Seminar in Derived Categories

5. Resolutions

Sílvia Cavadas

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The main goal of this talk is to introduce projective and h-projective resolutions of complexes in an abelian category and to show that under certain circumstances one has an equivalence of categories between the derived category/full subcategory of bounded above complexes of the derived category and an appropriate subcategory of the homotopy category, given by assigning to each complex X an h-projective/projective resolution of X. Although I won't develop it here, there is a dual theory using injective/h-injective resolutions, which goes through mostly in the same way.

Throughout this talk:

$\mathcal{A}$ = an abelian category

$C\mathcal{A}$ = corresponding category of cochain complexes

$K\mathcal{A}$ = corresponding homotopy category of cochain complexes

$K^-\mathcal{A} \subset K\mathcal{A}$ = full subcategory of bounded above complexes

$K_a\mathcal{A} \subset K\mathcal{A}$ = full subcategory of acyclic complexes

$K_a^-\mathcal{A} \subset K\mathcal{A}$ = full subcategory of acyclic bounded above complexes

$K_p^-\mathcal{A} \subset K\mathcal{A}$ full subcategory of bounded above complexes of projectives

$D\mathcal{A} = K\mathcal{A}/K_a\mathcal{A}$, the derived category of $\mathcal{A}$

$D^-\mathcal{A} = K^-\mathcal{A}/K_a^-\mathcal{A}$, equivalently $D^-\mathcal{A}$ = full subcategory of $D\mathcal{A}$ of bounded above complexes

1 Preparation: projective resolutions of objects

Definition 1.1. Let X be an object in $\mathcal{A}$. A *projective resolution* of X is a complex of projectives P such that $P^n = 0$ for $n > 0$ together with an augmentation map $P^0 \to X$ such that

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to X \to 0$$

is an acyclic complex.

Definition 1.2. $\mathcal{A}$ has *enough projectives* if for every object X there is an epic map $P \twoheadrightarrow X$ where P is projective.

Proposition 1.3. *$\mathcal{A}$ has enough projectives iff every object has a projective resolution.*

Proof. "$\Leftarrow$": clear. "$\Rightarrow$": given an object X, construct a projective resolution of X as indicated in the following diagram, where the existence of the dashed arrows follows from the fact that $\mathcal{A}$ has enough projectives:

$$\begin{tikzcd}
\cdots \arrow[r] & P^2 \arrow[rr] \arrow[dr, dashed, two heads, "f_2"] & & P^1 \arrow[rr] \arrow[dr, dashed, two heads, "f^1"] & & P^0 \arrow[r, dashed, two heads, "f_0"] & X \\
& & \ker(f_1) \arrow[ur, hook] & & \ker(f_0) \arrow[ur, hook] & &
\end{tikzcd}$$

∎

The following two propositions are useful tools when working with projective resolutions.

Proposition 1.4 (Horseshoe Lemma). *Suppose $\mathcal{A}$ has enough projectives. Then, given a short exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ in $\mathcal{A}$ and projective resolutions P_1 of X_1 and P_3 of X_3, there is a projective resolution P_2 of X_2 such that the diagram*

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1^{-1} \arrow[r] \arrow[d] & P_2^{-1} \arrow[r] \arrow[d] & P_3^{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1^{0} \arrow[r] \arrow[d] & P_2^{0} \arrow[r] \arrow[d] & P_3^{0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & 0
\end{tikzcd}
$$

commutes and has exact rows.

Proof. Construct P_2 as indicated in the diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1^{-1} \arrow[r] \arrow[d, two heads] \arrow[dr] & P_1^{-1} \oplus P_3^{-1} \arrow[r] \arrow[d, dashed, "\exists \varepsilon"] & P_3^{-1} \arrow[r] \arrow[d, two heads] \arrow[dl, dashed, "\exists l"] & 0 \\
0 \arrow[r] & Z_1^{0} \arrow[r] \arrow[d, hook] & Z_2^{0} \arrow[r] \arrow[d, hook] & Z_3^{0} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & P_1^{0} \arrow[r] \arrow[d, two heads] \arrow[dr] & P_1^{0} \oplus P_3^{0} \arrow[r] \arrow[d, dashed, "\exists \varepsilon"] & P_3^{0} \arrow[r] \arrow[d, two heads] \arrow[dl, dashed, "\exists l"] & 0 \\
0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & 0
\end{tikzcd}
$$

where l exists by projectivity of P_3^{-q} for $q \in \mathbb{N}$, and ε exists by the universal property of the coproduct. $\square$

