EQUIVARIANT SHEAVES ON FLAG VARIETIES

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Abstract. We show that the Borel-equivariant derived category of sheaves on the flag variety of a complex reductive group is equivalent to the perfect derived category of dg modules over the extension algebra of the direct sum of the simple equivariant perverse sheaves. This proves a conjecture of Soergel and Lunts in the case of flag varieties.

Contents

1. Introduction

Let G be a complex connected reductive affine algebraic group and $B \subset P \subset G$ a Borel and a parabolic subgroup. The main result of this article is an algebraic description of the B-equivariant (bounded, constructible) derived category $\mathcal{D}^{\mathrm{b}}_{B,c}(X)$ (see [BL94]) of sheaves of real vector spaces on the partial flag variety $X := G/P$.

Let $\mathcal{S}$ be the stratification of X into B-orbits and $\mathcal{IC}_B(S) \in \mathcal{D}^{\mathrm{b}}_{B,c}(X)$ the equivariant intersection cohomology complex of the closure of the stratum $S \in \mathcal{S}$. The $(\mathcal{IC}_B(S))_{S\in\mathcal{S}}$ are the simple equivariant perverse sheaves on X. Let $\mathcal{IC}_B(\mathcal{S})$ be their direct sum and $\mathcal{E} = \mathrm{Ext}(\mathcal{IC}_B(\mathcal{S}))$ its graded algebra of self-extensions in $\mathcal{D}^{\mathrm{b}}_{B,c}(X)$. We consider $\mathcal{E}$ as a differential graded (dg) algebra with differential $d = 0$. Let $\mathrm{dgDer}(\mathcal{E})$ be the derived category of (right) dg $\mathcal{E}$-modules (see [Kel94]) and $\mathrm{dgPer}(\mathcal{E})$ the perfect derived category, i. e. the smallest strict full triangulated subcategory of $\mathrm{dgDer}(\mathcal{E})$ containing $\mathcal{E}$ and closed under forming direct summands. We give alternative descriptions of $\mathrm{dgPer}(\mathcal{E})$ below.

Theorem 1 (cf. Theorem 71)**.** *There is an equivalence of triangulated categories*

$$\mathcal{D}^{\mathrm{b}}_{B,c}(X) \cong \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}_B(\mathcal{S}))).$$

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Similar equivalences between equivariant derived categories and categories of dg modules over the extension algebra of the simple equivariant perverse sheaves are known for a connected Lie group acting on a point ([BL94, 12.7.2]), for a torus acting on an affine or projective normal toric variety ([Lun95]), and for a complex semisimple adjoint group acting on a smooth complete symmetric variety ([Gui05]). The key point in the proof of these equivalences is the formality of some dg algebra whose cohomology is the extension algebra.

Conjecturally ([Lun95, 0.1.3], [Soe01, 4]), the analog of Theorem 1 should hold for the equivariant derived category of a complex reductive group acting on a projective variety with a finite number of orbits. (Theorem 1 is a special case of this conjecture since $\mathcal{D}^{\mathrm{b}}_{G,c}(G \times_B X)$ and $\mathcal{D}^{\mathrm{b}}_{B,c}(X)$ are equivalent by the induction equivalence.) Besides the above mentioned known results, there are other indications ([BF08, BY08]) that this conjecture is true.

Let $\mathcal{D}^{\mathrm{b}}(X)$ be the (bounded) derived category of sheaves of real vector spaces on $X = G/P$, and $\mathcal{D}^{\mathrm{b}}(X, \mathcal{S})$ the full subcategory of $\mathcal{S}$-constructible objects. Let $\mathcal{IC}(\mathcal{S})$ be the direct sum of the (non-equivariant) simple $\mathcal{S}$-constructible perverse sheaves on X, and $\mathcal{F} = \mathrm{Ext}(\mathcal{IC}(\mathcal{S}))$ its graded algebra of self-extensions in $\mathcal{D}^{\mathrm{b}}(X)$. The category $\mathrm{dgPer}(\mathcal{F})$ is defined similarly as $\mathrm{dgPer}(\mathcal{E})$ above. In the course of the proof of Theorem 1 we obtain as a special case (of Theorem 3 below) the following non-equivariant analog.

Theorem 2 (cf. Theorem 37)**.** *There is an equivalence of triangulated categories*

$$\mathcal{D}^{\mathrm{b}}(X, \mathcal{S}) \cong \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{S}))).$$

This Theorem 2 can also be obtained using localization and a shadow of Koszul duality, using results of [BGS96]. Our proof, however, is much more straightforward and works in greater generality.

The category $\mathrm{Perv}_B(X)$ of equivariant perverse sheaves on X is the heart of the perverse t-structure on $\mathcal{D}^{\mathrm{b}}_{B,c}(X)$, and similarly for $\mathrm{Perv}(X, \mathcal{S}) \subset \mathcal{D}^{\mathrm{b}}(X, \mathcal{S})$. The corresponding t-structure on $\mathrm{dgPer}(\mathcal{E})$ and $\mathrm{dgPer}(\mathcal{F})$ can be defined for a more general class of dg algebras; we explain this below. It turns out that the heart of such a t-structure is equivalent to a full abelian subcategory dgFlag of the abelian category of dg modules. The equivalences in Theorems 1 and 2 are in fact t-exact and induce equivalences

$$\begin{aligned} \mathrm{Perv}_B(X) &\cong \mathrm{dgFlag}(\mathrm{Ext}(\mathcal{IC}_B(\mathcal{S}))), \\ \mathrm{Perv}(X, \mathcal{S}) &\cong \mathrm{dgFlag}(\mathrm{Ext}(\mathcal{IC}(\mathcal{S}))), \end{aligned}$$

i. e. algebraic descriptions of the categories of (equivariant) perverse sheaves. The simple object $\mathcal{IC}_B(S)$ is mapped to $e_S\mathcal{E}$ (where $e_S \in \mathcal{E}$ is the projector from $\mathcal{IC}_B(\mathcal{S})$ onto the direct summand $\mathcal{IC}_B(S)$), which is an indecomposable projective dg $\mathcal{E}$-module. This seems to be part of a Koszul duality (cf. [BGS96, 1.2.6]).

The forgetful functor $\mathrm{For} : \mathcal{D}^{\mathrm{b}}_{B,c}(X) \to \mathcal{D}^{\mathrm{b}}(X, \mathcal{S})$ induces a surjective morphism $\mathcal{E} \to \mathcal{F}$ of dg algebras and an extension of scalars functor $(? \overset{L}{\otimes}_{\mathcal{E}} \mathcal{F}) : \mathrm{dgPer}(\mathcal{E}) \to \mathrm{dgPer}(\mathcal{F})$. These two functors provide a connection between the equivalences in Theorems 1 and 2, i. e. there is a commutative (up to natural isomorphism) square

(see Remark 73)

$$\begin{array}{ccc}\mathcal{D}^{\mathrm{b}}_{B,c}(X) & \xrightarrow{\sim} & \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}_B(\mathcal{S}))) \\ \downarrow{\scriptstyle \mathrm{For}} & & \downarrow{\scriptstyle (?\overset{L}{\otimes}_{\mathcal{E}}\mathcal{F})} \\ \mathcal{D}^{\mathrm{b}}(X,\mathcal{S}) & \xrightarrow{\sim} & \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{S}))).\end{array}$$

Let us comment on some purely algebraic results concerning certain perfect derived categories of dg modules mentioned above (see [Sch08]). Let $\mathcal{A} = (A = \bigoplus_{i\geq 0} A^i, d)$ be a positively graded dg algebra with A^0 a semisimple ring and $d(A^0) = 0$ (i.e. A^0 is a dg subalgebra). Let $(L_x)_{x\in W}$ be the finite collection of non-isomorphic simple (right) A^0-modules, and dgPrae($\mathcal{A}$) the smallest strict full triangulated subcategory of the derived category dgDer($\mathcal{A}$) of dg $\mathcal{A}$-modules that contains all $\widehat{L}_x := L_x \otimes_{A^0} \mathcal{A}$ (where L_x is concentrated in degree zero). Let dgMod($\mathcal{A}$) be the abelian category of dg $\mathcal{A}$-modules, and dgFlag($\mathcal{A}$) the full subcategory of dgMod($\mathcal{A}$) consisting of objects that have an $\widehat{L}_x$-flag, i.e. a finite filtration with subquotients isomorphic to objects of $\{\widehat{L}_x\}_{x\in W}$ (without shifts). Then dgPrae($\mathcal{A}$) coincides with dgPer($\mathcal{A}$) and carries a natural bounded t-structure. Moreover dgFlag($\mathcal{A}$) is a full abelian subcategory of dgMod($\mathcal{A}$) and naturally equivalent to the heart of this t-structure. Let us note that there is another equivalent full subcategory of dgPer($\mathcal{A}$) consisting of certain filtered dg modules that is quite accessible to computations (cf. Theorem 56).

These remarks apply in particular to the dg algebras $\mathcal{E}$ and $\mathcal{F}$ defined above and make the categories of dg modules appearing in our main equivalences quite explicit. They also show that the categories of dg modules appearing in the main equivalences of [Lun95] and [Gui05] are in fact of the form dgPer.

Assume for this paragraph that we work with sheaves of complex vector spaces. Our main Theorems 1 and 2 remain true (see subsection 3.13 and Remark 72). Assume now in addition that G is semisimple and that $P = B$. Then the extension algebras are isomorphic to morphism spaces of Soergel's bimodules (see [Soe01, Soe92, Soe90]). These bimodules are isomorphic to the (B-equivariant) intersection cohomologies of Schubert varieties and can be described using the moment graph picture (see [BM01]). Thus, if $T \subset B$ is a maximal torus, the B-equivariant derived category of the flag variety G/B only depends on the moment graph associated to T acting on G/B.

Let us describe in more detail our approach to prove Theorem 1. We use notation from subsequent sections without further explanation.

Step 1 (see section 3). Let X be a complex variety with a stratification $\mathcal{T}$ into cells (i.e. $T \cong \mathbb{C}^{d_T}$ for each $T \in \mathcal{T}$). Under some purity assumptions explained below we will establish an equivalence

$$\mathcal{D}^{\mathrm{b}}(X,\mathcal{T}) \cong \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{T}))) \tag{1}$$

of triangulated categories, of which Theorem 2 is a special case. Note that we could write equivalently dgPrae on the right hand side. The proof works as follows. Since $\mathcal{T}$ is a cell-stratification, there is an equivalence

$$\mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X,\mathcal{T})) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(X,\mathcal{T}).$$

There are enough projective objects in $\mathrm{Perv}(X,\mathcal{T})$, so we find projective resolutions $P_T \to \mathcal{IC}(T)$ of finite length

$$\ldots \to P_T^{-2} \to P_T^{-1} \to P_T^0 \to \mathcal{IC}(T) \to 0.$$

Let $P \to \mathcal{IC}(\mathcal{T})$ be the direct sum of these resolutions and $\mathcal{B} = \mathcal{E}\mathrm{nd}\,(P)$ the dg algebra of endomorphisms of P. The functor $\mathcal{H}\mathrm{om}\,(P,?)$ induces an equivalence

$$\mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X,\mathcal{T})) \xrightarrow{\sim} \mathrm{dgPrae}_{\mathcal{B}}(\{e_T\mathcal{B}\}_{T\in\mathcal{T}}).$$

Note that the cohomology of $\mathcal{B}$ is isomorphic to $\mathrm{Ext}(\mathcal{IC}(\mathcal{T}))$. Thus we obtain equivalence (1) if $\mathcal{B}$ is formal. In order to prove formality, we need to choose the resolutions $P_T \to \mathcal{IC}(T)$ more carefully.

Each $\mathcal{IC}(T)$ is the underlying perverse sheaf of a mixed Hodge module $\widetilde{\mathcal{IC}}(T)$ that is pure of weight d_T. We construct resolutions $\widetilde{P}_T \to \widetilde{\mathcal{IC}}(T)$ in the category of mixed Hodge modules so that the underlying resolutions $P_T \to \mathcal{IC}(T)$ are projective resolutions as considered above. From these resolutions we get a dg algebra of mixed Hodge structures with underlying dg algebra $\mathcal{B} = \mathcal{E}\mathrm{nd}\,(P)$. If each $\widetilde{\mathcal{IC}}(T)$ is $\mathcal{T}$-pure of weight d_T (i. e. all restrictions to strata in $\mathcal{T}$ are pure of weight d_T), this additional structure on $\mathcal{B}$ enables us to construct a dg subalgebra $\mathrm{Sub}(\mathcal{B})$ of $\mathcal{B}$ and quasi-isomorphisms

$$\mathcal{B} \hookleftarrow \mathrm{Sub}(\mathcal{B}) \twoheadrightarrow \mathrm{H}(\mathcal{B})$$

of dg algebras, establishing the formality of $\mathcal{B}$.

We will need the following slightly more general statement than equivalence (1), with essentially the same proof.

Theorem 3 (cf. Theorem 31)**.** *Let $(X,\mathcal{S})$ be a stratified complex variety with irreducible and simply connected strata. Let $\mathcal{T}$ be a cell-stratification refining $\mathcal{S}$. If $\widetilde{\mathcal{IC}}(S)$ is $\mathcal{T}$-pure of weight d_S for each $S \in \mathcal{S}$, there is a triangulated equivalence*

$$\mathcal{D}^{\mathrm{b}}(X,\mathcal{S}) \cong \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{S}))).$$

Step 2 (see section 4). Let $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ be stratified complex varieties with irreducible and simply connected strata. Let $i : Y \to X$ be a closed embedding so that $\mathcal{S} \to \mathcal{T}$, $S \mapsto S \cap Y$, is bijective and $i|_{\overline{S}\cap Y} : \overline{S} \cap Y \to \overline{S}$ is a normally nonsingular inclusion of a fixed codimension c for all $S \in \mathcal{S}$. Then

$$[-c]i^*(\widetilde{\mathcal{IC}}(S)) \xrightarrow{\sim} \widetilde{\mathcal{IC}}(S \cap Y) \tag{2}$$

for all $S \in \mathcal{S}$. If both stratifications $\mathcal{S}$ and $\mathcal{T}$ have compatible refinements by cell-stratifications satisfying the purity conditions of Theorem 3, we obtain the vertical equivalences in the following diagram.

$$\begin{tikzcd} \mathcal{D}^{\mathrm{b}}(X,\mathcal{S}) \arrow[r, "{[-c]i^*}"] \arrow[d, "\sim"] & \mathcal{D}^{\mathrm{b}}(Y,\mathcal{T}) \arrow[d, "\sim"] \\ \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{S}))) \arrow[r, "{? \overset{L}{\underset{\mathrm{Ext}(\mathcal{IC}(\mathcal{S}))}{\otimes}} \mathrm{Ext}(\mathcal{IC}(\mathcal{T}))}"'] & \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{T}))) \end{tikzcd} \tag{3}$$

The extension of scalars functor in the lower row is induced by the isomorphisms (2). This diagram is commutative (up to natural isomorphism). Unfortunately the proof is rather technical.

Step 3 (see section 6). Let $X = G/P$ be a partial flag variety with stratification $\mathcal{S}$ into B-orbits as before. We construct a sequence

$$E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \to \ldots \to E_n \xrightarrow{f_n} E_{n+1} \to \ldots$$

of resolutions $p_n : E_n \to X$ of X satisfying several nice properties. For example, each p_n is smooth and n-acyclic (in the classical topology), the quotient morphisms $q_n : E_n \to \overline{E}_n := B\backslash E_n$ are Zariski locally trivial fiber bundles, and each $\mathcal{S}_n := \{q_n(p_n^{-1}(S)) \mid S \in \mathcal{S}\}$ is a stratification of $\overline{E}_n$. The induced morphisms $\overline{f}_n : \overline{E}_n \to \overline{E}_{n+1}$ define functors $\overline{f}_n^* : \mathcal{D}^{\mathrm{b}}(\overline{E}_{n+1}, \mathcal{S}_{n+1}) \to \mathcal{D}^{\mathrm{b}}(\overline{E}_n, \mathcal{S}_n)$, and we obtain a sequence of categories whose inverse limit is equivalent to the category we want to describe,

$$\mathcal{D}^{\mathrm{b}}_{B,c}(X) \cong \varprojlim \mathcal{D}^{\mathrm{b}}(\overline{E}_n, \mathcal{S}_n).$$

Moreover, the morphisms $\overline{f}_n$ satisfy the assumptions of Step 2 (in particular, the stratifications $\mathcal{S}_n$ admit refinements where the purity conditions hold), and the obtained commutative diagrams of the form (3) provide an equivalence

$$\varprojlim \mathcal{D}^{\mathrm{b}}(\overline{E}_n, \mathcal{S}_n) \xrightarrow{\sim} \varprojlim \operatorname{dgPer}(\operatorname{Ext}(\mathcal{IC}(\mathcal{S}_n))).$$

Finally, the obvious morphisms $\mathcal{E} = \operatorname{Ext}(\mathcal{IC}_B(\mathcal{S})) \to \operatorname{Ext}(\mathcal{IC}(\mathcal{S}_n))$ of dg algebras induce an equivalence

$$\operatorname{dgPer}(\operatorname{Ext}(\mathcal{IC}_B(\mathcal{S}))) \xrightarrow{\sim} \varprojlim \operatorname{dgPer}(\operatorname{Ext}(\mathcal{IC}(\mathcal{S}_n))).$$

This finishes the sketch of proof of Theorem 1.

It would be nice to know whether the analog of Theorem 1 is true for $\mathcal{D}^{\mathrm{b}}_{Q,c}(G/P)$ if Q is a parabolic subgroup of G containing B. Theorem 37 shows that the non-equivariant version holds, i. e. we can replace the stratification $\mathcal{S}$ in Theorem 2 by the stratification into Q-orbits. We expect that our methods can be generalized to affine flag varieties.

This article is organized as follows: In section 2 we introduce the main categories of dg modules, show how dg modules can be used to describe certain triangulated categories and prove an elementary but crucial result establishing the formality of some dg algebras with an additional grading. Sections 3, 4 and 6 contain essentially the results explained above in Steps 1, 2 and 3 respectively. However the methods are developed in a broader context and may be applied to other situations. Section 5 contains some results on inverse limits of categories (of dg modules) used in Step 3.

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2. Differential Graded Modules

2.1. **DG Modules.** We review the language of differential graded (dg) modules over a dg algebra (see [Kel94, Kel98, BL94]).

Let k be a commutative ring and $\mathcal{A} = (A = \bigoplus_{i \in \mathbb{Z}} A^i, d)$ a differential graded k-algebra (= dg algebra). A dg (right) module over $\mathcal{A}$ will also be called an $\mathcal{A}$-module or a dg module if there is no doubt about the dg algebra. We often write M for a dg module (M, d_M). We consider the category $\operatorname{dgMod}(\mathcal{A})$ of dg modules, the homotopy category $\operatorname{dgHot}(\mathcal{A})$ and the derived category $\operatorname{dgDer}(\mathcal{A})$ of dg modules. We denote the shift functor on all these categories by $M \mapsto \{1\}M$, e.g. $(\{1\}M)^i = M^{i+1}$, $d_{\{1\}M} = -d_M$. We define $\{n\} = \{1\}^n$ for $n \in \mathbb{Z}$. Both $\operatorname{dgHot}(\mathcal{A})$ and $\operatorname{dgDer}(\mathcal{A})$ are triangulated categories.

A dg module P is called **homotopically projective** ([Kel98]), if it satisfies one of the following equivalent conditions ([BL94, 10.12.2.2]):

(a) $\operatorname{Hom}_{\operatorname{dgHot}}(P, ?) = \operatorname{Hom}_{\operatorname{dgDer}}(P, ?)$, i.e. for all dg modules M, the canonical map $\operatorname{Hom}_{\operatorname{dgHot}}(P, M) \to \operatorname{Hom}_{\operatorname{dgDer}}(P, M)$ is an isomorphism.
(b) $\operatorname{Hom}_{\operatorname{dgHot}}(P, M) = 0$ for each acyclic dg module M.

In [Kel94, 3.1] such a module is said to have *property (P)*, in [BL94, 10.12.2] the term *$\mathcal{K}$-projective* is used. For example, $\mathcal{A}$ and each direct summand of $\mathcal{A}$ is homotopically projective.

Let $\operatorname{dgHotp}(\mathcal{A})$ be the full subcategory of $\operatorname{dgHot}(\mathcal{A})$ consisting of homotopically projective dg modules. The quotient functor $\operatorname{dgHot}(\mathcal{A}) \to \operatorname{dgDer}(\mathcal{A})$ induces a triangulated equivalence ([Kel94, 3.1, 4.1])

$$\operatorname{dgHotp}(\mathcal{A}) \xrightarrow{\sim} \operatorname{dgDer}(\mathcal{A}). \tag{4}$$

Let $\operatorname{dgPer}(\mathcal{A})$ be the perfect derived category, i.e. the smallest strict (= closed under isomorphisms) full triangulated subcategory of $\operatorname{dgDer}(\mathcal{A})$ containing $\mathcal{A}$ and closed under forming direct summands.

Each morphism of dg algebras (**dga-morphism**) $f : \mathcal{A} \to \mathcal{B}$ induces on cohomology a dga-morphism $\mathrm{H}(f) : \mathrm{H}(\mathcal{A}) \to \mathrm{H}(\mathcal{B})$. If $\mathrm{H}(f)$ is an isomorphism, f is called a **dga-quasi-isomorphism**. Two dg algebras $\mathcal{A}$ and $\mathcal{B}$ are **equivalent** if there is a sequence $\mathcal{A} \leftarrow \mathcal{C}_1 \to \mathcal{C}_2 \leftarrow \ldots \to \ldots \leftarrow \mathcal{C}_n \to \mathcal{B}$ of dga-quasi-isomorphisms. A dg algebra $\mathcal{A}$ is **formal** if it is equivalent to a dg algebra with differential $d = 0$. In this case, $\mathcal{A}$ is equivalent to $\mathrm{H}(\mathcal{A})$.

If $\mathcal{A} \to \mathcal{B}$ is a morphism of dg algebras, we have the **extension of scalars** functor ([BL94, 10.11])

$$\operatorname{prod}_{\mathcal{A}}^{\mathcal{B}} = (? \otimes_{\mathcal{A}} \mathcal{B}) : \operatorname{dgMod}(\mathcal{A}) \to \operatorname{dgMod}(\mathcal{B}).$$

It descends to a triangulated functor $\operatorname{prod}_{\mathcal{A}}^{\mathcal{B}} = (? \otimes_{\mathcal{A}} \mathcal{B})$ between the homotopy categories and has the left derived functor $\operatorname{prod}_{\mathcal{A}}^{\mathcal{B}} = (? \overset{L}{\otimes}_{\mathcal{A}} \mathcal{B})$ on the level of derived categories. This left derived functor is an equivalence if $\mathcal{A} \to \mathcal{B}$ is a dga-quasi-isomorphism ([Kel94, 6.1]).

2.2. **Differential Graded Graded Algebras and Formality.** We show that some dg algebras with an extra grading are formal. Let k be a commutative ring and $\mathcal{R}$ a differential graded graded (dgg) algebra, i.e. a $\mathbb{Z}^2$-graded associative k-algebra $R = \bigoplus_{i,j \in \mathbb{Z}} R^{ij}$ endowed with a k-linear differential $d : R \to R$ that is homogeneous of degree $(1,0)$ and satisfies the Leibniz rule $d(ab) = (da)b + (-1)^i a db$ for all $a \in R^{ij}$, $b \in R^{kl}$. A dgg module $\mathcal{M} = (M, d)$ over $\mathcal{R}$ is a $\mathbb{Z}^2$-graded right R-module $M = \bigoplus_{i,j \in \mathbb{Z}} M^{ij}$ with a k-linear differential $d : M \to M$ of degree $(1,0)$ satisfying $d(ma) = (dm)a + (-1)^i m da$ for all $m \in M^{ij}$, $a \in R^{kl}$. Morphisms of dgg modules are morphisms of the underlying $\mathbb{Z}^2$-graded R-modules of degree $(0,0)$ that commute with the differentials. We denote the category of dgg modules over $\mathcal{R}$ by $\operatorname{dggMod}(\mathcal{R})$.

The cohomology of a dgg module over a dgg algebra $\mathcal{R}$ is a dgg module over the the dgg algebra $\mathrm{H}(\mathcal{R})$. Morphisms of dgg algebras (**dgga-morphisms**) are algebra homomorphisms that are morphisms of dgg modules. The meanings of **dgga-quasi-isomorphism**, **equivalent** and **formal** are the obvious generalizations from dg algebras.

A $\mathbb{Z}^2$-graded k-module $M = \bigoplus M^{ij}$ is **pure of weight** w, if $M^{ij} \neq 0$ implies $j = i + w$. A dgg module or algebra is pure of weight w if the underlying bigraded module is pure of weight w. Every pure dgg algebra $\mathcal{R}$ of weight $w \neq 0$ is the zero algebra, since $1 \in R^{00}$; hence it is also pure of weight 0.

Let M be in $\mathrm{dggMod}(\mathcal{R})$. We define a bigraded k-submodule $\Gamma(M)$ of M by

$$\Gamma(M)^{ij} = \begin{cases} M^{ij} & \text{if } i < j, \\ \ker(d^{ij} : M^{ij} \to M^{i+1,j}) & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases} \tag{5}$$

The differential of M restricts to a differential of $\Gamma(M)$. The multiplication on $\mathcal{R}$ restricts to a multiplication on $\Gamma(\mathcal{R})$ and $\Gamma(\mathcal{R})$ becomes a dgg algebra. Similarly, $\Gamma(M)$ is a dgg module over $\Gamma(\mathcal{R})$. In fact, we obtain a functor

$$\Gamma : \mathrm{dggMod}(\mathcal{R}) \to \mathrm{dggMod}(\Gamma(\mathcal{R})). \tag{6}$$

Proposition 4. *If the cohomology* $\mathrm{H}(\mathcal{R})$ *of a dgg algebra* $\mathcal{R}$ *is pure of weight* 0*, then* $\mathcal{R}$ *is formal. More precisely,* $\mathcal{R} \hookleftarrow \Gamma(\mathcal{R}) \twoheadrightarrow \mathrm{H}(\mathcal{R})$ *are dgga-quasi-isomorphisms where* $\Gamma(\mathcal{R}) \hookrightarrow \mathcal{R}$ *is the obvious inclusion and* $\Gamma(\mathcal{R}) \twoheadrightarrow \mathrm{H}(\mathcal{R})$ *the componentwise projection.*

Proof. Let $\mathcal{R}$ be an arbitrary dgg algebra. We include the following picture illustrating the morphisms $\mathcal{R} \hookleftarrow \Gamma(\mathcal{R}) \twoheadrightarrow \mathrm{H}(\mathcal{R})$. The differentials go to the right, the cocycle and cohomology modules of the complexes R^{*j} are denoted by Z^{ij} and H^{ij} respectively.

$$\left|\begin{matrix} R^{01} & \to & R^{11} \\ R^{00} & \to & R^{10} \end{matrix}\right| \hookleftarrow \left|\begin{matrix} R^{01} & \to & \mathrm{Z}^{11} \\ \mathrm{Z}^{00} & \to & 0 \end{matrix}\right| \twoheadrightarrow \left|\begin{matrix} 0 & \to & \mathrm{H}^{11} \\ \mathrm{H}^{00} & \to & 0 \end{matrix}\right|$$

The proposition results from the following evident statements.

(a) The dgga-inclusion $\Gamma(\mathcal{R}) \hookrightarrow \mathcal{R}$ induces on cohomology an isomorphism in degrees (i, j) with $i \leq j$ (above the diagonal).
(b) If $\mathrm{H}(\mathcal{R})$ vanishes in degrees (i, j) with $i < j$ (the cohomology lives below the diagonal), componentwise projection $\Gamma(\mathcal{R}) \to \mathrm{H}(\mathcal{R})$ is a well-defined dgga-morphism and induces on cohomology an isomorphism in degrees (i, i) (on the diagonal).

□

We generalize Proposition 4 slightly to dgg algebras that look like matrices. Let $\mathcal{R}$ be a dgg algebra and $\{e_\alpha\}_{\alpha \in I}$ a finite set of orthogonal idempotent elements of R^{00} satisfying $1 = \sum_{\alpha \in I} e_\alpha$ and $d(e_\alpha) = 0$ for all $\alpha \in I$. We get a direct sum decomposition $R = \bigoplus R_{\alpha\beta}$ where $R_{\alpha\beta} := e_\alpha R e_\beta$ for α, $\beta \in I$. The differential of $\mathcal{R}$ induces differentials on each component $R_{\alpha\beta}$. In particular, we can consider the cohomologies $\mathrm{H}(R_{\alpha\beta})$.

Proposition 5. *Let* $\mathcal{R}$ *and* $\{e_\alpha\}_{\alpha \in I}$ *be as above. If there are integers* $(n_\alpha)_{\alpha \in I}$ *such that each* $\mathrm{H}(\mathcal{R}_{\alpha\beta})$ *is pure of weight* $n_\alpha - n_\beta$*, then* $\mathcal{R}$ *is formal. More precisely, there are a dgg subalgebra* $\mathcal{S}$ *of* $\mathcal{R}$ *containing all* $\{e_\alpha\}_{\alpha \in I}$ *and quasi-isomorphisms* $\mathcal{R} \hookleftarrow \mathcal{S} \twoheadrightarrow \mathrm{H}(\mathcal{R})$ *of dgg algebras.*

Proof. Define $S = \bigoplus S^{ij}_{\alpha\beta} \subset R$ by

$$S^{ij}_{\alpha\beta} = \begin{cases} R^{ij}_{\alpha\beta} & \text{if } i + n_\alpha - n_\beta < j, \\ \ker(d^{ij}_{\alpha\beta} : R^{ij}_{\alpha\beta} \to R^{i+1,j}_{\alpha\beta}) & \text{if } i + n_\alpha - n_\beta = j, \\ 0 & \text{if } i + n_\alpha - n_\beta > j. \end{cases}$$

With the induced multiplication and differential, S becomes a dgg subalgebra $\mathcal{S}$ of $\mathcal{R}$. The inclusion $\mathcal{S} \hookrightarrow \mathcal{R}$ and the obvious projection $\mathcal{S} \twoheadrightarrow \mathrm{H}(\mathcal{R})$ are easily seen to be quasi-isomorphisms of dgg algebras. □

2.3. **Subcategories of Triangulated Categories.** Let $\mathcal{T}$ be a triangulated category, with shift $X \mapsto [1]X$. If M is a set of objects of $\mathcal{T}$, we denote by $\operatorname{tria}(M) = \operatorname{tria}(M, \mathcal{T})$ the smallest strict full triangulated subcategory of $\mathcal{T}$ that contains all objects of M, and by $\operatorname{thick}(M) = \operatorname{thick}(M, \mathcal{T})$ the closure of $\operatorname{tria}(M)$ under taking direct summands. If X is an object of $\mathcal{T}$, we abbreviate $\operatorname{tria}(\{X\})$ by $\operatorname{tria}(X)$, and similarly for thick.

Lemma 6 (Beilinson's Lemma)**.** *Let $F : \mathcal{T} \to \mathcal{T}'$ be a triangulated functor between triangulated categories, and let M be a set of objects of $\mathcal{T}$. If F induces isomorphisms*

$$\operatorname{Hom}_{\mathcal{T}}(X, [i]Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{T}'}(F(X), [i]F(Y)),$$

for all X, Y in M and all $i \in \mathbb{Z}$, it induces a triangulated equivalence

$$\operatorname{tria}(M) \xrightarrow{\sim} \operatorname{tria}(F(M)),$$

where $F(M) = \{F(X) \mid X \in M\}$.

