GROMOV–WITTEN/HURWITZ WALL-CROSSING

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ABSTRACT. For a target variety X and a nodal curve C, we introduce a one-parameter stability condition, termed ϵ-admissibility, for maps from nodal curves to $X \times C$. If X is a point, ϵ-admissibility interpolates between moduli spaces of stable maps to C relative to some fixed points and moduli spaces of admissible covers with arbitrary ramifications over the same fixed points and simple ramifications elsewhere on C.

Using Zhou's calibrated tails, we prove wall-crossing formulas relating invariants for different values of ϵ. If X is a surface, we use this wall-crossing in conjunction with author's quasimap wall-crossing to show that relative Pandharipande–Thomas/Gromov–Witten correspondence on $X \times C$ and Ruan's extended crepant resolution conjecture of the pair $X^{[n]}$ and $[X^{(n)}]$ are equivalent up to explicit wall-crossings. We thereby prove crepant resolution conjecture for 3-point genus-0 invariants in all classes, if X is a toric del Pezzo surface.

Contents

1. Introduction

1.1. **Overview.** Inspired by the theory of quasimaps to GIT quotients of [CKM14], a theory of quasimaps to moduli spaces of sheaves was introduced in [Nes21a]. When applied to Hilbert schemes of n-points $S^{[n]}$ of a surface S, moduli spaces of ϵ-stable quasimaps interpolate between moduli spaces of stable maps to $S^{[n]}$ and Hilbert schemes of 1-dimensional subschemes of a relative geometry $S \times C_{g,N}/\overline{M}_{g,N}$,

$$\overline{M}_{g,N}(S^{[n]}, \beta) \overset{\epsilon}{\leftarrow\!\!-\!\!-\!\!\rightarrow} \mathrm{Hilb}_{n,\check{\beta}}(S \times C_{g,N}/\overline{M}_{g,N}), \tag{1}$$

where $C_{g,N} \to \overline{M}_{g,N}$ is the universal curve of a moduli space of stable marked curves.

This interpolation gives rise to wall-crossing formulas, which therefore relate Gromov–Witten (GW) theory of $S^{[n]}$ and relative Donaldson–Thomas (DT) theory of $S \times C_{g,N}/\overline{M}_{g,N}$. Alongside with results of [Nes21b], the quasimap wal-crossing was used to prove various correspondences, among which is the wall-crossing part of Igusa cusp form conjecture of [OP16]. For more details, we refer to [Nes21a, Nes21b].

In this article we introduce a notion of ϵ-admissibility, depending on a parameter $\epsilon \in \mathbb{R}_{\leq 0}$, for maps

$$P \to X \times C$$

relative to $X \times \mathbf{x}$, where P is a nodal curve, $(C, \mathbf{x})$ is a marked nodal curve and X is a smooth projective variety.

As the value of ϵ varies, moduli spaces of ϵ-admissible maps interpolate between moduli spaces of stable twisted maps to an orbifold symmetric product $[X^{(n)}]$ and moduli space of stable maps to a relative geometry $X \times C_{g,N}/\overline{M}_{g,N}$,

$$\mathcal{K}_{g,N}([X^{(n)}], \beta) \leftarrow\!\text{-}\epsilon\text{-}\!\rightarrow \overline{M}^{\bullet}_{\mathsf{h}}(X \times C_{g,N}/\overline{M}_{g,N}(\gamma, n)),$$

such that the various discrete data on both sides, like genus or degree of a map, determine each other, as is explained in Section 2.3.

Using Zhou's theory of calibrated tails from [Zho22], we establish wall-crossing formulas which relates the associated invariants for different values of $\epsilon \in \mathbb{R}_{\leq 0}$. This wall-crossing is completely analogous to the quasimap wall-crossing. The result is an equivalence of orbifold GW theory of $[X^{(n)}]$ and relative[1] GW theory $X \times C_{g,N}/\overline{M}_{g,N}$ for an arbitrary smooth projective target X, which can be expressed in terms of a change of variables applied to certain generating series. The change of variables involves so-called *I-functions*, which are defined via localised GW theory of $X \times \mathbb{P}^1$ with respect to $\mathbb{C}^*$-action coming from the $\mathbb{P}^1$-factor. This wall-crossing can be termed Gromov–Witten/Hurwitz (GW/H) wall-crossing [2], because if X is a point, the moduli spaces of ϵ-admissible maps interpolates between Gromov–Witten and Hurwitz spaces of a curve C.

In conjunction with the quasimap wall-crossing of [Nes21a], GW/H wall-crossing establishes the square of theories for a smooth surface S, illustrated in Figure 1. The square relates crepant resolution conjecture (C.R.C.), proposed in [Rua06] and refined in [BG09, CCIT09], and Pandharipande–Thomas/Gromov–Witten correspondence (PT/GW), proposed in [MNOP06a, MNOP06a]. The square has some similarities with Landau–Ginzburg/Calabi–Yau correspondence, as it is explained in Section 1.6.2.

[1] By relative, we mean a theory with relative insertions.

[2] Note that our terminology unintentionally resembles the terminology used in [OP06]. However, we do not know, if the two phenomena have any relation.

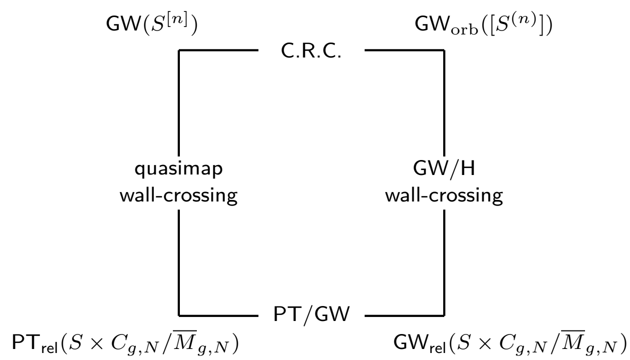

FIGURE 1. The square

With the help of the square, we establish the following results:

- 3-point genus-0 crepant resolution conjecture in the sense of [BG09] for the pair $S^{[n]}$ and $[S^{(n)}]$ in all classes, if S is a toric del Pezzo surface.
- the geometric origin of $y = -e^{iu}$ in $\mathsf{PT/GW}$ through $\mathsf{C.R.C.}$

Moreover, a cycle-valued version of the wall-crossing should have applications in the theory of double ramifications cycles of [JPPZ17], comparison results for the TQFT's from [Cav07] and [BP08], etc. This will be addressed in a future work.

Various instances of the vertical sides of the square were studied on the level of invariants in numerous articles, mainly for $\mathbb{C}^2$ and $\mathcal{A}_m$ - [OP10a], [OP10b], [OP10c], [BP08], [PT19a], [Mau09], [MO09], [Che09] and etc. The wall-crossings, however, provide a geometric justification for these phenomena.

1.2. **Analogy.** Let us illustrate how the theory of quasimaps sheds light on a seemingly unrelated theme of admissible covers.

1.2.1. *ϵ-stable quasimaps.* A map from a nodal curve C,

$$f : C \to S^{[n]},$$

is determined by its graph

$$\Gamma_f \subset S \times C.$$

If the curve C varies, the pair (C, Γ_f) can degenerate in two ways:

(i) the curve C degenerates;
(ii) the graph Γ_f degenerates.

By a degeneration of Γ_f we mean that it becomes non-flat[3] over C as a subscheme of $S \times C$, which is due to

- floating points;
- non-dominant components.

Two types of degenerations of a pair (C, Γ_f) are related. GW theory of $S^{[n]}$ proposes that C sprouts out a rational tail (C degenerates), whenever non-flatness arises (Γ_f degenerates). DT theory, on the other hand, allows non-flatness, since it is interested in arbitrary 1-dimensional subschemes, thereby restricting degenerations of C to semistable ones (no rational tails).

A non-flat graph Γ does not define a map to $S^{[n]}$, but it defines a quasimap to $S^{[n]}$. Hence the motto of quasimaps:

Trade rational tails for non-flat points and vice versa.

The idea of ϵ-stability is to allow both rational tails and non-flat points, restricting their degrees. The moduli spaces involved in (1) are given by the extremal values of ϵ.

1.2.2. *ϵ-admissible maps.* The motto of GW/H wall-crossing is the following one:

Trade rational tails for branching points and vice versa.

Let us explain what we mean by making an analogy with quasimaps. Let

$$f \colon P \to C$$

be an admissible cover with simple ramifications introduced in [HM82, Chapter 4]. If the curve C varies, the pair (C, f) can degenerate in two ways:

(i) the curve C degenerates;
(ii) the cover f degenerates.

The degenerations of f arise due to

- ramifications of higher order;
- contracted components and singular points mapping to smooth locus.

As previously, these two types of degenerations of a pair (C, f) are related. Hurwitz theory of a varying curve C proposes that C sprouts out rational tails, whenever f degenerates in the sense above. On the other hand, Gromov–Wittengm theory of a varying curve C allows f to degenerate and therefore restricts the degenerations of C to semistable ones. The purpose of ϵ-admissible maps is to interpolate between these Hurwitz and Gromov–Witten cases.

[3]A 1-dimensional subscheme $\Gamma \subset S \times C$ is a graph, if and only if it is flat.

1.3. **Definition.** Let us now sketch the definition of ϵ-admissibility. Let $f\colon P \to C$ be a degree-n map between nodal curves, such that it is admissible at nodes (see [HM82, Chapter 4] for admissibility at the nodes) and $g(P) = \mathsf{h}$, $g(C) = g$. We allow P to be disconnected, requiring that each connected component is mapped non-trivially to C. Following [FP02], we define the branching divisor

$$\mathsf{br}(f) \in \mathrm{Div}(C),$$

it is an effective divisor which measures the degree of ramification away from nodes and the genera of contracted components of P. If C is smooth, the branching divisor $\mathsf{br}(f)$ is given by associating to the 0-dimensional complex

$$Rf_*[f^*\Omega_C \to \Omega_P]$$

its support weighted by Euler characteristics. Otherwise, we need to take the part of the support which is contained in the regular locus of C.

Remark 1.1. To establish that the branching divisor behaves well in families for maps between singular curves, we have to go through an auxiliary (at least for the purposes of this work) notion of twisted ϵ-admissible map, as is explained in Section 2. The construction of br for families in (6) and (7) is the only place where we use twisted maps.

Using the branching divisor br, we now can define ϵ-admissibility by the weighted stability of the pair $(C, \mathsf{br}(f))$, considered in [Has03]. Similar stability was considered in [Deo14], where the source curve P is allowed to have more degenerate singularities instead of contracted components. However, the moduli spaces of [Deo14] do not have a perfect obstruction theory.

Definition. *Let $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$. A map f is ϵ-admissible, if*

- $\omega_C(e^{1/\epsilon} \cdot \mathsf{br}(f))$ *is ample;*
- $\forall p \in C$, $\mathrm{mult}_p(\mathsf{br}(f)) \leq e^{-1/\epsilon}$*;*

Remark 1.2. Note that the presence of exponential $e^{1/\epsilon}$ in the definition is mostly conventional, we could also make the definition with ϵ instead of $e^{1/\epsilon}$. The reason is that we would like ϵ-admissibility to be defined for $\epsilon \in \mathbb{R}_{\leq 0}$, because ϵ-stability of quasimaps is defined for $\epsilon \in \mathbb{R}_{>0}$. In this way we can view both theories as a part of one theory which is defined for $\epsilon \in \mathbb{R}$. This is useful for the purposes of crepant resolution conjecture.

One can readily verify that for $\epsilon = -\infty$, an ϵ-admissible map is an admissible cover with simple ramifications. For $\epsilon = 0$, an ϵ-admissible map is a stable[4] map, such that the target curve C is semistable. Hence ϵ-admissibility provides an interpolation between the moduli space of admissible covers with simple ramifications $Adm_{g,\mathsf{h},n}$ and the moduli space of stable maps $\overline{M}^{\bullet}_{\mathsf{h}}(C_g/\overline{M}_g, n)$,

$$Adm_{g,\mathsf{h},n} \xleftarrow{\;\epsilon\;}\!\!\xrightarrow{\;\;} \overline{M}^{\bullet}_{\mathsf{h}}(C_g/\overline{M}_g, n)$$

[4]When the target curve C is singular, by a stable map we will mean a stable map which is admissible at nodes.

After introducing markings $\mathbf{x} = (x_1, \dots, x_N)$ on C and requiring maps to be admissible over these markings, ϵ-admissibility interpolates between admissible covers with arbitrary ramifications over markings and relative stable maps. As it is explained in [ACV03], sometimes it is more convenient to consider the normalisation of the moduli space of admissible covers - the moduli space of stable twisted maps to BS_n, denoted by $\mathcal{K}_{g,N}(BS_n, \mathsf{h})$. For the purposes of enumerative geometry (virtual intersection theory of moduli spaces), the interpolation above can therefore be equally considered as the following one

$$\mathcal{K}_{g,N}(BS_n, \mathsf{h}) \xleftarrow{\ \epsilon\ } \!\!\! \dashrightarrow \overline{M}^{\bullet}_{\mathsf{h}}(C_{g,N}/\overline{M}_{g,N}, n).$$

In fact, this point of view is more appropriate, if one wants to make an analogy with quasimaps.

1.4. **Higher-dimensional case.** We can upgrade the set-up even further by adding a map $f_X \colon P \to X$ for some target variety X. This leads us to the study of ϵ-admissibility of the data

$$(P, C, \mathbf{x}, f_X \times f_C),$$

which can be represented as a correspondence

$$\begin{array}{ccc} P & \xrightarrow{f_X} & X \\ {\scriptstyle f_C}\downarrow & & \\ (C, \mathbf{x}) & & \end{array}$$

In this case, ϵ-admissibility also takes into account the degree of the components of P with respect to the map f_X, see Definition 2.4. If X is a point, we get the set-up discussed previously.

Let $\beta = (\gamma, \mathsf{h}) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$ be an *extended* degree[5]. For $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$, we then define

$$Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$$

to be the moduli space of the data

$$(P, C, \mathbf{x}, f_X \times f_C),$$

such that $g(P) = \mathsf{h}$, $g(C) = g$, $|\mathbf{x}| = N$ and the map $f_X \times f_C$ is of degree (γ, n). The notation is slightly misleading, as ϵ-admissible maps are not maps to $X^{(n)}$. However, it is justified by the analogy with quasimaps and is more natural with respect to our notions of degree of an ϵ-admissible map (see also Section 2.3).

[5]By a version of Riemann–Hurwitz formula, Lemma 2.9, the degree of the branching divisor $\mathsf{br}(f) = \mathsf{m}$ and the genus h determine each other, latter we will use m instead of h.

As in the case of X is a point, we obtain the following description of these moduli spaces for extremal values of ϵ,

$$\begin{aligned}&\overline{M}^{\bullet}_{\mathsf{h}}(X \times C_{g,N}/\overline{M}_{g,N}(\gamma,n)) = Adm^{0}_{g,N}(X^{(n)},\beta),\\ &\mathcal{K}_{g,N}([X^{(n)}],\beta) \xrightarrow{\rho} Adm^{-\infty}_{g,N}(X^{(n)},\beta),\end{aligned}$$

such that the map ρ is a virtual normalisation in the sense of the diagram (16), which makes two spaces equivalent from the perspective of enumerative geometry. We therefore get an interpolation,

$$\mathcal{K}_{g,N}([X^{(n)}],\beta) \overset{\epsilon}{\dashleftarrow\dashrightarrow} \overline{M}^{\bullet}_{\mathsf{h}}(X \times C_{g,N}/\overline{M}_{g,N}(\gamma,n)),$$

which is completely analogous to (1).

1.5. **Wall-crossing.** The invariants of $\overline{M}^{\bullet}_{\mathsf{h}}(X \times C_{g,N}/\overline{M}_{g,N}(\gamma,n))$ that can be related to orbifold invariants of $\mathcal{K}_{g,N}([X^{(n)}],\beta)$ are the *relative* GW invariants taken with respect to the markings of the target curve C. More precisely, for all ϵ, there exist natural evaluations maps

$$\mathrm{ev}_i \colon Adm^{\epsilon}_{g,N}(X^{(n)},\beta) \to \bar{\mathcal{I}}X^{(n)},\ i = 1,\dots,N,$$

where $\bar{\mathcal{I}}X^{(n)}$ is a rigidified version of the inertia stack $\mathcal{I}X^{(n)}$. We define

$$\langle \tau_{m_1}(\gamma_1),\dots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon}_{g,N,\beta} := \int_{[Adm^{\epsilon}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}}} \prod_{i=1}^{i=N} \psi_i^{m_i}\mathrm{ev}_i^*(\gamma_i,),$$

where $\gamma_1,\dots,\gamma_N$ are classes in orbifold cohomology $H^*_{\mathrm{orb}}(X^{(n)})$ and $\psi_1,\dots,\psi_N$ are ψ-classes associated to the markings of the source curves. By Lemma 2.17, these invariants specialise to orbifold GW invariants associated to a moduli space $\mathcal{K}_{g,N}([X^{(n)}],\beta)$ and relative GW invariants associated to a moduli space $\overline{M}^{\bullet}_{\mathsf{h}}(X \times C_{g,N}/\overline{M}_{g,N}(\gamma,n))$ for corresponding choices of ϵ.

