QUASIMAPS TO MODULI SPACES OF SHEAVES ON A K3 SURFACE

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Abstract. In this article, we study quasimaps to moduli spaces of sheaves on a K3 surface S. We construct a surjective cosection of the obstruction theory of moduli spaces of ϵ-stable quasimaps. We then establish reduced wall-crossing formulas which relate reduced Gromov–Witten theory of moduli spaces of sheaves on S and reduced Donaldson–Thomas theory of $S \times C$, where C is a nodal curve.

As applications, we prove the wall-crossing part of Igusa cusp form conjecture; higher-rank/rank-one relative Donaldson–Thomas correspondence on $S \times C$, if $g(C) \leq 1$; relative Donaldson–Thomas/Pandharipande–Thomas correspondence on $S \times \mathbb{P}^1$.

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1. Introduction

1.1. **Overview.** In [Nes21], ϵ-stable quasimaps to a moduli space of sheaves on a surface S were introduced. When applied to a punctorial Hilbert scheme of points $S^{[n]}$, moduli spaces of ϵ-stable quasimaps $Q^{\epsilon}_{g,N}(S^{[n]}, \beta)$ interpolate between moduli spaces of stable maps to $S^{[n]}$ and moduli spaces of 1-dimensional subschemes on $S \times C$,

$$\overline{M}_{g,N}(S^{[n]}, \beta) \overset{\epsilon}{\leftarrow\!-\!-\!\rightarrow} \mathrm{Hilb}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}), \tag{1}$$

under variation of the stability parameter $\epsilon \in \mathbb{R}_{>0}$. Using Zhou's master-space technique, moduli spaces of ϵ-stable quasimaps therefore lead to wall-crossing formulas, which relate Gromov–Witten (GW) theory of $S^{[n]}$ and relative Donaldson–Thomas (DT) theory of $S \times C_{g,N}/\overline{M}_{g,N}$.

The case of moduli spaces of sheaves on a K3 surface requires a special treatment due to the presence of a holomorphic symplectic form and, consequently, vanishing of the standard virtual fundamental class of $\overline{M}_{g,N}(S^{[n]},\beta)$. In more concrete terms, the vanishing is due to existence of a surjective cosection of the obstruction-theory complex $\mathbb{E}^\bullet$,

$$\sigma\colon \mathbb{E}^\bullet \twoheadrightarrow \mathcal{O}[-1].$$

A non-trivial *reduced* enumerative theory is obtained by taking the cone of σ. The same phenomenon happens on DT side - obstruction theories of moduli spaces $\mathrm{Hilb}_{n,\check{\beta}}(S\times C_{g,N}/\overline{M}_{g,N})$ admit surjective cosections, therefore *reduction* is also necessary. In order to compare *reduced* GW theory of $S^{[n]}$ and *reduced* DT theory of $S\times C_{g,N}/\overline{M}_{g,N}$, we have to furnish $Q^{\epsilon}_{g,N}(S^{[n]},\beta)$ with a surjective cosection and, consequently, with a *reduced* obstruction theory - this is the principle aim of the present work.

Once the reduced wall-crossing formula is established in Theorem 3.3, we proceed to prove following results:

- (reduced) quantum cohomology of $S^{[n]}$ is determined by relative Pandharipande–Thomas theory of $S\times\mathbb{P}^1$, if S is a K3 surface, conjectured in [Obe19];
- the wall-crossing part of Igusa cusp form conjecture, conjectured in [OP10];
- relative higher-rank/rank-one DT correspondence for $S\times\mathbb{P}^1$ and $S\times E$, if S is a K3 surface and E is an elliptic curve;
- relative Donaldson–Thomas/Pandharipande–Thomas correspondence for $S\times\mathbb{P}^1$, if S is a K3 surface.

1.1.1. *Vanishing.* Let S be a K3 surface and M be a projective moduli space of stable sheaves on S. To give a short motivation for our forthcoming considerations, let us recall the origin of reduced perfect obstruction theory of GW theory of M. Since M is hyper-Kähler, for any algebraic curve class $\beta\in H_2(M,\mathbb{Z})$ there exists a first-order *twistor family*

$$\mathcal{M}\to \operatorname{Spec}\mathbb{C}[\epsilon]/\epsilon^2$$

of M, for which the horizontal lift of β is of (k,k)-type only at the central fiber. In particular, the standard GW invariants vanish. To get a non-trivial enumerative theory, we have to remove obstructions that arise via such deformations of M. However, in the case of ϵ-stable quasimaps, we need twistor families not only of the moduli space M but of the pair $(M,\mathfrak{Coh}_r(S)_{\mathbf{v}})$. Such twistor families can be given by non-commutative deformations of S. Let us now elaborate on this point by slightly changing the point of view.

1.1.2. *Cosection.* For simplicity, assume $M=S^{[1]}=S$. A map $f\colon C\to S$ of degree β is determined by its graph on $S\times C$. Let I be the associated ideal sheaf of this graph. The deformation theories of I, as a sheaf with fixed determinant, and of f are equivalent. Assuming C is smooth and $\beta\neq 0$, the existence of a first-order twistor family associated to the class β is therefore equivalent to the surjectivity of the following composition

$$H^1(T_S) \hookrightarrow H^1(T_{S\times C}) \xrightarrow{\cdot \mathrm{At}(I)} \mathrm{Ext}^2(I,I)_0 \xrightarrow{\sigma_I} H^3(\Omega^1_{S\times C}) \cong \mathbb{C}, \qquad (2)$$

i.e. to the existence of a class $\kappa_\beta \in H^1(T_S)$ whose image is non-zero with respect to the composition above, where $\sigma_I := \mathrm{tr}(* \cdot -\mathrm{At}(I))$ for the Atiyah class $\mathrm{At}(I) \in \mathrm{Ext}^1(I, I \otimes \Omega^1_{S\times C})$. To see this, recall that the second map gives the obstruction to deform I along a first-order deformation $\kappa \in H^1(T_S)$, while the third map, called *semiregularity map* [BF03], relates obstructions of deforming I to the obstructions of

$$\mathrm{ch}_2(I) = (-\beta, 1) \in H^4(S\times C, \mathbb{Z}) = H^2(S,\mathbb{Z}) \oplus \mathbb{Z}$$

to stay of (k,k)-type. With these interpretations in mind, it is not difficult to grasp that κ_β is indeed our first-order twistor family associated to β.

The semiregularity map σ_I globalises, i.e. there exists a cosection

$$\sigma \colon \mathbb{E}^\bullet \twoheadrightarrow \mathcal{O}[-1]$$

of the obstruction-theory complex of the moduli space of ideals $\mathcal{M}$ on $S\times C$. This cosection σ is surjective by the existence of first-order twistor families, if the second Chern character of ideals is equal to (β, n) for $\beta \neq 0$. By localisation-by-cosection technique introduced by Kiem–Li [KL13], the standard virtual fundamental class therefore vanishes. To make the enumerative theory non-trivial, we have to consider the reduced obstruction-theory complex $\mathbb{E}_{\mathrm{red}} := \mathrm{cone}(\sigma)[-1]$. Proving that $\mathbb{E}_{\mathrm{red}}$ defines an obstruction theory

$$\mathbb{T}_{\mathcal{M}} \to \mathbb{E}_{\mathrm{red}}$$

is sometimes difficult. Instead, [KL13] provides a construction of the reduced virtual fundamental class without an obstruction theory.

Let us come back to the case of a general moduli space of sheaves M. By construction of M, the deformation theory of quasimaps to M is equivalent to the one of sheaves on threefolds of the type $S\times C$, see [Nes21, Proposition 5.5)] for more details. The obstruction theory of higher-rank sheaves on $S\times C$ also admits a cosection given by the semiregularity map. We want to show it is surjective. However, already for $M = S^{[n]}$ with $n > 1$, there is a problem with the argument presented above. If the degree of $f \colon C \to S^{[n]}$ is equal to a multiple of the exceptional curve class[1], then (2) is zero. Indeed, in this case $\mathrm{ch}_2(I) = (0, n)$ and the composition (2) is equal to the contraction $\langle -, \mathrm{ch}_2(I)\rangle$, which therefore pairs trivially with classes in $H^1(T_S)$. The geometric interpretation of this issue is that the exceptional curve class of $S^{[n]}$ stays Hodge along the commutative deformations of S, because punctorial Hilbert schemes deform to punctorial Hilbert schemes

[1]The curve class dual to a multiple of the exceptional divisor associated to the resolution of singularities $S^{[n]} \to S^{(d)}$.

under commutative deformations of S. To fix the argument, we have to consider classes not only in $H^1(T_S)$, but in a larger space

$$H^0(\wedge^2 T_S) \oplus H^1(T_S) \oplus H^2(\mathcal{O}_S),$$

i.e. we have to consider non-commutative first-order twistor families to prove the surjectivity of the semiregularity map.

1.1.3. *Strategy.* For surjectivity of the semiregularity map, we will largely follow [BF03, Section 4] and [MPT10, Proposition 11] with few extra layers of complication. Firstly, since our threefold $S \times C$ might be singular (because C is nodal), we have to consider Atiyah classes valued in $\Omega^1_S \boxplus \omega_C$,

$$\mathrm{At}_\omega(F) \in \mathrm{Ext}^1(F, F \otimes (\Omega^1_S \boxplus \omega_C)),$$

instead of $\Omega^1_S \boxplus \mathbb{L}_C = \Omega^1_S \boxplus \Omega^1_C$, as the latter does not behave well under degenerations. Chern characters of sheaves are then defined via the Atiyah class of the form as above. Secondly, after establishing the expected correspondence between degrees of quasimaps and Chern characters of sheaves, we allow contractions with classes in $H^0(\wedge^2 T_S) \oplus H^1(T_S) \oplus H^2(\mathcal{O}_S)$ instead of only $H^1(T_S)$, unlike in [BF03, Section 4]. Proposition 2.3 is a vast extension of [MPT10, Proposition 11] and implies surjectivity of the global semiregularity map, Corollary 3.2.