Proof. This follows by a standard dévissage argument. □

2.4. **Derived Categories and DG Modules.** Let $\mathcal{A}$ be an abelian category. We denote by $\operatorname{Ket}(\mathcal{A})$, $\operatorname{Hot}(\mathcal{A})$ and $\operatorname{Der}(\mathcal{A})$ (or $\mathcal{D}(\mathcal{A})$) the category of (cochain) complexes in $\mathcal{A}$, the homotopy category of complexes in $\mathcal{A}$ and the derived category of $\mathcal{A}$ respectively, with shift functor $A \mapsto [1]A$. We often consider $\mathcal{A}$ as a full subcategory of these categories, consisting of complexes (with cohomology) concentrated in degree zero. If I is a subset of $\mathbb{Z}$, we write $\operatorname{Der}^I(\mathcal{A})$ for the full subcategory of $\operatorname{Der}(\mathcal{A})$ with objects whose cohomology vanishes in degrees outside I. For objects A and B in the derived category of $\mathcal{A}$, we write $\operatorname{Ext}^n_{\mathcal{A}}(A, B)$ for $\operatorname{Hom}^n_{\operatorname{Der}(\mathcal{A})}(A, B) := \operatorname{Hom}_{\operatorname{Der}(\mathcal{A})}(A, [n]B)$ and $\operatorname{Ext}_{\mathcal{A}}(A, B)$ for the direct sum of all $\operatorname{Ext}^n_{\mathcal{A}}(A, B)$, $n \in \mathbb{Z}$. We call

$$\operatorname{Ext}(A) := \operatorname{Ext}_{\mathcal{A}}(A, A) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Ext}^n_{\mathcal{A}}(A, A). \tag{7}$$

the **extension algebra** of A.

If M, N are complexes in $\mathcal{A}$, let $\mathcal{H}\mathrm{om}\,(M, N)$ or $\mathcal{H}\mathrm{om}_{\mathcal{A}}(M, N)$ denote the complex of abelian groups with n-th component

$$\mathcal{H}\mathrm{om}^{\,n}(M, N) = \prod_{i+j=n} \operatorname{Hom}_{\mathcal{A}}(M^{-i}, N^j)$$

and differential $df = d \circ f - (-1)^n f \circ d$ for each homogeneous f of degree n. The n-th cohomology of this complex is $\operatorname{Hom}_{\operatorname{Hot}(\mathcal{A})}(M, [n]N)$. With the obvious composition, $\mathcal{H}\mathrm{om}\,(M, M)$ becomes a dg algebra that we denote by $\mathcal{E}\mathrm{nd}\,(M)$. The functor

$$\mathcal{H}\mathrm{om}\,(M, ?) : \operatorname{Ket}(\mathcal{A}) \to \operatorname{dgMod}(\mathcal{E}\mathrm{nd}\,(M)),$$

induces a triangulated functor between the homotopy categories.

Recall the category $\mathrm{dgPer}(\mathcal{R})$ for a dg algebra $\mathcal{R}$ from subsection 2.1. By definition, it is equal to $\mathrm{thick}(\mathcal{R}, \mathrm{dgDer}(\mathcal{R}))$. If M is a set of $\mathcal{R}$-modules, we define

$$\mathrm{dgPrae}_{\mathcal{R}}(M) := \mathrm{tria}(M, \mathrm{dgDer}(\mathcal{R})).$$

Proposition 7. *Let $\mathcal{A}$ be an abelian category, and $\{P_\alpha\}_{\alpha\in I}$ a finite set of complexes in $\mathcal{A}$ such that the canonical maps*

$$\mathrm{Hom}_{\mathrm{Hot}(\mathcal{A})}(P_\alpha, [n]P_\beta) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(P_\alpha, [n]P_\beta) \tag{8}$$

are isomorphisms for all $n \in \mathbb{Z}$ and all α, $\beta \in I$. (For example, all P_α could be bounded above complexes of projective objects of $\mathcal{A}$.) Define $P = \bigoplus P_\alpha$ and $\mathcal{R} = \mathcal{E}\mathrm{nd}\,(P)$. Let $e_\alpha \in \mathcal{R}^0$ be the projector from P onto its direct summand P_α. Then the functor $\mathcal{H}\mathrm{om}\,(P, ?)$ induces a triangulated equivalence

$$\mathrm{tria}(\{P_\alpha\}_{\alpha\in I}, \mathcal{D}(\mathcal{A})) \xrightarrow{\sim} \mathrm{dgPrae}_{\mathcal{R}}(\{e_\alpha\mathcal{R}\}_{\alpha\in I}). \tag{9}$$

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{tria}(\{P_\alpha\}_{\alpha\in I}, \mathrm{Hot}(\mathcal{A})) \arrow[r, "\mathcal{H}\mathrm{om}\,(P,?)"] \arrow[d] & \mathrm{tria}(\{e_\alpha\mathcal{R}\}_{\alpha\in I}, \mathrm{dgHot}(\mathcal{R})) \arrow[d] \\
\mathrm{tria}(\{P_\alpha\}_{\alpha\in I}, \mathcal{D}(\mathcal{A})) & \mathrm{dgPrae}_{\mathcal{R}}(\{e_\alpha\mathcal{R}\}_{\alpha\in I})
\end{tikzcd}$$

with obvious vertical functors. We claim that all arrows are equivalences. For the arrow on the left this follows from (8) and Lemma 6. Since all $e_\alpha\mathcal{R}$ are homotopically projective dg modules, equivalence (4) restricts to the equivalence on the right. Since $\mathrm{Hom}_{\mathrm{Hot}(\mathcal{A})}(P_\alpha, [n]P_\beta)$ and $\mathrm{Hom}_{\mathrm{dgHot}(\mathcal{R})}(e_\alpha\mathcal{R}, [n]e_\beta\mathcal{R})$ are both naturally identified with $\mathrm{H}^n(e_\beta\mathcal{R}e_\alpha)$ and these identifications are compatible with the functor $\mathcal{H}\mathrm{om}\,(P, ?)$, Lemma 6 proves our claim. $\square$

Remark 8. Let P be a complex in an abelian category $\mathcal{A}$ with endomorphism complex $\mathcal{R} = \mathcal{E}\mathrm{nd}\,(P)$. If the composition

$$\mathrm{Hot}(\mathcal{A}) \xrightarrow{\mathcal{H}\mathrm{om}\,(P,?)} \mathrm{dgHot}(\mathcal{R}) \to \mathrm{dgDer}(\mathcal{R}) \tag{10}$$

vanishes on acyclic complexes, it factors through $q : \mathrm{Hot}(\mathcal{A}) \to \mathrm{Der}(\mathcal{A})$ to a triangulated functor

$$\mathcal{H}\mathrm{om}\,(P, ?) : \mathrm{Der}(\mathcal{A}) \to \mathrm{dgDer}(\mathcal{R}). \tag{11}$$

This is the case, for example, if P is a bounded above complex of projective objects of $\mathcal{A}$.

If we keep the assumptions of Proposition 7 and assume that the composition (10) vanishes on acyclic complexes, then the restriction of (11) yields directly equivalence (9).

2.5. **Perfect DG Modules.** We recall some results from [Sch08]. We assume in this subsection that $\mathcal{A} = (A, d)$ is a dg algebra satisfying the following conditions:

- (P1) A is positively graded, i. e. $A^i = 0$ for $i < 0$;
- (P2) A^0 is a semisimple ring;
- (P3) the differential of $\mathcal{A}$ vanishes on A^0, i. e. $d(A^0) = 0$.

The semisimple ring A^0 has only a finite number of non-isomorphic simple (right) modules $(L_x)_{x\in W}$. We view A^0 as a dg subalgebra $\mathcal{A}^0$ of $\mathcal{A}$ and the L_x as $\mathcal{A}^0$-modules concentrated in degree zero. Extension of scalars yields $\mathcal{A}$-modules $\widehat{L}_x := L_x \otimes_{\mathcal{A}^0} \mathcal{A}$. Define

$$\mathrm{dgPrae}(\mathcal{A}) := \mathrm{dgPrae}_{\mathcal{A}}(\{L_x\}_{x\in W}).$$

Let $\mathrm{dgPer}^{\leq 0}$ (and $\mathrm{dgPer}^{\geq 0}$ resp.) be the full subcategories of $\mathrm{dgPer}(\mathcal{A})$ consisting of objects $\mathcal{M}$ such that $\mathrm{H}^i(\mathcal{M}\overset{L}{\otimes}_{\mathcal{A}}\mathcal{A}^0)$ vanishes for $i > 0$ (for $i < 0$ respectively). Let $\mathrm{dgFlag}(\mathcal{A}) \subset \mathrm{dgMod}(\mathcal{A})$ be the full subcategory consisting of objects that have an $\widehat{L}_x$-flag, i.e. a finite filtration with subquotients isomorphic to objects of $\{\widehat{L}_x\}_{x\in W}$ (without shifts).

Theorem 9 ([Sch08])**.** *Let $\mathcal{A}$ be a dg algebra satisfying (P1)-(P3).*

(1) Then $\mathrm{dgPrae}(\mathcal{A}) = \mathrm{dgPer}(\mathcal{A})$, i.e. $\mathrm{dgPrae}(\mathcal{A})$ is closed under taking direct summands.
(2) $(\mathrm{dgPer}^{\leq 0}, \mathrm{dgPer}^{\geq 0})$ defines a bounded (hence non-degenerate) t-structure on $\mathrm{dgPer}(\mathcal{A})$.
(3) Its heart dgPer^0 is equivalent to $\mathrm{dgFlag}(\mathcal{A})$. More precisely, $\mathrm{dgFlag}(\mathcal{A})$ is a full abelian subcategory of $\mathrm{dgMod}(\mathcal{A})$ and the obvious functor $\mathrm{dgMod}(\mathcal{A}) \to \mathrm{dgDer}(\mathcal{A})$ induces an equivalence $\mathrm{dgFlag}(\mathcal{A}) \xrightarrow{\sim} \mathrm{dgPer}^0$.
(4) Any object in the heart dgPer^0 has finite length, and the simple objects in dgPer^0 are (up to isomorphism) the $\{\widehat{L}_x\}_{x\in W}$.

3. Formality of Derived Categories

3.1. **Sheaves and Perverse Sheaves.** We only consider complex (algebraic) varieties. Let X be a variety. We denote by $\mathrm{Sh}(X)$ the abelian category of sheaves of real vector spaces with respect to the classical topology on X and by $\mathcal{D}^{\mathrm{b}}(X) = \mathrm{Der}^{\mathrm{b}}(\mathrm{Sh}(X))$ its bounded derived category. Let $\mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ be the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(X)$, consisting of complexes with algebraically constructible cohomology ([BBD82, 2.2.1]).

Any morphism $f : X \to Y$ of varieties gives rise to functors f^*, f_*, $f_!$, $f^!$ relating $\mathcal{D}^{\mathrm{b}}(X)$ and $\mathcal{D}^{\mathrm{b}}(Y)$. These functors would classically be written f^{-1}, Rf_*, $Rf_!$ and $f^!$ respectively. Similarly we write $\otimes$ and $\mathscr{H}\mathrm{om}$ for the derived functors of tensor product and sheaf homomorphisms. We denote the constant sheaf with stalk $\mathbb{R}$ on X by $\underline{X}$. Verdier duality is defined by $\mathbb{D} = \mathbb{D}_X = \mathscr{H}\mathrm{om}(?, c^!(\underline{\mathrm{pt}}))$ where $c : X \to \mathrm{pt}$ is the unique map to the final object pt in the category of varieties. We have $\mathbb{D}f_* = f_!\mathbb{D}$, $\mathbb{D}f^* = f^!\mathbb{D}$, and $\mathbb{D}^2 = \mathrm{id}$ on $\mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(X)$.

An algebraic stratification of X is a finite partition $\mathcal{S}$ of X into non-empty locally closed subvarieties, called strata, such that the closure of each stratum is a union of strata. If $S \in \mathcal{S}$ is a stratum, we denote by l_S the inclusion of S in X. From now on, if we speak about stratifications, we always mean algebraic Whitney stratifications. In particular, all strata are nonsingular. We assume in the following that all strata are irreducible varieties. The (complex) dimension of a stratum S is denoted by d_S. A **cell-stratification** is a stratification such that each stratum S is isomorphic to an affine linear space, so $S \cong \mathbb{C}^{d_S}$.

A sheaf $F \in \mathrm{Sh}(X)$ is called **smooth (along a stratification $\mathcal{S}$)** or **$\mathcal{S}$-constructible**, if $l_S^*(F)$ is a local system on S, for all $S \in \mathcal{S}$. Let $\mathrm{Sh}(X, \mathcal{S}) \subset \mathrm{Sh}(X)$

be the full subcategory of such sheaves. An object F of $\mathcal{D}^{\mathrm{b}}(X)$ is called **smooth (along $\mathcal{S}$)** or **$\mathcal{S}$-constructible**, if all $\mathrm{H}^i(F)$ are in $\mathrm{Sh}(X, \mathcal{S})$.

Let $(X, \mathcal{S})$ be a stratified variety. The full subcategory $\mathcal{D}^{\mathrm{b}}(X, \mathcal{S}) \subset \mathcal{D}^{\mathrm{b}}(X)$ of $\mathcal{S}$-constructible objects is a triangulated subcategory and closed under taking direct summands. Middle perversity defines perverse t-structures on $\mathcal{D}^{\mathrm{b}}(X, \mathcal{S})$ and $\mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, see [BBD82, 2.1, 2.2]. Their hearts $\mathrm{Perv}(X, \mathcal{S})$ and $\mathrm{Perv}(X)$ are the categories of smooth perverse sheaves and of perverse sheaves respectively. We have $\mathrm{Perv}(X, \mathcal{S}) = \mathrm{Perv}(X) \cap \mathcal{D}^{\mathrm{b}}(X, \mathcal{S})$. Since any object of $\mathcal{D}^{\mathrm{b}}(X, \mathcal{S})$ has perverse cohomology in finitely many degrees only ([BBD82, 2.1.2.1]), perverse truncation shows the non-trivial inclusion in

$$\mathcal{D}^{\mathrm{b}}(X, \mathcal{S}) = \mathrm{tria}(\mathrm{Perv}(X, \mathcal{S}), \mathcal{D}^{\mathrm{b}}(X)). \tag{12}$$

There is a triangulated equivalence of categories (see [Beĭ87, BBD82])

$$\mathrm{real} = \mathrm{real}_X : \mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X)) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(X). \tag{13}$$

We denote this functor often by $A \mapsto \underline{A} := \mathrm{real}(A)$. In particular,

$$\mathrm{real} : \mathrm{Ext}^n_{\mathrm{Perv}(X)}(A, B) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathrm{Sh}(X)}(\underline{A}, \underline{B}) \tag{14}$$

is an isomorphism for all A, $B \in \mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X))$. The corresponding statement for sheaves that are smooth along a fixed stratification $\mathcal{S}$ is usually false.

If $\mathcal{S}$ is a cell-stratification and $S \in \mathcal{S}$ a stratum, we define $\Delta_S = l_{S*}([d_S]\underline{S})$. Since l_S is affine, Δ_S belongs to $\mathrm{Perv}(X, \mathcal{S})$. The objects isomorphic to some Δ_S are called **standard objects**.

Theorem 10 ([BGS96, 3.2, 3.3])**.** *Let $(X, \mathcal{S})$ be a cell-stratified variety. Then the category $\mathrm{Perv}(X, \mathcal{S})$ is artinian and has enough projective and injective objects. Each projective object has a finite filtration with standard subquotients. Each object has a projective resolution of finite length. There is a triangulated equivalence*

$$\mathrm{real} = \mathrm{real}_{X,\mathcal{S}} : \mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X, \mathcal{S})) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(X, \mathcal{S}); \tag{15}$$

(this functor is constructed in [BBD82, 3.1]*) we denote it by $A \mapsto \underline{A}$.*

3.2. **Mixed Hodge Structures.** The following definitions and results are taken from [Del71, Del94, DMOS82].

A (real) mixed Hodge structure M consists of

(a) a real vector space $M_{\mathbb{R}}$ of finite dimension,
(b) a finite increasing filtration W on $M_{\mathbb{R}}$, called weight filtration,
(c) a finite decreasing filtration F on the complexification $M_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} M_{\mathbb{R}}$, called Hodge filtration,

such that the filtration $W_{\mathbb{C}}$, obtained by extension of scalars, the filtration F and its complex conjugate filtration $\overline{F}$ form a system of three opposed filtrations on $M_{\mathbb{C}}$, i. e. $\mathrm{gr}^p_F \mathrm{gr}^q_{\overline{F}} \mathrm{gr}^{W_{\mathbb{C}}}_n(M_{\mathbb{C}}) = 0$ if $n \neq p + q$. A morphism $f : M \to N$ of mixed Hodge structures is an $\mathbb{R}$-linear map $f_{\mathbb{R}} : M_{\mathbb{R}} \to N_{\mathbb{R}}$ that is compatible with the weight filtrations and whose complexification $f_{\mathbb{C}}$ is compatible with the Hodge filtration.

A mixed Hodge structure M has weights $\leq n$ (resp. $\geq n$), if $\mathrm{gr}^W_{j,\mathbb{R}}(M) := \mathrm{gr}^W_j(M_{\mathbb{R}}) = W_j M_{\mathbb{R}}/W_{j-1} M_{\mathbb{R}} = 0$ for $j > n$ (resp. $j < n$). It is pure of weight n, if it is of weight $\leq n$ and of weight $\geq n$.

Let $\mathbb{R}(n)$ be the Tate structure of weight $-2n$. It is a pure Hodge structure of weight $-2n$, with $\mathbb{R}(n)_{\mathbb{R}} = (2\pi i)^n \mathbb{R} \subset \mathbb{C} = \mathbb{R}(n)_{\mathbb{C}}$.

The category MHS of mixed Hodge structures is a rigid abelian $\mathbb{R}$-linear tensor category. It admits the fiber functor "underlying vector space" $\omega_0 : \mathrm{MHS} \to \mathbb{R}\text{-}\mathrm{mod}$ to the category of finite dimensional real vector spaces and is hence neutral tannakian. A mixed Hodge structure M is polarizable, if each graded piece $\mathrm{gr}_n^W(M)$ is a polarizable Hodge structure ([Del71, 2.1.16]). The polarizable mixed Hodge structures are a rigid tensor subcategory of MHS.

The functor $\mathrm{gr}_{\mathbb{R}}^W : \mathrm{MHS} \to \mathbb{R}\text{-}\mathrm{gmod}$, $M \mapsto \bigoplus_{n\in\mathbb{Z}} \mathrm{gr}_{n,\mathbb{R}}^W(M)$, is an exact faithful $\mathbb{R}$-linear tensor functor to the category of finite dimensional graded real vector spaces. We denote the composition of $\mathrm{gr}_{\mathbb{R}}^W$ with the functor "underlying vector space" $\eta : \mathbb{R}\text{-}\mathrm{gmod} \to \mathbb{R}\text{-}\mathrm{mod}$ by ω_W. This functor $\omega_W : \mathrm{MHS} \to \mathbb{R}\text{-}\mathrm{mod}$ is a fiber functor and there is an isomorphism of fiber functors ([Del94, p. 513])

$$a : \omega_0 \xrightarrow{\sim} \omega_W. \tag{16}$$

3.3. **Mixed Hodge Modules.** We denote by $\mathrm{MHM}(X)$ the abelian category of mixed Hodge modules (over $\mathbb{R}$) on a complex variety X (see [Sai94, Sai89, BGS96]). Instead of mixed Hodge module we also say Hodge sheaf. There is a faithful and exact functor $\mathrm{rat} : \mathrm{MHM}(X) \to \mathrm{Perv}(X)$. It induces a triangulated functor $\mathrm{rat} : \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X)) \to \mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X))$. Objects and morphisms in $\mathrm{MHM}(X)$ or in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ are sometimes denoted by a letter with a tilde, and omission of the tilde means application of rat, e. g. $\widetilde{M} \mapsto M = \mathrm{rat}(\widetilde{M})$.

There are functors $\mathscr{H}\mathrm{om}$, $\otimes$ and Verdier duality $\mathbb{D}$. For $f : X \to Y$ a morphism of complex varieties, we have functors f^*, f_*, $f_!$, $f^!$ relating $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ and $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(Y))$. We have the usual adjunctions (f^*, f_*) and $(f_!, f^!)$ between these functors, and $\mathbb{D}f_* = f_!\mathbb{D}$, $\mathbb{D}f^* = f^!\mathbb{D}$ and $\mathbb{D}^2 = \mathrm{id}$. All these functors "commute" with the composition

$$v := \mathrm{real} \circ \mathrm{rat} : \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X)) \to \mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X)) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}_{\mathrm{c}}(X),$$

where real is the equivalence (13).

The Hodge sheaves on the point pt are the polarizable mixed Hodge structures ([Sai89, 1.4]). Each Tate structure $\mathbb{R}(n)$ is in $\mathrm{MHM}(\mathrm{pt})$.

Each Hodge sheaf $M \in \mathrm{MHM}(X)$ has a finite increasing filtration W in $\mathrm{MHM}(X)$ called weight filtration. This filtration is functorial, and $M \mapsto \mathrm{gr}_n^W(M)$ is an exact functor ([Sai89, 1.5]). A Hodge sheaf M has weights $\leq n$ (resp. $\geq n$), if $\mathrm{gr}_j^W(M) = 0$ for $j > n$ (resp. $j < n$). More generally, a complex of Hodge sheaves M has weights $\leq n$ (resp. $\geq n$), if each $\mathrm{H}^i(M)$ has weights $\leq n+i$ (resp. $\geq n+i$). It is called pure of weight n, if it has weights $\leq n$ and $\geq n$.

We give some properties of mixed Hodge modules.

- (M1) If $M \in \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ is of weight $\leq w$ (resp. $\geq w$), so are $f_!M$, f^*M (resp. f_*M, $f^!M$) ([Sai89, 1.7]).
- (M2) M is of weight $\leq w$ if and only if $\mathbb{D}M$ is of weight $\geq -w$.
- (M3) For any $M \in \mathrm{MHM}(X)$, every $\mathrm{gr}_n^W(M)$ is a semisimple object of $\mathrm{MHM}(X)$ ([Sai89, 1.9]).
- (M4) If $M \in \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ is pure of weight n, we have a noncanonical isomorphism $M \cong \bigoplus_{j\in\mathbb{Z}}[-j]\,\mathrm{H}^j(M)$ ([Sai89, 1.11]).

In the following, $f : X \to Y$ is a morphism of complex varieties, M, N, A, B, C, D are objects of $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ or $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(Y))$, and $c : X \to \mathrm{pt}$ is the constant map.

- (M5) We have $f^*(A \otimes B) = f^*A \otimes f^*B$.

(M6) The Tate twist $M(n)$ of M is defined by $M(n) = M \otimes c^*(\mathbb{R}(n))$ ([Sai89, 1.15]), satisfies $M(0) = M$ and commutes with all functors f^*, $f^!$, f_*, $f_!$.

(M7) If M is of weight $\leq w$, then $M(n)$ is of weight $\leq w - 2n$, and $[n]M$ is of weight $\leq w + n$. The same statement with $\leq$ replaced by $\geq$.

(M8) The adjunction $(? \otimes B, \mathscr{H}\mathrm{om}\,(B, ?))$ ([Sai90, 2.9]) yields the composition morphism

$$\mathscr{H}\mathrm{om}\,(B, C) \otimes \mathscr{H}\mathrm{om}\,(A, B) \to \mathscr{H}\mathrm{om}\,(A, C)$$

and in combination with the symmetry of the tensor product a morphism

$$\mathscr{H}\mathrm{om}\,(A, B) \otimes \mathscr{H}\mathrm{om}\,(C, D) \to \mathscr{H}\mathrm{om}\,(A \otimes C, B \otimes D).$$

(M9) We have ([Sai90, 2.9.3])

$$\mathscr{H}\mathrm{om}\,(A, B) = \mathbb{D}(A \otimes \mathbb{D}B).$$

(M10) From (M9) and (M5) we get

$$f^! \mathscr{H}\mathrm{om}\,(A, B) = \mathbb{D}(f^*A \otimes \mathbb{D}f^!B) = \mathscr{H}\mathrm{om}\,(f^*A, f^!B).$$

(M11) If f is smooth of relative (complex) dimension n, we have

$$[2n]f^*(M)(n) = f^!(M).$$

Let $(X, \mathcal{S})$ be a stratified variety and $\mathrm{MHM}(X, \mathcal{S})$ the full abelian subcategory of $\mathrm{MHM}(X)$ consisting of Hodge sheaves M satisfying $\mathrm{rat}(M) \in \mathrm{Perv}(X, \mathcal{S})$. We denote by $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X), \mathcal{S})$ the full subcategory of $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ consisting of complexes M satisfying $v(M) \in \mathcal{D}^{\mathrm{b}}(X, \mathcal{S})$ (or, equivalently $\mathrm{H}^i(M) \in \mathrm{MHM}(X, \mathcal{S})$, for all $i \in \mathbb{Z}$). Objects of $\mathrm{MHM}(X, \mathcal{S})$ and $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X), \mathcal{S})$ are called smooth (along $\mathcal{S}$).

Proposition 11. *Let $S = \mathbb{C}^n$ for some $n \in \mathbb{N}$ and $c : S \to pt$ the constant map. If $M \in \mathrm{MHM}(S, \{S\})$ is a pure Hodge sheaf of weight w and smooth along the trivial stratification, there is a pure Hodge structure $E \in \mathrm{MHM}(pt)$ of weight $w - n$ such that $M \cong [n]c^*(E)$.*

Proof. By [Sai89, 2.2 Theorem], M corresponds to a polarizable variation V of Hodge structure of weight $w - n$ on $S = \mathbb{C}^n$. The fiber V_0 of V at $0 \in \mathbb{C}^n$ is a polarizable Hodge structure of weight $w-n$. We denote its constant extension to $\mathbb{C}^n$ by $\underline{V_0}$. Obviously, there is an isomorphism $V \xrightarrow{\sim} \underline{V_0}$ of the underlying local systems that respects the Hodge filtration at $0 \in \mathbb{C}^n$. By the Rigidity Theorem ([Sch73, 7.24], see also [CMSP03, 13.1.9, 13.1.10]), this isomorphism is an isomorphism of polarizable variations of Hodge structures of weight $w - n$. We obtain $M \cong [n]c^*(V_0)$, where we now consider V_0 as a polarizable Hodge structure of weight $w - n$ on pt, in particular as an element of MHM(pt). □

If Y is a variety, we define $\widetilde{\underline{Y}} = c^*(\mathbb{R}(0)) \in \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(Y))$, so $v(\widetilde{\underline{Y}}) = \underline{Y}$. Let X be an irreducible variety of dimension d_X and $j : U \to X$ the inclusion of a nonsingular affine open dense subset. The intersection cohomology complexes of X are defined by ([Sai89, 1.13])

$$\mathcal{IC}(X) := \mathrm{im}(j_!([d_X]\underline{U}) \to j_*([d_X]\underline{U})) \in \mathrm{Perv}(X) \text{ and}$$

$$\widetilde{\mathcal{IC}}(X) := \mathrm{im}(j_!([d_x]\widetilde{\underline{U}}) \to j_*([d_x]\widetilde{\underline{U}})) \in \mathrm{MHM}(X).$$

This definition does not depend on the choice of U, $\widetilde{\mathcal{IC}}(X)$ is simple and pure of weight $d_X := \dim_{\mathbb{C}} X$ and satisfies $\mathrm{rat}(\widetilde{\mathcal{IC}}(X)) = \mathcal{IC}(X)$.

If $l_{\overline{S}} : \overline{S} \to X$ is the inclusion of the closure of a stratum S in a stratified variety $(X, \mathcal{S})$, we denote $l_{\overline{S}*}(\mathcal{IC}(\overline{S}))$ by $\mathcal{IC}_S$, and similarly for $\widetilde{\mathcal{IC}}_S$. These objects are smooth, $\widetilde{\mathcal{IC}}_S$ is simple and pure of weight d_S, and we have $\mathrm{rat}(\widetilde{\mathcal{IC}}_S) = \mathcal{IC}_S$. If $\mathcal{S}$ is a cell-stratification, the $(\mathcal{IC}_S)_{S\in\mathcal{S}}$ are precisely the simple objects of $\mathrm{Perv}(X, \mathcal{S})$.

In the introduction, we wrote $\mathcal{IC}(S)$ and $\widetilde{\mathcal{IC}}(S)$ instead of $\mathcal{IC}_S$ and $\widetilde{\mathcal{IC}}_S$. We will use this notation later on again.

3.4. **Construction of Epimorphisms from Projective Objects.** In this subsection we describe an algorithm for constructing an epimorphism from a projective object onto a given object. This algorithm will be used in subsection 3.5 in order to show that there are enough perverse-projective mixed Hodge modules.

Let $\mathcal{A}$ be an artinian k-category, where k is a field. We write Hom, End, Ext, $\otimes$ instead of $\mathrm{Hom}_{\mathcal{A}}$, $\mathrm{End}_{\mathcal{A}}$, $\mathrm{Ext}_{\mathcal{A}}$, $\otimes_k$, respectively. We make the following assumptions:

(E1) $\mathrm{End}(L) = k$ for all simple objects L in $\mathcal{A}$.
(E2) There are enough projective objects in $\mathcal{A}$.

Note that (E1) implies that $\mathrm{Hom}(M, N)$ is finite dimensional, for all M, $N \in \mathcal{A}$. Then (E2) shows that $\mathrm{Ext}^1(M, N)$ is finite dimensional, for all M, $N \in \mathcal{A}$.

The following algorithm keeps extending simple objects to a given object until this is no longer possible. In doing this, only non-trivial extensions are used.

Step 1: Take an object $A \in \mathcal{A}$ as input datum.
Step 2: Set $i = 0$ and $A_0 = A$.
Step 3: While there is a simple object $L \in \mathcal{A}$ with $\mathrm{Ext}^1(A_i, L) \neq 0$
  Step 3.1: Take a simple object $L \in \mathcal{A}$ with $E := \mathrm{Ext}^1(A_i, L) \neq 0$.
  Step 3.2: The element $\mathrm{id} \in E^* \otimes E = \mathrm{Ext}^1(A_i, E^* \otimes L)$ gives rise[1] to an extension $E^* \otimes L \hookrightarrow A_{i+1} \twoheadrightarrow A_i$.
  Step 3.3: Increase i by 1.
Step 4: Define $Q = A_i$ and return the epimorphism $Q = A_i \twoheadrightarrow A_0 = A$.

Proposition 12. *Given any $A \in \mathcal{A}$, the above algorithm terminates after finitely many steps and returns an epimorphism $Q \to A$ from a projective object Q.*

Proof. We denote the length of an object $X \in \mathcal{A}$ by $\lambda(X)$. Assume that our algorithm does not stop. Then it constructs a sequence $\ldots \twoheadrightarrow A_2 \twoheadrightarrow A_1 \twoheadrightarrow A_0$ of objects and epimorphisms with $\lambda(A_0) < \lambda(A_1) < \lambda(A_2) < \ldots$. By (E2), there are a projective object P and an epimorphism $\pi_0 : P \to A_0 = A$. Since $A_1 \to A_0$ is epimorphic, there is a morphism $\pi_1 : P \to A_1$ lifting $\pi_0 : P \to A_0$. Proceeding in this manner we obtain liftings $\pi_i : P \to A_i$ of π_{i-1} for all $i > 0$. Now Lemma 13 below shows that all these π_i are epimorphisms. In particular, we get the contradiction $\lambda(A_i) \leq \lambda(P)$ for all i. So our algorithm stops. The returned object Q is projective since $\mathrm{Ext}^1(Q, L) = 0$ for all simple objects $L \in \mathcal{A}$. $\square$

[1] Perhaps we should explain what we mean by the object $E^* \otimes L$. Let Nat be the following category: Its objects are the natural numbers $\mathbb{N}$; if m, n are objects of Nat, we define $\mathrm{Hom}_{\mathrm{Nat}}(m, n)$ to be the set of $n \times m$-matrices over k; composition is matrix multiplication. We fix an equivalence of categories $\phi : k\text{-}\mathrm{mod} \xrightarrow{\sim} \mathrm{Nat}$ between the category of finite dimensional vector spaces over k and Nat. There is an obvious functor $(?\otimes?) : \mathrm{Nat} \times \mathcal{A} \to \mathcal{A}$, $(n, M) \mapsto n \otimes M := M^{\oplus n}$. If V is a finite dimensional vector space and M is in $\mathcal{A}$, we define $V \otimes M := \phi(V) \otimes M$.

