Let $H : \mathrm{Thick}_\ell(G) \longrightarrow k\text{-mod}$ be a functor.
Let C_ℓ be the following category:

<u>Objects:</u> pairs (A, x), with $A \in \mathrm{Thick}_\ell(G)$ and $x \in H(A)$

<u>Morphisms:</u> A map $(A, x) \longrightarrow (B, y)$ is a morphism $A \xrightarrow{f} B$ in Thick_ℓ so that $H(f)(x) = y$.

The next exercises prove that if

(*) H is <u>homological</u> and ~~$H(X)$ is~~ $\bigoplus_{i \in \mathbb{Z}} H(\Sigma^i X)$ is a <u>finite</u> k-module for all $X \in \mathrm{Thick}_\ell(G)$

then the category C_ℓ has a weakly initial object.

<u>Problem 1</u>: Prove the case $\ell = 1$.

<u>Problem 2</u>: Suppose true for C_ℓ, we want to prove for $C_{2\ell}$. Let H be a functor on $C_{\ell+1}$ satisfying (*). Then H restricts to a functor on $C_\ell \subset C_{2\ell}$ satisfying *, have there exists a weakly initial object in C_ℓ. Call it (A, x).

by Yoneda this gives an epimorphism of functors on Thick_ℓ

$$\text{Hom}(A,-) \longrightarrow H(-)$$

In the short exact sequence

$$0 \longrightarrow H'(-) \longrightarrow \text{Hom}(A,-) \longrightarrow H(-) \longrightarrow 0$$

the functor $H'(-)$ also satisfies $(*)$, and by induction we get an object $B \in \text{Thick}_\ell$ and an exact sequence

$$\text{Hom}(B,-) \longrightarrow \text{Hom}(A,-) \longrightarrow H(-) \longrightarrow 0$$

Complete $A \longrightarrow B$ to a triangle

$$C \longrightarrow A \longrightarrow B.$$

Prove that there ~~way~~ is an element ~~of~~ $y \in H(C)$ so that ~~$x+y$~~ $(A \oplus C,\ x+y)$ is a weak generator of $\mathcal{C}_{2\ell}$

Let $Z \subset X$ be a closed set, and let $\mathcal{L}$ be a line bundle.

Problem 1 There exist finitely many sections of $H^0(\mathcal{L}^n)$ vanishing exactly on Z.

Problem 2 Let $\sigma_i : \mathcal{O}_X \longrightarrow \mathcal{L}^n$ be the sections of Problem 1. Then

$$K = \bigotimes_{i} \left(\mathcal{O}_X \xrightarrow{\sigma_i} \mathcal{L}^n \right)$$ is

a compact object supported at Z.

Problem 3 Show that if $\mathcal{S} \neq 0$ is supported on Z then

$$R\mathcal{H}om(K, \mathcal{S}) \neq 0.$$

[Hint: Up to suspensions and tensoring with a power of $\mathcal{L}$, we have $R\mathcal{H}om(K, \mathcal{S}) \simeq K \otimes \mathcal{S}$

It suffices to show that

$$\mathcal{S} \otimes [\mathcal{O} \xrightarrow{\sigma_i} \mathcal{L}^n] = 0 \implies \mathcal{S} = 0]$$

~~Hence~~

<u>Problem 4</u> Deduce that the objects $\{K \otimes \mathcal{L}^n \mid n \in \mathbb{Z}\}$ weakly generate the category $D_Z(X)$.

<u>Problem 5</u> Show that

$$\mathrm{Loc}\{K \otimes \mathcal{L}^n \mid n \in \mathbb{Z}\} = D_Z(X),$$ and

hence $\dfrac{D(X)}{\mathrm{Loc}\{K \otimes \mathcal{L}^n \mid n \in \mathbb{Z})} \cong D(U)$.