Having constructed a surjective cosection of the obstruction theory, ideally one would like to reduce the obstruction theory. However, due to the involvement of non-commutative geometry in our considerations, we can reduce the obstruction theory only under a certain assumption, which is nevertheless not unnatural, see Proposition A.1 for more details. However, we do not use our reduced obstruction theory for the construction of the reduced virtual fundamental class due to the limitations of our assumption. We instead choose to work with reduced classes of [KL13].

1.2. **Applications of the quasimap wall-crossing.**

1.2.1. *Enumerative geometry of* $K3^{[n]}$. In [Obe21b], the wall-crossing terms are shown to be virtual Euler numbers of certain Quot schemes, which are computed for $S^{[n]}$, if S is a K3 surface. Therefore, using the results of [Obe21b] together with reduced quasimap wall-crossing for $S^{[n]}$, we obtain the wall-crossing part of the *Igusa cusp form conjecture* [OP16, Conjecture A], thereby completing the proof of the conjecture along with [OS20] and [OP18].

Genus-0 3-point Gromov–Witten theory of $S^{[n]}$ is shown to be equivalent to Pandharipande–Thomas (PT) theory of $S \times \mathbb{P}^1$ with three relative vertical insertions. Together with PT/GW correspondence of [Obe21a], this confirms the conjecture proposed in [Obe19].

In [Obe22], a holomorphic anomaly equation is established for $S^{[n]}$ for genus-0 GW invariants with at most 3 markings. The proof crucially uses the quisimap wall-crossing, which relates genus-0 GW invariants of $S^{[n]}$ to PT invariants of $S \times \mathbb{P}^1$ and then to GW invariants of $S \times \mathbb{P}^1$ by [Obe21a].

1.2.2. *Higher-rank/rank-one DT wall-crossing for $K3 \times C$.* Assume M satisfies various assumptions of [Nes21] which are listed in Section 3.1. Since M is deformation equivalent to $S^{[n]}$, we can prove certain higher-rank/rank-one DT wall-crossing statements for threefolds $S \times C$, using the quasimap wall-crossing on both sides, as it is represented in Figure 1.

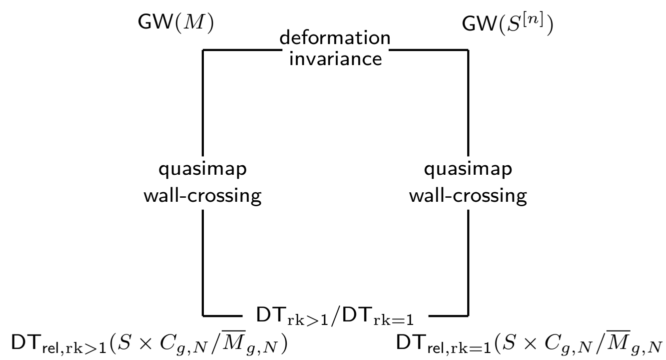

FIGURE 1. Higher-rank/rank-one DT

If $g = 0, N = 3$, the wall-crossing is trivial. We therefore obtain that higher-rank invariants with three relative vertical insertions associated to moduli spaces of sheaves which are stable at a general fiber, exactly match rank-one invariants on $S \times \mathbb{P}^1$,

$$\mathsf{DT}_{\mathsf{rel},\mathrm{rk}=1}(S \times \mathbb{P}^1/S_{0,1,\infty}) = \mathsf{GW}_{0,3}(S^{[n]}) = \mathsf{DT}_{\mathsf{rel},\mathrm{rk}>1}(S \times \mathbb{P}^1/S_{0,1,\infty}),$$

where $S_{0,1,\infty} = S \times \{0, 1, \infty\} \subset S \times \mathbb{P}^1$. We want to stress that equality above is equality of invariants, not just generating series.

However, the result is not optimal, since stability of sheaf at a general fiber over a curve is shown to be equivalent to stability of the sheaf only under some assumptions. Namely, as is shown in [Nes21, Proposition A.4], we require $\mathrm{rk} \leq 2$ and M to be a projective moduli space of *slope* stable sheaves. However, if the assumptions of [Nes21, Proposition A.4] are satisfied, we really get the equality of standard DT invariants associated to moduli of stable sheaves. An example of such M is discussed in Remark 3.1.

In the case of $S \times E$, where E is an elliptic curve, we get a wall-crossing statement for absolute invariants and equality of certain relative invariants.

1.2.3. *DT/PT correspondence for $K3 \times C$.* Using both standard and perverse quasimap wall-crossings, we can reduce Donaldson–Thomas/Pandharipande–Thomas correspondence (DT/PT) for a relative geometry of the form

$$S \times C_{g,N} \to \overline{M}_{g,N}$$

to DT/PT of wall-crossing invariants, as it is illustrated in Figure 2.

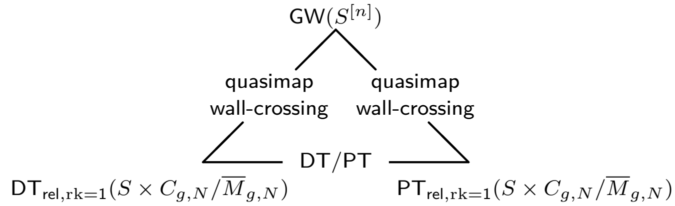

FIGURE 2. DT/PT

As before, if $g = 0, N = 3$, then the wall-crossing is trivial. We therefore obtain the following

$$\mathsf{DT}_{\mathsf{rel},\mathrm{rk}=1}(S \times \mathbb{P}^1/S_{0,1,\infty}) = \mathsf{GW}_{0,3}(S^{[n]}) = \mathsf{PT}_{\mathsf{rel},\mathrm{rk}=1}(S \times \mathbb{P}^1/S_{0,1,\infty}).$$

Again, equality above is equality of invariants, not just generating series. Such equality can be reasonably expected due to the nature of *reduced* virtual fundamental classes. In general, one can only expect equality of certain generating series, which also account for the wall-crossing, as it is conjectured in [PT09, Conjecture 3.3] for Calabi–Yau threefolds and proven in [Bri11, Theorem 1.1].

Note that we are in the setting of non-Calabi-Yau relative geometry, hence the techniques of wall-crossings in derived categories of [KS08] and [JS12] cannot be applied to prove wall-crossing statements as above.

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1.4. **Notation and conventions.** We work over the field of complex numbers $\mathbb{C}$. By $S^{[n]}$ we denote the Hilbert scheme of length-n points on a surface S.

By convention we set $e_{\mathbb{C}^*}(\mathbb{C}_{\mathrm{std}}) = -z$, where $\mathbb{C}_{\mathrm{std}}$ is the standard representation of $\mathbb{C}^*$ on a vector space $\mathbb{C}$.

Let N be a semigroup and $\beta \in N$ be its generic element. By $\mathbb{Q}[\![q^\beta]\!]$ we will denote the (completed) semigroup algebra $\mathbb{Q}[\![N]\!]$. In our case, N will be various semigroups of effective curve classes.

After fixing an ample line bundle $\mathcal{O}_S(1)$ on a surface S, we define $\deg(F)$ to be the degree of a sheaf F with respect to $\mathcal{O}_S(1)$. By a *general fiber* of a sheaf F on $S \times C$, we will mean a fiber of F over some dense open subset of C.

2. Semiregularity map

2.1. **Preliminaries.** In what follows, we assume S to be a K3 surface. Let F be a sheaf on $S \times C$ flat over a nodal curve C, such that fibers of F have Chern character $\mathbf{v} \in H^*(S, \mathbb{Q})$. We start with some preparations. Consider the Atiyah class

$$\mathrm{At}(F) \in \mathrm{Ext}^1(F, F \otimes \Omega^1_{S\times C}),$$

represented by the canonical exact sequence

$$0 \to F \otimes \Omega^1_{S\times C} \to \mathcal{P}^1(F) \to F \to 0,$$

where $\mathcal{P}^1(F)$ is the sheaf of principle parts. Composing the Atiyah class with the natural map

$$\Omega^1_{S\times C} = \Omega^1_S \boxplus \Omega^1_C \to \Omega^1_S \boxplus \omega_C,$$

we obtain a class

$$\mathrm{At}_\omega(F) \in \mathrm{Ext}^1(F, F \otimes (\Omega^1_S \boxplus \omega_C)).$$

We then define the Chern character of a sheaf F on $S \times C$ for possibly singular C as follows

$$\mathrm{ch}_k(F) := \mathrm{tr}\left(\frac{(-1)^k}{k!}\mathrm{At}_\omega(F)^k\right) \in H^k(\wedge^k(\Omega^1_S \boxplus \omega_C)). \tag{3}$$

If C is smooth, it agrees with the standard definition of the Chern character. Using the canonical identification $H^1(\omega_C) \cong \mathbb{C}$ and

$$\wedge^k(\Omega^1_S \boxplus \omega_C) \cong \Omega^k_S \boxplus (\Omega^{k-1}_S \boxtimes \omega_S),$$

we get a Künneth's decomposition of the cohomology

$$H^k(\wedge^k(\Omega^1_S \boxplus \omega_C)) \cong H^k(\Omega^k_S) \oplus H^{k-1}(\Omega^{k-1}_S),$$

therefore

$$\bigoplus_k H^k(\wedge^k(\Omega^1_S \boxplus \omega_C)) \cong \Lambda \oplus \Lambda, \tag{4}$$

where

$$\Lambda := \bigoplus_k H^{k,k}(S).$$

With respect to this decomposition above, the Chern character $\mathrm{ch}(F)$ has two components

$$\mathrm{ch}(F) = (\mathrm{ch}(F)_{\mathrm{f}}, \mathrm{ch}(F)_{\mathrm{d}}) \in \Lambda \oplus \Lambda.$$

If C is smooth, it was shown in [Nes21, Lemma 3.2] that

$$(\mathrm{ch}(F)_{\mathrm{f}}, \mathrm{ch}(F)_{\mathrm{d}}) = (\mathbf{v}, \check{\beta}),$$

where β is the degree of a quasimap associated to F and $\check{\beta}$ is its dual class in $H^*(S, \mathbb{Q})$, defined in [Nes21, Definition 3.3]. We would like to establish the same result with respect to the definition of the Chern character given in (3), if C is singular. Let

$$\pi \colon S \times \tilde{C} \to S \times C$$

be the normalisation morphism and π^*F_i be the restriction of π^*F to the connected components $\tilde{C}_i$ of $\tilde{C}$. The above decomposition of the Chern character then satisfies the following property.