Lemma 13. *Let $\pi : P \twoheadrightarrow A$ be an epimorphism from an object P (not necessarily projective) onto A. Let L be a simple object, $E = \operatorname{Ext}^1(A, L)$ and*

$$0 \to E^* \otimes L \xrightarrow{i} M \xrightarrow{c} A \to 0 \tag{17}$$

the extension defined by $\mathrm{id} \in E^* \otimes E = \operatorname{Ext}^1(A, E^* \otimes L)$. Let $\widetilde{\pi} : P \to M$ be a morphism such that $c \circ \widetilde{\pi} = \pi$. Then $\widetilde{\pi}$ is an epimorphism.

Proof. We have to show that

$(*)_N$ The map $\widetilde{\pi}^* : \operatorname{Hom}(M, N) \to \operatorname{Hom}(P, N)$ is injective.

holds for all objects N. If (N', N, N'') is a short exact sequence and $(*)_{N'}$ and $(*)_{N''}$ hold, then it is easy to see that $(*)_N$ is satisfied. So it is enough to prove $(*)_N$ for simple objects N.

Let N be simple. By applying $\operatorname{Hom}(?, N)$ to the exact sequence (17), we obtain the exact sequence

$$0 \to \operatorname{Hom}(A, N) \xrightarrow{c^*} \operatorname{Hom}(M, N) \xrightarrow{i^*} \operatorname{Hom}(E^* \otimes L, N) \xrightarrow{\delta} \operatorname{Ext}^1(A, N).$$

Our claim is that c^* is bijective. For $N \ncong L$ this is clear, since $\operatorname{Hom}(E^* \otimes L, N)$ vanishes. If $N = L$ there is an obvious map $\mathrm{can} : E \to \operatorname{Hom}(E^* \otimes L, L)$ such that $\delta \circ \mathrm{can} = \mathrm{id}$. Since $\operatorname{Hom}(L, L) = k$ by (E1), can and hence δ are isomorphisms. This shows that c^* is an isomorphism if $N = L$ or $N \cong L$.

We now apply $\operatorname{Hom}(?, N)$ to $c \circ \widetilde{\pi} = \pi$ and obtain $\widetilde{\pi}^* \circ c^* = \pi^*$. Since c^* is bijective and π^* is injective, $\widetilde{\pi}^*$ is injective and $(*)_N$ is true. □

3.5. **Existence of Enough Perverse-Projective Hodge Sheaves.** Let $(X, \mathcal{S})$ be a cell-stratified complex variety. A smooth Hodge sheaf $\widetilde{P} \in \operatorname{MHM}(X, \mathcal{S})$ is called **perverse-projective**, if the underlying perverse sheaf $P = \operatorname{rat}(\widetilde{P})$ is a projective object of $\operatorname{Perv}(X, \mathcal{S})$. A complex in $\operatorname{MHM}(X, \mathcal{S})$ is called **perverse-projective**, if all its components are perverse-projective.

Proposition 14. *If $(X, \mathcal{S})$ is a cell-stratified complex variety, there are enough perverse-projective objects in $\operatorname{MHM}(X, \mathcal{S})$, i. e. for every smooth Hodge sheaf $\widetilde{A} \in \operatorname{MHM}(X, \mathcal{S})$, there is a perverse-projective smooth Hodge sheaf $\widetilde{P} \in \operatorname{MHM}(X, \mathcal{S})$ and an epimorphism $\widetilde{P} \twoheadrightarrow \widetilde{A}$.*

Proof. Postponed to the end of this subsection. □

Corollary 15. *Every smooth Hodge sheaf $\widetilde{A} \in \operatorname{MHM}(X, \mathcal{S})$ has a* ***perverse-projective resolution*** *$\widetilde{P} \to \widetilde{A}$, i. e. there is a perverse-projective complex $\widetilde{P} = (\widetilde{P}^n, d^n)$ in $\operatorname{MHM}(X, \mathcal{S})$ with $\widetilde{P}^n = 0$ for $n > 0$ and a quasi-isomorphism $\widetilde{P} \to \widetilde{A}$ in $\operatorname{Ket}(\operatorname{MHM}(X, \mathcal{S}))$. Moreover, we can assume that this resolution is* ***of finite length****, i. e. $\widetilde{P}^n = 0$ for $n \ll 0$.*

Proof. The first statement is obvious from the proposition, the second one follows from Theorem 10. □

If $\widetilde{M}, \widetilde{N} \in \operatorname{MHM}(X)$ are Hodge sheaves on X with underlying perverse sheaves M and N, there is a (polarizable) mixed Hodge structure on all $\operatorname{Ext}^i_{\operatorname{Perv}(X)}(M, N)$, defined as follows. Let $c : X \to \mathrm{pt}$ be the constant map and $\underline{M} = \operatorname{real}(M)$, $\underline{N} = \operatorname{real}(N)$. On a point, perverse cohomology and ordinary cohomology coincide, and we get

$$v(\mathrm{H}^i\, c_* \mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N})) = \mathrm{H}^i\, c_* \mathscr{H}\mathrm{om}\,(\underline{M}, \underline{N}) = \operatorname{Ext}^i_{\operatorname{Sh}(X)}(\underline{M}, \underline{N}).$$

Thus, we obtain a natural mixed Hodge structure on $\mathrm{Ext}^i_{\mathrm{Sh}(X)}(\underline{M},\underline{N})$. We transfer this structure to $\mathrm{Ext}^i_{\mathrm{Perv}(X)}(M,N)$ (using (14)) and denote it by $\mathrm{Ext}^i_{\mathrm{Perv}(X)}(\widetilde{M},\widetilde{N})$ (and by $\mathrm{Hom}_{\mathrm{Perv}(X)}(\widetilde{M},\widetilde{N})$ for $i=0$). If $\widetilde{M}$, $\widetilde{N}$ are smooth along our cell-stratification, the analogous argument equips $\mathrm{Ext}^i_{\mathrm{Perv}(X,\mathcal{S})}(M,N)$ with a mixed Hodge structure $\mathrm{Ext}^i_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{M},\widetilde{N})$.

Remark 16. Our construction defines bifunctors

$$\mathrm{Ext}^i_{\mathrm{Perv}(X)}(?,?) : \mathrm{MHM}(X)^{\mathrm{op}} \times \mathrm{MHM}(X) \to \mathrm{MHS}\,.$$

The usual long exact Ext-sequences in both variables underlie exact sequences of mixed Hodge structures. Furthermore, if $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ are Hodge sheaves, composition defines a morphism of mixed Hodge structures

$$\mathrm{Ext}^i_{\mathrm{Perv}(X)}(\widetilde{B},\widetilde{C}) \otimes \mathrm{Ext}^j_{\mathrm{Perv}(X)}(\widetilde{A},\widetilde{B}) \to \mathrm{Ext}^{i+j}_{\mathrm{Perv}(X)}(\widetilde{A},\widetilde{C}). \tag{18}$$

This can be seen as follows. If F, G are in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ there is a natural morphism $c_*F\otimes c_*G \to c_*(F\otimes G)$. We compose this morphism for $F = \mathscr{H}\mathrm{om}\,(\widetilde{B},\widetilde{C})$ and $G = \mathscr{H}\mathrm{om}\,(\widetilde{A},\widetilde{B})$ with c_* of the morphism $\mathscr{H}\mathrm{om}\,(\widetilde{B},\widetilde{C}) \otimes \mathscr{H}\mathrm{om}\,(\widetilde{A},\widetilde{B}) \to \mathscr{H}\mathrm{om}\,(\widetilde{A},\widetilde{C})$ (cf. (M8)) and get a morphism

$$c_*\mathscr{H}\mathrm{om}\,(\widetilde{B},\widetilde{C}) \otimes c_*\mathscr{H}\mathrm{om}\,(\widetilde{A},\widetilde{B}) \to c_*\mathscr{H}\mathrm{om}\,(\widetilde{A},\widetilde{C}).$$

Now we take the $(i+j)$-th cohomology and use the obvious morphism of mixed Hodge structures $\mathrm{H}^i(c_*F)\otimes \mathrm{H}^j(c_*G) \to \mathrm{H}^{i+j}(c_*F\otimes c_*G)$ in order to get morphism (18). The analogous remarks are valid for $\mathrm{Ext}^i_{\mathrm{Perv}(X,\mathcal{S})}$.

Lemma 17. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact faithful functor between abelian categories, $F : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ its derived functor, and f a morphism in $\mathcal{D}(\mathcal{A})$. Then f is an isomorphism if and only if $F(f)$ is an isomorphism.*

Proof. This is an easy exercise. □

Lemma 18. *Let A, $B \in \mathrm{MHM}(X)$ be Hodge sheaves on X, let $M \in \mathrm{MHM}(pt)$ be a polarizable mixed Hodge structure and $c : X \to pt$ the constant map. Then there are natural isomorphisms*

$$M \otimes c_*A \xrightarrow{\sim} c_*(c^*M \otimes A) \tag{19}$$

$$c^*M \otimes \mathscr{H}\mathrm{om}\,(A,B) \xrightarrow{\sim} \mathscr{H}\mathrm{om}\,(A, c^*M \otimes B). \tag{20}$$

in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(pt))$ and $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ respectively.

Proof. The morphism (19) is the image of the identity morphism of $c^*M\otimes A$ under the chain of obvious morphisms

$$\begin{aligned}\mathrm{Hom}(c^*M\otimes A, c^*M\otimes A) &\to \mathrm{Hom}(c^*M\otimes c^*c_*A, c^*M\otimes A)\\ &= \mathrm{Hom}(c^*(M\otimes c_*A), c^*M\otimes A)\\ &= \mathrm{Hom}(M\otimes c_*A, c_*(c^*M\otimes A)),\end{aligned}$$

where $\mathrm{Hom} = \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathrm{MHM})}$. Since $v(M)$ is a finite dimensional vector space, $v(19)$ is an isomorphism, and Lemma 17, applied to rat, shows that (19) is an isomorphism.

The morphism (20) comes from the identifications ((M6), (M9))

$$c^*M = \mathbb{D}(c^*\mathbb{R}(0)\otimes \mathbb{D}c^*M) = \mathscr{H}\mathrm{om}\,(c^*\mathbb{R}(0), c^*M)$$

and the morphism ((M8), (M6))

$$\begin{aligned}\mathscr{H}\mathrm{om}\,(c^*\mathbb{R}(0), c^*M) \otimes \mathscr{H}\mathrm{om}\,(A, B) &\to \mathscr{H}\mathrm{om}\,(c^*\mathbb{R}(0) \otimes A, c^*M \otimes B)\\ &= \mathscr{H}\mathrm{om}\,(A, c^*M \otimes B).\end{aligned}$$

All these morphisms are mapped to isomorphisms by the functor v, so (20) is an isomorphism by Lemma 17 and equivalence (13). □

Lemma 19. *Let $\widetilde{A}$, $\widetilde{B} \in \mathrm{MHM}(X, \mathcal{S})$, $\widetilde{M} \in \mathrm{MHM}(pt)$, and let $c : X \to pt$ be the constant map. Then $c^*\widetilde{M} \otimes \widetilde{B} \in \mathrm{MHM}(X, \mathcal{S})$, and there is a natural isomorphism*

$$\widetilde{M} \otimes \mathrm{Ext}^i_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{A}, \widetilde{B}) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{A}, c^*\widetilde{M} \otimes \widetilde{B})$$

of (polarizable) mixed Hodge structures, for all $i \in \mathbb{Z}$.

Proof. The first statement is obvious. Isomorphisms (19) and (20) yield an isomorphism

$$\widetilde{M} \otimes c_*\mathscr{H}\mathrm{om}\,(\widetilde{A}, \widetilde{B}) \xrightarrow{\sim} c_*\mathscr{H}\mathrm{om}\,(\widetilde{A}, c^*\widetilde{M} \otimes \widetilde{B})$$

Taking the i-th cohomology and using the exactness of the functor $(\widetilde{M}\otimes ?)$ finishes the proof. □

Proof of Proposition 14. If $\widetilde{M}$, $\widetilde{N} \in \mathrm{MHM}(X)$ are Hodge sheaves, there is a short exact sequence (see [Sai90])

$$\begin{aligned}0 \to \mathrm{H}^1_{\mathrm{MHM(pt)}}(\mathrm{Hom}_{\mathrm{Perv}(X)}(\widetilde{M}, \widetilde{N})) &\to \mathrm{Ext}^1_{\mathrm{MHM}(X)}(\widetilde{M}, \widetilde{N})\\ &\to \mathrm{H}^0_{\mathrm{MHM(pt)}}(\mathrm{Ext}^1_{\mathrm{Perv}(X)}(\widetilde{M}, \widetilde{N})) \to 0,\end{aligned}$$

where $\mathrm{H}^i_{\mathrm{MHM(pt)}}$ is the absolute Hodge cohomology functor: For $A \in \mathrm{MHM(pt)}$, it is defined by $\mathrm{H}^i_{\mathrm{MHM(pt)}}(A) := \mathrm{Ext}^i_{\mathrm{MHM(pt)}}(\mathbb{R}(0), A)$. The categories $\mathrm{MHM}(X, \mathcal{S})$ and $\mathrm{Perv}(X, \mathcal{S})$ are closed under extensions in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ and $\mathcal{D}^{\mathrm{b}}(X)$ ([BBD82, 1.3.6, 3.1.17]). Thus, for smooth $\widetilde{M}$, $\widetilde{N} \in \mathrm{MHM}(X, \mathcal{S})$, there is a short exact sequence

$$\begin{aligned}0 \to \mathrm{H}^1_{\mathrm{MHM(pt)}}(\mathrm{Hom}_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{M}, \widetilde{N})) &\to \mathrm{Ext}^1_{\mathrm{MHM}(X,\mathcal{S})}(\widetilde{M}, \widetilde{N})\\ &\to \mathrm{H}^0_{\mathrm{MHM(pt)}}(\mathrm{Ext}^1_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{M}, \widetilde{N})) \to 0.\end{aligned} \tag{21}$$

Let $\widetilde{M}$, $\widetilde{N} \in \mathrm{MHM}(X, \mathcal{S})$ and consider the polarizable mixed Hodge structure $\widetilde{E} = \mathrm{Ext}^1_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{M}, \widetilde{N})$. The map $\mathrm{can} : \mathbb{R}(0) \to \widetilde{E}^* \otimes \widetilde{E}$, $1 \mapsto \mathrm{id}_{\widetilde{E}}$, is a morphism of polarizable mixed Hodge structures, i. e. an element $\mathrm{can} \in \mathrm{H}^0_{\mathrm{MHM(pt)}}(\widetilde{E}^* \otimes \widetilde{E})$. Lemma 19 yields an isomorphism

$$\widetilde{E}^* \otimes \widetilde{E} = \widetilde{E}^* \otimes \mathrm{Ext}^1_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{M}, \widetilde{N}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{M}, c^*(\widetilde{E}^*) \otimes \widetilde{N})$$

of polarizable mixed Hodge structures. The exact sequence (21) shows that there is an extension of smooth Hodge sheaves

$$c^*(\widetilde{E}^*) \otimes \widetilde{N} \hookrightarrow \widetilde{K} \twoheadrightarrow \widetilde{M}$$

such that the underlying extension of perverse sheaves is given by the element $\mathrm{id}_E \in E^* \otimes E \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Perv}(X,\mathcal{S})}(M, c^*(E^*) \otimes N)$.

We now use the following algorithm in order to prove our proposition.

Step 1: Take an object $\widetilde{A} \in \mathrm{MHM}(X, \mathcal{S})$ as input datum.
Step 2: Set $i = 0$ and $\widetilde{A}_0 = \widetilde{A}$.

Step 3: While there is a stratum $S \in \mathcal{S}$ with $\operatorname{Ext}^1_{\operatorname{Perv}(X,\mathcal{S})}(\widetilde{A}_i, \widetilde{\mathcal{IC}}_S) \neq 0$

Step 3.1: Take a stratum $S \in \mathcal{S}$ with $\widetilde{E} = \operatorname{Ext}^1_{\operatorname{Perv}(X,\mathcal{S})}(\widetilde{A}_i, \widetilde{\mathcal{IC}}_S) \neq 0$.

Step 3.2: Choose, as explained above, an extension $c^*(\widetilde{E}^*) \otimes \widetilde{\mathcal{IC}}_S \hookrightarrow \widetilde{A}_{i+1} \twoheadrightarrow \widetilde{A}_i$ of smooth Hodge sheaves such that the underlying extension of perverse sheaves is given by $\mathrm{id} \in E^* \otimes E$.

Step 3.3: Increase i by 1.

Step 4: Define $\widetilde{P} = \widetilde{A}_i$ and return the epimorphism $\widetilde{P} = \widetilde{A}_i \twoheadrightarrow \widetilde{A}_0 = \widetilde{A}$ of smooth Hodge sheaves.

The underlying algorithm is the algorithm from subsection 3.4 for $\mathcal{A} = \operatorname{Perv}(X,\mathcal{S})$. Since $\mathcal{S}$ is a cell-stratification, this choice of $\mathcal{A}$ is justified by Theorem 10. Thus, Proposition 12 shows that our algorithm terminates and returns an epimorphism $\widetilde{P} \twoheadrightarrow \widetilde{A}$ from a perverse-projective smooth Hodge sheaf $\widetilde{P}$. □

3.6. **Comparison of Mixed Hodge Structures.** Let $\widetilde{A}$, $\widetilde{B} \in \operatorname{MHM}(X,\mathcal{S})$ be smooth Hodge sheaves on a cell-stratified variety $(X,\mathcal{S})$, with underlying smooth perverse sheaves A and B. As explained in subsection 3.5, there is a (polarizable) mixed Hodge structure $\operatorname{Ext}^i_{\operatorname{Perv}(X,\mathcal{S})}(\widetilde{A}, \widetilde{B})$ on $\operatorname{Ext}^i_{\operatorname{Perv}(X,\mathcal{S})}(A, B)$.

Now assume that $\widetilde{P} \to \widetilde{A}$ and $\widetilde{Q} \to \widetilde{B}$ are perverse-projective resolutions of finite length (cf. Corollary and Definition 15), with underlying projective resolutions $P \to A$ and $Q \to B$. We apply $\operatorname{Hom}_{\operatorname{Perv}(X,\mathcal{S})}(?,?)$ to $\widetilde{P}$ and $\widetilde{Q}$ and get a double complex of (polarizable) mixed Hodge structures (see Remark 16) with (i,j)-component $\operatorname{Hom}_{\operatorname{Perv}(X,\mathcal{S})}(\widetilde{P}^{-i}, \widetilde{Q}^j)$. We denote its simple complex by $\mathcal{H}\mathrm{om}_{\operatorname{Perv}(X,\mathcal{S})}(\widetilde{P}, \widetilde{Q})$ or simply by $\mathcal{H}\mathrm{om}(\widetilde{P}, \widetilde{Q})$. (We use this notation also for arbitrary complexes $\widetilde{P}$ and $\widetilde{Q}$ in $\operatorname{MHM}(X,\mathcal{S})$.) The underlying complex of real vector spaces is the complex $\mathcal{H}\mathrm{om}(P,Q) = \mathcal{H}\mathrm{om}_{\operatorname{Perv}(X,\mathcal{S})}(P,Q)$ from subsection 2.4. The n-th cohomology $\mathrm{H}^n(\mathcal{H}\mathrm{om}_{\operatorname{Perv}(X,\mathcal{S})}(\widetilde{P}, \widetilde{Q}))$ is a mixed Hodge structure, and its underlying vector space is

$$\begin{aligned}\mathrm{H}^n(\mathcal{H}\mathrm{om}_{\operatorname{Perv}(X,\mathcal{S})}(P,Q)) &= \operatorname{Hom}_{\operatorname{Hot}(\operatorname{Perv}(X,\mathcal{S}))}(P,[n]Q)\\ &= \operatorname{Ext}^n_{\operatorname{Perv}(X,\mathcal{S})}(P,Q) \xrightarrow{\sim} \operatorname{Ext}^n_{\operatorname{Perv}(X,\mathcal{S})}(P,B)\\ &\xleftarrow{\sim} \operatorname{Ext}^n_{\operatorname{Perv}(X,\mathcal{S})}(A,B).\end{aligned}$$

Proposition 20. *The (polarizable) mixed Hodge structures*

$$\operatorname{Ext}^n_{\operatorname{Perv}(X,\mathcal{S})}(\widetilde{A}, \widetilde{B}) \text{ and } \mathrm{H}^n(\mathcal{H}\mathrm{om}_{\operatorname{Perv}(X,\mathcal{S})}(\widetilde{P}, \widetilde{Q}))$$

with underlying vector space $\operatorname{Ext}^n_{\operatorname{Perv}(X,\mathcal{S})}(A,B)$ *are isomorphic.*

Proof. We write Hom, $\mathcal{H}\mathrm{om}$ and Ext instead of $\operatorname{Hom}_{\operatorname{Perv}(X,\mathcal{S})}$, $\mathcal{H}\mathrm{om}_{\operatorname{Perv}(X,\mathcal{S})}$ and $\operatorname{Ext}_{\operatorname{Perv}(X,\mathcal{S})}$ respectively, and show the existence of isomorphisms of mixed Hodge structures

$$\operatorname{Ext}^n(\widetilde{A}, \widetilde{B}) \xleftarrow{\sim} \mathrm{H}^n(\mathcal{H}\mathrm{om}(\widetilde{P}, \widetilde{B})) \xleftarrow{\sim} \mathrm{H}^n(\mathcal{H}\mathrm{om}(\widetilde{P}, \widetilde{Q})). \tag{22}$$

Let us construct the isomorphism on the left in (22). We decompose $\widetilde{P} \to \widetilde{A}$ into short exact sequences as follows. For $i \leq 0$, let $\widetilde{K}^i$ be the image of the differential $\widetilde{P}^{i-1} \to \widetilde{P}^i$, and define $\widetilde{K}^1 = \widetilde{A}$. For each $i \leq 0$, we get a short exact sequence $(\widetilde{K}^i, \widetilde{P}^i, \widetilde{K}^{i+1})$. The associated long exact Ext-sequence in the first variable gives an exact sequence of mixed Hodge modules (see Remark 16)

$$
\begin{aligned}
(23) \qquad & 0 \to \mathrm{Hom}(\widetilde{K}^{i+1}, \widetilde{B}) \to \mathrm{Hom}(\widetilde{P}^{i}, \widetilde{B})) \to \mathrm{Hom}(\widetilde{K}^{i}, \widetilde{B}) \\
& \to \mathrm{Ext}^1(\widetilde{K}^{i+1}, \widetilde{B}) \to 0
\end{aligned}
$$

and isomorphisms

$$
(24) \qquad \mathrm{Ext}^j(\widetilde{K}^{i}, \widetilde{B}) \xrightarrow{\sim} \mathrm{Ext}^{j+1}(\widetilde{K}^{i+1}, \widetilde{B})
$$

for all $j \geq 1$. The 0-th cohomology of the complex $\widetilde{C} := \mathcal{H}\mathrm{om}\,(\widetilde{P}, \widetilde{B})$ is

$$
\mathrm{H}^0(\widetilde{C}) = \ker\left(\mathrm{Hom}(\widetilde{P}^0, \widetilde{B}) \to \mathrm{Hom}(\widetilde{K}^0, \widetilde{B})\right) \xrightarrow{\sim} \mathrm{Hom}(\widetilde{A}, \widetilde{B})
$$

by (23) for $i = 0$. For $m \geq 0$ we have

$$
\begin{aligned}
\mathrm{H}^{m+1}(\widetilde{C}) &= \mathrm{cok}\left(\mathrm{Hom}(\widetilde{P}^{-m}, \widetilde{B}) \to \mathrm{Hom}(\widetilde{K}^{-m}, \widetilde{B})\right) \\
&\xrightarrow{\sim} \mathrm{Ext}^1(\widetilde{K}^{-m+1}, \widetilde{B}) \\
&\xrightarrow{\sim} \mathrm{Ext}^{m+1}(\widetilde{A}, \widetilde{B})
\end{aligned}
$$

by (23) and repeated use of (24). This establishes the isomorphism on the left in (22). The isomorphism on the right is a consequence of the following Lemma 21 applied to the quasi-isomorphism $\widetilde{Q} \to \widetilde{B}$. □

Lemma 21. *Let $\widetilde{P}$, $\widetilde{Q}$ and $\widetilde{R}$ be complexes in* $\mathrm{MHM}(X, \mathcal{S})$*, and assume that $\widetilde{P}$ is perverse-projective and bounded above. Then any quasi-isomorphism $f : \widetilde{Q} \to \widetilde{R}$ in* $\mathrm{Ket}(\mathrm{MHM}(X, \mathcal{S}))$ *induces a quasi-isomorphism $\mathcal{H}\mathrm{om}\,(\widetilde{P}, \widetilde{Q}) \to \mathcal{H}\mathrm{om}\,(\widetilde{P}, \widetilde{R})$.*

Proof. It suffices to prove that $\mathrm{rat}(f) : Q \to R$ in $\mathrm{Ket}(\mathrm{Perv}(X, \mathcal{S}))$ induces a quasi-isomorphism $\mathcal{H}\mathrm{om}\,(P, Q) \to \mathcal{H}\mathrm{om}\,(P, R)$. But this follows from Remark 8. □

Remark 22. Assume that $\widetilde{P} \to \widetilde{A}$, $\widetilde{Q} \to \widetilde{B}$ and $\widetilde{R} \to \widetilde{C}$ are perverse-projective resolutions of finite length. Then $\mathcal{H}\mathrm{om}\,(\widetilde{P}, \widetilde{Q})$ and $\mathcal{H}\mathrm{om}\,(\widetilde{Q}, \widetilde{R})$ are complexes of (polarizable) mixed Hodge structures. The obvious composition map

$$
\mathcal{H}\mathrm{om}\,(\widetilde{Q}, \widetilde{R}) \otimes \mathcal{H}\mathrm{om}\,(\widetilde{P}, \widetilde{Q}) \to \mathcal{H}\mathrm{om}\,(\widetilde{P}, \widetilde{R})
$$

is a morphism of complexes of mixed Hodge structures. It induces a morphism of mixed Hodge structures

$$
\mathrm{H}^i(\mathcal{H}\mathrm{om}\,(\widetilde{Q}, \widetilde{R})) \otimes \mathrm{H}^j(\mathcal{H}\mathrm{om}\,(\widetilde{P}, \widetilde{Q})) \to \mathrm{H}^{i+j}(\mathcal{H}\mathrm{om}\,(\widetilde{P}, \widetilde{R})).
$$

Under the identifications from (the proof of) Proposition 20, this morphism corresponds to the morphism (18) in Remark 16. The reason for this fact is that both morphisms are just the composition of morphisms in the derived category of perverse sheaves, if we forget about the mixed Hodge structures.

3.7. **Local-to-Global Spectral Sequence.** If X is a complex variety and M a complex of sheaves (of Hodge sheaves respectively) on X, the hypercohomology $\mathbb{H}(M) := \mathrm{H}(X; M) := \mathrm{H}(c_* M)$ is a complex (with differential zero) of vector spaces (of mixed Hodge structures respectively). Here $c : X \to \mathrm{pt}$ is the constant map. If $l : Y \hookrightarrow X$ is a locally closed subvariety, the local hypercohomology of M along Y is $\mathbb{H}_Y(M) := \mathbb{H}(l^! M) = \mathbb{H}(l_* l^! M)$.

Consider now a complex variety X filtered by closed subvarieties $X = X_0 \supset X_1 \supset \cdots \supset X_r = \emptyset$. If $M \in \mathcal{D}^{\mathrm{b}}(X)$ is a complex of sheaves, there is a local-to-global spectral sequence with E_1-term $E_1^{p,q} = \mathbb{H}^{p+q}_{X_p - X_{p+1}}(M)$ converging to $E_\infty^{p,q} =$

$\mathrm{gr}^p(\mathbb{H}^{p+q}(M))$. It can be constructed from an injective resolution of M, cf. [BGS96, 3.4].

Even though there are not enough injective Hodge sheaves, we shall construct a similar spectral sequence of mixed (polarizable) Hodge structures, if M is a complex of Hodge sheaves on our filtered variety X. In order to do so, we need the following technical proposition.

Proposition 23. *Let $\mathcal{A}$ be an abelian category and*

$$A^0 \xleftarrow{a_1} A^1 \xleftarrow{a_2} \cdots \xleftarrow{a_{r-1}} A^{r-1} \xleftarrow{a_r} A^r = 0$$

a finite sequence of objects and morphisms in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$. Then there is a bounded complex K in $\mathcal{A}$ with a finite filtration F by subcomplexes, $K = F^0K \supset F^1K \supset \cdots \supset F^{r-1}K \supset F^rK = 0$, and quasi-isomorphisms $u_p : F^pK \to A^p$ in $\mathrm{Ket}(\mathcal{A})$ such that the diagram

$$\begin{tikzcd}
A^0 & A^1 \arrow[l, "a_1"'] & A^2 \arrow[l, "a_2"'] & \cdots \arrow[l, "a_3"'] & A^{r-1} \arrow[l, "a_{r-1}"'] & A^r \arrow[l, "a_r"'] \\
F^0K \arrow[u, "u_0"] & F^1K \arrow[l, "k_1"'] \arrow[u, "u_1"] & F^2K \arrow[l, "k_2"'] \arrow[u, "u_2"] & \cdots \arrow[l, "k_3"'] & F^{r-1}K \arrow[l, "k_{r-1}"'] \arrow[u, "u_{r-1}"] & F^rK \arrow[l, "k_r"'] \arrow[u, "u_r"]
\end{tikzcd}$$

commutes in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$. Here $k_p : F^pK \hookrightarrow F^{p+1}K$ denotes the inclusion.

Proof. This seems to be well known, cf. the similar statement given in [BBD82, 3.1.2.7] without proof. For an explicit proof see [Sch07, Prop. 26] □

Now let $M \in \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ be a complex of Hodge sheaves on X, where X is a complex variety, filtered by closed subvarieties $X = X_0 \supset X_1 \supset \cdots \supset X_r = \emptyset$. If we let $i_p : X_p \hookrightarrow X$ denote the inclusion, the adjunctions $(i_{p*} = i_{p!}, i_p^!)$ yield a sequence in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$

$$M = i_{0*}i_0^!M \longleftarrow i_{1*}i_1^!M \longleftarrow \cdots \longleftarrow i_{r-1*}i_{r-1}^!M \longleftarrow i_{r*}i_r^!M = 0.$$

We apply c_* to this sequence, where $c : X \to \mathrm{pt}$. Proposition 23 shows that there is a diagram

$$\begin{tikzcd}
c_*M & c_*i_{1*}i_1^!M \arrow[l] & c_*i_{2*}i_2^!M \arrow[l] & \cdots \arrow[l] & 0 \arrow[l] \\
K = F^0K \arrow[u] & F^1K \arrow[l] \arrow[u] & F^2K \arrow[l] \arrow[u] & \cdots \arrow[l] & F^rK = 0, \arrow[l] \arrow[u]
\end{tikzcd}$$

where the lower horizontal row is a finite filtration on a complex K in MHS, the vertical maps are quasi-isomorphisms in $\mathrm{Ket}^{\mathrm{b}}(\mathrm{MHS})$, and the diagram commutes in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHS})$. (We could write MHM(pt) instead of MHS.)

By [Lan02, Proposition XX.9.3], there exists a spectral sequence $(E_r, d_r)_{r\geq 0}$ with $E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{gr}_F^p(K))$ and $E_\infty^{p,q} = \mathrm{gr}^p(\mathrm{H}^{p+q}(K))$. Using standard techniques it is easy to identify the E_1-term of this spectral sequence with $\mathbb{H}^{p+q}_{X_p - X_{p+1}}(M)$ (for details see [Sch07, 2.11]). This proves

Proposition 24. *Let X be a complex variety, filtered by closed subvarieties $X = X_0 \supset X_1 \supset \cdots \supset X_r = \emptyset$, and $M \in \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ a complex of Hodge sheaves on*

X. Then there is a spectral sequence $(E_r, d_r)_{r\geq 0}$ of (polarizable) mixed Hodge structures with E_1-term $E_1^{p,q} = \mathbb{H}^{p+q}_{X_p - X_{p+1}}(M)$ that converges to $E_\infty^{p,q} = \mathrm{gr}^p(\mathbb{H}^{p+q}(M))$ (where $\mathbb{H}(M)$ is filtered by the images of the obvious maps $\mathbb{H}_{X_p}(M) \to \mathbb{H}(M)$).