Proposition 1.5. *Let P and A be complexes such that P^n is projective for $n \leq 0$ and $H^n(A) = 0$ for $n \leq 1$. Then, given a map $f\colon P^1 \to A^1$ such that the composition $P^1 \to A^1 \to A^2$ is zero, there is up to homotopy a unique map $F\colon P \to A$ satisfying $F^1 = f$ and $F^n = 0$ for $n > 1$. We call such F a "lift" of f.*

$$
\begin{tikzcd}
\cdots \arrow[r] & P^{-1} \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[d] & P^1 \arrow[r] \arrow[d, "f"] & P^2 \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & A^{-1} \arrow[r] & A^0 \arrow[r] & A^1 \arrow[r] & A^2 \arrow[r] & \cdots
\end{tikzcd}
$$

Proof. Existence: We construct F inductively. For $n \geq 1$, if F^n is already defined we define F^{n-1} as indicated in the diagram

$$
\begin{tikzcd}
P^{n-1} \arrow[rr] \arrow[dr, dashed, "\phi"] \arrow[dd, dashed, "F^{n-1}"'] & & P^n \arrow[r] \arrow[dd, "F^n"] & P^{n+1} \arrow[dd] \\
 & \ker(d_A^n) \arrow[dr, hook] & & \\
A^{n-1} \arrow[rr] \arrow[ur, two heads] & & A^n \arrow[r, "d_A^n"] & A^{n+1}
\end{tikzcd}
$$

where ϕ exists by the universal property of the kernel and F^{n-1} exists by projectivity of P^{n-1}.

Uniqueness: By passing to the difference of two lifts F and F' we reduce to showing that any lift of $f = 0$ is null homotopic. Supposing $f = 0$, we construct a homotopy h between F and the zero map. Let $h^n : P^n \to A^{n-1}$ be the zero map for $n \geq 2$. Now we define the rest of the homotopy inductively. If h_{n+1} is already defined for $n \leq 1$, h_n is defined as indicated in the diagram

$$
\begin{tikzcd}
 & & P^n \arrow[r, "d_P^n"] \arrow[dl, dashed, "\phi"'] \arrow[dd, "F^n - h_{n+1} d_P^n"] \arrow[ddll, dashed, bend right, "h^n"'] & P^{n+1} \arrow[dd] \\
 & \ker(d_A^n) \arrow[dr, hook] & & \\
A^{n-1} \arrow[rr] \arrow[ur, two heads] & & A^n \arrow[r, "d_A^n"] & A^{n+1}
\end{tikzcd}
$$

where ϕ exists by the universal property of the kernel since

$$d_A^n(F^n - h_{n+1}d_P^n) = d_A^n F^n - d_A^n h_{n+1} d_P^n = d_A^n F^n - (F^{n+1} - h^{n+2}d_P^{n+1})d_P^n = d_A^n F^n - F^{n+1}d_P^n = 0$$

and h^{n-1} exists by projectivity of P^n.

□

Corollary 1.6. *Suppose $\mathcal{A}$ has enough projectives. Then there exists up to natural isomorphism a unique functor $p\colon \mathcal{A} \to K\mathcal{A}$ sending each object X to a projective resolution of X and each morphism $f\colon X \to Y$ to the unique map $pX \to pY$ which fits into the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & pX^{-1} \arrow[r] \arrow[d] & pX^0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & pY^{-1} \arrow[r] & pY^0 \arrow[r] & Y \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

2 Projective resolutions of bounded above complexes

Definition 2.1. Let X be a bounded above complex. A *projective resolution* of X is a bounded above complex of projectives P together with a quasi-isomorphism $P \to X$.

Remark. Definition 1.1 is a particular case of Definition 2.1 considering the object X as a complex concentrated in degree 0 and requiring that the complex P satisfies $P^n = 0$ for $n > 0$.

