QUASIMAPS TO MODULI SPACES OF SHEAVES

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Abstract. In this article, we develop a theory of quasimaps to moduli spaces of sheaves on a surface S. Under some natural assumptions, we prove that moduli spaces of quasimaps are proper and carry a perfect obstruction theory. Moreover, moduli spaces of quasimaps are naturally isomorphic to relative moduli spaces of sheaves. Using Zhou's theory of calibrated tails, we establish a wall-crossing formula which relates Gromov–Witten theory of $S^{[n]}$ and relative Donaldson–Thomas theory of $S \times C$, where C is a nodal curve.

As an application, we prove that quantum cohomology of $S^{[n]}$ is determined by relative Pandharipande–Thomas theory of $S \times \mathbb{P}^1$, if S is a del Pezzo surface, conjectured by Maulik.

Contents

1. Introduction

1.1. **Overview.** Quasimaps were first considered in an unpublished work by Drinfeld in early 80's in the context of geometric representation theory, see Braverman [Bra06] for an account of the representation-theoretic side of the theory. Their importance in a different but not unrelated field of enumerative geometry was also already understood. In subsequent years the enumerative side of quasimaps was studied as an alternative to Gromov–Witten theory in the case of certain GIT targets by many people (e.g. [MOP11], [Tod11]), leading to the work of Ciocan-Fontanine–Kim–Maulik [CKM14], where the theory was given the most general treatment.

Moduli spaces of stable quasimaps and stable maps are different compactifications of moduli spaces of stable maps with smooth domains. There also exists a mixed theory of ϵ-stable quasimaps that interpolates between the two, thereby giving rise to a wall-crossing, which provides an effective way to compute Gromov–Witten invariants in terms of quasimap invariants, which in many cases are more accessible. Moreover, it turned out that the quasimap wall-crossing is related to enumerative mirror symmetry. For example, in [CK20] it was shown that for a quintic threefold the generating series of quasimap invariants exactly matches the B-model series, while the quasimap wall-crossing is the mirror transformation.

Quasimaps then found their applications beyond numbers in the enumerative geometry of Nakajima quiver varieties (see e.g. [Oko17]), which are naturally GIT quotients. This brought quasimaps back to their roots, since enumerative geometry is inseparable from geometric representation theory in this context. It also brings us to the theme of quasimaps to moduli spaces of sheaves. Already for the simplest example of a Nakajima quiver variety - a punctorial Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of the affine plane $\mathbb{C}^2$ - one can consider five enumerative theories, among which there is the GIT quasimap theory:

GW - Gromov–Witten theory of $(\mathbb{C}^2)^{[n]}$;

Q - GIT quasimap theory of $(\mathbb{C}^2)^{[n]}$;

$\mathsf{GW}_{\mathrm{orb}}$ - orbifold Gromov–Witten theory of $[(\mathbb{C}^2)^{(n)}]$;

$\mathsf{GW}_{\mathrm{rel}}$ - relative Gromov–Witten theory of $\mathbb{C}^2 \times C_{g,N}/\overline{M}_{g,N}$;

$\mathsf{PT}_{\mathrm{rel}}$ - relative Pandharipande–Thomas theory of $\mathbb{C}^2 \times C_{g,N}/\overline{M}_{g,N}$,

where $C_{g,N} \to \overline{M}_{g,N}$ is the universal curve over a moduli space of stable marked curves. These enumerative theories are related in the following ways:

- GIT quasimap wall-crossing between GW and Q, proposed in [CK14], proved in [Zho22];
- the moduli spaces of Q and $\mathsf{PT}_{\mathrm{rel}}$ are naturally isomorphic and virtual fundamental classes coincide, [Oko17, Excercise 4.3.22].
- analytic continuation and a change of variables between GW and $\mathsf{GW}_{\mathrm{orb}}$ provided by crepant resolution conjecture ($\mathsf{C.R.C.}$), proposed in [Rua06], refined in [BG09, CIT09];
- analytic continuation and a change of variables between $\mathsf{GW}_{\mathrm{rel}}$ and $\mathsf{PT}_{\mathrm{rel}}$ provided by Pandharipande–Thomas/Gromov–Witten correspondence ($\mathsf{PT/GW}$), proposed in [MNOP06a, MNOP06b].

Moreover, all of those correspondences are equivalences - the generating series of invariants of the theories above are equal up to a change of variables. The last two correspondences were established in a series of articles [BP08,BG09,OP10a,OP10c,OP10b], culmination of which were [PT19a] and [PT19b], where they were shown to hold on the level of cohomological field theories. Establishment of these correspondences is fundamental for many developments in the field of modern enumerative geometry, like the proof of $\mathsf{PT/GW}$ for a quintic 3-fold in [PP17].

Similar correspondences can be formulated for an arbitrary surface S with one exception - the theory of the type Q does not make sense in the form it is stated for $\mathbb{C}^2$, because, in general, $S^{[n]}$ does not admit a natural GIT presentation[1], despite being constructed with the help of GIT techniques. On the other hand, $S^{[n]}$ is naturally embedded into a *rigidified*[2] stack of coherent sheaves $\mathfrak{Coh}(S)/\!\!/\mathbb{C}^*$. More generally, any moduli space M of Gieseker stable sheaves on S with Chern character $\mathbf{v} \in H^*(S,\mathbb{Q})$ is naturally embedded into a rigidified stack of all coherent sheaves in the class $\mathbf{v}$,

$$M \subset \mathfrak{Coh}_r(S)_{\mathbf{v}} := \mathfrak{Coh}(S)_{\mathbf{v}}/\!\!/\mathbb{C}^*.$$

In this article, we will be interested in quasimaps to a pair

$$(M, \mathfrak{Coh}_r(S)_{\mathbf{v}}),$$

which we define to be maps from nodal curves to $\mathfrak{Coh}_r(S)_{\mathbf{v}}$ which generically map to M, see Definition 3.1. It will be shown that our quasimap theory is naturally equivalent to the theory of the type $\mathsf{PT}_{\mathrm{rel}}$ already on the level of moduli spaces, i.e. the corresponding moduli spaces are naturally isomorphic.

We then introduce a notion of ϵ-stability for quasimaps to moduli spaces of sheaves, which depends on a parameter $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$. Moduli spaces of ϵ-stable qusimaps therefore interpolate between theories of types GW and $\mathsf{PT}_{\mathrm{rel}}$. We prove that their moduli spaces are proper and admit a perfect obstruction theory. Using the theory of calibrated tails of Zhou introduced in [Zho22], we establish a wall-crossing formula which relates invariants for different values of $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$. The result is an equivalence of the theories of type GW and $\mathsf{PT}_{\mathrm{rel}}$ in a general context: for all surfaces, all positive ranks and all curve classes. The wall-crossing invariants are given by the virtual intersection theory of moduli spaces of flags of sheaves on a surface S. They were thoroughly studied in [Obe21] in the case of K3 surfaces, where they were shown to be virtual Euler numbers of certain Quot schemes.

The correspondence between $\mathsf{PT}_{\mathrm{rel}}$ and GW was already considered on the level of invariants, e.g. for $(\mathbb{C}^2)^{[n]}$ in [OP10b] and more recently for $\mathcal{A}_m^{[n]}$ in [Liu21]. It was also expected to hold in a more general context. In particular, the conjectures of Oberdieck–Phandharipande [OP16, Conjecture A] and Oberdieck [Obe19, Conjecture 1] regarding such relation for K3 surfaces served as our main motivating goal.

This article is the first in a series of three articles, being its technical heart. The two others are [Nes21,Nes22b]. Here, we lay down the foundation of the

[1]There is no natural choice of a GIT stack, whose stable locus is $S^{[n]}$, apart from $S^{[n]}$ itself, which is not interesting for our purposes.

[2]Rigidification amounts to taking quotient of the usual stack $\mathfrak{Coh}(S)$ by the scaling $\mathbb{C}^*$-action, the quotient affects the automorphisms of the objects but not the isomorphism classes of the objects. We refer to Section 2.1 for more details.

theory of quasimaps to moduli spaces of sheaves. While in [Nes21], we focus on quasimaps to moduli spaces of sheaves on a K3 surface. Moduli spaces of sheaves on K3 surfaces require a special treatment due to the presence of a holomorphic symplectic form and, consequently, vanishing of standard virtual fundamental classes of the relevant enumerative theories.

In [Nes22b], we study a correspondence between $\mathsf{GW}_{\mathrm{orb}}$ and $\mathsf{GW}_{\mathrm{rel}}$ for an arbitrary smooth projective target X. Influenced by the ideas from the theory of quasimaps, we introduce a stability, termed ϵ-admissibility, for maps from nodal curves to $X \times C_{g,N}/\overline{M}_{g,N}$, which depends on the parameter $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$. Moduli spaces of ϵ-admissible maps interpolate between theories $\mathsf{GW}_{\mathrm{orb}}$ and $\mathsf{GW}_{\mathrm{rel}}$ for X. Using Zhou's theory of calibrated tails, we establish a wall-crossing formula relating the invariants for different values of $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$, which is completely analogous to quasimap wall-crossing formulas. The result is an equivalence of the theories of type $\mathsf{GW}_{\mathrm{orb}}$ and $\mathsf{GW}_{\mathrm{rel}}$ for an arbitrary smooth projective target X. This wall-crossing can be termed Gromov–Witten/Hurwitz ($\mathsf{GW/H}$) wall-crossing, because if X is a point, the moduli spaces of ϵ-admissible maps interpolate between Gromov–Witten and Hurwitz spaces.

Together, quasimap and $\mathsf{GW/H}$ wall-crossings can be represented by the square in Figure 1.

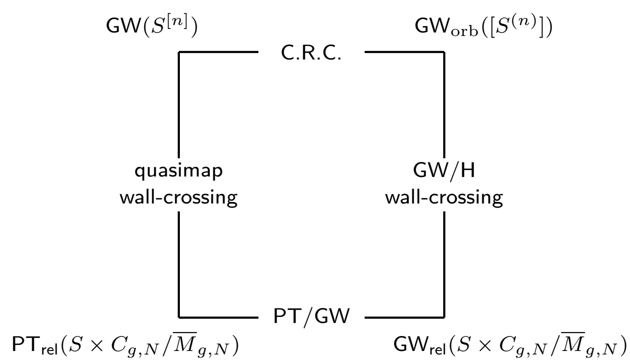

FIGURE 1. The Square

The square seems to be a natural framework for both $\mathsf{C.R.C.}$ and $\mathsf{PT/GW}$ in this context. It was considered for the first time in [BG09] for $S = \mathbb{C}^2$ and $C = \mathbb{P}^1$, and was established with the help of [OP10a, OP10b, BP08]. We, however, provide a geometric justification for the square. Moreover, the vertical sides of the square hold in greater generality - quasimap wall-crossing holds for moduli spaces of sheaves of an arbitrary rank; $\mathsf{GW/H}$ wall-crossing holds for targets of an arbitrary dimension.

Using the wall-crossings, we establish the following results. Here, we prove:

- quantum cohomology of $S^{[n]}$ is determined by relative Pandharipande–Thomas theory of $S \times \mathbb{P}^1$, if S is a del Pezzo surface, conjectured by Davesh Maulik;

In [Nes21]:

- quantum cohomology of $S^{[n]}$ is determined by relative Pandharipande–Thomas theory of $S\times\mathbb{P}^1$, if S is a K3 surface, conjectured in [Obe19];
- the wall-crossing part of Igusa cusp form conjecture, conjectured in [OP10b];
- relative higher-rank/rank-one Donaldson–Thomas correspondence for $S \times \mathbb{P}^1$ and $S \times E$, if S is a K3 surface and E is an elliptic curve;
- relative Donaldson–Thomas/Pandharipande–Thomas correspondence for $S \times \mathbb{P}^1$, if S is a K3 surface;

In [Nes22b]:

- 3-point genus-0 $\mathsf{C.R.C.}$ in the sense of [BG09] for the pair $S^{[n]}$ and $[S^{(n)}]$, if S is a toric del Pezzo surface.
- the geometric origin of $y = -e^{iu}$ in PT/GW through $\mathsf{C.R.C.}$

Moreover, the quasimap wall-crossing played a crucial role in establishing a holomorphic anomaly equation for $\mathrm{K3}^{[n]}$ in [Obe22].

From the perspective of mathematical physics, the quasimap wall-crossing is related to so-called *dimensional reduction*. For example, it was used in [KW07]. In fact, our quasimap wall-crossing for moduli spaces of rank-0 sheaves is one of the algebro-geometric aspects of [KW07] (observed independently by [SW22]). This will be addressed in the forthcoming work - [Nes22a]. For more on dimensional reduction in a mathematical context, we refer to [GLSY18], and in a physical context - to [BJSV95].

1.2. **Quasimaps and sheaves.** Let us explain the correspondence between quasimaps to a moduli space of sheaves on a surface and sheaves on threefolds. For simplicity, let our moduli space be $S^{[n]}$ for a smooth surface S over the field of complex numbers $\mathbb{C}$, satisfying $q(S) := h^{1,0}(S) = 0$. Then we have a natural embedding

$$S^{[n]} \subset \mathfrak{Coh}_r(S)_{\mathbf{v}},$$

such that the complement of $S^{[n]}$ is the locus of non-torsion-free sheaves with Chern character $\mathbf{v} = (1, 0, -n) \in H^*(S, \mathbb{Q})$.

For the choice of $\mathbf{v}$ as above, the stack $\mathfrak{Coh}_r(S)_{\mathbf{v}}$ has a canonical universal family. For a smooth curve C, the relation between torsion-free sheaves on a threefold $S \times C$ and quasimaps from C to the pair $(S^{[n]}, \mathfrak{Coh}_r(S)_{\mathbf{v}})$ then becomes apparent. Indeed, by the moduli problem of sheaves on S, the later is given by a family of sheaves on S over C, i.e. a sheaf on $S \times C$,

$$f\colon C \to \mathfrak{Coh}_r(S)_{\mathbf{v}} \Leftrightarrow F \in \mathrm{Coh}(S \times C),$$

where F is flat over C. The rigidification of the stack amounts to requiring the determinant of F to be trivial. The general fiber of F over C is torsion-free by the definition of a quasimap. Therefore F is torsion-free itself. Being of rank 1 and having a trivial determinant, F is an ideal sheaf of a 1-dimensional subscheme. Conversely, any ideal sheaf of 1-dimensional subscheme defines a quasimap in the above sense.

The degree of a quasimap to a pair $(S^{[n]}, \mathfrak{Coh}_r(S)_{\mathbf{v}})$ is defined by evaluating it at determinant line bundles over $\mathfrak{Coh}_r(S)_{\mathbf{v}}$. In this way, the degree is determined by the Chern character of the corresponding family and vice versa,

$$\text{degree } \beta \text{ of } f \Leftrightarrow \mathrm{ch}(F) = (n, \check{\beta}).$$

For more about the notation of Chern character, we refer to Section 3 and Section 4.1.

1.3. **Stability.** We import ϵ-stability from the GIT set-up to ours, Definition 3.10. This will allow us to interpolate between GW theory and stable[3] quasimap theory. The idea of ϵ-stability can be summarised as follows. In the stable quasimap theory we trade rational tails (which are allowed in GW theory) for *base points*[4] (which are prohibited in GW theory) for the sake of properness of moduli spaces. On the other hand, ϵ-stability allows both rational tails and base points, putting numerical restrictions on their degrees. The parameter $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$ is the measure of that degree, see Definition 3.10. When $\epsilon = 0^+$, quasimaps do not have any rational tails but have base points of all degrees. When $\epsilon = \infty$, quasimaps do not have any base points but have rational tails of all degrees.

In the language of one-dimensional subschemes on threefolds, ϵ-stability controls non-flatness of a subscheme on $S \times C$ over C. Non-flatness arises due to the presence of non-dominant components or floating points. ϵ-stability requires that a weighted[5] sum of the degree and the Euler characterstics of either floating points or non-dominant components must not exceed $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$. If it becomes larger than ϵ in the limit, then a curve sprouts a rational tail, like in relative DT theory. In addition, ϵ-stability also controls the degree and the Euler characteristics of components of the subscheme which lie on rational tails. See Corollary 4.1 for more precise statements.

1.4. **Properness.** Having defined ϵ-stability, we then use the relation between sheaves and quasimaps to prove Proposition 3.21, where it is shown that moduli spaces of ϵ-stable quasimaps are proper for projective moduli spaces of stable sheaves M, for which there exists a class $u \in K_0(S)$, such that

$$\int_S \mathbf{v} \cdot \mathrm{ch}(u) \cdot \mathrm{td}_S = 1.$$

[3]By which we mean 0^+-stable quasimaps.

[4]Those points that are mapped outside of the stable locus.

[5]The degree is weighted more than the Euler characterstics.

In particular, M must be a fine moduli space. The stack $\mathfrak{Coh}_r(S)_{\mathbf{v}}$ is not bounded, but the stability of quasimaps suffices to guarantee the boundedness of moduli spaces of ϵ-stable quasimaps. However, it is essential to consider the entire stack $\mathfrak{Coh}_r(S)_{\mathbf{v}}$, because with the increase of the degree, the more of the stack becomes relevant for the properness of moduli spaces. This is one of the reasons why GIT point-of-view breaks[6] down here, at least for a projective surface. Nevertheless, we closely follow the proof of properness in the GIT set-up, and it roughly consists of two steps.

The first step is to prove that the number of components of the domain of a quasimap is bounded after fixing the degree and the genus. This is achieved with a line bundle on the stack $\mathfrak{Coh}_r(S)_{\mathbf{v}}$ which is positive with respect to quasimaps, see Section 3.1. Our construction of such line bundle crucially exploits the geometry of coherent sheaves, in particular, Langton's semistable reduction, [Lan75].

The second step is to show that quasimaps are bounded for a fixed curve. For this, in Lemma 3.11 we reverse the Langton's semistable reduction and prove that there is bounded amount of choices to obtain a stable quasimap of a fixed degree from a stable map. Boundedness also implies that families of sheaves corresponding to quasimaps are stable for a *suitable* stability. In Appendix A the converse is shown to be true for moduli spaces of slope stable sheaves with $\mathrm{rk} \leq 2$. In the case of $S^{[n]}$, it is not difficult to see, as sheaves are of rank one, therefore being stable is equivalent to being torsion-free. We also expect it to be true in general. We then prove a variant of *Hartog's property* for sheaves on families of nodal curves over a discrete valuation ring (DVR), Lemma 3.19, which allows us to conclude the proof of properness of the moduli spaces in the same way as it is done in the GIT case, [CKM14, Section 4.3].

On the way we establish a precise relation between quasimaps and sheaves. Namely, in the case of $S^{[n]}$, the moduli space of stable quasimaps is naturally isomorphic to a relative Hilbert scheme

$$Q_{g,N}(S^{[n]},\beta) \cong \mathrm{Hilb}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}).$$

More specifically, the moduli space on the right parametrises triples $(I, S \times C, \mathbf{x})$, where I is an ideal on $S \times C$ and $\mathbf{x}$ is a marking of C. Stability of such triples consists of the following data:

- the curve $(C, \mathbf{x})$ is *semistable*, in particular, it does not have rational tails;
- the subscheme corresponding to the ideal is flat over nodes and marked points [7];

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[6]More precisely, the stack $\mathfrak{Coh}(S)$ is locally a GIT stack. However, it is unbounded and (very) singular. Moreover, those GIT charts, through which our quasimaps factor for a fixed degree, are not stacky quotients of affine schemes. Therefore results from the theory of GIT quasimaps are not applicable.

[7]This is a usual stability condition in relative DT theory, referred to as *predeformability*.

- the ideal is fixed only by finitely many automorphisms of the curve $(C, \mathbf{x})$.

A moduli space of ϵ-stable quasimaps $Q^{\epsilon}_{g,N}(S^{[n]},\beta)$ similarly admits a purely ideal-theoretic formulation, such that some rational tails are allowed and some subschemes with vertical components are prohibited, as is shown in Corollary 4.1. These moduli spaces therefore provide an interpolation,

$$\overline{M}_{g,N}(S^{[n]},\beta) \leftarrow\!\!\epsilon\!\!\rightarrow \mathrm{Hilb}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}). \tag{1}$$

The higher-rank case also admits an identification with relative moduli spaces of sheaves, see Definition 3.14. The resulting moduli spaces of sheaves have some interesting features. For example, they have an exotic determinant-line-bundle condition, which behaves well with respect to degenerations. As a result, we can define higher-rank DT theory on $S \times C$ relative to a vertical divisor (a divisor of the type $S \times p \subset S \times C$). DT theory relative to a divisor has not been defined in the literature for any threefold. Among other things, this permits to apply degeneration techniques to higher-rank DT theory, at least in this set-up. For more details, we refer to Remark 3.16.

1.5. **Perverse quasimaps.** A variant of the quasimap theory for a moduli space of sheaves is given by considering the stack of objects in a perverse heart $\mathrm{Coh}(S)^{\sharp}$. For $S^{[n]}$, one can therefore consider a perverse pair,

$$S^{[n]} \subset \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}},$$

see Section 4.2 for precise definitions. The moduli space of stable perverse quasimaps is isomorphic to the relative moduli stack of stable pairs (we refer to [PT09] for the theory of stable pairs in enumerative geometry),

$$Q_{g,N}(S^{[n]},\beta)^{\sharp} \cong \mathrm{P}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}).$$

In Section 4.3, we discuss the case of $S = \mathbb{C}^2$, for which the moduli stack of perverse sheaves with a framing is naturally isomorphic to the GIT stack associated to $(\mathbb{C}^2)^{[n]}$ (including the unstable part) viewed as Nakajima quiver variety, thereby making GIT quasimaps and moduli-of-sheaves quasimaps equivalent in this case. This provides a conceptual geometric explanation for the equivalences of different enumerative theories that were previously observed on the level of invariants, e.g. in [OP10b].

1.6. **Obstruction theory.** An obstruction theory of $Q_{g,N}(S^{[n]},\beta)$ is given by the deformation theory of maps from curves to a derived enhancement of $\mathbb{R}\mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$ of $\mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}}$. The former exists by [TV07]. In fact, we consider a modification of the standard enhancement - we take a derived fiber over derived Picard stack, as is explained in [ST15], to obtain the enhancement, whose virtual tangent bundle is given by the traceless obstruction theory of sheaves. The relative sheaf-theoretic obstruction theory of $\mathrm{Hilb}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N})$ can be shown to be isomorphic to the map-theoretic obstruction theory of $Q_{g,N}(S^{[n]},\beta)$, as is explained in Section 5.2. With this comparison result, we can relate GW theory and relative DT theory via quasimap wall-crossing.

The moduli space of ϵ-stable quasimaps has all the necessary structure, such as the evaluation maps, to define invariants via the virtual fundumental class in the spirit of GW theory,

$$\langle \tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{M,\epsilon}_{g,\beta} := \int_{[Q^{\epsilon}_{g,N}(M,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{i=N} \psi_i^{m_i}\mathrm{ev}_i^*\gamma_i,$$

where $\gamma_1,\dots,\gamma_N$ are classes in $H^*(S^{[n]})$ and $\psi_1,\dots,\psi_N$ are ψ-classes associated to the markings of curves. In the language of ideals on threefolds, the primary quaismap insertions correspond exactly to relative DT insertions. We similarly define perverse invariants ${}^{\sharp}\langle \tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon}_{g,N,\beta}$ associated to the pair $(S^{[n]},\mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}})$.

1.7. **Wall-crossing.** Invoking recent results of [Zho22], we then establish the quasimap wall-crossing. However, for this part of the present work we mostly refer to Zhou's article, as his arguments carry over to our case almost verbatim. We now briefly explain what is meant by the quasimap wall-crossing.