Lemma 2.1. *Under the identification (4) the following holds*

$$\mathrm{ch}(F) = (\mathbf{v}, \sum_i \mathrm{ch}(\pi^*F_i)_{\mathrm{d}}) \in \Lambda \oplus \Lambda.$$

In other words, if the quasimap associated to F is of degree β, then

$$\mathrm{ch}(F)_{\mathrm{d}} = \check{\beta}.$$

Proof. Firstly, there exist canonical maps making the following diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & \pi^*F \otimes \pi^*\Omega^1_{S\times C} \arrow[r] \arrow[d] & \pi^*\mathcal{P}^1(F) \arrow[r] \arrow[d] & \pi^*F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi^*F \otimes \Omega^1_{S\times \tilde{C}} \arrow[r] & \mathcal{P}^1(\pi^*F) \arrow[r] & \pi^*F \arrow[r] & 0
\end{tikzcd}$$

where the first row is exact on the left, because $L\pi^* \cong F$, since[2] F is flat over C. The diagram above implies that the pullback of the Atiyah class $\pi^*\mathrm{At}(F)$ is mapped to $\mathrm{At}(\pi^*F)$ with respect to the map

$$\mathrm{Ext}^1(\pi^*F, \pi^*F \otimes \pi^*\Omega^1_{S\times C}) \to \mathrm{Ext}^1(\pi^*F, \pi^*F \otimes \Omega^1_{S\times \tilde{C}}).$$

The same holds for $\pi^*\mathrm{At}^k(F)$. Consider now the following commutative diagram

$$\begin{tikzcd}
R\mathcal{H}om(F, F \otimes \Omega^k_{S\times C}) \arrow[rr] \arrow[d] & & \Omega^k_{S\times C} \arrow[dd] \arrow[ddr] & \\
\pi_*R\mathcal{H}om(\pi^*F, \pi^*F \otimes \pi^*\Omega^k_{S\times C}) \arrow[d] & & & \\
\pi_*R\mathcal{H}om(\pi^*F, \pi^*F \otimes \Omega^k_{S\times \tilde{C}}) \arrow[rr] & & \pi_*\Omega^k_{S\times\tilde{C}} \arrow[r] & \wedge^k(\Omega^1_S \boxplus \omega_C)
\end{tikzcd}$$

such that the first vertical map is the composition

$$\begin{aligned}
&R\mathcal{H}om(F, F \otimes \Omega^k_{S\times C}) \to \pi_*L\pi^*R\mathcal{H}om(F, F \otimes \Omega^k_{S\times C}) = \\
&= \pi_*R\mathcal{H}om(\pi^*F, \pi^*F \otimes L\pi^*\Omega^k_{S\times C}) \to \pi_*R\mathcal{H}om(\pi^*F, \pi^*F \otimes \pi^*\Omega^k_{S\times C}),
\end{aligned}$$

where we used that $L\pi^*F \cong \pi^*F$. Taking cohomology of the diagram above and using the exactness of π_*, we can therefore factor the map

$$\mathrm{Ext}^k(F, F \otimes \Omega^k_{S\times C}) \to H^k(\wedge^k(\Omega^1_S \boxplus \omega_C))$$

[2]To see that, one can use a standard locally-free resolution for a flat sheaf; these resolutions are functorial with respect to pullbacks.

as follows

$$\begin{aligned}\operatorname{Ext}^k(F, F\otimes\Omega^k_{S\times C}) &\to \operatorname{Ext}^k(\pi^*F, \pi^*F\otimes\pi^*\Omega^k_{S\times C}) \to \operatorname{Ext}^k(\pi^*F, \pi^*F\otimes\Omega^k_{S\times\tilde{C}})\\ &\to H^k(\Omega^k_{S\times\tilde{C}}) \cong H^k(\Omega^k_S) \oplus \bigoplus_i H^{k-1}(\Omega^{k-1}_S)\otimes H^1(\omega_{\tilde{C}_i})\\ &\to H^k(\Omega^k_S) \oplus H^{k-1}(\Omega^{k-1}_S)\otimes H^1(\omega_C) \cong H^k(\wedge^k(\Omega^1_S \boxplus \omega_C)).\end{aligned}$$

Under the natural identifications $H^1(\omega_{\tilde{C}_i}) \cong \mathbb{C}$ and $H^1(\omega_C) \cong \mathbb{C}$, the last map in the sequence above becomes

$$H^k(\Omega^k_S) \oplus \bigoplus_i H^{k-1}(\Omega^{k-1}_S) \xrightarrow{(\mathrm{id},+)} H^k(\Omega^k_S) \oplus H^{k-1}(\Omega^{k-1}_S).$$

The claim then follows by tracking the powers of the Atiyah class $\operatorname{At}^k(F)$ along the maps above. The fact that

$$\sum_i \operatorname{ch}(\pi^*F_i)_{\mathrm{d}} = \check{\beta}$$

follows from the definition of $\check{\beta}$, [Nes21, Definition 3.3]. □

2.2. **Semiregularity map.** By pulling back classes in

$$HT^2(S) := H^0(\wedge^2 T_S) \oplus H^1(T_S) \oplus H^2(\mathcal{O}_S)$$

to $S\times C$, we will treat $HT^2(S)$ as classes on $S \times C$. Let

$$\sigma_i := \operatorname{tr}(* \cdot \frac{(-1)^i}{i!}\operatorname{At}_\omega(F)^i)\colon \operatorname{Ext}^2(F,F) \to H^{i+2}(\wedge^i(\Omega^1_S \boxplus \omega_C))$$

be a semiregularity map.

Lemma 2.2. *The following diagram commutes*

$$\begin{tikzcd} H^{2-k}(\wedge^k T_S) \arrow[rr, "\cdot\frac{(-1)^k}{k!}\operatorname{At}_\omega(F)^k"] \arrow[dr, "\langle *,\operatorname{ch}_{k+i}(F)\rangle"'] & & \operatorname{Ext}^2(F,F) \arrow[dl, "\sigma_i"] \\ & H^{i+2}(\wedge^i(\Omega^1_S \boxplus \omega_C)) & \end{tikzcd}$$

Proof. If $i = 0$, then $\sigma_0 = \operatorname{tr}$ and the commutativity is implied by the following statement

$$\langle \kappa, \operatorname{tr}(\operatorname{At}_\omega(F)^k))\rangle = \operatorname{tr}\langle \kappa, \operatorname{At}_\omega(F)^k\rangle,$$

whose proof is presented in [BF03, Proposition 4.2] for $k = 1$ and is the same for other values of k.

If $i = 1$, then for the commutativity of the digram we have to prove that

$$\langle \kappa, \operatorname{tr}(\frac{\operatorname{At}_\omega(F)^{k+1}}{k+1!})\rangle = \operatorname{tr}(\langle \kappa, \frac{\operatorname{At}^k_\omega(F))}{k!}\rangle \cdot \operatorname{At}_\omega(F)).$$

If $\kappa \in H^2(\mathcal{O}_S)$, the equality follows trivially, since there is no contraction. The case of $\kappa \in H^1(T_S)$ is treated in [BF03, Proposition 4.2]. For $\kappa \in$

$H^0(\wedge^2 T_S)$ we use the derivation property for contraction with a 2-vector field

$$\langle \xi, \mathrm{At}^3_\omega(F)\rangle = 3\langle \xi, \mathrm{At}^2_\omega(F)\rangle \cdot \mathrm{At}_\omega(F),$$

which can be checked locally on a 2-vector field of the form $V \wedge W$. $\square$

Due to the decomposition

$$H^i(\wedge^i(\Omega^1_S \boxplus \omega_C)) \cong H^i(\Omega^i_S) \oplus H^{i-1}(\Omega^{i-1}_S),$$

there are two ways to contract a class in $H^i(\wedge^i(\Omega^1_S \boxplus \omega_C))$ with a class in $H^{2-k}(\wedge^k T_S)$: either via the first component of the decomposition above or via the second. Hence due to the wedge degree or the cohomological degree, only one component of $H^i(\wedge^i(\Omega^1_S \boxplus \omega_C))$ pairs non-trivially with $H^{2-k}(\wedge^k T_S)$ for a fixed k. It is not difficult to check that contraction with the Chern character

$$H^{2-k}(\wedge^k T_S) \xrightarrow{\langle -, \mathrm{ch}_{k+i}(F)\rangle} H^{i+2}(\wedge^i(\Omega^1_S \boxplus \omega_C))$$

is therefore equal to $\langle -, \mathrm{ch}(F)_{\mathrm{f}}\rangle$ for $i = 0$ and to $\langle -, \mathrm{ch}(F)_{\mathrm{d}}\rangle$ for $i = 1$. Moreover, using the identification

$$H^{i+2}(\wedge^i(\Omega^1_S \boxplus \omega_C)) \cong H^2(\mathcal{O}_S),$$

contraction $\langle -, \mathrm{ch}(F)_{\mathrm{d/f}}\rangle$ with classes on $S \times C$ is identified with contraction with classes on S.

Proposition 2.3. *Assume*

$$\mathrm{ch}(F)_{\mathrm{f}} \wedge \mathrm{ch}(F)_{\mathrm{d}} \neq 0,$$

then there exists $\kappa \in HT^2(S)$, such that

$$\langle \kappa, \mathrm{ch}(F)_{\mathrm{f}}\rangle = 0 \quad \textit{and} \quad \langle \kappa, \mathrm{ch}(F)_{\mathrm{d}}\rangle \neq 0.$$

Hence the restriction of the semiregularity map to the traceless part

$$\sigma_1\colon \mathrm{Ext}^2(F,F)_0 \to H^3(\Omega^1_S \boxplus \omega_C)$$

is non-zero.

