D}$, it is a triangulated category again, i. e. it contains all objects necessary to complete $\mathcal{C}$-morphisms to triangles, and the embedding into $\mathcal{D}$ is a triangulated functor;*
2. *$\mathcal{C}$ is Σ-stable, i. e. $\Sigma\mathcal{C} = \mathcal{C}$, and for any triangle $X \to Y \to Z \to \Sigma X$ with $X, Y \in \mathcal{C}$ also $Z \in \mathcal{C}$.*

In particular, the triangles in a triangulated subcategory $\mathcal{C}$ are precisely the triangles in $\mathcal{D}$ consisting of objects in $\mathcal{C}$. As for morphisms of an additive category, it is useful to define the kernel of a triangulated functor.

Definition 1.3. *Given a triangulated functor $F : \mathcal{D} \to \mathcal{D}'$, its* kernel *is the full subcategory of $\mathcal{D}$ consisting of objects $\{X \in \mathcal{D} : FX \cong 0\}$.*

By using the natural isomorphism coming along with F and the additivity of F, it is immediate that $\ker F$ admits the following property:

Definition 1.4. *A triangulated subcategory $\mathcal{C} \subseteq \mathcal{D}$ is called* thick *if $\forall X, Y \in \mathcal{D} : X \oplus Y \in \mathcal{C} \Rightarrow X, Y \in \mathcal{C}$, i. e. it contains all direct summands of its objects.*

It will turn out that $\ker F$ as given by Verdier's theorem is the smallest thick subcategory of $\mathcal{D}$ containing $\mathcal{C}$.

2. The collection of morphisms $\mathrm{Mor}_{\mathcal{C}}$

Now, after having explained all termini used in the theorem, some facts about the category $\mathrm{Mor}_{\mathcal{C}}$ have to be recapitulated before constructing $\mathcal{D}/\mathcal{C}$.

Definition 2.1. *Given a triangulated subcategory $\mathcal{C} \subseteq \mathcal{D}$, let $\mathrm{Mor}_{\mathcal{C}}$ be the collection of morphisms $f : X \to Y$ in $\mathcal{D}$*

$$\mathrm{Mor}_{\mathcal{C}} := \{f \in \mathrm{Hom}_{\mathcal{D}}(X, Y) | X, Y \in \mathcal{D}, \operatorname{cone} f \in \mathcal{C}\},$$

, i. e. it contains precisely those morphisms $f : X \to Y$ of $\mathcal{D}$ for which, if the morphism is completed to a triangle $X \to Y \to Z \to \Sigma X$, we have $Z \in \mathcal{C}$.

This collection is well-defined because such a diagram always exists in a triangulated category, and since the third element Z of the triangle us inuque up to an isomorphism, this definition does not depend on the choice of Z.

Remark 2.2. *The collection $\mathrm{Mor}_{\mathcal{C}}$ satisfies the following properties:*

[1] i. e. it is closed under isomorphisms.

- *Since* $0 \in \mathcal{C}$ *as it is additive,* $\mathrm{Mor}_{\mathcal{C}}$ *contains all isomorphism, i. p. all identities.*

- *by the octahedral axiom,* $\mathrm{Mor}_{\mathcal{C}}$ *satisfies a two-of-three-rule, i. e. if two of the morphisms* f*,* g *and* gf *are morphisms in* $\mathrm{Mor}_{\mathcal{C}}$*, then the third is as well*

- *In particular, it is possible to compose morphisms in* $\mathrm{Mor}_{\mathcal{C}}$*.*

- $\mathrm{Mor}_{\mathcal{C}}$ *contains all* homotopy pullbacks *and* pushouts *of its morphisms.*

Actually, as a side remark, $\mathrm{Mor}_{\mathcal{C}}$ *together with all objects of* $\mathcal{D}$ *forms a category.*

For the kast point, we introduce the following notion:

Reminder 2.3. *A commutative square*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g"] & A \arrow[d, "g'"] \\ B \arrow[r, "f'"] & Q \end{tikzcd}$$

is called homotopy bicartesian *if there is a triangle* $P \xrightarrow{\binom{g}{-f}} B \oplus A \to Q \to \Sigma P$*. If so,* P *and* Q *are called* homotopy pullback *and* homotopy pushout *respectively*[2] *Homotopy pullbacks and pushouts always do exist.*

With this notion in mind, the statement that $\mathrm{Mor}_{\mathcal{C}}$ contains all homotopy pullbacks and pushouts means that, in the notation above, f or g are contained in $\mathrm{Mor}_{\mathcal{C}}$ iff f' or g' are, respectively. We state the following result that in particular will apply to the Verdier quotient category:

Lemma 2.4. *Once defined, the functor* $F : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ *as well as any other triangulated functor* $\mathcal{D} \to \mathcal{D}'$ *whose kernel contains* $\mathcal{C} \subseteq \mathcal{D}$ *maps* $\mathrm{Mor}_{\mathcal{C}}$ *to isomorphisms.*

Proof. We know that, in any triangulated category, a morphism is an isomorphism iff it fits into a triangle isomorphic to $X \to Y \to 0 \to \Sigma X$. Applying F to an arbitrary triangle where $Z \in \mathcal{C}$ yields

$$\begin{aligned} &(X \to Y \to Z \to \Sigma X) \\ \xmapsto{F} &(FX \to FY \to FZ \to \Sigma FX) \\ \cong &(FX \xrightarrow{\cong} FY \to 0 \to \Sigma FX). \end{aligned}$$

Hence $FX \cong FY$. □

3. Construction, Categoriality

Slogan: Endow the objects of $\mathcal{D}$ with enhanced morphisms. Although these will not form a category, after diving out an appropriate equivalence relation, this will become a category.