To relate invariants for different values of ϵ, we use the master space technique developed by Zhou in [Zho22] for the purposes of quasimaps. We establish the properness of the master space in our setting in Section 3, following the strategy of Zhou.

To state compactly the wall-crossing formula, we define

$$F^{\epsilon}_{g}(\mathbf{t}(z)) = \sum_{n=0}^{\infty}\sum_{\beta}\frac{q^{\beta}}{N!}\langle \mathbf{t}(\psi_1),\dots,\mathbf{t}(\psi_N)\rangle^{\epsilon}_{g,N,\beta},$$

where $\mathbf{t}(z) \in H^*_{\mathrm{orb}}(X^{(n)},\mathbb{Q})[z]$ is a generic element, and the unstable terms are set to be zero. There exists an element

$$\mu(z) \in H^*_{\mathrm{orb}}(X^{(n)})[z] \otimes \mathbb{Q}[\![q^{\beta}]\!],$$

defined in Section 4.1 as a truncation of an I-function. The I-function is in turn defined via the virtual localisation on the space of stable maps to $X \times \mathbb{P}^1$ relative to $X \times \{\infty\}$. The element $\mu(z)$ provides the change of variables, which relates generating series for extremal values of ϵ.

Theorem. *For all $g \geq 1$, we have*

$$F_g^0(\mathbf{t}(z)) = F_g^{-\infty}(\mathbf{t}(z) + \mu(-z)).$$

For $g = 0$, the same equation holds modulo constant and linear terms in $\mathbf{t}$.

The change of variables above is the consequence of a wall-crossing formula across each wall between extremal values of ϵ, see Theorem 4.3.

1.6. **Applications.**

1.6.1. *The square.* For a del Pezzo surface S, we compute the wall-crossing invariants in Section 5. A computation for analogous quasimap wall-crossing invariants is given in [Nes21a, Proposition 6.10].

The wall-crossing invariants can easily be shown to satisfy $\mathsf{PT/GW}$. Hence when both quasimap and $\mathsf{GW/H}$ wall-crossings are applied, $\mathsf{C.R.C.}$ becomes equivalent to $\mathsf{PT/GW}$. For precise statements of both in this setting, we refer to Section 6.1. This is expressed in terms of the square of theories in Figure 1.

In [PP17], $\mathsf{PT/GW}$ is established for $S \times \mathbb{P}^1$ relative to $S \times \{0, 1, \infty\} \subset S \times \mathbb{P}^1$, if S is toric. Alongside with [PP17], the square therefore gives us the following result.

Theorem. *If S is a toric del Pezzo surface, $g = 0$ and $N = 3$, then* $\mathsf{C.R.C.}$ *(in the sense of [BG09]) holds for $S^{[n]}$ for all $n \geq 1$ and in all classes.*

Previously, the theorem above was established for $n = 2$ and $S = \mathbb{P}^2$ in [Wis11, Section 6]; for an arbitrary n and an arbitrary toric surface, but only for exceptional curve classes, in [Che13]; for an arbitrary n and a simply connected S, but only for exceptional curve classes and in sense of [Rua06], in [LQ16]. We believe that with a little bit of effort, all of the results above can be given a more natural proof by reducing them to $\mathsf{PT/GW}$ for $S \times \mathbb{P}^1$ by means of our wall-crossings, as $\mathsf{PT/GW}$ is a more computationally accessible side.

If $S = \mathbb{C}^2$, $\mathsf{C.R.C.}$ was proved for all genera and any number of markings on the level of cohomological field theories in [PT19b]. If $S = \mathcal{A}_n$, it was proved in genus-0 case and for any number of markings in [CCIT09] in the sense of [CIR14]. For surfaces with $\mathrm{c}_1(S) = 0$, $\mathsf{C.R.C.}$ was established in [FG03] with representation-theoretic methods, as GW invariants vanish in this case. Crepant resolution conjecture was also proved for resolutions other than those that are of Hilbert–Chow type. The list is too long to mention them all.

The theorem can also be restated as an isomorphism of quantum cohomologies,

$$QH^*_{\mathrm{orb}}(S^{(n)}) \cong QH^*(S^{[n]}),$$

we refer to Section 6.2 for more details. The result is very appealing, because the underlying cohomologies with classical multiplications are not isomorphic for surfaces with $\mathrm{c}_1(S) \neq 0$, but the quantum cohomologies are. In particular, the classical multiplication on $H^*_{\mathrm{orb}}(S^{(n)})$ is a non-trivial quantum deformation of the classical multiplication on $H^*(S^{[n]})$.

We want to stress that $\mathsf{C.R.C.}$ should be considered as a more fundamental correspondence than $\mathsf{PT/GW}$, because it relates theories which are closer to each other. Moreover, as [BG09] points out, $\mathsf{C.R.C.}$ explains the origin of the change of variables,

$$y = -e^{iu}, \tag{2}$$

it arises due to the following features of $\mathsf{C.R.C.}$,

(i) analytic continuation of generating series from 0 to -1;
(ii) factor $i = \sqrt{-1}$ in the identification of cohomologies of $S^{[n]}$ and $S^{(d)}$;
(iii) the divisor equation in $\mathsf{GW}(S^{[n]})$;
(iv) failure of the divisor equation in $\mathsf{GW}_{\mathrm{orb}}([S^{(n)}])$.

More precisely, (i) is responsible for the minus sign in (2); (iii) and (iv) are responsible for the exponential; (ii) is responsible for i in the exponential. More conceptual view on $\mathsf{C.R.C.}$ is presented in works of Iritani, e.g. [Iri09].

1.6.2. *LG/CY vs C.R.C.* We will now draw certain similarities between $\mathsf{C.R.C.}$ and Landau-Ginzburg/Calabi-Yau correspondence ($\mathsf{LG/CY}$), which are schematically illustrated in Table 1. For all details and notation on $\mathsf{LG/CY}$, we refer to [CIR14].

$\mathsf{LG/CY}$ consists of two types of correspondences - A-model and B-model correspondences. The B-model correspondence is the statement of equivalence of two categories - matrix factorisation categories and derived categories. While the A-model correspondence is the statement of equality of generating series of certain curve-counting invariants after an analytic continuation and a change of variables. Moreover, there exists a whole family of enumerative theories depending on a stability parameter $\epsilon \in \mathbb{R}$. For $\epsilon \in \mathbb{R}_{>0}$ it gives the theory of GIT quasimaps, while for $\epsilon \in \mathbb{R}_{\leq 0}$ it gives FJRW (Fan–Jarvis–Ruan–Witten) theory. GLSM (Gauged Linear Sigma Model) formalism, defined mathematically in [FJR18], allows to unify quasimaps and FJRW theory. The analytic continuation occurs, when one crosses the wall at $\epsilon = 0$.

In the case of $\mathsf{C.R.C.}$ we have a similar picture. B-model correspondence is given by an equivalence of categorises, $\mathrm{D}^b(S^{[n]})$ and $\mathrm{D}^b([S^{(n)}])$. A-model correspondence is given by an analytic continuation of generating series and subsequent application of a change of variables, as it is stated in Section 6. There also exists a family of enumerative theories depending on a parameter $\epsilon \in \mathbb{R}$. For $\epsilon \in \mathbb{R}_{>0}$, it is given by ϵ-stable quasimaps to a moduli space of sheaves, while for $\epsilon \in \mathbb{R}_{\leq 0}$ it is given by ϵ-admissable maps. It would be interesting to know, if a unifying theory exists in this case (like GLSM in $\mathsf{LG/CY}$).

The above comparison is not a mere observation about structural similarities of two correspondences. In fact, both correspondences are instances of the same phenomenon. Namely, in both cases there should exist *Kähler* *moduli spaces*, $\mathcal{M}_{\mathsf{LG/CY}}$ and $\mathcal{M}_{\mathsf{C.R.C.}}$, such that two geometries in question correspond to two different cusps of these moduli spaces (e.g. $S^{[n]}$ and $[S^{(n)}]$ correspond to two different cusps of $\mathcal{M}_{\mathsf{C.R.C.}}$). B-models do not vary across these moduli spaces, hence the relevant categories are isomorphic. On the other hand, A-models vary in the sense that there exist non-trivial global quantum D-modules, $\mathcal{D}_{\mathsf{LG/CY}}$ and $\mathcal{D}_{\mathsf{C.R.C.}}$, which specialise to relevant enumerative invariants around cusps. For more details on this point of view, we refer to [CIR14] in the case of $\mathsf{LG/CY}$, and to [Iri10] in the case of $\mathsf{C.R.C.}$

	B-model	A-model
$\mathsf{LG/CY}$	$\mathrm{D^b}(X_W) \cong \mathrm{MF}(W)$	$\mathsf{GW}(X_W) \xleftarrow{\epsilon\leq 0} \vert_0 \xrightarrow{\epsilon>0} \mathsf{FJRW}(\mathbb{C}^n, W)$
$\mathsf{C.R.C.}$	$\mathrm{D^b}(S^{[n]}) \cong \mathrm{D^b}([S^{(n)}])$	$\mathsf{GW}(S^{[n]}) \xleftarrow{\epsilon\leq 0} \vert_0 \xrightarrow{\epsilon>0} \mathsf{GW}_{\mathrm{orb}}([S^{(n)}])$

TABLE 1. $\mathsf{LG/CY}$ vs $\mathsf{C.R.C}$

1.7. **Acknowledgments.** First and foremost I would like to thank Georg Oberdieck for the supervision of my PhD. In particular, I am grateful to Georg for pointing out that ideas of quasimaps can be applied to orbifold symmetric products.

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A great intellectual debt is owed to Yang Zhou for his theory of calibrated tails, without which the wall-crossings would not be possible.

1.8. **Notation and conventions.** We work over the field of complex numbers $\mathbb{C}$. Given a variety X, by $[X^{(n)}]$ we denote a stacky symmetric product $[X^n/S_n]$ and by $X^{(n)}$ its coarse quotient. We denote a Hilbert scheme of n-points by $X^{[n]}$. For a partition μ of n, let $\ell(\mu)$ denote the length of μ and $\mathrm{age}(\mu) = n - \ell(\mu)$.

For a possibly disconnected twisted curve $\mathcal{C}$ with the underlying coarse curve C, we define $g(\mathcal{C}) := 1 - \chi(\mathcal{O}_{\mathcal{C}}) = 1 - \chi(\mathcal{O}_C)$.

We set $e_{\mathbb{C}^*}(\mathbb{C}_{\mathrm{std}}) = -z$, where $\mathbb{C}_{\mathrm{std}}$ is the standard representation of $\mathbb{C}^*$ on a vector space $\mathbb{C}$.

Let N be a semigroup and $\beta \in N$ be its generic element. By $\mathbb{Q}[\![q^\beta]\!]$ we will denote the (completed) semigroup algebra $\mathbb{Q}[\![N]\!]$. In our case, N will be various semigroups of effective curve classes.

2. ϵ-ADMISSIBLE MAPS

Let X be a smooth projective variety, $(\mathcal{C}, \mathbf{x})$ be a twisted[6] marked nodal curve, defined in [AGV08], and let $\mathcal{P}$ be a possibly disconnected orbifold nodal curve.

Definition 2.1. For a map

$$f = f_X \times f_{\mathcal{C}} \colon \mathcal{P} \to X \times \mathcal{C},$$

the data $(\mathcal{P}, \mathcal{C}, \mathbf{x}, f)$ is called a *twisted pre-admissible* map, if

- $f_{\mathcal{C}}$ is étale over marked points and nodes;
- $f_{\mathcal{C}}$ is a representable;
- f is non-constant on each connected component.

We will refer to $\mathcal{P}$ and $\mathcal{C}$ as *source* and *target* curves, respectively. Note that by all the conditions above, $\mathcal{P}$ itself must be a twisted nodal curve with orbifold points over nodes and marked points of $\mathcal{C}$.

Consider now the following complex

$$Rf_{\mathcal{C}*}[f_{\mathcal{C}}^*\mathbb{L}_{\mathcal{C}} \to \mathbb{L}_{\mathcal{P}}] \in \mathrm{D}^b(\mathcal{C}),$$

where the morphism $f_{\mathcal{C}}^*\mathbb{L}_{\mathcal{C}} \to \mathbb{L}_{\mathcal{P}}$ is the one which is naturally associated to the map $f_{\mathcal{C}}$. The complex is supported at finitely many points of the non-stacky smooth locus, which we call *branching* points. They arise either due to ramification points or contracted components of the map $f_{\mathcal{C}}$. Following [FP02], to the complex above, we can associate an effective Cartier divisor

$$\mathsf{br}(f) \in \mathrm{Div}(\mathcal{C})$$

by taking the support of the complex weighted by its Euler characteristics. This divisor will be referred to as a *branching divisor*.

Let us give a more explicit expression for the branching divisor. Let $\mathcal{P}_\circ \subseteq \mathcal{P}$ be the maximal subcurve of $\mathcal{P}$ which contracted by the map $f_{\mathcal{C}}$. Let $\mathcal{P}_\bullet \subseteq \mathcal{P}$ be the complement of $\mathcal{P}_\circ$, i.e. the maximal subcurve which is not contracted by the map $f_{\mathcal{C}}$. By $\widetilde{\mathcal{P}}_\bullet$ we denote its normalisation at the nodes which are mapped into a regular locus of $\mathcal{C}$. Note that the restriction of $f_{\mathcal{C}}$ to $\widetilde{\mathcal{P}}_\bullet$ is a ramified cover, the branching divisor of which is therefore given by points of ramifications.

By $\widetilde{\mathcal{P}}_{\circ,i}$ we denote the connected components of the normalisation $\widetilde{\mathcal{P}}_\circ$ and by $p_i \in \mathcal{C}$ their images in $\mathcal{C}$. Finally, let $N \subset \mathcal{P}$ be the locus of nodal points which are mapped into regular locus of $\mathcal{C}$. Following [FP02, Lemma 10, Lemma 11], the branching divisor $\mathsf{br}(f)$ can be expressed as follows.

Lemma 2.2. *With the notation from above we have*

$$\mathsf{br}(f) = \mathsf{br}(f_{|\widetilde{\mathcal{P}}_\bullet}) + \sum_i (2g(\widetilde{\mathcal{P}}_{\circ,i}) - 2)[p_i] + 2f_*(N).$$

[6] By a twisted nodal curve we will always mean a balanced twisted nodal curve.

Proof. By the definition of twisted pre-admissibility, all the branching takes place away from orbifold points and nodes. Since the branching of a map can be determined locally, we therefore can assume that both $\mathcal{C}$ and $\mathcal{P}$ are ordinary nodal curves C and P.

Let

$$v\colon \widetilde{C} \to C$$

be the normalisation of C at N, and let $\tilde{f} = f \circ v$. Recall that $\mathbb{L}_C \cong \Omega_C$. By composing the normalisation morphism $\mathbb{L}_C \to v_* v^* \mathbb{L}_C$ with the natural morphism $v_* v^* \mathbb{L}_C \to v_* \mathbb{L}_{\widetilde{C}}$, we obtain following exact sequence

$$0 \to \mathcal{O}_N \to \mathbb{L}_C \to v_* \mathbb{L}_{\widetilde{\mathcal{C}}} \to 0, \tag{3}$$

which, in particular, implies that

$$\chi(\mathbb{L}_C) = \chi(\omega_C). \tag{4}$$

On the other hand, since N is mapped to the regular locus of P and $\mathbb{L}_P$ is locally free at regular points, we obtain

$$0 \to f^* \mathbb{L}_P \to v_* \tilde{f}^* \mathbb{L}_P \to \mathcal{O}_N \otimes f^* \mathbb{L}_P \to 0. \tag{5}$$

With the sequences (3) and (5), the proof of [FP02, Lemma 10] in our setting is exactly the same. So is the proof of [FP02, Lemma 11] with (4). □

Remark 2.3. One could also use $\omega_{\mathcal{C}}$ instead of $\mathbb{L}_{\mathcal{C}}$. For a smooth curve $\mathcal{C}$, the map $\phi\colon f_{\mathcal{C}}^* \omega_{\mathcal{C}} \to \omega_{\mathcal{P}}$ is constructed in [FP02, Lemma 8]. If $\mathcal{C}$ is not smooth, then $f_{\mathcal{C}}$ is étale over nodal points, therefore in a neighbourhood U of a nodal point we have a natural isomorphism $\phi_{|U}\colon f_{\mathcal{C}}^* \omega_{\mathcal{C}|U} \xrightarrow{\simeq} \omega_{\mathcal{P}|f^{-1}U}$. In general, the map ϕ of [FP02, Lemma 8] over smooth locus of a nodal twisted curve $\mathcal{C}$ glues with a map $\phi_{|U}$. In this way we obtain a map $\phi\colon f_{\mathcal{C}}^* \omega_{\mathcal{C}} \to \omega_{\mathcal{P}}$ for a nodal twisted curve $\mathcal{C}$.