Proof. Using a symplectic form on S, we have the following identifications

$$\wedge^2 T_S \cong \mathcal{O}_S, \quad T_S \cong \Omega^1_S, \quad \mathcal{O}_S \cong \Omega^2_S.$$

After applying the identifications and taking cohomology, the pairing

$$HT^2(S) \otimes H\Omega_0(S) \to H^2(\mathcal{O}_S), \tag{5}$$

which is given by contraction, becomes the intersection pairing

$$H\Omega_0(S) \otimes H\Omega_0(S) \to H^2(\Omega^2_S),$$

where $H\Omega_0(S) = \bigoplus H^i(\Omega^i)$. In particular, the pairing (5) is non-degenerate. Hence $\mathrm{ch}(F)^\perp_{\mathrm{d}}$ and $\mathrm{ch}(F)^\perp_{\mathrm{f}}$ are distinct, if and only if $\mathrm{ch}(F)_{\mathrm{d}}$ is not a multiple of $\mathrm{ch}(F)_{\mathrm{f}}$, therefore there exists a class $\kappa \in HT^2(S)$ with the properties stated in the lemma. By Lemma 2.2 and the discussion afterwards, the property $\langle \kappa, \mathrm{ch}(F)_{\mathrm{f}}\rangle = 0$ implies that

$$\kappa \cdot \exp(-\mathrm{At}_\omega(F) \in \mathrm{Ext}^2(F,F)_0,$$

while the property $\langle \kappa, \mathrm{ch}(F)_{\mathrm{d}}\rangle = 0$ implies that the restriction of the semiregularity map to $\mathrm{Ext}^2(F,F)_0$ is non-zero, as it is non-zero when applied to the element $\kappa \cdot \exp(-\mathrm{At}_\omega(F))$. □

Remark 2.4. From the point of view of quasimaps, the condition

$$\mathrm{ch}(F)_{\mathrm{f}} \wedge \mathrm{ch}(F)_{\mathrm{d}} \neq 0,$$

is equivalent to the fact that the quasimap $f\colon C \to \mathfrak{Coh}_r(S)$ associated to F is not constant.

A geometric interpretation of the above result is the following one. With respect to Hochschild–Kostant–Rosenberg (HKR) isomorphism

$$HT^2(S) \cong HH^2(S),$$

the space $HT^2(S)$ parametrises first-order non-commutative deformations of S, i.e. deformations of $\mathrm{D}^b(S)$. Given a first-order deformation $\kappa \in HT^2(S)$, the unique horizontal lift of $\mathrm{ch}(F)_{\mathrm{d/f}}$ relative to some kind of Gauss-Manin connection associated to κ should stay Hodge, if and only if $\langle \kappa, \mathrm{ch}(F)_{\mathrm{d/f}}\rangle = 0$. On the other hand, $\langle \kappa, \exp(-\mathrm{At}_\omega(F))$ gives obstruction for deforming F on $S \times C$ in direction κ. Therefore by Lemma 2.2 the semiregularity map σ_i relates obstruction to deform F along κ with the obstruction that $\mathrm{ch}(F)_{\mathrm{d/f}}$ stays Hodge. Proposition 2.3 states that there exists a deformation $\kappa \in HT^2(S)$, for which $\mathrm{ch}(F)_{\mathrm{f}}$ stays Hodge, but $\mathrm{ch}(F)_{\mathrm{d}}$ does not. From the point of view of quasimaps, this means that the moduli space of stable sheaves M on S deforms along κ, but the quasimap associated to F does not, if its degree is non-zero. For example, let S be a K3 surface associated to a cubic 4-fold Y, such that the Fano variety of lines $F(Y)$ is isomorphic to $S^{[2]}$. Then if we deform Y away from the Hassett divisor (see [Has00]), $F(Y)$ deforms along, but the point class of S does not. Therefore such deformation of Y will give the first-order non-commutative deformation $\kappa \in HT^2(S)$ of S, such that $\mathbf{v} = (1, 0, -2)$ stays Hodge, but $\check{\beta} = (0, 0, k)$ does not. Note that $\check{\beta} = (0, 0, k)$ corresponds to multiplies of the exceptional curve class in $S^{[2]}$. Indeed, there are no commutative deformations of S that will make $(0, 0, k)$ non-Hodge, because the exceptional divisor deforms along with $S^{[2]}$.

3. Reduced wall-crossing

3.1. **Surjective cosection.** Now let M be a moduli space of Gieseker stable sheaves on S with Chern character $\mathbf{v} \in H^*(S, \mathbb{Q})$, subject to the following assumptions:

- $\mathrm{rk}(\mathbf{v}) > 0$;
- there are no strictly Gieseker semistable sheaves;

- there exists $u \in K_0(S)$, such that $\int_S \mathbf{v} \cdot \mathrm{ch}(u) \cdot \mathrm{td}_S = 1$.

Note that the second assumption implies that M is smooth and projective, while the third one implies that M is a fine moduli space (might be even equivalent, but the author could not find a reference).

Remark 3.1. An example of a moduli space M which satisfies the assumptions above will be a moduli space of sheaves in the class $\mathbf{v} = (2, \alpha, 2k+1)$ for a polarisation such that $\deg(\alpha)$ is odd (or a generic polarisation that is close to a polarisation for which $\deg(\alpha)$ is odd). Firstly, $\mathrm{rk}(\mathbf{v})$ and $\deg(\mathbf{v})$ are coprime, therefore there are no strictly slope semistable sheaves. The class $u = [\mathcal{O}_S] - (k+2)[\mathcal{O}_{\mathrm{pt}}] \in K_0(S)$ has the property $\chi(\mathbf{v} \cdot u) = 1$. Moreover, [Nes21, Proposition A.4] holds in this case, therefore the space $M_{\mathbf{v},\check{\beta},u}(S \times C)$ is a moduli space of all stable sheaves for some *suitable* polarisation. More specifically, such set-up can be arranged on an elliptic K3 surface.

By [Nes21, Lemma 3.15], there exists an identification between a space of quasimaps $Q^{\epsilon}_{g,N}(M, \beta)$ and a certain relative moduli space of sheaves $M^{\epsilon}_{\mathbf{v},\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$,

$$Q^{\epsilon}_{g,N}(M, \beta) \cong M^{\epsilon}_{\mathbf{v},\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N}),$$

such that the naturally defined corresponding obstructions theories are isomorphic under the identification above. Let

$$\begin{aligned} &\pi\colon S \times \mathfrak{C}_{g,N} \times_{\mathfrak{M}_{g,N}} Q^{\epsilon}_{g,N}(M, \beta) \to Q^{\epsilon}_{g,N}(M, \beta), \\ &\mathbb{F} \in \mathrm{Coh}(S \times \mathfrak{C}_{g,N} \times_{\mathfrak{M}_{g,N}} Q^{\epsilon}_{g,N}(M, \beta)) \end{aligned}$$

be the canonical projection and the universal sheaf, which is defined via the identifation $Q^{\epsilon}_{g,N}(M, \beta) \cong M^{\epsilon}_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$. Let

$$\mathbb{E}^{\bullet} := R\mathcal{H}om_{\pi}(\mathbb{F}, \mathbb{F})_0[1]$$

be the obstruction-theory complex relative to the moduli stack of nodal curves $\overline{M}_{g,N}$. We construct a surjective cosection as follows. There exists a global relative semiregularity map

$$\mathsf{sr}\colon \mathbb{E}^{\bullet} \to R^3\pi_*(\Omega^1_S \boxplus \omega_{\mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}})[-1],$$

and since

$$R^3\pi_*(\Omega^1_S \boxplus \omega_{\mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}}) \cong H^2(\mathcal{O}_S) \otimes \mathcal{O}_{Q^{\epsilon}_{g,N}(M,\beta)},$$

we obtain a cosection of the obstruction theory

$$\mathsf{sr}\colon \mathbb{E}^{\bullet} \to H^2(\mathcal{O}_S) \otimes \mathcal{O}_{Q^{\epsilon}_{g,N}(M,\beta)}[-1] \cong \mathcal{O}_{Q^{\epsilon}_{g,N}(M,\beta)}[-1], \tag{6}$$

surjectivity of the cosection follows from the preceding results.

Corollary 3.2. *Assuming $\beta \neq 0$, the semiregularity map* sr *is surjective.*

Proof. Under the given assumption the surjectivity of sr follows from Proposition 2.3 and Lemma 2.1. $\square$

Consider now the composition

$$\mathrm{Ext}^1_C(\Omega_C, \mathcal{O}_C(-\sum p_i)) \to \mathrm{Ext}^2(F,F)_0 \xrightarrow{\sigma_1} H^3(\Omega^1_S \boxplus \omega_C), \tag{7}$$

where the first map defined by the following composition

$$\mathrm{Ext}^1_C(\Omega_C, \mathcal{O}_C(-\sum p_i)) \to \mathrm{Ext}^1_C(\omega_C, \mathcal{O}_C) \xrightarrow{\cdot - \mathrm{At}_\omega(F)} \mathrm{Ext}^2(F,F)_0.$$

The composition (7) is zero by the same arguments as those which are presented in Lemma 2.2. Therefore the semiregularity map descends to the absolute obstruction theory

$$\begin{tikzcd} \mathbb{T}_{\mathfrak{M}_{g,N}}[-1] \arrow[r] & \mathbb{E}^\bullet \arrow[r] \arrow[d, "\mathsf{sr}"] & \mathbb{E}^\bullet_{\mathrm{abs}} \arrow[dl, dashed, "\mathsf{sr}_{\mathrm{abs}}"] \\ & \mathcal{O}_{Q^\epsilon_{g,N}(M,\beta)}[-1] & \end{tikzcd}$$

so the results of [KL13] apply.

The cosection of the obstruction theory of the master space $MQ^{\epsilon_0}_{g,N}(M,\beta)$ (see [Nes21, Section 6.3] for the definition of the master space) is constructed in the similar way as for $Q^\epsilon_{g,N}(M,\beta)$ - by viewing it as a relative moduli space of sheaves. In what follows, we use Kiem–Li's construction of reduced virtual fundamental classes via localisation by a cosection, see [KL13]. Kiem–Li's classes can be seen as virtual fundamental classes associated to the reduced obstruction-theory complex $\mathbb{E}_{\mathrm{red}}$, defined as the cone of the cosection,

$$\mathbb{E}^\bullet_{\mathrm{red}} = \mathrm{cone}(\mathsf{sr})[-1] \to \mathbb{E}^\bullet_{\mathrm{abs}} \xrightarrow{\mathsf{sr}_{\mathrm{abs}}} \mathcal{O}[-1].$$

However, showing that $\mathbb{E}_{\mathrm{red}}$ really defines an obstruction theory is difficult. This is addressed in Appendix A. We need the virtual localisation theorem for the proof of the wall-crossing formulas, we therefore refer to [CKL17] for the virtual localisation of Kiem–Li's reduced classes.