[2] Compare this to the short exact sequence $0 \to A \times_Q B \xrightarrow{\binom{g}{-f}} B \oplus A \to Q \to 0$ an ordinary pullback in an abelian category fits into.

Definition 3.1. *For* X, Y *objects of* $\mathcal{D}$ *we define collections of morphisms*

$$\widehat{\mathrm{Hom}}_{\mathcal{D}}(X, Y) := \left\{ X \xleftarrow{f} W \to Y : f \in \mathrm{Mor}_{\mathcal{C}} \right\}.$$

Diagrams in $\widehat{\mathrm{Hom}}_{\mathcal{D}}(X, Y)$ *are called* roofs.

Since $\mathrm{Mor}_{\mathcal{C}}$ contains all identities, $\forall X \in \mathcal{D} : (X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X) \in \widehat{\mathrm{Hom}}_{\mathcal{D}}(X, X,)$. In the following, let X, Y denote arbitrary objects of $\mathcal{D}$.

Definition and Lemma 3.2. *The composition of roofs in* $\widehat{\mathrm{Hom}}_{\mathcal{D}}(X, Y)$ *given by*

$$(Y \leftarrow W \to Z) \circ (X \leftarrow W' \to Y)$$

$$:= \left(\begin{tikzcd}[column sep=small] & & W \times_Y W' \arrow[dl] \arrow[dr] & & \\ & W \arrow[dl] \arrow[dr] & & W' \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd} \right)$$

is associative. i. e. a composition of two roofs is given by a $\mathcal{D}$*-homotopy pullback above the "middle" object.*

Note that, although the pullback always can be taken, it is only unique up to a non-canonical isomorphism [Nee01, p. 54]. Hence, the composition of roofs is only defined up to isomorphisms.

Definition and Lemma 3.3. *On* $\widehat{\mathrm{Hom}}_{\mathcal{D}}(X, Y)$ *there is a equivalence relation* $\diamond$ *which is compatible with the just defined composition, given by* $(X \leftarrow Z \to Y) \diamond (X \leftarrow Z' \to Y)$ *iff there is a roof* $(X \leftarrow W \to Y) \in \widehat{\mathrm{Hom}}_{\mathcal{D}}(X, Y)$ *such that the diagram*

$$\begin{tikzcd} & Z' \arrow[dl] \arrow[dr] & \\ X & W \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\ & Z \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes. In particular the vertical morphisms are contained in $\mathrm{Mor}_{\mathcal{C}}$*. The equivalence class of* $X \xleftarrow{f} W \xrightarrow{g} Y$ *is denoted by* gf^{-1}*.*

Proof. We have to show that the relation is an equivalence relation compatible with the composition.

- Vertical morphisms are contained in $\mathrm{Mor}_{\mathcal{C}}$ because of the two-of-three-rule.
- Symmetry and reflexivity are obvious.
- For checking transitivity, let $(X \leftarrow Z_1 \to Y) \diamond (X \leftarrow Z_2 \to Y) \diamond (X \leftarrow Z_3 \to Y)$. If this is spelled out, the resulting diagram

$$\begin{tikzcd} & & Z_1 \arrow[ddl] \arrow[ddr] & \\ & & Z \arrow[u] \arrow[d] \arrow[dl] \arrow[dr] & \\ Z'' \arrow[urr, dashed] \arrow[drr, dashed] \arrow[r, dotted] \arrow[rrr, dotted, bend right] & X & Z_2 \arrow[l] \arrow[r] & Y \\ & & Z' \arrow[u] \arrow[d] \arrow[ul] \arrow[ur] & \\ & & Z_2 \arrow[uul] \arrow[uur] & \end{tikzcd}$$

can be completed by the homotopy pullback Z''. Since the vertical lines and thus their homotopy pullback are contained in $\mathrm{Mor}_{\mathcal{C}}$, composing the appropriate morphisms leads the desired diagram exhibiting $(X \leftarrow Z_1 \rightarrow Y) \diamondplus (X \leftarrow Z_3 \rightarrow Y)$.