We fix $L \in \operatorname{Pic}(X)$, an ample line bundle on X, such that for all effective curve classes $\gamma \in H_2(X, \mathbb{Z})$,

$$\deg(\gamma) = \mathrm{c}_1(L) \cdot \beta \gg 0.$$

Let $(\mathcal{P}, \mathcal{C}, \mathbf{x}, f)$ be a twisted pre-admissible map. For a point $p \in \mathcal{C}$, let

$$f^* L_p := f_X^* L_{|f_{\mathcal{C}}^{-1}(p)},$$

we set $\deg(f^* L_p) = 0$, if $f_{\mathcal{C}}^{-1}(p)$ is 0-dimensional. For a component $\mathcal{C}' \subseteq \mathcal{C}$, let

$$f^* L_{|\mathcal{C}'} := f_X^* L_{|f_{\mathcal{C}}^{-1}(\mathcal{C}')}.$$

Recall that a *rational tail* of a curve $\mathcal{C}$ is a component isomorphic to $\mathbb{P}^1$ with one special point (a node or a marked point). A *rational bridge* is a component isomorphic to $\mathbb{P}^1$ with two special points.

Definition 2.4. Let $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$. A twisted pre-admissible map f is twisted ϵ-admissible, if (**i**) for all points $p \in \mathcal{C}$,

$$\mathrm{mult}_p(\mathsf{br}(f)) + \deg(f^*L_p) \leq e^{-1/\epsilon};$$

(**ii**) for all rational tails $T \subseteq (\mathcal{C}, \mathbf{x})$,

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^*L_{|T}) > e^{-1/\epsilon};$$

(**iv**)

$$|\mathrm{Aut}(f)| < \infty.$$

Lemma 2.5. *The condition of twisted ϵ-admissability is an open condition.*

Proof. The conditions of twisted ϵ-admissibility are constructable. Hence we can use the valuative criteria for openness. Given a discrete valuation ring R with a fraction field K, we therefore need to show that if a pre-admissible map

$$(\mathcal{P}, \mathcal{C}, \mathbf{x}, f) \in \mathfrak{M}(X \times \mathfrak{C}^{\mathrm{tw}}_{g,N}/\mathfrak{M}^{\mathrm{tw}}_{g,N}, (\gamma, n))(R)$$

is ϵ-admissible at a closed fiber $\operatorname{Spec}\mathbb{C}$ of $\operatorname{Spec} R$, then it is ϵ-admissible at the generic fiber. In fact, each of conditions of ϵ-admissibility is an open condition. For example, let

$$T \subseteq (\mathcal{C}, \mathbf{x})$$

be a family of subcurves of $(\mathcal{C}, \mathbf{x})$ such that the generic fiber $T_{|\operatorname{Spec} K}$ is a rational tail that does not satisfy the condition (**ii**). Then the central fiber $T_{|\operatorname{Spec}\mathbb{C}}$ of T will be a tree of rational curves, whose rational tails do not satisfy the condition (**ii**), because the degree of both $\mathsf{br}(f)$ and f_X^*L can only decrease on rational tails of $T_{|\operatorname{Spec}\mathbb{C}}$. Note we use that $\mathsf{br}(f)$ is defined for families of pre-admissible twisted maps to conclude that the degree of $\mathsf{br}(f)$ is constant in families.

Other conditions of ϵ-admissibility can be shown to be open in a similar way. □

A family of twisted ϵ-admissable maps over a base scheme B is given by two families of twisted B-curves $\mathcal{P}$ and $(\mathcal{C}, \mathbf{x})$ and a B-map

$$f = f_X \times f_{\mathcal{C}} \colon \mathcal{P} \to X \times (\mathcal{C}, \mathbf{x}),$$

whose fibers over geometric points of B are ϵ-admissable maps. An isomorphism of two families

$$\Phi = (\phi_1, \phi_2) \colon (\mathcal{P}, \mathcal{C}, \mathbf{x}, f) \cong (\mathcal{P}', \mathcal{C}', \mathbf{x}', f')$$

is given by the data of isomorphisms of the source and target curves

$$(\phi_1, \phi_2) \in \mathrm{Isom}_B(\mathcal{P}, \mathcal{P}') \times \mathrm{Isom}_B((\mathcal{C}, \mathbf{x}), (\mathcal{C}', \mathbf{x}')),$$

which commute with the maps f and f',

$$f' \circ \phi_1 \cong \phi_2 \circ f.$$

Definition 2.6. Given an element
$$\beta = (\gamma, \mathsf{m}) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z},$$
we say that a twisted ϵ-admissible map is of degree β to $X^{(n)}$, of genus g with N markings, if

- f is of degree (γ, n) and $\deg(\mathsf{br}(f)) = \mathsf{m}$;
- $g(C) = g$ and $|\mathbf{x}| = N$.

We define
$$\begin{aligned} Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)^{\mathrm{tw}} \colon (Sch/\mathbb{C})^{\circ} &\to (Grpd) \\ S &\mapsto \{\text{families of } \epsilon\text{-admissable maps over } B\} \end{aligned}$$
to be the moduli space of twisted ϵ-admissible to $X^{(n)}$ maps of degree β and genus g with N markings. Recall that $\mathbb{L}_{\mathcal{C}}$ is a perfect complex, since $\mathcal{C}$ is l.c.i., so is $\mathbb{L}_{\mathcal{P}}$. Hence following [FP02, Section 3.2] (see also [Deo14, Theorem 3.8]), one can construct the universal branching divisor
$$\mathsf{br} : Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)^{\mathrm{tw}} \to \mathfrak{M}_{g,N,\mathsf{m}}. \tag{6}$$
The space $\mathfrak{M}_{g,N,\mathsf{m}}$ is an Artin stack which parametrises triples
$$(C, \mathbf{x}, D),$$
where $(C, \mathbf{x})$ is a genus-g curve with N markings; D is an effective divisor of degree m disjoint from markings $\mathbf{x}$. An isomorphism of triples is an isomorphism of curves which preserve markings and divisors.

There is another moduli space related to $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)^{\mathrm{tw}}$, which is obtained by associating to a twisted ϵ-admissible map the corresponding map between the coarse moduli spaces of the twisted curves. This association defines the following map
$$p \colon Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)^{\mathrm{tw}} \to \mathfrak{M}(X \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}, (\beta, n)),$$
where $\mathfrak{M}(X \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}, (\beta, n))$ is the relative moduli space of stable maps to the relative geometry
$$X \times \mathfrak{C}_{g,N} \to \mathfrak{M}_{g,N},$$
where $\mathfrak{C}_{g,N} \to \mathfrak{M}_{g,N}$ is the universal curve. By Lemma 2.5, the image of p is open.

Definition 2.7. We denote the image of p with its natural open-substack structure by $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$.

The closed points of $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$ are relative stable maps with restricted branching away from marked points and nodes, to which we refer as *ϵ-admissable maps.* On can similarly define *pre-admissable maps.* As in Definition 2.1, we denote the data of a pre-admissible map by
$$(P, C, \mathbf{x}, f).$$
The moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$ will be the central objects of our study.

Remark 2.8. The difference between the moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ and $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)^{\mathrm{tw}}$ is the same as the one between admissible covers and twisted bundles of [ACV03]. We prefer to work with $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$, because it is more convenient to work with schemes than with stacks for the purposes of deformation theory and of analysis of the basic properties of the moduli spaces. Moreover, the enumerative geometries of these two moduli spaces are equivalent, at least for the relevant values of ϵ. For more details, see Section 2.3 and Section 2.6.

Since $\mathsf{br}(f)$ is supported away from stacky points, the branching-divisor map descends,

$$\mathsf{br} : Adm^{\epsilon}_{g,N}(X^{(n)},\beta) \to \mathfrak{M}_{g,N,\mathsf{m}}. \tag{7}$$

The moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ also admit a disjoint-union decomposition

$$Adm^{\epsilon}_{g,N}(X^{(n)},\beta) = \coprod_{\underline{\mu}} Adm^{\epsilon}_{g,N}(X^{(n)},\beta,\underline{\mu}), \tag{8}$$

where $\underline{\mu} = (\mu^1,\dots,\mu^N)$ is a N-tuple of ramifications profiles of f_C over the markings $\mathbf{x}$.

Riemann–Hurwitz formula extends to the case of pre-admissible maps.

Lemma 2.9. *If $f\colon P \to (C,\mathbf{x})$ is a degree-n pre-admissible map with ramification profiles $\underline{\mu} = (\mu^1,\dots,\mu^N)$ at the markings $\mathbf{x} \subset C$, then*

$$2g(P) - 2 = n \cdot (2g(C) - 2) + \deg(\mathsf{br}(f)) + \sum_i \mathrm{age}(\mu^{\mathrm{i}}).$$

Proof. Using Lemma 2.2 and the standard Riemann–Hurwitz formula, one can readily check that the above formula holds for pre-admissible maps. □

2.1. **Properness.** We now establish the properness of $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$, starting with the following result.

Proposition 2.10. *The moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ are quasi-separated Deligne-Mumford stacks of finite type.*

Proof. By ϵ-admissibility condition, the map br factors through a quasi-separated substack of finite type. Indeed, $(C,\mathbf{x},\mathsf{br}(f))$ is not stable (i.e. has infinitely many automorphisms), if one of the following holds

- (**i**) there is a rational tail $T \subseteq (C,\mathbf{x})$, such that $\mathrm{supp}(\mathsf{br}(f)_{|T})$ is at most a point;
- (**ii**) there is a rational bridge $B \subseteq (C,\mathbf{x})$, such that $\mathrm{supp}(\mathsf{br}(f)_{|B})$ is empty.

Up to a change of coordinates, the restriction of f_C to T or B must of the form

$$z^{\underline{n}}\colon (\sqcup^k \mathbb{P}^1) \sqcup_0 P' \to \mathbb{P}^1 \tag{9}$$

at each connected component of P over T or B. Let us clarify the notation of (9). The curve $\sqcup^k \mathbb{P}^1$ is disjoint union of k distinct $\mathbb{P}^1$. A possibly disconnected marked nodal curve $(P', \mathbf{p})$ is attached via markings to the disjoint union $\sqcup^k \mathbb{P}^1$ at points $0 \in \mathbb{P}^1$ at each connected component of the disjoint union; P' is contracted to $0 \in \mathbb{P}^1$ in the target $\mathbb{P}^1$; while on i'th $\mathbb{P}^1$ in the disjoint union, the map is given by z^{n_i} for $\underline{n} = (n_1, \dots, n_k)$.

The fact that the restriction of f_C is given by a map of such form can be seen as follows. The condition (**i**) or (**ii**) implies that the restriction of f_C to T or B has at most two[7] branching points, which in turn implies that the source curve must be $\mathbb{P}^1$ by Riemann–Hurwitz theorem. A map from $\mathbb{P}^1$ to itself with two ramifications points is given by $z^m \colon \mathbb{P}^1 \to \mathbb{P}^1$ up to change of coordinate. For a rational tail T, there might also be a contracted component P' attached to the ramification point.

In the case of (**ii**), the ϵ-admissibility condition then says that

$$\deg(f^* L_{|B}) > 0.$$

While in the case of (**i**),

$$\deg(\mathsf{br}(f)_{|T}) = \mathrm{mult}_p(\mathsf{br}(f))$$

for a unique point $p \in T$ which is not a node. Hence ϵ-admissibility says that

$$\deg(f^* L_{|T}) - \deg(f^* L_p) > 0.$$

Since we fixed the class β, the conclusions above bound the number of components T or B by $\deg(\beta)$. Hence the image of br is contained in a quasi-compact substack of $\mathfrak{M}_{g,N,\mathsf{m}}$, which is therefore quasi-separated and of finite type, because $\mathfrak{M}_{g,N,\mathsf{m}}$ is quasi-separated and locally of finite type.

The branching-divisor map br is of finite type and quasi-separated, since the fibers of br are sub-loci of stable maps to $X \times C$ for some nodal curve C. The moduli space $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$ is of finite type and quasi-separated itself, because br is of finite type and quasi-seperated and factors through a quasi-separated substack of finite type. $\square$

Lemma 2.11. *Given a pre-admissable map $(P, C, \mathbf{x}, f)$. Let $(P', C', \mathbf{x}', f')$ be given by contraction of a rational tail $T \subseteq (C, \mathbf{x})$ and stabilisation of the induced map*

$$f \colon P \to X \times C'.$$

Let $p \in C'$ be the image of contraction of T. Then the following holds

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^* L_{|T}) = \mathrm{mult}_p(\mathsf{br}(f')) + \deg(f'^* L_p).$$

Proof. By Lemma 2.9,

$$2g(P_{|T}) - 2 = -2d + \deg(\mathsf{br}(f)) + d - \ell(p),$$

[7]Remember that branching might also be present at the nodes.

where $\ell(p)$ is the number of points in fiber above p, from which it follows that

$$\begin{aligned}\deg(\mathsf{br}(f)) &= 2g(P_{|T}) - 2 + 2d - d + \ell(p) \\ &= 2g(P_{|T}) - 2 + d + \ell(p).\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}\mathrm{mult}_p(\mathsf{br}(f)) &= 2g(P_{|T}) - 2 + 2\ell(p) + d - \ell(p) \\ &= 2g(P_{|T}) - 2 + d + \ell(p).\end{aligned}$$

It is also clear by definition, that

$$\deg(f^*L_{|T}) = \deg(f^*L_p),$$

the claim then follows. □

Definition 2.12. Let R be a discrete valuation ring. Given a pre-admissible map $(P, C, \mathbf{x}, f)$ over $\operatorname{Spec} R$. A *modification* of $(P, C, \mathbf{x}, f)$ is a pre-admissible map $(\widetilde{P}, \widetilde{C}, \widetilde{\mathbf{x}}, \widetilde{f})$ over $\operatorname{Spec} R'$ such that

$$(\widetilde{P}, \widetilde{C}, \widetilde{\mathbf{x}}, \widetilde{f})_{|\operatorname{Spec} K'} \cong (P, C, \mathbf{x}, f)_{|\operatorname{Spec} K'},$$

where R' is a finite extension of R with a fraction field K'.

A modification of a family of curves C over a discrete valuation ring is given by three operations:

- blow-ups of the central fiber of C;
- contractions of rational tails and rational bridges in the central fiber of C;
- base changes with respect to finite extensions of discrete valuation rings.

A modification of a pre-admissible map is therefore given by an appropriate choice of three operations above applied to both target and source curves, such that the map f can be extended as well.

Theorem 2.13. *The moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$ are proper Deligne-Mumford stacks.*

Proof. We will now use the valuative criteria of properness for quasi-separated Deligne-Mumford stacks. Let

$$(P^*, C^*, \mathbf{x}^*, f^*) \in Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)(K)$$

be a family of ϵ-admissable maps over the fraction field K of a discrete valuation ring R. The strategy of the proof is to separate P^* into two components $P^*_{\circ}$ and $P^*_{\bullet}$, the contracted component and non-contracted one of f^*_C, respectively (as it was done for Lemma 2.2). We then take a limit of $f^*_{|P^*_{\bullet}}$ preserving it as a cover over the target curve, and a limit of $f^*_{|P^*_{\circ}}$ as a stable map. We then glue the two limits back and perform a series of modifications to get rid of points or rational components that do not satisfy ϵ-admissibility.