From now on, by a virtual fundamental class we always will mean a *reduced* virtual fundamental class, except for $\beta = 0$, since, in this case, the standard virtual fundamental class does not vanish. The arguments of this section apply both to standards invariants $\langle \tau_{m_1}(\gamma_1), \ldots, \tau_{m_N}(\gamma_N)\rangle^\epsilon_{g,N,\beta}$ and perverse invariants $\langle \tau_{m_1}(\gamma_1), \ldots, \tau_{m_n}(\gamma_N)\rangle^{\natural,\epsilon}_{g,N,\beta}$, if $M = S^{[n]}$. We therefore state and prove everything only for the standard invariants.

3.2. **Wall-crossing.** We start with derivation of a more explicit formula for wall-crossing invariants $\mu_\beta(z)$, [Nes21, Definition 6.1], by using virtual localisation on a graph space $GQ_{0,1}(M,\beta)$. As it is explained in [Nes21, Section 6.1], the $\mathbb{C}^*$-fixed components of $GQ_{0,1}(M,\beta)$ are identified with certain products. The reduced virtual fundamental class of a product splits as a product of reduced and non-reduced classes on its factors (cf. [MPT10, Section 3.9]). Assuming the marking is over ∞, the virtual class is therefore non-zero only for F_β and $F^{0,0}_{1,\beta}$ in the notation of [Nes21, Section 6.1]. Now let $\{B^i\}$ be a basis of $H^*(M,\mathbb{Q})$ and $\{B_i\}$ be its dual basis with respect to intersection pairing. Let

$$B_ip_\infty := B_i \boxtimes p_\infty \in H^*_{\mathbb{C}^*}(M\times \mathbb{P}^1),$$

where $p_\infty \in H^*_{\mathbb{C}^*}(\mathbb{P}^1)$ is the equivariant class of $\infty \in \mathbb{P}^1$. Then by the virtual localisation formula, we have the following identity

$$\sum_i B^i \int_{[GQ_{0,1}(M,\beta)]^{\mathrm{vir}}} \mathsf{ev}^*B_ip_\infty = \sum_i B^i \int_{[F_\beta]^{\mathrm{vir}}} \frac{-z\mathsf{ev}^*B_i}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})} \\ + \sum_i B^i \langle B_i, \frac{\mathbb{1}}{-z-\psi}\rangle^{0+}_{0,2,\beta}, \quad (8)$$

where we used that

$$e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{0,0}_{1,\beta}/GQ_{0,1}(M,\beta)}) = z(z+\psi)$$

and

$$p_\infty|_0 = 0, \quad p_\infty|_\infty = -z,$$

which also implies that only fixed components with markings over ∞ contribute to the integral (8). The other fixed components of $GQ_{0,1}(M,\beta)$, described in [Nes21, Section 6.1], do not contribute, because the reduced class of a product splits as a product of reduced and non-reduced classes on its factors (cf. [MPT10, Section 3.9]). See also [CK14, Section 5.3].

The virtual dimension of $GQ_{0,1}(M,\beta)$ is $\dim(M)+2$, while the virtual dimension of $Q^{0+}_{0,2}(M,\beta)$ is $\dim(M)$. On the other hand, the cohomological degree of B_ip_∞ is at most $\dim(M)+1$. Hence the left-hand side of (8) is zero and the second term on the right-hand side is non-zero only for $B_i = [\mathrm{pt}]$. We therefore get that

$$-z\mathsf{ev}_*\left(\frac{[F_\beta]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta/QG_{0,1}(M,\beta)})}\right) = \frac{\mathbb{1}}{z}\langle [\mathrm{pt}], \mathbb{1}\rangle^{0+}_{0,2,\beta} \in A^*(M)[z^{\pm}].$$

In particular, we obtain that

$$\mu_\beta(z) = \mathbb{1}\langle [\mathrm{pt}], \mathbb{1}\rangle^{0+}_{0,2,\beta} \in A^*(M)[z]. \quad (9)$$

Theorem 3.3. *Assuming* $2g-2+N+\epsilon_0\deg(\beta)>0$*, we have*

$$\langle \tau_{m_1}(\gamma_1),\dots,\tau_{m_n}(\gamma_N)\rangle^{\epsilon_-}_{g,N,\beta} - \langle \tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon_+}_{g,N,\beta} \\ = \langle [\mathrm{pt}], \mathbb{1}\rangle^{0+}_{0,2,\beta}\cdot \langle \tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N), \mathbb{1}\rangle_{g,N+1,0},$$

if $\deg(\beta)=d_0$*, and*

$$\langle \tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon_-}_{g,N,\beta} = \langle \tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon_+}_{g,N,\beta},$$

otherwise.

Sketch of the proof. As in the case of [Nes21, Theorem 6.3], we have to refer mostly to [Zho22, Section 6]. The difference with is that we use reduced classes now.

The fixed components of the master space which contribute to the wall-crossing formula are of the form (up to some finite gerby structure and finite coverings)

$$\widetilde{Q}^{\epsilon^+}_{g,N+k}(M,\beta') \times_{M^k} \prod_{i=1}^{k} F_{\beta_i},$$

where $\beta = \beta' + \beta_1 + \cdots + \beta_k$ and $\deg(\beta_i) = d_0$. Recall that $\widetilde{Q}^{\epsilon^+}_{g,N+k}(M,\beta')$ is just a base change of $Q^{\epsilon^+}_{g,N}(M,\beta)$ from $\mathfrak{M}_{g,N}$ to $\widetilde{\mathfrak{M}}_{g,N,d}$, where the latter is the moduli space of curves with entangled tails. The reduced class of a product splits as a product of reduced and non-reduced classes on its factors (cf. [MPT10, Section 3.9]). Hence by Corollary 3.2 and [KL13], it vanishes, unless $\beta' = 0$ and $k = 1$, in which case

$$\widetilde{Q}^{\epsilon^+}_{g,N+1}(M,0) = Q^{\infty}_{g,N+1}(M,0) = \overline{M}_{g,N+1}(M,0).$$

Using the explicit expression of $\mu_\beta(z)$ from (9) and the analysis presented in [Zho22, Section 7], we get that contribution of this component to the wall-crossing is

$$\langle [\mathrm{pt}], \mathbb{1}\rangle^{0^+}_{0,2,\beta} \cdot \langle \tau_{m_1}(\gamma_1), \ldots, \tau_{m_N}(\gamma_N), \mathbb{1}\rangle^{\infty}_{g,N+1,0},$$

this concludes the argument. $\square$

Corollary 3.4. *For all $g \geq 1$ we have*

$$F^{0^+}_g(\mathbf{t}(z)) = F^{\infty}_g(\mathbf{t}(z)) + F_{\mathrm{wall}}(\mathbf{t}(z))$$

where

$$F_{\mathrm{wall}}(\mathbf{t}(z)) = \mu(q) \cdot \left(\sum_{n=0}^{\infty} \frac{1}{N!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi), \mathbb{1}\rangle^{\infty}_{g,N+1,0} \right)$$

and

$$\mu(q) = \sum_{\beta > 0} \langle [\mathrm{pt}], \mathbb{1}\rangle^{0^+}_{0,2,\beta} q^{\beta}.$$

For $g = 0$, the equation holds modulo constant and linear terms in $\mathbf{t}$.

There are invariants that are not covered by the results above and of great interest for us - those of a fixed genus-1 curve. We deal with them now. Let E be a smooth genus-1 curve and $Q^{\epsilon}_{E/E}(M,\beta)$ be the fiber of

$$Q^{\epsilon}_{1,0}(M,\beta) \to \overline{M}_{1,0}$$

over the stacky point $[E]/E \in \overline{M}_{1,0}$. In other words, $Q^{\epsilon}_{E/E}(M,\beta)$ is the moduli space of ϵ-stable quasimaps, whose smoothing of the domain is E. Maps are considered up translations of E.

Theorem 3.5. *Assuming $\beta \neq 0$, we have*

$$\int_{[Q^{\epsilon_-}_{E/E}(M,\beta)]^{\mathrm{vir}}} 1 = \int_{[Q^{\epsilon_+}_{E/E}(M,\beta)]^{\mathrm{vir}}} 1 + \chi(M)\langle [\mathrm{pt}], \mathbb{1}\rangle^{0+}_{0,2,\beta},$$

if $\deg(\beta) = d_0$*, and*

$$\int_{[Q^{\epsilon_-}_{E/E}(M,\beta)]^{\mathrm{vir}}} 1 = \int_{[Q^{\epsilon_+}_{E/E}(M,\beta)]^{\mathrm{vir}}} 1,$$

otherwise.

Sketch of the proof. As in Theorem 3.3, the only case when the contribution from the wall-crossing components is non-zero is the one of $\beta' = 0$ and $k = 1$. In this case

$$\widetilde{Q}^{\epsilon_+}_{(E,0)}(M, 0) \cong M,$$

and the obstruction bundle is given by the tangent bundle T_M. Hence the virtual fundamental class is $\chi(M)[\mathrm{pt}]$, then by (9) the wall-crossing term is

$$\chi(M)\langle [\mathrm{pt}], \mathbb{1}\rangle^{0+}_{0,2,\beta},$$

this concludes the argument. □

4. Applications

4.1. Enumerative geometry of $S^{[n]}$.

4.1.1. *Genus-0 invariants.* We start with genus-0 3-point quasimap invariants of $S^{[n]}$.