- For proving that the relation is compatible with the composition in $\widehat{\mathrm{Hom}}_{\mathcal{D}}(X,Y)$, consider the following postcomposition of two roofs that are in relation with a third one, as well as the respective compositions:

$$\begin{tikzcd}
& W_1 \arrow[dl] \arrow[dr] & V_1 \arrow[l] \arrow[dr] \\
X & W \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y & V \arrow[l] \arrow[r] & Z \\
& W_2 \arrow[ul] \arrow[ur] & V_2 \arrow[l] \arrow[ur]
\end{tikzcd}$$

where $V_i = W_i \times_Y V$. Taking the respective pullbacks

$$\begin{tikzcd}
& U_1 \arrow[dr, dashed] \arrow[rr, dashed, bend left] & W_1 \arrow[dl] \arrow[dr] & V_1 \arrow[l] \arrow[dr] \\
U \arrow[ur, dashed] \arrow[dr, dashed] & X & W \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y & V \arrow[l] \arrow[r] & Z \\
& U_2 \arrow[ur, dashed] \arrow[rr, dashed, bend right] & W_2 \arrow[ul] \arrow[ur] & V_2 \arrow[l] \arrow[ur]
\end{tikzcd}$$

defined by $U = W \times_Y V_1 \times_W W \times_Y V_2 = V_1 \times_Y V_2$ shows that both compositions are in relation. Hence the relation is compatible with composition and thus forms a equivalence relation.

□

Definition 3.4. *The* Verdier quotient $\mathcal{D}/\mathcal{C}$ *is the category defined by the following data:*

$$\mathrm{Obj}(\mathcal{D}/\mathcal{C}) \coloneqq \mathrm{Obj}(\mathcal{D})$$
$$\mathrm{Hom}_{\mathcal{D}/\mathcal{C}}(X,Y) \coloneqq \widehat{\mathrm{Hom}}_{\mathcal{D}}(X,Y)/\diamondplus\,.$$

Additionally, we the obvious functor

$$F : \mathcal{D} \to \mathcal{D}/\mathcal{C}$$
$$X \mapsto X$$
$$(X \xrightarrow{f} Y) \mapsto (X \xleftarrow{1} X \xrightarrow{f} X)$$

is called Verdier localisation map.

The functor F will turn out to be our triangulated functor whose existence is claimed by Verdier's theorem. Since the composition in $\widehat{\mathrm{Hom}}_{\mathcal{D}}(X,Y)$ is defined up to non-canonical isomorphisms, it is well-defined in $\mathcal{D}/\mathcal{C}$. We have already seen that the composition of roofs is associative, and that contains all identities.

4. Properties of $\mathcal{D}/\mathcal{C}$, Universality of F

The functor F and the composition law immediately yield the following result:

Lemma 4.1. *If $g \in \mathrm{Mor}_{\mathcal{C}}$, then gf^{-1} is invertible, and its inverse is given by $(gf^{-1})^{-1} = fg^{-1}$. Furthermore, every morphism gf^{-1} can be written as $gf^{-1} = F(g)F(f)^{-1}$.*

In particular, $F(f)$ is an isomorphism if $f \in \mathrm{Mor}_{\mathcal{C}}$. However, the following result shows the importance of the functor F:

Proposition 4.2. *The functor $F : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is universal among all functors mapping $\mathrm{Mor}_{\mathcal{C}}$ to isomorphisms, i. e. every functor $T : \mathcal{D} \to \mathcal{E}$ sending $\mathrm{Mor}_{\mathcal{C}}$ to isomorphisms factors as $\mathcal{D} \xrightarrow{F} \mathcal{D}/\mathcal{C} \to \mathcal{E}$.*

Proof. We can extend F to roofs in the obvious way. Applying T the diagram of two equivalent roofs yields

$$\begin{tikzcd}
& TW \arrow[dl, "\cong"'] \arrow[dr] \\
TX & TW' \arrow[l] \arrow[r] \arrow[u, "\cong"'] \arrow[d, "\cong"] & TY \\
& TW'' \arrow[ul, "\cong"] \arrow[ur]
\end{tikzcd}$$

showing all the left half of the diagram being isomorphic. Hence, T maps equivalent morphisms to the same in $\mathcal{E}$. □

To show Verdier's theorem, we have yet to show that $\mathcal{D}/\mathcal{C}$ is a triangulated category and that F is a triangulated functor.

Proposition 4.3. *The category $\mathcal{D}/\mathcal{C}$ is additive.*

Proof. The following properties have to be shown:

Existence of a zero-object It comes to mind that $0 \in \mathcal{D}$ is a good candidate for the zero object. Indeed, $X \xleftarrow{1} X \to 0$ is a morphism into 0, and taking an arbitrary morphism $X \xleftarrow{f} W \to 0$, the commutative diagram

$$\begin{tikzcd}
& P \arrow[dl] \arrow[dr] \arrow[dd, "f"] \\
X & & 0 \\
& X \arrow[ul] \arrow[ur]
\end{tikzcd}$$

shows that $01^{-1} \diamondplus 0f^{-1}$. Hence for each object X there is exactly one morphism into 0.