3.8. **Purity.** Let $(X, \mathcal{S})$ be a stratified complex variety, $\widetilde{M} \in \mathrm{MHM}(X)$ and $w \in \mathbb{Z}$. We say that $\widetilde{M}$ is **$\mathcal{S}$-$*$-pure of weight** w, if, for all strata $S \in \mathcal{S}$, the restrictions $l_S^* \widetilde{M}$ are pure of weight w. It is **$\mathcal{S}$-!-pure of weight** w, if all restrictions $l_S^! \widetilde{M}$ are pure of weight w, and **$\mathcal{S}$-pure of weight** w, if it is $\mathcal{S}$-$*$-pure and $\mathcal{S}$-!-pure of weight w.

Theorem 25. *Let $(X, \mathcal{S})$ be a cell-stratified complex variety, $\widetilde{M}$, $\widetilde{N} \in \mathrm{MHM}(X, \mathcal{S})$ smooth Hodge sheaves, and m, $n \in \mathbb{Z}$. If $\widetilde{M}$ is $\mathcal{S}$-$*$-pure of weight m and $\widetilde{N}$ is $\mathcal{S}$-!-pure of weight n, the complex (with differential zero) of (polarizable) mixed Hodge structures*

$$\mathrm{Ext}_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{M}, \widetilde{N}) = \bigoplus_{i\in\mathbb{N}} \mathrm{Ext}^i_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{M}, \widetilde{N})$$

is pure of weight $n - m$.

Proof. Recall that the mixed Hodge structure $\mathrm{Ext}^i_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{M}, \widetilde{N})$ was defined from $\mathrm{H}^i(c_* \mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N})) = \mathbb{H}^i(\mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N}))$. We define X_p to be the union of all strata whose codimension in X is greater or equal to p, $X_p = \bigcup_{S\,\in\,\mathcal{S},\ d_S\,+\,p\,\leq\,\dim_{\mathbb{C}} X} S$. This defines a filtration of X by closed subvarieties, $X = X_0 \supset X_1 \supset \cdots \supset X_r = \emptyset$, where $r = \dim_{\mathbb{C}} X + 1$. Proposition 24 shows that there is a spectral sequence of mixed Hodge structures with E_1-term $E_1^{p,q} = \mathbb{H}^{p+q}_{X_p - X_{p+1}}(\mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N}))$ converging to $E_\infty^{p,q} = \mathrm{gr}^p(\mathbb{H}^{p+q}(\mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N})))$. Lemma 26 below shows that $E_1^{p,q}$ is a pure Hodge structure of weight $p+q+n-m$. There are no non-zero morphisms between pure Hodge structures of different weights, hence our spectral sequence degenerates at the E_1-term, i.e. $E_1 = E_2 = \ldots = E_\infty$. Furthermore, $\mathbb{H}^{p+q}(\mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N}))$ is pure of weight $p + q + n - m$, since it has a finite filtration with successive subquotients that are pure and of the same weight (it is in fact isomorphic to the direct sum of these subquotients, see (M3)). □

Lemma 26. *Under the assumptions of Theorem 25 and with the notation introduced in its proof, $E_1^{p,q} = \mathbb{H}^{p+q}_{X_p - X_{p+1}}(\mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N}))$ is a pure Hodge structure of weight $p + q + n - m$.*

Proof. The decomposition $X_p - X_{p+1} = \bigcup_{S\,\in\,\mathcal{S},\ d_S\,+\,p\,=\,\dim_{\mathbb{C}} X} S$ into strata of codimension p is the decomposition of $X_p - X_{p+1}$ into connected components. Therefore, we have

$$\mathbb{H}^{p+q}_{X_p - X_{p+1}}(\mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N})) = \bigoplus_{S\,\in\,\mathcal{S},\ d_S\,+\,p\,=\,\dim_{\mathbb{C}} X} \mathbb{H}^{p+q}(l_S^! \mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N})),$$

where $l_S : S \hookrightarrow X$ is the inclusion of the stratum $S \in \mathcal{S}$. For each $S \in \mathcal{S}$, we have by property (M10)

$$l_S^! \mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N}) = \mathbb{D}(l_S^* \widetilde{M} \otimes \mathbb{D} l_S^! \widetilde{N}).$$

The restrictions $l_S^* \widetilde{M}$ and $l_S^! \widetilde{N}$ are pure, so (M4) yields isomorphisms $l_S^* \widetilde{M} \cong \bigoplus_{i\in\mathbb{Z}} [-i]\, \mathrm{H}^i(l_S^* \widetilde{M})$ and $l_S^! \widetilde{N} \cong \bigoplus_{j\in\mathbb{Z}} [-j]\, \mathrm{H}^j(l_S^! \widetilde{N})$.

Fix i, $j \in \mathbb{Z}$. Since $\mathrm{H}^i(l_S^* \widetilde{M})$ is in $\mathrm{MHM}(S, \{S\})$ and pure of weight $m + i$, Proposition 11 shows that there is $A' \in \mathrm{MHM}(\mathrm{pt})$ pure of weight $m + i - d_S$ such that $\mathrm{H}^i(l_S^*\widetilde{M}) \cong [d_S]c^*A'$. Then $A := [d_S - i]A' \in \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(\mathrm{pt}))$ is pure of weight m and we have $[-i]\,\mathrm{H}^i(l_S^*\widetilde{M}) \cong c^*(A)$. If we proceed similarly and use (M7) and (M11), we find $B \in \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(\mathrm{pt}))$ pure of weight n such that $[-j]\,\mathrm{H}^j(l_S^!\widetilde{N}) \cong c^!(B)$. Note that A and B are up to shift objects of $\mathrm{MHM}(\mathrm{pt})$. Using (M10), (M11) and the adjunction isomorphism $\mathrm{id} \xrightarrow{\sim} c_*c^*$, we obtain

$$\begin{aligned}\mathbb{H}^{p+q}\big(\mathbb{D}(c^*(A) \otimes \mathbb{D}(c^!B))\big) &= \mathrm{H}^{p+q}\left(c_*c^!\mathscr{H}\mathrm{om}\,(A,B)\right)\\ &= \mathrm{H}^{p+q}\left([2d_S]c_*c^*\mathscr{H}\mathrm{om}\,(A,B)(d_S)\right)\\ &\xleftarrow{\sim} \mathrm{H}^{p+q}\left([2d_S]\mathscr{H}\mathrm{om}\,(A,B)(d_S)\right),\end{aligned}$$

and this is pure of weight $p+q+n-m$ by (M2), (M7) and (M9). □

3.9. **Formality of some DG Algebras.** Let X be a complex variety with a cell-stratification $\mathcal{S}$, and $\widetilde{M} \in \mathrm{MHM}(X,\mathcal{S})$ a smooth Hodge sheaf. By Corollary 15, there is a perverse-projective resolution $\widetilde{P} \to \widetilde{M}$ of finite length, with underlying projective resolution $P \to M$. As in subsection 3.6, we consider the complex

$$\widetilde{A} := \mathcal{E}\mathrm{nd}\,(\widetilde{P}) := \mathcal{H}\mathrm{om}_{\,\mathrm{Perv}(X,\mathcal{S})}(\widetilde{P},\widetilde{P}).$$

of (polarizable) mixed Hodge structures. Remark 22 shows that the multiplication (= composition map) $\widetilde{A} \otimes \widetilde{A} \to \widetilde{A}$ is a morphism of complexes of mixed Hodge structures. Note that a complex of mixed Hodge structures is the same as an object of the tensor category dgMHS of differential graded mixed Hodge structures. So $\widetilde{A}$ is a (unital) ring object in dgMHS (a “dg algebra of mixed Hodge structures”).

The exact faithful $\mathbb{R}$-linear tensor functors “underlying vector space” ω_0, “associated graded vector space” $\mathrm{gr}^W_{\mathbb{R}}$, “underlying vector space” η and “underlying vector space of the associated graded vector space” $\omega_W = \eta \circ \mathrm{gr}^W_{\mathbb{R}}$ (see subsection 3.2) induce tensor functors (denoted by the same symbol):

$$\begin{tikzcd} \mathrm{dgMHS} \arrow[r, "\mathrm{gr}^W_{\mathbb{R}}"] \arrow[d, "\omega_0"] \arrow[dr, "\omega_W"] & \mathrm{dggMod}(\mathbb{R}) \arrow[d, "\eta"] \\ \mathrm{dgMod}(\mathbb{R}) & \mathrm{dgMod}(\mathbb{R}) \end{tikzcd} \tag{25}$$

Here we consider $\mathbb{R}$ as a dg $\mathbb{R}$-algebra concentrated in degree 0 and as a dgg $\mathbb{R}$-algebra concentrated in degree $(0,0)$ (see subsections 2.1 and 2.2). More elementary, $\mathrm{dgMod}(\mathbb{R})$ and $\mathrm{dggMod}(\mathbb{R})$ are the categories of dg real vector spaces and dg graded real vector spaces respectively. The isomorphism a from (16) induces an isomorphism

$$a : \omega_0 \xrightarrow{\sim} \omega_W \tag{26}$$

between the induced functors. Then $A = \omega_0(\widetilde{A})$ is the dg algebra $\mathcal{E}\mathrm{nd}\,(P)$. Its cohomology is the extension algebra $\mathrm{Ext}_{\mathrm{Perv}(X,\mathcal{S})}(P)$ and isomorphic to $\mathrm{Ext}_{\mathrm{Perv}(X,\mathcal{S})}(M)$.

Theorem 27. *Let $(X,\mathcal{S})$, $\widetilde{P} \to \widetilde{M}$, $\widetilde{A}$ and A be as above, and w an integer. If $\widetilde{M}$ is $\mathcal{S}$-pure of weight w, then A is formal. More precisely, there are a dg subalgebra $\mathrm{Sub}(A)$ of A and dga-quasi-isomorphisms $A \hookleftarrow \mathrm{Sub}(A) \twoheadrightarrow \mathrm{H}(A)$.*

Proof. Consider the dgg algebra $\widetilde{R} := \mathrm{gr}^W_{\mathbb{R}}(\widetilde{A})$. Its graded components are $\widetilde{R}^{ij} = \mathrm{gr}^W_{j,\mathbb{R}}(\widetilde{A}^i)$. By Proposition 20 and Theorem 25, the complexes (with differential zero) of mixed Hodge structures $\mathrm{H}(\widetilde{A})$ and $\mathrm{Ext}_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{M},\widetilde{M})$ are isomorphic and pure of weight 0. So

$$\mathrm{gr}^W_{j,\mathbb{R}}(\mathrm{H}^i(\widetilde{A})) = \mathrm{H}^i(\mathrm{gr}^W_{j,\mathbb{R}}(\widetilde{A})) = \mathrm{H}^i(\widetilde{R}^{*j}) = (\mathrm{H}(\widetilde{R}))^{ij}$$

vanishes for $i \neq j$, and the dgg algebra $\mathrm{H}(\widetilde{R})$ is pure of weight 0. Proposition 4 shows the existence of dgga-quasi-isomorphisms $\widetilde{R} \hookleftarrow \Gamma(\widetilde{R}) \twoheadrightarrow \mathrm{H}(\widetilde{R})$. We define $R := \eta(\widetilde{R})$ and $\Gamma(R) := \eta(\Gamma(\widetilde{R}))$, apply η to $\widetilde{R} \hookleftarrow \Gamma(\widetilde{R}) \twoheadrightarrow \mathrm{H}(\widetilde{R})$ and obtain the dga-quasi-isomorphisms in the first row in the following diagram

$$\begin{tikzcd}
\omega_W(\widetilde{A}) = R & \Gamma(R) \arrow[l, hook'] \arrow[r, two heads] & \mathrm{H}(R) = \omega_W(\mathrm{H}(\widetilde{A})) \\
\omega_0(\widetilde{A}) = A \arrow[u, "\sim"', "a_{\widetilde{A}}"] & \mathrm{Sub}(A) \arrow[l, hook'] \arrow[u, "\sim"'] & \mathrm{H}(A) = \omega_0(\mathrm{H}(\widetilde{A})), \arrow[u, "\sim"', "a_{\mathrm{H}(\widetilde{A})}"]
\end{tikzcd} \tag{27}$$

where the vertical isomorphisms come from the natural isomorphism (26) and the dg subalgebra $\mathrm{Sub}(A) \subset A$ is defined as the pull-back, i. e. $\mathrm{Sub}(A) := a^{-1}(\Gamma(R))$. All vertical (horizontal) morphisms in this diagram are dga-(quasi-)isomorphisms. □

We generalize Theorem 27 slightly as follows. Let $(X,\mathcal{S})$ be as above, I a finite set and $\widetilde{P}_\alpha \to \widetilde{M}_\alpha$ perverse-projective resolutions of finite length of smooth Hodge sheaves $\widetilde{M}_\alpha$ $(\alpha \in I)$. Let $\widetilde{P}$ be the direct sum of the $(\widetilde{P}_\alpha)_{\alpha\in I}$, $\widetilde{A} = \mathcal{E}\mathrm{nd}\,(\widetilde{P})$, $A = \omega_0(\widetilde{A}) = \mathcal{E}\mathrm{nd}\,(P)$ the dg algebra underlying the "dg algebra of mixed Hodge structures" $\widetilde{A}$, and $e_\alpha \in A^0$ the projector from P onto the direct summand P_α. The cohomology $\mathrm{H}(A)$ of A is isomorphic to the extension algebra $\mathrm{Ext}_{\mathrm{Perv}(X,\mathcal{S})}(M)$, where M is the direct sum of the underlying perverse sheaves $(M_\alpha)_{\alpha\in I}$.

Theorem 28. *Let X, $\mathcal{S}$, I, $\widetilde{P}_\alpha \to \widetilde{M}_\alpha$, $\widetilde{A}$, A, e_α be as above. Let $w_\alpha \in \mathbb{Z}$ $(\alpha \in I)$ be integers. If $\widetilde{M}_\alpha$ is $\mathcal{S}$-pure of weight w_α, for all $\alpha \in I$, then A is formal. More precisely, there are a dg subalgebra* $\mathrm{Sub}(A)$ *of A containing all $(e_\alpha)_{\alpha\in I}$ and quasi-isomorphisms $A \hookleftarrow \mathrm{Sub}(A) \twoheadrightarrow \mathrm{H}(A)$ of dg algebras.*

Proof. The proof is very similar to that of Theorem 27. Mainly, we use Proposition 5 instead of Proposition 4. □

3.10. **Formality of Cell-Stratified Varieties.**

Theorem 29. *Let X be a complex variety with a cell-stratification $\mathcal{S}$, $(\widetilde{M}_\alpha)_{\alpha\in I}$ a finite number of smooth Hodge sheaves $\widetilde{M}_\alpha \in \mathrm{MHM}(X,\mathcal{S})$ with direct sum $\widetilde{M} = \bigoplus \widetilde{M}_\alpha$. Denote by $\mathrm{Ext}(\underline{M}) := \mathrm{Ext}_{\mathrm{Sh}(X)}(\underline{M})$ the extension algebra of $\underline{M} = v(\widetilde{M})$, a dg algebra with differential $d = 0$. Let $e_\alpha \in \mathrm{Ext}^0(\underline{M}) = \mathrm{End}_{\mathrm{Sh}(X)}(\underline{M})$ be the projector from $\underline{M}$ onto the direct summand $\underline{M}_\alpha = v(\widetilde{M}_\alpha)$. If there are integers $(w_\alpha)_{\alpha\in I}$, such that $\widetilde{M}_\alpha$ is $\mathcal{S}$-pure of weight w_α, for all $\alpha \in I$, there is an equivalence of triangulated categories*

$$\mathrm{tria}(\{\underline{M}_\alpha\}_{\alpha\in I}, \mathcal{D}^{\mathrm{b}}(X)) \xrightarrow{\sim} \mathrm{dgPrae}_{\mathrm{Ext}(\underline{M})}(\{e_\alpha\,\mathrm{Ext}(\underline{M})\}_{\alpha\in I}). \tag{28}$$

Under the equivalence we construct in the proof, the objects $\underline{M}_\alpha$ and $e_\alpha\,\mathrm{Ext}(\underline{M})$ correspond. We do not emphasize similar obvious correspondences in the following.

Due to the equivalences (13) and (15) we can replace $\mathrm{Ext}(\underline{M})$ by $\mathrm{Ext}_{\mathrm{Perv}(X,\mathcal{S})}(M)$ or $\mathrm{Ext}_{\mathrm{Perv}(X)}(M)$, and also the left hand side of (28) by $\mathrm{tria}(\{M_\alpha\}_{\alpha\in I})$ formed in $\mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X,\mathcal{S})))$ or in $\mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X))$.

Proof. Let $\widetilde{P}_\alpha \to \widetilde{M}_\alpha$ be perverse-projective resolutions of finite length (Corollary 15) and $P_\alpha \to M_\alpha$ the underlying projective resolutions in $\mathrm{Perv}(X,\mathcal{S})$. Let $\widetilde{P} = \bigoplus \widetilde{P}_\alpha$ and $P = \bigoplus P_\alpha$. As in the second part of subsection 3.9, we define $\widetilde{A} = \mathcal{E}\mathrm{nd}\,(\widetilde{P})$ and $A = \omega_0(\widetilde{A}) = \mathcal{E}\mathrm{nd}\,(P)$. Theorem 28 yields a dg subalgebra $\mathrm{Sub}(A)$ of A and dga-quasi-isomorphisms $A \hookleftarrow \mathrm{Sub}(A) \twoheadrightarrow \mathrm{H}(A)$. We claim that

$$\text{tria}(\{\underline{M}_\alpha\}, \mathcal{D}^{\mathrm{b}}(X)) \xleftarrow{\mathrm{real}_{X,\mathcal{S}}} \text{tria}(\{M_\alpha\}, \mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X,\mathcal{S}))) \tag{29}$$

$$\xrightarrow{\mathcal{H}\mathrm{om}\,(P,?)} \mathrm{dgPrae}_A(\{e_\alpha A\}) \tag{30}$$

$$\xleftarrow{? \overset{L}{\underset{\mathrm{Sub}(A)}{\otimes}} A} \mathrm{dgPrae}_{\mathrm{Sub}(A)}(\{e_\alpha \,\mathrm{Sub}(A)\}) \tag{31}$$

$$\xrightarrow{? \overset{L}{\underset{\mathrm{Sub}(A)}{\otimes}} \mathrm{H}(A)} \mathrm{dgPrae}_{\mathrm{H}(A)}(\{e_\alpha \,\mathrm{H}(A)\}) \tag{32}$$

$$\xrightarrow{? \overset{L}{\underset{\mathrm{H}(A)}{\otimes}} \mathrm{Ext}(\underline{M})} \mathrm{dgPrae}_{\mathrm{Ext}(\underline{M})}(\{e_\alpha \mathrm{Ext}(\underline{M})\}). \tag{33}$$

is a sequence of triangulated equivalences. By Theorem 10, (29) is an equivalence. The isomorphisms $P_\alpha \to M_\alpha$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X,\mathcal{S}))$, Proposition 7 and Remark 8 show that (30) is an equivalence. (The e_α here are in fact representatives of the e_α in the theorem, but we do not care about this too much.) The dga-quasi-isomorphisms $A \hookleftarrow \mathrm{Sub}(A) \twoheadrightarrow \mathrm{H}(A)$ induce equivalences between the respective derived categories dgDer and restrict to the equivalences (31) and (32). The last equivalence is similarly induced by the dga-isomorphism (cf. (22))

$$\mathrm{H}(A) = \mathrm{H}(\mathcal{E}\mathrm{nd}\,(P)) = \mathrm{Ext}(P) \xrightarrow[\sim]{P \xrightarrow{\sim} M} \mathrm{Ext}(M) \xrightarrow{\mathrm{real}} \mathrm{Ext}(\underline{M}).$$

□

Remark 30. We use the notation $\mathrm{Form}_{\widetilde{P}\to\widetilde{M}}$ for the "formality equivalence" (28) constructed in the proof of Theorem 29 to indicate that it mainly depends on the perverse-projective resolutions $\widetilde{P}_\alpha \to \widetilde{M}_\alpha$.

3.11. **Formality and Intersection Cohomology Complexes.** Let $(X,\mathcal{T})$ be a cell-stratified complex variety, $\mathcal{E} = \mathrm{Ext}(\mathcal{IC}(\mathcal{T}))$ the extension algebra of the direct sum $\mathcal{IC}(\mathcal{T})$ of the $(\mathcal{IC}_T)_{T\in\mathcal{T}}$, and e_T the projector from this direct sum onto $\mathcal{IC}_T$. Then the dg algebra $\mathcal{E}$ satisfies the conditions (P1)-(P3), hence

$$\mathrm{dgPrae}_{\mathcal{E}}(\{e_T\mathcal{E}\}_{T\in\mathcal{T}}) = \mathrm{dgPrae}(\mathcal{E}) = \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{T}))$$

thanks to Theorem 9. If $\widetilde{\mathcal{IC}}_T$ is $\mathcal{T}$-pure of weight d_T, for all $T \in I$, these equalities, Theorem 29 and (12) yield an equivalence

$$\mathcal{D}^{\mathrm{b}}(X,\mathcal{T}) \cong \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{T}))).$$

Similarly, if $\mathcal{T}'$ is a subset of $\mathcal{T}$ and all $\widetilde{\mathcal{IC}}_{T'}$ are $\mathcal{T}$-pure of weight $d_{T'}$, for $T' \in \mathcal{T}'$, we get by Theorem 29 an equivalence

$$\mathrm{tria}(\{\mathcal{IC}_{T'}\}_{T'\in\mathcal{T}'}, \mathcal{D}^{\mathrm{b}}(X)) \cong \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{T}'))). \tag{34}$$

Theorem 31. *Let $(X, \mathcal{S})$ be a stratified variety with simply connected strata. Let $\mathcal{T}$ be a cell-stratification refining $\mathcal{S}$. If $\widetilde{\mathcal{IC}}_S$ is $\mathcal{T}$-pure of weight d_S for all $S \in \mathcal{S}$, there is a triangulated equivalence*

$$\mathcal{D}^{\mathrm{b}}(X, \mathcal{S}) \xrightarrow{\sim} \operatorname{dgPer}(\operatorname{Ext}(\mathcal{IC}(\mathcal{S}))). \tag{35}$$

This equivalence is t-exact with respect to the perverse t-structure on $\mathcal{D}^{\mathrm{b}}(X, \mathcal{S})$ and the t-structure from Theorem 9 on dgPer. *Restriction to the respective hearts yields an equivalence*

$$\operatorname{Perv}(X, \mathcal{S}) \cong \operatorname{dgFlag}(\operatorname{Ext}(\mathcal{IC}(\mathcal{S}))).$$

Proof. Since each $S \in \mathcal{S}$ is irreducible, it contains a (unique) dense stratum $T(S) \in \mathcal{T}$; then $\mathcal{IC}_S = \mathcal{IC}_{T(S)}$. We apply equivalence (34) to the set $\mathcal{T}'$ of these dense strata. Then we use (12) and the fact that the $(\mathcal{IC}_S)_{S\in\mathcal{S}}$ are the simple objects of $\operatorname{Perv}(X, \mathcal{S})$. This show equivalence (35).

Since $\mathcal{IC}(\mathcal{S})$ is mapped to $e_S \operatorname{Ext}(\mathcal{IC}(\mathcal{S}))$ (where e_S is the obvious projector), the remaining statements follow from Theorem 9. □

Remark 32. We use the notation $\operatorname{Form}^{\mathcal{T}}_{\widetilde{P}\to\widetilde{\mathcal{IC}}(\mathcal{S})}$ for equivalence (35) to indicate its dependence on the refinement $\mathcal{T}$ and the perverse-projective resolutions $\widetilde{P}_S \to \widetilde{\mathcal{IC}}_S$ (cf. Remark 30).

Remark 33. In Theorem 31, it is sufficient to require that each $\widetilde{\mathcal{IC}}_S$ is $\mathcal{T}$-$*$-pure of weight d_S: If S is a stratum of a stratified variety and l the inclusion of a subvariety, we have $\mathbb{D}(\widetilde{\mathcal{IC}}_S) \cong \widetilde{\mathcal{IC}}_S(d_S)$ and obtain

$$l^!(\widetilde{\mathcal{IC}}_S) = \mathbb{D}(l^*(\mathbb{D}(\widetilde{\mathcal{IC}}_S))) \cong \mathbb{D}(l^*(\widetilde{\mathcal{IC}}_S(d_S))) = \mathbb{D}(l^*(\widetilde{\mathcal{IC}}_S))(-d_S).$$

So if $l^*(\widetilde{\mathcal{IC}}_S)$ is pure of weight d_S, then $l^!(\widetilde{\mathcal{IC}}_S)$ is pure of weight d_S.

3.12. **Formality of Partial Flag Varieties.** Let G be a complex connected reductive affine algebraic group and a $B \subset G$ Borel subgroup. Let P, Q be parabolic subgroups of G containing B. The Bruhat decomposition of the flag variety G/B into B-orbits is a cell-stratification. More generally, the B-orbits on the partial flag variety G/P form a cell-stratification, and the Q-orbits on G/P form a stratification. (These stratifications are indeed Whitney stratifications thanks to [Kal05, Thm. 2].)

Proposition 34. *The Q-orbits in G/P are simply connected.*

This is probably well-known but we could not find a proof in the literature.

Proof. Let $Y \subset G/P$ be a Q-orbit and $T \subset B \subset G$ a maximal torus. Then $Y = QwP/P$ for some w in the normalizer of T in G. The stabilizer S of wP/P in Q is $Q \cap wPw^{-1}$ and is connected as the intersection of two parabolic subgroups of a connected reductive group ([Bor91, 14.22]). The exact sequence of homotopy groups associated to the S-principal fiber bundle $Q \to Y$, $q \mapsto qwP/P$, shows that it is sufficient to prove surjectivity of $\pi_1(S) \to \pi_1(Q)$. Note that $T \subset S$. If L_Q is a Levi subgroup of Q, then $T \subset L_Q$ and $\pi_1(Q) = \pi_1(L_Q)$. Hence surjectivity is a consequence of the following Lemma 35. □

Lemma 35. *If T is a maximal torus in a connected reductive group L, then $\pi_1(T) \to \pi_1(L)$ is surjective.*

Proof. Let $A \subset L$ be a Borel subgroup containing T. Then $\pi_1(T) = \pi_1(A)$ and the long exact homotopy sequence for the A-principal fiber bundle $L \to L/A$ yields an exact sequence $\pi_1(A) \to \pi_1(L) \to \pi_1(L/A) \to \pi_0(A)$. Since A is connected and fundamental groups of topological groups are abelian, $\pi_1(L/A)$ is abelian and vanishes since the flag variety L/A has only even cohomology. So $\pi_1(A) \to \pi_1(L)$ is surjective. □

Theorem 36. *Let $F \in \mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(G/P))$ be pure and smooth along the stratification by B-orbits. Let $l : Y \to G/P$ be the inclusion of a B-orbit. Then $l^*(F)$ and $l^!(F)$ are pure as well, of the same weight as F.*

Proof. Let $\pi : G/B \to G/P$ be the obvious projection and $Z \subset \pi^{-1}(Y)$ the unique Bruhat cell such that π induces an isomorphism $Z \xrightarrow{\sim} Y$:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "\pi", "\sim"'] & \pi^{-1}(Y) \arrow[r] \arrow[d, "\pi"] & G/B \arrow[d, "\pi"] \\
Y \arrow[r, equal] & Y \arrow[r, "l"] & G/P.
\end{tikzcd}$$

Since π is smooth of relative dimension $n = \dim_{\mathbb{C}}(P/B)$, $\pi^*(F) = [-2n]\pi^!(F)(-n)$ and $\pi^!(F)$ are pure of the same weight as F (use (M1), (M7) and (M11)). Obviously, $\pi^*(F)$ is smooth along the stratification by Bruhat cells, and so is $\pi^!(F)$. Hence we deduce from [Soe89, Parabolic Purity Theorem] that $l^*(F)$ and $l^!(F)$ are pure of the same weight as F. □

Theorem 37. *Let $\mathcal{Q}$ be the stratification of G/P into Q-orbits. Then there is a t-exact equivalence $\mathcal{D}^{\mathrm{b}}(G/P, \mathcal{Q}) \cong \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{Q})))$ inducing an equivalence $\mathrm{Perv}(G/P, \mathcal{Q}) \cong \mathrm{dgFlag}(\mathrm{Ext}(\mathcal{IC}(\mathcal{Q})))$.*

Proof. The strata of $\mathcal{Q}$ are simply connected by Proposition 34. The cell-stratification $\mathcal{T}$ of G/P into B-orbits refines $\mathcal{Q}$, and every $\widetilde{\mathcal{IC}}_Q$ is $\mathcal{T}$-pure of weight d_Q, for $Q \in \mathcal{Q}$ (Theorem 36). Hence we can apply Theorem 31. □

3.13. **Complex coefficients.** Let $(X, \mathcal{S})$ be a cell-stratified variety. We denote the derived category of sheaves of complex vector spaces on X by $\mathcal{D}^{\mathrm{b}}(X)_{\mathbb{C}}$ and use similar notation in the following. The obvious extension of scalars functor $\mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(X)_{\mathbb{C}}$, $N \mapsto N_{\mathbb{C}}$ restricts to a functor $\mathcal{D}^{\mathrm{b}}(X, \mathcal{S}) \to \mathcal{D}^{\mathrm{b}}(X, \mathcal{S})_{\mathbb{C}}$. This functor is t-exact with respect to the perverse t-structure and maps projective objects of $\mathrm{Perv}(X, \mathcal{S})$ to projective objects of $\mathrm{Perv}(X, \mathcal{S})_{\mathbb{C}}$. If M, N are in $\mathcal{D}^{\mathrm{b}}(X)$ we have a canonical isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(X)}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(X)_{\mathbb{C}}}(M_{\mathbb{C}}, N_{\mathbb{C}}). \tag{36}$$

Under the assumptions of Theorem 29 the complexified version

$$\mathrm{tria}(\{\underline{M_{\alpha}}_{\mathbb{C}}\}_{\alpha \in I}, \mathcal{D}^{\mathrm{b}}(X)_{\mathbb{C}}) \xrightarrow{\sim} \mathrm{dgPrae}_{\mathrm{Ext}(\underline{M}_{\mathbb{C}})}(\{e_{\alpha}\,\mathrm{Ext}(\underline{M}_{\mathbb{C}})\}_{\alpha \in I}).$$

of equivalence 28 is true: With the notation used in the proof of Theorem 29, $(P_{\alpha})_{\mathbb{C}} \to (M_{\alpha})_{\mathbb{C}}$ is a projective resolution in $\mathrm{Perv}(X, \mathcal{S})_{\mathbb{C}}$. From (36) we see that $\mathbb{C} \otimes_{\mathbb{R}} A$ and $\mathcal{E}\mathrm{nd}\,(P_{\mathbb{C}})$ are isomorphic as dg algebras. Since A is formal, the same is true for $\mathcal{E}\mathrm{nd}\,(P_{\mathbb{C}})$. Now it is easy to adapt the sequence of equivalences (29)-(33) to the case of complex coefficients.

In particular, Theorem 37 is also true for complex coefficients.

4. Formality and Closed Embeddings

We formulate in subsection 4.1 the goal of this section and explain in subsection 4.2 the main application. The proof of the goal is divided into several parts and given in the following subsections.

4.1. **The Goal of the Section.** Let $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ be cell-stratified complex varieties, and $i : Y \to X$ a closed embedding such that $i(\mathcal{T}) := \{i(T) \mid T \in \mathcal{T}\} \subset \mathcal{S}$. We say for short that $i : (Y,\mathcal{T}) \to (X,\mathcal{S})$ is a **closed embedding of cell-stratified varieties**. (If $\mathcal{S}$ and $\mathcal{T}$ are merely stratifications, the term **closed embedding of stratified varieties** is defined similarly.)