Definition 4.1. Given classes

$$\gamma_1, ..., \gamma_N \in H^*(M, \mathbb{Q}) \text{ and } \gamma'_1, ..., \gamma'_{N'} \in H^*(S \times C, \mathbb{Q}).$$

We define

$$\langle \gamma_1, ..., \gamma_N \mid \gamma'_1, ..., \gamma'_{N'})\rangle^{S\times C}_{\mathbf{v},\check{\beta}} \in \mathbb{Q}.$$

to be DT invariants associated to the moduli space of sheaves $M_{\mathbf{v},\check{\beta},u}(S \times C/S_{\mathbf{x}})$, defined in [Nes21, Definition 3.14]. On the left we put relative primary insertions, on the right - the absolute primary ones (cf. [Nes21, Remark 5.8]). Moduli spaces $M_{\mathbf{v},\check{\beta},u}(S\times C/S_{\mathbf{x}})$ can be identified for different choices of $u \in K_0(S)$, we therefore drop u from the notation. If $M = S^{[n]}$, then we define

$$\langle \gamma_1, ..., \gamma_N \mid \gamma'_1, ..., \gamma'_{N'})\rangle^{\sharp, S\times C}_{n,\check{\beta}} \in \mathbb{Q}$$

to be invariants associated to a moduli space of stable pairs $\mathrm{P}_{n,\check{\beta}}(S\times C/S_{\mathbf{x}})$.

The moduli space of $\mathbb{P}^1$ with three marked points is a point. Hence by fixing markings, [Nes21, Corollary 4.7] implies that

$$Q^{0+}_{0,3}(S^{[n]}, \beta)^{\sharp} \cong \mathrm{P}_{n,\check{\beta}}(S \times \mathbb{P}^1/S_{0,1,\infty}). \tag{10}$$

Moreover, by Theorem 3.3 and the string equation, the wall-crossing is trivial, if $g = 0$ and $N = 3$. We therefore obtain that

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle^{\sharp, S^{[n]}, 0^+}_{0,3,\beta} = \langle \gamma_1, \gamma_2, \gamma_3 \rangle^{S^{[n]},\infty}_{0,3,\beta}.$$

Applying the identification (10), we get the following result, which together with PT/GW of [Obe21a] confirms the conjecture proposed in [Obe19].

Corollary 4.2.

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle^{S^{[n]},\infty}_{0,3,\beta} = \langle \gamma_1, \gamma_2, \gamma_3 \rangle^{S\times\mathbb{P}^1}_{n,\check{\beta}}.$$

More generally, the results above can be restated for a relative geometry

$$S \times C_{g,N} \to \overline{M}_{g,N},$$

such that $2g - 2 + N > 0$. In this case, by the string equation, the wall-crossing is also trivial.

4.1.2. *Genus-1 invariants, Igusa cusp form conjecture.* Let us firstly establish a more precise relation between β and $\check{\beta}$. Let E be a smooth genus 1 curve. Given a generic K3 surface S, then by global Torelli theorem,

$$\mathrm{Eff}(S) = \langle \beta_h \rangle, \quad \beta_h^2 = 2h - 2.$$

Using Nakajima operators, we obtain

$$\mathrm{Eff}(S^{[n]}) = \langle C_{\beta_h}, \mathsf{A} \rangle,$$

where A is the exceptional curve class associated to the Hilbert–Chow morphism $S^{[n]} \to S^{(n)}$. The class C_{β_h} is given by moving a point on a smooth curve representing the class β_h and keeping $n-1$ points fixed. For more on Nakajima basis in the relevant context, we refer to [Obe18a].

Using [Nes21, Corollary 3.13], we obtain a correspondence between degrees of quasimaps and classes on the threefold $S \times E$,

$$(n, -\check{\beta}) \colon \mathrm{Eff}(S^{[n]}, \mathfrak{Coh}^{\sharp}_r(S)) \hookrightarrow \mathrm{Eff}(S \times E) \oplus H^6(S \times E).$$

For $n > 1$, its restriction to $\mathrm{Eff}(S^{[n]}) \subset \mathrm{Eff}(S^{[n]}, \mathfrak{Coh}_r(S))$ is given by

$$k_1 C_{\beta_h} + k_2 A \mapsto ((n, k_1\beta_h), k_2).$$

While for $n = 1$, the class β_h is sent to $((\beta_h, 1), 0)$. Note that we changed the sign of classes on $S \times E$, which amounts to considering the class of the subscheme rather than its ideal. A general class in $\mathrm{Eff}(S^{[n]}, \mathfrak{Coh}_r(S))$ can therefore be identified with $k_1 C_{\beta_h} + k_2 A$ for possibly negative k_2.

By [Nes21, Corollary 4.7], we have the following identification of moduli spaces

$$Q^{0^+}_{E/E}(S^{[n]}, C_{\beta_h} + kA)^{\sharp} \cong [\mathrm{P}_{(n,\beta_h),k}(S \times E)/E],$$

such that the natural obstruction theories match. As before, the subscript notation of the moduli on the left indicates that we consider maps up to translations of E. For the same reason we take the quotient by E on the left. On the other hand,

$$Q^{\infty}_{E/E}(S^{[n]}, C_{\beta_h}+kA)^{\sharp} = \overline{M}_{E/E}(S^{[n]}, C_{\beta_h}+kA).$$

Consider now the following two generating series

$$\begin{aligned}\mathsf{PT}(p,q,\tilde{q}) &:= \sum_{n\geq 0}\sum_{h\geq 0}\sum_{k\in\mathbb{Z}} p^k q^{h-1}\tilde{q}^{n-1}\int_{[\mathrm{P}_{(n,\beta_h),k}(S\times E)/E]^{\mathrm{vir}}} 1\\ \mathsf{GW}(p,q,\tilde{q}) &:= \sum_{n>0}\sum_{h\geq 0}\sum_{k\geq 0} p^k q^{h-1}\tilde{q}^{n-1}\int_{[\overline{M}_{E/E}(S^{[n]},C_{\beta_h}+kA)]^{\mathrm{vir}}} 1.\end{aligned}$$

The series are well-defined, because (S,β) and (S',β') are deformation equivalent, if and only if

$$\beta^2=\beta'^2 \quad \text{and} \quad \mathrm{div}(\beta)=\mathrm{div}(\beta'),$$

where $\mathrm{div}(\beta)$ is the divisibility of the class, in our case β_h's are primitive by definition. In [OP18], it was proven that

$$\mathsf{PT}(p,q,\tilde{q}) = \frac{1}{-\chi_{10}(p,q,\tilde{q})},$$

where $\chi_{10}(p,q,\tilde{q})$ is the *Igusa cusp form*, hence the name of the conjecture. By the discussion above, we can view both series as generating series of quasimaps for $\epsilon\in\{0^+,\infty\}$. Using Theorem 3.5, we obtain

$$\begin{aligned}\mathsf{PT}(p,q,\tilde{q}) = \mathsf{GW}(p,q,\tilde{q})&\\ +\sum_{n\geq 0}\sum_{h\geq 0}\sum_{k\in\mathbb{Z}} p^k q^{h-1}\tilde{q}^{n-1}\chi(S^{[n]})&\langle[\mathrm{pt}],\mathbb{1}\rangle^{\sharp,S^{[n]},0^+}_{C_{\beta_h}+kA}.\end{aligned}$$

Remark 4.3. The effective quasimap cone $\mathrm{Eff}(S^{[n]},\mathfrak{Coh}_r(S))$ is strictly bigger than the effective cone $\mathrm{Eff}(S^{[n]})$. For a class, which is not in $\mathrm{Eff}(S^{[n]})$, the moduli space of ∞-stable quasimaps will be just empty. Nevertheless, the wall-crossing formula still applies but with zero contribution from $\epsilon=\infty$.

The invariants $\langle[\mathrm{pt}],\mathbb{1}\rangle^{\sharp,S^{[n]},0^+}_{C_{\beta_h}+kA}$ are so-called *rubber* PT invariants on $S\times\mathbb{P}^1$. These are invariants associated to the moduli of stable pairs $\mathrm{P}_{n,\check{\beta}}(S\times\mathbb{P}^1/S_{0,\infty})$ up to the $\mathbb{C}^*$-action coming from $\mathbb{P}^1$-factor which fixes 0 and ∞. Said differently, these are invariants associated to the quotient

$$[\mathrm{P}_{n,\check{\beta}}(S\times\mathbb{P}^1/S_{0,\infty})/\mathbb{C}^*].$$

These invariants can be *rigidified* to standard relative PT invariants with absolute insertions as follows.

Lemma 4.4.

$$\langle[\mathrm{pt}],\mathbb{1}\rangle^{\sharp,S^{[n]},0^+}_{C_{\beta_h}+kA} = \langle[\mathrm{pt}],\mathbb{1}\mid D\boxtimes\omega\rangle^{\sharp,S\times\mathbb{P}^1}_{(n,\beta_h),k},$$

where $D\in H^2(S,\mathbb{Q})$ is some class such that $D\cdot\beta_h=1$ and $\omega\in H^2(\mathbb{P}^1,\mathbb{Z})$ is the point class.

Proof. See [MO09, Lemma 3.3]. □

The wall-crossing invariants can also be given a different and more sheaf-theoretic interpretation as virtual Euler numbers of Quot schemes, as it is explained in [Obe21b]. In the same article, wall-crossing invariants are also explicitly computed for $S^{[n]}$. Therefore we obtain the following corollary, which completes the proof of the Igusa cusp conjecture.

Corollary 4.5.

$$\mathsf{PT}(p,q,\tilde{q}) = \mathsf{GW}(p,q,\tilde{q}) + \frac{1}{F^2\Delta}\cdot\frac{1}{\tilde{q}}\prod_{n\geq 1}\frac{1}{(1-(\tilde{q}\cdot G)^n)^{24}}.$$

For the definition of the generating series on the right, we refer to [OP16, Section 2].

4.2. **Higher-rank DT invariants.** A moduli space M is deformation equivalent to a punctorial Hilbert scheme $S^{[n]}$, where $2n = \dim(M)$. Hence GW theory of M is equivalent to the one of $S^{[n]}$. Applying quasimap wall-crossing both to M and to $S^{[n]}$, we can therefore express higher-rank DT invariants of a threefold $S \times C$ in terms of rank-one DT invariants and wall-crossing invariants.

