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Equivariant Sheaves on Flag Varieties

Olaf M. Schnürer

The aim of our talk was to give an algebraic description of the Borel-equivariant derived category of sheaves on the flag variety of a connected reductive algebraic group.

Let G be a complex algebraic group acting on a complex variety X. We introduced the G-equivariant (bounded, constructible) derived category $\mathcal{D}^b_{G,c}(X)$ of sheaves of real or complex vector spaces on X (see [BL94]). It carries the perverse t-structure with heart the category of G-equivariant perverse sheaves. If G acts with finitely many orbits, there are only finitely many simple objects in this heart; we denote their direct sum by $\mathcal{IC}$. The extension algebra of this object is

$$\operatorname{Ext}(\mathcal{IC}) := \bigoplus_{n\in\mathbb{N}} \operatorname{Hom}(\mathcal{IC}, \mathcal{IC}[n]).$$

We view this graded algebra as a differential graded (dg) algebra with differential $d = 0$.

Let $\mathcal{A}$ be a dg algebra. We defined the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ (see e. g. [Kel98]). The perfect derived category $\operatorname{Perf}(\mathcal{A})$ of $\mathcal{A}$ is the thick subcategory of $\mathcal{D}(\mathcal{A})$ generated by $\mathcal{A}$ (i. e. the smallest full triangulated subcategory that contains $\mathcal{A}$ and is closed under taking direct summands). Its objects are precisely the compact objects in $\mathcal{D}(\mathcal{A})$.

The following conjecture of Soergel and Lunts (cf. [Lun95]) relates the geometric category $\mathcal{D}^b_{G,c}(X)$ and the algebraic category $\mathrm{Perf}(\mathrm{Ext}(\mathcal{IC}))$: If a complex reductive group G acts on a projective variety X with finitely many orbits, there is an equivalence of triangulated categories

$$\mathcal{D}^b_{G,c}(X) \cong \mathrm{Perf}(\mathrm{Ext}(\mathcal{IC})).$$

This conjecture (or a similar statement) is known to be true for a connected Lie group acting on a point ([BL94, 12.7.2]), for a torus acting on an affine or projective normal toric variety ([Lun95]), and for a complex semisimple adjoint group acting on a smooth complete symmetric variety (in the sense of de Concini and Procesi) ([Gui05]). We recently became aware of a related result for the loop rotation equivariant derived Satake category of the affine loop Grassmannian in [BF08]. Our main result is:

Theorem 1 ([Sch08])**.** Let G be a complex connected reductive affine algebraic group, $B \subset G$ a Borel subgroup, and $X = G/B$ the flag variety. Then there is an equivalence of triangulated categories

$$\mathcal{D}^b_{B,c}(X) \cong \mathrm{Perf}(\mathrm{Ext}(\mathcal{IC})).$$

We conclude with some remarks:

- Note that $\mathcal{D}^b_{B,c}(X)$ is equivalent to $\mathcal{D}^b_{G,c}(G \times_B X)$ or $\mathcal{D}^b_{G,c}(X \times X)$ by the induction equivalence. Hence our result fits into the setting of the conjecture.
- The perverse t-structure on $\mathcal{D}^b_{B,c}(X)$ corresponds to a t-structure on the perfect derived category $\mathrm{Perf}(\mathrm{Ext}(\mathcal{IC}))$ that can be described for a more general class of dg algebras (see [Sch08a]). This yields an algebraic description of the category of B-equivariant perverse sheaves on X.
- The algebra $\mathrm{Ext}(\mathcal{IC})$ is isomorphic to the endomorphism algebra of the B-equivariant hypercohomology of $\mathcal{IC}$ ([Soe01]); this hypercohomology can be described using Soergel's bimodules or the moment graph picture ([BM01]). In particular, the category $\mathcal{D}^b_{B,c}(X)$ depends only on the corresponding combinatorial data.
- The non-equivariant analog of this theorem is also true. In fact, we prove the theorem as a limit of equivalences that are similar to the non-equivariant analog. For the proof of the non-equivariant version we need the formality of a carefully constructed dg algebra; to obtain this formality we use mixed Hodge modules and purity results on intersection cohomology complexes.

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MV-polytopes/cycles and affine buildings

Michael Ehrig

1. Basic Notations and Definitions

We want to give a combinatorial construction of MV-polytopes. This is done by using the LS-gallery model by Gaussent and Littelmann [1], a discrete and building-theoretic version of Littelmann's path model. This gives a construction of MV-polytopes alternativ to the one given by Kamnitzer in [2] and [3] and independent of the type of the algebraic group. We start by fixing the basic notations.

Notation 1.1. By G we denote a complex, simply-connected, semi-simple algebraic group. We fix $B \subset G$ a Borel subgroup, $T \subset B$ a maximal torus, and denote by W its Weyl group. In addition we denote by B^- the Borel subgroup opposite to B, i.e., the Borel subgroup such that $B \cap B^- = T$, and by U^- its unipotent radical. Finally we denote by $\mathcal{O} = \mathbb{C}[[t]]$ the ring of formal power series and by $\mathcal{K} = \mathbb{C}((t))$ its field of fraction, the field of formal Laurent series.

Using these we have a number of associated objects.

Notation 1.2. Let us denote by $\mathcal{G} = G(\mathcal{K})/G(\mathcal{O})$ the affine Grassmannian, by $X^\vee$ the coweight lattice of G, and by $X^\vee_+$ the dominant coweights.

Let us now look at the basic geometric set-up:.

$$X^\vee \xhookrightarrow{\ i\ } \mathcal{G} \hookrightarrow \mathbb{P}(V) \xrightarrow{\ \boldsymbol{\mu}\ } X^\vee \otimes \mathbb{R}\ .$$

The inclusion i is an inclusion as T-fixed points and we denote the image of a coweight λ by t^λ, $\mathbb{P}(V)$ is a projective space over a suitable representation of the affine Kac-Moody group $\hat{\mathcal{L}}(G)$ corresponding to G, and the map $\boldsymbol{\mu}$ is its usual moment map.