The big aim of this secton is to show that, under the assumption that $\mathcal{A}$ has enough projectives, every bounded above complex has a projective resolution and moreover this gives an equivalence of categories $D^-\mathcal{A} \simeq K_p^-\mathcal{A}$. A crucial piece will be the following property of bounded above complexes of projectives, which follows directly from Proposition 1.5:

Proposition 2.2. *Let P be a bounded above complex of projectives. Then* $\mathrm{Hom}_{K\mathcal{A}}(P, A) = 0$ *for every acyclic complex A.*

Also central is the following notion:

Definition 2.3. Let X be a complex. A *Cartan-Eilenberg (projective) resolution* of X is a complex of complexes $P^{\bullet,\bullet} = \quad \cdots \to P^{-2,\bullet} \to P^{-1,\bullet} \to P^{0,\bullet}$ together with a map of complexes $P^{0,\bullet} \to X$ such that

(i) if $X^q = 0$, then $P^{\bullet,q} = 0$ is zero

(ii) for every q, the sequences

$\cdots \to B^q(P^{-2,\bullet}) \to B^q(P^{-1,\bullet}) \to B^q(P^{0,\bullet}) \to B^q(X)$
$\cdots \to H^q(P^{-2,\bullet}) \to H^q(P^{-1,\bullet}) \to H^q(P^{0,\bullet}) \to H^q(X)$
$\cdots \to Z^q(P^{-2,\bullet}) \to Z^q(P^{-1,\bullet}) \to Z^q(P^{0,\bullet}) \to Z^q(X)$
$\cdots \to P^{-2,q} \to P^{-1,q} \to P^{0,q} \to X$

are projective resolutions.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & P^{-1,-2} \arrow[r] \arrow[d] & P^{-1,-1} \arrow[r] \arrow[d] & P^{-1,0} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P^{0,-2} \arrow[r] \arrow[d] & P^{0,-1} \arrow[r] \arrow[d] & P^{0,0} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X^{-2} \arrow[r] & X^{-1} \arrow[r] & X^0 \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 2.4. *If $\mathcal{A}$ has enough projectives, every complex has a Cartan-Eilenberg resolution.*

Proof. For every $q \in \mathbb{Z}$ fix projective resolutions of $B^q(X)$ and $H^q(X)$. By the horseshoe lemma, there is a projective resolution of $Z^q(X)$ fitting in a commutative diagram with exact rows

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_B^{-1q} \arrow[r] \arrow[d] & P_Z^{-1q} \arrow[r] \arrow[d] & P_H^{-1q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_B^{0q} \arrow[r] \arrow[d] & P_Z^{0q} \arrow[r] \arrow[d] & P_H^{0q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^q(X) \arrow[r] & Z^q(X) \arrow[r] & H^q(X) \arrow[r] & 0
\end{tikzcd}$$

Similarly we get a projective resolution of X^q fitting in a commutative diagram with exact rows

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_Z^{-1q} \arrow[r] \arrow[d] & P^{-1q} \arrow[r] \arrow[d] & P_B^{-1,q+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_Z^{0q} \arrow[r] \arrow[d] & P^{0q} \arrow[r] \arrow[d] & P_B^{0,q+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^q(X) \arrow[r] & X^q \arrow[r] & B^{q+1}(X) \arrow[r] & 0
\end{tikzcd}$$

From this construction we obtain a map $P^{\bullet,q} \to P_B^{\bullet,q+1} \to P_Z^{\bullet,q+1} \to P^{\bullet,q+1}$.
Check: the double complex $P^{\bullet,\bullet}$ so defined is indeed a Cartan-Einlenberg resolution. $\square$

Proposition 2.5. *Suppose $\mathcal{A}$ has enough projectives. Then*

1) For every bounded above complex X there is a triangle

$$P \to X \to A \to SP$$

in $K\mathcal{A}$ such that P is a bounded above complex of projectives and A is an acyclic complex. In particular, by a long exact sequence argument $P \to X$ is a quasi-isomorphism, i.e. P is a projective resolution of X.

2) If $P \to X \to A \to SP$ and $P' \to Y \to A' \to SP'$ are triangles such that P, P' are bounded above complexes of projectives and A, A' are acyclic, for every morphism $f : X \to Y$ there is a unique morphism of triangles

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d] & SP \arrow[d] \\
P' \arrow[r] & Y \arrow[r] & A' \arrow[r] & SP'
\end{tikzcd}$$

extending f.