Existence, Step 1. Let
$$(P^*_\circ, \mathbf{q}^*_\circ) \subseteq P^*$$
be the maximal subcurve contracted by $f^*_{C^*}$, the markings $\mathbf{q}^*_\circ$ are given by the nodes of P^* disconnecting $P^*_\circ$ from the rest of the curve. By
$$(P^*_\bullet, \mathbf{q}^*_\bullet) \subseteq P^*$$
we denote the complement of $P^*_\circ$ with similar markings. Let
$$(\widetilde{P}^*_\bullet, \mathbf{t}^*, \mathbf{t}'^*)$$
be the normalisation of $P^*_\bullet$ at nodes which are mapped by $f^*_{C^*}$ to the regular locus of C^*, the markings $\mathbf{t}^*$ and $\mathbf{t}'^*$ are given by the preimages of those nodes. The induced map
$$\tilde{f}^*_{\bullet,C^*} \colon \widetilde{P}^*_\bullet \to C^*$$
is an admissible cover. By properness of admissible covers, there exists, possibly after a finite base change[8], an extension
$$((P_\bullet, \mathbf{q}_\bullet, \mathbf{t}, \mathbf{t}'), (C, \mathbf{x}), \tilde{f}_{\bullet,C}) \in \mathcal{A}dm(R),$$
where $\mathcal{A}dm$ is the moduli space of stable admissible covers with fixed ramification profiles, such that both source and target curves are marked, and markings of the source curve are not allowed to map to the markings of the target curve. The ramification profiles are given by the ramification profiles of $\tilde{f}^*_{\bullet,C^*}$. If necessary, we then take a finite base change and modify the central fibers of source and target curves to obtain a map
$$f_\bullet \colon P_\bullet \to X \times C,$$
such that $f_{\bullet,C}$ is still an admissible cover (possibly unstable)[9]. Now let
$$f_\circ \colon (P_\circ, \mathbf{q}_\circ) \to X \times C$$
be the extension of
$$f^* \colon (P^*_\circ, \mathbf{q}^*_\circ) \to X \times C$$
to $\operatorname{Spec} R$. It exists, possibly after a finite base change, by properness of the moduli space of stable marked maps. If necessary, we modify the curve C to avoid contracted components mapping to the markings $\mathbf{x}$. If we do so, we modify $P_\bullet$ accordingly to make $f_{\bullet,C}$ admissible cover (again, possibly unstable). We then glue back $P_\circ$ and $P_\bullet$ at the markings $(\mathbf{q}_\circ, \mathbf{q}_\bullet)$ and $(\mathbf{t}, \mathbf{t}')$ to obtain a map
$$f \colon P \to X \times C.$$
Let
$$(P, C, \mathbf{x}, f) \tag{10}$$

[8]For this proof, if we take a finite extension $R \to R'$, we relabel R' by R.

[9]The map $f_\bullet$ can be constructed differently. One can lift $\tilde{f}^*_\bullet \colon \widetilde{P}^*_\bullet \to X \times C^*$ to an element of the moduli of twisted stable map $\mathcal{K}_{g,N}([\mathrm{Sym}^n X])$ after passing from admissible covers to twisted stable maps and then take a limit there.

be the corresponding pre-admissible map. We now perform a series of modification to the map above to obtain an ϵ-admissible map.

Existence, Step 2. Let us analyse $(P, C, \mathbf{x}, f)$ in relation to the conditions of ϵ-admissibility.

(**i**) Let $p_0 \in C_{|\operatorname{Spec}\mathbb{C}}$ be a point in the central fiber of C that does not satisfy the condition (**i**) of ϵ-admissibility. There must be a contracted component over p_0, because $f_{\bullet,C}$ was constructed as an admissible cover, preserving the ramifications profiles. We then blow-up the family C at the point $p_0 \in C$. The map f_C lifts to a map $\tilde{f}_C$,

$$\begin{tikzcd} & P \arrow[dl, "\tilde{f}_C"'] \arrow[d, "f_C"] \\ \mathrm{Bl}_{p_0}C \arrow[r] & C \end{tikzcd}$$

by the universal property of a blow-up, since the preimage of the point p_0 is a contracted curve (which is a Cartier divisor inside P). The map f_X is left unchanged. Let $T \subset \mathrm{Bl}_{p_0}C$ be the exceptional curve, which is also a rational tail of the central fiber of $\mathrm{Bl}_{p_0}C$ attached at p_0 to $C_{|\operatorname{Spec}\mathbb{C}}$. By Lemma 2.11, we obtain that

$$\deg(\mathsf{br}(\tilde{f})_{|T}) + \deg(\tilde{f}^*L_{|T}) = \mathrm{mult}_{p_0}(\mathsf{br}(f)) + \deg(f^*L_{p_0}) \tag{11}$$

and, for all points $p \in T$,

$$\mathrm{mult}_p(\mathsf{br}(\tilde{f})) + \deg(\tilde{f}^*L_p) < \mathrm{mult}_{p_0}(\mathsf{br}(f)) + \deg(f^*L_{p_0}). \tag{12}$$

We repeat this process inductively for all points of the central fiber for which the part (**i**) of ϵ-admissibility is not satisfied. By (11) and (12), this procedure will terminate and we will arrive at the map which satisfies the part (**i**) of ϵ-admissibility. Moreover, the procedure does not create rational tails which do not satisfy the part (**ii**) of ϵ-admissibility.

(**ii**) If a rational tail $T \subseteq (C_{|\operatorname{Spec}\mathbb{C}}, \mathbf{x}_{|\operatorname{Spec}\mathbb{C}})$ does not satisfy the condition (**ii**) of ϵ-admissibility, we contract it

$$\begin{tikzcd} & P \arrow[dl, "\tilde{f}_C"'] \arrow[d, "f_C"] \\ C \arrow[r] & \mathrm{Con}_T C \end{tikzcd}$$

The map f_X is left unchanged. Let $p_0 \in \mathrm{Con}_T C$ be the image of the contracted rational tail T. Since

$$\deg(\mathsf{br}(\tilde{f})_{|P}) + \deg(\tilde{f}^*L_{|P}) = \mathrm{mult}_{p_0}(\mathsf{br}(f)) + \deg(f^*L_{p_0}),$$

the central fiber satisfies the condition (**i**) of ϵ-admissibility at the point $p_0 \in \mathrm{Con}_T C$. We repeat this process until we get rid of all rational tails that do not satisfy the condition (**ii**) of ϵ-admissibility.

(**iii**) By the construction of the family (10), all the rational bridges of C satisfy the condition (**iii**) of ϵ-admissibility.

Uniqueness. Assume we are given two families of ϵ-admissible maps over $\operatorname{Spec} R$

$$(P_1, C_1, \mathbf{x}_1, f_1) \text{ and } (P_2, C_2, \mathbf{x}_2, f_2),$$

which are isomorphic over $\operatorname{Spec} K$. Possibly after a finite base change, there exists a family of pre-admissible maps

$$(\widetilde{P}, \widetilde{C}, \tilde{\mathbf{x}}, \tilde{f})$$

which dominates both families in the sense that there exists a commutative square

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\tilde{f}} & X \times \widetilde{C} \\ \downarrow & & \downarrow \\ P_i & \xrightarrow{f_i} & X \times C_i \end{array} \tag{13}$$

We take a minimal family $(\widetilde{P}, \widetilde{C}, \tilde{\mathbf{x}}, \tilde{f})$ with such property. The vertical maps are given by contraction of rational tails. Then by the equality

$$\deg(\mathsf{br}(f)_{|P}) + \deg(L_{|P}) = \mathrm{mult}_{p_0}(\mathsf{br}(f)) + \deg(L_{p_0}),$$

those rational tails cannot satisfy the condition (**ii**) of ϵ-admissibility, otherwise the image point of contraction of a rational tail will not satisfy the condition (**i**) of ϵ-admissibility. But $(P_i, C_i, \mathbf{x}_i, f_i)$'s are ϵ-admissible by assumption. Hence the target curves are isomorphic. By separatedness of the moduli space of maps to a fixed target, it must be that

$$(P_1, C_1, \mathbf{x}_1, f_1) \cong (\widetilde{P}, \widetilde{C}, \tilde{\mathbf{x}}, \tilde{f}) \cong (P_2, C_2, \mathbf{x}_2, f_2).$$

□

2.2. **Obstruction theory.** The obstruction theory of $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta)$ is defined via the obstruction theory of relative maps in the spirit of [GV05, Section 2.8] with the difference that we have a relative target geometry $X \times \mathfrak{C}_{g,N}/\mathfrak{M}_{g,N}$. There exists a complex $E^{\bullet}$, which defines a perfect obstruction theory relative to $\mathfrak{M}_{h,N'} \times \mathfrak{M}_{g,N}$,

$$\phi : E^{\bullet} \to \mathbb{L}_{Adm^{\epsilon}_{g,N}(X^{(n)}, \beta))/\mathfrak{M}_{h,N'} \times \mathfrak{M}_{g,N}},$$

where $\mathfrak{M}_{h,N'}$ is the moduli space of source curves with markings at the fibers over marked points of the target curves; and $\mathfrak{M}_{g,N}$ is the moduli space of target curves. More precisely, such complex exists at each connected component $Adm^{\epsilon}_{g,N}(X^{(n)}, \beta, \underline{\mu})$.

Proposition 2.14. *The morphism ϕ is a perfect obstruction theory.*

Proof. This is a relative version of [GV05, Section 2.8]. □

2.3. **Relation to other moduli spaces.** Let us now relate the moduli spaces of ϵ-admissible maps for the extremal values of $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$ to more familiar moduli spaces.

2.3.1. $\epsilon = -\infty$. In this case the first two conditions of Definition 2.4 are

(**i**) for all points $p \in C$,

$$\mathrm{mult}_p(\mathsf{br}(f)) + \deg(f^*L_p) \leq 1;$$

(**ii**) for all rational tails $T \subseteq (\mathcal{C}, \mathbf{x})$,

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^*L_{|T}) > 1.$$

Since multiplicity and degree take only integer values, by Lemma 2.2 and the choice of L, there is only possibility for which the condition (**i**) is satisfied. Namely, if $f_{\mathcal{C}}$ does not contract any irreducible components and has only simple ramifications.

To unpack the condition (**ii**), recall that a non-constant ramified map from a smooth curve to $\mathbb{P}^1$ has at least two ramification points; it has precisely two ramification points, if it is given by

$$z^2 \colon \mathbb{P}^1 \to \mathbb{P}^1 \tag{14}$$

up to a change of coordinates. Hence

$$\mathrm{mult}_p(\mathsf{br}(f)) + \deg(f^*L_p) = 1,$$

if and only if $f_C = z^2$ and f_X is constant. In this case, $|\mathrm{Aut}(f)| = \infty$. In the light of the condition (**iii**) of ϵ-admissibility, the condition (**ii**) is therefore automatically satisfied.

We obtain that the data of a $-\infty$-admissible map $(P, C, \mathbf{x}, f)$ can be represented by the following correspondence

$$\begin{array}{ccc} P & \xrightarrow{f_X} & X \\ {\scriptstyle f_C}\downarrow & & \\ (C, \mathbf{x}, \mathbf{p}) & & \end{array}$$

where f_C is a degree-n admissible cover with arbitrary ramifications over the marking $\mathbf{x}$ and with simple ramifications over the unordered marking $\mathbf{p} = \mathsf{br}(f)$, such that $|\mathrm{Aut}(f)| < \infty$. Hence the moduli space $Adm^{-\infty}_{g,N}(X^{(n)}, \beta)$ admits a projection from the moduli space of twisted stable maps with *extended degree* (see [BG09, Section 2.1] for the definition) to the orbifold $[X^{(n)}]$,

$$\rho \colon \mathcal{K}_{g,N}([X^{(n)}], \beta) \to Adm^{-\infty}_{g,N}(X^{(n)}, \beta), \tag{15}$$

which is given by passing from twisted curves to their coarse moduli spaces. Indeed, an element of $\mathcal{K}_{g,N}([X^{(n)}], \beta)$ is given by a data of

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f_X} & X \\ {\scriptstyle f_{\mathcal{C}}}\downarrow & & \\ (\mathcal{C}, \mathbf{x}, \mathbf{p}) & & \end{array}$$

where $f_{\mathcal{C}}$ is a representable degree-n étale cover over twisted marked curve $(C, \mathbf{x}, \mathbf{p})$. The additional marking $\mathbf{p}$ is unordered, over this marking the map $f_{\mathcal{C}}$ must have simple ramifications after passing to coarse moduli spaces. The map f_X has to be fixed by only finitely many automorphisms of the cover $f_{\mathcal{C}}$. Passing to coarse moduli space, the above data becomes the data of a $-\infty$-admissible map.

The virtual fundamental classes of two moduli spaces are related as follows.

Lemma 2.15.

$$\rho_*[\mathcal{K}_{g,N}([X^{(n)}], \beta)]^{\mathrm{vir}} = [Adm^{-\infty}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}}.$$

Proof. Let $\mathfrak{K}_{g,N}(BS_n, \mathsf{m})$ be the moduli stacks of twisted maps to BS_n (not necessarily stable) and $\mathfrak{Adm}_{g,\mathsf{m},n,N}$ be the moduli stack of admissible covers (again not necessarily stable). There exists the following pull-back diagram,

$$\begin{array}{ccc} \mathcal{K}_{g,N}([X^{(n)}], \beta) & \xrightarrow{\rho} & Adm^{-\infty}_{g,N}(X^{(n)}, \beta) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} \\ \mathfrak{K}_{g,N}(BS_n, \mathsf{m}) & \longrightarrow & \mathfrak{Adm}_{g,\mathsf{m},n,N} \end{array} \tag{16}$$

The bottom arrow is a normalisation map, therefore it is of degree 1. By [Cos06, Theorem 5.0.1], we therefore obtain the claim for virtual fundamental classes given by the relative obstruction theories,

$$\begin{aligned} &\rho_*[\mathcal{K}_{g,N}([X^{(n)}], \beta)/\mathfrak{K}_{g,N}(BS_n, \mathsf{m})]^{\mathrm{vir}} \\ &\qquad = [Adm^{-\infty}_{g,N}(X^{(n)}, \beta)/\mathfrak{Adm}_{g,\mathsf{m},n,N}]^{\mathrm{vir}}. \end{aligned} \tag{17}$$

The moduli space $\mathfrak{K}_{g,N}(BS_n, \mathsf{m})$ is smooth and $\mathfrak{Adm}_{g,\mathsf{m},n,N}$ is a locally complete intersection (see [ACV03, Proposition 4.2.2]), which implies that their naturally defined obstruction theories are given by cotangent complexes. Using virtual pull-backs of [Man12], one can therefore express the virtual fundamental classes given by absolute perfect obstruction theories as follows

$$\begin{aligned} [Adm^{-\infty}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}} &= (p \circ \pi_2)^![\mathfrak{M}_{g,N}] \\ &= \pi_2^! p^! [\mathfrak{M}_{g,N}] \\ &= \pi_2^! [\mathfrak{Adm}_{g,\mathsf{m},n,N}] \\ &= [Adm^{-\infty}_{g,N}(X^{(n)}, \beta)/\mathfrak{Adm}_{g,\mathsf{m},n,N}]^{\mathrm{vir}}, \end{aligned}$$

where

$$p \colon \mathfrak{Adm}_{g,\mathsf{m},n,N} \to \mathfrak{M}_{g,N}$$

is the natural projection; we used that $p^![\mathfrak{M}_{g,N}] = [\mathfrak{Adm}_{g,\mathsf{m},n,N}]$, which is due to the fact that the obstruction theory is given by the cotangent complex. The same holds for $\mathcal{K}_{g,N}(BS_n, \mathsf{m})$, hence we obtain that

$$\rho_*[\mathcal{K}_{g,N}([X^{(n)}], \beta)]^{\mathrm{vir}} = [Adm^{-\infty}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}}.$$

$\square$

2.3.2. $\epsilon = 0$. By the first two conditions of Definition 2.4, the map f_C can have arbitrary ramifications and contracted components of arbitrary genera (more precisely, the two are only restricted by n, g, N and β). In conjunction with other conditions of Definition 2.4 we therefore obtain the following identification of moduli spaces

$$Adm^0_{g,N}(X^{(n)},\beta) = \overline{M}^{\bullet}_{\mathsf{m}}(X \times C_{g,N}/\overline{M}_{g,N},(\gamma,n)), \tag{18}$$

where the space on the right is the moduli space of relative stable maps with disconnected domains to the relative geometry

$$X \times C_{g,N} \to \overline{M}_{g,N},$$

where $C_{g,N} \to \overline{M}_{g,N}$ is the universal curve and where the markings play the role of relative divisors. Instead of fixing the genus of source curves, we fix the degree m of the branching divisor. At each component $Adm^0_{g,N}(X^{(n)},\beta,\underline{\mu})$ of the decomposition (8), the genus of the source curve and the degree of the branching divisor are related by Lemma 2.9.

The obstruction theories of two moduli spaces are equal, since the obstruction theory of the space $Adm^0_{g,N}(X^{(n)},\beta)$ was defined via the obstruction theory of relative stable maps.