Assume that we are in the setting of Theorem 29 on X and on Y. More precisely, let $(\widetilde{M}_\alpha)_{\alpha\in I}$ and $(\widetilde{N}_\alpha)_{\alpha\in I}$ be finite collections of smooth Hodge sheaves on X and on Y. Assume that there are integers $(w_\alpha)_{\alpha\in I}$ (resp. $(v_\alpha)_{\alpha\in I}$) such that $\widetilde{M}_\alpha$ (resp. $\widetilde{N}_\alpha$) is $\mathcal{S}$-pure (resp. $\mathcal{T}$-pure) of weight w_α (resp. v_α), for all $\alpha \in I$.

Let μ be an integer (in our applications, μ will be the negative complex codimension of the inclusion $i : Y \to X$, and we will have $w_\alpha + \mu = v_\alpha$ for all $\alpha \in I$). Suppose that there are isomorphisms

$$\widetilde{\sigma}_\alpha : [\mu] i^*(\widetilde{M}_\alpha) \xrightarrow{\sim} \widetilde{N}_\alpha \tag{37}$$

in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(Y))$, for all $\alpha \in I$. Let $\widetilde{\sigma} : [\mu] i^*(\widetilde{M}) \xrightarrow{\sim} \widetilde{N}$ be the direct sum of these isomorphisms, where $\widetilde{M} = \bigoplus \widetilde{M}_\alpha$ and $\widetilde{N} = \bigoplus \widetilde{N}_\alpha$.

Let $\widetilde{\pi}_\alpha : \widetilde{P}_\alpha \to \widetilde{M}_\alpha$ and $\widetilde{\rho}_\alpha : \widetilde{Q}_\alpha \to \widetilde{N}_\alpha$ be perverse-projective resolutions and $\widetilde{\pi} : \widetilde{P} \to \widetilde{M}$ and $\widetilde{\rho} : \widetilde{Q} \to \widetilde{N}$ their direct sum. The vertical equivalences in

$$\begin{tikzcd}
\mathrm{tria}(\{\underline{M}_\alpha\}, \mathcal{D}^{\mathrm{b}}(X)) \arrow[r, "{[\mu]i^*}"] \arrow[d, "\mathrm{Form}_{\widetilde{P}\to\widetilde{M}}"', "\sim"] & \mathrm{tria}(\{\underline{N}_\alpha\}, \mathcal{D}^{\mathrm{b}}(Y)) \arrow[d, "\mathrm{Form}_{\widetilde{Q}\to\widetilde{N}}"', "\sim"] \\
\mathrm{dgPrae}_{\mathrm{Ext}(\underline{M})}(\{e_\alpha \operatorname{Ext}(\underline{M})\}) \arrow[r, "? \overset{L}{\underset{\mathrm{Ext}(\underline{M})}{\otimes}} \mathrm{Ext}(\underline{N})"'] \arrow[ur, Rightarrow, "\sim"] & \mathrm{dgPrae}_{\mathrm{Ext}(\underline{N})}(\{e_\alpha \operatorname{Ext}(\underline{N})\})
\end{tikzcd} \tag{38}$$

come from Theorem 29, cf. Remark 30. The upper horizontal arrow is the restriction of $[\mu]i^* : \mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(Y)$, the lower one is the restriction of the extension of scalars functor coming from the composition

$$\operatorname{Ext}(\underline{M}) \xrightarrow{[\mu]i^*} \operatorname{Ext}([\mu]i^*(\underline{M})) \underset{\sim}{\xrightarrow{\sigma}} \operatorname{Ext}(\underline{N}), \tag{39}$$

where the σ above the arrow indicates that the isomorphism is constructed using the isomorphism $\sigma : [\mu]i^*(\underline{M}) \xrightarrow{\sim} \underline{N}$.

Theorem 38. *Keep the above assumptions. Then diagram* (38) *commutes up to the indicated natural isomorphism, i. e. there is a natural isomorphism (of triangulated functors)*

$$(? \overset{L}{\underset{\mathrm{Ext}(\underline{M})}{\otimes}} \operatorname{Ext}(\underline{N})) \circ \mathrm{Form}_{\widetilde{P}\to\widetilde{M}} \xrightarrow{\sim} \mathrm{Form}_{\widetilde{Q}\to\widetilde{N}} \circ [\mu]i^*.$$

Proof. Let $\widetilde{A} = \mathcal{E}\mathrm{nd}(\widetilde{P})$ and $\widetilde{B} = \mathcal{E}\mathrm{nd}(\widetilde{Q})$. We define A and $\mathrm{Sub}(A)$ as in the proof of Theorem 29, and B and $\mathrm{Sub}(B)$ accordingly. We only prove the theorem in the case that I is a singleton, the general case being an obvious generalization.

We expand diagram (38) according to the sequence of equivalences (29)-(33) to the following diagram.

(40)
$$
\begin{tikzcd}[column sep=large, row sep=huge]
\operatorname{tria}(\underline{M}, \mathcal{D}^{\mathrm{b}}(X)) \arrow[r, "{[\mu] i^*}"] & \operatorname{tria}(\underline{N}, \mathcal{D}^{\mathrm{b}}(Y)) \\
\operatorname{tria}(M, \mathcal{D}^{\mathrm{b}}(\operatorname{Perv}(X, \mathcal{S}))) \arrow[r, "{[\mu] L^p i^*}"] \arrow[u, "\operatorname{real}_{X,\mathcal{S}}", "\sim"'] \arrow[d, "\mathcal{H}om\,(P,?)"', "\sim"] & \operatorname{tria}(N, \mathcal{D}^{\mathrm{b}}(\operatorname{Perv}(Y, \mathcal{T}))) \arrow[u, "\sim", "\operatorname{real}_{Y,\mathcal{T}}"'] \arrow[d, "\sim"', "\mathcal{H}om\,(Q,?)"] \arrow[ul, Rightarrow, "\sim"', "\text{Proposition 41}"] \\
\operatorname{dgPrae}_A(A) \arrow[r, "{? \overset{L}{\underset{A}{\otimes}} \mathcal{H}om\,(Q, [\mu]^p i^*(P))}"] \arrow[ur, Rightarrow, "\sim", "\text{Corollary 43}"'] & \operatorname{dgPrae}_B(B) \\
\operatorname{dgPrae}_{\operatorname{Sub}(A)}(\operatorname{Sub}(A)) \arrow[r, "{? \overset{L}{\underset{\operatorname{Sub}(A)}{\otimes}} \operatorname{Sub}(\mathcal{H}om\,(Q, [\mu]^p i^*(P)))}"] \arrow[u, "? \overset{L}{\underset{\operatorname{Sub}(A)}{\otimes}} A", "\sim"'] \arrow[d, "? \overset{L}{\underset{\operatorname{Sub}(A)}{\otimes}} \operatorname{H}(A)"', "\sim"] & \operatorname{dgPrae}_{\operatorname{Sub}(B)}(\operatorname{Sub}(B)) \arrow[u, "\sim", "? \overset{L}{\underset{\operatorname{Sub}(B)}{\otimes}} B"'] \arrow[d, "\sim"', "? \overset{L}{\underset{\operatorname{Sub}(B)}{\otimes}} \operatorname{H}(B)"] \arrow[ul, Rightarrow, "\sim"', "\text{Subsection 4.10}"] \arrow[dl, Rightarrow, "\sim"', "\text{Subsection 4.10}"] \\
\operatorname{dgPrae}_{\operatorname{H}(A)}(\operatorname{H}(A)) \arrow[r, "{? \overset{L}{\underset{\operatorname{H}(A)}{\otimes}} \operatorname{H}(\mathcal{H}om\,(Q, [\mu]^p i^*(P)))}"] \arrow[d, "? \overset{L}{\underset{\operatorname{H}(A)}{\otimes}} \operatorname{Ext}(\underline{M})"', "\sim"] & \operatorname{dgPrae}_{\operatorname{H}(B)}(\operatorname{H}(B)) \arrow[d, "\sim"', "? \overset{L}{\underset{\operatorname{H}(B)}{\otimes}} \operatorname{Ext}(\underline{N})"] \arrow[dl, Rightarrow, "\sim"', "\text{Subsection 4.11}"] \\
\operatorname{dgPrae}_{\operatorname{Ext}(\underline{M})}(\operatorname{Ext}(\underline{M})) \arrow[r, "{? \overset{L}{\underset{\operatorname{Ext}(\underline{M})}{\otimes}} \operatorname{Ext}(\underline{N})}"] & \operatorname{dgPrae}_{\operatorname{Ext}(\underline{N})}(\operatorname{Ext}(\underline{N}))
\end{tikzcd}
$$

The horizontal functors in this diagram will be explained in the rest of this section. It will result from the specified Proposition, Corollary and subsections that all squares commute up to natural isomorphism, as indicated by the diagonal arrows. Since all vertical arrows are equivalences, this proves the theorem. □

4.2. **Formality and Normally Nonsingular Inclusions.** If $f : Y \to X$ is a closed embedding of irreducible varieties and, in the classical topology, a normally nonsingular inclusion of (complex) codimension c (cf. [GM88, I.1.11]), we have a canonical isomorphism ([GM83, 5.4.1], and [BBD82, 0] for the different normalization)

$$[-c]f^*(\mathcal{IC}(X)) \xrightarrow{\sim} \mathcal{IC}(Y) \qquad \text{in } \operatorname{Perv}(Y). \tag{41}$$

It comes from a canonical isomorphism

$$[-c]f^*(\widetilde{\mathcal{IC}}(X)) \xrightarrow{\sim} \widetilde{\mathcal{IC}}(Y) \qquad \text{in } \operatorname{MHM}(Y). \tag{42}$$

Let $i : Y \to X$ be a closed embedding of varieties and assume that we have stratifications $\mathcal{S}$ of X and $\mathcal{T}$ of Y (with irreducible strata). Suppose that $\mathcal{S} \to \mathcal{T}$, $S \mapsto i^{-1}(S) = S \cap Y$, is bijective and that $i|_{\overline{S}\cap Y} : \overline{S} \cap Y \to \overline{S}$ is a normally nonsingular inclusion of a fixed codimension c, for all $S \in \mathcal{S}$. The isomorphisms (41) and (42) induce isomorphisms

$$[-c]i^*(\mathcal{IC}_S) \xrightarrow{\sim} \mathcal{IC}_{S\cap Y} \qquad \text{in } \operatorname{Perv}(Y, \mathcal{T}) \text{ and} \tag{43}$$

$$[-c]i^*(\widetilde{\mathcal{IC}}_S) \xrightarrow{\sim} \widetilde{\mathcal{IC}}_{S\cap Y} \qquad \text{in } \operatorname{MHM}(Y, \mathcal{T}). \tag{44}$$

Theorem 39. *Let $i : Y \to X$ and $\mathcal{S}$, $\mathcal{T}$ be as above. Assume that*

(a) all strata in $\mathcal{S}$ and $\mathcal{T}$ are simply connected,
(b) there are cell-stratifications $\mathcal{S}'$ and $\mathcal{T}'$ refining $\mathcal{S}$ and $\mathcal{T}$ such that
- *$\widetilde{\mathcal{IC}}_S$ is $\mathcal{S}'$-pure of weight d_S, for all $S \in \mathcal{S}$,*
- *$\widetilde{\mathcal{IC}}_T$ is $\mathcal{T}'$-pure of weight d_T, for all $T \in \mathcal{T}$, and*
- *$i : (Y, \mathcal{T}') \to (X, \mathcal{S}')$ is a closed embedding of cell-stratified varieties.*

Let $\widetilde{P}_S \to \widetilde{\mathcal{IC}}_S$ and $\widetilde{Q}_T \to \widetilde{\mathcal{IC}}_T$ be perverse-projective resolutions (smooth along the cell-stratifications), for $S \in \mathcal{S}$ and $T \in \mathcal{T}$. Then diagram

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}(X, \mathcal{S}) \arrow[r, "{[-c]i^*}"] \arrow[d, "\operatorname{Form}^{\mathcal{S}'}_{\widetilde{P} \to \widetilde{\mathcal{IC}}(\mathcal{S})}"', "\sim"] & \mathcal{D}^{\mathrm{b}}(Y, \mathcal{T}) \arrow[d, "\sim"', "\operatorname{Form}^{\mathcal{T}'}_{\widetilde{Q} \to \widetilde{\mathcal{IC}}(\mathcal{T})}"] \\
\operatorname{dgPer}(\operatorname{Ext}(\mathcal{IC}(\mathcal{S}))) \arrow[r, "? \overset{L}{\underset{\operatorname{Ext}(\mathcal{IC}(\mathcal{S}))}{\otimes}} \operatorname{Ext}(\mathcal{IC}(\mathcal{T}))"] & \operatorname{dgPer}(\operatorname{Ext}(\mathcal{IC}(\mathcal{T})))
\end{tikzcd}$$

is commutative (up to natural isomorphism). The vertical functors are given by Theorem 31 (cf. Remark 32), the extension of scalars functor is induced by the isomorphisms (43). *All functors in this diagram are t-exact (with respect to the perverse t-structure and the t-structure from Theorem 9).*

Proof. Except for the t-exactness of the horizontal functors, this is a consequence of Theorem 38 and the results of subsection 3.11.

It is obvious that the extension of scalars functor is t-exact. The t-exactness of $[-c]i^*$ can be proved as follows: Since all strata in $\mathcal{S}$ are simply connected, the $(\mathcal{IC}_S)_{S\in\mathcal{S}}$ are the simple objects of $\operatorname{Perv}(X, \mathcal{S})$. Every object of $\operatorname{Perv}(X, \mathcal{S})$ has finite length. Now the isomorphisms (43) and the long exact perverse cohomology sequence show the t-exactness. $\square$

4.3. **Closed Embeddings and (Perverse) Sheaves.** Let $i : (Y, \mathcal{T}) \to (X, \mathcal{S})$ be a closed embedding of stratified varieties. Since i_* is perverse t-exact [BBD82, 1.3.17, 1.4.16], ${}^p i_* : \mathrm{Perv}(Y, \mathcal{T}) \to \mathrm{Perv}(X, \mathcal{S})$ is exact and induces the functor ${}^p i_*$ in diagram

$$\begin{tikzcd} \mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(Y,\mathcal{T})) \arrow[r, "\mathrm{real}_{Y,\mathcal{T}}"] \arrow[d, "{}^p i_*"] & \mathcal{D}^{\mathrm{b}}(Y,\mathcal{T}) \arrow[d, "i_*"] \\ \mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X,\mathcal{S})) \arrow[r, "\mathrm{real}_{X,\mathcal{S}}"] & \mathcal{D}^{\mathrm{b}}(X,\mathcal{S}). \end{tikzcd} \tag{45}$$

The horizontal functors were introduced in (15).

Proposition 40. *Keep the above assumptions. Then diagram* (45) *commutes up to natural isomorphism.*

Proof. This follows from the definition of the functor $\mathrm{real}_{X,\mathcal{S}}$ given in [BBD82, 3.1]; for details see [Sch07, 3.3, 3.4]. □

Let $i : (Y, \mathcal{T}) \hookrightarrow (X, \mathcal{S})$ be a closed embedding of cell-stratified varieties. Theorem 10 shows that the right exact functor ${}^p i^* : \mathrm{Perv}(X, \mathcal{S}) \to \mathrm{Perv}(Y, \mathcal{T})$ has a left derived functor $L^p i^*$ between the bounded derived categories.

Proposition 41. *Let $i : (Y, \mathcal{T}) \hookrightarrow (X, \mathcal{S})$ be a closed embedding of cell-stratified varieties. Then there exists a natural isomorphism as indicated in the diagram*

$$\begin{tikzcd} \mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X,\mathcal{S})) \arrow[r, "\mathrm{real}_{X,\mathcal{S}}", "\sim"'] \arrow[d, "L^p i^*"] & \mathcal{D}^{\mathrm{b}}(X,\mathcal{S}) \arrow[d, "i^*"] \\ \mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(Y,\mathcal{T})) \arrow[r, "\mathrm{real}_{Y,\mathcal{T}}", "\sim"'] \arrow[ur, Rightarrow, "\sim"] & \mathcal{D}^{\mathrm{b}}(Y,\mathcal{T}). \end{tikzcd}$$

Proof. Both realization functors are equivalences of categories (Theorem 10) and up to these equivalences the functors ${}^p i_*$ and i_* coincide (Proposition 40). Now the statement is a consequence of the adjunctions adjunctions $(L^p i^*, {}^p i_*)$ and (i^*, i_*). □

4.4. **Tensor Product with a DG Bimodule.** By [Kel98, 8.1.1, 8.1.2, 8.2.5], there is a fully faithful functor $p : \mathrm{dgDer} \to \mathrm{dgHot}$ that is left adjoint to the quotient functor $q : \mathrm{dgHot} \to \mathrm{dgDer}$ and has image in dgHotp. If we consider p as a functor $\mathrm{dgDer} \to \mathrm{dgHotp}$, it is quasi-inverse to the equivalence (4).

Let $\mathcal{A}$ and $\mathcal{B}$ be dg algebras and X a dg $\mathcal{A}$-$\mathcal{B}$-bimodule (with $\mathcal{A}$ acting on the left and $\mathcal{B}$ on the right). This yields a triangulated functor

$$(? \otimes_{\mathcal{A}} X) : \mathrm{dgHot}(\mathcal{A}) \to \mathrm{dgHot}(\mathcal{B}).$$

Its left derived functor is the pair $((? \overset{L}{\otimes}_{\mathcal{A}} X), \sigma)$ (we use the definition of derived functors from [Del77, C.D.II.2.1.2, p. 301]), where

$$(? \overset{L}{\otimes}_{\mathcal{A}} X) := q \circ (? \otimes_{\mathcal{A}} X) \circ p : \mathrm{dgDer}(\mathcal{A}) \to \mathrm{dgDer}(\mathcal{B})$$

and σ is the natural transformation

$$\sigma : (? \overset{L}{\otimes}_{\mathcal{A}} X) \circ q \to q \circ (? \otimes_{\mathcal{A}} X) \tag{46}$$

coming from the adjunction (p, q).

4.5. **Passage from Geometry to DG Modules.** Let $I : \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories. We denote the induced functor $\mathrm{Hot}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{Hot}^{\mathrm{b}}(\mathcal{B})$ by the same symbol. Assume that each object of $\mathcal{A}$ has a projective resolution of finite length. Then I has a left derived functor $LI : \mathrm{Der}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{Der}^{\mathrm{b}}(\mathcal{B})$. Let P and Q be bounded complexes of projective objects in $\mathcal{A}$ and $\mathcal{B}$ respectively. Then $\mathcal{H}\mathrm{om}\,(Q, I(P))$ is obviously a dg $\mathcal{E}\mathrm{nd}\,(P)$-$\mathcal{E}\mathrm{nd}\,(Q)$-bimodule. It induces (see subsection 4.4) the lower horizontal arrow in diagram

$$\begin{tikzcd} \mathrm{Der}^{\mathrm{b}}(\mathcal{A}) \arrow[rr, "LI"] \arrow[d, "\mathcal{H}\mathrm{om}\,(P,?)"] & & \mathrm{Der}^{\mathrm{b}}(\mathcal{B}) \arrow[d, "\mathcal{H}\mathrm{om}\,(Q,?)"] \\ \mathrm{dgDer}(\mathcal{E}\mathrm{nd}\,(P)) \arrow[rr, "? \overset{L}{\underset{\mathcal{E}\mathrm{nd}\,(P)}{\otimes}} \mathcal{H}\mathrm{om}\,(Q,I(P))"'] \arrow[urr, Rightarrow, "\pi"] & & \mathrm{dgDer}(\mathcal{E}\mathrm{nd}\,(Q)). \end{tikzcd} \tag{47}$$

The vertical functors are restrictions of the functors (11) explained in Remark 8. We construct now a natural transformation π as indicated in the diagram. Since $\mathcal{H}\mathrm{om}\,(Q, ?) \circ LI$ is the left derived functor of

$$\mathcal{H}\mathrm{om}\,(Q, ?) \circ I : \mathrm{Hot}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{dgHot}(\mathcal{E}\mathrm{nd}\,(Q)),$$

it is enough, by the universal property of left derived functors, to construct a natural transformation $\widehat{\pi}$ as indicated by the diagram

$$\begin{tikzcd} \mathrm{Hot}^{\mathrm{b}}(\mathcal{A}) \arrow[rr, "I"] \arrow[d, "q"'] & & \mathrm{Hot}^{\mathrm{b}}(\mathcal{B}) \arrow[d, "\mathcal{H}\mathrm{om}\,(Q,?)"] \\ \mathrm{Der}^{\mathrm{b}}(\mathcal{A}) \arrow[d, "\mathcal{H}\mathrm{om}\,(P,?)"'] & & \mathrm{dgHot}(\mathcal{E}\mathrm{nd}\,(Q)) \arrow[d, "q"] \\ \mathrm{dgDer}(\mathcal{E}\mathrm{nd}\,(P)) \arrow[rr, "? \overset{L}{\underset{\mathcal{E}\mathrm{nd}\,(P)}{\otimes}} \mathcal{H}\mathrm{om}\,(Q,I(P))"'] \arrow[uurr, Rightarrow, "\widehat{\pi}"] & & \mathrm{dgDer}(\mathcal{E}\mathrm{nd}\,(Q)). \end{tikzcd}$$

Thus we define $\widehat{\pi}$ to be the composition

$$\begin{aligned} &\mathcal{H}\mathrm{om}\,(P, q(?)) \overset{L}{\underset{\mathcal{E}\mathrm{nd}\,(P)}{\otimes}} \mathcal{H}\mathrm{om}\,(Q, I(P)) \\ &\qquad = q(\mathcal{H}\mathrm{om}\,(P, ?)) \overset{L}{\underset{\mathcal{E}\mathrm{nd}\,(P)}{\otimes}} \mathcal{H}\mathrm{om}\,(Q, I(P)) \\ &\qquad \xrightarrow{\sigma \text{ from } (46)} q\Big(\mathcal{H}\mathrm{om}\,(P, ?) \underset{\mathcal{E}\mathrm{nd}\,(P)}{\otimes} \mathcal{H}\mathrm{om}\,(Q, I(P))\Big) \\ &\qquad \xrightarrow{I \text{ and composition}} q\big(\mathcal{H}\mathrm{om}\,(Q, I(?))\big). \end{aligned} \tag{48}$$

Proposition 42. *Assume in addition to the above assumptions that* $[\mu]I(P) \cong Q$ *in* $\mathrm{Der}^{\mathrm{b}}(\mathcal{B})$ *for some integer* μ*. Then diagram* (47) *induces the diagram*

$$
\begin{tikzcd}
\mathrm{tria}(P, \mathrm{Der}^{\mathrm{b}}(\mathcal{A})) \arrow[rr, "{[\mu]LI}"] \arrow[d, "\mathcal{H}\mathrm{om}\,(P,?)"] & & \mathrm{tria}(Q, \mathrm{Der}^{\mathrm{b}}(\mathcal{B})) \arrow[d, "\mathcal{H}\mathrm{om}\,(Q,?)"] \\
\mathrm{dgPrae}_{\mathcal{E}\mathrm{nd}\,(P)}(\mathcal{E}\mathrm{nd}\,(P)) \arrow[rr, "{? \overset{L}{\underset{\mathcal{E}\mathrm{nd}\,(P)}{\otimes}} \mathcal{H}\mathrm{om}\,(Q,[\mu]I(P))}"'] \arrow[urr, Rightarrow, "\widetilde{\pi}", "\sim"'] & & \mathrm{dgPrae}_{\mathcal{E}\mathrm{nd}\,(Q)}(\mathcal{E}\mathrm{nd}\,(Q))
\end{tikzcd}
\tag{49}
$$

and $\widetilde{\pi}$ *is a natural isomorphism.*

Proof. We have $LI(P) \cong I(P)$ in $\mathrm{Der}^{\mathrm{b}}(\mathcal{B})$ and $\mathcal{H}\mathrm{om}\,(Q, [\mu]I(P)) \cong \mathcal{E}\mathrm{nd}\,(Q)$ in $\mathrm{dgDer}(\mathcal{E}\mathrm{nd}\,(Q))$. Thus, after replacing the horizontal functors in diagram 47 by their composition with the shift $[\mu]$, this diagram restricts to (49).

In order to show that $\widetilde{\pi}$ is a natural isomorphism, it is sufficient to check that $\widetilde{\pi}_P$ or equivalently π_P is an isomorphism. Since π_P is obtained by plugging in P in (48) (P is a complex of projective objects), this follows from the obvious isomorphism

$$
\begin{aligned}
\mathcal{H}\mathrm{om}\,(P, q(P)) &\overset{L}{\underset{\mathcal{E}\mathrm{nd}\,(P)}{\otimes}} \mathcal{H}\mathrm{om}\,(Q, I(P)) \\
&= q\big(\mathcal{H}\mathrm{om}\,(P,P)) \otimes_{\mathcal{E}\mathrm{nd}\,(P)} \mathcal{H}\mathrm{om}\,(Q, I(P))\big) \\
&\xrightarrow{\sim} q(\mathcal{H}\mathrm{om}\,(Q, I(P))).
\end{aligned}
$$

□

Corollary 43. *Under the assumptions of subsection 4.1, the second square in diagram* (40) *commutes up to natural isomorphism.*

Proof. Take as $\mathcal{A}$ the category $\mathrm{Perv}(X, \mathcal{S})$ (using Theorem 10), as $I : \mathcal{A} \to \mathcal{B}$ the right exact functor ${}^p i^* : \mathrm{Perv}(X, \mathcal{S}) \to \mathrm{Perv}(Y, \mathcal{T})$ and as P and Q the complexes denoted by the same symbols in subsection 4.1. By assumption, we have an isomorphism $[\mu]i^*(\widetilde{M}) \xrightarrow{\sim} \widetilde{N}$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(Y))$. Hence $[\mu]i^*(\underline{M}) \xrightarrow{\sim} \underline{N}$ in $\mathcal{D}^{\mathrm{b}}(Y, \mathcal{T})$ or equivalently $[\mu]L^p i^*(M) \xrightarrow{\sim} N$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(Y, \mathcal{T}))$, by Proposition 41 and Theorem 10. But N is isomorphic to Q and $L^p i^*(M)$ is isomorphic to ${}^p i^*(P)$ since $P \to M$ is a projective resolution in $\mathrm{Perv}(X, \mathcal{S})$. □

4.6. **DG Bimodules and Transformations.**

Lemma 44. *Let* $B \to A$ *be a dga-morphism and* P *a homotopically projective dg* B*-module. Then* $P \otimes_B A$ *is homotopically projective.*

Proof. Since $? \otimes_B A$ is left adjoint to the restriction of scalars functor, we see that $\mathrm{Hom}_{\mathrm{dgHot}(A)}(P \otimes_B A, ?)$ vanishes on acyclic dg A-modules. □

Assume that we are given dga-morphisms $\phi : B \to A$ and $\psi : S \to R$. Let M be a dg A-R-bimodule. By restriction of scalars we view M as a dg B-S-module. Let $\chi : N \to M$ be a morphism of dg B-S-bimodules. We denote this situation as follows.

$$
\boxed{B \curvearrowright N \curvearrowleft S} \xrightarrow{(\phi,\, \chi,\, \psi)} \boxed{A \curvearrowright M \curvearrowleft R}
\tag{50}
$$

We get the following diagram

$$
\begin{tikzcd}
\mathrm{dgDer}(B) \arrow[r, "? \overset{L}{\underset{B}{\otimes}} A"] \arrow[d, "? \overset{L}{\underset{B}{\otimes}} N"'] & \mathrm{dgDer}(A) \arrow[d, "? \overset{L}{\underset{A}{\otimes}} M"] \\
\mathrm{dgDer}(S) \arrow[r, "? \overset{L}{\underset{S}{\otimes}} R"'] \arrow[ur, Rightarrow, "\theta"] & \mathrm{dgDer}(R)
\end{tikzcd}
\tag{51}
$$

and construct now the indicated natural transformation θ. From Lemma 44 we see that the obvious transformation

$$(? \overset{L}{\underset{A}{\otimes}} M) \circ (? \overset{L}{\underset{B}{\otimes}} A) \xrightarrow{\sim} (? \overset{L}{\underset{B}{\otimes}} M)$$

is an isomorphism. So it is sufficient to define a transformation

$$\widetilde{\theta} : (? \overset{L}{\underset{S}{\otimes}} R) \circ (? \overset{L}{\underset{B}{\otimes}} N) \to (? \overset{L}{\underset{B}{\otimes}} M)$$

But there is an obvious natural transformation

$$
\begin{aligned}
(? \overset{L}{\underset{S}{\otimes}} R) \circ (? \overset{L}{\underset{B}{\otimes}} N) \circ q &\xrightarrow{\sigma \text{ from (46) twice}} q \circ (? \underset{S}{\otimes} R) \circ (? \underset{B}{\otimes} N) \\
&\xrightarrow{\chi} q \circ (? \underset{B}{\otimes} M)
\end{aligned}
\tag{52}
$$

that induces, by the universal property of left derived functors, the transformation we want.

Proposition 45. *Keep the assumptions from above. Let $n \in N$ be an element such that the maps $f : S \to N$, $s \mapsto ns$ and $g : R \to M$, $r \mapsto \chi(n)r$ are quasi-isomorphisms of dg modules (so $n \in \mathrm{Z}(N)^0 := N^0 \cap \ker d_N$). Then diagram* (51) *restricts to*

$$
\begin{tikzcd}
\mathrm{dgPrae}_B(B) \arrow[r, "? \overset{L}{\underset{B}{\otimes}} A"] \arrow[d, "? \overset{L}{\underset{B}{\otimes}} N"'] & \mathrm{dgPrae}_A(A) \arrow[d, "? \overset{L}{\underset{A}{\otimes}} M"] \\
\mathrm{dgPrae}_S(S) \arrow[r, "? \overset{L}{\underset{S}{\otimes}} R"'] \arrow[ur, Rightarrow, "\theta|", "\sim"'] & \mathrm{dgPrae}_R(R)
\end{tikzcd}
\tag{53}
$$

and $\theta|$ is a natural isomorphism.

Proof. Since $S \cong N$ and $R \cong M$ in dgDer, diagram (51) restricts to (53). If N is a dg B-module, then θ_N is obtained by plugging in $p(N)$ in (52) (up to an isomorphism coming from the adjunction isomorphism $N \xrightarrow{\sim} q(p(N))$. In order to show that $\theta|$ is a natural isomorphism, it is enough to check that θ_B is an isomorphism. Since B is homotopically projective, we may assume $p(B) = B$. Then θ_B is given by

$$\big((B \overset{L}{\underset{B}{\otimes}} N) \overset{L}{\underset{S}{\otimes}} R\big) \xrightarrow{\sim} (q(N) \overset{L}{\underset{S}{\otimes}} R) \to q\big(N \underset{S}{\otimes} R\big) \xrightarrow{\chi} q(M) \tag{54}$$

We may assume that the adjunction morphism $p(q(N)) \to N$ is given by $f : S \to N$. Then the composition of the last two maps in (54) is identified with the isomorphism $q(g) : q(R) \to q(M)$ in $\mathrm{dgDer}(R)$. $\square$

4.7. **DGG Bimodules.** In subsection 2.2 we considered dgg modules over a dgg algebra R. We defined a functor $\Gamma : \mathrm{dggMod}(R) \to \mathrm{dggMod}(\Gamma(R))$ (see (5), (6)) and used it to show that dgg algebras with pure cohomology are formal.

The construction of Γ is easily extended to bimodules. If A and B are dgg algebras and M is a dgg A-B-bimodule, then $\Gamma(M)$ becomes a dgg $\Gamma(A)$-$\Gamma(B)$-bimodule. We get the following situation similar to (50).

$$\boxed{\Gamma(A) \curvearrowright \Gamma(M) \curvearrowleft \Gamma(B)} \quad \subset \quad \boxed{A \curvearrowright M \curvearrowleft B} \tag{55}$$

Here the inclusion $\Gamma(M) \subset M$ is a morphism of dgg $\Gamma(A)$-$\Gamma(B)$-bimodules. The cohomology $\mathrm{H}(M)$ is a dgg $\mathrm{H}(A)$-$\mathrm{H}(B)$-bimodule. Assume now that the cohomologies of A, B and M vanish in degrees (i, j) with $i < j$. Then componentwise projection defines the following morphisms of dgg algebras and dgg bimodules:

$$\boxed{\Gamma(A) \curvearrowright \Gamma(M) \curvearrowleft \Gamma(B)} \to \boxed{\mathrm{H}(A) \curvearrowright \mathrm{H}(M) \curvearrowleft \mathrm{H}(B)} \tag{56}$$

We would like to apply Proposition 45 to the situations sketched in (55) and (56), i. e. we need an element $m \in \Gamma(M)$ inducing quasi-isomorphisms $\Gamma(B) \to \Gamma(M)$, $B \to M$ and $\mathrm{H}(B) \to \mathrm{H}(M)$.