4.2.1. $\mathrm{K3}\times\mathbb{P}^1$. Let us firstly consider invariants on $S\times\mathbb{P}^1$ relative to $S_{0,1,\infty}$. As previously, by fixing the markings, we obtain

$$Q^{0^+}_{0,3}(M,\beta) \cong M_{\mathbf{v},\check{\beta},u}(S\times\mathbb{P}^1/S_{0,1,\infty}).$$

Moreover, as in the case of $S^{[n]}$, there is no wall-crossing by Theorem 3.3 and the string equation, therefore

$$\langle\gamma_1,\gamma_2,\gamma_3\rangle^{M,0^+}_{0,3,\beta} = \langle\gamma_1,\gamma_2,\gamma_3\rangle^{M,\infty}_{0,3,\beta}.$$

Choose a deformation M to $S^{[n]}$, which keeps the class β algebraic. The deformation gives an identification of cohomologies

$$H^*(M,\mathbb{Q}) \cong H^*(S^{[n]},\mathbb{Q}).$$

Under this identification, we have

$$\begin{aligned}\langle\gamma_1,\gamma_2,\gamma_3\rangle^{M,0^+}_{0,3,\beta} &= \langle\gamma_1,\gamma_2,\gamma_3\rangle^{M,\infty}_{0,3,\beta} \\ &= \langle\gamma_1,\gamma_2,\gamma_3\rangle^{S^{[n]},\infty}_{0,3,\beta} = \langle\gamma_1,\gamma_2,\gamma_3\rangle^{S^{[n]},0^+}_{0,3,\beta}. \qquad (11)\end{aligned}$$

Passing from quasimaps to sheaves and using (11), we obtain the following result.

Corollary 4.6. *Given a deformation of (M,β) to $(S^{[n]},\beta)$ we have*

$$\langle\gamma_1,\gamma_2,\gamma_3\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check{\beta}} = \langle\gamma_1,\gamma_2,\gamma_3\rangle^{S\times\mathbb{P}^1}_{n,\check{\beta}}.$$

4.2.2. K3 $\times E$. Consider now $S\times E$ for a genus-1 curve E. Applying the same procedure as for $S\times \mathbb{P}^1$, we obtain

$$\int_{[M_{\mathbf{v},\check{\beta},u}(S\times E)/E]^{\mathrm{vir}}} 1 = \int_{[\mathrm{P}_{n,\check{\beta}}(S\times E)/E]^{\mathrm{vir}}} 1 + \chi(S^{[n]})\left(\langle [\mathrm{pt}], \mathbb{1}\rangle^{S^{[n]},0^+}_{0,2,\beta} - \langle [\mathrm{pt}], \mathbb{1}\rangle^{M,0^+}_{0,2,\beta}\right).$$

The invariants $\langle [\mathrm{pt}], \mathbb{1}\rangle^{M,0^+}$ are *rubber* invariants associated to the moduli space $M_{\mathbf{v},\check{\beta},u}(S\times \mathbb{P}^1/S_{0,\infty})$ up to $\mathbb{C}^*$-action coming from $\mathbb{P}^1$-factor. As in Lemma 4.4, one can *rigidify* these invariants.

Lemma 4.7.

$$\langle [\mathrm{pt}], \mathbb{1}\rangle^{M,0^+}_{0,2,\beta} = \langle [\mathrm{pt}], \mathbb{1} \mid D\boxtimes\omega)\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check{\beta}},$$

where $D\in H^2(S,\mathbb{Q})$ is some class such that $\mathrm{c}_1(\check{\beta})\cdot D=1$, and $\omega\in H^*(\mathbb{P}^1,\mathbb{Z})$ is the point class.

Proof. The proof is exactly the same as in [MO09, Lemma 3.3]. $\square$

By degenerating $\mathbb{P}^1$ to $\mathbb{P}^1\cup_0 \mathbb{P}^1$ and applying degeneration formula of [MNOP06, Section 3.4] (see also [LW15]), we obtain

$$\langle [\mathrm{pt}], \mathbb{1} \mid D\boxtimes\omega \mid\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check{\beta}} = \langle [\mathrm{pt}] \mid D\boxtimes\omega\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check{\beta}},$$

which is the consequence of the fact that reduced class restricts to a reduced and non-reduced classes (which vanishes, unless $\beta=0$) on irreducible components. Putting everything together, we get the following wall-crossing expression for higher-rank DT invariants.

Corollary 4.8.

$$\int_{[M_{\mathbf{v},\check{\beta},u}(S\times E)/E]^{\mathrm{vir}}} 1 = \int_{[\mathrm{P}_{n,\check{\beta}}(S\times E)/E]^{\mathrm{vir}}} 1 + \chi(S^{[n]})\left(\langle [\mathrm{pt}] \mid D\boxtimes\omega\rangle^{S\times\mathbb{P}^1}_{n,\check{\beta}} - \langle [\mathrm{pt}] \mid D\boxtimes\omega\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check{\beta}}\right).$$

Using same arguments as in [MO09, Lemma 3.3], we get the following rigidification of the genus-1 invariant

$$\int_{[M_{\mathbf{v},\check{\beta},u}(S\times E)/E]^{\mathrm{vir}}} 1 = \langle D\boxtimes\omega\rangle^{S\times E}_{\mathbf{v},\check{\beta}}.$$

By degenerating E to $E\cup_0\mathbb{P}^1$ and applying the degeneration formula, we obtain

$$\langle D\boxtimes\omega\rangle^{S\times E}_{\mathbf{v},\check{\beta}} = \langle \mathbb{1} \mid D\boxtimes\omega\rangle^{S\times E}_{\mathbf{v},\check{\beta}} + \chi(M)\langle [\mathrm{pt}] \mid D\boxtimes\omega\rangle^{S\times\mathbb{P}^1}_{\mathbf{v},\check{\beta}},$$

the second term on the right is the wall-crossing term.

Corollary 4.9.

$$\langle \mathbb{1} \mid D\boxtimes\omega\rangle^{S\times E}_{\mathbf{v},\check{\beta}} = \langle \mathbb{1} \mid D\boxtimes\omega\rangle^{S\times E}_{n,\check{\beta}}.$$

Using the Igusa cusp form conjecture, we therefore have an explicit expression for these higher-rank relative DT invariants.

Moreover, by [Nes21, Lemma 3.22], the higher-rank invariants associated to the moduli space $M_{\mathbf{v},\check{\beta},u}(S\times C)$ can be related to invariants associated to moduli spaces of sheaves with fixed determinant, $M_{\mathbf{v},\check{\beta},L}(S\times C)$,

$$\int_{[M_{\mathbf{v},\check{\beta},L}(S\times E)/E]^{\mathrm{vir}}} 1 = \int_{[M_{\mathbf{v},\check{\beta},u}(S\times E)/E]^{\mathrm{vir}}} \mathrm{rk}(\mathbf{v})^2.$$

4.3. **DT/PT correspondence.** Using the wall-crossing for the standard pair $(S^{[n]}, \mathfrak{Coh}_r(S))$ and the perverse pair $(S^{[n]}, \mathfrak{Coh}_r^{\sharp}(S))$, we also can relate rank-one PT invariants to rank-one DT invariants. If $g=0$ and $N=3$, by Theorem 3.3 and the string equation, there is no wall-crossing in both case.

$$\langle\gamma_1,\gamma_2,\gamma_3\rangle_{0,3,\beta}^{\sharp,S\times\mathbb{P}^1} = \langle\gamma_1,\gamma_2,\gamma_3\rangle_{0,3,\beta}^{S^{[n]},\infty} = \langle\gamma_1,\gamma_2,\gamma_3\rangle_{0,3,\beta}^{\sharp,S\times\mathbb{P}^1},$$

Unlike for standard virtual fundamental classes, we really have equality of invariants in the reduced case. In general, one can only expect equality of certain generating series, which also account for wall-crossings. Equality of two theories also holds for $S\times E$, as was shown in [OS19, Theorem 3] and [Obe18b]. However, in order to apply our methods to this case, we have to show that DT/PT correspondence holds for the wall-crossing invariants.

Appendix A. Reduced obstruction theory

Let $(\mathbb{E}^{\bullet}_{\mathrm{red}})^{\vee}$ be the cone of the dual of the relative semiregularity map $\mathsf{sr}^{\vee}$ from (6). In this section, we will show existence of the obstruction-theory morphism

$$(\mathbb{E}^{\bullet}_{\mathrm{red}})^{\vee} \to \mathbb{L}_{Q^{\epsilon}_{g,N}(M,\beta)/\mathfrak{M}_{g,N}},$$

under certain assumptions. The proof closely follows [KT18].

Proposition A.1. *Given $(\mathbf{v},\check{\beta})\in\Lambda\oplus\Lambda$, assume a first-order deformation $\kappa_S\in HT^2(S)\cong HH^2(S)$ from Proposition 2.3 is represented by a $\mathbb{C}[\epsilon]/\epsilon^2$-linear admissible subcategory*

$$\mathcal{C}\subseteq \mathrm{D_{perf}}(\mathcal{Y}),$$

where $\mathcal{Y}\to B=\operatorname{Spec}\mathbb{C}[\epsilon]/\epsilon^2$ is flat. Then there exists an obstruction theory morphism

$$(\mathbb{E}^{\bullet}_{\mathrm{red}})^{\vee} \to \mathbb{L}_{Q^{\epsilon}_{g,N}(M,\beta)/\mathfrak{M}_{g,N}}.$$

Proof. Firstly, by taking the central fiber, we get that

$$\mathrm{D_{perf}}(S)\subsetneqq \mathrm{D_{perf}}(Y)$$

is an admissible subcategory, where Y is the central fiber of $\mathcal{Y}$. Therefore there is an isomorphism of moduli stacks

$$\mathfrak{Coh}(S)\cong\mathfrak{D}_{\mathrm{Coh}(S)}(Y), \tag{12}$$

where $\mathfrak{D}_{\mathrm{Coh}(S)}(Y)$ is the moduli stack of objects on Y which are contained in the subcategory $\mathrm{Coh}(S)$. This also implies that the quasimap moduli stacks are isomorphic,

$$Q^{\epsilon}_{g,N}(M, \mathfrak{Coh}(S), \beta) \cong Q^{\epsilon}_{g,N}(M, \mathfrak{D}_{\mathrm{Coh}(S)}(Y), \beta).$$

Let

$$M_Y := M^{\epsilon}_{\mathbf{v},\check{\beta}}(Y \times C_{g,N}/\overline{M}_{g,N}) \cong M^{\epsilon}_{\mathbf{v},\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}) =: M_S$$

be the relative moduli spaces of objects which are associated to moduli spaces $Q^{\epsilon}_{g,N}(M, \mathfrak{D}_{\mathrm{Coh}(S)}(Y), \beta)$ and $Q^{\epsilon}_{g,N}(M, \mathfrak{Coh}(S), \beta)$, respectively.