Existence of finite biproducts The obvious candidate is $X \oplus_{\mathcal{D}/\mathcal{C}} Y = X \oplus_{\mathcal{D}} Y$, which indeed satisfies the universal properties of the product and coproduct in $\mathcal{D}/\mathcal{C}$.

Group structure The addition given by $f + g \coloneqq X \to X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \to Y$ has to yield a group law, i. e. it has to contain additive inverses: Since composition on $\mathcal{D}$ is linear, one can compute $gf^{-1} + (-g)f^{-1} = (g - g)f^{-1} = 0$.

□

5. Isomorphisms

Before being able to proof Verdier's theorem, the following utterly boring technical results are yet to be shown.

Lemma 5.1. *For any morphism gf^{-1} in $\mathcal{D}/\mathcal{C}$ satisfying $gf^{-1} \oplus 1$ one has $g \in \mathrm{Mor}_{\mathcal{C}}$.*

Proof. Via the commutative diagram associated to the equivalence,

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & W \arrow[l] \arrow[r] \arrow[u, "\varpi"] \arrow[d, "\xi"] & X \\
& X \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where ϖ and $\xi = g\varpi$ are contained in $\mathrm{Mor}_{\mathcal{C}}$, yielding $f \in \mathrm{Mor}_{\mathcal{C}}$. □

Lemma 5.2. *Two morphisms $f, g : X \to Y$ in $\mathcal{D}$ are mapped to the same $\mathcal{D}/\mathcal{C}$-roof by F iff there exists an equalising map $E \to X$ in $\mathrm{Mor}_{\mathcal{C}}$.*

Proof. Being mapped to the same morphism means

$$\begin{tikzcd}
& X \arrow[dl, "1"'] \arrow[dr, "f"] & \\
X & E \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\
& X \arrow[ul, "1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

hence $E \to X$ is the desired equalising morphism, Conversely, if such an equalising morphism exists, it fits into this diagram. □

Lemma 5.3. *We can characterise isomorphisms and the kernel of F as follows:*

1. *The morphism gf^{-1} is an isomorphism iff $\exists l, r \in \mathcal{D}$: $lg, gr \in \mathrm{Mor}_{\mathcal{C}}$.*
2. *Given an object $X \in \mathcal{D}$, then $FX \cong 0 \in \mathcal{D}/\mathcal{C} \Leftrightarrow \exists X' \in \mathcal{D} : X \oplus X' \in \mathcal{C}$.*

Proof.
1. ($\Leftarrow$) The morphisms $F(lg)$ and $F(gr)$ are invertible since $lg, gr \in \mathrm{Mor}_{\mathcal{C}}$. Hence $F(g)$ admits a left and a right inverse and is thus invertible. Therefore, also gf^{-1} is invertible.

   ($\Rightarrow$) Assuming gf^{-1} is an isomorphism, in particular g is invertible. A right rs^{-1} inverse such that $grs^{-1} \oplus 1$ yields that $gr \in \mathrm{Mor}_{\mathcal{C}}$. The morphism l can be constructed analogously.
2. ($\Rightarrow$) If $X \to 0$ is an isomorphism in $\mathcal{D}/\mathcal{C}$, then rotating the split triangle $Y \to X \oplus Y \to X \xrightarrow{0} \Sigma Y$ yields $X \xrightarrow{0} \Sigma Y \to \Sigma(X \oplus Y) \to \Sigma X$ which exhibits $\Sigma(X \oplus Y)$ and hence $X \oplus Y$ as a member of $\mathcal{C}$.

   ($\Leftarrow$) Assuming $X \oplus Y \in \mathcal{C}$, we want to find left and right factors that make $X \to 0$ a member of $\mathcal{C}$. The zero morphism $0 \to X$ yields the isomorphism $(0 \to X \to 0) \in \mathrm{Mor}_{\mathcal{C}}$, and the zero morphism $X \to 0 \to \Sigma Y$ fits into the triangle $X \xrightarrow{0} \Sigma Y \to \Sigma(X \oplus Y) \to \Sigma X$, such that $(X \xrightarrow{0} \Sigma Y) \in \mathrm{Mor}_{\mathcal{C}}$. □

Proposition 5.4. *The functor F maps a morphism $X \to Y$ to an isomorphism iff for any triangle $(X \to Y \to Z \to \Sigma X)$, $\exists Z' : Z \oplus Z' \in \mathcal{C}$.*

Proof. Omitted. The rather technical proof can be found in [Nee01, p. 92]. □

In particular, this shows that $\ker F$ is the smallest thick subcategory of $\mathcal{D}$ category containing the direct summands of objects in $\mathcal{D}$.