Proof. 1) Let X be a bounded above complex; suppose without loss of generality that $X^n = 0$ for $n > 0$. Let $P^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of X. By our hypothesis about X this is a third quadrant double complex, hence we can consider be the total complex P of $P^{\bullet,\bullet}$ and the total complex A of the augmented double complex $P^{\bullet,\bullet} \to X$.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & P^{-2,-2} \arrow[r] \arrow[d] & P^{-2,-1} \arrow[r] \arrow[d] & P^{-2,0} \arrow[d] \\
\cdots \arrow[r] & P^{-1,-2} \arrow[r] \arrow[d] & P^{-1,-1} \arrow[r] \arrow[d] & P^{-1,0} \arrow[d] \\
\cdots \arrow[r] & P^{0,-2} \arrow[r] \arrow[d] & P^{0,-1} \arrow[r] \arrow[d] & P^{0,0} \arrow[d] \\
\cdots \arrow[r] & X^{-2} \arrow[r] & X^{-1} \arrow[r] & X^{0}
\end{tikzcd}$$

The augmentation map induces a map $P \to X$ and, unwinding the definitions, one sees that the mapping cone of this map is precisely A. Thus we have a triangle $P \to X \to A \to SP$ in $K\mathcal{A}$. It remains to show:

i) *P is a bounded above complex of projectives*: since $P^{\bullet,\bullet}$ is a third quadrant double complex, $P^n = 0$ for $n > 0$. Moreover, every component of the total complex P is a sum of projectives and hence projective.

ii) *A is acyclic*: follows from the following fact (presented here without proof):

Lemma 2.6. *The total complex of a third quadrant double complex whose columns are acyclic is also acyclic.*

2) Apply the cohomological functor $\mathrm{Hom}_{K\mathcal{A}}(P, -)$ to the triangle $P' \to Y \to A' \to SP'$ to get an exact sequence

$$0 = \mathrm{Hom}_{K\mathcal{A}}(P, A) \to \mathrm{Hom}_{K\mathcal{A}}(P, P') \to \mathrm{Hom}_{K\mathcal{A}}(P, Y) \to \mathrm{Hom}_{K\mathcal{A}}(P, A) = 0.$$

It follows that $\mathrm{Hom}_{K\mathcal{A}}(P, P') \to \mathrm{Hom}_{K\mathcal{A}}(P, Y)$ is an isomorphism, so there is a unique map $g : P \to P'$ making the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, dashed, "\exists! g"] & X \arrow[d, "f"] \\ P' \arrow[r] & Y \end{tikzcd} \tag{1}$$

commute. Similarly, applying $\mathrm{Hom}(-, A')$ to the triangle $P \to X \to A \to SP$ gives an exact sequence

$$0 = \mathrm{Hom}_{K\mathcal{A}}(SP, A') \to \mathrm{Hom}_{K\mathcal{A}}(A, A') \to \mathrm{Hom}_{K\mathcal{A}}(X, A') \to \mathrm{Hom}_{K\mathcal{A}}(P, A') = 0,$$

so we conclude that $\mathrm{Hom}_{K\mathcal{A}}(A, A') \to \mathrm{Hom}_{K\mathcal{A}}(X, A')$ is an isomorphism and thus there is a unique map $h : A \to A'$ making the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & A \arrow[d, dashed, "\exists! h"] \\ Y \arrow[r] & A' \end{tikzcd} \tag{2}$$

commute. By the axioms of a triangulated category, the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "g"] & X \arrow[r] \arrow[d, "f"] & A \arrow[r] & SP \arrow[d] \\ P' \arrow[r] & Y \arrow[r] & A' \arrow[r] & SP' \end{tikzcd}$$

can be completed to a morphism of triangles and it follows that this morphism has to be

$$\begin{tikzcd} P \arrow[r] \arrow[d, "g"] & X \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, "h"] & SP \arrow[d] \\ P' \arrow[r] & Y \arrow[r] & A' \arrow[r] & SP' \end{tikzcd}$$

$\square$

In particular, we get a (unique up to natural isomorphism) functor

$$p\colon K^-\mathcal{A} \to K^-\mathcal{A}$$

sending each object X to a projective resolution of X and each morphism $f\colon X \to Y$ to the dashed morphism indicated in (1).

Proposition 2.7. *Let P be a bounded above complex of projectives. Then for every complex X the canonical map $\mathrm{Hom}_{K\mathcal{A}}(P, X) \to \mathrm{Hom}_{D\mathcal{A}}(P, X)$ is an isomorphism.*

Proof. First note that, if $X \to Y$ is a quasi-isomorphism, the canonical map $\mathrm{Hom}_{K\mathcal{A}}(P, X) \to \mathrm{Hom}_{K\mathcal{A}}(P, Y)$ is an isomorphism (apply $\mathrm{Hom}_{K\mathcal{A}}(P, -)$ to the triangle $X \to Y \to \mathrm{cone}(f) \to SX$ where $\mathrm{cone}(f)$ is acyclic and use a long exact sequence argument). We now show that $\mathrm{Hom}_{K\mathcal{A}}(P, X) \to \mathrm{Hom}_{D\mathcal{A}}(P, X)$ is bijective.

