1. Problems on Lectures I & II

In all the problems below $\mathcal{T}$ will be a triangulated category.

Problem 1.1. Let A be an object of $\mathcal{T}$. Prove that the representable functor $\mathrm{Hom}(A, -)$ is a homological functor from $\mathcal{T}$ to the category of abelian groups. [Hint: If $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ is a triangle, use TR2 to construct maps from rotations of the triangle $0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0$ to it].

Problem 1.2. Prove that $X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow \Sigma X$ is a triangle if and only if f is an isomorphism.

Definitions 1.3 and 1.4 are not really of any intrinsic interest, but may be helpful in Problem 1.7.

Definition 1.3. *A* candidate triangle *is a sequence $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in $\mathcal{T}$ so that the composites vu, wv and $(\Sigma u)w$ all vanish.*

Definition 1.4. *A* pretriangle *is a candidate triangle such that any product-preserving homological functor from $\mathcal{T}$ to abelian groups takes it to an exact sequence.*

Problem 1.5. Prove that every triangle is a pretriangle, and that products and direct summands of pretriangles (when they exist) are pretriangles.

Problem 1.6. Assume

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & {\scriptstyle h}\big\downarrow & & {\scriptstyle \Sigma f}\big\downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'
\end{array}$$

is a morphism of pretriangles, ie all the squares commute. If f and g are isomorphisms prove that so is h. [Hint: show that every product-preserving homological functor takes h to an isomorphism, and apply Problem 1.1].

Problem 1.7. Prove that coproducts and products of triangles (when they exist) are triangles, and that direct summands of triangles are triangles. [Hint: it suffices to consider products and direct summands, coproducts being dual to products. By Problem 1.5 they are pretriangles, and by Problem 1.6 it suffices to produce morphisms to or from triangles to them where f and g are isomorphisms.]

Problem 1.8. If $X \longrightarrow Y \longrightarrow Z \xrightarrow{0} \Sigma X$ is a triangle, ie if the map $Z \longrightarrow \Sigma X$ is zero, then prove that it is isomorphic to the direct sum of the triangles $X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$ and $0 \longrightarrow Z \xrightarrow{1} Z \longrightarrow 0$. Deduce that a morphism $Y \longrightarrow Z$ is epi if and only if it is split epi, that is of the form $X \oplus Z \longrightarrow Z$.

For the remaining exercises assume $\mathcal{T}$ has countable coproducts.

Definition 1.9. *Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \dots$ be a sequence of objects and connecting morpisms in $\mathcal{T}$. The* homotopy colimit *is defined to be the third edge of the triangle*

$$\coprod_{i=1}^{\infty} X_i \xrightarrow{1-f} \coprod_{i=1}^{\infty} X_i \longrightarrow \underrightarrow{\mathrm{Hocolim}}\, X_i \longrightarrow \Sigma\left(\coprod_{i=1}^{\infty} X_i\right)$$

Problem 1.10. Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \dots$ be a sequence in $\mathcal{T}$. Prove:

(i) If $f_i = 0$ for all i then $\underrightarrow{\mathrm{Hocolim}}\, X_i = 0$.

(ii) If f_i is an isomorphism for every i then the natural map $X_1 \longrightarrow \underrightarrow{\mathrm{Hocolim}}\, X_i$ is an isomorphism.

[Hint: for (ii) you might wish to show that in the triangle

$$\coprod_{i=1}^{\infty} X_i \xrightarrow{1-f} \coprod_{i=1}^{\infty} X_i \longrightarrow \underrightarrow{\mathrm{Hocolim}}\, X_i \longrightarrow \Sigma\left(\coprod_{i=1}^{\infty} X_i\right)$$

the map $1 - f$ is a split mono.]

Problem 1.11. If $e : X \longrightarrow X$ is an idempotent, that is if $e^2 = e$, prove that X splits as $X \cong Y \oplus Z$ so that e is isomorphic to the map $1 \oplus 0\colon Y \oplus Z \longrightarrow Y \oplus Z$. [Hint: let Y be the homotopy colimit of the sequence $X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \dots$ and let Z be the homotopy limit of the sequence $X \xrightarrow{1-e} X \xrightarrow{1-e} X \xrightarrow{1-e} \dots$].

Problem 1.12. Let C be a compact object of $\mathcal{T}$, and let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \dots$ be a sequence. Prove that

$$\mathrm{Hom}(C, \underrightarrow{\mathrm{Hocolim}}\, X_i) \cong \underrightarrow{\mathrm{colim}}\big[\mathrm{Hom}(C, X_i)\big] .$$

Problem 1.13. Let $\mathcal{A}$ be an abelian category and $\mathbf{D}(\mathcal{A})$ its derived category. Let P be a projective object in $\mathcal{A}$.

For any objects $X, Y \in \mathbf{D}(\mathcal{A})$ the homological functor H induces a map

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X, Y) \xrightarrow{\varphi} \mathrm{Hom}_{\mathcal{A}}\big(H^0(X), H^0(Y)\big)$$

Prove that if X is the complex P then φ is an isomorphism. Deduce that, if P is a projective generator for $\mathcal{A}$, then it is also a weak generator for $\mathbf{D}(\mathcal{A})$.

Open question: is P necessarily a generator?

Problem 1.14. Let R be a ring, and let $\mathbf{D}(R) = \mathbf{D}(R\text{–Mod})$ be the derived category of the abelian category of R–modules. Prove that R is a compact generator for $\mathbf{D}(R)$. [Hint: use Problem 1.13].

Problem 1.15. Let X be a quasiprojective scheme and let $\mathcal{L}$ be an ample line bundle. Prove that the objects $\{\mathcal{L}^n \mid n \in \mathbb{Z}\}$ are compact generators.