6. Commutative Diagrams

Lemma 6.1. *Any commutative square in $\mathcal{D}/\mathcal{C}$ is isomorphic to the F-image of a $\mathcal{D}$-commutative square.*

Proof. Given a commutative $\mathcal{D}/\mathcal{C}$-square

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z,
\end{tikzcd}$$

the composed morphisms can be lifted to a composition of morphisms in $\mathcal{D}$ (just consider the pullback diagram for the roof composition)

$$\begin{tikzcd}
& & & & W_1 \arrow[rr] \arrow[dl] & & X' \arrow[dl] \arrow[ddl, bend left] \\
W' \arrow[r] & W_3 \arrow[urrr] \arrow[rr, dashed] \arrow[dr] & & W \arrow[rr, "\shortmid" marking] \arrow[dd, "-" marking] & & X \arrow[dd, "-" marking] & \\
& & W_2 \arrow[ur] \arrow[dd] & & & & \\
& & & Y \arrow[rr, "\shortmid" marking] & & Z & \\
& & Y' \arrow[ur] \arrow[urrr, bend right] & & & &
\end{tikzcd}$$

that is isomorphic to the original one in $\mathcal{D}/\mathcal{C}$. In this diagram, roofs are marked by the ticks crossing the arrows. The object W_3 is taken as the homotopy pullback.

Now, the inner square is no diagram in $\mathcal{D}$, hence the outer diagram is not necessarily $\mathcal{D}$-commutative. But since the inner square is $\mathcal{D}/\mathcal{C}$-commutative, there is a $\mathcal{D}$-equalizer in $\mathrm{Mor}_{\mathcal{C}}$ for both paths, denoted W'. Hence we obtained

$$\begin{tikzcd}
W' \arrow[r] \arrow[d] & X' \arrow[d] \\
Y' \arrow[r] & Z
\end{tikzcd}$$

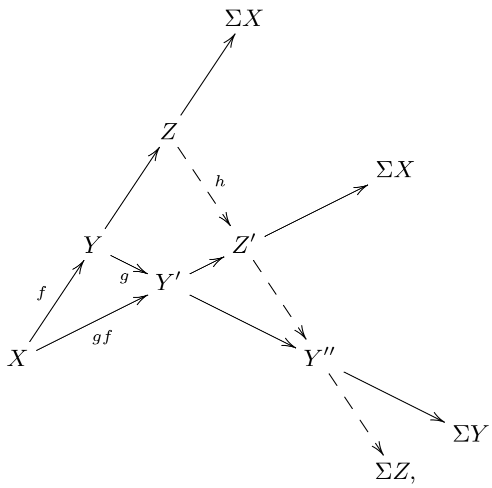
as a $\mathcal{D}$-commutative square. □

Proposition 6.2. *Given a commutative diagram of two $\mathcal{D}$-triangles and two isomorphisms in $\mathcal{D}/\mathcal{C}$ (dotted),*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dotted] & Y \arrow[r] \arrow[d, dotted, "g"] & Z \arrow[r] & \Sigma X \arrow[d, dotted] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X',
\end{tikzcd}$$

the diagram can be completed to a commutative diagram by a missing $\mathcal{D}/\mathcal{C}$-isomorphism $Z \to Z'$.

Proof. We assume that $X' = X$. Using the octahedral axiom we obtain the triangle in



hence Y'' is a direct summand of some object in $\mathcal{C}$, hence h is a $\mathcal{D}/\mathcal{C}$-isomorphism by applying the previous proposition twice. The general case relies on this one and can be found in [Nee01, p. 95]. $\square$

7. Triangulated Structure

In order to turn $\mathcal{D}/\mathcal{C}$ into a triangulated category, we need to define the distinguished triangles:

Definition 7.1. *Let the suspension functor Σ of $\mathcal{D}$ be extended to $\mathcal{D}/\mathcal{C}$ via*

$$\begin{aligned} \Sigma : \mathcal{D}/\mathcal{C} &\to \mathcal{D}/\mathcal{C} \\ X &\mapsto \Sigma_{\mathcal{D}} X \\ [X \xleftarrow{f} W \xrightarrow{Y}] &\mapsto [\Sigma X \xleftarrow{\Sigma f} \Sigma W \xrightarrow{\Sigma g} \Sigma Y]. \end{aligned}$$

Let a candidate triangle in $\mathcal{D}/\mathcal{C}$ be distinguished if it is isomorphic to the image of a $\mathcal{D}$-triangle under F.

Proposition 7.2. *This turns $\mathcal{D}/\mathcal{C}$ into a triangulated category.*

Proof. We have to make sure that

(TR0) candidate triangles isomorphic to triangles are distinguished: immediate.

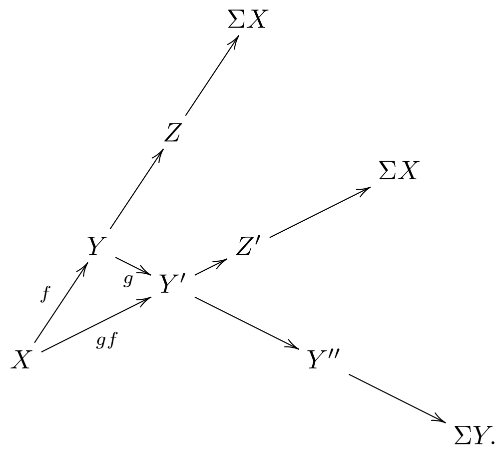
(TR2) "rotating" triangles yields triangles: immediate from (TR2) in $\mathcal{D}$.