2.4. **Inertia stack.** We would like to define evaluation maps of moduli spaces $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ to a certain rigidification of the inertia stack $\mathcal{I}X^{(n)}$ of $[X^{(n)}]$, for that we need a few observations.

The inertia stack can be defined as follows

$$\mathcal{I}X^{(n)} = \coprod_{[g]} [X^{n,g}/C(g)],$$

where the disjoint union is taken over conjugacy classes $[g]$ of elements of S_n, $X^{n,g}$ is the fixed locus of g acting on X^n and $C(g)$ is the centraliser subgroup of g. Recall that conjugacy classes of elements of S_n are in one-to-one correspondence with partitions μ of n. Let us express a partition μ in terms to repeating parts and their multiplicities,

$$\mu = (\underbrace{\eta_1,\cdots,\eta_1}_{m_1},\cdots,\underbrace{\eta_s,\cdots,\eta_s}_{m_s}).$$

We define

$$C(\mu) := \prod_{t=1}^{s} C_{\eta_t} \wr S_{m_t}, \tag{19}$$

here C_{η_t} is a cyclic group and "$\wr$" is a *wreath product* defined as

$$C_{\eta_t} \wr S_{m_t} := C_{\eta_t}^{\Omega_t} \rtimes S_{m_t},$$

where $\Omega_t = \{1,\ldots,m_t\}$; S_{m_t} acts on $C_{\eta_t}^{\Omega_t}$ by permuting the factors. There exist two natural subgroups of $C(\mu)$

$$\mathrm{Aut}(\mu) := \prod_{t=1}^{s} S_{m_t} \quad \text{and} \quad N(\mu) := \prod_{t=1}^{s} C_{\eta_t}^{\Omega_t} \tag{20}$$

as the notation suggests, $\mathrm{Aut}(\mu)$ coincides with the automorphism group of the partition μ. The inclusion $\mathrm{Aut}(\mu) \hookrightarrow C(\mu)$ splits the following the sequence from the right

$$1 \to N(\mu) \to C(\mu) \to \mathrm{Aut}(\mu) \to 1. \tag{21}$$

Viewing a partition μ as a partially ordered[10] set, we define X^{μ} as the self-product of X over the set μ. More precisely,

$$X^{\mu} \cong X^{\ell(\mu)},$$

where $\ell(\mu)$ is the length of the partition μ. The group $C(\mu)$ acts on X^{μ} as follows. The products of cyclic groups $C_{\eta_t}^{\Omega}$ acts trivially on corresponding factors of X^{μ}, while S_{m_t} permutes the factors corresponding to the same part η_t. These actions are compatible with the wreath product.

Given an element $g \in S_n$ in a conjugacy class corresponding to a partition μ, we have the following identifications

$$C(g) \cong C(\mu) \quad \text{and} \quad X^{n,g} \cong X^{\mu},$$

such that the group actions match. In particular, with the notation introduced above the inertia stack can be re-expressed,

$$\mathcal{I}X^{(n)} = \coprod_{\mu} [X^{\mu}/C(\mu)], \tag{22}$$

and by the splitting of (21) we obtain that

$$\mathcal{I}X^{(n)} = \coprod_{\mu} [X^{\mu}/\mathrm{Aut}(\mu)] \times BN(\mu). \tag{23}$$

We thereby define a rigidified version of $\mathcal{I}X^{(n)}$,

$$\bar{\mathcal{I}}X^{(n)} := \coprod_{\mu} [X^{\mu}/\mathrm{Aut}(\mu)].$$

Note, however, that this is not a rigidified inertia stack in the sense of [AGV08, Section 3.3], $\bar{\mathcal{I}}X^{(n)}$ is a further rigidifiction of $\mathcal{I}X^{(n)}$.

Recall that as a graded vector space, the orbifold cohomology is defined as follows

$$H^*_{\mathrm{orb}}(X^{(n)}, \mathbb{Q}) := H^{*-2\mathrm{age}(\mu)}(\mathcal{I}X^{(n)}, \mathbb{Q}).$$

By (22), we therefore get that

$$H^*_{\mathrm{orb}}(X^{(n)}, \mathbb{Q}) = H^{*-2\mathrm{age}(\mu)}(\mathcal{I}X^{(n)}, \mathbb{Q}) = H^{*-2\mathrm{age}(\mu)}(\bar{\mathcal{I}}X^{(n)}, \mathbb{Q}). \tag{24}$$

---

[10] $\mu_i \geq \mu_j, \iff j \geq i$.

2.5. **Invariants.** Let $\overrightarrow{Adm}^{\epsilon}_{g,N}(X^{(n)},\beta)$ be the moduli space obtained from $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ by putting the *standard order*[11] on the fibers over marked points of the source curve. The two moduli spaces are related as follows

$$\coprod_{\underline{\mu}}[\overrightarrow{Adm}^{\epsilon}_{g,N}(X^{(n)},\beta,\underline{\mu})/\prod \mathrm{Aut}(\mu^i)] = \overline{M}^{\epsilon}_{g,N}(X^{(n)},\beta). \tag{25}$$

There exist naturally defined evaluation maps at marked points

$$\mathrm{ev}_i\colon \overrightarrow{Adm}^{\epsilon}_{g,N}(X^{(n)},\beta) \to \coprod_{\mu} X^{\mu}, \quad i=1,\ldots,N.$$

By (20), (22) and (25), we can define evaluation maps

$$\mathrm{ev}_i\colon Adm^{\epsilon}_{g,N}(X^{(n)},\beta) \to \bar{\mathfrak{I}}X^{(n)}, \quad i=1,\ldots,N, \tag{26}$$

as the composition

$$Adm^{\epsilon}_{g,N}(X^{(n)},\beta) = \coprod_{\underline{\mu}}[\overrightarrow{Adm}^{\epsilon}_{g,N}(X^{(n)},\beta,\underline{\mu})/\prod \mathrm{Aut}(\mu^i)] \xrightarrow{\mathrm{ev}_i}$$

$$\xrightarrow{\mathrm{ev}_i} \coprod_{\mu}[X^{\mu}/\mathrm{Aut}(\mu)] = \bar{\mathfrak{I}}X^{(n)}.$$

For universal markings

$$s_i\colon Adm^{\epsilon}_{g,N}(X^{(n)},\beta) \to \mathcal{C}_{g,N}$$

to the universal *target* curve

$$\mathcal{C}_{g,N} \to Adm^{\epsilon}_{g,N}(X^{(n)},\beta),$$

we also define cotangent line bundles

$$\mathcal{L}_i := s_i^*(\omega_{\mathcal{C}_{g,N}/Adm^{\epsilon}_{g,N}(X^{(n)},\beta)}), \quad i=1,\ldots,N,$$

where $\omega_{\mathcal{C}_{g,N}/Adm^{\epsilon}_{g,N}(X^{(n)},\beta)}$ is the universal relative dualising sheaf. We denote

$$\psi_i := \mathrm{c}_1(\mathcal{L}_i).$$

With above structures at hand we can define ϵ-admissible invariants.

Definition 2.16. The *descendent ϵ-admissible invariants* are

$$\langle \tau_{m_1}(\gamma_1),\ldots,\tau_{m_N}(\gamma_N)\rangle^{\epsilon}_{g,\beta} := \int_{[Adm^{\epsilon}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}}} \prod_{i=1}^{i=N} \psi_i^{m_i}\mathrm{ev}_i^*(\gamma_i,),$$

where $\gamma_1,\ldots,\gamma_N \in H^*_{\mathrm{orb}}(X^{(n)})$ and $m_1,\ldots m_N$ are non-negative integers.

2.6. **Relation to other invariants.** We will now explore how ϵ-admissible invariants are related to the invariants associated to the spaces discussed in Section 2.3.

[11] We order the points in a fiber in accordance with their ramification degrees.

2.6.1. *Classes.* Let $\{\delta_1, \dots \delta_{m_S}\}$ be an ordered basis of $H^*(X, \mathbb{Q})$. Let

$$\vec{\mu} = ((\mu_1, \delta_{\ell_1}), \dots, (\mu_k, \delta_{\ell_k}))$$

be a cohomology-weighted partition of n with the standard ordering, i.e.

$$(\mu_i, \delta_{\ell_i}) > (\mu_{i'}, \delta_{\ell_{i'}}),$$

if $\mu_i > \mu_{i'}$, or if $\mu_i = \mu_{i'}$ and $\ell_i > \ell_{i'}$. The underlying partition will be denoted by μ. For each $\vec{\mu}$, we consider a class

$$\delta_{l_1} \otimes \dots \otimes \delta_{l_k} \in H^*(X^\mu, \mathbb{Q}),$$

we then define

$$\lambda(\vec{\mu}) := \overline{\pi}_*(\delta_{l_1} \otimes \dots \otimes \delta_{l_k}) \in H^*_{\mathrm{orb}}(S^{(d)}, \mathbb{Q}),$$

where

$$\overline{\pi} \colon \coprod_\mu X^\mu \to \bar{\mathcal{I}} X^{(n)}$$

is the natural projection. More explicitly, as an element of

$$H^*(X^\mu, \mathbb{Q})^{\mathrm{Aut}(\mu)} \subseteq H^*_{\mathrm{orb}}(X^{(n)}, \mathbb{Q}),$$

the class $\lambda(\vec{\mu})$ is given by the following formula

$$\sum_{h \in \mathrm{Aut}(\mu)} h^*(\delta_{l_1} \otimes \dots \otimes \delta_{l_k}) \in H^*(X^\mu, \mathbb{Q})^{\mathrm{Aut}(\mu)}.$$

The importance of these classes is due to the fact they form a basis of $H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{Q})$, see Proposition 6.1.

2.6.2. *Comparison.* Given weighted partitions

$$\vec{\mu}^i = ((\mu^i_1, \delta^i_1), \dots, (\mu^i_{k_i}, \delta^i_{k_i})), \quad i = 1, \dots, N,$$

the relative GW descendent invariants associated to the moduli space $\overline{M}^\bullet_{\mathsf{m}}(X \times C_{g,N}/\overline{M}_{g,N}, (\gamma, n))$ are usually[12] defined as

$$\int_{[\overline{M}^\bullet_{\mathsf{m}}(X \times C_{g,N}/\overline{M}_{g,N}, (\gamma,n))]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{m_i} \prod_{j=1}^{k_i} \mathrm{ev}^*_{i,j} \delta^i_j,$$

such that the product is ordered according to the standard ordering of weighted partitions and

$$\mathrm{ev}_{i,j} \colon \overline{M}^\bullet_{\mathsf{m}}(X \times C_{g,N}/\overline{M}_{g,N}, (\gamma, n)) \to X, \quad i = 1, \dots, N, j = 1, \dots, k_i,$$

are evaluation maps defined by sending a corresponding point in a fiber over a marked point.

In the case of $\mathcal{K}_{g,N}([X^{(n)}], \beta)$, we define evaluation maps as the composition

$$\mathrm{ev}_i \colon \mathcal{K}_{g,N}([X^{(n)}], \beta) \to \mathcal{I}X^{(n)} \to \bar{\mathcal{I}}X^{(n)}, \quad i = 1, \dots N,$$

where we used (23).

[12]Note that sometimes the factor $1/|\mathrm{Aut}(\vec{\mu})|$ is introduced, in this case we add such factor for all classes defined previously.

The next lemma concludes the comparison initiated in Section 2.3. In what follows, by a ψ-class on $\mathcal{K}_{g,N}([X^{(n)}],\beta)$ we will mean a *coarse* ψ-class. Orbifold ψ-classes are rational multiples of coarse ones.

Lemma 2.17.

$$\langle \tau_{m_1}(\lambda(\vec{\mu}^1)),\dots,\tau_{m_N}(\lambda(\vec{\mu}^N))\rangle^{0}_{g,\beta} = \int_{[\overline{M}^{\bullet}_{\mathsf{m}}(X\times C_{g,N}/\overline{M}_{g,N},(\gamma,n))]^{\mathrm{vir}}} \prod_{i=1}^{N}\psi_i^{m_i}\prod_{j=1}^{k_i}\mathrm{ev}^*_{i,j}\delta^i_j$$

$$\langle \tau_{m_1}(\lambda(\vec{\mu}^1)),\dots,\tau_{m_N}(\lambda(\vec{\mu}^N))\rangle^{-\infty}_{g,\beta} = \int_{[\mathcal{K}_{g,N}([X^{(n)}],\beta)]^{\mathrm{vir}}} \prod_{i=1}^{N}\psi_i^{m_i}\mathrm{ev}_i^*\lambda(\vec{\mu}^i).$$

Proof. In the light of our conventions, it is a straightforward application of projection and pullback-pushforward formulas. □

3. Master space

3.1. **Definition of the master space.** The space $\mathbb{R}_{\leq 0}\cup\{-\infty\}$ of ϵ-stabilities is divided into chambers, inside of which the moduli space $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$ stays the same, and as ϵ crosses a wall between chambers, the moduli space changes discontinuously. Let $\epsilon_0\in\mathbb{R}_{\leq 0}\cup\{-\infty\}$ be a wall and ϵ_+, ϵ_- be some values that are close to ϵ_0 from the left and the right of the wall, respectively. We set

$$d_0 = e^{-1/\epsilon_0} \quad\text{and}\quad \deg(\beta) := \mathsf{m} + \deg(\gamma) = d.$$

From now on, we assume

$$2g-2+N+1/d_0\cdot\deg(\beta)\geq 0$$

and

$$1/d_0\cdot\deg(\beta) > 2,$$

if $(g,N)=(0,0)$.

Definition 3.1. A pre-admissible map $(P,C,f,\mathbf{x})$ is called ϵ_0-*pre-admissible*, if

(**i**) for all points $p\in\mathcal{C}$,

$$\mathrm{mult}_p(\mathsf{br}(f)) + \deg(f^*L_p)\leq e^{-1/\epsilon_0};$$

(**ii**) for all rational tails $T\subseteq\mathcal{C}$,

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^*L_{|T})\geq e^{-1/\epsilon_0};$$

(**iii**) for all rational bridges $B\subseteq\mathcal{C}$,

$$\deg(\mathsf{br}(f)_{|B}) + \deg(f^*L_{|B}) > 0;$$

We denote by $\mathfrak{Adm}^{\epsilon_0}_{g,N}(X^{(n)},\beta)$ the moduli space of ϵ_0-pre-admissible maps. Let $\mathfrak{M}^{ss}_{g,N,d}$ be the moduli space of weighted semistable curves defined in [Zho22, Definition 2.1.2]. There exists a map

$$\mathfrak{Adm}^{\epsilon_0}_{g,N}(X^{(n)},\beta)\to\mathfrak{M}^{ss}_{g,N,d}$$
$$(P,C,f,\mathbf{x})\mapsto(C,\mathbf{x},\underline{d}),$$

where the value of $\underline{d}$ on a subcurve $C' \subseteq C$ is defined as follows

$$\underline{d}(C') = \deg(\mathsf{br}(f_{|C'})) + \deg(f^*L_{|C'}).$$

By $M\mathfrak{Adm}^{\epsilon_0}_{g,N}(X^{(n)},\beta)$ we denote the moduli space of ϵ_0-pre-admissible maps with calibrated tails, defined as the fiber product

$$M\mathfrak{Adm}^{\epsilon_0}_{g,N}(X^{(n)},\beta) = \mathfrak{Adm}^{\epsilon_0}_{g,N}(X^{(n)},\beta) \times_{\mathfrak{M}^{ss}_{g,N,d}} M\widetilde{\mathfrak{M}}_{g,N,d},$$

where $M\widetilde{\mathfrak{M}}_{g,N,d}$ is the moduli space of curves with calibrated tails introduced in [Zho22, Definition 2.8.2], which is a projective bundle over the moduli spaces of curves with entangled tails over a moduli space of curves with entangled tails, $\widetilde{\mathfrak{M}}_{g,N,d}$, see [Zho22, Section 2.2]. The latter is constructed by induction on the integer k by a sequence of blow-ups at the loci of curves with at least k rational tails of degree d_0.

Definition 3.2. Given a pre-admissible map $(P, C, f, \mathbf{x})$. We say a rational tail $T \subseteq (C, \mathbf{x})$ is of degree d_0, if

$$\deg(\mathsf{br}(f)_{|T}) + \deg(f^*L_{|T}) = d_0.$$

We say a branching point $p \in C$ is of degree d_0, if

$$\mathrm{mult}_p(\mathsf{br}(f)) + \deg(f^*L_p) = d_0.$$

Definition 3.3. We say a rational tail $T \subseteq (C, \mathbf{x})$ is *constant*, if

$$|\mathrm{Aut}((P, C, f, \mathbf{x})_{|T})| = \infty.$$

In other words, a rational tail $T \subseteq (C, \mathbf{x})$ is constant, if at each connected component of $P_{|T}$, the map $f_{C|T}$ is equal to

$$z^{\underline{n}} \colon (\sqcup^k \mathbb{P}^1) \sqcup_0 P' \to \mathbb{P}^1$$

up to a change of coordinates. The notation is the same as in (9).

