The map (1) is defined by passing to the Hochschild homology map induced by this functor (the original definition of [1] used a different analytic approach).

Up until recently the explicit computations of the FJRW classes were only done in the so called *concave case*, i.e., when certain line bundles on the universal curve over $S_g(\gamma_1, \ldots, \gamma_n)$ have no global sections when restricted to each particular curve. In the talk I discussed the recent work of Guéré [2] where the FJRW classes were calculated in many nonconcave cases with W being *invertible* (i.e., such that the number of monomials in W is equal to the number of variables).

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Matrix factorizations, semiorthogonal decompositions, and motivic measures

Olaf M. Schnürer

(joint work with Valery A. Lunts)

Let k be an algebraically closed field of characteristic zero. The Grothendieck group $K_0(\mathrm{Var}_k)$ of varieties over k is the free abelian group on isomorphism classes $[X]$ of varieties X over k modulo the subgroup generated by the "scissor relations" $[X] - [X \setminus Y] - [Y]$ whenever Y is a closed subvariety of a variety X over k. It becomes a commutative unital ring by defining $[X] \cdot [Y] = [X \times Y]$. In order to understand this Grothendieck ring of k-varieties better one may construct **motivic measures**, i. e. morphisms of rings from $K_0(\mathrm{Var}_k)$ to some other ring.

Consider the map that sends a smooth projective k-variety X to its bounded derived category $D^b(\mathrm{Coh}(X))$ of coherent sheaves. A beautiful result due to A. Bondal, M. Larsen and V. Lunts says that this map can be turned (uniquely) into a motivic measure $K_0(\mathrm{Var}_k) \to K_0(\mathrm{sat})$ if one replaces $D^b(\mathrm{Coh}(X))$ by its "injective" enhancement (see [2]). Here $K_0(\mathrm{sat})$ denotes the Grothendieck group of saturated (= proper, smooth, and triangulated) differential $\mathbb{Z}$-graded (k-)categories with relations coming from semiorthogonal decompositions. Its ring structure is induced by the tensor product of differential $\mathbb{Z}$-graded categories (and by passing to the triangulated envelope).

Our aim is to establish a similar motivic measure using categories of matrix factorizations. Let X be a smooth quasi-projective variety over k together with a morphism $W\colon X \to \mathbb{A}^1 = \mathbb{A}^1_k$. We define the **category of singularities of** W as

$$\mathbf{MF}(W) = \prod_{a \in k} \mathbf{MF}(X, W - a).$$

Here $\mathbf{MF}(X, W-a)$ is the category of (global) matrix factorizations of $W - a$ on X. We have $\mathbf{MF}(W) = 0$ if and only if W is smooth. We denote by $\mathbf{MF}(W)^{\mathrm{dg}}$

a suitable enhancement (in the differential $\mathbb{Z}_2$-graded setting) of $\mathbf{MF}(W)$ (for example defined using injective quasi-coherent sheaves), and by $\mathbf{MF}(W)^{\mathrm{dg},\natural}$ the triangulated envelope of $\mathbf{MF}(W)^{\mathrm{dg}}$.

Consider the Grothendieck group $K_0(\mathrm{Var}_{\mathbb{A}^1})$ of varieties over $\mathbb{A}^1$ defined similarly as the group $K_0(\mathrm{Var}_k)$ above. It is turned into a commutative unital ring by defining

$$[X \xrightarrow{W} \mathbb{A}^1] \cdot [Y \xrightarrow{V} \mathbb{A}^1] := [X \times Y \xrightarrow{W*V} \mathbb{A}^1]$$

where $(W*V)(x,y) = W(x)+V(y)$. On the other hand we consider the Grothendieck ring $K_0(\mathrm{sat}_2)$ of saturated differential $\mathbb{Z}_2$-graded categories defined similarly as $K_0(\mathrm{sat})$ above. Now we can state our main theorem.

Theorem 1 (see [4])**.** There is a unique morphism

$$K_0(\mathrm{Var}_{\mathbb{A}^1}) \to K_0(\mathrm{sat}_2)$$

of rings (= a Landau-Ginzburg motivic measure) that maps $[X \xrightarrow{W} \mathbb{A}^1]$ to the class of $\mathbf{MF}(W)^{\mathrm{dg},\natural}$ whenever X is a smooth variety and $W\colon X \to \mathbb{A}^1$ is a proper morphism.