Lemma 46. *Let B be a dgg algebra, M a dgg B-module and $f : B \to M$ a quasi-isomorphism of (right) dgg B-modules. Then $m := f(1) \in \Gamma(M)^{00}$ and the multiplication maps $(m\cdot) : B \to M$, $(m\cdot) : \Gamma(B) \to \Gamma(M)$, and $([m]\cdot) : \mathrm{H}(B) \to \mathrm{H}(M)$ are quasi-isomorphisms of dgg modules (over B, $\Gamma(B)$ and $\mathrm{H}(B)$), where we denote by $[m]$ the class of m in $\mathrm{H}(M)$.*

Proof. Since $1 \in B^{00} \cap \ker d_B$, we have $m = f(1) \in M^{00} \cap \ker d_M = \Gamma(M)^{00}$. The functor Γ is a "truncation functor" and maps quasi-isomorphisms to quasi-isomorphisms, so $\Gamma(f)$ is a quasi-isomorphism. But $f = (m\cdot)$, $\Gamma(f) = (m\cdot)$ and $\mathrm{H}(f) = ([m]\cdot)$. $\square$

4.8. **Triangulated Functors on Objects.** Let $\mathcal{A}$ be an abelian category. The stupid truncation functors $\sigma_{\leq i}$, $\sigma_{\geq i} : \mathrm{Ket}(\mathcal{A}) \to \mathrm{Ket}(\mathcal{A})$, for $i \in \mathbb{Z}$, are defined as follows: $\sigma_{\leq i}$ preserves all components in degrees $\leq i$ and replaces all components in degrees $> i$ by zero; similarly for $\sigma_{\geq i}$. There are obvious transformations $\mathrm{id} \to \sigma_{\leq i}$ and $\sigma_{\geq i} \to \mathrm{id}$.

Proposition 47. *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories and $F : \mathrm{Der}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{Der}^{\mathrm{b}}(\mathcal{B})$ a triangulated functor. Let*

$$\ldots \to 0 \to P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \to \ldots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{p} M \to 0 \to \ldots$$

be a bounded exact complex in $\mathcal{A}$ (a resolution of M). Assume that $F(P^i)$ is an object of $\mathcal{B}$, for all $i = -n, \ldots, 0$, where we identify $\mathcal{B}$ with the heart of the standard t-structure on $\mathrm{Der}^{\mathrm{b}}(\mathcal{B})$. Let $\widehat{F}(P)$ be the complex

$$\ldots \to 0 \to F(P^{-n}) \xrightarrow{F(d^{-n})} F(P^{-n+1}) \to \ldots \xrightarrow{F(d^{-1})} F(P^0) \to 0 \to \ldots$$

in $\mathcal{B}$. Then $F(M)$ and $\widehat{F}(P)$ are isomorphic in $\mathrm{Der}^{\mathrm{b}}(\mathcal{B})$.

Proof. We write Hom instead of $\mathrm{Hom}_{\mathrm{Der}(\mathcal{B})}$. The transformation $\sigma_{\geq 0} \to \mathrm{id}$ yields an inclusion $s : F(P^0) = \sigma_{\geq 0}(\widehat{F}(P)) \to \widehat{F}(P)$ in $\mathrm{Ket}^{\mathrm{b}}(\mathcal{B})$. By induction on n, we prove the following more precise statement: There is an isomorphism $\alpha \in \mathrm{Hom}(\widehat{F}(P), F(M))$ such that $\alpha \circ s = F(p)$ in $\mathrm{Der}(\mathcal{B})$.

For $n = 0$, this is obvious. Assume that $n \geq 1$. Consider the morphism $f : [-1]\sigma_{\leq -1}(\widehat{F}(P)) \to \sigma_{\geq 0}(\widehat{F}(P)) = F(P^0)$ in $\mathrm{Ket}(\mathcal{B})$, given by $F(d^{-1})$ in degree zero. Its mapping cone is $\widehat{F}(P)$, and we get a distinguished triangle

$$[-1]\sigma_{\leq -1}(\widehat{F}(P)) \xrightarrow{f} F(P^0) \xrightarrow{s} \widehat{F}(P) \xrightarrow{[1]} \tag{57}$$

in $\mathrm{Der}^{\mathrm{b}}(\mathcal{B})$. Similarly, we get a distinguished triangle

$$[-2]\sigma_{\leq -2}(\widehat{F}(P)) \xrightarrow{g} F(P^{-1}) \xrightarrow{t} [-1]\sigma_{\leq -1}(\widehat{F}(P)) \xrightarrow{[1]} \tag{58}$$

in $\mathrm{Der}^{\mathrm{b}}(\mathcal{B})$, where t is the obvious inclusion, defined similarly as s above, and g is a morphism in $\mathrm{Ket}(\mathcal{A})$, given by $F(d^{-2})$ in degree zero.

We factorize $d^{-1} : P^{-1} \to P^0$ as $P^{-1} \xrightarrow{a} K \xrightarrow{b} P^0$, where $K = \ker p = \operatorname{im} d^{-1}$. By induction, applied to the exact complex

$$(\ldots \to 0 \to P^{-n} \xrightarrow{d^{-n}} P^{-n+1} \to \ldots \to P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{a} K \to 0 \to \ldots),$$

there is an isomorphism $\beta \in \mathrm{Hom}([-1]\sigma_{\leq -1}\widehat{F}(P), F(K))$ such that $\beta \circ t = F(a)$ in $\mathrm{Der}(\mathcal{B})$.

Consider now the diagram

$$\begin{tikzcd}
{[-1]\sigma_{\leq -1}(\widehat{F}(P))} \arrow[r, "f"] \arrow[d, "\beta"', "\sim"] & F(P^0) \arrow[r, "s"] \arrow[d, equal] & \widehat{F}(P) \arrow[r, "{[1]}"] & {} \\
F(K) \arrow[r, "F(b)"] & F(P^0) \arrow[r, "F(p)"] & F(M) \arrow[r, "{[1]}"] & {}
\end{tikzcd} \tag{59}$$

Both rows are distinguished triangles, the upper one is (57), and the lower one comes from the short exact sequence $K \xrightarrow{b} P^0 \xrightarrow{p} M$. We claim that this diagram is commutative. If this is the case, we can complete the partial morphism (β, id) of distinguished triangles in (59) by some morphism $\alpha \in \mathrm{Hom}(\widehat{F}(P), F(M))$ to a morphism of distinguished triangles, and any such α is an isomorphism.

So let us show that $f = F(b) \circ \beta$. If we apply $\mathrm{Hom}(?, F(P^0))$ to (58) and use $\mathrm{Hom}([-1]\sigma_{\leq -2}(\widehat{F}(P)), F(P^0)) = 0$, we get an injection

$$(? \circ t) : \mathrm{Hom}([-1]\sigma_{\leq -1}(\widehat{F}(P)), F(P^0)) \hookrightarrow \mathrm{Hom}(F(P^{-1}), F(P^0)).$$

Hence it is enough to check the equality $f \circ t = F(b) \circ \beta \circ t$. But $F(b) \circ \beta \circ t = F(b) \circ F(a) = F(d^{-1}) = f \circ t$. □

4.9. **Restriction of Projective Objects.** Let $i : (Y, \mathcal{T}) \to (X, \mathcal{S})$ be a closed embedding of cell-stratified varieties.

Lemma 48. *If V is an object of $\mathrm{Perv}(X, \mathcal{S})$ with a finite filtration with standard subquotients, then $i^*(V)$ is in $\mathrm{Perv}(Y, \mathcal{T})$, so $i^*(V) = {}^p i^*(V)$. If V is a projective object of $\mathrm{Perv}(X, \mathcal{S})$, then the restriction $i^*(V)$ is a projective object of $\mathrm{Perv}(Y, \mathcal{T})$.*

Proof. For standard objects $\Delta_S = l_{S!}([d_S]\underline{S})$, we have $i^*(\Delta_S) = \Delta_S$ if $S \in \mathcal{T}$ and $i^*(\Delta_S) = 0$ otherwise. Thus $i^*(\Delta_S)$ is in $\mathrm{Perv}(Y, \mathcal{T})$. Since $\mathrm{Perv}(Y, \mathcal{T})$ is stable by extensions ([BBD82, 1.3.6]) this proves the first statement. The second statement follows from Theorem 10 and the fact that ${}^p i^*$ is left adjoint to the exact functor ${}^p i_*$. □

Corollary 49. *If $\widetilde{V} \in \mathrm{MHM}(X, \mathcal{S})$ is perverse-projective, then $i^*(\widetilde{V})$ is an object of $\mathrm{MHM}(Y, \mathcal{T})$ and perverse-projective, where we consider $\mathrm{MHM}(Y, \mathcal{T})$ as the heart of the standard t-structure on $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(Y), \mathcal{T})$.*

Proof. Lemma 48 shows that $v(i^*(\widetilde{V}))$ is a projective object of $\mathrm{Perv}(Y, \mathcal{T})$. This implies that $i^*(\widetilde{V})$ is in $\mathrm{MHM}(Y, \mathcal{T})$, since $\mathrm{rat} : \mathrm{MHM}(X) \to \mathrm{Perv}(X)$ is exact and faithful. □

Remark 50. We define ${}^p i^*(\widetilde{V}) := i^*(\widetilde{V})$ for perverse-projective $\widetilde{V}$ in $\mathrm{MHM}(X, \mathcal{S})$. This notation is justified by $\mathrm{rat}({}^p i^*(\widetilde{V})) \cong {}^p i^*(\mathrm{rat}(\widetilde{V}))$ (cf. Corollary 49, Proposition 41).

In subsection 3.5 we have defined a mixed Hodge structure $\mathrm{Ext}^i_{\mathrm{Perv}(X)}(\widetilde{M}, \widetilde{N})$ on $\mathrm{Ext}^i_{\mathrm{Perv}(X)}(M, N)$, for objects $\widetilde{M}$, $\widetilde{N}$ of $\mathrm{MHM}(X)$. The same definition also works for objects $\widetilde{M}$, $\widetilde{N}$ of $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$. Consider the obvious composition in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(X))$ provided by several adjunction morphisms

$$\mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N}) \to i_* i^* \mathscr{H}\mathrm{om}\,(\widetilde{M}, \widetilde{N}) \to i_* \mathscr{H}\mathrm{om}\,(i^*(\widetilde{M}), i^*(\widetilde{N})).$$

We take hypercohomology and obtain morphisms of (polarizable) mixed Hodge structures

$$\mathrm{Ext}^j_{\mathrm{Perv}(X)}(\widetilde{M}, \widetilde{N}) \to \mathrm{Ext}^j_{\mathrm{Perv}(Y)}(i^*(\widetilde{M}), i^*(\widetilde{N})).$$

These morphisms are natural in $\widetilde{M}$ and $\widetilde{N}$. In particular, if $\widetilde{P}$ and $\widetilde{Q}$ are smooth perverse-projective Hodge sheaves, then $i^*(\widetilde{P})$ and $i^*(\widetilde{Q})$ are smooth perverse-projective Hodge sheaves and we get a morphism

$$\mathrm{Hom}_{\mathrm{Perv}(X,\mathcal{S})}(\widetilde{P}, \widetilde{Q}) \to \mathrm{Hom}_{\mathrm{Perv}(Y,\mathcal{T})}({}^p i^*(\widetilde{P}), {}^p i^*(\widetilde{Q})) \tag{60}$$

of mixed Hodge structures (see Corollary 49 and Remark 50).

4.10. **Passage to Cohomology Algebras.** We combine our results in order to prove that the third and fourth square in diagram (40) commute up to natural isomorphism.

Assume that we are in the setting described in subsection 4.1 (with I a singleton). So we are given a perverse-projective resolution of finite length $\widetilde{P} \to \widetilde{M}$. Let

$${}^p i^*(\widetilde{P}) := (\ldots \to \ldots \to {}^p i^*(\widetilde{P}^{-1}) \to {}^p i^*(\widetilde{P}^0) \to 0 \to \ldots)$$

be the complex obtained by applying ${}^p i^*$ to $\widetilde{P}$ (cf. Corollary 49 and Remark 50). We may and will assume that this complex ${}^p i^*(\widetilde{P})$ is a complex in $\mathrm{MHM}(Y, \mathcal{T})$. The underlying complex of smooth projective perverse sheaves is denoted by ${}^p i^*(P)$.

The definition of the complex $\widetilde{A} = \mathcal{E}\mathrm{nd}\,(\widetilde{P})$ in subsection 3.6 and the comments around (60) at the end of subsection 4.9 show that there is a morphism of dg algebras "of mixed Hodge structures"

$$\widetilde{A} = \mathcal{E}\mathrm{nd}\,(\widetilde{P}) \to \mathcal{E}\mathrm{nd}\,({}^p i^*(\widetilde{P})) = \mathcal{E}\mathrm{nd}\,([\mu]^p i^*(\widetilde{P})). \tag{61}$$

Recall that $\widetilde{B} = \mathcal{E}\mathrm{nd}\,(\widetilde{Q})$. Hence the dg $\mathcal{E}\mathrm{nd}\,([\mu]^p i^*(\widetilde{P})$-$\mathcal{E}\mathrm{nd}\,(\widetilde{Q})$-bimodule

$$\widetilde{V} := \mathcal{H}\mathrm{om}\,(\widetilde{Q}, [\mu]^p i^*(\widetilde{P}))$$

becomes a dg $\widetilde{A}$-$\widetilde{B}$-bimodule. Note that $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{V}$ are complexes of mixed Hodge structures, and the differentials, multiplications and operations are morphisms of mixed Hodge structures.

We apply the tensor functors ω_0, $\mathrm{gr}^W_{\mathbb{R}}$, and $\omega_W = \eta \circ \mathrm{gr}^W_{\mathbb{R}}$ from subsection 3.9 (cf. diagram (25)) to $\widetilde{A}$, $\widetilde{B}$ and the $\widetilde{A}$-$\widetilde{B}$-bimodule $\widetilde{V}$ and call the obtained dg(g) algebras and bimodules as shown here:

$$\begin{tikzcd} \boxed{\widetilde{A} \curvearrowright \widetilde{V} \curvearrowleft \widetilde{B}} \arrow[r, mapsto, "\mathrm{gr}^W_{\mathbb{R}}"] \arrow[d, mapsto, "\omega_0"] \arrow[dr, mapsto, "\omega_W"] & \boxed{\widetilde{R} \curvearrowright \widetilde{W} \curvearrowleft \widetilde{S}} \arrow[d, mapsto, "\eta"] \\ \boxed{A \curvearrowright V \curvearrowleft B} \arrow[r, "a", "\sim"'] & \boxed{R \curvearrowright W \curvearrowleft S}. \end{tikzcd} \tag{62}$$

The isomorphism of dg algebras and bimodules indicated by the lower horizontal arrow comes from the natural isomorphism (26).

Proposition 51. *There is a quasi-isomorphism $\widetilde{f} : \widetilde{S} \to \widetilde{W}$ of (right) dgg $\widetilde{S}$-modules.*

Proof. By Proposition 47 (using Corollary 49) and assumption (37) we have isomorphisms

$$[\mu]^p i^*(\widetilde{P}) \xrightarrow{\sim} [\mu] i^*(\widetilde{M}) \xrightarrow{\sim} \widetilde{N}$$

in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(Y))$; recall $\widetilde{N} \in \mathrm{MHM}(Y, \mathcal{T})$. Hence $\mathrm{H}^j([\mu]^p i^*(\widetilde{P}))$ vanishes for $j \neq 0$ and we get a sequence

$$\widetilde{Q} \to \widetilde{N} \xleftarrow{\sim} \mathrm{H}^0([\mu]^p i^*(\widetilde{P}) \leftarrow \tau_{\leq 0}([\mu]^p i^*(\widetilde{P})) \to [\mu]^p i^*(\widetilde{P})$$

of quasi-isomorphisms in $\mathrm{Ket}^{\mathrm{b}}(\mathrm{MHM}(X, \mathcal{S}))$, where $\tau_{\leq 0}$ is the intelligent truncation functor as defined, for example, in [KS94, 1.3]. We apply $\mathcal{H}\mathrm{om}\,(\widetilde{Q}, ?)$ to this sequence and obtain, using Lemma 21, a sequence of quasi-isomorphisms of $\widetilde{B}$-modules connecting

$$\widetilde{B} = \mathcal{H}\mathrm{om}\,(\widetilde{Q}, \widetilde{Q}) \text{ and } \widetilde{V} = \mathcal{H}\mathrm{om}\,(\widetilde{Q}, [\mu]^p i^*(\widetilde{P})).$$

Hence $\mathrm{H}(\widetilde{S})$ and $\mathrm{H}(\widetilde{W})$ are isomorphic as dgg $\mathrm{H}(\widetilde{S})$-modules. Choose $w \in \mathrm{Z}(\widetilde{W})^{00}$ such that $\mathrm{H}(\widetilde{S}) \to \mathrm{H}(\widetilde{W})$, $s \mapsto [w]s$, is an isomorphism. Then $\widetilde{f} : \widetilde{S} \to \widetilde{W}$, $s \mapsto ws$, is a quasi-isomorphism of dgg $\widetilde{S}$-modules. $\square$

Lemma 46 shows that multiplication by $w := \widetilde{f}(1) \in \Gamma(\widetilde{W})^{00}$ defines quasi-isomorphisms $\widetilde{S} \to \widetilde{W}$, $\Gamma(\widetilde{S}) \to \Gamma(\widetilde{W})$ and $\mathrm{H}(\widetilde{S}) \to \mathrm{H}(\widetilde{W})$ of dgg modules.

We apply η to the morphisms of dgg algebras and bimodules

$$\boxed{\widetilde{R} \curvearrowright \widetilde{W} \curvearrowleft \widetilde{S}} \supset \boxed{\Gamma(\widetilde{R}) \curvearrowright \Gamma(\widetilde{W}) \curvearrowleft \Gamma(\widetilde{S})}$$

and denote the resulting situation by

$$\boxed{R \curvearrowright W \curvearrowleft S} \supset \boxed{\Gamma(R) \curvearrowright \Gamma(W) \curvearrowleft \Gamma(S)}. \tag{63}$$

Multiplication by $w \in \mathrm{Z}(\Gamma(W))^0$ still defines quasi-isomorphisms $S \to W$ and $\Gamma(S) \to \Gamma(W)$ of dg modules, hence we can apply Proposition 45 and obtain a natural isomorphism

$$
\begin{tikzcd}
\mathrm{dgPrae}_{\Gamma(R)}(\Gamma(R)) \arrow[r, "? \overset{L}{\underset{\Gamma(R)}{\otimes}} R"] \arrow[d, "? \overset{L}{\underset{\Gamma(R)}{\otimes}} \Gamma(W)"'] & \mathrm{dgPrae}_R(R) \arrow[d, "? \overset{L}{\underset{R}{\otimes}} W"] \\
\mathrm{dgPrae}_{\Gamma(S)}(\Gamma(S)) \arrow[r, "? \overset{L}{\underset{\Gamma(S)}{\otimes}} S"'] \arrow[ur, Rightarrow, "\sim"'] & \mathrm{dgPrae}_S(S).
\end{tikzcd}
\tag{64}
$$

Since the cohomologies $\mathrm{H}(\widetilde{R})$, $\mathrm{H}(\widetilde{S})$ and hence $\mathrm{H}(\widetilde{W})$ are pure of weight zero (as shown in the proof of Theorem 27), componentwise projection defines well-defined morphisms of dgg algebras and modules

$$\boxed{\Gamma(\widetilde{R}) \curvearrowright \Gamma(\widetilde{W}) \curvearrowleft \Gamma(\widetilde{S})} \to \boxed{\mathrm{H}(\widetilde{R}) \curvearrowright \mathrm{H}(\widetilde{W}) \curvearrowleft \mathrm{H}(\widetilde{S})}$$

with underlying morphisms of dg algebras and modules

$$\boxed{\Gamma(R) \curvearrowright \Gamma(W) \curvearrowleft \Gamma(S)} \to \boxed{\mathrm{H}(R) \curvearrowright \mathrm{H}(W) \curvearrowleft \mathrm{H}(S)}.$$

Multiplication by $w \in \mathrm{Z}(\Gamma(W))^0$ and its class $[w] \in \mathrm{H}(W)^0$ defines quasi-isomorphisms $\Gamma(S) \to \Gamma(W)$ and $\mathrm{H}(S) \to \mathrm{H}(W)$ of dg modules, so application of Proposition 45 yields a natural isomorphism

$$
\begin{tikzcd}
\mathrm{dgPrae}_{\Gamma(R)}(\Gamma(R)) \arrow[r, "? \overset{L}{\underset{\Gamma(R)}{\otimes}} \mathrm{H}(R)"] \arrow[d, "? \overset{L}{\underset{\Gamma(R)}{\otimes}} \Gamma(W)"'] & \mathrm{dgPrae}_{\mathrm{H}(R)}(\mathrm{H}(R)) \arrow[d, "? \overset{L}{\underset{\mathrm{H}(R)}{\otimes}} \mathrm{H}(W)"] \\
\mathrm{dgPrae}_{\Gamma(S)}(\Gamma(S)) \arrow[r, "? \overset{L}{\underset{\Gamma(S)}{\otimes}} \mathrm{H}(S)"'] \arrow[ur, Rightarrow, "\sim"'] & \mathrm{dgPrae}_{\mathrm{H}(S)}(\mathrm{H}(S))
\end{tikzcd}
\tag{65}
$$

Let

$$\boxed{A \curvearrowright V \curvearrowleft B} \supset \boxed{\mathrm{Sub}(A) \curvearrowright \mathrm{Sub}(V) \curvearrowleft \mathrm{Sub}(B)}$$

be the inverse image of (63) under the isomorphism a in (62) (cf. diagram (27) in the proof of Theorem 27). Then diagrams (64) and (65) get transformed in the third and forth square in diagram (40).

4.11. **Passage to Extension Algebras.** We prove now that the fifth square in diagram (40) commutes up to natural isomorphism. The setting is as in subsection 4.10. Recall that $M = \mathrm{rat}(\widetilde{M})$ and $\underline{M} = \mathrm{real}(M) = v(\widetilde{M})$ and similarly for $\widetilde{N}$.

We define in the following isomorphisms of dg algebras and bimodules (with all differentials equal to zero)

$$
\begin{aligned}
&\boxed{\mathrm{H}(A) \curvearrowright \mathrm{H}(V) \curvearrowleft \mathrm{H}(B)} \xrightarrow{(\phi,\chi,\psi)} \boxed{\mathrm{Ext}(M) \curvearrowright \mathrm{Ext}(N) \curvearrowleft \mathrm{Ext}(N)} \\
&\xrightarrow{(\mathrm{real},\mathrm{real},\mathrm{real})} \boxed{\mathrm{Ext}(\underline{M}) \curvearrowright \mathrm{Ext}(\underline{N}) \curvearrowleft \mathrm{Ext}(\underline{N})},
\end{aligned}
\tag{66}
$$

where we omit some indices $\mathrm{Perv}(X, \mathcal{S})$ and $\mathrm{Perv}(Y, \mathcal{T})$ in the second box. The right module structures on these bimodules and the isomorphisms real are the obvious ones. For the definition of the left module structures and the morphisms ϕ, χ and

ψ we have to recall and establish several isomorphisms. (The left module structures on $\mathrm{H}(V)$ and $\mathrm{Ext}(\underline{M})$ were already defined, but we repeat the definition.)

There is an isomorphism $\widetilde{\sigma} : [\mu]i^*(\widetilde{M}) \xrightarrow{\sim} \widetilde{N}$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{MHM}(Y))$ by assumption (see (37)). Define $\underline{\sigma} := v(\widetilde{\sigma})$ and let τ be the natural isomorphism from Proposition 41. We obtain isomorphisms

$$[\mu]\underline{L^p i^*(M)} \xrightarrow{[\mu]\tau_M} [\mu]i^*(\underline{M}) \xrightarrow{\underline{\sigma}} \underline{N}.$$

The equivalence real : $\mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X,\mathcal{S})) \to \mathcal{D}^{\mathrm{b}}(X,\mathcal{S})$ shows that there is a unique isomorphism $\lambda : [\mu]L^p i^*(M) \xrightarrow{\sim} N$ in $\mathcal{D}^{\mathrm{b}}(\mathrm{Perv}(X,\mathcal{S}))$ such that $\underline{\mathrm{real}(\lambda)} = \underline{\sigma} \circ [\mu]\tau_M$.

We denote the perverse-projective resolutions $\widetilde{P} \to \widetilde{M}$ and $\widetilde{Q} \to \widetilde{N}$ by $\widetilde{\pi}$ and $\widetilde{\rho}$, and their underlying projective resolutions as $\pi : P \to M$ and $\rho : Q \to N$. Since π is a projective resolution, we may assume that $L^p i^*(P)$ and $L^p i^*(M)$ are identical to ${}^p i^*(P)$ and that $L^p i^*(\pi)$ is the identity. Hence we may consider λ also as an isomorphism

$$\lambda : [\mu]^p i^*(P) \xrightarrow{\sim} N.$$

Instead of $\bigoplus \mathrm{Hom}^n_{\mathrm{Hot}(\mathrm{Perv}(X,\mathcal{S}))}$ and $\bigoplus \mathrm{Hom}^n_{\mathrm{Der}(\mathrm{Perv}(X,\mathcal{S}))}$ we write $\mathrm{Hom}_{\mathrm{Hot}}$ and $\mathrm{Hom}_{\mathrm{Der}}$, and similarly for $(Y,\mathcal{T})$. The above isomorphisms give rise to dga-morphisms

$$\mathrm{H}(A) = \mathrm{Hom}_{\mathrm{Hot}}(P) \xrightarrow{[\mu]^p i^*} \mathrm{Hom}_{\mathrm{Hot}}([\mu]^p i^*(P)),$$

$$\mathrm{Ext}(M) \xrightarrow{[\mu]L^p i^*} \mathrm{Ext}([\mu]L^p i^*(M) \underset{\sim}{\xrightarrow{\lambda}} \mathrm{Ext}(N),$$

$$\mathrm{Ext}(\underline{M}) \xrightarrow{[\mu]i^*} \mathrm{Ext}([\mu]i^*(\underline{M})) \underset{\sim}{\xrightarrow{\underline{\sigma}}} \mathrm{Ext}(\underline{N}).$$

The first morphism comes from (61), the last one coincides with (39). These morphisms and the obvious left multiplications define the left operations shown in (66). The morphism ϕ is the composition

$$\mathrm{H}(A) = \mathrm{Hom}_{\mathrm{Hot}}(P) = \mathrm{Hom}_{\mathrm{Der}}(P) = \mathrm{Ext}(P) \underset{\sim}{\xrightarrow{\pi}} \mathrm{Ext}(M),$$

ψ is defined analogously, and χ is given by

$$\mathrm{H}(V) = \mathrm{Hom}_{\mathrm{Hot}}(Q, [\mu]^p i^* P) = \mathrm{Ext}(Q, [\mu]^p i^*(P)) \underset{\sim}{\xrightarrow{\rho,\lambda}} \mathrm{Ext}(N).$$

It is easy to check that all morphisms in (66) are isomorphisms of dg algebras and dg bimodules respectively. We apply Proposition 45 to the situation (66) and the element $n \in \mathrm{H}(V)$ corresponding to $\mathrm{id} \in \mathrm{Ext}(\underline{N})$ and obtain the commutativity (up to natural isomorphism) of the fifth square in diagram (40).

5. Inverse Limits

5.1. **Inverse Limits of Categories.** We exhibit a definition of inverse limit of a sequence of categories that will enable us to consider inverse limits of dg categories (subsection 5.3) and to obtain a description of the equivariant derived category (subsection 6.1).

Let $\mathcal{C}_0 \xleftarrow{F_0} \mathcal{C}_1 \leftarrow \ldots \leftarrow \mathcal{C}_n \xleftarrow{F_n} \mathcal{C}_{n+1} \leftarrow \ldots$ or in short $((\mathcal{C}_n),(F_n))$ be a sequence of categories (and functors). We call the following category the inverse limit of this sequence and denote it by $\varprojlim \mathcal{C}_n$:

- Objects are sequences $((M_n)_{n\in\mathbb{N}}, (\phi_n)_{n\in\mathbb{N}})$ of objects M_n in $\mathcal{C}_n$ and isomorphisms $\phi_n : F_n(M_{n+1}) \xrightarrow{\sim} M_n$.

- Morphisms $\alpha : ((M_n), (\phi_n)) \to ((N_n), (\psi_n))$ are sequences $(\alpha_n)_{n\in\mathbb{N}}$ of morphisms $\alpha_n : M_n \to N_n$ such that $\psi_n \circ F_n(\alpha_{n+1}) = \alpha_n \circ \phi_n$, for all $n \in \mathbb{N}$.

Lemma 52. *Let $N \in \mathbb{N}$ and assume that $F_n : \mathcal{C}_{n+1} \to \mathcal{C}_n$ is an equivalence for all $n \geq N$. Then the obvious projection functor $\mathrm{pr}_N : \varprojlim \mathcal{C}_n \to \mathcal{C}_N$ is an equivalence.*

Proof. Obvious. □

A morphism of sequences $((\mathcal{C}_n), (F_n))$ and $((\mathcal{D}_n), (G_n))$ of categories is a sequence $\nu = (\nu_n)$ of functors $\nu_n : \mathcal{C}_n \to \mathcal{D}_n$ such that $\nu_n \circ F_n$ and $G_n \circ \nu_{n+1}$ coincide (up to natural isomorphism) for each $n \in \mathbb{N}$. Any such morphism ν obviously defines a functor $\varprojlim \nu_n : \varprojlim \mathcal{C}_n \to \varprojlim \mathcal{D}_n$.

In the following, we describe a setting in which this functor is an equivalence. Let $(\mathcal{I}, \leq)$ be a directed (i. e. for all I, $J \in \mathcal{I}$ there is $K \in \mathcal{I}$ with $I \leq K$, $J \leq K$) partially ordered set (e. g. the set of segments in $\mathbb{Z}$, partially ordered by inclusion). An $\mathcal{I}$-filtered category is a category $\mathcal{C}$ together with strict full subcategories $(\mathcal{C}^I)_{I\in\mathcal{I}}$ such that $\mathcal{C}^I \subset \mathcal{C}^J$ for $I \leq J$. We say that $\mathcal{C}$ is the union of the $\mathcal{C}^I$ if any object of $\mathcal{C}$ is contained in some $\mathcal{C}^I$. A morphism $\mathcal{C} \to \mathcal{D}$ of $\mathcal{I}$-filtered categories ($\mathcal{I}$-filtered morphism) is a functor $F : \mathcal{C} \to \mathcal{D}$ inducing functors $F^I : \mathcal{C}^I \to \mathcal{D}^I$ for all $I \in \mathcal{I}$.

If $((\mathcal{C}_n), (F_n))$ is a sequence of $\mathcal{I}$-filtered categories (and $\mathcal{I}$-filtered morphisms), the inverse limit $\varprojlim \mathcal{C}_n$ is filtered by the $\varprojlim \mathcal{C}_n^I$. We will use the following conditions on a sequence of $\mathcal{I}$-filtered categories $((\mathcal{C}_n), (F_n))$.

(F1) For each $I \in \mathcal{I}$ there is $N_I \in \mathbb{N}$ such that, for all $n \geq N_I$, $F_n^I : \mathcal{C}_{n+1}^I \to \mathcal{C}_n^I$ is an equivalence.

(F2) $\varprojlim \mathcal{C}_n$ is the union of the $\varprojlim \mathcal{C}_n^I$.