Surjectivity. Consider a morphism $\phi\colon P \to X$ in $D\mathcal{A}$ represented by a diagram $P \xrightarrow{f} Y \xleftarrow{t} X$ where t is a quasi-isomorphism. Then the following diagram shows that ϕ can be represented by a morphism $g : P \to X$ in $K\mathcal{A}$ whose existence comes from the surjectivity of the map $\mathrm{Hom}_{K\mathcal{A}}(P, X) \to \mathrm{Hom}_{K\mathcal{A}}(P, Y)$:

$$\begin{tikzcd}
 & Y \arrow[d, "\mathrm{id}"] & \\
P \arrow[ur, "f"] \arrow[r, "f"] \arrow[dr, dashed, "\exists g"'] & Y & X \arrow[ul, "t"'] \arrow[l, "t"] \arrow[dl, "\mathrm{id}"] \\
 & X \arrow[u, "t"] &
\end{tikzcd}$$

Injectivity. If $f, g\colon P \to X$ in $K\mathcal{A}$ represent the same map in $D\mathcal{A}$, we have a diagram

$$\begin{tikzcd}
 & X \arrow[d, "h"] & \\
P \arrow[ur, "f"] \arrow[r] \arrow[dr, "g"'] & Y & X \arrow[ul, "\mathrm{id}"'] \arrow[l, "h"] \arrow[dl, "\mathrm{id}"] \\
 & X \arrow[u, "h"] &
\end{tikzcd}$$

From the injectivity of the map $\mathrm{Hom}_{K\mathcal{A}}(P, X) \to \mathrm{Hom}_{K\mathcal{A}}(P, Y)$ follows $f = g$.

□

Suppose for the rest of this section that $\mathcal{A}$ has enough projectives.

Proposition 2.8. *The canonical functor $q\colon K_p^-\mathcal{A} \to D^-\mathcal{A}$ gives an equivalence of categories.*

Proof. q is essentially surjective because every bounded above complex X is isomorphic in $D^-\mathcal{A}$ to pX, and Proposition 2.7 shows that q is fully faithful. □

Proposition 2.9. *$p\colon K^-A \to K^-\mathcal{A}$ descends to a functor $p\colon D^-\mathcal{A} \to K^-\mathcal{A}$*

Proof. We show that p sends quasi-isomorphisms to isomorphisms: if $f : X \to Y$ is a quasi-isomorphism between bounded above complexes, we have a square

$$\begin{tikzcd}
pX \arrow[r] \arrow[d, "pf"] & X \arrow[d, "f"] \\
pY \arrow[r] & Y
\end{tikzcd}$$

where f and the two horizontal maps are quasi-isomorphisms, so also pf is a quasi-isomorphism. Then pf corresponds to an isomorphism under the equivalence $K_p^-\mathcal{A} \xrightarrow{\sim} D^-\mathcal{A}$ and hence also pf is an isomorphism. □

Proposition 2.10. *We have an adjunction $p : D^-\mathcal{A} \to K^-\mathcal{A} : q$. In particular, $p\colon D^-\mathcal{A} \to K_p^-\mathcal{A}$ is a left adjoint and quasi-inverse of $q\colon K_p^-\mathcal{A} \to D^-\mathcal{A}$.*

Proof. Given $X \in D^-\mathcal{A}$ and $Y \in K^-\mathcal{A}$, we have natural isomorphisms

$$\mathrm{Hom}_{K\mathcal{A}}(pX, Y) \xrightarrow{\sim} \mathrm{Hom}_{D\mathcal{A}}(pX, Y) \xleftarrow{\sim} \mathrm{Hom}_{D\mathcal{A}}(X, Y)$$

where the first map is the canonical one, which is an isomorphism by the previous proposition, and the second one is induced from the canonical map $pX \to X$ which is an isomorphism in $D\mathcal{A}$. Checking naturality, i.e. commutativity of the following diagrams for each morphism $X \to X'$ and $Y \to Y'$, is straightforward.