We prove first that $\mathbf{MF}(W)^{\mathrm{dg},\natural}$ is indeed saturated if X is a smooth variety and $W\colon X \to \mathbb{A}^1$ is a proper morpism. Additivity is based on an alternative description of $K_0(\mathrm{Var}_{\mathbb{A}^1})$ in terms of "blow-up relations" (see [1]) and on semiorthogonal decompositions for categories of matrix factorizations on blowing-ups and projective space bundles (see [3] and below). Multiplicativity needs a Thom-Sebastiani result for such categories of singularities and some compactification argument.

Let us explain the semiorthogonal decompositions obtained from blowing-ups in more detail. Let $\pi\colon \widetilde{X} \to X$ be the blowing-up of a smooth quasi-projective variety X along a smooth connected closed subvariety Y of codimension r. Let $j\colon E \hookrightarrow \widetilde{X}$ be the inclusion of the exceptional divisor, and let $p\colon E \to Y$ be the obvious morphism. The usual construction of the blowing-up endows $\widetilde{X}$ with a line bundle $\mathcal{O}_{\widetilde{X}}(1)$. We denote its restriction to E by $\mathcal{O}_E(1)$. Let $W\colon X \to \mathbb{A}^1$ be a morphism. Denote its pullback functions to Y and $\widetilde{X}$ by the same symbol. The following theorem is the main result of the article [3] and the analog of a well-known result for bounded derived categories of coherent sheaves.

Theorem 2 (see [3])**.** The category $\mathbf{MF}(\widetilde{X}, W)$ has the following semiorthogonal decomposition into admissible subcategories,

$$\begin{aligned}\mathbf{MF}(\widetilde{X}, W) = \big\langle & j_*(\mathcal{O}_E(-r+1) \otimes p^*(\mathbf{MF}(Y,W))), \dots, \\ & j_*(\mathcal{O}_E(-1) \otimes p^*(\mathbf{MF}(Y,W))), \pi^*(\mathbf{MF}(X,W))\big\rangle.\end{aligned}$$

In our talk we also discussed the relation between the motivic measure from [2] and the Landau-Ginzburg motivic measure from Theorem 1. For more details and our future plans we refer the reader to the articles [3, 4].

References

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Matrix factorizations and homological projective duality in physics

Eric Sharpe

(joint work with Tony Pantev, others)

'Gauged linear sigma models' (GLSM's) are one of the central tools used by physicists to describe strings propagating on spaces. They were originally developed about twenty years ago by E. Witten [1], but have recently undergone a revolution. For example, prior to around 2007, it was believed that gauged linear sigma models

- could only describe geometries presented as global complete intersections,
- in which those geometries were realized as the critical locus of a 'superpotential,'
- and any two geometries related by a GLSM were necessarily birational.

Over the last few years, counterexamples to all of these claims have been found (see *e.g.* [2, 3, 4] for some early work), and the more subtle ideas replacing them revolve around aspects of Kuznetsov's homological projective duality [5, 6, 7].

In this talk we will give a basic introduction to some of these phenomena and their consequences, largely following [4]. Instead of working with GLSM's, we will instead translate to 'Landau-Ginzburg (LG) models,' which are defined by a complex Kähler manifold X together with a holomorphic function $W : X \to \mathbb{C}$ known as the superpotential. Another set of theories, known as 'nonlinear sigma models' (NLSM's), are defined just by specifying just a complex Kähler manifold, without a superpotential. String propagation on a space is described by a nonlinear sigma model. Given a Landau-Ginzburg model, we can sometimes (though not always) construct a nonlinear sigma model by an operation called 'renormalization group flow,' which generates an effective theory describing just the low-energy fluctuations of the Landau-Ginzburg model.

As a warm-up, let us describe a Landau-Ginzburg model associated to a quintic Calabi-Yau hypersurface in $\mathbb{P}^4$. The Landau-Ginzburg model is defined on

$$\mathrm{Tot}\left(\mathcal{O}(-5) \xrightarrow{\pi} \mathbb{P}^4\right),$$

with superpotential $W = p\pi^* s$, $s \in \Gamma(\mathcal{O}(5))$, p a fiber coordinate. This theory contains a potential V of the form

$$V = |dW|^2 = |s|^2 + |pds|^2,$$