Weak Weyl's law for congruence subgroups

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1. The problem

- G connected semisimple algebraic group over $\mathbb{Q}$
- $\Gamma \subset G(\mathbb{Q})$ arithmetic subgroup
- $K_\infty \subset G(\mathbb{R})$ maximal compact subgroup

$\phi \in L^2(\Gamma\backslash G(\mathbb{R}))$ is called cusp form $\Leftrightarrow$

1) $D\phi = \lambda(D)\phi, \quad D \in Z(\mathfrak{g}_{\mathbb{C}})$;

2) ϕ is K_∞-finite;

3)

$$\int_{N_P(\mathbb{R})\cap\Gamma\backslash N_P(\mathbb{R})} \phi(ng)\, dn = 0$$

for all proper rational parabolic subgroups $P \subset G$, N_P the unipotent radical of P.

Problem: Existence and construction of cusp forms

- A convenient way to count cusp forms is to count the Casimir eigenvalues of cusp forms containing a fixed K_∞-type.

- $L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R})) \subset L^2(\Gamma\backslash G(\mathbb{R}))$ space of cusp forms

- $\sigma : K_\infty \to \mathrm{GL}(V_\sigma)$ irreducible unitary representation.

$$H^\Gamma_{\text{cusp}}(\sigma) := \left(L^2_{\text{cusp}}(\Gamma\backslash G(\mathbb{R})) \otimes V_\sigma\right)^{K_\infty}$$

Space of Γ-cusp forms of "weight" σ.

- $\Omega \in Z(\mathfrak{g}_\mathbb{C})$ Casimir element.

- Δ_σ selfadjoint operator in $H^\Gamma_{\text{cusp}}(\sigma)$ induced by $-\rho_\infty(\Omega) \otimes \mathrm{Id}$, ρ_∞ regular representation of $G(\mathbb{R})$.

Geometric interpretation: Assume that Γ is torsion free. Let

$$X = G(\mathbb{R})/K_\infty$$

be the Riemannian symmetric space and let $\tilde{E}_\sigma \to X$ be the homogeneous vector bundle attached to σ. Set

$$E_\sigma = \Gamma\backslash\tilde{E}_\sigma \to \Gamma\backslash X.$$

Then

$$\left(L^2(\Gamma\backslash G(\mathbb{R})) \otimes V_\sigma\right)^{K_\infty} \cong L^2(\Gamma\backslash X, E_\sigma)$$

and

$$\Delta_\sigma = (\nabla^\sigma)^* \nabla^\sigma - \lambda_\sigma \mathrm{Id},$$

where ∇^σ is the canonical invariant connection of $\tilde{E}_\sigma$ and λ_σ the Casimir eigenvalue of σ.

- Δ_σ has pure point spectrum in $H^\Gamma_{\mathrm{cusp}}(\sigma)$:

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty.$$

- cuspidal spectrum of "weight" σ.

Counting function:

$$N^\Gamma_{\mathrm{cusp}}(\lambda, \sigma) := \#\left\{i \colon |\lambda_i| \le \lambda\right\}.$$

Let

$$X = G(\mathbb{R})/K_\infty \quad \text{and} \quad d = \dim X.$$

Weyl's constant:

$$C_\Gamma := \frac{\mathrm{Vol}(\Gamma\backslash X)}{(4\pi)^{d/2}\Gamma(d/2+1)}.$$

Conjecture(Sarnak, 1984):

$$N^{\Gamma}_{\text{cusp}}(\lambda,\sigma) \sim \dim(\sigma) C_{\Gamma}\, \lambda^{d/2}$$

as $\lambda \to \infty$.

2. Results

1. Special cases

The conjecture has been proved in the following cases:

- A. Selberg, 1954: $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ congruence subgroup, $\sigma = 1$.

- I. Efrat, 1987: $\Gamma \subset \mathrm{SL}(2,\mathbb{R})^n$ Hilbert modular group, $\sigma = 1$.

- A. Reznikov, 1993: $\Gamma \subset \mathrm{SO}_0(n,1)$ congruence subgroup, $\sigma = 1$.

- St. Miller, 2001: $\Gamma = \mathrm{SL}(3,\mathbb{Z})$, $\sigma = 1$.

- M., 2003: $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$ principal congruence subgroup, σ arbitrary.

2. General results

Theorem(Donnelly). G semisimple algebraic group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{R})$ lattice. Then

$$\limsup_{\lambda \to \infty} \frac{N^{\Gamma}_{\text{cusp}}(\lambda, \sigma)}{\lambda^{d/2}} \leq \dim(\sigma) C_{\Gamma}.$$

Theorem(Piatetski-Shapiro). Let $\sigma = 1$. For every Γ there exists a normal subgroup of finite index Γ' of Γ such that

$$\lim_{\lambda \to \infty} N^{\Gamma'}_{\text{cusp}}(\lambda, 1) = \infty.$$

- A.B. Venkov, $G = \text{SL}(2)$.

- Let S be a finite set of primes containing at least two finite primes. There exists $C_{\Gamma}(S) \leq 1$ with $0 < C_{\Gamma}(S)$ for Γ a deep enough congruence subgroup.