Secondly, the inclusion

$$\mathrm{D^b}(S) \hookrightarrow \mathrm{D_{perf}}(Y)$$

induces a map between Hochschild cohomologies

$$HH^2(Y) \to HH^2(S), \tag{13}$$

given by restricting the natural transformation of functors

$$\mathrm{id}_{\mathrm{D_{perf}(Y)}} \to [2].$$

This map sends κ_Y to κ_S (see e.g. [Per, Lemma 4.6]), where κ_Y is the class associated to the deformation $\mathcal{Y} \to B$. Moreover, for a complex $F \in \mathrm{D^b}(S \times C)$ the class

$$\kappa(F) \in \mathrm{Ext}^2(F, F),$$

which is given by applying the natural transformation associated to $\kappa \in HH^2(S)$ to F, is the obstruction to deform F in κ-direction. By [Tod09, Proposition 5.2] and [Căl05], it agrees with obstruction class given by composing Kodaira–Spencer class with Atiyah class

$$\kappa(F) = \kappa \cdot \exp(-\mathrm{At}(F))$$

after applying HKR isomorphism

$$HH^2(S) \cong HT^2(S).$$

We now identify a sheaf $F \in \mathrm{Coh}(S \times C)$ with its image in $\mathrm{D_{perf}}(Y \times C)$, then the following triangle commutes

$$\begin{tikzcd} HH^2(S) \arrow[r] & \mathrm{Ext}^2(F,F) \\ HH^2(Y) \arrow[u] \arrow[ur] & \end{tikzcd}$$

Hence by the choice of κ_S, the deformation of sheaves in the class $(\mathbf{v}, \check{\beta})$ viewed as complexes on $Y \times C$ is obstructed in κ_Y-direction, because the obstruction class is non-zero by the construction of κ_S.

We now closely follow [KT18, Section 3.2]. By the above discussion the inclusion of the central fiber over B

$$M_Y \hookrightarrow M_{\mathcal{Y}/B}$$

is an isomorphism. The obstruction complexes of M_Y and M_S are isomorphic under the natural identifications of the moduli spaces

$$R\mathcal{H}om_{\pi_S}(\mathbb{F}_S, \mathbb{F}_S) \cong R\mathcal{H}om_{\pi_Y}(\mathbb{F}_Y, \mathbb{F}_Y), \tag{14}$$

because both complexes can be defined just in terms of $\mathrm{D}^{\mathrm{b}}(S)$, where $\mathbb{F}_{S/Y}$ are universal families of $M_{S/Y}$ with $\pi_{S/Y}$ being the obvious projections. Note that the trace map on $Y \times C$ has no effect on Ext^2, since $H^2(\mathcal{O}_{Y\times C}) = 0$. In certain sense, a semiregularity map σ_i on $S \times C$ corresponds to a semiregularity map σ_{i+1} on $Y \times C$, not σ_i. In particular, $R\mathcal{H}om_{\pi_S}(\mathbb{F}_S, \mathbb{F}_S)_0$ and $R\mathcal{H}om_{\pi_Y}(\mathbb{F}_Y, \mathbb{F}_Y)_0$ are not isomorphic. Nevertheless, we *claim* that the following holds.

Claim. *The composition*

$$(\mathbb{E}^\bullet)^\vee = (R\mathcal{H}om_{\pi_S}(\mathbb{F}_S, \mathbb{F}_S)_0[-1])^\vee \to (R\mathcal{H}om_{\pi_Y}(\mathbb{F}_Y, \mathbb{F}_Y)_0[-1])^\vee \\ \to \mathbb{L}_{M_{\mathcal{Y}/B}/B} \tag{15}$$

where the first map is given by identification (14), while the second is given by the Atiyah class on $\mathcal{Y} \times M_{\mathcal{Y}/B}$, is a perfect obstruction theory.

Proof of the Claim. For the proof of the claim we plan to use the criteria from [BF97, Theorem 4.5].

For any B-scheme Z_0, a B-map $Z_0 \to M_{\mathcal{Y}/B}$ factors though the central fiber. Hence the B-structure map $Z_0 \to B$ factors through the closed point of B. Let $\mathcal{F}_0$ be the sheaf associated to the map $Z_0 \to M_{\mathcal{Y}/B}$. The morphism

$$(R\mathcal{H}om_{\pi_Y}(\mathbb{F}_Y, \mathbb{F}_Y)_0[-1])^\vee \to \mathbb{L}_{M_{\mathcal{Y}/B}/B}$$

is an obstruction theory. By [BF97, Theorem 4.5], to prove that (15) is an obstruction theory, it suffices to prove that the image of a non-zero obstruction class $\varpi(\mathcal{F}_0) \in \mathrm{Ext}^2_{Y\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_Y^* I)$ with respect to the map

$$\mathrm{Ext}^2_{Y\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_Y^* I) \cong \mathrm{Ext}^2_{S\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_S^* I) \\ \to \mathrm{Ext}^2_{S\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_S^* I)_0 \tag{16}$$

is non-zero for any square-zero B-extension Z of Z_0 given by an ideal I, where $p_Y \colon Y \times_B Z_0 = Y \times Z_0 \to Z_0$ and $p_S \colon S \times Z_0 \to Z_0$ are the natural projections.

Given a square-zero B-extension Z of Z_0, there are two possibilities:

(i) the B-structure map $Z \to B$ factors through the closed point;
(ii) the B-structure map $Z \to B$ does not factor through the closed point.

(i) In this case, the obstruction of lifting the map to $Z \to M_{\mathcal{Y}/B}$ coincides with the obstruction of lifting the map to $Z \to M_Y \cong M_S$, hence if $\varpi(\mathcal{F}_0)$ is non-zero, its image with respect (16) is non-zero.

(ii) In this case, a lift to $Z \to M_{\mathcal{Y}/B}$ is always obstructed, and the obstruction is already present at a single fiber of p_Y in the following sense. By assumption there exists a section $B \to Z$ which is an immersion (we can find an open affine subscheme $U \subset Z$ such that $U \to B$ is flat, but then $U \cong U_0 \times B$, because first-order deformations of affine schemes are trivial, thereby we get a section). Let $z \in Z$ be image of the closed point of B of the section, then the restriction

$$\mathrm{Ext}^2_{Y\times S_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_Y^* I) \to \mathrm{Ext}^2_{Y\times z}(\mathcal{F}_{0,z}, \mathcal{F}_{0,z} \otimes p_Y^* I_z)$$

applied to the obstruction class $\varpi(\mathcal{F}_0)$ is non-zero and is the obstruction to lift the sheaf $\mathcal{F}_{0,z}$ on Y to a sheaf on $\mathcal{Y}$, hence due to the following commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^2_{Y\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_Y^* I) & \longrightarrow & \mathrm{Ext}^2_{Y\times z}(\mathcal{F}_{0,z}, \mathcal{F}_{0,z} \otimes p_Y^* I_z) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^2_{S\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p_S^* I)_0 & \longrightarrow & \mathrm{Ext}^2_{S\times z}(\mathcal{F}_{0,z}, \mathcal{F}_{0,z} \otimes p_S^* I_z)_0 \end{array}$$

we conclude that the image of $\varpi(\mathcal{F}_0)$ in $\mathrm{Ext}^2_{S\times Z_0}(\mathcal{F}_0, \mathcal{F}_0 \otimes p^* I)_0$ is non-zero, because the image of $\varpi(\mathcal{F}_{0,z})$ is non-zero in $\mathrm{Ext}^2_{S\times z}(\mathcal{F}_{0,z}, \mathcal{F}_{0,z} \otimes p^* I_z)_0$. This establishes claim.

The absolute perfect obstruction theory $(\mathbb{H}^\bullet)^\vee$ is then defined by taking the cone of $(\mathbb{E}^\bullet)^\vee \to \Omega_B[1]$, so that we have the following diagram

$$\begin{array}{ccccc} (\mathbb{H}^\bullet)^\vee & \longrightarrow & (\mathbb{E}^\bullet)^\vee & \longrightarrow & \Omega_B[1] \\ \downarrow & & \downarrow & & \| \\ \mathbb{L}_{M_Y} & \longrightarrow & \mathbb{L}_{M_{\mathcal{Y}/B}/B} & \longrightarrow & \Omega_B[1] \end{array}$$

By the same argument as in [KT18, Section 2.3], the composition

$$(\mathbb{H}^\bullet)^\vee \to (\mathbb{E}^\bullet)^\vee \to (\mathbb{E}^\bullet_{\mathrm{red}})^\vee$$

is an isomorphism, hence the proposition follows. □

For example, if $M = S^{[n]}$ and $\mathrm{c}_1(\check{\beta}) \neq 0$ (i.e. the curve class is not exceptional), we can use a commutative deformation given by the infinitesimal twistor family $\mathcal{S} = \mathcal{Y} \to B$ with respect to the class $\mathrm{c}_1(\check{\beta})$.

The situation becomes more complicated already in the case of $S^{[n]}$ and $\mathrm{c}_1(\check{\beta}) = 0$ (i.e. an exceptional curve class), a commutative first-order deformation can no longer satisfy the property stated in Proposition 2.3. If $n = 2$ and $S^{[2]}$ is isomorphic to a Fano variety of lines of some special cubic fourfold (e.g. see [Has00, Theorem 1.0.3]), then

$$D_{\mathrm{perf}}(Y) = \langle D_{\mathrm{perf}}(S), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$$

and the family $\mathcal{Y} \to B$ is given by deformation of Y away from the Hassettt divisor.

Remark A.2. In [Tod09], Toda constructed geometric realisations of infinitesimal non-commutative deformations in $HH^2(X)$ for a smooth projective X. However, it is not clear, if they are of the type required by Proposition

A.1. In principle, there should be no problem in proving Proposition A.1, dropping the assumption. For that, one has to show that Toda's infinitesimal deformations behave well under base change.

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