$$\begin{tikzcd}
\mathrm{Hom}_{K\mathcal{A}}(pX, Y) \arrow[r] & \mathrm{Hom}_{D\mathcal{A}}(pX, Y) & \mathrm{Hom}_{D\mathcal{A}}(X, Y) \arrow[l] \\
\mathrm{Hom}_{K\mathcal{A}}(pX', Y) \arrow[r] \arrow[u] & \mathrm{Hom}_{D\mathcal{A}}(pX', Y) \arrow[u] & \mathrm{Hom}_{K\mathcal{A}}(X', Y) \arrow[l] \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_{K\mathcal{A}}(pX, Y) \arrow[r] \arrow[d] & \mathrm{Hom}_{D\mathcal{A}}(pX, Y) \arrow[d] & \mathrm{Hom}_{D\mathcal{A}}(X, Y) \arrow[l] \arrow[d] \\
\mathrm{Hom}_{K\mathcal{A}}(pX, Y') \arrow[r] & \mathrm{Hom}_{D\mathcal{A}}(pX, Y') & \mathrm{Hom}_{K\mathcal{A}}(X, Y') \arrow[l]
\end{tikzcd}$$

□

3 Homotopically projective resolutions of complexes

Noticing that in the previous section the only property of bounded above complexes of projectives that was essentially used was the fact that all the maps from such a complex to an acyclic complex are null homotopic, we are led to the following natural generalization of bounded above complexes of projectives:

Definition 3.1. A complex X is called *homotopically projective* (*h-projective*) if $\mathrm{Hom}_{K\mathcal{A}}(X, A) = 0$ for every acyclic complex A.

Examples. 1) Bounded above complexes of projectives are homotopically projective.

2) (Arbitrary) sums of homotopically projective complexes are homotopically projective.

3) Complexes of projectives with vanishing differential are h-projective.

Definition 3.2. Let X be a complex. An *h-projective resolution* of X is an h-projective complex P together with a quasi-isomorphism $P \to X$.

Let $K_{hp}\mathcal{A} \subset K\mathcal{A}$ be the full subcategory of h-projective complexes. The big aim of this secton is to show, in resemblance with the last section, that, in the category of R-modules, every complex has an h-projective resolution and moreover this gives an equivalence of categories $D\mathcal{A} \simeq K_{hp}\mathcal{A}$. We follow precisely the same steps as in last section, ommiting proofs when they go through basically unchanged.

Lemma 3.3. *Let $X \to Y \to Z \to SX$ be a triangle in $K\mathcal{A}$. If two of the complexes X,Y,Z are h-projective, so is the third.*

Proof. For each acyclic complex A, apply the cohomological functor $\mathrm{Hom}_{K\mathcal{A}}(-, A)$ to the triangle to obtain a long exact sequence

$$\cdots \to \mathrm{Hom}_{K\mathcal{A}}(X, A) \to \mathrm{Hom}_{K\mathcal{A}}(Y, A) \to \mathrm{Hom}_{K\mathcal{A}}(Z, A) \to \mathrm{Hom}_{K\mathcal{A}}(SX, A) \to \mathrm{Hom}_{K\mathcal{A}}(SY, A) \to \cdots$$

If two of the complexes are h-projective, two terms in every three in the sequence are zero, and it follows that the remaining terms are also zero. Hence the third complex is also h-projective. □

Lemma 3.4. *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $C\mathcal{A}$ which is split in every component. If two of the complexes X,Y,Z are h-projective, so is the third.*

Proof. Follows from the previous lemma using the fact (seen in the first talk) that a componentwise split short exact sequence in $C\mathcal{A}$ induces a triangle in $K\mathcal{A}$. □

From now on we set $\mathcal{A} = R$-mod for some ring R.

Proposition 3.5. *Let*

$$P_0 \xrightarrow{i_0} P_1 \xrightarrow{i_1} \cdots \to P_q \xrightarrow{i_q} \cdots$$

be a sequence of complexes in $C\mathcal{A}$ such that each morphism i_q has split injective components and all the subquotients P_{q+1}/P_q are h-projective. Then $\varinjlim P_q$ is h-projective.

Proof. First we show by induction that every P_q is h-projective.

Base case: $P_0 = P_0/0 = P_0/P_{-1}$ is h-projective by hypothesis.

Induction step: For each $q \in \mathbb{N}$, $0 \longrightarrow P_q \xrightarrow{i_q} P_{q+1} \longrightarrow P_{q+1}/P_q \longrightarrow 0$ is a component-wise split short exact sequence. Since P_{q+1}/P_q is h-projective by hypothesis, if P_q is h-projective then Lemma 3.4 implies that also P_{q+1} is h-projective.