The space $\mathbb{R}_{>0}\cup\{0^+,\infty\}$ of ϵ-stabilities is divided into chambers, in which the moduli space $Q^{\epsilon}_{g,N}(M,\beta)$ stays the same, and as ϵ crosses a wall between chambers the moduli space changes discontinuously. The quasimap wall-crossing relates invariant for different values of ϵ, it involves certain DT-type invariants that are defined via the virtual localisation $S\times\mathbb{P}^1$ with respect to the $\mathbb{C}^*$-action on $\mathbb{P}^1$,

$$t[x,y]=[tx,y],\quad t\in\mathbb{C}^*.$$

These invariants are assembled together in so-called *I-functions*, which is defined in Section 6.1. By convention we set

$$e_{\mathbb{C}^*}(\mathbb{C}_{\mathrm{std}})=-z,$$

where $\mathbb{C}_{\mathrm{std}}$ is the standard representation of $\mathbb{C}^*$ on $\mathbb{C}$. Then in the case of $S^{[n]}$, the I-function is

$$I(q,z)=1+\sum_{\beta>0} q^{\beta}\mathsf{ev}_*\left(\frac{[F_\beta]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})}\right)\in A^*(S^{[n]})[z^{\pm}]\otimes_{\mathbb{Q}}\mathbb{Q}[\![q^{\beta}]\!],$$

where $F_\beta\subset \mathrm{Hilb}_{n,\check{\beta}}(S\times\mathbb{P}^1)$ is the distinguished $\mathbb{C}^*$-fixed component consisting of subschemes which are supported on fibers of $S\times\mathbb{P}^1\to S$ and over $0\in\mathbb{P}^1$, and which are non-flat only over $0\in\mathbb{P}^1$. The evaluation

$$\mathsf{ev}\colon F_\beta\to S^{[n]}$$

is defined by taking the fiber of the subscheme over $\infty\in\mathbb{P}^1$. We define

$$\mu(z):=[zI(q,z)-z]_+\in A^*(S^{[n]})[z],$$

where $[\dots]_+$ is the truncation by taking only non-negative powers of z. To state the wall-crossing formula in the most efficient way, we assemble invariants in the following generating series

$$F_g^{\epsilon}(\mathbf{t}(z)) = \sum_{n=0}^{\infty}\sum_{\beta}\frac{q^{\beta}}{N!}\langle \mathbf{t}(\psi_1),\dots,\mathbf{t}(\psi_N)\rangle^{\epsilon}_{g,N,\beta},$$

where $\mathbf{t}(z) \in H^*(X^{[n]},\mathbb{Q})[z]$ is a generic element, and the unstable terms are set to be zero.

Theorem. *For all $g \geq 1$ we have*

$$F_g^{0^+}(\mathbf{t}(z)) = F_g^{\infty}(\mathbf{t}(z)+\mu(-z)).$$

For $g=0$, the same equation holds modulo constant and linear terms in $\mathbf{t}$.

The change of variables above is a consequence of the wall-crossing formula across each wall between extremal values of ϵ, see Theorem 6.3. Moreover, by evoking the identification $\mathbb{C}^*$-equivariant sheaves on $S\times\mathbb{C}$ with flags of sheaves on S, one can express the wall-crossing invariants in terms of integrals on moduli spaces of flags of sheaves. For more details on this relation we refer to [Obe21], where the case of K3 surfaces is treated, leading to a beautiful connection between different enumerative theories.

1.8. **Applications of the quasimap wall-crossing.**

1.8.1. *Enumerative geometry of $S^{[n]}$.* The quasimap wall-crossing can be effectively used to study the enumerative geometry of $S^{[n]}$ by relating it to PT theory of $S\times C$ and, consequently, to GW theory of $S\times C$ by PT/GW correspondence. The latter is much simpler to deal with, as $S\times C$ is a 3-dimensional product variety. Such approach proved to be very useful in establishing a holomorphic anomaly equation for $\mathrm{K3}^{[n]}$ in [Obe21].

In Section 6.4, the wall-crossing invariants of $S^{[n]}$ for a del Pezzo surface S are explicitly computed. To sate the result, recall that using Nakajima operators, for $n>1$ we have the following identification

$$H_2(S^{[n]},\mathbb{Z}) \cong H_2(S,\mathbb{Z})\oplus\mathbb{Z}\cdot\mathsf{A},$$

where A is the exceptional curve class of the Hilbert–Chow morphism

$$S^{[n]}\to S^{(n)}.$$

Invoking the identification above, we assemble invariants in classes

$$(\gamma,\mathsf{mA})\in H_2(S,\mathbb{Z})\oplus\mathbb{Z}\cdot\mathsf{A}$$

in the following generating series

$$^{\sharp}\langle\gamma_1,\dots,\gamma_N\rangle^{S^{[n]},\epsilon}_{0,\gamma} := \sum_{\mathsf{m}} {}^{\sharp}\langle\gamma_1,\dots,\gamma_N\rangle^{S^{[n]},\epsilon}_{0,(\gamma,\mathsf{mA})}y^{\mathsf{m}}.$$

Assuming $2g-2+N \geq 0$, the quasimap wall-crossing then gives us

$$^{\sharp}\langle \gamma_1, \dots, \gamma_N \rangle_{0,\gamma}^{S^{[n]},0^+} = (1+y)^{\mathrm{c}_1(S)\cdot\gamma} \cdot {}^{\sharp}\langle \gamma_1, \dots, \gamma_N \rangle_{0,\gamma}^{S^{[n]},\infty}.$$

After applying identification of moduli spaces of perverse quasimaps with moduli spaces of stable pairs on $S\times\mathbb{P}^1$, the above result relates the quantum cohomology of $S^{[n]}$ to the ring, whose structure constants are given by PT theory of $S\times\mathbb{P}^1$. The change of variables as above was predicted[8] by Davesh Maulik.

1.8.2. *DT/PT for $S\times C$.* Using both standard and perverse quasimap wall-crossings, we can reduce Donaldson–Thomas/Pandharipande–Thomas correspondence (DT/PT) for a relative geometry of the form

$$S \times C_{g,N} \to \overline{M}_{g,N}$$

to DT/PT of wall-crossing invariants, as it is illustrated in Figure 2.

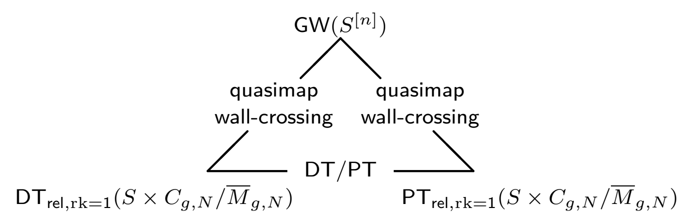

FIGURE 2. DT/PT

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[8]Communicated to the author by Georg Oberdieck.

1.10. **Notation and conventions.** We work over the field of complex numbers $\mathbb{C}$.

We set $e_{\mathbb{C}^*}(\mathbb{C}_{\mathrm{std}}) = -z$, where $\mathbb{C}_{\mathrm{std}}$ is the standard representation of $\mathbb{C}^*$ on a vector space $\mathbb{C}$.

Let N be a semigroup and $\beta \in N$ be its generic element. By $\mathbb{Q}[\![q^\beta]\!]$ we will denote the (completed) semigroup algebra $\mathbb{Q}[\![N]\!]$. In our case, N will be various semigroups of effective curve classes.

After fixing an ample line bundle $\mathcal{O}_S(1)$ on a surface S, we define $\deg(F)$ to be the degree of a sheaf F with respect to $\mathcal{O}_S(1)$. By a *general fiber* of a sheaf F on $S \times C$, we will mean a fiber of F over some dense open subset of C.

2. Stack of coherent sheaves

2.1. **Rigidification.** Let S be a smooth projective surface. Let $\mathcal{O}_S(1) \in \mathrm{Pic}(S)$ be a very ample line bundle and $\mathbf{v} \in K_{\mathrm{num}}(S)$ be a class, such that

- $\mathrm{rk}(\mathbf{v}) > 0$;
- there are no strictly Gieseker semistable sheaves.

We will frequently identify $\mathbf{v}$ with its Chern character. Let

$$\mathfrak{Coh}(S)_{\mathbf{v}} \colon (Sch/\mathbb{C})^\circ \to (Grpd)$$

be the stack of coherent sheaves on S in the class $\mathbf{v}$. We will usually drop $\mathbf{v}$ from the notation, as we will be working with a fixed class, unless we want to emphasise some particular choice of the class. There is a locus of Gieseker $\mathcal{O}_S(1)$-stable sheaves in the class $\mathbf{v}$,

$$\mathcal{M} \hookrightarrow \mathfrak{Coh}(S),$$

which is a $\mathbb{C}^*$-gerbe over a projective scheme M. The $\mathbb{C}^*$-automorphisms come from multiplication by scalars. In fact, we can quotient out $\mathbb{C}^*$-automorphisms of the entire stack $\mathfrak{Coh}(S)$, as explained in [AGV08, Appendix C], thereby obtaining a *rigidified* stack

$$\mathfrak{Coh}_r(S) := \mathfrak{Coh}(S) /\!\!/ \mathbb{C}^*.$$

A B-valued point of $\mathfrak{Coh}_r(S)$ can be represented by a pair $(\mathcal{G}, \phi)$, where $\mathcal{G}$ is a $\mathbb{C}^*$-gerbe over B and $\phi \colon \mathcal{G} \to \mathfrak{Coh}(S)$ is a $\mathbb{C}^*$-equivariant map (here we will ignore 2-categorical technicalities, see [AGV08, Appendix C.2] for more details). The moduli space M canonically embeds into the stack $\mathfrak{Coh}_r(S)$, giving rise to the following square

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[d, "\mathbb{C}^*-\text{gerbe}"] & \mathfrak{Coh}(S) \arrow[d, "\mathbb{C}^*-\text{gerbe}"] \\ M \arrow[r, hook] & \mathfrak{Coh}_r(S) \end{tikzcd}$$

Now let $(X, \mathfrak{X})$ be one of the following pairs, $(M, \mathfrak{Coh}_r(S))$ or $(\mathcal{M}, \mathfrak{Coh}(S))$. Abusing the notation, we define

$$\mathrm{Pic}(\mathfrak{X}) := \lim_{\mathcal{U} \subset \mathfrak{X}} \mathrm{Pic}(\mathcal{U}),$$

where the limit is taken over open substacks of finite type. The stack $\mathfrak{Coh}(S)$ is not of finite type, therefore this definition of Picard group might not agree with the standard one. However, for our purposes it is the most suitable one. We will refer to the elements of $\mathrm{Pic}(\mathfrak{X})$ as *line bundles.* The need for this definition of the Picard group is justified in Remark 2.1.

2.2. **Determinant line bundles.** Let $\mathcal{F}$ be the universal sheaf on $S \times \mathfrak{Coh}(S)$, then for each open substack of finite type $\mathcal{U} \subset \mathfrak{Coh}(S)$, we have naturally defined maps

$$\lambda_{|\mathcal{U}} \colon K_0(S) \xrightarrow{p_S^!} K_0(S \times \mathcal{U}) \xrightarrow{\cdot [\mathcal{F}_{|\mathcal{U}}]} K_0(S \times \mathcal{U}) \xrightarrow{p_{\mathcal{U}!}} K_0(\mathcal{U}) \xrightarrow{\det} \mathrm{Pic}(\mathcal{U})$$

which are compatible with respect to inclusions $\mathcal{U}' \subset \mathcal{U}$, hence we have the induced map

$$\lambda \colon K_0(S) \to \mathrm{Pic}(\mathfrak{Coh}(S)).$$

Remark 2.1. The construction of $\lambda_{|\mathcal{U}}$ requires a locally free resolutions of $\mathcal{F}_{|\mathcal{U}}$, the ranks of terms of the resolution grow with $\mathcal{U}$. Hence, to the best knowledge of the author, determinant line bundles cannot be easily defined as honest line bundles on $\mathfrak{Coh}(S)$, but only as elements of $\mathrm{Pic}(\mathfrak{Coh}(S))$ in the sense of our definition of $\mathrm{Pic}(\mathfrak{Coh}(S))$.

In general, the $\mathbb{C}^*$-weight of the line bundle $\lambda(u)$ is equal to the Euler characteristics $\chi(\mathbf{v} \cdot u)$,

$$w_{\mathbb{C}^*}(\lambda(u)) = \chi(\mathbf{v} \cdot u),$$

so there are two types of classes that we will be of interest to us. A class $u \in K_0(S)$, such that $\chi(\mathbf{v} \cdot u) = 1$, gives a trivilisation of the $\mathbb{C}^*$-gerbe

$$\mathfrak{Coh}(S) \to \mathfrak{Coh}_r(S)$$

over each open substack of finite type $\mathcal{U}$, or, in other words, a universal family on $\mathfrak{Coh}_r(S)$. While for a class $u \in K_0(S)$, such that $\chi(\mathbf{v} \cdot u) = 0$, the line bundle $\lambda(u)_{|\mathcal{U}}$ descends to $\mathcal{U} /\!\!/ \mathbb{C}^*$. Let

$$K_{\mathbf{v}}(S) := \mathbf{v}^{\perp} = \{u \in K_0(S) \mid \chi(\mathbf{v} \cdot u) = 0\} \subset K_0(S),$$

then λ restricted to $K_{\mathbf{v}}(S)$ descends to a map to $\mathrm{Pic}(\mathfrak{Coh}_r(S))$,

$$\lambda_{\mathbf{v}} \colon K_{\mathbf{v}}(S) \to \mathrm{Pic}(\mathfrak{Coh}_r(S)).$$

The class $\mathbf{v}$ will be frequently dropped from the notation in $\lambda_{\mathbf{v}}$, when it is clear what stack is considered. We define

$$\mathrm{Pic}_{\lambda}(\mathfrak{Coh}(S)) := \mathrm{Im}(\lambda), \quad \mathrm{Pic}_{\lambda}(\mathfrak{Coh}_r(S)) := \mathrm{Im}(\lambda_{\mathbf{v}}).$$

There exists a particular class of elements in $K_{\mathbf{v}}(S)$, which deserve a special mention and will be used extensively later,

$$u_i := -\mathrm{rk}(\mathbf{v}) \cdot h^i + \chi(\mathbf{v} \cdot h^i) \cdot [\mathcal{O}_x],$$
$$\mathcal{L}_i := \lambda(u_i),$$

where $\mathcal{O}_x$ is a structure sheaf of a point $x \in S$, and $h = [\mathcal{O}_H]$ for a hyperplane $H \in |\mathcal{O}_S(1)|$. More generally, let us fix a $\mathbb{Q}$-basis $\{L_1, \dots, L_{\rho(S)}\}$ of $NS(S)$ consisting of ample $\mathbb{Q}$-line bundles, such that L_i's and $\mathcal{O}_S(1)$ are in the same chamber of Gieseker stabilities. Let $\{\mathcal{L}_{1,1}, \dots, \mathcal{L}_{1,\rho(S)}\}$ be the corresponding determinant $\mathbb{Q}$-line bundles defined in the same way as $\mathcal{L}_1$. The importance of these classes is due to the following theorem.

Theorem 2.2. *The line bundles $\mathcal{L}_1$ and $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ are nef and ample respectively on fibers of $M \to \mathrm{Pic}(S)$ for all $m \gg 0$. The same holds for $\mathcal{L}_{1,\ell}$. Moreover, their restrictions to the fibers are independent of a point $x \in S$.*

Proof. See [HL97, Chapter 8]. □

3. Quasimaps

From now on, we assume that

$$q(S) := h^{1,0}(S) = 0.$$

The case of surfaces with $q(S) \neq 0$ is discussed in Section 3.6.

Definition 3.1. A map $f\colon (C, \mathbf{x}) \to \mathfrak{X}$ is a *quasimap* to $(X, \mathfrak{X})$ of genus g and of degree $\beta \in \mathrm{Hom}(\mathrm{Pic}_\lambda(\mathfrak{X}), \mathbb{Z})$ if

- $(C, \mathbf{x})$ is a marked nodal connected curve of genus g,
- $\mathcal{L} \cdot_f C := \deg(f^*\mathcal{L}) = \beta(\mathcal{L})$ for all $\mathcal{L} \in \mathrm{Pic}_\lambda(\mathfrak{X})$;
- $|\{p \in C |\ f(p) \in \mathfrak{X} \setminus X\}| < \infty$.

We will refer to the set $\{p \in C |\ f(p) \in \mathfrak{X} \setminus X\}$ as *base* points. A quasimap f is *prestable* if

- $\{t \in C |\ f(t) \in \mathfrak{X} \setminus X\} \cap \{\text{nodes}, \mathbf{x}\} = \emptyset$.

We define

$$\mathrm{Eff}(X, \mathfrak{X}) \subset \mathrm{Hom}(\mathrm{Pic}_\lambda(\mathfrak{X}), \mathbb{Z})$$

to be the cone of classes of quasimaps.

Let

$$\Lambda := \bigoplus_p H^{p,p}(S).$$

be the (p,p)-part of the cohomology of the surface S. For a smooth connected curve C, we then have a Künneth's decomposition of (p,p)-part of the cohomology of the threefold $S \times C$,

$$\bigoplus_p H^{p,p}(S \times C) = \Lambda \otimes H^0(C, \mathbb{C}) \oplus \Lambda \otimes H^2(C, \mathbb{C}) = \Lambda \oplus \Lambda. \tag{2}$$

Let $f\colon C \to \mathfrak{Coh}(S)$ be a quasimap of degree β. By the definition of $\mathfrak{Coh}(S)$, the quasimap f is given by a sheaf F on $S \times C$ which is flat over C. The

Chern character of F has two components with respect to the decomposition in (2),

$$\mathrm{ch}(F) = (\mathrm{ch}(F)_{\mathrm{f}}, \mathrm{ch}(F)_{\mathrm{d}}) \in \Lambda \oplus \Lambda,$$

where the subscripts "f" and "d" stand for *fiber* and *degree*, respectively. As the notation suggests,

$$\mathrm{ch}(F)_{\mathrm{f}} = \mathbf{v},$$

which can be seen by pulling back $\mathrm{ch}(F)$ to a fiber over C and using the flatness of F. Consider now the linear extension

$$\mathrm{Eff}(\mathcal{M}, \mathfrak{Coh}(S)) \to \Lambda, \quad \beta \mapsto \mathrm{ch}(F)_{\mathrm{d}} \tag{3}$$

of the map given by associating the degree part of the Chern character to the degree of the quasimap on a smooth curve. By relating β to $\mathrm{ch}(F)_{\mathrm{d}}$ in more explicit terms in the following lemma, we show that the association above is indeed well-defined, i.e. a degree β cannot have a presentation by two different $\mathrm{ch}(F)_{\mathrm{d}}$'s. Later, in Corollary 3.13, it will be shown that the map is even injective, i.e. the degree of f and the Chern character of the corresponding family F determine each other, thereby justifying the subscript "d" in $\mathrm{ch}(F)_{\mathrm{d}}$.

Lemma 3.2. *The map (3) is well-defined.*

Proof. By the functoriality of the determinant line bundle construction,

$$\beta(\lambda(u)) = \deg(\lambda_F(u)),$$

where $\lambda_F(u)$ is the determinant line bundle associated to the family F and a class $u \in K_0(S)$. Using Grothendieck–Riemann–Roch and projection formulae we obtain

$$\begin{aligned}
\deg(\lambda_F(u)) &= \int_C \mathrm{ch}(p_{C!}(p_S^! u \cdot [F])) \\
&= \int_{S\times C} \mathrm{ch}(p_S^! u \cdot [F]) \cdot p_S^* \mathrm{td}_S \\
&= \int_S \mathrm{ch}(u) \cdot p_{S*}\mathrm{ch}(F) \cdot \mathrm{td}_S \\
&= \int_S \mathrm{ch}(u) \cdot \mathrm{ch}(F)_{\mathrm{d}} \cdot \mathrm{td}_S.
\end{aligned}$$

Now let $\beta_\Lambda \colon \Lambda \to \mathbb{Q}$ be the descend of $(\beta \circ \lambda)_{\mathbb{Q}} \colon K_0(S)_{\mathbb{Q}} \to \mathbb{Q}$ to Λ via Chern character,

$$\begin{tikzcd}
\Lambda \arrow[r, dashed, "\beta_\Lambda"] & \mathbb{Q} \\
K_0(S)_{\mathbb{Q}} \arrow[u, "\mathrm{ch}"] \arrow[ur, "(\beta\circ\lambda)_{\mathbb{Q}}"'] &
\end{tikzcd}$$

which exists by the above formula for the degree of a determinant line bundle. The formula also shows that the descend β_Λ and β determine each other. We thereby obtain an expression of $\mathrm{ch}(F)_{\mathrm{d}}$ in terms of β_Λ,

$$\mathrm{ch}(F)_{\mathrm{d}} = \beta_\Lambda^\vee \cdot \mathrm{td}_S^{-1},$$

where $\beta_\Lambda^\vee$ is the dual of β_Λ with respect to the cohomological intersection pairing on Λ. Using non-degeneracy of the intersection pairing over algebraic classes and the above expression of $\mathrm{ch}(F)_{\mathrm{d}}$, we obtain that (3) is indeed well-defined. Moreover, if $\mathrm{ch}(F)_{\mathrm{d}} = 0$, then $\beta = 0$. $\square$

Definition 3.3. Following the notation of the proof of Lemma 3.2, we define

$$\check{\beta} := \beta_\Lambda^\vee \cdot \mathrm{td}_S^{-1}.$$

Note that

$$\check{\beta} = \sum_i \mathrm{ch}(F_i)_{\mathrm{d}},$$

where F_i are restrictions of F to the connected components of the normalisation of C. If C is smooth, then we obtain an expression of the Chern character of the family F,

$$\mathrm{ch}(F) = (\mathbf{v}, \check{\beta}) \in \Lambda \oplus \Lambda,$$

which can also be considered as a definition of $\mathrm{ch}(F)$ in the case of a singular curve C.

Remark 3.4. Another justification for the use of $\mathrm{Pic}_\lambda(\mathfrak{X})$ is the following one: $\lambda_{\mathbb{Q}|M}$ is surjective for Hilbert schemes of points of surfaces with $q(S) = 0$, all projective moduli of stable sheaves on a K3 surface and expected to be surjective for all projective moduli of stable sheaves over surfaces with $q(S) = 0$ (see e.g. [HL97, Theorem 8.1.6]). Heuristically speaking, we care only about curve classes on M, we therefore can throw away some obscure classes on $\mathfrak{X}$, leaving $\mathrm{Hom}(\mathrm{Pic}_\lambda(\mathfrak{X}), \mathbb{Z})$, which is good enough for our purposes.

3.1. **Positivity.** The aim of this section is prove the positivity for certain line bundles - Proposition 3.7. We start with the following result, which is inspired by [BM14, Proposition 4.4].

Lemma 3.5. *Let F be the sheaf on $S \times C$ associated to a map $f\colon C \to \mathfrak{Coh}(S)$, then*

$$\begin{aligned} \mathcal{L}_1 \cdot_f C &= \deg(\mathbf{v})\mathrm{rk}(p_{S*}F) - \mathrm{rk}(\mathbf{v})\deg(p_{S*}F), \\ \mathcal{L}_0 \cdot_f C &= \chi(\mathbf{v})\mathrm{rk}(p_{S*}F) - \mathrm{rk}(\mathbf{v})\chi(p_{S*}F), \end{aligned}$$

where $\deg(\mathbf{v})$ *is the degree of* $\mathbf{v}$ *with respect to* $\mathcal{O}_S(1)$.

Proof. By the proof of Lemma 3.2,

$$\mathcal{L}_i \cdot_f C = \chi(u_i \cdot p_{S!}[F]), \quad i = 0, 1.$$

The claim then follows from the following computation

$$\begin{aligned}
\chi(u_1 \cdot B) &= -\mathrm{rk}(\mathbf{v})\chi(B \cdot h) + \chi(\mathbf{v} \cdot h)\chi([\mathcal{O}_{\mathrm{pt}}] \cdot B) \\
&= -\mathrm{rk}(\mathbf{v})(\deg(B) - \frac{\mathrm{rk}(B)}{2}H^2 - \frac{\mathrm{rk}(B)}{2}H \cdot \mathrm{c}_1(S)) \\
&+ (\deg(\mathbf{v}) - \frac{\mathrm{rk}(\mathbf{v})}{2}H^2 - \frac{\mathrm{rk}(\mathbf{v})}{2}H \cdot \mathrm{c}_1(S))\mathrm{rk}(B) \\
&= \mathrm{rk}(B)\deg(\mathbf{v}) - \mathrm{rk}(\mathbf{v})\deg(B); \\
\chi(u_0 \cdot B) &= -\mathrm{rk}(\mathbf{v})\chi(B) + \chi(\mathbf{v})\chi([\mathcal{O}_{\mathrm{pt}}] \cdot B) \\
&= \chi(\mathbf{v})\mathrm{rk}(B) - \mathrm{rk}(\mathbf{v})\chi(B).
\end{aligned}$$

□

The relation between quasimaps to $\mathfrak{Coh}(S)$ and sheaves on $S \times C$ is the central for our study of quasimaps. Since we are interested in quasimaps to the rigidified stack $\mathfrak{Coh}_r(S)$, we would also like to extend that relation to this setting, which is done in the following lemma.