Definition 3.4. A B-family family of ϵ_0-pre-admissible maps with calibrated tails

$$(P, C, \mathbf{x}, f, e, \mathcal{L}, v_1, v_2)$$

is ϵ_0-admissible, if

1) any constant tail is an entangled tail;
2) if a geometric fiber C_b of C over a point $b \in B$ has rational tails of degree d_0, then those rational tails contain all the degree-d_0 branching points;
3) if $v_1(b) = 0$, then $(P, C, \mathbf{x}, f)_b$ is ϵ_--admissible;
4) if $v_2(b) = 0$, then $(P, C, \mathbf{x}, f)_b$ is ϵ_+-admissible.

Let

$$MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta) \subset M\mathfrak{Adm}^{\epsilon_0}_{g,N}(X^{(n)},\beta)$$

denote the moduli space of genus-g, N-marked, ϵ_0-admissable maps with calibrated tails.

3.2. **Obstruction theory.** The obstruction theory of $MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)$ is defined in the same way as the one of $Adm^{\epsilon}_{g,N}(X^{(n)},\beta)$. There exists a complex $E^\bullet$, which defines a perfect obstruction theory relative to $\mathfrak{M}_{h,N'} \times M\widetilde{\mathfrak{M}}_{g,N,d}$,

$$\phi : E^\bullet \to \mathbb{L}_{MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)/\mathfrak{M}_{h,N'}\times M\widetilde{\mathfrak{M}}_{g,N}}.$$

The fact that it is indeed a perfect obstruction theory is a relative version of [GV05, Section 2.8].

3.3. **Properness.**

Theorem 3.5. *The moduli space $MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)$ is a quasi-separated Deligne-Mumford stack of finite type.*

Proof. The proof is the same as in [Zho22, Proposition 4.1.11]. □

We now deal with properness of $MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)$ with the help of valuative criteria of properness. We will follow the strategy of [Zho22, Section 5]. Namely, given a discrete valuation ring R with the fraction field K. Let

$$\xi^* = (P^*, C^*, \mathbf{x}^*, f^*, e^*, \mathcal{L}^*, v_1^*, v_2^*) \in MAdm^{\epsilon_0}_{g,N}(X^{(n)},\beta)(K)$$

be a family of ϵ_0-admissable map with calibrated tails over $\operatorname{Spec} K$. We will classify all the possible ϵ_0-pre-admissible extensions of ξ^* to R up to a finite base change. There will be a unique one which is ϵ_0-admissible.

3.3.1. $(g,N,d) \neq (0,1,d_0)$. Assume $(g,N,d) \neq (0,1,d_0)$ and η^* does not have rational tails of degree d_0. Let

$$\eta^* = (P^*, C^*, \mathbf{x}^*, f^*) \text{ and } \lambda^* = (e^*, \mathcal{L}^*, v_1^*, v_2^*)$$

be the underlying pre-admissable map and the calibration data of η^*, respectively. Let

$$\xi_- = (\eta_-, \lambda_-) \in M\mathfrak{M}^{\epsilon_0}_{g,N}(X^{(n)},\beta)(R')$$

be family over degree-r extension R' of R, where the ϵ_--pre-admissible map

$$\eta_- = (P_-, C_-, \mathbf{x}_-, f_-).$$

is constructed according to the same procedure as (10). More precisely, we apply modifications of *Step 2* with respect to ϵ_--stability; we leave the degree-d_0 branching points which are limits of degree-d_0 branching points of the generic fiber untouched. The family η_- is the one closest to being ϵ_--admissible limit of η^*. The calibration λ_- is given by a unique extension of λ^* to the curve C_-, which exists by [Zho22, Lemma 5.1.1 (1)].

Let

$$\{p_i \mid i = 1, \ldots, \ell\}$$

be an ordered set, consisting of nodes of degree-d_0 rational tails and degree-d_0 branching points of the central fiber

$$p_i \in C_{-|\operatorname{Spec}\mathbb{C}} \subset C_-.$$

We now define

$$b_i \in \mathbb{R}_{>0} \cup \{\infty\}, \ i = 1, \dots, \ell$$

as follows. Set b_i to be ∞, if p_i is a degree-d_0 branching point. If p_i is a node of a rational tail, then we define b_i via the singularity type of C_- at p_i. Namely, if the family C_- has a A_{b-1}-type singularity at p_i, we set $b_i = b/r$.

We now classify all ϵ_0-pre-admissible modifications of ξ_- in the sense of Definition 2.12. By [Zho22, Lemma 5.1.1 (1)], it is enough to classify the modifications of η_-.

All the modifications of η_- are given by blow-ups and blow-downs around the points p_i after taking base-changes with respect to finite extensions of R. The result of these modifications will be a change of singularity type of η_- around p_i. Hence the classification will depend on an array of rational numbers

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Q}^\ell_{\geq 0},$$

the nominator of which keeps track of the singularity type around p_i, while the denominator is responsible for the degree of an extension of R. The precise statement is the following lemma.

Lemma 3.6. *For each $\underline{\alpha} = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Q}^\ell_{\geq 0}$, such that $\underline{\alpha} \leq \underline{b}$, there exists a ϵ_0-pre-admissible modification $\eta_{\underline{\alpha}}$ of η_- with following properties*

- *up to a finite base change,*
$$\eta_{\underline{\alpha}} \cong \eta_{\underline{\alpha}'} \iff \underline{\alpha} = \underline{\alpha}';$$
- *given a ϵ_0-pre-admissible modification $\tilde{\eta}$ of η_-, then there exists $\underline{\alpha}$ such that*
$$\tilde{\eta} \cong \eta_{\underline{\alpha}}$$
up to a finite base change.
- *the central fiber of $\eta_{\underline{\alpha}}$ is ϵ_--stable, if and only if $\underline{a} = \underline{b}$.*

Proof. Let us choose a fractional presentation of $(a_1, \dots, a_\ell)$ with a common denominator

$$(a_1, \dots, a_\ell) = (\frac{a'_1}{rr'}, \dots, \frac{a'_\ell}{rr'}).$$

Take a base change of η_- with respect to a degree-r' extension of R'. We then construct $\eta_{\underline{\alpha}}$ by applying modifications η_- around each point p_i, the result of which is a family

$$\eta_{\alpha_i} = (P_{\alpha_i}, C_{\alpha_i}, \mathbf{x}_{\alpha_i}, f_{\alpha_i}),$$

which is constructed as follows.

Case 1. If p_i is a node of a degree-d_0 rational tail and $a_i \neq 0$, we blow-up C_- at p_i,

$$\mathrm{Bl}_{p_i}(C_-) \to C_-.$$

The map f_{C_-} then defines a rational map

$$f_{C_-}\colon P_- \dashrightarrow \mathrm{Bl}_{p_i}(C_-).$$

We can eliminate the indeterminacies of the map above by blowing-up P_- to obtain an everywhere-defined map

$$f_{\mathrm{Bl}_{p_i}(C_-)}\colon \widetilde{P}_- \to \mathrm{Bl}_{p_i}(C_-),$$

we take a minimal blow-up with such property. The exceptional curve E of $\mathrm{Bl}_{p_i}(C_-)$ is a chain of $r'b_i$ rational curves. The exceptional curve of $\widetilde{P}_-$ is a disjoint union $\sqcup E_j$, where each E_j is a chain of rb_i rational curves mapping to E without contracted components. We blow-down all the rational curves but the a_i'-th ones in both E and E_j for all j. The resulting families are C_{α_i} and P_{α_i}, respectively. The family C_{α_i} has an $A_{\alpha_i'-1}$-type singularity at p_i. The marking $\mathbf{x}_-$ clearly extends to a marking $\mathbf{x}_{\alpha_i}$ of C_{α_i}. The map $f_{\mathrm{Bl}_{p_i}(C_-)}$ descends to a map

$$f_{C_{\alpha_i}}\colon P_{\alpha_i} \to C_{\alpha_i}.$$

The map $f_{-,X}$ is carried along with all those modifications to a map

$$f_{\alpha_i,X}\colon P_{\alpha_i} \to X,$$

because exceptional divisors are of degree 0 with respect to $f_{-,X}$, hence the contraction of curves in the exceptional divisors does not introduce any indeterminacies. We thereby constructed the family η_{α_i}.

Case 2. Assume now that p_i is a node of a degree-d_0 rational tail, but $a_i = 0$. The family C_{α_i} is then given by the contraction of that degree-d_0 rational tail, it is smooth at p_i. The marking $\mathbf{x}_-$ extends to a marking $\mathbf{x}_{\alpha_i}$ of C_{α_i}. The family P_{α_i} is set to be equal to P_-. The map f_{α_i} is the composition of the contraction and f_-.

Case 3. If p_i is a branching point, we blow-up C_- inductively a_i' times, starting with a blow-up at p_i and then continuing with a blow-up at a point of the exceptional curve of the previous blow-up. We then blow-down all rational curves in the exceptional divisor but the last one. The resulting family is C_{α_i}, it has an $A_{a_i'}$-type singularity at p_i. The marking $\mathbf{x}_-$ extends to the marking $\mathbf{x}_{\alpha_i}$ of C_{α_i}. The map f_{C_-} then defines a rational map

$$f_{C_-}\colon P_- \dashrightarrow C_{\alpha_i}.$$

We set

$$f_{C_{\alpha_i}}\colon P_{\alpha_i} \to C_{\alpha_i}$$

to be the minimal resolution of indeterminacies of the rational map above. More specifically, P_{α_i} is obtained by consequently blowing-up P_- and blowing-down all the rational curves in the exceptional divisor but the last one. The map $f_{-,X}$ is carried along, as in *Case 1.*

It is not difficult to verify that the central fiber of $\eta_{\underline{\alpha}}$ is indeed ϵ_0-pre-admissible. Up to a finite base change, the resulting family is uniquely determined by $\underline{\alpha} = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Q}^{\ell}_{\geq 0}$ and independent of its fractional presentation, because of the singularity types at points p_i and the degree of an extension R.

Given now an arbitrary ϵ_0-pre-admissible modification

$$\eta = (P, C, \mathbf{x}, f)$$

of η_-. Possibly after a finite base change, there exists a modification

$$\tilde{\eta} = (\widetilde{P}, \widetilde{C}, \tilde{\mathbf{x}}, \tilde{f})$$

that dominates both η and η_- in the sense of (13). We take a minimal modification with such property. The family $\tilde{\eta}$ is given by blow-ups of C_- and P_-. By the assumption of minimality and ϵ_0-pre-admissibility of η, these are blow-ups at p_i. By ϵ_0-pre-admissibility of η, the projections

$$\widetilde{C} \to C \quad \text{and} \quad \widetilde{P} \to P$$

are given by contraction of degree-d_0 rational tails or rational components which do not satisfy ϵ_0-pre-admissibility. These are exactly the operations described in *Steps 1,2,3* of the proof. Uniqueness of of maps follows from seperatedness of the moduli space of maps to a fixed target. Hence we obtain that

$$\eta \cong \eta_{\underline{\alpha}}$$

for some $\underline{\alpha} = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Q}^{\ell}_{\geq 0}$, where $\underline{\alpha}$ is determined by the singularity types of η at points p_i. □

3.3.2. $(g, N, d) = (0, 1, d_0)$. We now assume that $(g, N, d) = (0, 1, d_0)$. In this case the calibration bundle is the relative cotangent bundle along the unique marking. Moreover, there is no entanglement. Given a family of pre-admissible maps $(P, C, \mathbf{x}, f)$, we will denote the calibration bundle by M_C. Therefore the calibration data λ is given just by a rational section s of M_C.

Let

$$\xi_- = (\eta_-, \lambda_-) \in M\mathfrak{Adm}^{\epsilon_0}_{0,1}(X^{(n)}, \beta)(R')$$

be the family over degree-r extension R' of R, such that η_- is again given by (10), if there is no degree-d_0 branching point in η^*. Otherwise, let η_- be any pre-admissible limit. The calibration data λ_- is given by a rational section s_- which is an extension of the section s^* of M_{C^*} to M_{C_-}.

Given a modification $\widetilde{\eta}$ of η_- over a degree-r' extension of R', the section s^* extends to a rational section $\tilde{s}$ of $M_{\widetilde{C}}$.

Definition 3.7. The order of the modification $\widetilde{\eta}$ is defined to be $\mathrm{ord}(\tilde{s})/r$ at the closed point of $\operatorname{Spec} R'$.

We set $b = \mathrm{ord}(s_-)/r$, of there is no degree-d_0 branching point in the generic fiber of η^*. Otherwise set $b = -\infty$.

Lemma 3.8. *For each $\alpha \in \mathbb{Q}$, such that $\alpha \leq b$, there exists a ϵ_0-pre-admissible modification η_α of η_- of order α with following properties*

- *up to a finite base change,*
$$\eta_\alpha \cong \eta_{\alpha'} \iff \alpha = \alpha';$$
- *given a ϵ_0-pre-admissible modification $\tilde{\eta}$ of η_-, then there exists α such that*
$$\tilde{\eta} \cong \eta_\alpha$$
up to a finite base change.
- *the central fiber of η_α is ϵ_--stable, if and only if $\alpha = b$.*

Proof. Assume η^* does not have a degree-d_0 branching point. We choose a fractional presentation $\alpha = \alpha'/rr'$. We take a base change of η_- with respect to a degree-r' extension of R'. We blow-up consequently α' times the central fiber at the unique marking. We then blow-down all rational curves in the exceptional divisor but the last one. The resulting family with markings is $(C_\alpha, \mathbf{x}_\alpha)$. We do the same with the family P_- at the points in the fiber over the marked point to a obtain the family P_α and the map

$$f_{P_\alpha} \colon P_\alpha \to \widetilde{C},$$

the map $f_{-,X}$ is carried along with blow-ups and blow-downs. The resulting family of ϵ_0-pre-admissible maps is of order α.

Assume now that the generic fiber has a degree-d_0 branching point. We take a base change of η_- with respect to a degree-r' extension of R'. After choosing some trivialisation of M_{C^*}, we have that

$$s^* = \pi^{r'\alpha_-} \in K',$$

where α_- is the order of vanishing of s_- before the base-change and π is some uniformiser of R'. Because of automorphisms of $\mathbb{P}^1$ which fix a branching point and a marked point, we have an isomorphisms of ϵ_0-pre-admissible maps with calibrated tails,

$$(\eta^*, s^*) \cong (\eta^*, \pi^c \cdot s^*)$$

for arbitrary $c \in \mathbb{Z}$. Hence we can multiply the section s_- with $\pi^{\alpha' - r'\alpha_-}$ to obtain a modification of order α.

The fact that these modifications classify all possible modifications follow from the same arguments as in the case $(g, N, d) \neq (0, 1, d_0)$. $\square$

Theorem 3.9. *The moduli space $MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)$ is proper.*

Proof. With the classifications of modifications of η_- of Lemma 3.6 and Lemma 3.8, the proof of properness follows from the same arguments as in [Zho22, Proposition 5.0.1].

$\square$

4. Wall-crossing

4.1. **Graph space.** For a class $\beta = (\beta, \mathsf{m}) \in H_2(X,\mathbb{Z}) \oplus \mathbb{Z}$ consider now

$$\overline{M}^{\bullet}_{\mathsf{m}}(X \times \mathbb{P}^1/X_\infty, (\gamma, n)),$$

the space of relative stable maps with disconnected domains of degree (γ, n) to $X \times \mathbb{P}^1$ relative to

$$X_\infty := X \times \{\infty\} \subset X \times \mathbb{P}^1.$$

One should refer to this moduli space as *graph space*, as it will play the same role, as the graph space in the quasimap wall-crossing. Note that we fix the degree of the branching divisor m instead of the genus h, the two are determined by Lemma 2.9.