In particular, the sum of all the P_q is h-projective. Now consider the short exact sequence

$$0 \longrightarrow \bigoplus_{p\in\mathbb{N}} P_p \xrightarrow{\phi} \bigoplus_{q\in\mathbb{N}} P_q \longrightarrow \varinjlim P_q \longrightarrow 0 \tag{3}$$

where

$$\phi = \begin{bmatrix} 1 & & & & \\ -i_0 & 1 & & & \\ & -i_1 & 1 & & \\ & & -i_2 & 1 & \\ & & & & \ddots \end{bmatrix}$$

This sequence splits component-wise: if $s_q : P_{q+1} \to P_q$ (not necessarily a morphism of complexes) is a component-wise left inverse of i_q, then setting

$$\sigma = \begin{bmatrix} 0 & -s_0 & -s_0 s_1 & -s_0 s_1 s_2 & \\ & 0 & -s_1 & -s_1 s_2 & \\ & & 0 & -s_2 & \\ & & & & \ddots \end{bmatrix}$$

(again not necessarily a map of complexes) we get $\sigma\phi = \mathrm{id}$, so σ is a component-wise split of (3). Hence, by lemma 3.3 it follows that $\varinjlim P_q$ is h-projective.

$\square$

Proposition 3.6.

1) For every complex X there is a triangle

$$pX \to X \to aX \to SpX$$

in $K\mathcal{A}$ such that pX is h-projective and aX is acyclic. In particular, pX is an h-projective resolution of X.

2) If $P \to X \to A \to SP$ and $P' \to Y \to A' \to SP'$ are triangles such that P, P' are h-projective and A, A' are acyclic, for every morphism $f : X \to Y$ there is a unique morphism of triangles

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d] & SP \arrow[d] \\ P' \arrow[r] & Y \arrow[r] & A' \arrow[r] & SP' \end{tikzcd}$$

extending f.

Proof. 1) Again consider a Cartan-Eilenberg resolution $P^{\bullet,\bullet}$ of X. Since in R-mod there are arbitrary sums, we can form the total complex P of $P^{\bullet,\bullet}$ and the total complex A of the augmented double complex $P^{\bullet,\bullet} \to X$.

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & P^{-2,-2} \arrow[r] \arrow[d] & P^{-2,-1} \arrow[r] \arrow[d] & P^{-2,0} \arrow[r] \arrow[d] & P^{2,1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & P^{-1,-2} \arrow[r] \arrow[d] & P^{-1,-1} \arrow[r] \arrow[d] & P^{-1,0} \arrow[r] \arrow[d] & P^{1,1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & P^{0,-2} \arrow[r] \arrow[d] & P^{0,-1} \arrow[r] \arrow[d] & P^{0,0} \arrow[r] \arrow[d] & P^{0,1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & X^{-2} \arrow[r] & X^{-1} \arrow[r] & X^{0} \arrow[r] & X^{1} \arrow[r] & \cdots \end{tikzcd}$$

As before we have a triangle $P \to X \to A \to SP$ in $K\mathcal{A}$. It remains to show:

i) *P is h-projective*: Let $P^{\leq p}$ be the total complex of the double complex

$$\begin{tikzcd} \cdots \arrow[r] & P^{p,-2} \arrow[r] \arrow[d] & P^{p,-1} \arrow[r] \arrow[d] & P^{p,0} \arrow[r] \arrow[d] & P^{p,1} \arrow[r] \arrow[d] & \cdots \\ & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & P^{-1,-2} \arrow[r] \arrow[d] & P^{-1,-1} \arrow[r] \arrow[d] & P^{-1,0} \arrow[r] \arrow[d] & P^{1,1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & P^{0,-2} \arrow[r] & P^{0,-1} \arrow[r] & P^{0,0} \arrow[r] & P^{0,1} \arrow[r] & \cdots \end{tikzcd}$$

obtained by truncating $P^{\bullet,\bullet}$ at the pth row. Then P is the colimit of the sequence

$$P^{\leq 0} \subset P^{\leq 1} \subset P^{\leq 2} \subset \cdots$$

where each inclusion has split injective components and each quotient $P^{\leq p}/P^{\leq p-1}$ is isomorphic to $P^{p,\bullet}$. By Proposition 3.5, to conclude that P is h-projective it is enough to show