Lemma 3.6. *Any quasimap $f\colon C \to \mathfrak{Coh}_r(S)$ admits a lift to $\mathfrak{Coh}(S)$. Different lifts are related by tensoring the corresponding sheaf on $S \times C$ with a line bundle from C.*

Proof. By [AGV08, Appendix C.2] a map $C \to \mathfrak{Coh}_r(S)$ is given by a $B\mathbb{C}^*$-gerbe $\mathcal{G}$ over C with an $\mathbb{C}^*$-equivariant map $\phi\colon \mathcal{G} \to \mathfrak{Coh}(S)$. It can be checked that

$$H^2_{\mathrm{fppf}}(C, \mathcal{O}_C^*) = H^2_{\text{ét}}(C, \mathcal{O}_C^*) = 0$$

by passing to the normalisation of C and using the exponential sequence. Therefore $\mathcal{G}$ is a trivial gerbe. Choose some trivialisation

$$\mathcal{G} \cong C \times B\mathbb{C}^*.$$

By the moduli problem of sheaves a $\mathbb{C}^*$-equivariant map $\phi\colon C \times B\mathbb{C}^* \to \mathfrak{Coh}(S)$ is given by a $\mathbb{C}^*$-equivariant sheaf F on $S \times C$ such that the $\mathbb{C}^*$-equivariant structure is the one given by multiplication by scalars applied to the sheaf F viewed as a sheaf on $S \times C$. In particular, $\mathbb{C}^*$-equivariant structure is unique and determined by F alone. The sheaf F defines a lift $f\colon C \to \mathfrak{Coh}(S)$. Given another lift $f'\colon C \to \mathfrak{Coh}(S)$ with an associated sheaf F' on $S \times C$, then by the properties the rigidification (see [AGV08, Appendix C.2]) there exists an automorphism of the trivial gerbe

$$\psi\colon C \times B\mathbb{C}^* \cong C \times B\mathbb{C}^*,$$

such that $f \cong f' \circ \psi$, therefore $(\mathrm{id}_S \times \psi)^* F' \cong F$. Automorphisms of a trivial gerbe admit the following description

$$\mathrm{Aut}_C(\mathcal{G}) \cong \mathrm{Pic}(C), \quad \psi \mapsto L_\psi,$$

which can be easily proven after recalling that maps to $B\mathbb{C}^*$ are given by line bundles. Moreover, the pullback of a sheaf by ψ is isomorphic to the sheaf tensored by L_ψ. Hence we obtain that

$$F \cong (\mathrm{id} \times \psi)^* F' \cong F' \otimes p_C^* L_\psi.$$

□

Proposition 3.7. *Let $f \colon C \to \mathfrak{Coh}_r(S)$ be a prestable quasimap. Fix $\mathbf{v}$ and $\mathcal{L}_{1,\ell} \cdot_f C$ for all ℓ. Then there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ the quasimap is non-constant, if and only if*

$$\mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_f C > 0.$$

The same holds for all subcurves C' and the induced maps for the same choice of m.

For the illustration of the method, which will be used to prove the claim, we will firstly prove that

$$\mathcal{L}_1 \cdot_f C \geq 0 \tag{4}$$

under the same assumption. The proof of the inequality (4) also contains the essential ingredients for the proof of the proposition.

Warm-up for Proposition 3.7. By Lemma 3.6, any $f \colon C \to \mathfrak{Coh}_r(S)$ can be lifted to $\mathfrak{Coh}(S)$ and intersections with $\mathcal{L}_i$'s are independent of the lift. Let F be a family of sheaves on $S \times C$ associated to a lift of f. Assume for simplicity that f has one base point $b \in C$. By Langton's semistable reduction, [Lan75], the sheaf F can be modified at a point b to a sheaf which is stable over b and is isomorphic to F away from $S \times b \subset S \times C$. The modification is given by a finite sequence of short exact sequences

$$0 \to F^1 \to F^0 \to Q^1 \to 0,$$
$$\vdots$$
$$0 \to F^k \to F^{k-1} \to Q^k \to 0,$$

where $F^0 = F$, the sheaf F^k is stable over $b \in C$ and Q^i is the maximally destabilising quotient sheaf of F_b^{i-1} (if F_b^{i-1} has torsion, then Q^i is the quotient by the maximal torsion subsheaf). In particular, for all i

$$\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) \geq 0. \tag{5}$$

Applying derived pushforward p_{S*} to each sequence, we get distinguished triangles

$$p_{S*}(F^i) \to p_{S*}(F^{i-1}) \to Q^i \to .$$

By Lemma 3.5 we obtain that

$$\mathcal{L}_1 \cdot_{f^{i-1}} C = \mathcal{L}_1 \cdot_{f^i} C + \deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i), \tag{6}$$

where f^i is the quasimap associated to F^i. The line bundle $\mathcal{L}_1$ is nef on M by Theorem 2.2 and the assumption $q(S) = 0$, therefore

$$\mathcal{L}_1 \cdot_{f^k} C \geq 0,$$

because f^k does not have base points. The property of $\mathcal{L}_1$ stated in (4) now follows from (5) and (6). $\square$

Proof of Proposition 3.7. We now deal with the claim in the proposition. By Lemma 3.5

$$\begin{aligned} &\mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_f C = \mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_{f^k} C \\ &+ m \sum_i \deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) + \sum_i \chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i), \end{aligned} \tag{7}$$

therefore for $\mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_f C$ to be positive for some big enough m, the terms

$$\chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i)$$

have to be bounded from below. We will now split our analysis, depending on whether (5) is positive or zero.

Consider firstly the case of Q^i's, such that

$$\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) > 0.$$

We plan to use Lemma 3.8. The sheaves Q^i sit in filtrations (see e.g. [HL97, Theorem 2.B.1]) inside F_b^m,

$$Q^1 \subset Q^2 \subset \cdots \subset Q^m \subset F_b^m. \tag{8}$$

Since F_b^m is stable, we have a bound for $\mu_{\max}(Q^i)$,

$$\mu_{\max}(Q^i) \leq \mu(\mathbf{v}).$$

By (5) and (6), the degrees of such Q^i can be bounded,

$$\frac{\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathcal{L}_1 \cdot_f C}{\mathrm{rk}(\mathbf{v})} \leq \deg(Q^i) < \frac{\deg(\mathbf{v})\mathrm{rk}(Q^i)}{\mathrm{rk}(\mathbf{v})}, \tag{9}$$

we therefore get a uniform bound on $\deg(Q^i)$ for all such Q^i depending on the sign of $\deg(\mathbf{v})$,

$$\begin{aligned} -\frac{\mathcal{L}_1 \cdot_f C}{\mathrm{rk}(\mathbf{v})} \leq \deg(Q^i) < \deg(\mathbf{v}), \quad &\text{if } \deg(\mathbf{v}) \geq 0 \ , \\ \deg(\mathbf{v}) - \frac{\mathcal{L}_1 \cdot_f C}{\mathrm{rk}(\mathbf{v})} < \deg(Q^i) < \frac{\deg(\mathbf{v})}{\mathrm{rk}(\mathbf{v})}, \quad &\text{if } \deg(\mathbf{v}) < 0. \end{aligned}$$

If $\rho(S) > 1$, then we can get the similar bounds for $\mathcal{L}_{1,\ell}$ for all ℓ, thereby bounding $\mathrm{c}_1(Q^i)$. Hence by Lemma 3.8, we obtain that

$$\mathrm{ch}_2(Q^i) < \mathsf{A}',$$

where the constant A' depends only on $\mathrm{rk}(\mathbf{v}), \deg(\mathbf{v})$ and $\mathcal{L}_{1,\ell} \cdot_f C$, we therefore can also uniformly bound $\chi(Q^i)$,

$$\chi(Q^i) < \mathsf{A}.$$

We conclude that

$$\chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i) > \chi(\mathbf{v}) - \mathsf{A}\cdot \mathrm{rk}(\mathbf{v}), \quad \text{if } \chi(\mathbf{v}) \geq 0;$$
$$\chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i) > \chi(\mathbf{v})\mathrm{rk}(\mathbf{v}) - \mathsf{A}\cdot \mathrm{rk}(\mathbf{v}), \quad \text{if } \chi(\mathbf{v}) < 0.$$

Consider now the case of Q^i's, such that

$$\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) = 0.$$

By (8) and stability of F_b^m, it must be that

$$\chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i) > 0.$$

Now let $m_0 \in \mathbb{N}$ be such that $\mathcal{L}_0 \otimes \mathcal{L}_1^{m_0}$ is ample on M (possible by Theorem 2.2) and

$$m_0 \cdot (\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i)) > \mathsf{A}\cdot \mathrm{rk}(\mathbf{v}) - \chi(\mathbf{v})$$

for all Q^i, such that $\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) > 0$, if $\chi(\mathbf{v}) \geq 0$. Similarly, if $\chi(\mathbf{v}) < 0$. By (7), the proposition then follows for quasimaps with one base point. Note that all the bounds do not depend on a base point $b \in C$ and therefore are the same for all base points, hence we can safely drop the assumption that there is one base point.

The dependence of m_0 on $\mathbf{v}$ and $\mathcal{L}_{1,\ell} \cdot_f C$ follows from bounds presented in (9). The fact, that positivity of the line bundle $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ holds for all subcurves for the same choice of m, follows from the proof itself. □

Lemma 3.8. *Let F be a torsion-free sheaf of rank r on a smooth projective surface S with Picard rank $\rho(S) = 1$, such that $\mu_{\max}(F) < \mathsf{B}$. Then $\mathrm{ch}_2(F)$ is bounded from above by a number that depends only on $r, \mathrm{c}_1(F)$ and B.*

The same holds, if $\rho(S) \neq 1$ and we view $\mu_{\max}(F)$ and B as functions on a neighbourhood $U \subset \mathrm{Amp}(S)$ around $\mathcal{O}_S(1)$.

Proof. We present the proof for $\rho(S) = 1$, the case of $\rho(S) \neq 1$ follows from the same argument. Let

$$0 = HN_0(F) \subset HN_1(F) \subset \cdots \subset HN_k(F) = F$$

be the Harder-Narasimhan filtration of F. Slopes of the graded pieces of the filtration satisfy

$$\mu_{\max}(F) = \mu(gr_1^{HN}) \geq \cdots \geq \mu(gr_k^{HN}),$$

therefore

$$\deg(gr_i^{HN}) < \mathsf{B}\cdot \mathrm{rk}(gr_i^{HN})$$

and

$$\begin{aligned}\deg(gr_i^{HN}) = \deg(F) - \sum_{j\neq i}\deg(gr_j^{HN}) &> \deg(F) - \mathsf{B}\cdot\sum_{j\neq i}\mathrm{rk}(gr_j^{HN}) \\ &= \deg(F) - \mathsf{B}\cdot(r - \mathrm{rk}(gr_i^{HN})).\end{aligned}$$

Hence we get a uniform bound for all i,

$$\deg(F) - \mathsf{B}\cdot r < \deg(gr_i^{HN}) < \mathsf{B}\cdot r, \quad \text{if } \mathsf{B}\geq 0;$$
$$\deg(F) < \deg(gr_i^{HN}) < 0, \quad \text{if } \mathsf{B} < 0,$$

which implies that $\mathrm{c}_1(gr_i^{HN})$ is uniformly bounded, since $\rho(S)=1$. So there exists $\mathsf{A}'\in\mathbb{Z}$, which depends only on B, r and $\mathrm{c}_1(F)$, such that

$$\mathrm{c}_1(gr_i^{HN})^2 < \mathsf{A}', \text{ for all } i,$$

then by semistablity of gr_i^{HN} and Bogomolov-Gieseker inequality

$$\mathrm{ch}_2(gr_i^{HN}) \leq \mathrm{c}_1(gr_i^{HN})^2/2\mathrm{rk}(gr_i^{HN}),$$

so we get

$$\mathrm{ch}_2(gr_i^{HN}) < \mathsf{A} = \begin{cases} \mathsf{A}' & \text{if } \mathsf{A}'\geq 0 \\ \mathsf{A}'/2r & \text{if } \mathsf{A}' < 0 \end{cases}$$

Finally, by $\mathrm{ch}_2(F) = \sum \mathrm{ch}_2(gr_i^{HN})$ and by the fact that there are at most r pieces in the filtration, we get the desired bound

$$\mathrm{ch}_2(F) < r\cdot \mathsf{A}.$$

□

3.2. **Stable quasimaps.** For all $\beta\in \mathrm{Eff}(M, \mathfrak{Coh}_r(S))$, we fix once and for ever a line bundle[9],

$$\mathcal{L}_\beta := \mathcal{L}_0\otimes\mathcal{L}_1^m \in \mathrm{Pic}_\lambda(\mathfrak{Coh}_r(S)),$$

for some $m\in\mathbb{N}$, such that $\mathcal{L}_0\otimes\mathcal{L}_1^m$ satisfies the conclusion of Proposition 3.7.

Given a quasimap $f\colon C\to\mathfrak{Coh}_r(S)$ of a degree β and a point $p\in C$ in the regular locus of C. By Langton's semistable reduction, we can modify the quasimap f at the point p, to obtain a quasimap

$$f_p\colon C\to \mathfrak{Coh}_r(S),$$

which maps to the stable locus M at p (if p is not a base point, then $f_p = f$). In other words, because M is proper and C is spectrum of a discrete valuation ring at p, one can eliminate the indeterminacy of f at p, if we view it as a rational map to M. We refer to f_p as *stabilisation* of f at p.

Definition 3.9. We define the *length* of a point $p\in C$ to be

$$\ell(p) := \mathcal{L}_\beta\cdot_f C - \mathcal{L}_\beta\cdot_{f_p} C.$$

By the proof of Proposition 3.7, $\ell(p)\geq 0$; and $\ell(p)=0$, if and only if p is not a base point.

In what follows, by 0^+ we will denote a number $A\in\mathbb{R}_{>0}$, such that $A\ll 1$.

[9]Such line bundle indeed depends on β, because the conclusions of Proposition 3.7 depend on β via the intersection numbers $\mathcal{L}_{1,\ell}\cdot_f C$ for all ℓ.

Definition 3.10. Given $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$, a prestable quasimap $f\colon (C, \mathbf{x}) \to \mathfrak{Coh}_r(S)$ of degree β is ϵ-stable, if

(i) $\omega_C(\mathbf{x}) \otimes f^*\mathcal{L}_\beta^\epsilon$ is positive;

(ii) $\epsilon \ell(p) \leq 1$ for all $p \in C$.

We will refer to 0^+-stable and ∞-stable quasimaps as stable quasimaps and stable maps respectively.

A *family* of quasimap over a base B is a family of nodal curves $\mathcal{C}$ over B with a map $f\colon \mathcal{C} \to \mathfrak{Coh}_r(S)$ such that the geometric fibers of f over B are quasimaps.

Let

$$
\begin{aligned}
Q^\epsilon_{g,N}(M,\beta)\colon (Sch/\mathbb{C})^\circ &\to (Grpd) \\
B &\mapsto \{\text{families of } \epsilon\text{-stable quasimaps over B}\}
\end{aligned}
$$

be the moduli space of ϵ-stable quasimaps of genus g and the degree β with N marked points.

3.3. **Quasi-compactness.** We firstly show that a moduli space $Q^\epsilon_{g,N}(M,\beta)$ is quasi-compact and then, in Section 3.4, that it is proper. The first step on the way to proving quasi-compactness of the moduli space is the following lemma.

Lemma 3.11. *Let $\beta \in \mathrm{Eff}(M, \mathfrak{Coh}_r(S))$ and a nodal curve C be fixed. The family of quasimaps of degree β from C to M is quasi-compact.*

Proof. Choose a lift of f to $\mathfrak{Coh}(S)$, let F^0 be the associated family. The semistable reduction applied to all base points at once gives a sequence of short exact sequences

$$
\begin{gathered}
0 \to F^1 \to F^0 \to Q^1 \to 0, \\
\vdots \\
0 \to F^k \to F^{k-1} \to Q^k \to 0,
\end{gathered}
$$

such that F^k defines a map $f^k\colon C \to M$. To establish the claim of the lemma, we plan to reverse the semistable reduction, i.e. we start with some map from C to M and take consecutive extensions of the corresponding families of sheaves by sheaves supported scheme-theoretically on fibers. For that we have to show that there is bounded number of possibilities. In particular, we have to show that

(i) the number of steps in the semistable reduction is bounded, i.e. k is uniformly bounded;

(ii) the family of possible $f^k\colon C \to M$ is bounded;

(iii) the family of possible Q^i is bounded.

To be more precise, different lifts of a quasimap are related by tensoring a sheaf with a line bundle coming from C, hence a lift of f^k also determines a lift of f. Therefore if we fix lifts of maps to M, there will always be a lift of f, such that the lift of f^k is the one that we fixed, this will eliminate a potential unboundedness coming from different lifts.

(i) By Proposition 3.7 and its proof there are at most $\beta(\mathcal{L}^1)$ steps with

$$\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) > 0$$

and there are at most $\beta(\mathcal{L}_0 \otimes \mathcal{L}_1^m)$ steps with

$$\deg(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\deg(Q^i) = 0,$$

therefore

$$k \leq \beta(\mathcal{L}_1) + \beta(\mathcal{L}_0 \otimes \mathcal{L}_1^m).$$

(ii) By the proof of Proposition 3.7 the numerical degree of possible f^k with respect to an ample line bundle $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ is bounded in the following way

$$\beta(\mathcal{L}_0 \otimes \mathcal{L}_1^m) = \mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_f C > \mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_{f^k} C \geq 0.$$

Since the family of maps with a fixed domain of a given degree is bounded, the family of possible f^k must be bounded.

(iii) By the semistable reduction, sheaves Q^i's are subsheaves of stable sheaves in the class $\mathbf{v}$ (see [HL97, Theorem 2.B.1]). Chern classes of Q^i's are bounded by Lemma 3.5 and by the proof of Proposition 3.7. Therefore by boundedness of Quot schemes and stable sheaves, the family of possible Q^i's is also bounded. □

Corollary 3.12. *A moduli space $Q^{\epsilon}_{g,N}(M,\beta)$ is quasi-compact.*

Proof. The restriction of a stable quasimap to an unstable component (a rational bridge or a rational tail) must be non-constant by stability and it must pair positively with $\mathcal{L}_\beta$ by Proposition 3.7. Therefore the number of unstable components of the domain curve of a stable quasimap is bounded in terms of β. Hence the projection $Q^{\epsilon}_{g,N}(M,\beta) \to \mathfrak{M}_{g,N}$ factors through a substack of finite type. By Lemma 3.11, the projection is quasi-compact, therefore $Q^{\epsilon}_{g,N}(M,\beta)$ is quasi-compact. □

3.4. **Relative moduli spaces of sheaves and properness.** To continue further exploiting the geometry of sheaves, we need to be able to relate quasimaps to sheaves in families (Lemma 3.6 permits us to do it only pointwise). For that, we have to narrow down our scope. If the $\mathbb{C}^*$-gerbe $\mathfrak{Coh}(S) \to \mathfrak{Coh}_r(S)$ is trivial such that a trivialisation is given by a section

$$s \colon \mathfrak{Coh}_r(S) \to \mathfrak{Coh}(S),$$

then by composing quasimaps with s we can lift quasimaps from $\mathfrak{Coh}_r(S)$ to $\mathfrak{Coh}(S)$ in families. More generally, in order to lift quasimaps of fixed degree in families, the $\mathbb{C}^*$-gerbe has to be trivial only over any substack of finite type $\mathcal{U} \subset \mathfrak{Coh}_r(S)$. Indeed, a moduli space of quasimaps of fixed degree is quasi-compact, therefore the universal qusiamap factors through a substack of finite type.

A $\mathbb{C}^*$-gerbe is trivial, if and only if there exists a line bundle of $\mathbb{C}^*$-weight 1. In particular, if there exists a class $u \in K_0(S)$, such that $\chi(\mathbf{v} \cdot u) = 1$, then there is a section

$$s_{u|\mathcal{U}} \colon \mathcal{U} \to \mathfrak{Coh}(S),$$

which is given by the descend of the family $\mathcal{F} \otimes \lambda(u)^{-1}_{|\mathcal{U}}$ to $S \times \mathcal{U}$. Note that the section s_u is defined only over substacks of finite type, because $\lambda(u)$ is defined this way. We will only consider trivialisations that arise through determinant line bundles. In any case, they are the only ones that can be checked to exist in practice.

From now on, we make the following assumption, which implies that M is a *fine* moduli space and, in some sense, can be seen as slightly a stronger property than being fine. A more general case is discussed in Section 3.6.

Assumption.

$$\exists u \in K_0(S), \text{ such that } \chi(\mathbf{v} \cdot u) = 1.$$

We will identify a class β with its image with respect to the pushforward by the section s_u (more precisely, by the system of sections over substacks of finite type),

$$s_{u*} \colon \mathrm{Eff}(M, \mathfrak{Coh}_r(S)) \hookrightarrow \mathrm{Eff}(\mathcal{M}, \mathfrak{Coh}(S)).$$

Using (3), we can further identify $\mathrm{Eff}(M, \mathfrak{Coh}_r(S))$ with classes Λ, as shown in the following corollary.

Corollary 3.13. *The map*

$$\check{(\ldots)} \colon \mathrm{Eff}(M, \mathfrak{Coh}_r(S)) \to \Lambda,$$

defined as the restriction of (3) *to* $\mathrm{Eff}(M, \mathfrak{Coh}_r(S))$*, is injective.*

Proof. We need to show that $\beta \neq 0$ implies $\check{\beta} \neq 0$. By Proposition 3.7, a non-zero β intersects positively with a line bundle $\mathcal{L}_0 \otimes \mathcal{L}_1^m$ for some m. Hence by the definition of $\check{\beta}$ in (3.3) it also intersects positively with the corresponding class in Λ. Therefore it cannot be zero. ☐

Consider now the following map

$$\begin{aligned} Q^\epsilon_{g,N}(M, \beta) &\hookrightarrow \mathfrak{Coh}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}), \\ f &\mapsto s_u \circ f \mapsto F, \end{aligned} \tag{10}$$

where F is the family associated to the quasimap $s_u \circ f$, $\mathfrak{C}_{g,N} \to \mathfrak{M}_{g,N}$ is the universal curve over the moduli stack of nodal curves and $\mathfrak{Coh}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N})$ is the relative moduli stack of sheaves on $S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}$.

Definition 3.14. Let $M^\epsilon_{\mathbf{v},\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$ be the stacky image (the minimal substack through which the map factors) of the map (10). By

$M^{\epsilon}_{\mathbf{v},\check{\beta},u}(S \times C/S_{p_i})$ we will denote[10] a fiber of

$$M^{\epsilon}_{\mathbf{v},\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N}) \to \overline{M}_{g,N}$$

over a $\mathbb{C}$-valued point $[(C, \mathbf{x})] \in \overline{M}_{g,N}$ of the moduli space of stable marked curves. For a curve without markings, we denote the fiber by $M^{\epsilon}_{\mathbf{v},\check{\beta},u}(S \times C)$. Similarly, we define $Q^{\epsilon}_{(C,\mathbf{x})}(M, \beta)$ to be the fiber of

$$Q^{\epsilon}_{g,N}(M, \beta) \to \overline{M}_{g,N}$$

over a $\mathbb{C}$-valued point $[(C, \mathbf{x})] \in \overline{M}_{g,N}$. We will frequently drop $\mathbf{v}$ from the notation, as it is fixed; in the case of $\epsilon = 0^+$, we will drop 0^+.

The $\mathbb{C}$-valued points of the moduli space $M^{\epsilon}_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$ are just families of sheaves associated to quasimaps via the section s_u.

Lemma 3.15. *The map (10) is an isomorphism onto the image,*

$$Q^{\epsilon}_{g,N}(M, \beta) \xrightarrow{\sim} M^{\epsilon}_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N}).$$

Moreover, $M^{\epsilon}_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$ *is an open sublocus of sheaves, satisfying the condition*

$$\det(p_{C*}(p_S^* u \otimes F)) = \mathcal{O}_C$$

in families.