(TR1) any morphism $F(g)F(f)^{-1} : X \leftarrow W \to Y$ can be completed to a triangle: since $\mathcal{D}$ is triangulated, g can be completed to a $\mathcal{D}$-triangle $W \to Y \to Z \to \Sigma W$. By definition applying F yields the first row in the commutative diagram

$$\begin{tikzcd} FW \arrow[r, "g"] \arrow[d, "f"'] & FY \arrow[r] \arrow[d, "1"] & FZ \arrow[r] \arrow[d, "1"] & \Sigma FW \arrow[d, "\Sigma f"] \\ FX \arrow[r, "gf^{-1}"'] & FY \arrow[r] & FZ \arrow[r] & \Sigma FX \end{tikzcd}$$

exhibiting both rows being isomorphic.

(TR4) Consinder the following partially filled commutative diagram in $\mathcal{D}/\mathcal{C}$ representing the octahedral axiom:



We construct the missing morhpisms exihibing the missing triangle as follows; slogan: use the octahedral axiom in $\mathcal{D}$ by lifting all assertions from $\mathcal{D}/\mathcal{C}$ to $\mathcal{D}$. Throughout this proof, latin letters denote objects in $\mathcal{D}/\mathcal{C}$ whereas greek letters denote the corresponding objects in $\mathcal{D}$. Since triangles in $\mathcal{D}/\mathcal{C}$ are given as isomorphic images of $\mathcal{D}$-triangles, we have the following diagram where all arrows from back to front are in $\mathrm{Mor}_{\mathcal{C}}$:

$$\begin{tikzcd} & \Xi \arrow[rr] \arrow[dl] \arrow[dd, equal] & & \Upsilon \arrow[rr] \arrow[dl] & & \Omega \arrow[rr] \arrow[dl] & & \Sigma\Xi \arrow[dl] \arrow[dd, equal] \\ X \arrow[rr, crossing over] \arrow[dd, equal] & & Y \arrow[rr] \arrow[dd] & & Z \arrow[rr] & & \Sigma X \arrow[dd, equal] \\ & \Xi \arrow[rr] \arrow[dl] & & \Upsilon' \arrow[rr] \arrow[dl] & & \Omega' \arrow[rr] \arrow[dl] & & \Sigma\Xi \arrow[dl] \\ X \arrow[rr] & & Y' \arrow[rr] & & Z' \arrow[rr] & & \Sigma X' \end{tikzcd}$$

We also can lift the $\mathcal{D}/\mathcal{C}$-commutative square in the front to $\mathcal{D}$, hence

$$\begin{tikzcd} & \bar{\Xi} \arrow[rr] \arrow[dl] \arrow[dd, equal] & & \bar{\Upsilon} \arrow[rr] \arrow[dl] & & \bar{\Omega} \arrow[rr] & & \Sigma\bar{\Xi} \arrow[dl] \arrow[dd, equal] \\ X \arrow[rr, crossing over] \arrow[dd, equal] & & Y \arrow[rr] \arrow[dd] & & Z \arrow[rr] & & \Sigma X \arrow[dd, equal] \\ & \bar{\Xi} \arrow[rr] \arrow[dl] & & \bar{\Upsilon}' \arrow[rr, dashed] \arrow[dl] & & \bar{\Omega}' \arrow[rr] & & \Sigma\bar{\Xi} \arrow[dl] \\ X \arrow[rr] & & Y' \arrow[rr] & & Z' \arrow[rr] & & \Sigma X' \end{tikzcd}$$

Now we can use that $\bar{\Xi} \to \bar{\Upsilon} \to \bar{\Omega} \to \Sigma\bar{\Xi}$ and $\Xi \to \Upsilon \to \Omega \to \Sigma\Xi$ (as well as the bottom triangles in $\mathcal{D}$) can be made isomorphic in $\mathcal{D}/\mathcal{C}$ by the last proposition, which allows us to assume that we have a $\mathcal{D}/\mathcal{C}$-isomorphism of triangles

$$\begin{tikzcd} & \Xi \arrow[rr, "\varphi"] \arrow[dl] \arrow[dd, equal] & & \Upsilon \arrow[rr] \arrow[dl] \arrow[dd, "\gamma"] & & \Omega \arrow[rr] \arrow[dl] & & \Sigma\Xi \arrow[dl] \arrow[dd, equal] \\ X \arrow[rr, crossing over] \arrow[dd, equal] & & Y \arrow[rr] \arrow[dd] & & Z \arrow[rr] & & \Sigma X \arrow[dd, equal] \\ & \Xi \arrow[rr] \arrow[dl] & & \Upsilon' \arrow[rr] \arrow[dl] & & \Omega' \arrow[rr] \arrow[dl] & & \Sigma\Xi \arrow[dl] \\ X \arrow[rr] & & Y' \arrow[rr] & & Z' \arrow[rr] & & \Sigma X' \end{tikzcd}$$