Any morphism $(\nu_n) : ((\mathcal{C}_n), (F_n)) \to ((\mathcal{D}_n), (G_n))$ of sequences of $\mathcal{I}$-filtered categories induces an $\mathcal{I}$-filtered morphism $\varprojlim \nu_n : \varprojlim \mathcal{C}_n \to \varprojlim \mathcal{D}_n$.

Proposition 53. *Let $(\nu_n) : ((\mathcal{C}_n), (F_n)) \to ((\mathcal{D}_n), (G_n))$ be a morphism of sequences of $\mathcal{I}$-filtered categories and assume that both sequences satisfy condition (F2) and that $((\mathcal{C}_n), (F_n))$ satisfies condition (F1). If for all $I \in \mathcal{I}$ there is $N \in \mathbb{N}$ such that, for all $n \geq N$, $\nu_n^I : \mathcal{C}_n^I \to \mathcal{D}_n^I$ is an equivalence, then $\varprojlim \nu_n : \varprojlim \mathcal{C}_n \to \varprojlim \mathcal{D}_n$ is an equivalence and $((\mathcal{D}_n), (G_n))$ also satisfies condition (F1).*

Proof. Obviously $((\mathcal{D}_n), (G_n))$ also satisfies condition (F1). By condition (F2) it is sufficient to show that each $\varprojlim \nu_n^I$ is an equivalence. But this follows from condition (F1) and Lemma 52. □

Let $\mathcal{T}_0 \xleftarrow{F_0} \mathcal{T}_1 \xleftarrow{F_1} \dots \xleftarrow{F_{n-1}} \mathcal{T}_n \xleftarrow{F_n} \dots$ be a sequence of triangulated categories and triangulated functors. Then $\varprojlim \mathcal{C}_n$ is obviously additive and the shift functors of the various $\mathcal{T}_n$ induce an obvious shift functor [1] on $\varprojlim \mathcal{T}_n$. Assume that each $\mathcal{T}_n$ is an $\mathcal{I}$-filtered category and that all functors F_n are $\mathcal{I}$-filtered morphisms (the $\mathcal{T}_n^I$ are not assumed to be stable under the shift).

Proposition 54. *Let $((\mathcal{T}_n), (F_n))$ as above satisfy conditions (F1) and (F2) and assume that each $\mathcal{T}_n^I$ is closed under extensions in $\mathcal{T}_n$. Then there is a unique class $\mathcal{D}$ of triangles in $\varprojlim \mathcal{T}_n$ (considered as an additive category with shift functor* [1]*) such that $(\varprojlim \mathcal{T}_n, \mathcal{D})$ is a triangulated category and all projections $\mathrm{pr}_i : \varprojlim \mathcal{T}_n \to \mathcal{T}_i$ are triangulated ($i \in \mathbb{N}$). A triangle Σ is in $\mathcal{D}$ if and only if all $\mathrm{pr}_i(\Sigma)$ are distinguished ($i \in \mathbb{N}$).*

Proof. (Cf. [BL94, 2.5.2].) We denote by $\mathcal{E}$ the class of triangles Σ in $\varprojlim \mathcal{T}_n$ such that all $\mathrm{pr}_i(\Sigma)$ are distinguished and prove that $(\varprojlim \mathcal{T}_n, \mathcal{E})$ is a triangulated category. In all axioms of a triangulated category ([Ver96]), only a finite set F of objects is involved, and the existence of some objects and morphisms is asserted. So we may check these axioms in a suitable full subcategory $\varprojlim \mathcal{T}_n^I$ of $\varprojlim \mathcal{T}_n$ containing all $[k]X$, for $X \in F$ and $k = -1, 0, 1$. But this subcategory is equivalent to $\mathcal{T}_{N_I}^I$ by Lemma 52. (The condition that $\mathcal{T}_{N_I}^I$ is closed under extensions is used for constructing a distinguished triangle with a given base.)

If a class $\mathcal{D}$ of triangles satisfies the conditions of the proposition, then obviously $\mathcal{D} \subset \mathcal{E}$. If $\Sigma : X \xrightarrow{f} Y \to Z \to [1]X$ is in $\mathcal{E}$, there is a triangle $\Sigma' : X \xrightarrow{f} Y \to Z' \to [1]X$ in $\mathcal{D}$. All objects are in some $\varprojlim \mathcal{T}_n^I$. Since Σ and Σ' become isomorphic under $\mathrm{pr}_{N_I} : \varprojlim \mathcal{T}_n^I \xrightarrow{\sim} \mathcal{T}_{N_I}^I$, they are isomorphic in $\varprojlim \mathcal{T}_n$ and hence $\Sigma \in \mathcal{D}$. $\square$

Remark 55. We omit the obvious generalization of Proposition 53 to $\mathcal{I}$-filtered triangulated categories.

5.2. **Filtered DG Modules.** Let $\mathcal{A}$ be a dg algebra satisfying the conditions (P1)-(P3). We recall the definition of a certain equivalent subcategory of $\mathrm{dgPer}(\mathcal{A})$. This subcategory will enable us to prove the concise statement of Proposition 58.

Recall the $\mathcal{A}$-modules $\{\widehat{L}_x\}_{x\in W}$ from subsection 2.5. We consider the following full subcategory $\mathrm{dgFilt}(\mathcal{A})$ of $\mathrm{dgPer}(\mathcal{A})$: Its objects are $\mathcal{A}$-modules M admitting a finite filtration $0 = F_0(M) \subset F_1(M) \subset \cdots \subset F_n(M) = M$ by dg submodules with subquotients $F_i(M)/F_{i-1}(M) \cong \{l_i\}\widehat{L}_{x_i}$ in $\mathrm{dgMod}(\mathcal{A})$ for suitable $l_1 \geq l_2 \geq \cdots \geq l_n$ and $x_i \in W$.

Theorem 56 ([Sch08])**.** *If $\mathcal{A}$ is a dg algebra satisfying (P1)-(P3), then the inclusion* $\mathrm{dgFilt}(\mathcal{A}) \subset \mathrm{dgPer}(\mathcal{A})$ *is an equivalence of categories. Any object of* $\mathrm{dgFilt}(\mathcal{A})$ *is homotopically projective. An object M of* $\mathrm{dgFilt}(\mathcal{A})$ *lies in* $\mathrm{dgPer}^{\leq n}$ *(in* $\mathrm{dgPer}^{\geq n}$*) if and only if M is generated in degrees $\leq n$ (in degrees $\geq n$) as a graded A-module.*

5.3. **Inverse Limits of Categories of DG Modules.** Let $\mathcal{A}_\infty$ be a positively graded dg algebra with differential zero and A^0_∞ isomorphic to a finite product of division rings. If $\mathcal{A}_\infty$ is the inverse limit of a sequence of dg algebras $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of the same type that stabilizes in each degree, we show that $\mathrm{dgPer}(\mathcal{A}_\infty)$ is the inverse limit of the categories $\mathrm{dgPer}(\mathcal{A}_i)$. We first study the special case where only two dg algebras are involved, and generalize afterwards.

5.3.1. *Special Case.* Let $\mathcal{A} = (A = \bigoplus_{i\geq 0} A^i, d = 0)$ be a positively graded dg algebra with differential zero and $A^0 = \prod_{x\in W} e_x A^0$ a finite product of division rings (here e_x is the unit element of $e_x A^0$). In particular, $\mathcal{A}$ satisfies the conditions (P1)-(P3). The $e_x A^0$ are up to isomorphism the simple A^0-modules, so $\mathrm{dgPer}(\mathcal{A}) = \mathrm{dgPrae}_{\mathcal{A}}(\{e_x\mathcal{A}\}_{x\in W})$ thanks to Theorem 9. Let $\mathcal{B}$ be a dg algebra of the same type and $\phi : \mathcal{A} \to \mathcal{B}$ a dga-morphism. We assume that $\phi^0 : A^0 \to B^0$ is an isomorphism. Hence $B^0 = \prod e_x B^0$, where we write e_x instead of $\phi(e_x)$. The extension of scalars functor induces a triangulated functor

$$\mathrm{prod}_{\mathcal{A}}^{\mathcal{B}} = (? \overset{L}{\otimes}_{\mathcal{A}} \mathcal{B}) : \mathrm{dgPer}(\mathcal{A}) \to \mathrm{dgPer}(\mathcal{B}).$$

Since every object of $\mathrm{dgFilt}(\mathcal{A})$ is homotopically projective (Theorem 56) (and hence "homotopically flat") we can and will assume in the following that this functor is given by $M \mapsto M \otimes_{\mathcal{A}} \mathcal{B}$ on $\mathrm{dgFilt}(\mathcal{A})$. Using Theorem 56 it is then easy to see that this functor is t-exact with respect to the t-structures from Theorem 9.

By a segment we mean in the following a non-empty bounded interval in $\mathbb{Z}$. If $I = [a, b]$ is a segment (a, $b \in \mathbb{Z}$, $a \leq b$), we define $|I| = b - a$ and $\mathrm{dgPer}^I = \mathrm{dgPer}^{\geq a} \cap \mathrm{dgPer}^{\leq b}$. If $\mathcal{I}$ is the poset of all segments in $\mathbb{Z}$, partially ordered by inclusion, then dgPer is an $\mathcal{I}$-filtered category and the union of the dgPer^I. The functor $\mathrm{prod}_{\mathcal{A}}^{\mathcal{B}}$ is a morphism of $\mathcal{I}$-filtered categories (as defined in subsection 5.1).

Lemma 57. *Let $\mathcal{X}$ be in $\mathrm{dgPer}(\mathcal{A})$. If I is any segment, then $\mathcal{X}$ is in $\mathrm{dgPer}^I(\mathcal{A})$ if and only if $\mathrm{prod}_{\mathcal{A}}^{\mathcal{B}}(\mathcal{X})$ is in $\mathrm{dgPer}^I(\mathcal{B})$.*

Proof. This is a direct consequence of Theorem 56 and the assumption that $\phi^0 : A^0 \to B^0$ is an isomorphism. □

Proposition 58. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a morphism of dg algebras as above. If I is a segment and ϕ an isomorphism up to degree $|I| + 2$ (i. e. $\phi^i : A^i \to B^i$ is an isomorphism for all $i \leq |I| + 2$), then $\mathrm{prod}_{\mathcal{A}}^{\mathcal{B}} : \mathrm{dgPer}^I(\mathcal{A}) \to \mathrm{dgPer}^I(\mathcal{B})$ is an equivalence of categories.*

Proof. We may assume that $I = [0, b]$ for some $b \in \mathbb{N}$. The statement of the proposition is true since morphisms, homotopies and differentials are defined and can be defined in small degrees. But let us give the details.

$\mathrm{prod}_{\mathcal{A}}^{\mathcal{B}}$ **is fully faithful:** Let $\mathcal{X}$, $\mathcal{Y}$ be in $\mathrm{dgPer}^I(\mathcal{A})$. We have to show that

$$\mathrm{prod}_{\mathcal{A}}^{\mathcal{B}} : \mathrm{Hom}_{\mathrm{dgDer}(\mathcal{A})}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Hom}_{\mathrm{dgDer}(\mathcal{B})}(\phi(\mathcal{X}), \phi(\mathcal{Y}))$$

is an isomorphism, where we abbreviate $\phi(\mathcal{X}) = \mathrm{prod}_{\mathcal{A}}^{\mathcal{B}}(\mathcal{X})$ and $\phi(\mathcal{Y}) = \mathrm{prod}_{\mathcal{A}}^{\mathcal{B}}(\mathcal{Y})$. We may assume that $\mathcal{X}$ and $\mathcal{Y}$ are in dgFilt (Theorem 56) and that

$$\begin{aligned} X &= \{l_1\}e_{v_1}A \oplus \{l_2\}e_{v_2}A \oplus \cdots \oplus \{l_s\}e_{v_s}A, \\ Y &= \{m_1\}e_{w_1}A \oplus \{m_2\}e_{w_2}A \oplus \cdots \oplus \{m_t\}e_{w_t}A \end{aligned} \tag{67}$$

as graded A-modules, with v_i, $w_i \in W$, $0 \leq -l_i \leq b$ and $0 \leq -m_i \leq b$. Both $\phi(\mathcal{X}) = \mathcal{X} \otimes_{\mathcal{A}} \mathcal{B}$ and $\phi(\mathcal{Y})$ are given by the right hand side of (67), if we replace A by B there. Since objects of dgFilt are homotopically projective (Theorem 56) it is sufficient to show that

$$\mathrm{prod}_{\mathcal{A}}^{\mathcal{B}} : \mathrm{Hom}_{\mathrm{dgHot}(\mathcal{A})}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Hom}_{\mathrm{dgHot}(\mathcal{B})}(\phi(\mathcal{X}), \phi(\mathcal{Y})) \tag{68}$$

is an isomorphism.

We have

$$\mathrm{Hom}_{\mathrm{gMod}(A)}(\{l\}e_v A, \{m\}e_w A) = e_w A^{m-l} e_v$$

for v, $w \in W$, l, $m \in \mathbb{Z}$; here $\mathrm{gMod}(A)$ is the category of graded A-modules. Since the differentials $d_X : X \to \{1\}X$, $d_Y : Y \to \{1\}Y$ are morphisms of graded A-modules (the differential of $\mathcal{A}$ is zero), they are given by matrices x and y with entries in A. Similarly, morphisms $f \in \mathrm{Hom}_{\mathrm{dgMod}}(\mathcal{X}, \mathcal{Y})$ are matrices satisfying $yf = fx$, and homotopies $h : X \to \{-1\}Y$ are matrices. Each entry of these matrices is homogeneous, more precisely, we have $x_{ij} \in e_{v_i}A^{l_i+1-l_j}e_{v_j}$, $y_{ij} \in e_{w_i}A^{m_i+1-m_j}e_{w_j}$, $f_{ij} \in e_{w_i}A^{m_i-l_j}e_{v_j}$, and $h_{ij} \in e_{w_i}A^{m_i-1-l_j}e_{v_j}$. The differential of $\phi(\mathcal{X})$ is given by the matrix $\phi(x)$. The functor $\mathrm{prod}_{\mathcal{A}}^{\mathcal{B}} : \mathrm{dgMod}(\mathcal{A}) \to \mathrm{dgMod}(\mathcal{B})$ maps the matrix $f = (f_{ij})$ to $\phi(f) = (\phi(f_{ij}))$, and similarly for homotopies.

Surjectivity of (68): Let $\widetilde{f}$ be in $\mathrm{Hom}_{\mathrm{dgMod}}(\phi(\mathcal{X}), \phi(\mathcal{Y}))$. Since all entries of $\widetilde{f}$ are of degree $\leq b \leq |I| + 2$, there is a unique matrix f such that $\phi(f) = \widetilde{f}$ and

$\deg(f_{ij}) = \deg(\widetilde{f}_{ij})$ for all i, j (and $f_{ij} = 0$ if $\widetilde{f}_{ij} = 0$). This f defines an element of $\operatorname{Hom}_{\mathrm{dgMod}}(\mathcal{X}, \mathcal{Y})$ if and only if the matrix equation $yf = fx$ holds. All summands in every entry of this equation have degree $\leq b+1 \leq |I|+2$, and ϕ is an isomorphism up to degree $|I| + 2$. So it is enough to show that $\phi(y)\widetilde{f} = \widetilde{f}\phi(x)$. But this is true by assumption on $\widetilde{f}$.

Injectivity of (68): Assume that f in $\operatorname{Hom}_{\mathrm{dgMod}}(\mathcal{X}, \mathcal{Y})$ is mapped to $\phi(f) = 0$ in $\operatorname{Hom}_{\mathrm{dgHot}(\mathcal{B})}(\phi(\mathcal{X}), \phi(\mathcal{Y}))$. Then there is a homotopy $\widetilde{h} : \phi(X) \to \{-1\}\phi(Y)$ alias a matrix with entries in B, such that $\phi(f) = \widetilde{h}\phi(x) + \phi(y)\widetilde{h}$. Since all entries of $\widetilde{h}$ are homogeneous of degree $\leq b - 1 \leq |I| + 2$, there is a unique matrix h with $\phi(h) = \widetilde{h}$ and $\deg(h_{ij}) = \deg(\widetilde{h}_{ij})$. This matrix h defines a homotopy between f and 0, because ϕ is an isomorphism up to degree $|I| + 2$.

$\mathrm{prod}_{\mathcal{A}}^{\mathcal{B}}$ **is dense:** Let $\widetilde{\mathcal{X}}$ be an object of $\mathrm{dgPer}^I(\mathcal{B})$. We may assume that $\widetilde{\mathcal{X}}$ is in dgFilt and has, as a graded B-module, the form

$$\widetilde{X} = \{l_1\}e_{v_1}B \oplus \{l_2\}e_{v_2}B \oplus \dots \oplus \{l_s\}e_{v_s}B,$$

with $0 \leq -l_1 \leq -l_2 \leq \dots \leq -l_s \leq b$. The differential $d_{\widetilde{X}}$ is a matrix $\widetilde{x}$, with all entries in B of degree $\leq b+1 \leq |I|+2$. Let x be the unique matrix with entries in A such that $\phi(x) = \widetilde{x}$ and $\deg(x_{ij}) = \deg(\widetilde{x}_{ij})$ for all i, j. Define

$$X = \{l_1\}e_{v_1}A \oplus \{l_2\}e_{v_2}A \oplus \dots \oplus \{l_s\}e_{v_s}A.$$

The matrix x defines a differential on X if and only if $x^2 = 0$. In this matrix equation, all summands have degree $\leq b + 2 \leq |I| + 2$. But $\phi(x^2) = \widetilde{x}^2 = 0$ holds, and ϕ is an isomorphism in degrees $\leq |I| + 2$. Hence (X, x) is in dgFilt and the $\mathcal{A}$-module we are searching for. □

5.3.2. *General Case.* Let $\mathcal{A}_0 \xleftarrow{\phi_0} \mathcal{A}_1 \leftarrow \dots \leftarrow \mathcal{A}_n \xleftarrow{\phi_n} \mathcal{A}_{n+1} \leftarrow \dots$ be a sequence of dg algebras and dga-morphisms. Assume that

(S1) Each $\mathcal{A}_n = (A_n = \bigoplus_{i \geq 0} A_n^i, d = 0)$ is a positively graded dg algebra with differential zero.
(S2) $A_0^0 = \prod_{x \in W} e_x A_0^0$ is a finite product of division rings.
(S3) There is an increasing sequence $0 \leq r_0 \leq r_1 \leq \dots$ of non-negative integers (r_n) with $r_n \to \infty$ for $n \to \infty$, such that each ϕ_n is an isomorphism up to degree r_n. (In particular, all $\phi_n^0 : A_{n+1}^0 \to A_n^0$ are isomorphisms.)

The morphisms ϕ_n induce extension of scalars functors $\phi_n^* := \mathrm{prod}_{\mathcal{A}_{n+1}}^{\mathcal{A}_n}$ and we obtain a sequence $((\mathrm{dgPer}(\mathcal{A}_n)), (\phi_n^*))$ of categories or even of $\mathcal{I}$-filtered categories, where $\mathcal{I}$ is the poset of segments in $\mathbb{Z}$.

Proposition 59. *Under the above assumptions, $\varprojlim \mathrm{dgPer}(\mathcal{A}_n)$ has a natural structure of triangulated category with the following class of distinguished triangles: A triangle Σ is distinguished if and only if all $\mathrm{pr}_i(\Sigma)$ are distinguished ($i \in \mathbb{N}$).*

Proof. We want to deduce this from Proposition 54. It follows from (S3) and Proposition 58 that our sequence $((\mathrm{dgPer}(\mathcal{A}_n)), (\phi_n^*))$ satisfies condition (F1). Condition (F2) is fulfilled by Lemma 57. It is clear that each dgPer^I is closed under extensions in dgPer. □

Let $\mathcal{A}_\infty$ be a dg algebra with dga-morphism $\nu_n : \mathcal{A}_\infty \to \mathcal{A}_n$ ($n \in \mathbb{N}$) such that $\nu_n = \phi_n \circ \nu_{n+1}$ for all $n \in \mathbb{N}$. Assume that there is an increasing sequence $0 \leq s_0 \leq s_1 \leq \dots$ of non-negative integers (s_n) with $s_n \to \infty$ for $n \to \infty$, such that each ν_n is an isomorphism up to degree s_n. Equivalently, we could say that $\mathcal{A}_\infty$ is the inverse limit of our sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of dg algebras, i. e. $\mathcal{A}_\infty = \varprojlim \mathcal{A}_n$.

Proposition 60. *Under the above assumptions, the obvious functor* $\mathrm{dgPer}(\mathcal{A}_\infty) \to \varprojlim \mathrm{dgPer}(\mathcal{A}_n)$ *is a triangulated equivalence.*

Proof. This is a consequence of Proposition 53 and Remark 55 since $\mathrm{dgPer}(\mathcal{A}_\infty)$ is equivalent to the inverse limit of the constant sequence $((\mathrm{dgPer}(\mathcal{A}_\infty)), (\mathrm{id}))$. □

6. Formality of Equivariant Derived Categories

6.1. **Equivariant Derived Categories of Topological Spaces.** We introduce the equivariant derived category, following [BL94].

If Y is a topological space, we denote by $\mathrm{Sh}(Y)$ the category of sheaves of real vector spaces on Y and by $\mathcal{D}^+(Y)$ and $\mathcal{D}^{\mathrm{b}}(Y)$ its bounded below and bounded derived category.

Let $f : X \to Y$ be a continuous map and $n \in \mathbb{N} \cup \{\infty\}$. Given a base change $\widetilde{Y} \to Y$ we denote the induced map $X \times_Y \widetilde{Y} \to \widetilde{Y}$ by $\widetilde{f}$. We say that f is ***n*-acyclic** if for any base change $\widetilde{Y} \to Y$ and any sheaf $B \in \mathrm{Sh}(\widetilde{Y})$ the truncated adjunction morphism $B \to \tau_{\leq n}\widetilde{f}_*\widetilde{f}^*B$ is an isomorphism. Here $\tau_{\leq n}$ is the truncation functor for $n \in \mathbb{N}$ and $\tau_{\leq\infty} = \mathrm{id}$. The composition of n-acyclic maps is n-acyclic. A topological space X is called ***n*-acyclic** if the constant map $X \to \mathrm{pt}$ is n-acyclic.

Let G be a topological group. A ***G*-space** is a topological space X with a continuous G-action $G \times X \to X$. A ***G*-map** of G-spaces is a continuous G-equivariant map. A G-space X is **free** if it has a covering by open G-stable subspaces G-isomorphic to G-spaces of the form $G \times Y$ (for a suitable topological space Y) with G-action $g.(h, y) = (gh, y)$.

A **resolution** of a G-space X is a G-map from a free G-space to X. Morphisms of resolutions are G-maps over X. Let $p : P \to X$ be a resolution of X and $q : P \to G\backslash P$ the quotient map. The category $\mathcal{D}^{\mathrm{b}}_G(X, P)$ is defined as follows:

- Objects M are triples $(M_X, \overline{M}, \mu)$ where $M_X \in \mathcal{D}^{\mathrm{b}}(X)$, $\overline{M}$ in $\mathcal{D}^{\mathrm{b}}(G\backslash P)$ and $\mu : p^*(M_X) \xrightarrow{\sim} q^*(\overline{M})$ is an isomorphism in $\mathcal{D}^{\mathrm{b}}(P)$.
- Morphisms $\alpha : M \to N$ (where $M = (M_X, \overline{M}, \mu)$ and $N = (N_X, \overline{N}, \nu)$) are pairs $(\alpha_X, \overline{\alpha})$ where $\alpha : M_X \to N_X$ and $\overline{\alpha} : \overline{M} \to \overline{N}$ are morphisms in $\mathcal{D}^{\mathrm{b}}(X)$ and $\mathcal{D}^{\mathrm{b}}(G\backslash P)$ respectively such that $\nu \circ p^*(\alpha_X) = q^*(\overline{\alpha}) \circ \mu$.

We have two obvious functors: The forgetful functor $\mathrm{For} : \mathcal{D}^{\mathrm{b}}_G(X, P) \to \mathcal{D}^{\mathrm{b}}(X)$, $M \mapsto M_X$, and the functor $\gamma : \mathcal{D}^{\mathrm{b}}_G(X, P) \to \mathcal{D}^{\mathrm{b}}(G\backslash P)$, $M \mapsto \overline{M}$. If I is a segment in $\mathbb{Z}$, we let $\mathcal{D}^I_G(X, P)$ be the full subcategory of $\mathcal{D}^{\mathrm{b}}_G(X, P)$ consisting of objects M with $\mathrm{For}(M)$ in $\mathcal{D}^I(X)$. If p is surjective, this is equivalent to the condition $\gamma(M) \in \mathcal{D}^I(G\backslash P)$.

A resolution $p : P \to X$ is ***n*-acyclic** if the continuous map p is n-acyclic. Note that any n-acyclic map is surjective.

Proposition 61. *Let $\nu : P \to R$ be a morphism of n-acyclic resolutions $p : P \to X$ and $r : R \to X$, where $n \in \mathbb{N} \cup \{\infty\}$. If I is a segment with $n > |I|$, the obvious functor $\nu^* : \mathcal{D}^{\mathrm{b}}_G(X, R) \to \mathcal{D}^{\mathrm{b}}_G(X, P)$ restricts to an equivalence $\nu^* : \mathcal{D}^I_G(X, R) \to \mathcal{D}^I_G(X, P)$.*

Proof. Let $S = P \times_X R$ be the fiber product of P and R over X with projections $\pi_P : S \to P$ and $\pi_R : S \to R$. Then $\pi_P^* : \mathcal{D}^I_G(X, P) \to \mathcal{D}^I_G(X, S)$ and π_R^* :

$\mathcal{D}^I_G(X,R) \to \mathcal{D}^I_G(X,S)$ are equivalences of categories by [BL94, 2.2.2] (but with $n > |I|$). Let $(\mathrm{id}_P, \nu) : P \to S = P \times_X R$ be the unique morphism of resolutions with $\pi_P \circ (\mathrm{id}_P, \nu) = \mathrm{id}_P$ and $\pi_R \circ (\mathrm{id}_P, \nu) = \nu$. Then $(\mathrm{id}_P, \nu)^*$ is inverse to π_P^* and $\nu^* = (\mathrm{id}_P, \nu)^* \circ \pi_R^*$ is an equivalence on $\mathcal{D}^I_G$. □

If $P \to X$ and $R \to X$ are ∞-acyclic resolutions there is an equivalence of $\mathcal{D}^{\mathrm{b}}_G(X,P)$ and $\mathcal{D}^{\mathrm{b}}_G(X,R)$ that is defined up to a canonical natural isomorphism. The G-equivariant derived category $\mathcal{D}^{\mathrm{b}}_G(X)$ of X is defined to be $\mathcal{D}^{\mathrm{b}}_G(X,P)$, for $p : P \to X$ an ∞-acyclic resolution ([BL94, 2.7.2]). It is a triangulated category (cf. [BL94, 2.5.2]).

We give a description of the equivariant derived category as an inverse limit. Let $P_0 \to \ldots \to P_n \xrightarrow{f_n} P_{n+1} \to \ldots$ be a sequence of morphisms of resolutions $p_n : P_n \to X$. It gives rise to a sequence of categories and functors $((\mathcal{D}^{\mathrm{b}}_G(X,P_n)), (f_n^*))$. We consider the inverse limit $\varprojlim \mathcal{D}^{\mathrm{b}}_G(X,P_n)$ of these categories as defined in subsection 5.1. It is an additive category and has an obvious shift functor. We denote by γ_i the composition

$$\varprojlim \mathcal{D}^{\mathrm{b}}_G(X,P_n) \xrightarrow{\mathrm{pr}_i} \mathcal{D}^{\mathrm{b}}_G(X,P_i) \xrightarrow{\gamma} \mathcal{D}^{\mathrm{b}}(G\backslash P_i).$$

Proposition 62. *Keep the above assumptions and assume that $p_n : P_n \to X$ is n-acyclic, for each $n \in \mathbb{N}$. Then $\varprojlim \mathcal{D}^{\mathrm{b}}_G(X,P_n)$ carries a natural structure of triangulated category with the following class of distinguished triangles: A triangle Σ is distinguished if and only if all $\gamma_i(\Sigma)$ are distinguished ($i \in \mathbb{N}$). Moreover, the categories $\mathcal{D}^{\mathrm{b}}_G(X)$ and $\varprojlim \mathcal{D}^{\mathrm{b}}_G(X,P_n)$ are equivalent as triangulated categories.*

Proof. This follows from [BL94, 2] (more details in [Sch07, 5.2]). □

6.2. **Equivariant Derived Categories of Varieties.** Let G be an affine algebraic group (defined over $\mathbb{C}$, as all varieties in the following). The definitions of a G-variety and of a G-morphism between G-varieties are the obvious generalizations from the topological category.

A **Zariski principal fiber bundle (Zpfb) with structure group** G (or G-**Zpfb**) is a surjective G-morphism $q : E \to B$ between G-varieties with the following property: For every point in B, there is a Zariski open neighborhood U in B and a G-isomorphism $\tau : G \times U \xrightarrow{\sim} q^{-1}(U)$ (here the G-action on $G \times U$ is given by $g.(h,u) = (gh,u)$) such that $q \circ \tau$ is equal to the projection $\mathrm{pr}_U : G \times U \to U$. A map τ as above is called a **local trivialization** over U. By abuse of notation we often say that E is a Zpfb. A morphism $f : E \to E'$ of G-Zpfbs is a G-morphism $f : E \to E'$. It induces a morphism $f : B \to B'$ on quotient spaces.

Let X be a G-variety. A **Zariski resolution** of X is a datum $(B \xleftarrow{q} E \xrightarrow{p} X)$ consisting of a Zpfb $q : E \to B$ together with a G-morphism $p : E \to X$. We often omit $q : E \to B$ from the notation and say that $p : E \to X$ is a Zariski resolution or even that E is a Zariski resolution of X. Morphisms $E \to E'$ of Zariski resolutions of X are G-morphisms over X.

Let $n \in \mathbb{N} \cup \{\infty\}$. A variety (morphism of varieties) is called n**-acyclic**, if it is n-acyclic with respect to the classical topology. A Zariski resolution $(B \xleftarrow{q} E \xrightarrow{p} X)$ is n**-acyclic** if p is n-acyclic.

Let $E_0 \xrightarrow{f_0} E_1 \to \ldots \to E_n \xrightarrow{f_n} E_{n+1} \to \ldots$ be a sequence of Zariski resolutions of a G-variety X. If $p_n : E_n \to X$ is n-acyclic, Proposition 62 provides a triangulated equivalence

$$\mathcal{D}^{\mathrm{b}}_G(X) \cong \varprojlim \mathcal{D}^{\mathrm{b}}_G(X, E_n). \tag{69}$$

Let $(X, \mathcal{S})$ be a stratified G-variety (we do not assume that the strata are G-stable). We denote the full subcategory of $\mathcal{D}^{\mathrm{b}}_G(X)$ consisting of objects M with $\mathrm{For}(M) \in \mathcal{D}^{\mathrm{b}}(X, \mathcal{S})$ by $\mathcal{D}^{\mathrm{b}}_G(X, \mathcal{S})$. Similarly, we define $\mathcal{D}^{\mathrm{b}}_G(X, E, \mathcal{S}) \subset \mathcal{D}^{\mathrm{b}}_G(X, E)$, if E is a Zariski resolution of X. If I is a segment, we use the obvious definitions for $\mathcal{D}^I_G(X, \mathcal{S})$ etc.. Equivalence (69) restricts to a triangulated equivalence

$$\mathcal{D}^{\mathrm{b}}_G(X, \mathcal{S}) \cong \varprojlim \mathcal{D}^{\mathrm{b}}_G(X, E_n, \mathcal{S}). \tag{70}$$

If X is a G-variety, we denote by $\mathcal{D}^{\mathrm{b}}_{G,c}(X)$ the full subcategory of $\mathcal{D}^{\mathrm{b}}_G(X)$ consisting of objects F such that $\mathrm{For}(F)$ is constructible for some stratification of X.