Proof. Firstly, over some substack of finite type there is an indentification

$$\mathfrak{Coh}(S) \cong \mathfrak{Coh}_r(S) \times B\mathbb{C}^* \tag{11}$$

given by the section s_u. Hence, by Corollary 3.12, composition of a quasimap with the section s_u induces a closed immersion

$$Q^{\epsilon}_{g,N}(M, \beta) \hookrightarrow Q^{\epsilon}_{g,N}(\mathcal{M}, \beta) /\!\!/ B\mathbb{C}^*,$$
$$f \mapsto s_u \circ f.$$

With respect to (11), the space $Q^{\epsilon}_{g,N}(M, \beta)$ is a sublocus of the space $Q^{\epsilon}_{g,N}(\mathcal{M}, \beta) /\!\!/ B\mathbb{C}^*$, consisting of quasimaps which map to $B\mathbb{C}^*$-factor by a trivial line bundle. Here, $Q^{\epsilon}_{g,N}(\mathcal{M}, \beta)$ is the moduli space of ϵ-stable quasimaps to $\mathcal{M}$, defined in the same as in Definition 3.10. Moreover, by associating a family to a quasimap to $\mathcal{M}$, the space $Q^{\epsilon}_{g,N}(\mathcal{M}, \beta) /\!\!/ B\mathbb{C}^*$ is naturally an open sublocus of $\mathfrak{Coh}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}) /\!\!/ B\mathbb{C}^*$ (since being a family of sheaves on $S \times C$ is an open condition). The map (10) can therefore be factored as composition of a closed immersion with an open one,

$$Q^{\epsilon}_{g,N}(M, \beta) \hookrightarrow Q^{\epsilon}_{g,N}(\mathcal{M}, \beta) /\!\!/ B\mathbb{C}^* \hookrightarrow \mathfrak{Coh}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}) /\!\!/ B\mathbb{C}^*.$$

This proves the first claim.

By construction, the section s_u is given by the descend of the sheaf $\mathcal{F} \otimes \lambda(u)^{-1}$ from $\mathfrak{Coh}(S)$ to $\mathfrak{Coh}_r(S)$. A family F is a pullback of $\mathcal{F} \otimes \lambda(u)^{-1}$ by

[10] The notation is similar to the one of DT theory relative to divisors.

a map $f\colon \mathcal{C} \to \mathfrak{Coh}(S)$. By the definition of $\lambda(u)$, the family F therefore satisfies

$$\det(p_{C*}(p_S^* u \otimes F)) = \mathcal{O}_{\mathcal{C}}.$$

Since being a family of sheaves is a open condition, the second claim now follows. □

Remark 3.16. If $\epsilon = 0^+$, then $\mathbb{C}$-valued points of $M_{\mathbf{v},\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$ are triples $(C, \mathbf{x}, F)$, such that:

- $(C, \mathbf{x})$ is a prestable nodal curve;
- a sheaf F on $S \times C$ flat over C;
- $\mathrm{ch}(F)_{\mathrm{f}} = \mathbf{v}$, $\mathrm{ch}(F)_{\mathrm{d}} = \check{\beta}$;
- a general fiber of F is stable;
- fibers of F over nodes and markings $\mathbf{x}$ are stable;
- $\det(p_{C*}(p_S^* u \otimes F)) = \mathcal{O}_C$.

Our determinant-line-bundle condition is natural for families. The standard determinant-line-bundle condition would involve choice of a line bundle in families which might not even exist. For a fixed smooth curve, the two determinant-line-bundles conditions are not far from each other, as shown in Lemma 3.22.

By the definition of a quasimap, a general fiber of F over C is stable. The stability of a general fiber can be related to the stability of the sheaf F itself, as is shown in the following lemma.

Lemma 3.17. *There exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the moduli space $M^{\epsilon}_{\check{\beta},u}(S \times C) \subset M^{\epsilon}_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$ is an open sublocus of a moduli of Gieseker $\mathcal{O}_{S\times C}(1,k)$-stable*[11] *sheaves on $S \times C$, satisfying the condition* $\det(p_{C*}(p_S^* u \otimes F)) = \mathcal{O}_C$.

We will refer to the stability in the lemma as *suitable*. The converse of the lemma is more subtle. In Appendix A, it is proven in the rank-2 case for slope stabilities, rank-1 case holds trivially. Note that a sheaf, which is $\mathcal{O}_{S\times C}(1,k)$-stable for *all* $k \gg 0$, is stable at a general fiber. Hence proving the converse amounts to proving that there are no walls between $\mathcal{O}_{S\times C}(1,k)$-stabilities for $k \gg 0$.

Proof of Lemma 3.17. Given a sheaf $F \in M^{\epsilon}_{\check{\beta},u}(S \times C)$, a general fiber of F over C is stable. In particular, it is torsion-free, therefore F is torsion-free itself by Lemma 3.18. It also must be $\mathcal{O}_{S\times C}(1,k)$-stable for all $k \gg 0$, this can be seen as follows. Since a general fiber of F is stable, the difference between $\mathcal{O}_{S\times C}(1,k)$-Hilbert polynomials of F and of its subsheaves increases as k increases, because $\mathrm{c}_1(\mathcal{O}_C(k))$ pairs only with $\mathrm{ch}_{\mathrm{f}}(F)$. Since the family of $\mathcal{O}_{S\times C}(1,k)$-destabilising subsheaves of F is bounded for a fixed k, no

[11] $\mathcal{O}_{S\times C}(1,k)$ stands for $\mathcal{O}_S(1) \boxtimes \mathcal{O}_C(k)$.

subsheaves of F will be $\mathcal{O}_{S\times C}(1,k)$-destabilising for $k \gg 0$. Hence F is $\mathcal{O}_{S\times C}(1,k)$-stable for $k \gg 0$.

The moduli space $M^{\epsilon}_{\check{\beta},u}(S\times C)$ is quasi-compact, therefore there exists a uniform choice of k_0 for which the statement holds for all sheaves in $M^{\epsilon}_{\check{\beta},u}(S\times C)$. The fact that it is open follows from openness of stability of fibers. $\square$

Lemma 3.18. *Let F be a sheaf on $S\times C$ flat over C, such that F_p is torsion-free for a general $p \in C$, then F is torsion-free.*

Proof. Let $T(F) \subset F$ be the maximal torsion subsheaf. Firstly, $T(F) \neq F$, because $\mathrm{rk}(F) \neq 0$. It also cannot be supported on fibers of $S\times C \to C$ due to flatness of F over C, therefore $\mathrm{Supp}(T(F))$ intersects a general fiber. Since $F/T(F)$ is generically flat, restricting $T(F) \subset F$ to a general fiber $p \in C$, we get a torsion subsheaf of F_p for a general $p \in C$, which is zero, therefore $T(F)$ is zero. $\square$

The final ingredient for the proof of properness of the moduli space is the following lemma, *Hartogs' property* for families of nodal curves over a DVR.

Lemma 3.19. *Let $\mathcal{C} \to \Delta$ be a family of nodal curves over a DVR Δ and $\{p_i\} \subset \mathcal{C}$ be finitely many closed points in the regular locus of the central fiber. If there exists a class $u \in K_0(S)$, such that $\chi(u\cdot \mathbf{v}) = 1$, then any quasimap $\tilde{f}\colon \tilde{\mathcal{C}} = \mathcal{C}\setminus\{p_i\} \to \mathfrak{Coh}_r(S)$ extends to $f\colon \mathcal{C} \to \mathfrak{Coh}_r(S)$, which is unique up to unique isomorphism.*

Proof. Let $\tilde{F}$ be the family on $S\times\tilde{\mathcal{C}}$ corresponding to the lift of $\tilde{f}$ by s_u, we then extend $\tilde{F}$ to a coherent sheaf F on $S\times\mathcal{C}$, quotienting the torsion, if necessary. The sheaf F is therefore flat over Δ. The central fiber F_k of F defines a quasimap, if it is torsion-free, because $\mathcal{C}_k$ is regular at p_i. If F_k is not torsion-free, we can remove the torsion inductively as follows. Let $F^0 = F$ and F^i be defined by short exact sequences,

$$0 \to F^i \to F^{i-1} \to Q^i \to 0,$$

such that Q^i is the quotient of F^{i-1}_k by the maximal torsion subsheaf. It is not difficult to check, that at each step the torsion of F^i_k is supported at slices $S\times p_i$, therefore all F^i's are isomorphic to F^0 over $S\times\tilde{\mathcal{C}}$. By the standard argument (see e.g. [HL97, Theorem 2.B.1]), this process terminates, i.e. $F^i = F^{i+1}$ and F^i_k is torsion-free for $i \gg 0$. Let us redefine the sheaf F, we set $F = F^i$ for some $i \gg 0$, then the sheaf F induces a quasimap to $\mathfrak{Coh}(S)$, and composing it with the projection to $\mathfrak{Coh}_r(S)$, we thereby obtain an extension $f\colon \mathcal{C} \to \mathfrak{Coh}_r(S)$ of $\tilde{f}$.

Consider now another extension $f'\colon \mathcal{C} \to \mathfrak{Coh}_r(S)$, we lift both f and f' to $\mathfrak{Coh}(S)$ with s_u, then let F' and F be the corresponding families on $S\times\mathcal{C}$ (notice, F might differ from the previous F by a tensor with a line bundle), by Lemma 3.17 they define a family of stable sheaves relative to Δ in some relative moduli of sheaves $\mathcal{M}(S\times\mathcal{C}/\Delta)$, hence they must be isomorphic up to tensoring with a line bundle by separateness of the relative moduli of stable sheaves. The isomorphism becomes unique after projection to $\mathfrak{Coh}_r(S)$. $\square$

Remark 3.20. In general, Hartogs' property fails for sheaves on a surface. Hence the assumption that our surface is given by a family of curves $\mathcal{C} \to \Delta$ is necessary. This form of Hartogs' property is good enough for proving Theorem 3.21 in the spirit of [CKM14, Section 4].

Theorem 3.21. *If there exists a class $u \in K_0(S)$, such that $\chi(u \cdot \mathbf{v}) = 1$, then $Q^{\epsilon}_{g,N}(M, \beta)$ is a proper Deligne–Mumford stack.*

Proof. A relative moduli space of sheaves is known to be locally of finite type and quasi-separated over the base, e.g. see [Sta, Lemma 107.4.2]. Therefore by Lemma 3.12 and (3.15), a moduli space $Q^{\epsilon}_{g,N}(M, \beta)$ is of finite type and quasi-separated. By (**i**) of the quasimaps' stability (see Definition 3.1), ϵ-stable quasimaps have only finitely many automorphisms, therefore a moduli space $Q^{\epsilon}_{g,N}(M, \beta)$ is a quasi-separated Deligne–Mumford stack of finite type. Using the valuative criteria of properness for quasi-separated Deligne–Mumford stacks of finite type and Lemma 3.19, the proof of properness then proceeds as in the GIT case [CKM14, Section 4.3] . $\square$

3.5. **Sheaves with a fixed determinant.** We will now relate the moduli space $M_{\check{\beta},u}(S \times C)$ to a more familiar one - a moduli space of sheaves with a fixed determinant.

Assume C is a smooth. Let

$$L := \det(G) \in \mathrm{Pic}(S \times C)$$

be a determinant line bundle for some sheaf $G \in M_{\check{\beta},u}(S \times C)$. We define

$$M_{\check{\beta},L}(S \times C)$$

to be the moduli space of sheaves with a fixed determinant L in the class $(\mathbf{v}, \check{\beta})$, which satisfy the following assumptions:

- stable at a general fiber;
- fixed by at most finitely many automorphisms of C.

Note that the second condition is automatically satisfied, if $g(C) \geq 1$ and $\beta \neq 0$. There exists a projection that relates two moduli spaces,

$$p \colon M_{\check{\beta},L}(S \times C) \to M_{\check{\beta},u}(S \times C), \quad F \mapsto F \boxtimes \det(p_{C*}(p_S^* u \otimes F))^{-1} \tag{12}$$

The projection is well-defined by Lemma 3.17. In fact, it is étale, as shown in the following lemma.

Lemma 3.22. *Assuming C is a smooth curve and $M_{\check{\beta},u}(S \times C)$ is non-empty, the projection p is étale of degree* $\mathrm{rk}(\mathbf{v})^{2g}$.

Proof. The surjectivity can be seen as follows. Consider a sheaf $F \in M^{\epsilon}_{\check{\beta},u}(S \times C)$, then

$$L_0 := \det(F) \otimes L^{-1} \in \mathrm{Pic}_0(S \times C) = \mathrm{Pic}_0(C).$$

Now let $L_0^{\frac{1}{\mathrm{rk}(\mathbf{v})}}$ be a $\mathrm{rk}(\mathbf{v})^{\mathrm{th}}$ root of L_0 (recall that $\mathrm{Pic}_0(C)$ is a divisible group), then

$$\det(F \otimes L_0^{-\frac{1}{\mathrm{rk}(\mathbf{v})}}) = \det(F) \otimes L_0^{-1} = L,$$

therefore

$$F \otimes L_0^{-\frac{1}{\mathrm{rk}(\mathbf{v})}} \in M_{\check{\beta},L}(S \times C).$$

It can be easily verified that $F \otimes L_0^{-\frac{1}{\mathrm{rk}(\mathbf{v})}}$ is mapped to F via the map (12). This shows the surjectivity.

Now let $\mathrm{Pic}_0(C)[\mathrm{rk}(\mathbf{v})]$ be the $\mathrm{rk}(\mathbf{v})$-torsion points of $\mathrm{Pic}_0(C)$. The group $\mathrm{Pic}_0(C)[\mathrm{rk}(\mathbf{v})]$ acts on the moduli space $M_{\check{\beta},L}(S\times C)$, because $\det(F\otimes A) = \det(F) \otimes A^{\otimes \mathrm{rk}(F)}$ for a line bundle $A \in \mathrm{Pic}(S \times C)$. Orbits of the action are the fibers of (12). The action is free, because for a line bundle A, the following holds,

$$\det(p_{C*}(p_S^* u \otimes F \boxtimes A) \cong \det(p_{C*}(p_S^* u \otimes F) \otimes A,$$

which is due to $\chi(u \cdot \mathbf{v}) = 1$. In particular,

$$M_{\check{\beta},L}(S \times C)/\operatorname{Pic}_0(C)[\mathrm{rk}(\mathbf{v})] \xrightarrow{\sim} M_{\check{\beta},u}(S \times C),$$

the claim follows. □

3.6. **More general cases.**

3.6.1. *Non-trivial gerbe.* To establish properness of a moduli space $Q^{\epsilon}_{g,N}(M,\beta)$, we crucially rely on the identification of $\mathrm{t}Q^{\epsilon}_{g,N}(M,\beta)$ with the relative moduli space of sheaves $M^{\epsilon}_{\check{\beta},u}(S \times \mathfrak{C}/\mathfrak{M}_{g,N})$. To make it work in the case when $\mathbb{C}^*$-gerbe $\mathfrak{Coh}(S) \to \mathfrak{Coh}_r(S)$ is not trivial, one needs to consider twisted universal families. Given any $u \in K^0(S)$ such that

$$\mathrm{w} = \chi(u \cdot \mathbf{v}) \neq 0,$$

then over each finite type open substacks $\mathcal{U} \subset \mathfrak{Coh}(S)$ we can take a w^{th}-root stack associated to $\lambda(u)$ with the universal w^{th}-root $\lambda(u)^{\frac{1}{\mathrm{w}}}$ of $\lambda(u)$,

$$\mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}} \to \mathcal{U}, \quad \lambda(u)^{\frac{1}{\mathrm{w}}} \in \mathrm{Pic}(\mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}}).$$

Then $w_{\mathbb{C}^*}(\lambda(u)^{\frac{1}{\mathrm{w}}}) = 1$, therefore $\lambda(u)^{\frac{1}{\mathrm{w}}}$ defines a trivialisation of the $\mathbb{C}^*$-gerbe

$$\mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}} \to \mathfrak{Coh}_r(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}},$$

given by the descend of the twisted family $\mathcal{F} \otimes \lambda(u)^{-\frac{1}{\mathrm{w}}}$, where

$$\mathfrak{Coh}_r(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}} := \mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}} /\!\!/ \mathbb{C}^*.$$

We thereby obtain the desired section

$$s_{\frac{u}{\mathrm{w}}} : \mathfrak{Coh}_r(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}} \to \mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}_{|\mathcal{U}}.$$

The price we pay for this section is that the stable locus becomes a $\mathbb{Z}/\mathrm{w}\mathbb{Z}$-gerbe of M, which we denote by $M^{\frac{u}{\mathrm{w}}}$. In particular, we have to consider orbifold quasimaps for the sake of properness of the moduli space. All the definitions carry over to this setting verbatim, so let us consider now the quasimap theory of the pairs,

$$(M^{\frac{u}{\mathrm{w}}}, \mathfrak{Coh}_r(S)^{\frac{u}{\mathrm{w}}}) \quad \text{and} \quad (\mathcal{M}^{\frac{u}{\mathrm{w}}}, \mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}).$$

As in the case of untwisted case we can consider the following composition

$$Q^{\epsilon}_{g,N}(M^{\frac{u}{\mathrm{w}}}, \beta) \hookrightarrow Q^{\epsilon}_{g,N}(\mathcal{M}^{\frac{u}{\mathrm{w}}}, \beta) \to \mathfrak{Coh}(S \times \mathfrak{C}^{\mathrm{tw}}_{g,N}/\mathfrak{M}^{\mathrm{tw}}_{g,N}),$$
$$f \mapsto s_{\frac{u}{\mathrm{w}}} \circ f \mapsto F,$$

where $\mathfrak{M}^{\mathrm{tw}}_{g,N}$ is the moduli of twisted nodal curves with the universal family $\mathfrak{C}^{\mathrm{tw}}_{g,N}$. The second map is no longer an embedding, because the moduli problem of $\mathfrak{Coh}(S)^{\frac{u}{\mathrm{w}}}$ is now a pair

$$(F, \det(p_{C*}(p_S^* u \otimes F))^{\frac{1}{\mathrm{w}}}),$$

a sheaf and a w^{th}-root of $\det(p_{C*}(p_S^* u \otimes F))$. However, by the definition of the section $s_{\frac{u}{\mathrm{w}}}$, the w^{th}-root is fixed

$$(s_u \circ f)^* \lambda(u)^{\frac{1}{\mathrm{w}}} = \det(p_{C*}(p_S^* u \otimes F))^{\frac{1}{\mathrm{w}}} = \mathcal{O}_C,$$

hence the composition above is an embedding and $\det(p_{C*}(p_S^* u \otimes F)) = \mathcal{O}_C$. Let $M^{\epsilon}_{\check{\beta},u}(S \times C^{\mathrm{tw}}_{g,N}/\overline{M}^{\mathrm{tw}}_{g,N})$ be its image. We therefore have the desired identification,

$$Q^{\epsilon}_{g,N}(M^{\frac{u}{\mathrm{w}}}, \beta) \cong M^{\epsilon}_{\check{\beta},u}(S \times C^{\mathrm{tw}}_{g,N}/\overline{M}^{\mathrm{tw}}_{g,N}),$$

the rest goes as in the untwisted case. In principle, there are no obstacles for extension of all results including wall-crossing formulas. Using [AJT], we then can relate the twisted invariants to untwisted ones.

3.6.2. *Non-trivial Jacobian.* The case of a surface with $q(S) \neq 0$ can be tackled in the same manner. However, we need to adjust some definitions. Firstly, instead of the stack $\mathfrak{Coh}_r(S)_{\mathbf{v}}$ we have to take its fiber over $\mathrm{Pic}(S)$ with respect to the determinant morphism

$$\det \colon \mathfrak{Coh}_r(S)_{\mathbf{v}} \to \mathrm{Pic}(S),$$

where we slightly abuse the notation, because the morphism det is only defined over substacks of finite type.

Then for the definition of a degree, we have to take care of an extra summands in Künneth's decomposition of (p,p)-part of the cohomology on $S \times C$,

$$\bigoplus_i \bigoplus_{\substack{p+p'=i \\ q+q'=i}}^{p \neq q} H^{p,q}(S) \otimes H^{p',q'}(C).$$

The classes $\mathrm{Hom}(\mathrm{Pic}_\lambda(\mathfrak{X}), \mathbb{Z})$ are not sensitive to the piece of Künneth decomposition as above - Chern character $\mathrm{ch}(F)$ of a family F is not determined by the degree $\beta \in \mathrm{Hom}(\mathrm{Pic}_\lambda(\mathfrak{X}), \mathbb{Z})$ of the corresponding quasimap. On the GW side of M, this extra piece corresponds to extra classes that are not given by determinant line bundles. One could make the definition of the degree finer, by defining it as a class in $H_2(\mathfrak{X}, \mathbb{Z})$, but then we loose a direct connection of the degree with the Chern characters of sheaves on threefolds. One could also leave the definition as it is, thereby making the degree slightly coarser. For genus-0 invariants this, however, does not matter. Indeed the extra piece in Künneth decomposition of cohomology is not present, because $H^{1,0}(\mathbb{P}^1) = 0$.

Similarly, in the case of punctorial Hilbert schemes and the fixed-curve invariants, one can define the degree of a quasimap by the Chern character of the corresponding subscheme on a threefold, after contracting rational tails and projecting the subscheme to the component corresponding to the fixed curve.

4. Hilbert schemes

4.1. **Relative Hilbert schemes.** We now restrict to $\mathbf{v} = (1, 0, -n)$, i.e. $M = S^{[n]}$. Punctorial Hilbert schemes are special, because there exists a canonical trivialisation of $\mathfrak{Coh}(S)_{\mathbf{v}} \to \mathfrak{Coh}_r(S)_{\mathbf{v}}$ over any $\mathcal{U} \subset \mathfrak{Coh}_r(S)_{\mathbf{v}}$ of finite type. It is given by the determinant

$$\det(\mathcal{F}) \in \mathrm{Pic}(S \times \mathfrak{Coh}(S)_{\mathbf{v}})$$

of the universal sheaf $\mathcal{F}$ on $S \times \mathfrak{Coh}(S)_{\mathbf{v}}$. It is indeed a line bundle of weight 1, because $\mathcal{F}$ is of rank 1. Hence the family $\mathcal{F} \otimes \det(\mathcal{F}_{|\mathcal{U}})^{-1}$ descends to $S \times \mathcal{U}$, giving the canonical section

$$s_{\det|\mathcal{U}} \colon \mathcal{U} \to \mathfrak{Coh}(S)_{\mathbf{v}}$$

of the gerbe $\mathfrak{Coh}(S)_{\mathbf{v}} \to \mathfrak{Coh}_r(S)_{\mathbf{v}}$. By Corollary 3.12, there exists $\mathcal{U}$ of finite type through which the universal quasimap factors. Therefore the section $s_{\det|\mathcal{U}}$ gives us the embedding

$$Q^{\epsilon}_{g,N}(S^{[n]}, \beta) \hookrightarrow \mathfrak{Coh}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}),$$

which is defined as the one in (10). By the construction of the section, the sheaves in the image of the embedding satisfy

$$\det(F) = (\mathrm{id}_S \times f)^* \det(\mathcal{F}) = \mathcal{O}_{S \times \mathcal{C}}$$

over any base scheme B. Therefore the embedding factors through a relative Hilbert scheme,

$$Q^{\epsilon}_{g,N}(S^{[n]}, \beta) \hookrightarrow \mathrm{Hilb}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}).$$

Indeed, the above embedding factors through the relative moduli of sheaves of rank 1 with trivial determinant by the construction of the section $s_{\det}$. This moduli is in turn isomorphic to the moduli of ideals, because there exits a natural embedding $F \hookrightarrow F^{\vee\vee} \cong \mathcal{O}_{S\times C}$. It is a stack but not a scheme, because $S \times \mathfrak{C}_{g,N} \to \mathfrak{M}_{g,N}$ is a stack.

We denote the image of the embedding above by $\mathrm{Hilb}^{\epsilon}_{n,\check{\beta}}(S\times C_{g,N}/\overline{M}_{g,N})$, where the subscript "$n, \check{\beta}$" is the shortening of

$$((1,0,-n),\check{\beta}) \in \Lambda \otimes \Lambda.$$

The image can be described more explicitly in terms of ideals, or, equivalently, in terms of the corresponding one-dimensional subschemes. Firstly, the automorphisms of a quasimap f admit the following description

$$\mathrm{Aut}(f) = \mathrm{Aut}_{(C,\mathbf{x})}(I) = \mathrm{Aut}_{(C,\mathbf{x})}(\Gamma),$$

where I is the corresponding ideal sheaf, $\Gamma \subset S \times C$ is the associated subscheme and

$$\mathrm{Aut}_{(C,\mathbf{x})}(I) = \{\psi\colon (C,\mathbf{x}) \cong (C,\mathbf{x}) | (\mathrm{id}_S \times \psi)^*I = I\},$$

similarly for $\mathrm{Aut}_{(C,\mathbf{x})}(\Gamma)$. The quasimap ϵ-stability therefore requires the group $\mathrm{Aut}_{(C,\mathbf{x})}(I)$ to be finite.