There is a standard $\mathbb{C}^*$-action on $\mathbb{P}^1$ given by

$$t[x,y] = [tx, y], \ t \in \mathbb{C}^*,$$

which induces a $\mathbb{C}^*$-action on $\overline{M}^{\bullet}_{\mathsf{m}}(X \times \mathbb{P}^1/X_\infty, (\gamma, n))$. Let

$$F_\beta \subset \overline{M}^{\bullet}_{\mathsf{m}}(X \times \mathbb{P}^1/X_\infty, (\gamma, n))^{\mathbb{C}^*}$$

be the distinguished $\mathbb{C}^*$-fixed component consisting of maps to $X \times \mathbb{P}^1$ (no expanded degenerations). Said differently, F_β is the moduli space of maps, which are admissible over $\infty \in \mathbb{P}^1$ and whose degree lies entirely over $0 \in \mathbb{P}^1$ in the form of a branching point. Other $\mathbb{C}^*$-fixed components admit exactly the same description as in the case of quasimaps in [Nes21a, Section 6.1].

The virtual fundamental class of F_β,

$$[F_\beta]^{\mathrm{vir}} \in A_*(F_\beta),$$

is defined via the fixed part of the perfect obstruction theory of

$$\overline{M}^{\bullet}_{\mathsf{m}}(X \times \mathbb{P}^1/X_\infty, (\gamma, n)).$$

The virtual normal bundle $N^{\mathrm{vir}}_{F_\beta}$ is defined by the moving part of the obstruction theory. There exists an evaluation map

$$\mathsf{ev}\colon F_\beta \to \bar{\mathfrak{I}}X^{(n)}$$

defined in the same way as (26).

Definition 4.1. We define an I-function to be

$$I(q,z) = 1 + \sum_{\beta \neq 0} q^\beta \mathsf{ev}_* \left(\frac{[F_\beta]}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta})} \right) \in H^*_{\mathrm{orb}}(X^{(n)})[z^{\pm}] \otimes_{\mathbb{Q}} \mathbb{Q}[\![q^\beta]\!].$$

Let

$$\mu(z) \in H^*_{\mathrm{orb}}(X^{(n)})[z] \otimes_{\mathbb{Q}} \mathbb{Q}[\![q^\beta]\!]$$

be the truncation $[zI(q,z) - z]_+$ by taking only non-negative powers of z. Let

$$\mu_\beta(z) \in H^*_{\mathrm{orb}}(X^{(n)})[z]$$

be the coefficient of $\mu(z)$ at q^β.

For later, it is also convenient to define

$$\mathcal{I}_\beta := \frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta})} \in A^*(F_\beta)[z^{\pm}].$$

4.2. **Wall-crossing formula.** From now on, we assume that

$$2g - 2 + n + 1/d_0 \deg(\beta) > 0,$$

for $(g, N, d_0) = (0, 1, d_0)$ we refer to [Zho22, Section 6.4]. There exists a natural $\mathbb{C}^*$-action on the master space $MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)$ given by

$$t \cdot (P, C, \mathbf{x}, f, e, \mathcal{L}, v_1, v_2) = (P, C, \mathbf{x}, f, e, \mathcal{L}, t \cdot v_1, v_2), \quad t \in \mathbb{C}^*.$$

By arguments presented in [Zho22, Section 6], the fixed locus admits the following expression

$$MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)^{\mathbb{C}^*} = F_+ \sqcup F_- \sqcup \coprod_{\vec{\beta}} F_{\vec{\beta}},$$

we will now explain the meaning of each term in the union above, giving a description of virtual fundamental classes and virtual normal bundles.

4.2.1. F_+. This is a simplest component,

$$F_+ = Adm^{\epsilon_+}_{g,N}(X^{(n)}, \beta), \quad N^{\mathrm{vir}}_{F_+} = \mathbb{M}^\vee_+,$$

where $\mathbb{M}^\vee_+$ is the dual of the calibration bundle $\mathbb{M}_+$ on $Adm^{\epsilon_+}_{g,N}(X^{(n)}, \beta)$, with a trivial $\mathbb{C}^*$-action of weight -1, cf. [Zho22]. The obstruction theories also match, therefore

$$[F_+]^{\mathrm{vir}} = [Adm^{\epsilon_+}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}}$$

with respect to the identification above.

4.2.2. F_-. We define

$$\widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta) := Adm^{\epsilon_-}_{g,N}(X^{(n)}, \beta) \times_{\mathfrak{M}_{g,N,d}} \widetilde{\mathfrak{M}}_{g,N,d},$$

then

$$F_- = \widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta), \quad N^{\mathrm{vir}}_{F_-} = \mathbb{M}_-,$$

where, as previously, $\mathbb{M}_-$ is the calibration bundle on $\widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta)$ with trivial $\mathbb{C}^*$-action of weight 1. The obstruction theories also match and

$$p_*[\widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}} = [Adm^{\epsilon_-}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}},$$

where

$$p \colon \widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta) \to Adm^{\epsilon_-}_{g,N}(X^{(n)}, \beta)$$

is the natural projection.

4.2.3. $F_{\vec{\beta}}$. These are the wall-crossing components, which will be responsible for wall-crossing formulas. Let

$$\overrightarrow{\beta} = (\beta', \beta_1, \dots, \beta_k)$$

be a $k+1$-tuple of classes in $H_2(X,\mathbb{Z}) \oplus \mathbb{Z}$, such that $\beta = \beta' + \beta_1 + \dots + \beta_k$ and $\deg(\beta_i) = d_0$. Then a component $F_{\vec{\beta}}$ is defined as follows

$$\begin{aligned} F_{\vec{\beta}} = \{\xi \mid \xi \text{ has exactly } k \text{ entangled tails,} \\ \text{which are all fixed tails, of degree } \beta_1, \dots, \beta_k\}. \end{aligned}$$

Let

$$\mathcal{E}_i \underset{p_i}{\rightleftarrows} F_{\vec{\beta}} \qquad i = 1, \dots, k,$$

be the universal i-th entangled rational tail with the universal marking p_i given by the node. We define $\psi(\mathcal{E}_i)$ to be the ψ-class associated to the marking p_i. Let

$$\widetilde{\mathrm{gl}}_k \colon \widetilde{\mathfrak{M}}_{g,N+k,d-kd_0} \times (\mathfrak{M}_{0,1,d_0})^k \to \widetilde{\mathfrak{M}}_{g,N,d}$$

be the gluing morphism, cf. [Zho22, Section 2.4]. Let

$$\mathfrak{D}_i \subset \widetilde{\mathfrak{M}}_{g,N,d}$$

be a divisor defined as the closure of the locus of curves with exactly $i+1$ entangled tails. Finally, let

$$Y \to \widetilde{Adm}^{\epsilon_-}_{g,N}(X^{(n)}, \beta')$$

be the stack of k-roots of $\mathbb{M}^{\vee}_{-}$.

Proposition 4.2. *There exists a canonical isomorphism*

$$\widetilde{\mathrm{gl}}_k^* F_{\vec{\beta}} \cong Y \times_{(\bar{\mathcal{I}}X^{(n)})^k} \prod_{i=1}^{i=k} F_{\beta_i}.$$

With respect to the identification above we have

$$\begin{aligned} [\widetilde{\mathrm{gl}}_k^* F_{\vec{\beta}}]^{\mathrm{vir}} =& [Y]^{\mathrm{vir}} \times_{(\bar{\mathcal{I}}X^{(n)})^k} \prod_{i=1}^{i=k} [F_{\beta_i}]^{\mathrm{vir}}, \\ \frac{1}{e_{\mathbb{C}^*}(\widetilde{\mathrm{gl}}_k^* N^{\mathrm{vir}}_{F_{\vec{\beta}}})} =& \frac{\prod_{i=1}^k (z/k + \psi(\mathcal{E}_i))}{-z/k - \psi(\mathcal{E}_1) - \psi_{n+1} - \sum_{i=k}^{\infty} \mathfrak{D}_i} \cdot \prod_{i=1}^{k} \mathcal{I}_{\beta_i}(z/k + \psi(\mathcal{E}_i)). \end{aligned}$$

Proof. See [Zho22, Lemma 6.5.6]. □

Theorem 4.3. *Assuming $2g-2+N+1/d_0 \cdot \deg(\beta) > 0$, we have*

$$\begin{aligned} &\langle \tau_{m_1}(\gamma_1), \dots, \tau_{m_n}(\gamma_N) \rangle^{\epsilon_+}_{g,\beta} - \langle \tau_{m_1}(\gamma_1), \dots, \tau_{m_n}(\gamma_N) \rangle^{\epsilon_-}_{g,\beta} \\ &= \sum_{k \geq 1} \sum_{\vec{\beta}} \frac{1}{k!} \int_{[Adm^{\epsilon_-}_{g,N+k}(X^{(n)},\beta')]^{\mathrm{vir}}} \prod_{i=1}^{i=N} \psi_i^{m_i} \mathrm{ev}_i^*(\gamma_i) \cdot \prod_{a=1}^{a=k} \mathrm{ev}^*_{n+a} \mu_{\beta_a}(z)|_{z=-\psi_{n+a}} \end{aligned}$$

where $\vec{\beta}$ runs through all the $(k+1)$-tuples of effective curve classes

$$\vec{\beta} = (\beta', \beta_1, \dots, \beta_k),$$

such that $\beta = \beta' + \beta_1 + \dots + \beta_k$ and $\deg(\beta_i) = d_0$ for all $i = 1, \dots, k$.

Sketch of Proof. We will just explain the master-space technique. For all the details we refer to [Zho22, Section 6]. By the virtual localisation formula we obtain

$$\begin{aligned}&[MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}} \\ &\quad = \left(\sum \iota_{F_\star *}\left(\frac{[F_\star]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\star})}\right)\right) \in A^{\mathbb{C}^*}_*(MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)) \otimes_{\mathbb{Q}[z]} \mathbb{Q}(z),\end{aligned}$$

where $F_\star$'s are the components of the $\mathbb{C}^*$-fixed locus of $MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)$. Let

$$\alpha = \prod_{i=1}^{i=N} \psi_i^{m_i} \mathrm{ev}_i^*(\gamma_i) \in A^*(MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta))$$

be the class corresponding to decedent insertions. After taking the residue[13] at $z = 0$ of the above formula, capping with α and taking the degree of the class, we obtain the following equality

$$\begin{aligned}&\int_{[Adm^{\epsilon_+}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}}} \alpha - \int_{[Adm^{\epsilon_-}_{g,N}(X^{(n)},\beta)]^{\mathrm{vir}}} \alpha \\ &\qquad\qquad = \deg\left(\alpha \cap \mathrm{Res}_{z=0}\left(\sum \iota_{F_\beta *}\left(\frac{[F_\beta]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta})}\right)\right)\right),\end{aligned}$$

where we used that there is no $1/z$-term in the decomposition of the class

$$[MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)]^{\mathrm{vir}} \in A^{\mathbb{C}^*}_*(MAdm^{\epsilon_0}_{g,N}(X^{(n)}, \beta)),$$

and that

$$\frac{1}{e_{\mathbb{C}^*}(\mathbb{M}_\pm)} = \frac{1}{z} + O(1/z^2).$$

The analysis of the residue on the right-hand side presented in [Zho22, Section 7] applies to our case. The resulting formula is the one claimed in the statement of the theorem. $\square$

We define

$$F^\epsilon_g(\mathbf{t}(z)) = \sum_{n=0}^{\infty} \sum_\beta \frac{q^\beta}{N!} \langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle^\epsilon_{g,N,\beta},$$

where $\mathbf{t}(z) \in H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{Q})[z]$ is a generic element, and the unstable terms are set to be zero. By repeatedly applying Theorem 4.3 we obtain.

[13] i.e. by taking the coefficient of $1/z$ of both sides of the equality.

Corollary 4.4. *For all $g \geq 1$ we have*

$$F_g^0(\mathbf{t}(z)) = F_g^{-\infty}(\mathbf{t}(z) + \mu(-z)).$$

For $g = 0$, the same equation holds modulo constant and linear terms in $\mathbf{t}(z)$.

For $g = 0$, the relation is true only moduli linear terms in $\mathbf{t}(z)$, because the moduli space $Adm_{0,1}^{\epsilon_+}(X^{(n)}, \beta)$ is empty, if $e^{1/\epsilon_+} \cdot \deg(\beta) \leq 1$. In particular, Theorem 4.3 does not hold. As for quasimaps, the wall-crossing takes a different form in this case. More specifically, [Nes21a, Theorem 6.6] and [Nes21a, Theorem 6.7] apply verbatim to the case of ϵ-admissible maps.

5. Del Pezzo

In this section we determine the I-function in the case $X = S$ is a del Pezzo surface. Firstly, consider the expansion

$$[zI(q,z) - z]_+ = I_1(q) + (I_0(q) - 1)z + I_{-1}(q)z^2 + I_{-2}(q)z^3 + \dots,$$

we will show that by the dimension constraint the terms I_{-k} vanish for all $k \geq 1$.

For this section we consider $H^*_{\mathrm{orb}}(X^{(n)})$ with its *naive*[14] grading. Let z be of cohomological degree 2 in $H^*_{\mathrm{orb}}(X^{(n)})[z^{\pm}]$. The virtual dimension of $\overline{M}^{\bullet}_{\mathsf{m}}(X \times \mathbb{P}^1/X_\infty, (\gamma, n), \mu)$ is equal to

$$\int_{\mathrm{c}_1(S)} \beta + n + \ell(\mu).$$

Hence, by the virtual localisation, the classes involved in the definition of I-function

$$\mathsf{ev}_* \left(\frac{[F_{\beta,\mu}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})} \right) \in H^*(S^\mu/\mathrm{Aut}(\mu))[z^{\pm}] \subseteq H^*_{\mathrm{orb}}(S^{(n)})[z^{\pm}],$$

have naive cohomological degree equal to

$$-2 \left(\int_{\mathrm{c}_1(S)} \beta + n - \ell(\mu) \right). \tag{27}$$

Since S is a del Pezzo surface, the above quantity is non-positive, which implies that

$$I_0 = 1 \text{ and } I_{-k} = 0$$

for all $-k \geq 1$, because cohomology is non-negatively graded. Moreover, the quantity (27) is zero, if and only if

$$\mu = (1, \dots, 1) \text{ and } \beta = (0, \mathsf{m}).$$

Let us now study $F_{\beta,\mu}$ for these values of μ and β. It is more convenient to put an ordering on fibers over $\infty \in \mathbb{P}^1$, so let $\overrightarrow{F}_{\beta,\mu}$ be the resulting space.

[14]We grade it with the cohomological grading of $H^*(\mathcal{I}S^{(d)}, \mathbb{Q})$.

We will not give a full description of $\overrightarrow{F}_{\beta,\mu}$, even though it is simple. We will only be interested in one type of components of $\overrightarrow{F}_{\beta,\mu}$,

$$\iota_i \colon \overline{M}_{\mathsf{h},p_i} \times S^n \hookrightarrow \overrightarrow{F}_{\beta,\mu}, \tag{28}$$

where $\overline{M}_{\mathsf{h},p_i}$ is the moduli spaces of stable genus-h curve with *one* marking labelled by p_i, $i = 1, \dots N$. The embedding ι_i is constructed as follows. Given a point

$$((C, \mathbf{x}), x_1, \dots, x_n)) \in \overline{M}_{\mathsf{h},p_i} \times S^n,$$

let

$$(\tilde{P}, p_1, \dots, p_n) = \coprod_{i=1}^{i=n} (\mathbb{P}^1, 0) \tag{29}$$

be an ordered disjoint union of $\mathbb{P}^1$ with markings at $0 \in \mathbb{P}^1$. We define a curve P by gluing $(\tilde{P}, p_1, \dots, p_n)$ with (C, p_i) at the marking with the same labelling. We define

$$f_{\mathbb{P}^1} \colon P \to \mathbb{P}^1$$

to be an identity on $\tilde{P}$ and contraction on C. We define

$$f_S \colon P \to S$$

by contracting j-th $\mathbb{P}^1$ in P (with the curve C, if $i = j$) to the point $x_j \in S$. We thereby defined the inclusion

$$\iota_i((C, p), x_1, \dots, x_n)) = (P, \mathbb{P}^1, f_{\mathbb{P}^1} \times f_S),$$

where the map $f_{\mathbb{P}^1}$ is clearly admissible at $\infty \in \mathbb{P}^1$.

By Lemma 2.9,

$$\mathsf{h} = \mathsf{m}/2, \tag{30}$$

in particular, m is even. More generally, any connected component of $\overrightarrow{F}_{\beta,\mu}$ admits a similar description with the difference that there might more markings on possibly disconnected C by which it attaches to $\widetilde{P}$, i.e. P has more nodes. These components are not relevant for our needs, as it will be explained below.

