

SOME ENHANCEMENTS OF DERIVED CATEGORIES OF COHERENT SHEAVES AND APPLICATIONS

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ABSTRACT. The following text is our contribution to the Oberwolfach report on the workshop "Interactions between Algebraic Geometry and Noncommutative Algebra" in May 2014.

Let X be a quasi-projective scheme over a field k . The unbounded derived category $D(\mathrm{Qcoh}(X))$ of quasi-coherent sheaves on X contains the bounded derived category $D^b(\mathrm{Coh}(X))$ of coherent sheaves and the category of perfect complexes $\mathfrak{P}erf(X)$ as full triangulated subcategories,

$$\mathfrak{P}erf(X) \subset D^b(\mathrm{Coh}(X)) \subset D(\mathrm{Qcoh}(X)).$$

The objects of $\mathfrak{P}erf(X)$ can be characterized as the compact objects of $D(\mathrm{Qcoh}(X))$.

Recall that an enhancement of a triangulated (k -)category \mathcal{T} is a pretriangulated dg (k -)category \mathcal{A} together with an equivalence $[\mathcal{A}] \cong \mathcal{T}$ where $[\mathcal{A}]$ is the homotopy category of \mathcal{A} . For example, the full dg subcategory $C(\mathrm{Qcoh}(X))_{\mathrm{h-inj}}$ of h-injective objects of the dg category $C(\mathrm{Qcoh}(X))$ of complexes of quasi-coherent sheaves forms an enhancement of $D(\mathrm{Qcoh}(X))$. The obvious full dg subcategories of $C(\mathrm{Qcoh}(X))_{\mathrm{h-inj}}$ define enhancements of $D^b(\mathrm{Coh}(X))$ and $\mathfrak{P}erf(X)$.

Fix a finite affine open covering $(U_i)_{i=1}^n$ of X and consider for any complex P of vector bundles on X its $*$ -Čech resolution

$$\mathcal{C}_*(P) = \left(\prod_i U_i P \rightarrow \prod_{i < j} U_{ij} P \rightarrow \dots \right)$$

where ${}_U P = u_* u^* P$ for $u: U \hookrightarrow X$. If P is a bounded complex of vector bundles the complex $\mathcal{C}_*(P)$ is defined as the obvious totalization.

For integral X and bounded complexes P and Q of vector bundles the canonical map

$$\mathrm{Hom}_{[C(\mathrm{Qcoh}(X))]}(\mathcal{C}_*(P), \mathcal{C}_*(Q)) \rightarrow \mathrm{Hom}_{D(\mathrm{Qcoh}(X))}(\mathcal{C}_*(P), \mathcal{C}_*(Q))$$

is an isomorphism and hence the full dg subcategory of $C(\mathrm{Qcoh}(X))$ consisting of $*$ -Čech resolutions $\mathcal{C}_*(P)$ of bounded complexes P of vector bundles forms an enhancement of $\mathfrak{P}erf(X)$. For smooth projective X this enhancement was considered in [BLL04].

We found a modification of this construction which provides an enhancement of $\mathfrak{P}erf(X)$ for any quasi-projective scheme X over k . The objects of this enhancement are bounded complexes of vector bundles and the morphism spaces are suitably defined dg submodules of $\mathrm{Hom}_{C(\mathrm{Qcoh}(X))}(\mathcal{C}_*(P), \mathcal{C}_*(Q))$. Using $!$ -Čech resolutions we similarly define enhancements of $\mathfrak{P}erf(X)$ and $D^b(\mathrm{Coh}(X))$. We call these enhancements Čech enhancements.

Recall that a dg category \mathcal{A} is called k -smooth if the diagonal dg bimodule \mathcal{A} is a compact object of the derived category $D(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})$ of dg $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ -modules.

We say that $\mathfrak{P}erf(X)$ (resp. $D^b(\text{Coh}(X))$) is smooth over k if its h-injective enhancement is k -smooth as a dg category.

Using Čech enhancements we prove the following three theorems.

Theorem 1 (Homological versus geometric smoothness). *Let $\Delta: X \rightarrow X \times X$ be the diagonal immersion. The following three conditions are equivalent:*

- (1) $\mathfrak{P}erf(X)$ is smooth over k ;
- (2) $\Delta_*(\mathcal{O}_X) \in \mathfrak{P}erf(X \times X)$;
- (3) X is smooth over k .

Theorem 2. *If the field k is perfect then $D^b(\text{Coh}(X))$ is smooth over k .*

Theorem 3 (Fourier-Mukai kernels and dg bimodules). *Let X and Y be quasi-projective schemes over a field k and consider the projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$. Then there are dg algebras A and B and equivalences of triangulated categories*

$$\begin{aligned} \theta_X: D(\text{Qcoh}(X)) &\xrightarrow{\sim} D(A), \\ \theta_Y: D(\text{Qcoh}(Y)) &\xrightarrow{\sim} D(B), \\ \theta_{X \times Y}: D(\text{Qcoh}(X \times Y)) &\xrightarrow{\sim} D(A^{\text{op}} \otimes B), \end{aligned}$$

such that for any $K \in D(\text{Qcoh}(X \times Y))$ with corresponding $M = \theta_{X \times Y}(K)$ the diagram

$$\begin{array}{ccc} D(\text{Qcoh}(X)) & \xrightarrow{\mathbf{R}q_*(p^*(-) \otimes^{\mathbf{L}} K)} & D(\text{Qcoh}(Y)) \\ \theta_X \downarrow \sim & & \theta_Y \downarrow \sim \\ D(A) & \xrightarrow{- \otimes_A^{\mathbf{L}} M} & D(B) \end{array}$$

commutes up to an isomorphism of triangulated functors.

Some variants of these results are claimed in the literature without proof or with gaps in the proofs, cf. the discussion in [LS14].

All three theorems admit short heuristic arguments making them plausible. However, turning these arguments into rigorous proofs seems to be hard. The different functors involved (inverse image, tensor product, direct image, $\mathbf{R}Hom$) are usually computed via different types of replacements (h-flat, h-injective) which makes it difficult to treat them compatibly. It would be desirable to lift all these functors and the adjunctions among them from the derived level to the dg level of enhancements.

For simplicity we stated our results here for quasi-projective schemes over a field. This assumption can be weakened, see [LS14], where the above theorems and some more results characterizing properness are proved.

REFERENCES

- [BLL04] Alexey I. Bondal, Michael Larsen, and Valery A. Lunts, *Grothendieck ring of pretriangulated categories*, Int. Math. Res. Not. (2004), no. 29, 1461–1495.
- [LS14] Valery A. Lunts and Olaf M. Schnürer, *New enhancements of derived categories of coherent sheaves and applications*, in preparation (2014).