The part (**ii**) of ϵ-stability in Definition 3.10 can be rephrased in terms of Γ as follows. A sheaf I_p is an ideal for all $p \in C$, if and only if all irreducible components of the subscheme Γ are dominant over a component of C and there are not embedded points, if and only if Γ is flat over C. We call non-dominant components without embedded points *vertical*. Let

$$\Gamma^{h+v} \subseteq \Gamma$$

be the maximal subscheme without embedded ponts, then $\Gamma^{h+v} = \Gamma^h \cup \Gamma^v$, where Γ^h is *horizontal* part of Γ, which is dominant over C and therefore is the subscheme associated to the stabilisation of I, and Γ^v is the vertical part of Γ. We have the following equality

$$I^{h+v} = I^h \cap I^v,$$

because there are no embedded points. Therefore there is an exact sequence

$$0 \to I^{h+v} \to I^h \oplus I^v \xrightarrow{+} I_{\Gamma^h \cap \Gamma^v} \to 0, \tag{13}$$

such that I^h is stable over all $t \in C$. Now let $\Gamma^u_i \subset \Gamma$ be the maximal non-dominant subscheme (with embedded points) supported on $S \times b_i$ for a given base point b_i and Γ^v_i be its vertical component without embedded points, then by the part (**ii**) of Definition 3.10, Lemma 3.5 and the sequence above, these Γ^u_i's must satisfy

$$m \cdot \deg(\Gamma^u_i) + \chi(\Gamma^u_i) - \chi(I_{\Gamma^h \cap \Gamma^v_i}) \leq 1/\epsilon,$$

for some fixed m for which Proposition 3.7 holds.

Apart from the usual condition on finiteness of automorphisms, the part (**i**) of Definition 3.10 can be similarly translated into restriction of the 'size' of Γ on rational tails in terms of its degree and Euler characteristic: given a rational tail R_j of C, let $\deg(\Gamma_{|R_j}) := \deg(\mathrm{ch}(\Gamma_{|R_j})_{\mathrm{d}})$, then for all rational tails the following must be satisfied

$$m \cdot \deg(\Gamma_{|R_j}) + \chi(\Gamma_{|R_j}) > 1/\epsilon.$$

Finally, by stability of quasimaps, Γ has to be flat over nodes and marked points.

Punctorial Hilbert schemes $S^{[n]}$ clearly satisfy the assumption of Theorem 3.21, hence summing up the discussion above we obtain the following result.

Corollary 4.1. *The moduli stack $Q^{\epsilon}_{g,N}(S^{[n]},\beta)$ is a proper Deligne–Mumford stack. For some fixed $m \gg 0$, there exists a natural isomorphism of the moduli spaces*

$$Q^{\epsilon}_{g,N}(S^{[n]},\beta) \cong \mathrm{Hilb}^{\epsilon}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}),$$

where the stack on the right is the relative moduli stack of 1-dimensional subschemes, satisfying the following properties

- $|\mathrm{Aut}_{(C,\mathbf{x})}(\Gamma)| < \infty$;
- Γ *is flat over nodes and marked points;*
- $m \cdot \deg(\Gamma^u_i) + \chi(\Gamma^u_i) - \chi(\Gamma^s \cap \Gamma^v_i) \leq 1/\epsilon$ *for a component* Γ^u_i;
- $m \cdot \deg(\Gamma_{|C_j}) + \chi(\Gamma_{|R_j}) > 1/\epsilon$ *for a rational tail* R_j.

Remark 4.2. For a fixed smooth curve C with $g \geq 1$ and $\beta \neq 0$, we have

$$Q_C(S^{[n]},\beta) \cong \mathrm{Hilb}_{n,\check{\beta}}(S \times C),$$

by Corollary 4.1. On the other hand,

$$Q_{(C,p)}(S^{[n]},\beta) \cong \mathrm{Hilb}_{n,\check{\beta}}(S \times C/S_p).$$

Moreover, pulling back a class with a marking on the left is equivalent ot pulling back the class from a relative divisor on the right.

4.2. **Changing the t-structure.** Consider the following *torsion pair* in $\mathrm{Coh}(S)$,

$$\mathcal{T} = \{A \in \mathrm{Coh}(S) \mid \dim(A) = 0\},$$
$$\mathcal{T}^{\perp} = \{B \in \mathrm{Coh}(S) \mid \mathrm{Ext}^{\bullet}(A,B) = 0, \forall A \in \mathcal{T}\}.$$

Let $\mathrm{Coh}^{\sharp}(S) = \langle \mathcal{T}^{\perp}, \mathcal{T}[-1]\rangle$ be the corresponding tilted perverse heart, we refer to [HRS96] for the construction of tilted hearts associated to a torsion pair. Punctorial Hilbert schemes sit inside the rigidification of the corresponding moduli stack,

$$S^{[n]} \subset \mathfrak{Coh}^{\sharp}_r(S)_{\mathbf{v}} := \mathfrak{Coh}^{\sharp}(S)_{\mathbf{v}} /\!\!/ \mathbb{C}^*,$$

constructed, for example, in [Lie06].

Before proceeding further, let us introduce some terminology from [AP06]. Let $\mathcal{A} := \mathrm{Coh}^{\sharp}(S)$ and $\mathcal{A}_C$ be the Abramovich–Polishchuk heart in $\mathrm{D}_{\mathrm{perf}}(S \times C)$. An object $F \in \mathcal{A}_C$ is called *torsion*, if it is a pushforward of an object from $\mathrm{D}_{\mathrm{perf}}(S \times T)$, where $T \subset C$ is some proper subscheme. The object F

is *flat*, if $F_p := Li_p^* F \in \mathcal{A}$ for all $p \in C$, and it is *torsion-free*, if it does not contain any torsion subobjects.

Let $f\colon C \to \mathfrak{Coh}_r^\sharp(S)_\mathbf{v}$ be a quasimap to the pair $(S^{[n]}, \mathfrak{Coh}_r^\sharp(S))$, then, as in the case of the standard heart, we can lift it to $\mathfrak{Coh}^\sharp(S)_\mathbf{v}$ by the determinant section

$$s_{\det} : \mathcal{U} \to \mathfrak{Coh}^\sharp(S)_\mathbf{v},$$

defined over some substack $\mathcal{U} \subset \mathfrak{Coh}_r^\sharp(S)_\mathbf{v}$ of finite type. We now prove the following.

Proposition 4.3. *Let F be the family on $S \times C$ associated to the lift of $f\colon C \to \mathfrak{Coh}_r^\sharp(S)_\mathbf{v}$ via $s_{\det}$, then F is stable pair, i.e. $F \in \mathrm{P}(S \times C)$. Conversely, given a stable pair $I^\bullet \in \mathrm{P}(S \times C)$, then $I^\bullet \in \mathcal{A}_C$.*

Here and elsewhere, $\mathrm{P}(\dots)$ stands for a moduli space of stable pairs (see [PT09] for the theory of stable pairs in the context of enumerative geometry). To prove the proposition, we need the following lemma.

Lemma 4.4. *A flat object $F \in \mathcal{A}_C$ is torsion-free.*

Proof of Lemma 4.4. Let $\tilde{F}$ be the pullback of F to the normalisation $S \times \tilde{C}$. Let $T \subset \tilde{F}$ be the maximal torsion object, then $\tilde{F}' := \tilde{F}/T$ is a torsion-free object, hence it is flat by [AP06, Corollary 3.1.3]. Restricting to a fiber over some $p \in C$ we get an exact sequence

$$0 \to T_p \to \tilde{F}_p \to \tilde{F}'_p \to 0,$$

because $\tilde{F}$ is flat. Thus $T_p \in \mathrm{Coh}^\sharp(S)$ and $\mathrm{ch}(T_p) = 0$ for all $p \in C$, since $\mathrm{ch}(\tilde{F}_p) = \mathrm{ch}(\tilde{F}'_p)$, which implies that $T_p = 0$ for all $p \in C$, which in turn implies that $T = 0$. If F had torsion, it would produce torsion in $\tilde{F}$, hence F is torsion-free. $\square$

Proof of Proposition 4.3. Now let be F be an object corresponding to the lift of a quasimap $f\colon C \to \mathfrak{Coh}_r^\sharp(S)_\mathbf{v}$, by definition it is family of objects in $\mathcal{A}$, hence $F \in \mathcal{A}_C$ by [AP06] and F is flat. It is also clear that F is of rank 1, and that $\det(F) = \mathcal{O}_{S\times C}$ by the choice of the lift. By [Tod10, Lemma 3.11] to show that $F \in \mathrm{P}(S \times C)$, we have to establish the following properties:

(**i**) $\mathcal{H}^i(F) = 0$, for $i \neq 0, 1$;
(**ii**) $\mathcal{H}^0(F)$ is a rank-1 torsion-free sheaf and $\mathcal{H}^1(F)$ is 0-dimensional;
(**iii**) $\mathrm{Hom}(Q[-1], F) = 0$ for any 0-dimensional sheaf Q.

(**i**) Since F is a family of objects with amplitude $[0,1]$, F cannot be of amplitude wider than $[0,1]$. To see this, consider the two triangles, the object F fits in,

$$\tau_{<0}F \to F \to \tau_{\geq 0}F \to,$$

$$\tau_{<2}F \to F \to \tau_{\geq 2}F \to,$$

where the truncation is taken with respect to the standard t-structure. Taking fibers over $t \in C$ and considering long exact sequences of cohomologies in the standard heart we conclude that $\tau_{<0}F = 0$ and $\tau_{\geq 2}F = 0$.

(**ii**) Let $T(\mathcal{H}^0(F)) \subseteq \mathcal{H}^0(F)$ be the maximal torsion subsheaf, composition $T(\mathcal{H}^0(F)) \hookrightarrow \mathcal{H}^0(F) \to F$ is zero, because F is torsion-free, but in the standard heart the second map is just an inclusion of 0-th cohomology, hence the whole composition must be zero, therefore $T = 0$ and $\mathcal{H}^0(F)$ is torsion-free. Due to fact that F_t is an ideal for a general $p \in C$ and $F_p \in \mathcal{A}$ for all $p \in C$, $\mathcal{H}^1(F)$ must be 0-dimensional by the definition of $\mathcal{A}$.

(**iii**) The last property follows trivially, because F is torsion-free.

Conversely, given now a stable pair $I^\bullet \in \mathrm{P}(S \times C)$, by definition it sits in a triangle

$$\mathcal{H}^0(I^\bullet) \to I^\bullet \to \mathcal{H}^1(I^\bullet)[-1] \to,$$

such that $\mathcal{H}^0(I^\bullet)$ is an ideal sheaf and $\mathcal{H}^1(I^\bullet)$ is 0-dimensional. Applying $p_{S*}(- \otimes \mathcal{O}_C(m))$ for $m \gg 0$ to the triangle, we obtain that $p_{S*}(\mathcal{H}^0(I^\bullet) \otimes \mathcal{O}_C(m))$ is a torsion-free sheaf and $p_{S*}(\mathcal{H}^1(I^\bullet) \otimes \mathcal{O}_C(m))$ is 0-dimensional, therefore $p_{S*}(I^\bullet \otimes \mathcal{O}_C(m)) \in \mathcal{A}$ for $m \gg 0$, hence by the definition $I^\bullet \in \mathcal{A}_C$.

With a bit more work, one should be able to prove that

$$\mathcal{A}_C = \langle \mathcal{T}_C^{\perp}, \mathcal{T}_C[-1] \rangle,$$

where $\mathcal{T}_C = \{A \in \mathrm{Coh}(S \times C) | \dim(A) = 0\}$. $\square$

The determinant line bundle construction in this setting also defines the map $\lambda \colon K_0(S) \to \mathrm{Pic}(\mathfrak{Coh}^\sharp(S)_{\mathbf{v}})$. The line bundles $\mathcal{L}_0$ and $\mathcal{L}_1$ satisfy the same properties as in the case of the standard heart.

Lemma 4.5. *Let $f \colon C \to \mathfrak{Coh}^\sharp(S)$ be a semistable quasimap. Fix $\mathbf{v}$ and $\mathcal{L}_{1,\ell} \cdot_f C$ for all ℓ. There exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ the quasimap is non-constant, if and only if*

$$\mathcal{L}_0 \otimes \mathcal{L}_1^m \cdot_f C > 0.$$

The same holds for all subcurves C' and the induced maps for the same choice of m.

Proof. The proof is similar to the one of Proposition 3.7, but with one exception - the unstable locus of $\mathfrak{Coh}^\sharp(S)$ now contains objects which sit in a distinguished triangle

$$\mathcal{H}^0(A) \to A \to \mathcal{H}^1(A)[-1] \to$$

such that $\mathcal{H}^1(A)$ is a 0-dimension sheaf. When we apply semistable reduction to such objects the corresponding term $\chi(\mathbf{v})\mathrm{rk}(Q^i) - \mathrm{rk}(\mathbf{v})\chi(Q^i)$ is strictly negative. To get around this problem, for a pair $I^\bullet \in \mathrm{P}(S \times C)$ we firstly take its zeroth cohomology

$$\mathcal{H}^0(I^\bullet) \to I^\bullet \to \mathcal{H}^1(I^\bullet)[-1] \to$$

where $\mathcal{H}^0(I^\bullet)$ is an ideal sheaf and $\mathcal{H}^1(I^\bullet)$ is zero dimensional, and then run the Langton's semistable reduction for $\mathcal{H}^0(I^\bullet)$. $\square$

Now fixing a positive line bundle $\mathcal{L}_\beta$ from the Lemma 4.5 once and for ever for all $\beta \in \mathrm{Eff}(S^{[n]}, \mathfrak{Coh}^\sharp_r(S)_{\mathbf{v}})$, we can define the length of base point as previously. The definition of ϵ-stability carries over to this case verbatim. Given $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$, let

$$Q^\epsilon_{g,N}(S^{[n]}, \beta)^\sharp \colon (Sch/\mathbb{C})^\circ \to (Grpd)$$

be a moduli space of ϵ-stable perverse quasimaps to the pair $(S^{[n]}, \mathfrak{Coh}^\sharp_r(S)_{\mathbf{v}})$ for some $\beta \in \mathrm{Eff}(S^{[n]}, \mathfrak{Coh}^\sharp_r(S))$. The proof of boundedness of the moduli is exactly the same as in the case of the standard heart. Using Lemma 4.3, we obtain an immersion,

$$Q^\epsilon_{g,N}(S^{[n]}, \beta)^\sharp \hookrightarrow \mathrm{P}(S \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}),$$

where is the space on the right is the relative moduli space of stable pairs. We denote the stacky image of above embedding by $\mathrm{P}^\epsilon_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N})$. It can also be described more explicitly in terms of stable pairs just as in the case of a relative Hilbert scheme, Section 4.1.

For properness of $\mathrm{P}^\epsilon_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N})$, we need the following lemma. However, its proof is different from the one of the standard heart.

Lemma 4.6. *Let $\mathcal{C} \to \Delta$ be a family of nodal curves and $\{p_i\} \subset \mathcal{C}$ be finitely many closed in the regular locus of the central fiber. Then any quasimap $\tilde{u} \colon \tilde{\mathcal{C}} = \mathcal{C} \setminus \{p_i\} \to \mathfrak{Coh}^\sharp_r(S)_{\mathbf{v}}$ extends to $u \colon \mathcal{C} \to \mathfrak{Coh}^\sharp_r(S)_{\mathbf{v}}$, which is unique up to unique isomorphism.*

Proof. Employing the similar proof as the one of Lemma 3.19 is problematic in this case, as we do not know how to extend objects in the derived category (unlike sheaves), so we follow a different strategy.

Restricting $\tilde{u}$ to the generic fiber $\mathcal{C}^\circ$ of $\mathcal{C}$ over Δ, we obtain a relative family F° on $S \times \mathcal{C}^\circ$, which by properness of the relative moduli of stable pairs $\mathrm{P}(S \times \mathcal{C}/\Delta)$ can be completed to a family F on $S \times \mathcal{C}$. It may be non-flat only over nodes of the central fiber, therefore it defines a rational quasimap $u \colon \mathcal{C} \dashrightarrow \mathfrak{Coh}^\sharp_r(S)_{\mathbf{v}}$ possibly with indeterminacies only at the nodes of the central fiber. It also defines a rational map $u_{rat} \colon \mathcal{C} \dashrightarrow S^{[n]}$, so does $\tilde{u}$, $\tilde{u}_{rat} \colon \mathcal{C} \dashrightarrow S^{[n]}$, the corresponding graphs in $\mathrm{Hilb}(S^{[n]} \times \mathcal{C})$ agree generically, therefore by separateness of Hilbert schemes they are equal, i.e. $u_{rat} = \tilde{u}_{rat}$. Now if p_i is not a limit of base points of $\tilde{u}$, then there is a neighbourhood $U \subset \mathcal{C}$ around p_i, where

$$\tilde{u}_{|U/p_i} = \tilde{u}_{rat|U/p_i} = u_{rat|U/p_i} = u_{|U/p_i},$$

we then define $\tilde{u}_{|U} = u_{|U}$ (u is defined at p_i, because p_i is in the regular locus), since quasimaps to $\mathfrak{Coh}^\sharp_r(S)_{\mathbf{v}}$ do not have any internal automorphisms we can glue maps in a unique way thereby extending $\tilde{u}$ to p_i. If p_i is an limit of base points of $\tilde{u}$, let $B_i \subset \mathcal{C}$ be the section corresponding to these base points, then there is some neighbourhood U around B_i, such that

$$\tilde{u}_{|U/B_i} = \tilde{u}_{rat|U/B_i} = u_{rat|U/B_i} = u_{|U/B_i},$$

but since $\tilde{u}_{|\mathcal{C}^\circ} = u_{|\mathcal{C}^\circ}$, we conclude that $\tilde{u}_{U/p_i} = u_{U/p_i}$, again because quasimaps to $\mathfrak{Coh}^\sharp_r(S)_{\mathbf{v}}$ do not have any internal automorphisms and therefore glue in a unique away, we then proceed as before. Let $u' \colon \mathcal{C} \to \mathfrak{Coh}^\sharp_r(S)_{\mathbf{v}}$ be the resulting extension and F' be the associated family, then separateness of relative moduli of stable pairs implies that $F' = F$ and that the extension is unique. $\square$

Summing up the discussion above we obtain the following result.

Corollary 4.7. *The moduli stack $Q^\epsilon_g(S^{[n]}, \beta)^\sharp$ is a proper Deligne–Mumford stack, and there exists a natural isomorphism of the moduli stacks*

$$Q^\epsilon_{g,N}(S^{[n]}, \beta)^\sharp \cong \mathrm{P}^\epsilon_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}),$$

the stack on the right is the relative moduli stack of stable pairs, satisfying exactly the same conditions as in the case of the standard heart.

Proof. For the properness we again use Lemma 4.6 and the proof presented in [CKM14, Proposition 4.3.1.]. $\square$

Remark 4.8. As in the case of the standard heart, for a fixed smooth curve C with $g \geq 1$ and $\beta \neq 0$, we have

$$Q_C(S^{[n]}, \beta)^\sharp \cong \mathrm{P}_{n,\check{\beta}}(S \times C),$$

by Corollary 4.7. On the other hand,

$$Q_{(C,p)}(S^{[n]}, \beta)^\sharp \cong \mathrm{P}_{n,\check{\beta}}(S \times C/S_p).$$

4.3. **Affine plane.** A punctorial Hilbert scheme of the affine plane $\mathbb{C}^2$ admits two equivalent descriptions, one is a Nakajima variety of a quiver, which is a GIT construction,

$$(\mathbb{C}^2)^{[n]} = [\mu^{-1}(0)/\mathrm{GL}_n]^s \subset [\mu^{-1}(0)/\mathrm{GL}_n],$$

for the notation see [Gin12]. Another one is a moduli of framed sheaves on $\mathbb{P}^2$. Both descriptions sit in some bigger stack, but to match the unstable loci, one has to consider a non-standard heart of $\mathrm{D^b}(\mathbb{P}^2)$, namely $\mathrm{Coh}^\sharp(\mathbb{P}^2)$, then

$$(\mathbb{C}^2)^{[n]} \subset \mathfrak{Coh}^\sharp(\mathbb{P}^2, l^\infty)_{\mathbf{v}},$$

where on the right we consider framings with respect to the line at infinity, which in this case just kills $\mathbb{C}^*$-automorphisms. By [BFG06, Theorem 5.7], we have a canonical isomorphism

$$[\mu^{-1}(0)/\mathrm{GL}_n] \cong \mathfrak{Coh}^\sharp(\mathbb{P}^2, l^\infty)_{\mathbf{v}},$$

which identifies stable loci on both sides. Hence a GIT quasimap moduli space and a perverse-coherent-sheaves quasimap moduli spaces of $(\mathbb{C}^2)^{[n]}$ are isomorphic,

$$Q^{0^+}_{g,N}((\mathbb{C}^2)^{[n]},\beta)^{\mathrm{GIT}} \cong Q^{0^+}_{g,N}((\mathbb{C}^2)^{[n]},\beta)^{\sharp}.$$

Moreover, since $[\mu^{-1}(0)/GL_n]$ is l.c.i., an easy check of virtual dimensions shows that the obstruction theory on $\mathfrak{Coh}^\sharp(\mathbb{P}^2, l^\infty)$, given by

$$R\mathcal{H}om_\pi(\mathcal{F},\mathcal{F})_0[1]^\vee \to \mathbb{L}_{\mathfrak{Coh}^\sharp(\mathbb{P}^2,l^\infty)},$$

is an isomorphism, where $\mathcal{F}$ is the universal complex and $\mathbb{L}_{\mathfrak{Coh}^\sharp(\mathbb{P}^2,l^\infty)}$ is the *truncated* cotangent complex. Therefore the obstruction theories of both quasimap theories also match, (see Section 5.2 for the construction of the obstruction theory for perverse coherent-sheaves quasimaps). To match ϵ-stabilities, one would need to check that the naturally defined line bundles of both stacks agree. However, we will not be concerned with it here, since ϵ-stability is mostly an auxiliary tool to do the wall-crossing between $\epsilon = 0^+$ and $\epsilon = \infty$ chambers. The identification above is enough to conclude that the wall-crossing is same in both cases.

5. Obstruction theory

5.1. **Preparation.** From now on we fix a class $u \in K_0(S)$, such that

$$\chi(\mathbf{v}\cdot u) = 1,$$

to lift quasimaps with a section s_u. For punctorial Hilbert schemes, we use the determinant section $s_{\det}$. By a family associated to a quasimap $f\colon C \to \mathfrak{Coh}_r(S)$, we will mean the one that is obtained from the lift by this fixed section. The content of this section applies to the pair $(S^{[n]}, \mathfrak{Coh}_r(S))$ as well as to the pair $(S^{[n]}, \mathfrak{Coh}^\sharp_r(S))$, the arguments are exactly the same for both pairs, hence we will just state and prove everything for $(S^{[n]}, \mathfrak{Coh}_r(S))$.

Lemma 5.1. *Let $f\colon C \to \mathfrak{Coh}(S)$ be a quasimap, then the corresponding family F on $S \times C$ is perfect.*

Proof. Since F is a family of sheaves on a smooth S over C, which is of finite type, there exists a locally free resolution of finite length. □

Let

$$\mathrm{tr}\colon \mathcal{H}om(F, F\otimes L) \to L$$

be the trace morphism. We define

$$\mathrm{Ext}^i(F, F\otimes L)_0 := \ker H^i(\mathrm{tr}) \text{ for all i.}$$

Proposition 5.2. *Let $f\colon C \to \mathfrak{Coh}_r(S)$ be a prestable quasimap. Assume any of the following holds*

(i) $(M, \mathfrak{Coh}_r(S)) = (S^{[n]}, \mathfrak{Coh}_r(S))$ *or*
(ii) *S is a del Pezzo surface or*
(iii) *S is a K3 surface and $g(C) \leq 1$,*

then the corresponding family F satisfies the following

$$\mathrm{Ext}^i(F,F)_0 = 0 \text{ for } i \neq 1,2.$$

Proof. By Lemma 5.1 and by Serre duality we get

$$\mathrm{Ext}^i(F,F) = \mathrm{Ext}^{3-i}(F, F \otimes \omega_{S\times C}),$$

therefore $\mathrm{Ext}^i(F,F) = 0$ for $i \notin [0,3]$, because $S \times C$ is l.c.i. ($\omega_{S\times C}$ is a locally free sheaf). Since F is stable, it is simple, hence $\mathrm{Hom}(F,F)_0 = 0$. We therefore have to show that

$$\mathrm{Hom}(F, F \otimes \omega_{S\times C})_0 = 0.$$

And since the trace morphism has a section given by

$$\mathrm{id}_{\otimes} \colon \omega_{S\times C} \to \mathcal{H}om(F, F \otimes \omega_{S\times C}), \quad s \mapsto \mathrm{id}_F \otimes s$$

after taking cohomology, it is enough to show that $H^0(\mathrm{tr})$ is injective.