Lemma 3.7. *Each row $P^{p,\bullet}$ of a Cartan-Eilenberg resolution is h-projective.*

Proof. Since $P^{p,\bullet}$ has projective boundaries and homology, we have split short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z^q(P^{p,\bullet}) \arrow[r] & P^{p,q} \arrow[r] & B^{q+1}(P^{p,\bullet}) \arrow[l, bend right, dashed, "s"'] \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & B^q(P^{p,\bullet}) \arrow[r] & Z^q(P^{p,\bullet}) \arrow[r] & H^q(P^{p,\bullet}) \arrow[l, bend right, dashed, "t"'] \arrow[r] & 0 \end{tikzcd}$$

Hence $P^{p,q} = t(H^q(P^{p,\bullet})) \oplus B^q(P^{p,\bullet}) \oplus s(B^{q+1}(P^{p,\bullet}))$ and $P^{p,\bullet}$ is the sum of

$$\cdots \xrightarrow{0} t(H^{q-1}(P^{p,\bullet})) \xrightarrow{0} t(H^q(P^{p,\bullet})) \xrightarrow{0} t(H^{q+1}(P^{p,\bullet})) \xrightarrow{0} \cdots \tag{4}$$

with the family of complexes of the form

$$0 \longrightarrow s(B^q(P^{p,\bullet})) \longrightarrow B^q(P^{p,\bullet}) \longrightarrow 0 \tag{5}$$

for $q \in \mathbb{N}$. Since (4) is h-projective (being a complex of projectives with vanishing differential) and also each complex of the form (5) is h-projective (being a bounded complex of projectives), we conclude that the sum $P^{p,\bullet}$ is h-projective. $\square$

ii) A *is acyclic*: For each $q \in \mathbb{Z}$ let F^q be the total complex of the truncated augmented double complex

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ \cdots \arrow[r] & P^{-2,q-2} \arrow[r] \arrow[d] & P^{-2,q-1} \arrow[r] \arrow[d] & P^{-2,q} \arrow[r] \arrow[d] & Z^{q+1}(P^{-2,\bullet}) \arrow[d] \\ \cdots \arrow[r] & P^{-1,q-2} \arrow[r] \arrow[d] & P^{-1,q-1} \arrow[r] \arrow[d] & P^{-1,q} \arrow[r] \arrow[d] & Z^{q+1}(P^{-1,\bullet}) \arrow[d] \\ \cdots \arrow[r] & P^{0,q-2} \arrow[r] \arrow[d] & P^{0,q-1} \arrow[r] \arrow[d] & P^{0,q} \arrow[r] \arrow[d] & Z^{q+1}(P^{0,\bullet}) \arrow[d] \\ \cdots \arrow[r] & X^{q-2} \arrow[r] & X^{q-1} \arrow[r] & X^q \arrow[r] & Z^{q+1}(X) \end{tikzcd}$$

Then $A = \varinjlim F^q$ and so, by the following fact (presented here without proof), it is enough to show that each F^q is acyclic.

Lemma 3.8. *Let $\cdots \to X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots$ be a sequence of complexes of modules. Then, for every $i \in \mathbb{Z}$, $H_i(\varinjlim X_n) = \varinjlim H_i(X_n)$.*

But F^q is the total complex of a third quadrant double complex with acyclic columns and thus is acyclic by Lemma 2.6.

$\square$

In particular, we get a (unique up to natural isomorphism) functor

$$p \colon K\mathcal{A} \to K\mathcal{A}$$

sending each object X to an h-projective resolution of X.

Proposition 3.9. *Let P be an h-projective complex. Then for every complex X the canonical map* $\mathrm{Hom}_{K\mathcal{A}}(P, X) \to \mathrm{Hom}_{D\mathcal{A}}(P, X)$ *is an isomorphism.*

Proposition 3.10. *The canonical functor $q \colon K_{hp}\mathcal{A} \to D\mathcal{A}$ gives an equivalence of categories.*

Proposition 3.11. *$p \colon K\mathcal{A} \to K\mathcal{A}$ descends to a functor $p \colon D\mathcal{A} \to K\mathcal{A}$.*

Proposition 3.12. *We have an adjunction $p : D\mathcal{A} \rightleftarrows K\mathcal{A} : q$. In particular, $p \colon D\mathcal{A} \to K_{hp}\mathcal{A}$ is a left adjoint and quasi-inverse of $q \colon K_{hp}\mathcal{A} \to D\mathcal{A}$.*