Since the octahedral axiom holds in $\mathcal{D}$, we can construct

$$\begin{tikzcd}
 & & & \Sigma\Xi & & & & \\
 & & \Omega \arrow[ur] \arrow[ddr, dashed] & & & & & \\
 & & & & & \Sigma\Xi & & \\
 & \Upsilon \arrow[uur] \arrow[dr, "\gamma"] & & \Omega' \arrow[urr] \arrow[ddr, dashed] & & & & \\
 & & \Upsilon' \arrow[ur] \arrow[drr] & & & & & \\
\Xi \arrow[uur, "\varphi"] \arrow[urr, "\gamma\varphi"'] & & & & \Omega'' \arrow[ddrr] \arrow[dddr, dashed] & & & \\
 & & & & & & & \\
 & & & & & & \Sigma\Upsilon & \\
 & & & & & \Sigma\Omega, & &
\end{tikzcd}$$

which exhibits the morphism $Z \to Z'$. Now we can put Ω'' into a diagram

$$\begin{tikzcd}
 & \Xi \arrow[rr] \arrow[dl] \arrow[dd] & & \Upsilon' \arrow[rr] \arrow[dl] \arrow[dd, equal] & & \Omega' \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & \Sigma\Xi \arrow[dl] \arrow[dd] \\
X \arrow[rr] \arrow[dd, "f"'] & & Y' \arrow[rr] \arrow[dd, equal] & & Z \arrow[rr] & & \Sigma Y \arrow[dd] & \\
 & \Upsilon \arrow[rr] \arrow[dl] & & \Upsilon' \arrow[rr] \arrow[dl] & & \Omega'' \arrow[rr] & & \Sigma\Upsilon \arrow[dl] \\
Y \arrow[rr, "g"'] & & Y' \arrow[rr] & & Z'' \arrow[rr] & & \Sigma Y' &
\end{tikzcd}$$

Again, we can complete the $\mathcal{D}/\mathcal{C}$ isomorphism in the bottom row by the same argument as above (i. e. find the $\mathcal{D}$-triangle $\mathcal{D}/\mathcal{C}$-isomorphic to the bottom which is $\mathcal{D}/\mathcal{C}$-isomorphic to the bottom back triangle), hence we find a morphism $Z' \to Z''$ that fits into a triangle $Z \to Z' \to Z'' \to \Sigma Z$ in $\mathcal{D}/\mathcal{C}$. Putting all this together, we obtain

$$\begin{tikzcd}
 & & & \Sigma X & & & & \\
 & & Z \arrow[ur] \arrow[ddr, dashed] & & & & & \\
 & & & & & \Sigma X & & \\
 & Y \arrow[uur] \arrow[dr, "g"] & & Z' \arrow[urr] \arrow[ddr, dashed] & & & & \\
 & & Y' \arrow[ur] \arrow[drr] & & & & & \\
X \arrow[uur, "f"] \arrow[urr, "gf"'] & & & & Z'' \arrow[ddrr] \arrow[dddr, dashed] & & & \\
 & & & & & & & \\
 & & & & & & \Sigma Y & \\
 & & & & & \Sigma Z, & &
\end{tikzcd}$$

showing that the octahedral axiom holds in $\mathcal{D}/\mathcal{C}$. □

Corollary 7.3. *The functor F is triangulated and has the universal property claimed in Verdier's theorem, finally establishing its proof.*

Proof. Choosing the identity as a natural isomorphism $F\Sigma \cong \Sigma F$ immediately shows that F maps triangles to triangles since triangles in $\mathcal{D}/\mathcal{C}$ are defined to be isomorphic to images of triangles under F. Its universality has already been shown. □

8. Derived Categories

This section follows the construction given in [Wei94, sect. 10.1–10.4] and [Mur06, pp. 11-19]. We will apply the Verdier localisation procedure just developed to the homotopy category in order to eventually define the concept of the derived category.

Reminder 8.1. *Given an additive category $\mathcal{A}$, the category of cochain complexes with objects in $\mathcal{A}$ is denoted by* $\mathrm{Ch}(\mathcal{A})$*. Assuming that $\mathcal{A}$ is even abelian, the* homotopy category*, denoted $\mathcal{K}(\mathcal{A})$ or simply $\mathcal{K}$, is given by objects of* $\mathrm{Ch}(\mathcal{A})$ *and morphisms of* $\mathrm{Ch}(\mathcal{A})$ *up to homotopy. We saw that $\mathcal{K}$ is a triangulated category with suspension functor $\Sigma := [1]$ and distinguished triangles $X \xrightarrow{f} Y \to \mathrm{cone}(f) \to X[1]$.*

From now on, let $\mathcal{K}$ be the homotopy category of an *abelian* category $\mathcal{A}$.