(i) Assume that $(M, \mathfrak{Coh}_r(S)) = (S^{[n]}, \mathfrak{Coh}_r(S))$, then F is an ideal sheaf I of a curve $\Gamma \subset S \times C$. Let U be the complement of Γ and

$$\pi \colon S \times \tilde{C} \to S \times C, \quad D \subset S \times C$$

be the normalisation and the singular locus of $S \times C$, respectively. Then by applying $H^0(S \times C, -)$ and $H^0(U, -)$ to the exact sequence

$$0 \to \omega_{S\times C} \to \pi_*\pi^*\omega_{S\times C} \to \omega_{S\times C|D} \to 0,$$

we obtain

$$\begin{array}{ccccc} H^0(S\times C, \omega_{S\times C}) & \hookrightarrow & H^0(S\times C, \pi_*\pi^*\omega_{S\times C}) & \longrightarrow & H^0(D, \omega_{S\times C|D}) \\ \downarrow & & \| & & \| \\ H^0(U, \omega_{S\times C}) & \hookrightarrow & H^0(U, \pi_*\pi^*\omega_{S\times C}) & \longrightarrow & H^0(D\cap U, \omega_{S\times C|D}) \end{array}$$

The last two vertical arrows are bijective by Hartog‘s property for sections of locally free sheaves. Indeed, $\pi^*\omega_{S\times C}$ is locally free, Γ is of codimension 2, and D intersects properly with Γ. We conclude that

$$H^0(S\times C, \omega_{S\times C}) = H^0(U, \omega_{S\times C}).$$

Finally, since I is torsion free, the restriction of global sections

$$\begin{aligned} \mathrm{Hom}(I, I\otimes \omega_{S\times C}) \to \mathrm{Hom}(I_{|U}, I_{|U} \otimes \omega_{S\times C}) &= H^0(U, \omega_{S\times C}) \\ &= H^0(S\times C, \omega_{S\times C}) \end{aligned}$$

is injective. Moreover, it is equal to $H^0(\mathrm{tr})$ by the construction of tr, hence the claim follows.

(ii) Assume now that S is a del Pezzo surface, then the degree of a general fiber of $F \otimes \omega_{S\times C}$ is strictly smaller than the degree of a general fiber of F

by ampleness of the anti-canonical line bundle of S. Therefore by stability of a general giber of F, we have that

$$\mathrm{Hom}(F, F \otimes \omega_{S\times C}) = 0.$$

(iii) Finally, assume S is a K3 surface. We will show that

$$H^0(\mathrm{id}_{\otimes}) : H^0(S \times C, \omega_{S\times C}) \to \mathrm{Hom}(F, F \otimes \omega_{S\times C})$$

is surjective. By assumption, $\omega_{S\times C} \cong p_C^*\omega_C$, hence we have to show that all morphisms $\phi\colon F \to F \boxtimes \omega_C$ are of the form $\mathrm{id_F} \boxtimes s$ for some $s \in H^0(C, \omega_C,)$. By the normalisation sequence, it is enough to show it for

$$\pi^* F \to \pi^* F \boxtimes \pi_C^* \omega_C,$$

where

$$\pi = \mathrm{id} \times \pi_C \colon S \times \tilde{C} \to S \times C$$

is the normalisation map. We firstly establish the following result.

Lemma 5.3. *Let C be smooth. Given a sheaf F on $S \times C$, that defines a quasimap, and an effective divisor $D = \sum p_i$ on C, then any non-zero morphism*

$$\phi\colon F \to F(D) := F \boxtimes \mathcal{O}_C(D)$$

is injective. Moreover, if all p_i's are distinct, $\mathrm{supp}(\mathrm{coker}(\phi)) = S \times D$ *and F is stable over D, then $\phi = \mathrm{id}_F \boxtimes s$ for some $s \in H^0(\mathcal{O}(D), C)$.*

Proof of Lemma 5.3. Assume ϕ is not an inclusion, then the difference of Hilbert polynomials

$$p_{\mathcal{O}_{S\times C}(1,k)}(\mathrm{Im}(\phi)) - p_{\mathcal{O}_{S\times C}(1,k)}(F)$$

increases as k increases, because a general fiber of F is stable. Therefore $\mathrm{Im}(\phi)$ becomes $\mathcal{O}_{S\times C}(1,k)$-destabilising for $F(D)$ for some $k \gg 0$. However, by Lemma 3.17 the sheaf $F(D)$ is $\mathcal{O}_{S\times C}(1,k)$-stable for some $k \gg 0$, which is a contradiction. Hence ϕ must be an inclusion.

We now deal with the second part of the lemma. Consider the sequence

$$0 \to F \xrightarrow{\phi} F(D) \to \mathrm{coker}(\phi) \to 0,$$

restricting it to $S \times D$, we obtain

$$0 \to \mathrm{coker}(\phi) \to F_{|D} \xrightarrow{\phi_{|D}} F(D)_{|D} \to \mathrm{coker}(\phi) \to 0,$$

where we used that schematic support of $\mathrm{coker}(\phi)$ is $S \times D$. Since F_{p_i}'s are stable and $F_{|D} \cong F(D)_{|D}$, the map $\phi_{|D}$ must be zero. Therefore the map $F(D)_{|D} \to \mathrm{coker}(\phi)$ is an isomorphism. Consider now the following diagram

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\mathrm{id}_F \boxtimes s_D"] \arrow[d, dashed, "\psi"] & F(D) \arrow[r] \arrow[d, equal] & F(D)_{|D} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r, "\phi"] & F(D) \arrow[r] & \mathrm{coker}(\phi) \arrow[r] & 0,
\end{tikzcd}$$

where $s_D \in H^0(\mathcal{O}(D), C)$ is a defining section of D. The right square of the diagram is commutative, and the last two vertical arrows are isomorphisms, so we have

$$\phi = \psi \boxtimes s_D \text{ for some } \psi \in \mathrm{Aut}(F).$$

But F is stable and therefore simple, hence $\psi = c \cdot \mathrm{id}_F$ for some $c \in \mathbb{C}^*$. The claim now follows. $\square$

Continuation of the proof of Proposition 5.2. Recall that there is a natural isomorphism $\pi_C^*\omega_C \cong \omega_{\tilde{C}}(\sum q_i + q_i')$, where q_i and q_i' are preimages of a node of C. Given now a rational component $\tilde{C}_j$ of $\tilde{C}$ with at most two special points, then $\pi_j^*\omega_C \cong \mathcal{O}_{\mathbb{P}^1}(k)$ for $k \leq 0$. Both π_j^*F and $F_{|\tilde{C}_j} \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)$ are $\mathcal{O}_{S\times C}(1,n)$-stable for some $n \gg 0$ by Lemma 3.17. If $k < 0$, then Hilbert polynomials satisfy

$$p_{\mathcal{O}_{S\times C}(1,n)}(\pi_j^*F) > p_{\mathcal{O}_{S\times C}(1,n)}(\pi_j^*F \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)),$$

hence

$$\mathrm{Hom}(\pi_j^*F, \pi_j^*F \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)) = 0.$$

If $k = 0$, then $\pi_j^*F \cong \pi_j^*F \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)$. By induction we then conclude that the restriction of ϕ to all rational trees must be zero, and by the previous lemma the restriction of ϕ to their complement comes from box-tensoring a section. $\square$

Remark 5.4. All quasimaps are prestable in the case of punctorial Hilbert schemes, since an ideal I of a curve on a threefold $S \times C$ is stable over a node $s \in C$, if and only if it is flat over the node [12]. This can be seen as follows. The sheaf I_s is stable, if and only if it is torsion-free, which is equivalent to the injectivity on the left of the exact sequence

$$I_s \to \mathcal{O}_{S\times s} \to \Gamma_s \to 0,$$

which in turn is equivalent to $\mathrm{Tor}^1_{S\times C}(\mathcal{O}_\Gamma, \mathcal{O}_{S\times s}) = 0$, but by standard periodic resolution of a structure sheaf of a node,

$$\mathrm{Tor}^k_{S\times C}(\mathcal{O}_\Gamma, \mathcal{O}_{S\times s}) = \mathrm{Tor}^1_{S\times C}(\mathcal{O}_\Gamma, \mathcal{O}_{S\times s}) \text{ for all } k \geq 1.$$

If I is flat, then I is perfect, hence $\mathcal{O}_\Gamma$ is also perfect, so

$$\mathrm{Tor}^k_{S\times C}(\mathcal{O}_\Gamma, \mathcal{O}_{S\times s}) = 0 \text{ for some } k \gg 0,$$

which therefore implies that $\mathrm{Tor}^1_{S\times C}(\mathcal{O}_\Gamma, \mathcal{O}_{S\times s}) = 0$.

[12] In DT theory this condition is referred to as *predeformable*.

5.2. **Obstruction theory.** We have the following perfect obstruction theory over a substack of finite type $\mathcal{U} \subset \mathfrak{Coh}_r(S)$,

$$(\mathbb{T}^{\mathrm{vir}})^{\vee} := (R\mathcal{H}om_{\pi}(\mathcal{F}_{r|\mathcal{U}}, \mathcal{F}_{r|\mathcal{U}})_0[1])^{\vee} \to \mathbb{L}_{\mathcal{U}},$$

where $\mathcal{F}_r$ is the universal family on $S \times \mathfrak{Coh}_r(S)$, note that the complex $(\mathbb{T}^{\mathrm{vir}})^{\vee}$ is of amplitude [-1,1] due to the presence of non-discrete automorphisms of the unstable part of $\mathfrak{Coh}_r(S)$. Let

$$\pi_1 \colon \mathcal{C}_{g,N} \to Q^{\epsilon}_{g,N}(M,\beta)$$
$$\mathbb{f} \colon \mathcal{C}_{g,N} \to \mathfrak{Coh}_r(S),$$

be the canonical projection from the universal curve and the universal map. The universal map $\mathbb{f}$ factors through some substack of finite type, hence we can define the obstruction-theory complex $(\pi_* \mathbb{f}^* \mathbb{T}^{\mathrm{vir}})^{\vee}$. Let us show how it is related to obstruction-theory complex of a relative moduli of stable sheaves. Let

$$\pi_2 \colon S \times \mathcal{C}_{g,N} \to Q^{\epsilon}_{g,N}(M,\beta),$$
$$\mathbb{F} \in \mathrm{Coh}(S \times \mathcal{C}_{g,N} \times_{\mathfrak{M}_{g,N}} Q^{\epsilon}_{g,N}(M,\beta))$$

be the canonical projection and the universal sheaf, which is defined via the identification $Q^{\epsilon}_{g,N}(M,\beta) \cong M^{\epsilon}_{\check{\beta},u}(S \times C_{g,N}/\overline{M}_{g,N})$. We then take the traceless part of the relative derived self-hom complex

$$R\mathcal{H}om_{\pi_2}(\mathbb{F},\mathbb{F})_0[1],$$

and prove the following.

Proposition 5.5. *The complex* $(\pi_{1*}\mathbb{f}^*\mathbb{T}^{\mathrm{vir}})^{\vee}$ *is canonically isomorphic to the complex* $(R\mathcal{H}om_{\pi_2}(\mathbb{F},\mathbb{F})_0[1])^{\vee}$*.*

Proof. Consider the following diagram

$$\begin{tikzcd}
S \times \mathcal{C}_{g,N} \times Q^{\epsilon}_{g,N}(M,\beta) \arrow[r, "\mathrm{id}\times\mathbb{f}"] \arrow[d] \arrow[dd, bend right=60, "\pi_2"'] & S \times \mathcal{U} \arrow[d, "\pi_{\mathcal{U}}"] \\
S \times \mathcal{C}_{g,N} \times Q^{\epsilon}_{g,N}(M,\beta) \arrow[r, "\mathbb{f}"] \arrow[d, "\pi_1"] & \mathcal{U} \\
Q^{\epsilon}_{g,N}(M,\beta) &
\end{tikzcd}$$

the trace map $\mathrm{tr} \colon R\mathcal{H}om(\mathcal{F}_{r|\mathcal{U}}, \mathcal{F}_{r|\mathcal{U}}) \to \mathcal{O}_{\mathcal{U}}$ has a section given by the inclusion of identity $\mathcal{O}_{\mathcal{U}} \to R\mathcal{H}om(\mathcal{F}_{r|\mathcal{U}}, \mathcal{F}_{r|\mathcal{U}})$, therefore

$$R\mathcal{H}om(\mathcal{F}_{r|\mathcal{U}}, \mathcal{F}_{r|\mathcal{U}}) = R\mathcal{H}om(\mathcal{F}_{r|\mathcal{U}}, \mathcal{F}_{r|\mathcal{U}})_0 \oplus \mathcal{O}_{\mathcal{U}},$$

and by the moduli problem of $\mathfrak{Coh}_r(S)$, we get

$$(\mathbb{f} \times \mathrm{id})^* \mathcal{F}_r = \mathbb{F},$$

hence, by functoriality of the trace and the splitting above, we obtain that

$$(\mathbb{f} \times \mathrm{id})^* R\mathcal{H}om(\mathcal{F}_{r|\mathcal{U}}, \mathcal{F}_{r|\mathcal{U}})_0 = R\mathcal{H}om(\mathbb{F},\mathbb{F})_0.$$

By base change theorem,

$$R\mathcal{H}om_{\pi_2}(\mathbb{F},\mathbb{F})_0 = \pi_{1*}\mathbb{f}^* R\mathcal{H}om_{\pi_{\mathcal{U}}}(\mathcal{F}_{r|\mathcal{U}},\mathcal{F}_{r|\mathcal{U}})_0.$$

□

Corollary 5.6. *There exists an obstruction theory*

$$\phi\colon (\pi_{1*}\mathbb{f}^*\mathbb{T}^{\mathrm{vir}})^\vee \to \mathbb{L}_{Q^\epsilon_{g,N}(M,\beta)/\mathfrak{M}_{g,N}},$$

which is is perfect under the assumptions of Proposition 5.2. Moreover, if $\epsilon = 0^+$, the corresponding virtual fundamental classes coincide with those of DT theory.

Proof. Using the results of [TV07] and [ST15], the stack $\mathfrak{Coh}_r(S)$ can be naturally upgraded to a derived stack $\mathbb{R}\mathfrak{Coh}_r(S)$, such that

$$\tau_{\geq 0}\mathbb{R}\mathfrak{Coh}_r(S) = \mathfrak{Coh}_r(S),$$

and

$$\mathbb{L}_{\mathbb{R}\mathfrak{Coh}_r(S)} = (\mathbb{T}^{\mathrm{vir}})^\vee.$$

Recall that a derived enhancement gives rise to an obstruction theory of underlying classical stack, see [ST15, Section 1] for more details. The obstruction theory

$$\phi\colon (\pi_{1*}\mathbb{f}^*\mathbb{T}^{\mathrm{vir}})^\vee \to \mathbb{L}_{Q^\epsilon_{g,N}(M,\beta)/\mathfrak{M}_{g,N}}$$

is therefore given by a derived mapping stack of maps from curves to the derived stack $\mathbb{R}\mathfrak{Coh}_r(S)$, which exists by Lurie's representability theorem [Lur12] (see also [Toë09]). The obstruction theory is perfect by Proposition 5.5 and Proposition 5.2 .

By [Sie04], a virtual fundamental class depends only on Chern characters of the corresponding obstruction-theory complex. The second part of the claim therefore follows from Proposition 5.5. □

Let

$$[Q^\epsilon_{g,N}(M,\beta)]^{\mathrm{vir}} \in A_{\mathrm{vdim}}(Q^\epsilon_{g,N}(M,\beta))_{\mathbb{Q}}$$

be the associated virtual fundamental class. Invoking the identification presented in Lemma 3.2, the virtual dimension of the moduli space can be computed via the virtual dimension of the relative moduli space of sheaves,

$$\begin{aligned}
\mathrm{vdim} &= \sum (-1)^i \dim \mathrm{Ext}^i(F,F)_0 + (3g-3) + N \\
&= \int_{S\times C} (\mathrm{ch}(F)\cdot \mathrm{ch}(F)^\vee - 1)\cdot \mathrm{td}_{S\times C} + (3g-3) + N \\
&= \mathrm{rk}(\mathbf{v})\mathrm{c}_1(\check{\beta})\cdot \mathrm{c}_1(S) - \mathrm{rk}(\check{\beta})\mathrm{c}_1(\mathbf{v})\cdot \mathrm{c}_1(S) + (\dim(M)-3)(1-g) + N,
\end{aligned}$$

where $\mathrm{rk}(\check{\beta})$ and $\mathrm{c}_1(\check{\beta})$ are the components of $\check{\beta} \in \Lambda$ of cohomogical degrees 0 and 2, respectively.

By our definition of a degree β, it can only pair with determinant line bundles on the stack $\mathfrak{Coh}_r(S)$, and it is unclear, if the virtual anti-canonical line bundle is a determinant line bundle, even though it is the case over the stable locus in some very special instances. Therefore the above formula for the virtual dimension is the most reasonable one. We will treat the first two summands as the degree with respect to the virtual anti-canonical line bundle,

$$\beta(\det(\mathbb{T}^{\mathrm{vir}})) := \mathrm{rk}(\mathbf{v})\mathrm{c}_1(\check{\beta}) \cdot \mathrm{c}_1(S) - \mathrm{rk}(\check{\beta})\mathrm{c}_1(\mathbf{v}) \cdot \mathrm{c}_1(S).$$

The above formula is, however, dependent upon presentation of $Q^{\epsilon}_{g,N}(M,\beta)$ as a relative moduli space of sheaves, the virtual dimension itself is not though.

5.3. **Invariants.** The moduli spaces $Q^{\epsilon}_{g,N}(M,\beta)$ has the usual canonical structures to define the enumerative invariants:

- evaluation maps at marked points
$$ev_i \colon Q^{\epsilon}_{g,N}(M,\beta) \to M, \quad i = 1, \dots, N;$$
- cotangent line bundles
$$\mathcal{L}_i := s_i^*(\omega_{\mathcal{C}_{g,N}/Q^{\epsilon}_{g,N}(M,\beta)}), \quad i = 1, \dots, N,$$

where $s_i \colon Q^{\epsilon}_{g,N}(M,\beta) \to \mathcal{C}_{g,N}$ are universal markings. We denote

$$\psi_i := \mathrm{c}_1(\mathcal{L}_i), \quad i = 1, \dots, N.$$

Definition 5.7. The *descendent ϵ-invariants* are

$$\langle \tau^{m_1}(\gamma_1), \dots, \tau^{m_N}(\gamma_N)\rangle^{\epsilon}_{g,N,\beta} := \int_{[Q^{\epsilon}_{g,N}(M,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{i=N} \psi_i^{m_i} \mathrm{ev}_i^*(\gamma_i,),$$

where $\gamma_1, \dots, \gamma_N \in H^*(M, \mathbb{Q})$ and $m_1, \dots m_N$ are non-negative integers. We similarly define the perverse invariants $\langle \tau^{m_1}(\gamma_1), \dots, \tau^{m_N}(\gamma_N)\rangle^{\sharp,\epsilon}_{g,N,\beta}$.

Remark 5.8. We can also define another kind of invariants by the identification of quasimaps with the relative moduli of sheaves - relative DT descendent invariants (do not confuse with invariants relative to divisors), consider

$$\begin{tikzcd} & S \times \mathcal{C}_{g,N} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S \times \overline{M}_{g,N+1} & & Q^{\epsilon}_{g,N}(M,\beta) \end{tikzcd}$$

where for the π_1 we stabilise the curves and used the identification of $\overline{M}_{g,N+1}$ with the universal curve of $\overline{M}_{g,N}$, for the unstable values of g and N we set the product $S \times \overline{M}_{g,N+1}$ to be S. For a class $\bar{\gamma} \in H^*(S \times \overline{M}_{g,N+1}, \mathbb{Q})$ define the following operation on cohomology,

$$\mathrm{ch}_{k+2}(\bar{\gamma}) \colon H_*(Q^{\epsilon}_{g,N}(M,\beta), \mathbb{Q}) \to H_{*-2k+2-\ell}(Q^{\epsilon}_{g,N}(M,\beta), \mathbb{Q}),$$

$$\mathrm{ch}_{k+2}(\bar{\gamma})(\xi) = \pi_{2*}\left(\mathrm{ch}_{k+2}(\mathbb{F}) \cdot \pi_1^*(\bar{\gamma}) \cap \pi_2^*(\xi)\right).$$

The relative descendent invariants are then defined by

$$\begin{aligned}&\langle \tilde{\tau}_{k_1}(\bar{\gamma}_1), \dots, \tilde{\tau}_{k_r}(\bar{\gamma}_r)\rangle^{\epsilon}_{g,n,\beta} \\ &\qquad = (-1)^{k_1}\mathrm{ch}_{k_1+2} \ \circ \ \dots \ \circ \ (-1)^{k_r}\mathrm{ch}_{k_r+2}\left([Q^{\epsilon}_{g,N}(M,\beta)]^{\mathrm{vir}}\right),\end{aligned}$$

here we just transferred the definitions from rank-1 story, note that for higher ranks $\tilde{\tau}_{-1}(-)$ in the notation above might also be non-trivial. We can also define the mix of descendent GW invariants and relative DT invariants,

$$\langle \tilde{\tau}_{k_1}(\bar{\gamma}_1), \dots, \tilde{\tau}_{k_r}(\bar{\gamma}_r) \mid \tau_{m_1}(\gamma_1), \dots, \tau_{m_N}(\gamma_N)\rangle^{\epsilon}_{g,N,\beta},$$

which are essentially a mix of relative and absolute DT invariants of the relative geometry

$$S \times C_{g,N} \to \overline{M}_{g,N}$$

for different ϵ-stabilities. However, we will not be concerned with any of the DT-type invariants defined above in the present work.

The discussion in [CKM14, Section 6] also applies to ϵ-invariants in our setting. In particular, ϵ-invariants satisfy an analogue of the Splitting Axiom in GW theory, and there exists a projection to the moduli of stable nodal curves

$$p \colon Q^{\epsilon}_{g,N}(M,\beta) \to \overline{M}_{g,N}$$

by taking stabilisation of the domain of a quasimap, so that the classes

$$p_*(\prod_{i=1}^{i=N} \psi_i^{m_i} ev_i^*(\gamma_i,)) \in H^*(\overline{M}_{g,N}, \mathbb{Q})$$

gives rise to Cohomological field theory on $H^*(M,\mathbb{Q})$.

6. Wall-crossing

6.1. **Graph space.** As previously, all the results of this section apply to both standard and perverse quasimaps, if $M = S^{[n]}$. In the latter case, all the notations acquire the superscript '$\sharp$'. This subsection is largely a repetition of [CFK14, Section 4].

Given $\beta \in \mathrm{Eff}(M, \mathfrak{Coh}_r(S))$, let $\epsilon \in \mathbb{R}_{>0}$ and $k \in \mathbb{Z}_{>0}$ be such that $1/k < \epsilon < 1/\deg\beta$, then we define the *graph space*

$$QG_{0,1}(M,\beta) := Q^{\epsilon}_{0,1}(M \times \mathbb{P}^1, \beta + [\mathbb{P}^1]),$$

where we consider quasimaps to $\mathfrak{Coh}_r(S) \times \mathbb{P}^1$ and ϵ-stability on the right is given with respect to $\mathcal{L}_\beta \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)$. This is the moduli space of genus-0 quasimaps to M, whose domain has a unique parametrised rational tail, such that restriction of the quasimap to its complement satisfies ϵ-stability, which is equivalent to 0^+-stability by the choice of ϵ. The definition is independent of ϵ and k, as long as they satisfy the inequality above.