Proposition 63. *Let $(X, \mathcal{S})$ be a stratified G-variety. If each stratum is a G-orbit, then $\mathcal{D}^{\mathrm{b}}_G(X, \mathcal{S}) = \mathcal{D}^{\mathrm{b}}_{G,c}(X)$.*

Proof. Let M be in $\mathcal{D}^{\mathrm{b}}_{G,c}(X)$ and $l : S \to X$ the inclusion of an orbit. We have to prove that $\mathrm{H}^i(l^*(\mathrm{For}(M))) = \mathrm{For}(\mathrm{H}^i(l^*(M)))$ is a local system on S. But $\mathrm{H}^i(l^*(M))$ is in the heart of the standard t-structure on $\mathcal{D}^{\mathrm{b}}_{G,c}(S)$ and hence a G-equivariant constructible sheaf on the orbit S ([BL94, 2.5.3]). $\square$

A G-**stratification** of a G-variety is a stratification by G-stable strata. A G-**stratified** variety is a G-variety X with a G-stratification.

Let $(X, \mathcal{S})$ be a G-stratified variety. An **approximation** (E, f) of $(X, \mathcal{S})$ is a sequence $E_0 \to \ldots \to E_n \xrightarrow{f_n} E_{n+1} \to \ldots$ of Zariski resolutions $(B_n \xleftarrow{q_n} E_n \xrightarrow{p_n} X)$ of X. An approximation (E, f) is called an **A-approximation** if the following conditions hold.

- (A1) Each $(B_n \xleftarrow{q_n} E_n \xrightarrow{p_n} X)$ is n-acyclic.
- (A2) $\mathcal{S}_n := \{q_n(p_n^{-1}(S)) \mid S \in \mathcal{S}\}$ is a stratification of B_n, for all $n \in \mathbb{N}$. (In particular, each stratum of $\mathcal{S}_n$ is irreducible.)
- (A3) Every stratum in $\mathcal{S}_n$ is simply connected.

Any A-approximation $E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} \ldots$ yields a sequence $B_0 \xrightarrow{f_0} B_1 \xrightarrow{f_1} \ldots$ of varieties, and a sequence $((\mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n)), (f_n^*))$ of triangulated categories.

Proposition 64. *Let $(E, f) = (\ldots \to E_n \xrightarrow{f_n} E_{n+1} \to \ldots)$ be an A-approximation of a G-stratified variety $(X, \mathcal{S})$. Then there is a canonical triangulated equivalence*

$$\varprojlim \mathcal{D}^{\mathrm{b}}_G(X, E_n, \mathcal{S}) \xrightarrow{\sim} \varprojlim \mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n) \tag{71}$$

Proof. The functor $\mathcal{D}^{\mathrm{b}}_G(X, E_n) \to \mathcal{D}^{\mathrm{b}}(B_n)$, $(M_X, \overline{M}, \mu) \mapsto \overline{M}$, restricts to a functor $\nu_n : \mathcal{D}^{\mathrm{b}}_G(X, E_n, \mathcal{S}) \to \mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n)$, since q_n is locally trivial. The inverse limit $\varprojlim \nu_n$ is the functor in (71).

All categories $\mathcal{D}^{\mathrm{b}}_G(X, E_n, \mathcal{S})$ are $\mathcal{I}$-filtered by the $\mathcal{D}^I_G(X, E_n, \mathcal{S})$ (where $\mathcal{I}$ is the poset of segments in $\mathbb{Z}$), and are the union of these subcategories. Similarly for $\mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n)$. For $n > |I|$, restriction $f_n^* : \mathcal{D}^I_G(X, E_{n+1}, \mathcal{S}) \to \mathcal{D}^I_G(X, E_n, \mathcal{S})$ is an equivalence (Proposition 61).

We claim that $\nu_n^I : \mathcal{D}^I_G(X, E_n, \mathcal{S}) \to \mathcal{D}^I(B_n, \mathcal{S}_n)$ is an equivalence for $n > |I|$. By [BL94, Lemma 2.3.2] (but with $n > |I|$), ν_n^I is fully faithful and its essential image is closed under extensions in $\mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n)$. Let $S \in \mathcal{S}$ and $S_n := q_n(p_n^{-1}(S))$. The inclusions l_S and l_{S_n} and proper base change give rise to an object "extension by zero of the constant sheaf on S" in $\mathcal{D}^{[0,0]}_G(X, E_n)$ that is mapped to $l_{S_n!}\underline{S_n}$ under $\nu_n : \mathcal{D}^{\mathrm{b}}_G(X, E_n, \mathcal{S}) \to \mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n)$. It follows from Lemma 66 below that ν^I_n is dense.

Proposition 53 shows that (71) is an equivalence and that the $\mathcal{I}$-filtered category $\mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n)$ satisfies condition (F1). Proposition 54 equips $\varprojlim \mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n)$ with the structure of a triangulated category, and it is obvious that (71) is triangulated. □

Corollary 65. *Let (E, f) be an A-approximation, I a segment and $N \in \mathbb{N}$. If $N > |I|$, the obvious functor $\varprojlim \mathcal{D}^I(B_n, \mathcal{S}_n) \to \mathcal{D}^I(B_N)$ is fully faithful.*

Proof. The proof of Proposition 64 shows that $f^*_n : \mathcal{D}^I(B_{n+1}, \mathcal{S}_{n+1}) \to \mathcal{D}^I(B_n, \mathcal{S}_n)$ is an equivalence for $n > |I|$. Hence Lemma 52 shows that $\mathrm{pr}_N : \varprojlim \mathcal{D}^I(B_n, \mathcal{S}_n) \to \mathcal{D}^I(B_N, \mathcal{S}_N)$ is an equivalence for $N > |I|$. □

Lemma 66. *Let $(X, \mathcal{S})$ be a stratified variety with simply connected strata and I a segment in $\mathbb{Z}$. Then every $A \in \mathcal{D}^I(X, \mathcal{S})$ is an iterated extension of objects $l_{S!}\underline{S}[-i]$, for $S \in \mathcal{S}$ and $i \in I$.*

Proof. The shift [1] and the truncation functors for the standard t-structure allow us to assume that $A \in \mathrm{Sh}(X, \mathcal{S})$. If j is the inclusion of an open stratum U and i the inclusion of its closed complement, we get a distinguished triangle $(j_!j^*(A), A, i_*i^*(A))$. Since $j^*(A)$ is a finite direct sum of constant sheaves $\underline{U}$, an induction on the number of strata finishes the proof. □

6.3. **Equivariant Intersection Cohomology Complexes.** Let G be an affine algebraic group of complex dimension d_G and $(X, \mathcal{S})$ a G-stratified variety. On $\mathcal{D}^{\mathrm{b}}_G(X, \mathcal{S})$, there is the perverse t-structure whose heart is the category $\mathrm{Perv}_G(X, \mathcal{S})$ of equivariant perverse sheaves (smooth along $\mathcal{S}$) (see [BL94, 5]). If $\mathcal{L}$ is a G-equivariant local system on S, we have the equivariant intersection cohomology complex $\mathcal{IC}_G(\overline{S}, \mathcal{L}) = l_{S!*}([d_S]\mathcal{L})$ in $\mathrm{Perv}_G(X, \mathcal{S})$. We are mainly interested in the case of the constant G-equivariant local system $\underline{S}_G$ on S and define $\mathcal{IC}_G(S) := \mathcal{IC}_G(\overline{S}, \underline{S}_G)$. We will describe this object precisely using the following type of approximation.

An **AB-approximation** is an A-approximation (E, f) of $(X, \mathcal{S})$ such that the following conditions are satisfied.

(B1) Each morphism $p_n : E_n \to X$ is smooth of relative complex dimension d_{p_n}.
(B2) Each $f_n : B_n \to B_{n+1}$ is a closed embedding of varieties.
(B3) For all $n \in \mathbb{N}$ and $S \in \mathcal{S}$, $f_n : \overline{S}_n \to \overline{S}_{n+1}$ is a normally nonsingular inclusion of codimension c_n, where $S_n := q_n(p^{-1}_n(S))$; here c_n only depends on n.

Let (E, f) be an AB-approximation of $(X, \mathcal{S})$. If we consider $\mathcal{IC}_G(S)$ as an object of $\varprojlim \mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n)$, using equivalences (70) and (71), we have (cf. [BL94, 5.1], using B1)

$$\mathrm{pr}_n(\mathcal{IC}_G(S)) = [d_G - d_{p_n}]\mathcal{IC}(S_n) \in \mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n).$$

In order to avoid to many shifts, we replace $\varprojlim \mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n)$ by an equivalent category as follows. Consider the following morphism of sequences of triangulated categories

$$
\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}(B_0,\mathcal{S}_0) \arrow[d, "{[d_{p_0}-d_G]}"] & \mathcal{D}^{\mathrm{b}}(B_1,\mathcal{S}_1) \arrow[l, "f_0^*"'] \arrow[d, "{[d_{p_1}-d_G]}"] & \mathcal{D}^{\mathrm{b}}(B_2,\mathcal{S}_2) \arrow[l, "f_1^*"'] \arrow[d, "{[d_{p_2}-d_G]}"] & \cdots \arrow[l, "f_2^*"'] \\
\mathcal{D}^{\mathrm{b}}(B_0,\mathcal{S}_0) & \mathcal{D}^{\mathrm{b}}(B_1,\mathcal{S}_1) \arrow[l, "{[-c_0]f_0^*}"'] & \mathcal{D}^{\mathrm{b}}(B_2,\mathcal{S}_2) \arrow[l, "{[-c_1]f_1^*}"'] & \cdots . \arrow[l, "{[-c_2]f_2^*}"']
\end{tikzcd}
\tag{72}
$$

If we denote the inverse limit of the second row by $\varprojlim \mathcal{D}^{\mathrm{b}}[B_n,\mathcal{S}_n]$, this morphism induces a triangulated equivalence

$$\varprojlim \mathcal{D}^{\mathrm{b}}(B_n,\mathcal{S}_n) \xrightarrow{\sim} \varprojlim \mathcal{D}^{\mathrm{b}}[B_n,\mathcal{S}_n]. \tag{73}$$

Conditions A2, B2 and B3 show that each $f_n : B_n \to B_{n+1}$ meets the assumptions made before (43) and (44) in subsection 4.2. So we obtain isomorphisms

$$
\begin{aligned}
&\iota_{S,n} : [-c_n]f_n^*(\mathcal{IC}(S_{n+1})) \xrightarrow{\sim} \mathcal{IC}(S_n) && \text{in } \mathrm{Perv}(B_n,\mathcal{S}_n) \text{ and} \\
&\widetilde{\iota}_{S,n} : [-c_n]f_n^*(\widetilde{\mathcal{IC}}(S_{n+1})) \xrightarrow{\sim} \widetilde{\mathcal{IC}}(S_n) && \text{in } \mathrm{MHM}(B_n,\mathcal{S}_n).
\end{aligned}
\tag{74}
$$

As an object of $\varprojlim \mathcal{D}^{\mathrm{b}}[B_n,\mathcal{S}_n]$, the equivariant intersection cohomology complex $\mathcal{IC}_G(S)$ is $((\mathcal{IC}(S_n)),(\iota_{S,n}))$. Note that $\widetilde{\mathcal{IC}}_G(S) = ((\widetilde{\mathcal{IC}}(S_n)),(\widetilde{\iota}_{S,n}))$ is a natural "Hodge lift" of $\mathcal{IC}_G(S)$. The same argument as in the proof of Theorem 39 shows that all functors $[-c_n]f_n^*$ in (72) are t-exact with respect to the perverse t-structures.

6.4. **Better Approximations and Formality.** Let (E,f) be an approximation of a G-stratified variety $(X,\mathcal{S})$ and assume that we are given stratifications $\mathcal{T}_n$ of B_n, for each $n \in \mathbb{N}$. The triple $(E,f,\mathcal{T}=(\mathcal{T}_n))$ is called an **ABC-approximation** if (E,f) is an AB-approximation and the following conditions hold.

- (C1) Each $\mathcal{T}_n$ is a cell-stratification of B_n that is finer than the stratification $\mathcal{S}_n$.
- (C2) Each $f_n : (B_n,\mathcal{T}_n) \to (B_{n+1},\mathcal{T}_{n+1})$ is a closed embedding of (cell-)stratified varieties.
- (C3) The Hodge sheaf $\widetilde{\mathcal{IC}}(S_n)$ is $\mathcal{T}_n$-pure of weight d_{S_n}, for all $n \in \mathbb{N}$ and $S \in \mathcal{S}$. (By Remark 33 we can equivalently require $\mathcal{T}_n$-$*$-purity of weight d_{S_n}.)

In subsection 6.5 we show how to construct ABC-approximations.

For M, N in $\mathcal{D}^{\mathrm{b}}_G(X)$, define $\mathrm{Ext}^n(M,N) := \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}_G(X)}(M,[n]N)$ and

$$\mathrm{Ext}(M,N) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^n(M,N).$$

The (equivariant) extension algebra of M is $\mathrm{Ext}(M) := \mathrm{Ext}(M,M)$.

Theorem 67. *Let G be an affine algebraic group and $(X,\mathcal{S})$ a G-stratified variety. If $(X,\mathcal{S})$ has an ABC-approximation, there is a triangulated equivalence*

$$\mathcal{D}^{\mathrm{b}}_G(X,\mathcal{S}) \cong \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}_G(\mathcal{S}))), \tag{75}$$

where $\mathcal{IC}_G(\mathcal{S})$ is the direct sum of the $(\mathcal{IC}_G(S))_{S\in\mathcal{S}}$. This equivalence is t-exact with respect to the perverse t-structure on $\mathcal{D}^{\mathrm{b}}_G(X,\mathcal{S})$ and the t-structure from Theorem 9 on $\mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}_G(\mathcal{S})))$. *By restriction to the heart, it induces an equivalence of abelian categories*

$$\mathrm{Perv}_G(X,\mathcal{S}) \cong \mathrm{dgFlag}(\mathrm{Ext}(\mathcal{IC}_G(\mathcal{S}))). \tag{76}$$

Proof. Let $(E, f, \mathcal{T})$ be an ABC-approximation of $(X, \mathcal{S})$. By (70), (71) and (73) we have equivalences

$$\mathcal{D}^{\mathrm{b}}_G(X,\mathcal{S}) \cong \varprojlim \mathcal{D}^{\mathrm{b}}_G(X, E_n, \mathcal{S}) \xrightarrow{\sim} \varprojlim \mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n) \xrightarrow{\sim} \varprojlim \mathcal{D}^{\mathrm{b}}[B_n, \mathcal{S}_n]. \tag{77}$$

of triangulated categories.

Properties A3, C1 and C3 allow to apply Theorem 31 (cf. Remark 32), and we obtain equivalences

$$\mathrm{Form}_n := \mathrm{Form}^{\mathcal{T}_n}_{\widetilde{P}_n \to \widetilde{\mathcal{IC}}(\mathcal{S}_n)} : \mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n) \xrightarrow{\sim} \mathrm{dgPer}(\mathcal{E}_n)$$

of triangulated categories, where we fixed perverse-projective resolutions $\widetilde{P}_{n,S_n} \to \widetilde{\mathcal{IC}}(S_n)$ and where $\mathcal{E}_n := \mathrm{Ext}(\mathcal{IC}(\mathcal{S}_n))$. The isomorphisms (74) induce dga-morphisms $\phi_n : \mathcal{E}_{n+1} \to \mathcal{E}_n$. Properties A2, A3, B2, B3, C1, C2, C3 and Theorem 39 yield the following commutative (up to natural isomorphism) diagram with triangulated and t-exact functors:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}(B_{n+1}, \mathcal{S}_{n+1}) \arrow[r, "{[-c_n]f_n^*}"] \arrow[d, "\mathrm{Form}_{n+1}"', "\sim"] & \mathcal{D}^{\mathrm{b}}(B_n, \mathcal{S}_n) \arrow[d, "\mathrm{Form}_n"', "\sim"] \\
\mathrm{dgPer}(\mathcal{E}_{n+1}) \arrow[r, "? \overset{L}{\otimes}_{\mathcal{E}_{n+1}} \mathcal{E}_n"] & \mathrm{dgPer}(\mathcal{E}_n).
\end{tikzcd}$$

Hence the sequence $(\mathrm{Form}_n)_{n\in\mathbb{N}}$ defines a morphism between sequences of triangulated categories. Its inverse limit establishes an equivalence

$$\varprojlim \mathcal{D}^{\mathrm{b}}[B_n, \mathcal{S}_n] \xrightarrow{\sim} \varprojlim \mathrm{dgPer}(\mathcal{E}_n). \tag{78}$$

Define $\mathcal{E}_\infty := \mathrm{Ext}(\mathcal{IC}_G(\mathcal{S}))$. This is a positively graded dg algebra with differential zero and has as degree zero part the product of $|\mathcal{S}|$ copies of $\mathbb{R}$. From (77) we obtain dga-morphisms $\nu_n : \mathcal{E}_\infty \to \mathcal{E}_n$ such that $\nu_n = \phi_n \circ \nu_{n+1}$. Let J be a segment such that $\mathcal{IC}_G(\mathcal{S}) \in \mathcal{D}^J_G(X)$. We deduce from Corollary 65 that ν_n is an isomorphism up to degree $n - |J| - 1$. So our sequence of dg algebras $((\mathcal{E}_n), (\phi_n))$ satisfies (if we forget the first $|J| + 1$ members and renumerate, which is harmless for the following) the conditions (S1)-(S3) considered in subsection 5.3.2, and $\mathcal{E}_\infty$ is the inverse limit of this sequence. Proposition 59 shows that $\varprojlim \mathrm{dgPer}(\mathcal{E}_n)$ carries a natural structure of triangulated category. Since all functors Form_n are triangulated, it follows from Proposition 62 that equivalence (78) is triangulated. Finally, Proposition 60 provides a triangulated equivalence $\mathrm{dgPer}(\mathcal{E}_\infty) \xrightarrow{\sim} \varprojlim \mathrm{dgPer}(\mathcal{E}_n)$. This establishes (75). Since $\mathcal{IC}_G(S)$ is mapped to $e_S\mathcal{E}_\infty$, equivalence (75) is t-exact, and we obtain (76). □

6.5. **Existence of ABC-Approximations.** Let G be an affine algebraic group. An **ABCD-approximation** of a G-stratified variety $(X, \mathcal{S})$ is an ABC-approximation $(E, f, \mathcal{T})$ of $(X, \mathcal{S})$ satisfying the following condition.

(D) For each $n \in \mathbb{N}$, the Zpfb $q_n : E_n \to B_n$ can be trivialized around each stratum $T \in \mathcal{T}_n$ (this means that there is an open subvariety U of B_n containing T such that q_n has a local trivialization over U).

An **ABCD-approximation** for G is an ABCD-approximation of the G-stratified variety $(\mathrm{pt}, \{\mathrm{pt}\})$.

Proposition 68. *Every torus and any connected solvable affine algebraic group has an ABCD-approximation.*

Let us remark that ABCD-approximations also exist for $\mathrm{GL}_n(\mathbb{C})$ and parabolic subgroups of $\mathrm{GL}_n(\mathbb{C})$ (see [Sch07, 5.6]).

Proof. Let $q_i : E_i := \mathbb{C}^{i+1} \setminus \{0\} \to B_i := \mathbb{P}^i(\mathbb{C})$ be the obvious $\mathbb{C}^*$-Zpfb. The standard closed embeddings $\mathbb{C}^{i+1} \hookrightarrow \mathbb{C}^{i+2}$, $x \mapsto (x, 0)$ induce morphisms of Zpfbs $f_i : E_i \to E_{i+1}$. Let $\mathcal{T}_i$ be the standard cell-stratification of $B_i = \mathbb{P}^i(\mathbb{C})$, the strata being the orbits of the standard Borel subgroup of $\mathrm{GL}_{i+1}(\mathbb{C})$ under the natural action. Note that E_i is $2i$-acyclic (cf. [BL94, 3.1]). Thus $(B \xleftarrow{q} E, f)$ is an ABCD-approximation for $\mathbb{C}^*$ (for details see [Sch07, 5.6]). Taking the obvious product of this construction shows that any torus has an ABCD-approximation.

Now let G be a connected solvable group. Choose a maximal torus $T \subset G$ and let $U \subset G$ be the unipotent radical. Let $(B \xleftarrow{q^T} E^T, \mathcal{T}, f^T)$ be an ABCD-approximation for T. Define $E_i^G := \mathrm{ind}_T^G E_i^T = G \times_T E_i^T$, and let $f_i^G := \mathrm{ind}_T^G f_i^T$. The morphisms $G \times E_i^T \to B_i$, $(p, e) \mapsto q_i^T(e)$ induce G-Zpfbs $q_i^G : E_i^G \to B_i$. Since multiplication $U \times T \xrightarrow{\sim} G$ is an isomorphism we get an isomorphism of varieties $E_i^G = G \times_T E_i^T \xleftarrow{\sim} U \times E_i^T$. Since U is ∞-acyclic and E_i^T is i-acyclic, E_i^G is i-acyclic. So $(B \xleftarrow{q^G} E^G, \mathcal{T}, f^G)$ is an ABCD-approximation for G. □

Proposition 69. *Let G be an affine algebraic group and $(X, \mathcal{S})$ a G-stratified variety. Assume that*

(R1) $\mathcal{S}$ is a G-stratification into cells,
(R2) $\widetilde{\mathcal{IC}}(S)$ is $\mathcal{S}$-pure, for every $S \in \mathcal{S}$,
(R3) G has an ABCD-approximation,
(R4) there is a G-stratified variety $(Y, \widehat{\mathcal{S}})$ together with
  - *(a) a G-equivariant locally closed embedding $v : X \to Y$ satisfying $v(\mathcal{S}) := \{v(S) \mid S \in \mathcal{S}\} \subset \widehat{\mathcal{S}}$, and*
  - *(b) a G-equivariant closed embedding of Y in a smooth G-manifold M.*

Then $(X, \mathcal{S})$ has an ABCD-approximation.

Remark 70. Condition (R4) will be used for the proof of B3. Possibly it is redundant. It is satisfied for Schubert varieties and more generally for unions of Borel-orbits that are locally closed in the flag variety (take as Y and M the flag variety). For a normal projective G-variety Y and connected G, (R4) (b) is satisfied by [Sum74] or [Mum65].

Proof. If $(B \xleftarrow{r} E, f, \mathcal{T})$ is an ABCD-approximation for G, its i-th subdatum $(B_i \xleftarrow{r_i} E_i, \mathcal{T}_i)$ satisfies an obvious subset of the conditions A1-D.

So let $(B \xleftarrow{r} E, \mathcal{T})$ be the i-th subdatum of an ABCD-approximation for G. Consider the commutative diagram

(79)
$$\begin{tikzcd} E \times_G X & E \times X \arrow[l, "q"'] \arrow[r, "p"] \arrow[d, "\pi"] & X \arrow[d] \\ B \arrow[from=u, "\overline{\pi}"] & E \arrow[l, "r"'] \arrow[r, "c"] & \mathrm{pt.} \end{tikzcd}$$

Here q is the quotient map for the diagonal action of G on $E \times X$, p is the second projection, π the first projection and $\overline{\pi}$ the induced map on quotient spaces. We claim that the upper row of diagram (79) together with

$$r^{-1}(\mathcal{T}) \times_G \mathcal{S} := \{r^{-1}(T) \times_G S \mid T \in \mathcal{T}, S \in \mathcal{S}\}$$

satisfies the conditions imposed on the i-th subdatum of an ABCD-approximation of $(X, \mathcal{S})$.

The square on the left in (79) is cartesian, q is a Zpfb and $\overline{\pi}$ is a (Zariski locally trivial) fiber bundle with fiber X. These statements are also true in the classical topology. They can be deduced from local trivializations of r. If $\tau : G \times U \xrightarrow{\sim} r^{-1}(U)$ is such a trivialization (cf. subsection 6.2) over an open subvariety U of B, diagram (79) restricts to

$$\begin{tikzcd} U \times X \arrow[d, "\mathrm{pr}_U"] & G \times U \times X \arrow[l, "(u,g^{-1}x) \leftarrow\!\shortmid (g,u,x)"'] \arrow[r, "p"] \arrow[d, "\mathrm{pr}_{G\times U}"] & X \arrow[d] \\ U & G \times U \arrow[l, "\mathrm{pr}_U"'] \arrow[r, "c"] & \mathrm{pt.} \end{tikzcd}$$

Here we consider $U \times X$ as an open subvariety of $E \times_G X$, the inclusion given by $(u, x) \mapsto [\tau(1, u), x]$.

Since c is i-acyclic and smooth, the same holds for p (A1, B1). If $S \in \mathcal{S}$ is a stratum, the intersection of $E \times_G S = q(p^{-1}(S))$ with $U \times X \subset E \times_G X$ is $U \times S$. Hence $E \times_G \mathcal{S} := \{E \times_G S \mid S \in \mathcal{S}\}$ is a stratification of $E \times_G X$ (A2) (the irreducibility of the strata will be established below). The long exact sequence of homotopy groups for the fiber bundle $\overline{\pi} : E \times_G S \to B$ with fiber S shows that $E \times_G S$ is simply connected (A3), since S is a cell and B is simply connected by assumption.

Let $S \in \mathcal{S}$ and $T \in \mathcal{T}$ be strata. The intersection of $r^{-1}(T) \times_G S$ with $U \times X$ is $(T \cap U) \times S$, so $r^{-1}(\mathcal{T}) \times_G \mathcal{S}$ is a stratification of $E \times_G X$. By D, we find a local trivialization τ of r as above with $T \subset U$. Then $\tau \times \mathrm{id}_X$ is a local trivialization of q over $U \times X$, and $r^{-1}(T) \times_G S = T \times S \subset U \times X$ is a cell (D, C1).

By A2, B is irreducible. Let $T \in \mathcal{T}$ be dense in B. Then $r^{-1}(T)$ is dense in E and the cell $r^{-1}(T) \times_G S$ is dense in $E \times_G S$, for all $S \in \mathcal{S}$, showing the irreducibility of each stratum $E \times_G S$ (A2).

Let $S \in \mathcal{S}$. We prove C3. By Remark 33 is is sufficient to show that $\widetilde{\mathcal{IC}}(E \times_G S)$ is $(r^{-1}(\mathcal{T}) \times_G \mathcal{S})$-$*$-pure of weight $d_B + d_S$. Let $R \in \mathcal{S}$, $T \in \mathcal{T}$. We choose $U \subset B$ open as above containing T. Then the inclusion $r^{-1}(T) \times_G R \overset{l}{\hookrightarrow} E \times_G X$ looks like $T \times R \overset{t \times l_R}{\hookrightarrow} U \times X \overset{j}{\hookrightarrow} E \times_G X$. Since j is an open embedding, j^* preserves weights and we obtain

$$j^*(\widetilde{\mathcal{IC}}(E \times_G S)) \cong \widetilde{\mathcal{IC}}(U \times S) \cong \widetilde{\mathcal{IC}}(U) \boxtimes \widetilde{\mathcal{IC}}(S) \tag{80}$$

Let $l_T : T \hookrightarrow B$ be the inclusion and $\widetilde{\mathcal{IC}}(B)$ the Hodge intersection cohomology sheaf on B. Then $l_T^*(\widetilde{\mathcal{IC}}(B)) \cong t^*(\widetilde{\mathcal{IC}}(U))$, and restriction of (80) yields

$$l^*(\widetilde{\mathcal{IC}}(E \times_G S)) \cong t^*(\widetilde{\mathcal{IC}}(U)) \boxtimes l_R^*(\widetilde{\mathcal{IC}}(S)) \cong l_T^*(\widetilde{\mathcal{IC}}(B)) \boxtimes l_R^*(\widetilde{\mathcal{IC}}(S))$$

which is pure of weight $d_B + d_S$ by assumptions C3 and (R2).

Let $(B \overset{r}{\leftarrow} E, f, \mathcal{T})$ be an ABCD-approximation for G. Let

$$(E_i \times_G X \overset{q_i}{\longleftarrow} E_i \times X \overset{p_i}{\longrightarrow} X, r_i^{-1}(\mathcal{T}_i) \times_G \mathcal{S})$$

be the datum constructed from its i-th subdatum by the above method. We claim that the sequence of these data together with the sequence of morphisms $f_i \times \mathrm{id}_X : E_i \times X \to E_{i+1} \times X$ defines an ABCD-approximation of $(X, \mathcal{S})$. Conditions B2 and C2 are obviously satisfied. A slight modification of the above arguments shows that $(E_i \times_G Y, E_i \times_G \widehat{\mathcal{S}})$ is a stratified variety. Consider the diagram

$$\begin{tikzcd}
E_{i+1} \times_G X \arrow[r, "\mathrm{id} \times_G v"] & E_{i+1} \times_G Y \arrow[r] & E_{i+1} \times_G M \\
E_i \times_G X \arrow[r, "\mathrm{id} \times_G v"] \arrow[u, "f_i \times_G \mathrm{id}_X"] & E_i \times_G Y \arrow[r] \arrow[u, "f_i \times_G \mathrm{id}_Y"] & E_i \times_G M, \arrow[u, "f_i \times_G \mathrm{id}_M"]
\end{tikzcd}$$

where both squares are cartesian. In the smooth manifold $E_{i+1} \times_G M$, the smooth submanifold $E_i \times_G M$ is transverse to each stratum of the closed stratified variety $(E_{i+1} \times_G Y, E_{i+1} \times_G \widehat{\mathcal{S}})$. It follows from [GM88, I.1.11] that condition B3 is satisfied. (For each $S \in \mathcal{S}$, $f_i \times \mathrm{id}_{\overline{S}}$ obviously is a normally nonsingular inclusion of the same codimension as f_i. If this implies the same statement on quotient spaces, we can do without (R4).) □

6.6. **Formality of Equivariant Flag Varieties.** Let $G \supset P \supset B$ be respectively a complex connected reductive affine algebraic group, a parabolic and a Borel subgroup.

Theorem 71. *If $\mathcal{S}$ is the stratification of G/P into B-orbits, there is a t-exact equivalence of triangulated categories*

$$\mathcal{D}^{\mathrm{b}}_{B,c}(G/P) = \mathcal{D}^{\mathrm{b}}_B(G/P, \mathcal{S}) \cong \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}_B(\mathcal{S}))).$$

Restriction to the hearts induces an equivalence

$$\mathrm{Perv}_B(G/P) = \mathrm{Perv}_B(G/P, \mathcal{S}) \cong \mathrm{dgFlag}(\mathrm{Ext}(\mathcal{IC}_B(\mathcal{S}))).$$

Proof. The B-stratified variety $(G/P, \mathcal{S})$ has an ABCD-approximation by Theorem 36 and Propositions 68 and 69. Hence we can apply Theorem 67. The equalities follow from Proposition 63. □

Remark 72. The strategy from subsection 3.13 also shows that the complexified version of diagram (38) commutes. This implies that Theorem 71 is also true for complex coefficients.

Remark 73. We claim that the diagram

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{b}}_B(G/P, \mathcal{S}) \arrow[r, "\sim"] \arrow[d, "\mathrm{For}"] & \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}_B(\mathcal{S}))) \arrow[d, "? \overset{L}{\otimes}_{\mathrm{Ext}(\mathcal{IC}_B(\mathcal{S}))} \mathrm{Ext}(\mathcal{IC}(\mathcal{S}))"] \\
\mathcal{D}^{\mathrm{b}}(G/P, \mathcal{S}) \arrow[r, "\sim"] & \mathrm{dgPer}(\mathrm{Ext}(\mathcal{IC}(\mathcal{S})))
\end{tikzcd} \tag{81}$$

is commutative (up to natural isomorphism). The horizontal equivalences are those from Theorems 71 and 37, the vertical functors are the forgetful functor and the induced extension of scalars functor.

The upper horizontal equivalence was established as the inverse limit of a sequence of equivalences $(\mathrm{Form}_n)_{n \in \mathbb{N}}$ of categories (cf. proof of Theorem 67). The lower horizontal equivalence can be chosen equal to Form_0. (If we use our method for constructing the ABCD-approximation for $(G/P, \mathcal{S})$ we have $E_0 = B \times_T T \times G/P = B \times G/P$ for $T \subset B$ a maximal torus, and $B \backslash E_0 = G/P$ has the stratification $\mathcal{S}_0 = \mathcal{S}$.) Hence diagram (81) is commutative.

Since all functors in diagram (81) are triangulated and t-exact, we obtain by restriction a commutative diagram relating the hearts.

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