Let us now consider the virtual fundamental classes and the normal bundles of these components $\overline{M}_{\mathsf{h},p_i} \times S^n$. By standard arguments, we obtain that

$$\iota_i^* \frac{[F_{\beta,\mu}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})} = e(\pi_i^* T_S \otimes p^* \mathbb{E}_{\mathsf{h}}^\vee) \cdot \frac{e(\mathbb{E}^\vee z)}{z(z - \psi_1)},$$

where $\pi_i \colon \overline{M}_{\mathsf{h},p_i} \times S^n \to S$ is the projection to i-th factor of S^n and $p \colon \overline{M}_{\mathsf{h},p_i} \times S^n \to \overline{M}_{\mathsf{h},p_i}$ is the projection to $\overline{M}_{\mathsf{h},p_i}$; $\mathbb{E}$ is the Hodge bundle on $\overline{M}_{\mathsf{h},p_i}$

For other components of $\overrightarrow{F}_{\beta,\mu}$, the equivariant Euler classes $e_{\mathbb{C}^*}(N^{\mathrm{vir}})$ acquire factors

$$\frac{1}{z(z - \psi_i)}$$

for each marked point. This makes them irrelevant for purposes of determining the truncation of I-function. We therefore have to determine the following classes

$$\pi_*\left(e(\pi_i^* T_S \otimes p^*\mathbb{E}_{\mathsf{h}}^\vee)\cdot \frac{e(\mathbb{E}^\vee z)}{z(z-\psi_1)}\right) \in H^*(S^n)[z^{\pm}],$$

where $\pi\colon \overline{M}_{h,p_i}\times S^n \to S^n$ is the natural projection, which is identified with evaluation map ev via the inclusion (28).

Let ℓ_1 and ℓ_2 be the Chern roots of $\pi_i^* T_S$. Then we can rewrite the class above as follows

$$\int_{\overline{M}_{\mathsf{h},1}} \frac{\mathbb{E}^\vee(\ell_1)\cdot \mathbb{E}^\vee(\ell_2)\cdot \mathbb{E}^\vee(z)}{z(z-\psi_1)},$$

where

$$\mathbb{E}^\vee(z) := e(\mathbb{E}^\vee z) = \sum_{j=0}^{j=\mathsf{h}} (-1)^{g-j}\lambda_{g-j} z^j,$$

and similarly for $\mathbb{E}^\vee(\ell_1)$ and $\mathbb{E}^\vee(\ell_2)$.

By putting these Hodge integrals into a generating series, we obtain their explicit form. Note that below we sum over the degree m of the branching divisor, which in this case is related to the genus h by (30). The result was kindly communicated to the author by Maximilian Schimpf.

Proposition 5.1 (Maximilian Schimpf)**.**

$$1+\sum_{\mathsf{h}>0} u^{2\mathsf{h}} \int_{\overline{M}_{\mathsf{h},1}} \frac{\mathbb{E}^\vee(\ell_1)\cdot \mathbb{E}^\vee(\ell_2)\cdot \mathbb{E}^\vee(z)}{z(z-\psi_1)} = \left(\frac{\sin(u/2)}{u/2}\right)^{\frac{\ell_1+\ell_2}{z}}$$

Proof. The claim follows from the results of [FP00]. Firstly,

$$\begin{aligned} 1+\sum_{\mathsf{h}>0} u^{2\mathsf{h}} \int_{\overline{M}_{\mathsf{h},1}} &\frac{\mathbb{E}^\vee(\ell_1)\cdot \mathbb{E}^\vee(\ell_2)\cdot \mathbb{E}^\vee(z)}{z(z-\psi_1)} \\ &= 1+\sum_{\mathsf{h}>0} u^{2\mathsf{h}} \int_{\overline{M}_{\mathsf{h},1}} \frac{\mathbb{E}^\vee(\ell_1/z)\cdot \mathbb{E}^\vee(\ell_2/z)\cdot \mathbb{E}^\vee(1)}{1-\psi_1}. \end{aligned}$$

Now let

$$a=\ell_1/z, \quad b=\ell_2/z$$

and

$$F(a,b) = 1+\sum_{\mathsf{h}>0} u^{2\mathsf{h}} \int_{\overline{M}_{\mathsf{h},1}} \frac{\mathbb{E}^\vee(a)\cdot \mathbb{E}^\vee(b)\cdot \mathbb{E}^\vee(1)}{1-\psi_1}.$$

By using virtual localisation on a moduli space of stable maps to $\mathbb{P}^1$, we obtain the following identities

$$\begin{aligned} &F(a,b)\cdot F(-a,-b) = 1; \\ &F(a,b)\cdot F(-a,1-b) = F(0,1). \end{aligned}$$

These identities, with the fact $F(a,b)$ is symmetric in a and b, imply that

$$F(a,b) = F(a,b)^{a+b} \tag{31}$$

for integer values of a and b. Each coefficient of a power of u in $F(a,b)$ is a polynomial in a and b, hence the identity (31) is in fact a functional identity.

By the discussion in [FP02, Section 2.2] and by [FP02, Proposition 2], we obtain that

$$F(0,1) = \frac{\sin(u/2)}{u/2},$$

the claim now follows. □

Using the commutativity of the following diagram

$$\begin{array}{ccc} \overrightarrow{F}_{\beta,\mu} & \xrightarrow{\vec{\mathrm{ev}}} & S^n \\ \downarrow & & \downarrow \overline{\pi} \\ F_{\beta,\mu} & \xrightarrow{\mathrm{ev}} & [S^{(n)}] \end{array}$$

and Proposition 5.1, we obtain

$$I_1(q) = \log\left(\frac{\sin(u/2)}{u/2}\right) \cdot \frac{1}{d-1!}\overline{\pi}_*(\mathrm{c}_1(S) \otimes \cdots \otimes 1). \tag{32}$$

For $2g-2+N \geq 0$ we define

$$\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma} := \sum_k \langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,(\gamma,\mathsf{m})} u^{\mathsf{m}},$$

setting invariants corresponding to unstable values of g,N and β to zero. By repeatedly applying Theorem 4.3, we obtain that

$$\langle \gamma_1, \dots, \gamma_N \rangle^{0}_{g,\beta} = \sum_{k\geq 1} \frac{1}{k!} \left\langle \gamma_1, \dots, \gamma_N, \underbrace{I_1(q), \dots, I_1(q)}_{k} \right\rangle^{-\infty}_{g,\beta}.$$

Applying the divisor equation[15] and (32), we get following corollary.

Corollary 5.2. *Assuming* $2g-2+N \geq 0$,

$$\langle \gamma_1, \dots, \gamma_N \rangle^{0}_{g,\gamma} = \left(\frac{\sin(u/2)}{u/2}\right)^{\gamma\cdot \mathrm{c}_1(S)} \cdot \langle \gamma_1, \dots, \gamma_N \rangle^{-\infty}_{g,\gamma}.$$

6. Crepant resolution conjecture

To a cohomology-weighted partition

$$\vec{\mu} = ((\mu_1, \delta_{\ell_1}), \dots, (\mu_k, \delta_{\ell_k}))$$

we can also associate a class in $H^*(S^{[n]}, \mathbb{Q})$, using Nakajima operators,

$$\theta(\vec{\mu}) := \frac{1}{\prod_{i=1}^k \mu_i} P_{\delta_{\ell_1}}[\mu_1] \cdots P_{\delta_{\ell_k}}[\mu_k] \cdot 1 \in H^*(S^{[n]}, \mathbb{Q}),$$

[15]One can readily verify that an appropriate form of the divisor equation holds for classes in $H^*(S^{(d)}, \mathbb{Q}) \subseteq H^*_{\mathrm{orb}}(S^{(d)}, \mathbb{Q})$.

where operators are ordered according to the standard ordering (see Subsection 2.6.1). For more details on these classes, we refer to [Nak99, Chapter 8], or to [Obe18, Section 0.2] in a context more relevant to us.

Proposition 6.1. *There exists a graded isomorphism of vector spaces*

$$L\colon H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{C}) \simeq H^*(S^{[n]}, \mathbb{C}),$$

$$L(\lambda(\vec{\mu})) = (-i)^{\mathrm{age}(\mu)}\theta(\vec{\mu}).$$

Proof. See [FG03, Proposition 3.5]. □

Remark 6.2. The peculiar choice of the identification with a factor $(-i)^{\mathrm{age}(\mu)}$ is justified by crepant resolution conjecture - this factor makes the invariants match on the nose. See the next section for more details.

6.1. **Quasimaps and admissible covers.** From now one we assume that $2g-2+N \geq 0$. Using [Nes21a, Corollary 3.13], we obtain an identification

$$H_2(S^{[n]}, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \oplus \mathbb{Z}. \tag{33}$$

In the language of (quasi-)maps, it corresponds to association of the Chern character to the graph of a (quasi-)map . Given classes $\gamma_i \in H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{C})$, $i = 1, \dots N$, and a class

$$(\gamma, \mathsf{m}) \in H_2(S, \mathbb{Z}) \oplus \mathbb{Z},$$

for $\epsilon \in \mathbb{R}_{>0} \cup \{0^+, \infty\}$ we set

$$\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,(\gamma,\mathsf{m})} := {}^{\sharp}\langle L(\gamma_1), \dots, L(\gamma_N) \rangle^{\epsilon}_{g,(\gamma,\mathsf{m})} \in \mathbb{C},$$

the invariants on the right are defined in [Nes21a, Section 5.3] and L is defined in Proposition 6.1. We set

$$\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma} := \sum_{\mathsf{m}} \langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,(\gamma,\mathsf{m})} y^{\mathsf{m}}.$$

For $\epsilon = 0^+$, these are the relative PT invariants of the relative geometry $S \times C_{g,N} \to \overline{M}_{g,N}$. The summation over m with respect to the identification (33) corresponds to the summation over ch_3 of a subscheme.

Using wall-crossings, we will now show the compatibility of PT/GW and $\mathsf{C.R.C.}$ Let us firstly spell out our conventions.

- We sum over the degree of the branching divisor instead of the genus of the source curve. Assuming γ_i's are homogenous with respect to the age, the genus h and the degree m are related by Lemma 2.9,

  $$2\mathsf{h} - 2 = -2n + \mathsf{m} + \sum \mathrm{age}(\gamma_i).$$

  For $\epsilon \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$, let

  $${}^{\prime}\langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,\gamma} := \sum_{\mathsf{h}} \langle \gamma_1, \dots, \gamma_N \rangle^{\epsilon}_{g,(\gamma,\mathsf{h})} u^{2\mathsf{h}-2}$$

be generating series where the summation is taken over genus instead. Then two two generating series are are related as follows

$$'\langle\gamma_1,\dots,\gamma_N\rangle^{\epsilon}_{g,\gamma}=u^{2n-\sum \mathrm{age}(\gamma_i)}\cdot\langle\gamma_1,\dots,\gamma_N\rangle^{\epsilon}_{g,\gamma}.$$

- We sum over Chern character ch_3 instead of Euler characteristics χ. For $\epsilon\in\mathbb{R}_{>0}\cup\{0^+,\infty\}$, let

$$'\langle\gamma_1,\dots,\gamma_N\rangle^{\epsilon}_{g,\gamma}:=\sum_{\chi}{}^{\sharp}\langle\gamma_1,\dots,\gamma_N\rangle^{\epsilon}_{g,(\gamma,\chi)}y^{\chi}$$

be the generating series where the summation is taken over Euler characteristics instead. Then by Hirzebruch–Riemann–Roch, theorem the two generating series are related as follows

$$'\langle\gamma_1,\dots,\gamma_N\rangle^{\epsilon}_{g,\gamma}=y^{(g-1)n}\cdot\langle\gamma_1,\dots,\gamma_N\rangle^{\epsilon}_{g,\gamma}.$$

- The identification of Proposition 6.1 has a factor of $(-i)^{\mathrm{age}(\mu)}$.

Taking into account all the conventions above and Lemma 2.17, we obtain that [MNOP06b, Conjectures 2R, 3R] can be reformulated[16] as follows.

PT/GW. *The generating series* $\langle\gamma_1,\dots,\gamma_N\rangle^{0^+}_{g,\gamma}(y)$ *is a Taylor expansion of a rational function around* 0, *such that under the change of variables* $y=-e^{iu}$,

$$(-y)^{-\gamma\cdot\mathrm{c}_1(S)/2}\cdot\langle\gamma_1,\dots,\gamma_N\rangle^{0^+}_{g,\gamma}(y)=(-iu)^{\gamma\cdot\mathrm{c}_1(S)}\cdot\langle\gamma_1,\dots,\gamma_N\rangle^{0}_{g,\gamma}(u).$$

Assume now that S is a del Pezzo surface. Let us apply our wall-crossing formulas. Using Corollary 5.2, we obtain

$$(-iu)^{\gamma\cdot\mathrm{c}_1(S)}\cdot\langle\gamma_1,\dots,\gamma_N\rangle^{-\infty}_{g,\gamma}=(e^{iu/2}-e^{-iu/2})^{\gamma\cdot\mathrm{c}_1(S)}\cdot\langle\gamma_1,\dots,\gamma_N\rangle^{0}_{g,\gamma}. \quad (34)$$

Using [Nes21a, Corollary 6.11], we obtain

$$(-y)^{-\gamma\cdot\mathrm{c}_1(S)/2}\cdot\langle\gamma_1,\dots,\gamma_N\rangle^{\infty}_{g,\gamma}=(y^{1/2}-y^{-1/2})^{\gamma\cdot\mathrm{c}_1(S)}\cdot\langle\gamma_1,\dots,\gamma_N\rangle^{0^+}_{g,\gamma}. \quad (35)$$

Combining the two, we obtain the statement of C.R.C.

C.R.C. *The generating series* $\langle\gamma_1,\dots,\gamma_N\rangle^{\infty}_{g,\gamma}(y)$ *is a Taylor expansion of a rational function around 0, such that under the change of variables* $y=-e^{iu}$,

$$\langle\gamma_1,\dots,\gamma_N\rangle^{\infty}_{g,\gamma}(y)=\langle\gamma_1,\dots,\gamma_N\rangle^{-\infty}_{g,\gamma}(u).$$

By both wall-crossings, the statements of PT/GW and C.R.C. in the form presented above are equivalent.

Corollary 6.3.

$$\textbf{PT/GW} \iff \textbf{C.R.C.}$$

[16] We take the liberty to extend the statement of the conjecture in [MNOP06b] from a fixed curve to a moving one; and from one relative insertion to multiple ones.

6.2. **Quantum cohomology.** Let $g = 0, N = 3$. This is a particularly nice case, firstly because these invariants collectively are known as *quantum cohomology*. Secondly, the moduli space of genus-0 curves with 3 markings is a point. Hence the invariants $\langle \gamma_1, \gamma_2, \gamma_3 \rangle^{-\infty}_{0,\gamma}$ are relative PT invariants of $S \times \mathbb{P}^1$ relative to the vertical divisor $S_{0,1,\infty}$. In [PP17], PT/GW is established for $S \times \mathbb{P}^1$ relative to $S \times \{0, 1, \infty\}$, if S is toric. Corollary 6.3 then implies the following.

Corollary 6.4. *If S is toric del Pezzo, $g = 0$ and $N = 3$, then* **C.R.C.** *holds in all classes.*

The above result can also be stated as an isomorphism of quantum cohomologies with appropriate coefficient rings. Let

$$\begin{aligned} QH^*(S^{[n]}) &:= H^*(S^{[n]}, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\![q^\gamma]\!](y) \\ QH^*_{\mathrm{orb}}(S^{(n)}) &:= H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\![q^\gamma]\!](e^{iu}) \end{aligned}$$

be quantum cohomologies, where $\mathbb{C}[\![q^\gamma]\!](y)$ and $\mathbb{C}[\![q^\gamma]\!](e^{iu})$ are rings of rational functions with coefficients in $\mathbb{C}[\![q^\gamma]\!]$ and in variables y and e^{iu}, respectively. The latter we view as a subring of functions in the variable u. The quantum cohomologies are isomorphic by Corollary 6.4,

$$QL \colon QH^*_{\mathrm{orb}}(S^{(n)}) \cong QH^*(S^{[n]}),$$

where QL is given by a linear extension of L, defined in Proposition 6.1, from $H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{C})$ to $H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\![q^\gamma]\!]$ and by a change of variables $y = -e^{iu}$. In particular,

$$QL(\alpha \cdot q^\gamma \cdot y^k) = (-1)^k L(\alpha) \cdot q^\gamma \cdot e^{iku}$$

for an element $\alpha \in H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{C})$. Ideally, one would also like to specialise to $y = 0$ and $y = -1$, because in this way we recover the classical multiplications on $H^*_{\mathrm{orb}}(S^{(n)}, \mathbb{C})$ and $H^*(S^{[n]}, \mathbb{C})$, respectively. To do so, a more careful choice of coefficients is needed - we have to take rational functions with no poles at $y = 0$ and $y = -1$.

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