The obstruction theory of $QG_{0,1}(M,\beta)$ is given by

$$(R\pi_*\mathbb{f}^*(\mathbb{T}^{\mathrm{vir}} \boxplus T_{\mathbb{P}^1}))^{\vee} \to \mathbb{L}_{QG_{0,1}(M,\beta)/\mathfrak{M}_{0,1}}.$$

There is a $\mathbb{C}^*$-action on $\mathbb{P}^1$ given by

$$t[x, y] = [tx, y], \quad t \in \mathbb{C}^*,$$

which induces a $\mathbb{C}^*$-action on $QG_{0,1}(M, \beta)$. The fixed points of the action must have their entire degrees with the markings lie over either 0 or ∞. Assuming the marking is over ∞, there are two distinguished extremal fixed components

$$F_\beta \text{ and } F^{0,0}_{1,\beta} \cong Q^{0^+}_{0,1+\bullet}(M, \beta).$$

The former is the locus of quasimaps with entire degree β over 0 as a base point, while the latter is the locus of quasimaps with entire degree over ∞ in the form of rational components. If the degree splits non-trivially between 0 and ∞, then the fixed components are of the following form

$$F^{0,\beta_1}_{1,\beta_2} := F_{\beta_1} \times_M F^{0,0}_{1,\beta_2}, \tag{14}$$

where $\beta = \beta_1 + \beta_2$ and the fibered product is taken with respect to distinguished markings. The description of fixed components $F^{1,\beta_1}_{0,\beta_2}$ with the marking over 0 is exactly the same. The virtual fundamental classes $[F^{0,\beta_1}_{1,\beta_2}]^{\mathrm{vir}}$ and the virtual normal bundles $N^{\mathrm{vir}}_{F^{0,\beta_1}_{1,\beta_2}/QG_{0,1}(M,\beta)}$ are defined by fixed and moving parts of the obstruction theory of $QG_{0,1}(M, \beta)$. They are compatible with respect to the product expression above,

$$\begin{aligned} [F^{0,\beta_1}_{1,\beta_2}]^{\mathrm{vir}} &= [F_{\beta_1}]^{\mathrm{vir}} \times_M [F^{0,0}_{1,\beta_2}]^{\mathrm{vir}}, \\ N^{\mathrm{vir}}_{F^{0,\beta_1}_{1,\beta_2}/QG_{0,1}(M,\beta)} &= N^{\mathrm{vir}}_{F_{\beta_1}/QG_{0,1}(M,\beta)} \boxtimes_M N^{\mathrm{vir}}_{F^{0,0}_{1,\beta_2}/QG_{0,1}(M,\beta)}. \end{aligned}$$

Let

$$\mathsf{ev} \colon F_\beta \to M$$

be the evaluation map at the unique marking at $\infty \in \mathbb{P}^1$.

Definition 6.1. We define *I-function*

$$I(q, z) = 1 + \sum_{\beta > 0} -zq^\beta \mathsf{ev}_* \left(\frac{[F_\beta]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta/QG_{0,1}(M,\beta)})} \right) \in A^*(M)[z^{\pm}] \otimes_{\mathbb{Q}} \mathbb{Q}[\![q^\beta]\!],$$

where $-z := e_{\mathbb{C}^*}(\mathbb{C}_{\mathrm{std}})$, where $\mathbb{C}_{\mathrm{std}}$ is the standard representation of $\mathbb{C}^*$. We also define

$$\mu(z) := [zI(q, z) - z]_+ \in A^*(M)[z] \otimes_{\mathbb{Q}} \mathbb{Q}[\![q^\beta]\!],$$

where $[\dots]_+$ is the truncation given by taking only non-negative powers of z. Let

$$\mu_\beta(z) \in A^*(M)[z]$$

be the coefficients of q^β in $\mu(z)$.

6.2. **Graph space and sheaves.** There is a forgetful morphism

$$QG_{0,1}(M,\beta) \to \overline{M}_{0,1}(\mathbb{P}^1,1) \tag{15}$$

which is given by projecting a quasimap to its parametrised component, the graph space $QG_{0,1}(M,\beta)$ then admits a relative perfect obstruction

$$(R\pi_* \mathfrak{f}^* \mathbb{T}^{\mathrm{vir}})^\vee \to \mathbb{L}_{QC_{0,1}(M,\beta)/\overline{M}_{0,1}(\mathbb{P}^1,1)},$$

which sits in a distinguished triangle

$$\mathbb{L}_{\overline{M}_{0,1}(\mathbb{P}^1,1)} \to \mathbb{E}_{QG_{0,1}(M,\beta)} \to (R\pi_* \mathfrak{f}^* \mathbb{T}^{\mathrm{vir}})^\vee \to .$$

Restricting the sequence above to the fixed component F_β, we obtain that the morphism

$$\mathbb{E}^{\mathrm{f}}_{QG_{0,1}(M,\beta)} \to (R\pi_* \mathfrak{f}^* \mathbb{T}^{\mathrm{vir}})^{\vee,\mathrm{f}}$$

between fixed parts is an isomorphism and

$$e_{\mathbb{C}^*}((R\pi_* \mathfrak{f}^* \mathbb{T}^{\mathrm{vir}})^{\vee,\mathrm{mv}}) = -z e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta/QG_{0,1}(M,\beta)}),$$

because the restriction of $\mathbb{L}_{\overline{M}_{0,1}(\mathbb{P}^1,1)}$ is a trivial line bundle with the fiber being the cotangent space of $\mathbb{P}^1$ at ∞, which is not fixed and whose Euler class is equal to $-z$. Consider now the component

$$QG_{0,p_\infty}(M,\beta) \subset QG_{0,1}(M,\beta)$$

of quasimaps, whose marking is over ∞. In other words, this is the fiber of (15) over ∞. Then applying the identification of quasimaps with sheaves, we obtain

$$QG_{0,p_\infty}(M,\beta) \cong M_{\check{\beta},u}(S \times \mathbb{P}^1/S_\infty),$$

where we slightly abuse the notation, because the moduli spaces on the right is different from those defined in Definition 3.14. In Definition 3.14, we exclude sheaves which fixed by infinitely many automorphisms of a curve. Here, we include all sheaves.

Moreover, the obstruction theory $(R\pi_* F^* \mathbb{T}^{\mathrm{vir}})^\vee_{|QG_{0,p_\infty}(M,\beta)}$ matches the relative DT obstruction theory of $M_{\check{\beta},u}(S \times \mathbb{P}^1/S_\infty)$. Hence, for all purposes, the graph space can be replaced by $M_{\check{\beta},u}(S \times \mathbb{P}^1/S_\infty)$. The fixed component $F_\beta \subset M_{\check{\beta},u}(S \times \mathbb{P}^1/S_\infty)$ can then be expressed in terms of flags of sheaves on S by invoking the identifications between flags of sheaves and $\mathbb{C}^*$-equivariant sheaves on $S \times \mathbb{C}$.

6.3. **Master space and wall-crossing.** For the material discussed in this section we refer the reader to [Zho22]. Here we just glide over the machinery developed there, adjusting some minor details to our needs.

The space $\mathbb{R}_{>0} \cup \{0^+, \infty\}$ of ϵ-stabilities is divided into chambers, in which the moduli space $Q^\epsilon_{g,N}(M,\beta)$ stays the same, and as ϵ crosses the a wall between chambers, the moduli changes discontinuously. Let $\epsilon_0 = 1/d_0$ be a wall for a given $\beta \in \mathrm{Eff}(M, \mathfrak{Coh}_r(S))$ and ϵ^-, ϵ^+ be some values that are close to ϵ_0 from left and right of the wall respectively. Assuming $2g-2+N+\epsilon_0\deg(\beta)>0$, let

$$MQ^{\epsilon_0}_{g,N}(M,\beta)\to M\widetilde{\mathfrak{M}}_{g,N,d}$$

be the master space with the projection to the moduli of curves with calibrated tails constructed in [Zho22], the construction is carried over to our set-up varbatim. The space $M\widetilde{\mathfrak{M}}_{g,N,d}$ is a $\mathbb{P}^1$-bundle over $\widetilde{\mathfrak{M}}_{g,N,d}$, the latter is obtained by a series of blow-ups of a moduli space of semistable curves weighted by degree, $\mathfrak{M}^{\mathrm{ss}}_{g,N,d}$, with total degree $d=\deg(\beta)$. As in GIT case, the following holds.

Theorem 6.2. *$MQ^{\epsilon_0}_{g,N}(M,\beta)$ is a proper Deligne–Mumford stack.*

Proof. With Lemma 3.19 the proof is exactly the same as in GIT case, we therefore refer to [Zho22, Section 5]. □

The master space also carries a perfect obstruction theory, which is obtained in the same way as the one for $Q^{\epsilon}_{g,N}(M,\beta)$. Let

$$\mathbb{f}\colon MQ^{\epsilon_0}_{g,N}(M,\beta)\times_{M\widetilde{\mathfrak{M}}_{g,N,d}}\mathfrak{C}_{g,N}\to\mathfrak{Coh}_r(S),$$
$$\pi\colon MQ^{\epsilon_0}_{g,N}(M,\beta)\times_{M\widetilde{\mathfrak{M}}_{g,N,d}}\mathfrak{C}_{g,N}\to MQ^{\epsilon_0}_{g,N}(M,\beta)$$

be the universal quasimap and the canonical projection, then we have a relative perfect obstruction theory over $M\widetilde{\mathfrak{M}}_{g,N,d}$

$$\phi\colon\mathbb{E}^\bullet=(\pi_*\mathbb{f}^*\mathbb{T}^{\mathrm{vir}})^\vee\to\mathbb{L}_{MQ^{\epsilon_0}_{g,N}(M,\beta)/M\widetilde{\mathfrak{M}}_{g,N,d}},$$

which is constructed via the same identification as in Proposition 5.5. Using the master space, we can establish the wall-crossing formula.

Theorem 6.3. *Assuming $2g-2+N+\epsilon_0\deg(\beta)>0$, we have*

$$\langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon^-}_{g,N,\beta}-\langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon^+}_{g,N,\beta}$$
$$=\sum_{k\geq1}\sum_{\vec{\beta}}\frac{1}{k!}\int_{[Q^{\epsilon^+}_{g,N+k}(M,\beta')]^{\mathrm{vir}}}\prod_{i=1}^{i=N}\psi_i^{m_i}ev_i^*(\gamma_i)\cdot\prod_{a=1}^{a=k}ev^*_{N+a}\mu_{\beta_a}(z)|_{z=-\psi_{N+a}}$$

where $\vec{\beta}$ runs through all the $(k+1)$-tuples of effective curve classes

$$\vec{\beta}=(\beta',\beta_1,\dots,\beta_k),$$

such that $\beta=\beta'+\beta_1+\dots+\beta_k$ and $\deg(\beta_i)=d_0$ for all $i=1,\dots,k$, and ϵ_+-stability for the class β' is given by $\mathcal{L}_\beta$. The same holds for perverse quasimap invariants $\langle\tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\natural,\epsilon}_{g,N,\beta}$.

Sketch of the proof. Here we will sketch the proof, for all the details we refer to [Zho22, Section 6], as the proof in our case is exactly the same as the one for GIT quasimaps.

The master space $MQ^{\epsilon_0}_{g,N}(M,\beta)$ carries a natural $\mathbb{C}^*$-action, such that the fixed loci are following three types of spaces (up to finite coverings):

- $Q^{\epsilon^-}_{g,N}(M,\beta)$;
- $\widetilde{Q}^{\epsilon^+}_{g,N}(M,\beta)$, base change of $Q^{\epsilon^+}_{g,N}(M,\beta)$ from $\mathfrak{M}_{g,N,d}$ to $\widetilde{\mathfrak{M}}_{g,N,d}$;
- $Y \times_{M^k} \prod_{i=1}^k F_{\beta_i}$, a finite gerbe over $\widetilde{Q}^{\epsilon^+}_{g,N+k}(M,\beta') \times_{M^k} \prod_{i=1}^k F_{\beta_i}$.

Applying the virtual localisation formula and the taking equivariant residue, we obtain certain relations between the classes associated to the spaces above. Projecting everything to a point, we get the wall-crossing formula. All the effort goes into the careful construction of the master space and the analysis of moving and fixed parts of the obstruction theories at fixed loci. The latter task can be separated into two independent parts by splitting the restriction of the absolute obstruction theory $\mathbb{E}^{\bullet}_{MQ|F}$ of the master space to a fixed locus F (one of the spaces above) into the relative obstruction theory $\mathbb{E}^{\bullet}_{|F}$ and the restriction cotangent complex $\mathbb{L}_{M\widetilde{\mathfrak{M}}_{g,N,d}|F}$ of the moduli of calibrated curves,

$$\mathbb{L}_{M\widetilde{\mathfrak{M}}_{g,N,d}|F} \to \mathbb{E}^{\bullet}_{MQ|F} \to \mathbb{E}^{\bullet}_{|F} \to,$$

the analysis of $\mathbb{L}_{M\widetilde{\mathfrak{M}}_{g,N,d}|F}$ presented in [Zho22] is completely independent of what kind of quasimaps one considers, while the analysis of $\mathbb{E}^{\bullet}_{|F}$ does not use any special feature of the GIT set-up. For more details we refer the reader to [Zho22, Section 6]. $\square$

Remark 6.4. In the GIT set-up there are naturally defined maps $[W/G] \to [\mathbb{C}^{n+1}/\mathbb{C}^*]$, which induce $Q^{\epsilon}_{g,N}(W/G,\beta) \to Q^{\epsilon}_{g,N}(\mathbb{P}^n,d)$. This allows to give a more refined class-valued wall-crossing by pushforwarding the classes on $MQ^{\epsilon_0}_{g,N}(W/G,\beta)$ to $Q^{\epsilon^-}_{g,N}(\mathbb{P}^n,d)$ instead of a point. In our case this seems to be less natural. Even though $\mathfrak{Coh}_r(S)$ is Zariski-locally a GIT stack, we do not have these naturally defined maps, because it is unclear, if line bundles $\mathcal{L}_\beta$'s are actually ample on any of the GIT loci through which the universal quasimap factors. Moreover, for different β, these loci change.

It is also possible to pushforward the classes to $\overline{M}_{g,N}$ instead of $Q^{\epsilon}_{g,N}(\mathbb{P}^n,d)$. The problem with this approach is that the projection

$$Q^{\epsilon}_{g,N+k}(M,\beta) \to \overline{M}_{g,N}$$

involves stabilisation of a curve, which implies that ψ-classes do not pullback to ψ-classes. Consequently, the wall-crossing formula becomes inefficient to state.

Since our ϵ-stability depends on a class β, there are only two universally defined values - 0^+ and ∞, i.e. the values that correspond to stable quasimaps and stable maps. Let $\epsilon \in \{0^+,\infty\}$, we define

$$F^{\epsilon}_g(\mathbf{t}(z)) = \sum_{N=0}^{\infty}\sum_{\beta\geq 0}\frac{q^\beta}{N!}\langle \mathbf{t}(\psi),\ldots,\mathbf{t}(\psi)\rangle^{\epsilon}_{g,N,\beta},$$

where $\mathbf{t}(z) \in H^*(M,\mathbb{Q})[\![z]\!]$ is a generic element, and the unstable terms are set to be zero. By repeatedly applying Theorem 6.3 we obtain.

Corollary 6.5. *For all $g \geq 1$ we have*

$$F_g^{0^+}(\mathbf{t}(z)) = F_g^{\infty}(\mathbf{t}(z) + \mu(-z)).$$

For $g = 0$, the same equation holds modulo constant and linear terms in $\mathbf{t}$.

For $g = 0$ the relation holds only moduli linear terms in $\mathbf{t}(z)$, because the moduli space $Q^{\epsilon^-}_{0,1}(M, \beta)$ is empty, if $\epsilon^- \deg(\beta) \leq 1$. The wall-crossing formula takes a different form in this case.

Theorem 6.6. *For $\epsilon \in (\frac{1}{\deg(\beta)}, \frac{1}{\deg(\beta)-1})$ we have*

$$\mathsf{ev}_*\Big(\frac{[Q^{\epsilon^-}_{0,1}(M,\beta)]^{\mathrm{vir}}}{z(z-\psi_1)}\Big) = [I(q,z)]_{z^{\leq -2}, q^\beta},$$

where $[\dots]_{z^{-2}, q^\beta}$ means that we take a truncation up to z^{-2} and the coefficient of q^β.

Proof. See [Zho22, Lemma 7.2.1]. □

To express the wall-crossing formula above in terms of change of variables, we do the following. Let $\{B^i\}$ be a basis of $H^*(M, \mathbb{Q})$ and $\{B_i\}$ be its dual basis with respect to intersection pairing. Let

$$\begin{aligned} J^{0^+}(\mathbf{t}(z), q, z) = {} & \frac{\mathbf{t}(-z)}{z} + I(q,z) \\ & + \sum_{\beta \geq 0, N \geq 0} \frac{q^\beta}{N!} \sum_p B_i \langle \frac{B^i}{z(z-\psi)}, \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle^{M, 0^+}_{0,1+N,\beta}, \end{aligned}$$

where unstable terms are set to be zero, and let

$$\begin{aligned} J^{\infty}(\mathbf{t}(z), q, z) = {} & \frac{\mathbf{t}(-z)}{z} + 1 \\ & + \sum_{\beta \geq 0, N \geq 0} \frac{q^\beta}{N!} \sum_p B_i \langle \frac{B^i}{z(z-\psi)}, \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle^{M, \infty}_{0,1+N,\beta}, \end{aligned}$$

then genus-0 case admits the following wall-crossing formula.

Theorem 6.7. *We have*

$$J^{\infty}(\mathbf{t}(z) + \mu(-z)) = J^{0^+}(\mathbf{t}(z)).$$

Proof. We again refer to [Zho22, Section 7.4]. □

6.4. **Semi-positive targets.**

6.4.1. *I-function.* Using the virtual localisation on the graph space, we can obtain a more explicit expression for I-functions for *semi-positive* moduli of sheaves.

Definition 6.8. A pair $(M, \mathfrak{Coh}_r(S))$ is *semi-positive*, if for all classes $\beta \in \mathrm{Eff}(M, \mathfrak{Coh}_r(S))$ the following holds

$$\beta(\det(\mathbb{T}^{\mathrm{vir}})) \geq 0.$$

An example of a semi-positive target would be a moduli space of sheaves on a del Pezzo surface, e.g. $\mathbb{P}^2$. However, even a pair $(\mathbb{P}^2, \mathfrak{Coh}_r(S))$ is not Fano in the sense of quasimaps, i.e. there exists class a $\beta \in \mathrm{Eff}(\mathbb{P}^2, \mathfrak{Coh}_r(S))$, for which the following holds

$$\beta(\det(\mathbb{T}^{\mathrm{vir}})) = 0.$$

These are just the classes such that $\mathrm{c}_1(\check{\beta}) = 0$.

Consider now the expansion

$$[zI(q,z) - z]_+ = I_1(q) + (I_0(q) - 1)z + I_{-1}(q)z^2 + I_{-2}(q)z^3 + \dots,$$

we will show that all terms I_k with $k \geq -1$ vanish for a semi-positive target. The virtual dimension of $QG_{0,1}(M, \beta)$ is equal to $\dim(M) + 1 + \beta(\det(\mathbb{T}^{\mathrm{vir}}))$. Hence, by the virtual localisation theorem, degrees of the localised classes

$$-z\mathsf{ev}_* \left(\frac{[F_{\check{\beta}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_{\check{\beta}}/QG_{0,1}(M,\beta)})} \right) \in A^*(M)[z^{\pm}]$$

are equal to

$$-\beta(\det(\mathbb{T}^{\mathrm{vir}})).$$

On the other hand, again by the virtual localisation theorem, the degrees of these classes cannot be negative, since they are homogenous summands of an equivariant class. Hence, we obtain that

$$[zI(q,z) - z]_+ = I_1(q) + (I_0(q) - 1)z,$$

as claimed. The remaining terms can be viewed from two different perspectives. The first one is map-theoretic, and is shown in the following proposition.

Proposition 6.9. *For a semi-positive pair $(M, \mathfrak{Coh}_r(S))$ the following holds*

(i)

$$I_0(q)^{-1} = 1 + \sum_{\beta \neq 0} \sum_i q^\beta \langle \gamma_i, \mathbb{1}, \gamma^i \rangle^{0^+}_{0,3,\beta};$$

(ii)

$$I_1(q) = f_0(q)\mathbb{1} + \sum_j f_j(q) D_j,$$

where $\{D_j\}$ *is a basis of* $H^2(M,\mathbb{Q})$*, and*

$$\frac{f_0(q)}{I_0(q)} = \sum_{\beta\neq 0} q^\beta \langle [\mathrm{pt}], \mathbb{1}\rangle^{0+}_{0,2,\beta} \quad \frac{f_j(q)}{I_0(q)} = \sum_{\beta\neq 0}\sum_j q^\beta \langle D^j, \mathbb{1}\rangle^{0+}_{0,2,\beta}.$$

Proof. The proof is exactly the same as in [CK14, Section 5.5]. □

The second perspective is sheaf-theoretic. In the case of punctorial Hilbert schemes of del Pezzo surfaces, it allows us to explicitly determine the terms of the perverse I-function - $I_0^\sharp$ and $I_1^\sharp$. Let us firstly do some notational preparations. From now on, we assume that $M = S^{[n]}$.

By Corollary 3.13, we have an embedding

$$-\check{(...)}\colon \mathrm{Eff}(S^{[n]}, \mathfrak{Coh}^\sharp_r(S)) \hookrightarrow H^{1,1}(S)\oplus H^{2,2}(S), \tag{16}$$

here we change the sign of the classes, which amounts to considering classes of subschemes instead of classes of ideals on threefolds. Using this embedding, we identify β with its image $-\check{\beta}$. The class β can therefore be decomposed as

$$\beta = (\gamma, \mathsf{m}) \in H^{1,1}(S)\oplus H^{2,2}(S),$$

hence

$$\mathbb{Q}[\![q^\beta]\!] = \mathbb{Q}[\![q^\gamma]\!]\otimes \mathbb{Q}[\![y]\!], \quad q^\beta = q^\gamma\cdot y^{\mathsf{m}}.$$

On the side of $S^{[n]}$, the variable y keeps track of multiples of the exceptional curve class $\mathsf{A}\in H_2(S^{[n]},\mathbb{Z})$, and the above decomposition corresponds to the one of $H_2(S^{[n]},\mathbb{Z})$ given by Nakajima basis (images of Nakajima operators applied to classes on S),

$$H_2(S^{[n]},\mathbb{Z}) \cong H_2(S,\mathbb{Z})\oplus \mathbb{Z}\cdot\mathsf{A}.$$

More precisely, if $\Sigma\subset S$ is a curve, then we can define an associated curve $\Sigma_n\subset S^{[n]}$ given by letting one point move along Σ and keeping $n-1$ other distinct points fixed. The curve Σ_n then represents a class in $H_2(S,\mathbb{Z})\subset H_2(S^{[n]},\mathbb{Z})$ with respect to the identification above. For more on Nakajima basis in the relevant to us context, we refer to [Obe18].

We define $\mathrm{c}_1(S)_n \in H^2(S^{[n]},\mathbb{Z})$ to be the class associated to the class $\mathrm{c}_1(S)\in H^2(S,\mathbb{Z})$ as described above, after identifying homology with cohomology. With this notation, we have the following result, which was kindly communicated to the author by Georg Oberdieck.

Proposition 6.10 (Georg Oberdieck)**.** *Assume S is a del Pezzo surface, then for $M = S^{[n]}$ we have*

$$I_0^\sharp(q) = 1$$
$$I_1^\sharp(q) = \log(1+y)\mathrm{c}_1(S)_n.$$

Proof. By dimension constraints and the fact that there are no $\gamma\in\mathrm{Eff}(S)$ such that $\gamma\cdot\mathrm{c}_1(S) = 1$, the non-zero contributions to the I-function come only from classes of the form $\beta = (0,\mathsf{m})$. Let us firstly consider $I_0^\sharp$. Let

$P \in S^{[n]}$ be a point, then the preimage $\mathrm{ev}^{-1}(P) \subset F_{\check{\beta}}$ parametrizes stable pairs supported in $U \times \mathbb{P}^1$ where U is a local neighbourhood of the support of P. We can assume that U is the disjoint union's of $\mathbb{C}^2$, hence since $\mathbb{C}^2$ carries a symplectic form, the only non-vanishing contributions are therefore due to $\mathsf{m} = 0$. Hence $\langle I_0^\sharp, P\rangle = 1$, which implies that $I_0 = 1$.