Lemma 8.2. *The full subcategory $\mathcal{N} \in \mathcal{K}$ consisting of the acyclic cochain complexes is a triangulated subcategory. The corresponding collection* $\mathrm{Mor}_{\mathcal{N}}$ *consists of all quasi-isomorphisms.*

Proof. Since a chain map is a quasi-isomorphism iff its mapping cone is acyclic, the latter assertion is immediate. Furthermore, $\mathcal{N}$ is strictly full, additive and closed under $[\pm 1]$. Given a triangle $X \to Y \to Z \to X[1]$ where X and Y are exact, passing to the long exact sequence

$$\cdots \to \underbrace{H^i Y}_{0} \to H^i Z \to \underbrace{H^{i+1} X}_{0} \to \cdots$$

cohomology gives rise to immediately shows that also Z has to be exact. □

Definition 8.3. *The* derived category $\mathcal{D}(\mathcal{A})$ *of an abelian category $\mathcal{A}$ is given by the Verdier localisation $\mathcal{K}/\mathcal{N}$.*

9. Bounded Derived Category

This section follows the construction given in [Mur06, sect. 3.3]. It comes to mind not only to consider the localisation of cochain complexes along quasi-isomorphisms, but also to obtain a localisation of any of the following subcategory of $\mathrm{Ch}(\mathcal{A})$:

Definition 9.1. *Given an abelian category $\mathcal{A}$, besides the abelian category of cochain complexes* $\mathrm{Ch}(\mathcal{A})$ *we define full abelian subcategories of* $\mathrm{Ch}(\mathcal{A})$ *by*

$$X^\bullet \in \mathrm{Ch}^+(\mathcal{A}) \Leftrightarrow \forall n \gg 0 : X^n = 0$$
$$X^\bullet \in \mathrm{Ch}^-(\mathcal{A}) \Leftrightarrow \forall n \ll 0 : X^n = 0$$
$$X^\bullet \in \mathrm{Ch}^b(\mathcal{A}) \Leftrightarrow \forall |n| \gg 0 : X^n = 0$$

and are called bounded above, bounded below *and* bounded cochain complexes *respectively. Let $*$ denote either of $+$, $-$ and b.*

These subcategories give rise to the full subcategories $\mathcal{K}^*$ of $\mathcal{K}$ respectively. Note that $\mathcal{K}^*$ is not replete in general, and hence is called a *fragile triangulated subcategory*. The following lemma ensures that the very same construction of the derived category can be applied to $\mathcal{K}^*$:

Definition and Lemma 9.2. *The respective full subcategories $\mathcal{N}^* \subseteq \mathcal{K}^*$ consisting of acyclic cochain complexes are triangulated subcategories and thus give rise to the respective Verdier quotients denoted $\mathcal{D}^*$. In particular, $\mathcal{D}^b$ is called the* bounded derived category*.*

A. Localising via Multiplicative Systems

For those attendants who have also attended the homological algebra lecture, the following approach to localisation of categories via *multiplicative systems* may be more familiar. In fact, both formulations are equivalent as sketched below.

Definition A.1. *A* multiplicative system *S of a category $\mathcal{C}$ is a collection of morphisms such that*

MS1 *it contains all identity morphisms and is closed under composition;*

MS2 *any diagram of the form*

$$\begin{tikzcd} * \arrow[r, dashed, "g"] \arrow[d, dashed, "t\in S"'] & * \arrow[d, "s\in S"] \\ * \arrow[r, "f"'] & * \end{tikzcd}$$

can be completed if either the solid or dashed morphisms are provided;

MS3 *for morphisms $f, g : X \to Y$, the relation $\exists s \in S : fs = gs \Leftrightarrow \exists t \in S : tf = tg$ holds.*

If furthermore $\mathcal{C}$ is triangulated, S is called to be compatible with the triangulated structure on $\mathcal{C}$ *if it is stable under the suspension functor and the completion of morphisms of triangles in S.*

Obviously, the morphisms contained in $\mathrm{Mor}_{\mathcal{C}}$ form a multiplicative system. Another importand kind of multiplicative systems are those arising from cohomological functors:

Definition A.2. *A contravariant functor $F : \mathcal{D} \to A$ of a triangulated category into an abelian category is called* cohomological *if it maps triangles $X \to Y \to Z \to \Sigma X$ to long exact sequences*

$$\cdots \to F\Sigma X \to FZ \to FY \to FX \to F\Sigma^{-1}Z \to \cdots .$$

The following theorem shows the relation between multiplicative systems, cohomological functors and localisation along triangulated subcategories [Ver96, prop. 2.1.17], [Wei94, prop. 10.4.1].

Theorem A.3. *Given a cohomological functor $F : \mathcal{D} \to \mathcal{A}$, the collection of morphisms in $\mathcal{D}$ mapped to isomorphisms by F, denoted $S(F)$, is a multiplicative system, called the* multiplicative system arising from F*. Namely it is the multiplicative system induced by the full subcategory of $\mathcal{D}$ consisting of objects $\{X \in \mathcal{D} : FX \cong 0\}$.*

In particular, the cohomology functor H^0 gives rise to the multiplicative system consisting of all quasi-isomorphisms.

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