We now consider the term $I_1^\sharp$. With the same argument as above $\langle I_1^\sharp, \mathsf{A}\rangle = 0$. Now let us evaluate $I_1^\sharp$ at the classes in $H_2(S,\mathbb{Z}) \subset H_2(S^{[n]},\mathbb{Z})$. By the previous argument, the $n-1$ fixed points contribute 1 each, so that

$$\langle I_1^{\sharp, S^{[n]}}, [\Sigma_n]\rangle = \langle I_1^{\sharp, S}, [\Sigma]\rangle.$$

Hence we may assume $n = 1$. In this case, the moduli space $F_{\bullet,(0,\mathsf{m})}$ is isomorphic to S, parametrizing pairs (F,s) given by $I^\bullet = \mathcal{O}_{\mathbb{P}^1_x} \to \mathcal{O}_{\mathbb{P}^1_x}(D)$ where $\mathbb{P}^1_x = \mathbb{P}^1 \times x$ for a point $x \in S$, and $D = \mathsf{m}\cdot[\infty]$. The local model of $\mathrm{P}_{1,\beta}(S \times \mathbb{P}^1/S_0)$ near $F_{\check{\beta}}$ is $\mathrm{Sym}^{\mathsf{m}}(\mathbb{P}^1) \times S$. The obstruction theory was computed in [PT09, Section 4.2][13],

$$\mathrm{Def}_{I^\bullet} = H^0(\mathcal{O}_D(D))$$
$$\mathrm{Obs}_{I^\bullet} = H^0(\mathcal{O}_D(D) \otimes \omega_{S\times\mathbb{P}^1})^\vee = H^0(\mathcal{O}_D(D)\otimes\omega_{\mathbb{P}^1})^\vee \otimes \omega_{S|x}^\vee.$$

Consider now the $\mathbb{C}^*$-action on $\mathbb{P}^1$ by $t\cdot(x,y) = (tx,y)$. The coordinate $Y = y/x$ gets scaled by $t\cdot Y = t^{-1}Y$ hence has weight $-z$. Let us analyse the $\mathbb{C}^*$-equivariant structure the obstruction theory. Firstly,

$$H^0(\mathcal{O}(D)|_D) = (Y^{-\mathsf{m}}) \otimes \mathbb{C}[Y]/Y^{\mathsf{m}} = \mathbb{C}Y^{-\mathsf{m}} \oplus \mathbb{C}Y^{-\mathsf{m}+1} \oplus \ldots \oplus \mathbb{C}Y^{-1},$$

which therefore has weights $z, 2z, \ldots, \mathsf{m}z$ as a $\mathbb{C}$-module. Moreover, $\omega_{\mathbb{P}^1} = \mathbb{C}[Y]dY$, so since dY has weight $-z$ we get that $H^0(\mathcal{O}(D)|_D \otimes \omega_{\mathbb{P}^1})$ has weights $0, z, \ldots, (\mathsf{m}-1)z$, therefore its dual has weights $(-\mathsf{m}+1)z, \ldots, -z, 0$. Let $\mathrm{c}_1 = \mathrm{c}_1(S)$, we therefore obtain the following

$$\begin{aligned}
\mathrm{ev}_* \frac{[F_{\check{\beta}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})} &= p_{S*}\left(\frac{e_{\mathbb{C}^*}(\mathrm{Obs}^{\mathrm{mov}}_{I^\bullet})}{e_{\mathbb{C}^*}(\mathrm{Def}^{\mathrm{mov}}_{I^\bullet})} \cdot p_S^*\mathrm{c}_1\right) \\
&= \frac{(-z+\mathrm{c}_1)\cdots((-\mathsf{m}+1)z+\mathrm{c}_1)}{z\cdot 2z\cdots \mathsf{m}z}\cdot \mathrm{c}_1 \\
&= \frac{(-1)^{\mathsf{m}-1}(\mathsf{m}-1)!z^{\mathsf{m}-1}}{\mathsf{m}!z^{\mathsf{m}}}\cdot\mathrm{c}_1 + (\ldots)\cdot \mathrm{c}_1^2 \\
&= \frac{(-1)^{\mathsf{m}-1}}{\mathsf{m}z}\cdot \mathrm{c}_1 + (\ldots)\cdot\mathrm{c}_1^2,
\end{aligned}$$

this proves the claim. □

We now define

$$^\sharp\langle \gamma_1, \ldots, \gamma_N\rangle^{S^{[n]},\epsilon}_{g,\gamma} := \sum_{\mathsf{m}} {}^\sharp\langle\gamma_1,\ldots,\gamma_N\rangle^{S^{[n]},\epsilon}_{g,(\gamma,\mathsf{m})} y^{\mathsf{m}},$$

[13] The equivariantly correct obstruction theory is given in the latest arXiv version. The canonical line bundle $\omega_{\mathbb{P}^1}(D)|D = \omega_D$ is equivariantly not trivial.

then using the wall-crossing formula from Theorem 6.3, the string and divisor equations, one obtains the following result, which specialises to the result stated in Section 1.8 after enumerating the invariants with respect to classes on $S^{[n]}$ instead of $S \times C$.

Corollary 6.11. *Assume $2g-2+N \geq 0$. If S is a del Pezzo surface, then*

$$^{\sharp}\langle \gamma_1, \dots, \gamma_N \rangle^{0^+}_{g,\gamma} = (1+y)^{\mathrm{c}_1(S)\cdot\gamma} \cdot {}^{\sharp}\langle \gamma_1, \dots, \gamma_N \rangle^{\infty}_{g,\gamma}.$$

6.4.2. *DT/PT correspondence.* Using dilaton equation for GW invariants (see [CK20, Corollary 1.5]), one can restate the wall-crossing formula for $g \neq 1$ (for $g=1$ there is an extra constant term which we do not want to write down for the clarity of exposition, see [CK20, Corollary 1.5]) as follows

$$(I_0)^{2g-2} \cdot F^{0^+}_g(\mathbf{t}(z)) = F^{\infty}_g\left(\frac{\mathbf{t}(z)+I_1(q)}{I_0(q)}\right).$$

The same holds for the perverse generating series $F^{\sharp,\epsilon}_g(\mathbf{t}(z))$. Since the generating series are related by a change of variables, the above equation is equivalent to

$$(I_0)^{2g-2} \cdot F^{0^+}_g(I_0(q)\mathbf{t}(z) - I_1(q)) = F^{\infty}(\mathbf{t}(z)),$$

therefore perverse and non-perverse generating series are related in the following way

$$(I_0)^{2g-2} \cdot F^{0^+}_g(I_0(q)\mathbf{t}(z) - I_1(q)) = (I^{\sharp}_0)^{2g-2} \cdot F^{\sharp,0^+}_g(I^{\sharp}_0(q)\mathbf{t}(z) - I^{\sharp}_1(q)),$$

moving the change of variables to one side we, obtain

$$\frac{(I_0)^{2g-2}}{(I^{\sharp}_0)^{2g-2}} \cdot F^{0^+}_g\left(\frac{I_0(q)}{I^{\sharp}_0(q)} \cdot \left(\mathbf{t}(z) + I^{\sharp}_1(q)\right) - I_1(q)\right) = F^{\sharp,0^+}_g(\mathbf{t}(z)).$$

Passing from quasimaps to sheaves and establishing DT/PT for wall-crossing invariants, we would get DT/PT for the relative geometry

$$S \times C_{g,N} \to \overline{M}_{g,N},$$

such that $2g-2+N>0$ and $\mathrm{ch}(I)_{\mathrm{d}} \neq 0$. In particular, DT/PT relative to three vertical divisors on $S \times \mathbb{P}^1$ is reduced to the DT/PT of wall-crossing invariants.

Appendix A. Stability of fibers

The aim of this section is to prove Proposition A.4, the converse of Lemma 3.17. The proof is inspired by the proof of [Tho00, Proposition 4.2], which, however, contains a mistake in the direction

$$\text{stability} \implies \text{stability of a general fiber},$$

because a sheaf F on a threefold restricts to stable sheaf on the hyperplane section with respect to the stability that defines the hyperplane section, which is not necessarily suitable. If one adds fiber classes to the polarisation to make it suitable, then one has to take a hyperplane section of bigger degree, for which suitable polarisation may be different.

Let $X := S \times C \to C$ be a trivial surface fibration over a connected nodal curve C. Let us fix a very ample line bundle $\mathcal{O}_S(1)$. We denote a line bundle with specified degrees on each irreducible components $\mathcal{O}_S(1) \boxtimes \mathcal{O}_C(k_1, \dots, k_m)$ by L_{k_i}, and the degree of a sheaf F with respect to L_{k_i} by $\deg_{k_i}(F)$. Recall that for a possibly singular scheme X, the slope of a torsion-free sheaf F can be defined as follows

$$\mu(F) = \frac{a_{\dim(X)-1}(F)}{a_{\dim(X)}(F)},$$

where $a_i(F)$'s are the coefficients in a Hilbert polynomial

$$P(F, m) = \sum a_i(F) \frac{m^i}{i!}.$$

In what follows, by stability we will mean *slope* stability.

Proposition A.1. *Assume C is smooth. Fix a class $\beta \in H^*(S \times C, \mathbb{Q})$, such that $\mathrm{rk}(\beta) = 2$. There exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ and for all torsion free sheaves F with $\mathrm{ch}(F) = \beta$ the following statement holds: F is L_n-stable, if F_p is stable for a general $p \in C$.*

Proof. We will prove the proposition by restricting to a hyperplane section and then applying [HL97, Theorem 5.3.2], see also [Yos99, Lemma 1.2].

Firstly, consider the Künneth's decomposition,

$$H^2(S \times C, \mathbb{Q}) = H^2(S, \mathbb{Q}) \oplus H^1(S, \mathbb{Q}) \otimes H^1(C, \mathbb{Q}) \oplus \mathbb{Q},$$

the first Chern class of a sheaf can be expressed accordingly

$$\mathrm{c}_1(F) = \mathrm{c}_1(F_p) \oplus \alpha \oplus \mathrm{k}(F),$$

where each summand is in a corresponding Künneth component and F_p is a general fiber of F over $p \in C$. The intersection numbers with L_n's take the following form

$$\mathrm{c}_1(F) \cdot L_n \cdot L_m = d \cdot \mathrm{k}(F) + (n+m) \cdot \deg(F)_{\mathrm{f}}, \tag{17}$$

where $d = \mathcal{O}_S(1)^2$ and $\deg(F)_{\mathrm{f}} = \deg(F_p)$. In particular, slope-stability with respect to a curve class $L_1 \cdot L_{2n-1}$ coincides with slope-stability with respect to a curve class $L_n \cdot L_n$.

Consider now a general hyperplane section $H \in |\mathcal{O}_S(1) \boxtimes \mathcal{O}_C(1)|$, let $2n_0 - 1$ be the smallest odd integer such that [HL97, Theorem 5.3.2] holds for $H \to C$, the class $\beta_{|H}$ and a polarisation $L_{2n_0-1|H}$.

Assume F_p is unstable for all $p \in C$. Let $G \hookrightarrow F$ be a relative destabilising subsheaf (strictly speaking, it exists over some non-empty open subscheme $U \subseteq C$, we then extend over the entire C). Consider now the restriction to a general hyperplane section $G_{|H} \hookrightarrow F_{|H}$, it is destabilising by the proof of [HL97, Theorem 5.3.2] with respect to $L_{2n_0-1|H}$, therefore $G \hookrightarrow F$ is L_{n_0}-destabilising. $\square$

Remark A.2. The reason for the failure of the proof of Proposition A.1 for $\mathrm{rk} > 2$ is already present at the level of fibered surfaces. For a fibred surface the difference between $\mathrm{rk} = 2$ and $\mathrm{rk} > 2$ cases is that for the former a suitable polarisation has a stronger property, namely, a subsheaf is destabilising, if and only if it is destabilising on a fiber, as is shown in [HL97, Theorem 5.3.2]. However, the author could not establish such property of a suitable polarisation for $\mathrm{rk} > 2$. In this case, one can show that there are no walls between the fiber stability and L_n-stability for $n \gg 0$, which is a weaker property.

Corollary A.3. *Assume we are in the setting of Proposition A.1 and F is unstable at a general fiber, let $G \subset F$ be a relatively destabilising subsheaf, then*
$$\mathrm{rk}(G)\deg_n(F) - \mathrm{rk}(F)\deg_n(G) < 2(n_0 - n),$$
for all $n \geq n_0$, i.e. the difference of slopes can be made arbitrary negative by increasing n.

Proof. By the proof of Proposition A.1, $G \subset F$ is L_n-destabilising for all $n \geq n_0$, therefore
$$\begin{aligned} &\mathrm{rk}(G)\deg_n(F) - \mathrm{rk}(F)\deg_n(G) \\ &< \mathrm{rk}(G)\deg_n(F) - \mathrm{rk}(F)\deg_n(G) - (\mathrm{rk}(G)\deg_N(F) - \mathrm{rk}(F)\deg_{n_0}(G)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq 2(n_0 - n), \end{aligned}$$
where for the last inequality we used (17). $\square$

Now let C be a connected nodal curve and $\tilde{C}$ be its normalisation, by $\tilde{C}_i$ we will denote its connected components. For a sheaf F on a threefold $S \times C$ we denote its pullback to $X_i := S \times \tilde{C}_i$ by F_i.

Proposition A.4. *Fix classes $\beta_i \in H^*(S \times \tilde{C}_i, \mathbb{Q})$ with the same fiber component, such that $\mathrm{rk}(\beta_i) = 2$. There exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ and for all sheaves F flat over C with $\mathrm{ch}(F_i) = \beta_i$ the following statement holds: F is L_{nk_i}-stable, if F_p is stable for a general $p \in C$.*

We will prove the proposition for the case of C with one node, splitting the proof into two parts depending on whether the node is separating or non-separating. The proof easily generalises to the case of C with more nodes.

Proof (non-separating node). Let C be a connected nodal curve with one non-separating node $s \in C$ and $\pi \colon S \times \tilde{C} \to S \times C$ be the normalization map of the product. The sheaves F and π^*F are related by the normalisation sequence

$$0 \to F \to \pi_*\pi^*F \to F_s \to 0,$$

from which we obtain

$$a_3(F) = a_3(\pi^*F), \quad a_2(F) = a_2(\pi^*F) - a_2(F_s).$$

Now let $G \subset \pi^*F$ be a relatively destabilising subsheaf and $\tilde{G}$ be the kernel of the following composition

$$\pi_*G \hookrightarrow \pi_*\pi^*F \to F_s,$$

by construction $\tilde{G}$ is a subsheaf of F and

$$a_3(\tilde{G}) = a_3(G), \quad a_2(\tilde{G}) \leq a_2(G).$$

The difference of slopes of F and $\tilde{G}$ can then be bounded from above as follows

$$\frac{a_2(F)}{a_3(F)} - \frac{a_2(\tilde{G})}{a_3(\tilde{G})} \geq \frac{a_2(\pi^*F)}{a_3(\pi^*F)} - \frac{a_2(G)}{a_3(G)} - \frac{a_2(F_s)}{a_3(\pi^*F)}.$$

After multiplying by denominators, the right-hand side of the expression above is equal to

$$a_3(G) \cdot a_2(\pi^*F) - a_3(\pi^*F) \cdot a_2(G) - a_3(G) \cdot a_2(F_s) \qquad (*)$$

Recall that

$$a_2(F) = \deg_k(F) + \mathrm{rk}(F) \cdot a_2(\mathcal{O}_X),$$
$$a_3(F) = \mathrm{rk}(F) \cdot a_3(\mathcal{O}_X).$$

Substituting the above expressions into the summands of $(*)$, we obtain

$$(*) = a_3(\mathcal{O}_X) \cdot (\mathrm{rk}(G) \cdot \deg_k(\pi^*F) - \mathrm{rk}(F) \cdot \deg_k(G) - d \cdot \mathrm{rk}(F) \cdot \mathrm{rk}(G)),$$

where we also used that

$$a_2(F_s) = d \cdot \mathrm{rk}(F_s) = d \cdot \mathrm{rk}(F),$$

because F is flat over C. By Corollary A.3 the term

$$\mathrm{rk}(G) \cdot \deg_k(\pi^*F) - \mathrm{rk}(F) \cdot \deg_k(G)$$

can be made arbitrary negative by taking big enough power of $\mathcal{O}_C(k)$, thereby making the difference of slopes negative. Moreover, the choice of the power is uniform for all F. $\square$

Proof (separating node). Let $C = C_1 \cup C_2$ be a connected nodal curve with one separating node $s \in C$, and let $\mathcal{O}_C(k_1, k_2)$ be the ample line bundle with prescribed degrees on each component. The restrictions of F to $S \times C_i$ are related to F by the normalisation sequence

$$0 \to F \to F_1 \oplus F_2 \to F_s \to 0,$$

from which we obtain

$$a_3(F) = a_3(F_1) + a_3(F_2), \quad a_2(F) = a_2(F_1) + a_2(F_2) - a_2(F_s).$$

Now let $G_i \subset F_i$ be relatively destabilising subsheaves and $\tilde{G}$ be the kernel of the following composition

$$G_1 \oplus G_2 \hookrightarrow F_1 \oplus F_2 \to F_s,$$

by construction $\tilde{G}$ is a subsheaf of F and

$$a_3(\tilde{G}) = a_3(G_1) + a_3(G_2), \quad a_2(\tilde{G}) \leq a_2(G_1) + a_2(G_2).$$

The difference of slopes of F and $\tilde{G}$ then takes the following form

$$\frac{a_2(F)}{a_3(F)} - \frac{a_2(\tilde{G})}{a_3(\tilde{G})} \geq \frac{\sum a_2(F_i)}{\sum a_3(F_i)} - \frac{\sum a_2(G_i)}{\sum a_3(G_i)} - \frac{a_2(F_s)}{\sum a_3(F_i)}.$$

After multiplying by denominators, the right-hand side of the the expression above is equal to

$$\begin{aligned} &a_2(F_1) \cdot (a_3(G_1) + a_3(G_2)) - a_2(G_1) \cdot (a_3(F_1) + a_3(F_2)) \\ &+a_2(F_2) \cdot (a_3(G_1) + a_3(G_2)) - a_2(G_2) \cdot (a_3(F_1) + a_3(F_2)) \\ &-a_2(F_s) \cdot \sum a_3(G_i) \end{aligned}$$

We now group the summands in the following way

$$\begin{aligned} a_2(F_1) \cdot a_3(G_1) - a_2(G_1) \cdot a_3(F_1) + a_2(F_2) \cdot a_3(G_2) - a_2(G_2) \cdot a_3(F_2) \\ -a_2(F_s) \cdot \sum a_3(G_i) \end{aligned} \quad \textbf{(a)}$$

$$+a_2(F_1) \cdot a_3(G_2) - a_2(G_2) \cdot a_3(F_1) + a_2(F_2) \cdot a_3(G_1) - a_2(G_1) \cdot a_3(F_2) \quad \textbf{(b)}$$

We will analyse terms **(a)** and **(b)** separately.

Term (a). The term **(a)** is simple to deal, substituting

$$\begin{aligned} a_2(F_i) &= \deg_{k_i}(F_i) + \mathrm{rk}(F_i) \cdot a_2(\mathcal{O}_{X_i}) \\ a_3(F_i) &= \mathrm{rk}(F_i) \cdot a_3(\mathcal{O}_{X_i}) \end{aligned}$$

into **(a)** we obtain that

$$\textbf{(a)} = \sum a_3(\mathcal{O}_{X_i}) \cdot \left(\mathrm{rk}(G_i) \cdot \deg_{k_i}(F_i) - \mathrm{rk}(F) \cdot \deg_{k_i}(G_i) - \mathrm{rk}(F) \cdot \mathrm{rk}(G_i)\right),$$

since F is stable at a general fiber, the right-hand side can be made negative taking big enough power of $\mathcal{O}_C(k_1, k_2)$ by Corollary A.3.

Term (b). Making the same substitution into **(b)** we obtain

$$
\begin{aligned}
&\mathrm{rk}(G_2)\cdot\deg_{k_1}(F_1)\cdot a_3(\mathcal{O}_{X_2})-\mathrm{rk}(F_2)\cdot\deg_{k_1}(G_1)\cdot a_3(\mathcal{O}_{X_2})\\
&+\mathrm{rk}(G_1)\cdot\deg_{k_2}(F_2)\cdot a_3(\mathcal{O}_{X_1})-\mathrm{rk}(F_1)\cdot\deg_{k_2}(G_2)\cdot a_3(\mathcal{O}_{X_1}) \qquad (\mathbf{b.1})\\
&+\mathrm{rk}(F_1)\cdot\mathrm{rk}(G_2)\cdot a_2(\mathcal{O}_{X_1})\cdot a_3(\mathcal{O}_{X_2})-\mathrm{rk}(F_1)\cdot\mathrm{rk}(G_2)\cdot a_2(\mathcal{O}_{X_2})\cdot a_3(\mathcal{O}_{X_1})\\
&+\mathrm{rk}(F_2)\cdot\mathrm{rk}(G_1)\cdot a_2(\mathcal{O}_{X_2})\cdot a_3(\mathcal{O}_{X_1})-\mathrm{rk}(F_2)\cdot\mathrm{rk}(G_1)\cdot a_2(\mathcal{O}_{X_1})\cdot a_3(\mathcal{O}_{X_2}) \qquad (\mathbf{b.2})
\end{aligned}
$$

We again split the analysis in two parts. For the term $(\mathbf{b.1})$ we use that

$$
\begin{aligned}
\deg_{k_i}(F_i) &= d\cdot\mathrm{k}(F_i)+2k_i\cdot\deg(F_i)_{\mathrm{f}}\\
a_3(\mathcal{O}_{X_i}) &= d\cdot k_i
\end{aligned}
$$

to obtain

$$
\begin{aligned}
&2d\cdot k_1\cdot k_2\cdot\mathrm{rk}(G_2)\cdot\deg(F_1)_{\mathrm{f}}-2d\cdot k_1\cdot k_2\cdot\mathrm{rk}(F_2)\cdot\deg(G_1)_{\mathrm{f}}\\
&+2d\cdot k_1\cdot k_2\cdot\mathrm{rk}(G_1)\cdot\deg(F_2)_{\mathrm{f}}-2d\cdot k_1\cdot k_2\cdot\mathrm{rk}(F_1)\cdot\deg(G_2)_{\mathrm{f}}\\
&+d\cdot k_2\cdot d\cdot\mathrm{rk}(G_2)\cdot\mathrm{k}(F_1)-d\cdot k_2\cdot d\cdot\mathrm{rk}(F_2)\cdot\mathrm{k}(G_1)\\
&+d\cdot k_1\cdot d\cdot\mathrm{rk}(G_1)\cdot\mathrm{k}(F_2)-d\cdot k_1\cdot d\cdot\mathrm{rk}(F_1)\cdot\mathrm{k}(G_2)
\end{aligned}
$$

Let K_i be the smallest integer for which the proposition holds, then by (17)

$$d\cdot\mathrm{rk}(F)\cdot\mathrm{k}(G_i)>2K_i\cdot(\cdot\mathrm{rk}(G_i)\cdot\deg(F)_{\mathrm{f}}-\mathrm{rk}(F)\cdot\deg(G_i)_{\mathrm{f}})+d\cdot\mathrm{rk}(G_i)\cdot\mathrm{k}(F_i),$$

where we also used that

$$\mathrm{rk}(F_1)=\mathrm{rk}(F_2)=\mathrm{rk}(F).$$

Regrouping the summands and applying the above inequality, we obtain that

$$
\begin{aligned}
(\mathbf{b.1})<\sum d\cdot k_{i+1}\cdot(k_i-K_i)\cdot(\mathrm{rk}(G_i)\cdot\deg(F)_{\mathrm{f}}-\mathrm{rk}(F)\cdot\deg(G_i)_{\mathrm{f}})\\
+\sum d\cdot k_{i+1}\cdot d\cdot\mathrm{k}(F_i)\cdot(\mathrm{rk}(G_i)-\mathrm{rk}(G_{i+1})).
\end{aligned}
$$

For the term $(\mathbf{b.2})$ we use that

$$a_2(\mathcal{O}_{X_i})=d\cdot g_i+\frac{k_i\cdot\mathrm{c}_1(\mathcal{O}_S(1))\cdot\mathrm{c}_1(S)}{2},$$

where $g_i=g(C_i)$, then after some cancellations we obtain

$$(\mathbf{b.2})=\sum d\cdot k_i\cdot d\cdot g_{i+1}\cdot\mathrm{rk}(F)\cdot(\mathrm{rk}(G_i)-\mathrm{rk}(G_{i+1})),$$

now putting $(\mathbf{b.1})$ and $(\mathbf{b.2})$ together we see that if

$$
\begin{aligned}
(k_i-K_i)\cdot(\mathrm{rk}(G_i)\cdot\deg(F)_{\mathrm{f}}-\mathrm{rk}(F)\cdot\deg(G_i)_{\mathrm{f}})\\
<d\cdot(\mathrm{rk}(G_i)-\mathrm{rk}(G_{i+1}))\cdot(g_{i+1}\cdot\mathrm{rk}(F)-\mathrm{k}(F_i)),
\end{aligned}
$$

then $(\mathbf{b.1})+(\mathbf{b.2})$ is negative. The right-hand side of the above inequality can be bounded independently of F, therefore by taking high enough power of $\mathcal{O}_C(k_1,k_2)$ the term **(b)** is negative independently of F. □

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