Theorem(Labesse-M.). Let G be almost simple, connected and simply connected such that $G(\mathbb{R})$ is non compact. For every congruence subgroup $\Gamma \subset G(\mathbb{R})$ and every σ such that $\sigma|_{Z_{\Gamma}} =$Id we have

$$\dim(\sigma) C_{\Gamma} C_{\Gamma}(S) \leq \liminf_{\lambda \to \infty} \frac{N^{\Gamma}_{\text{cusp}}(\lambda, \sigma)}{\lambda^{d/2}}.$$

3. Methods

The method is a combination of the Arthur trace formula and the heat equation method.

a) $\mathsf{G} = \mathsf{SL}(2)$.

- Selberg's method

- $\Gamma \subset \mathsf{SL}(2, \mathbb{R})$ discrete, co-finite area, $\sigma = 1$.

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

$$\Delta \colon C_c^\infty(\Gamma \backslash H) \to L^2(\Gamma \backslash H)$$

essentially self-adjoint.

$$\mathrm{Spec}_{\mathrm{pp}}(\Delta) : 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

$$\sum_i e^{-t\lambda_i} = \mathrm{Tr}\left(e^{-t\Delta} \restriction L^2_{\mathrm{disc}}(\Gamma \backslash H) \right)$$

- Use Selberg trace formula to compute the trace

• $C(s) = (C_{ij}(s))$ scattering matrix

$$\phi(s) := \det C(s).$$

Selberg trace formula applied to the heat kernel gives

$$\sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\phi'}{\phi}(1/2+ir)e^{-(1/4+r^2)t}\,dr = \frac{\mathrm{Area}(\Gamma\backslash H)}{4\pi}t^{-1} + O\left(\frac{\log t}{\sqrt{t}}\right)$$

as $t \to 0+$.

• Karamata's Theorem implies Weyl's law

$$N^{\Gamma}_{\mathrm{disc}}(\lambda) - \frac{1}{4\pi}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\phi'}{\phi}(1/2+ir)\,dr \sim \frac{\mathrm{Area}(\Gamma\backslash H)}{4\pi}\lambda$$

as $\lambda \to \infty$.

$$N^{\Gamma}_{\mathrm{disc}}(\lambda) = N^{\Gamma}_{\mathrm{cusp}}(\lambda) + N^{\Gamma}_{\mathrm{res}}(\lambda), \quad N^{\Gamma}_{\mathrm{res}}(\lambda) \le C.$$

• $\Gamma = \mathrm{SL}(2,\mathbb{Z})$:

$$\phi(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

$$\left|\frac{\zeta'(1+ir)}{\zeta(1+ir)}\right| \le C\log(|r|)^6, \quad |r| \ge 2.$$

This implies

$$N_{\mathrm{cusp}}^{\Gamma}(\lambda) \sim \frac{\mathrm{Area}(\Gamma\backslash H)}{4\pi}\lambda, \quad \lambda \to \infty.$$

• similar for $\Gamma = \Gamma(N)$.

b) $G = \mathrm{SL}(n)$

• Selberg trace formula is replaced by (noninvariant) Arthur trace formula

$$\sum_{\chi\in\mathfrak{X}} J_\chi(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1).$$

The following facts are needed to prove Weyl's law:

• Weak version of Ramanujan conjecture (Luo, Rudnick, Sarnak)

• Analytic properties of Rankin-Selberg L-functions (Jacquet, Piatetski-Shapiro, Shalika, Shahidi, Mœglin, Waldspurger,...), bounds on the logarithmic derivatives.

A weaker result suffices: Let π_i, $i = 1, 2$, be a cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A})$. Set

$$\Lambda(s, \pi_1, \pi_2) = \frac{L(s, \pi_1 \times \tilde{\pi}_2)}{L(1 + s, \pi_1 \times \tilde{\pi}_2)\epsilon(s, \pi_1 \times \tilde{\pi}_2)}.$$

Then

$$\int_{-T}^{T} \left| \frac{\Lambda'}{\Lambda}(ir, \pi_1, \pi_2) \right| dr \leq C\, T \log(T + \nu(\pi_1 \times \tilde{\pi}_2))$$

for $T > 0$, where $\nu(\pi_1 \times \tilde{\pi}_2)$ is the analytic conductor.

• Description of the residual spectrum (Mœglin, Waldspurger).

• At present it seems to be out of reach to extend these results to other groups.

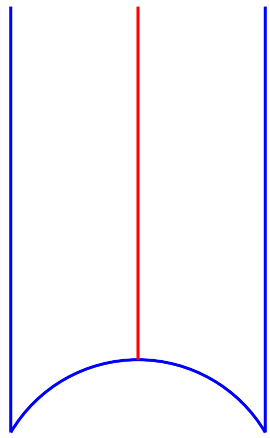
c) Weaker result

Define

$$T_{-1}\colon L^2(\Gamma\backslash H) \to L^2(\Gamma\backslash H)$$

by

$$T_{-1}(z) = -\overline{z}.$$



Then $T_{-1}^2 = \mathrm{Id}$ and

$$L^2(\Gamma\backslash H) = L^2_+(\Gamma\backslash H) \oplus L^2_-(\Gamma\backslash H)$$

- decomposition in even and odd functions.

- $T_{-1}E(z,s) = E(z,s)$.

- Δ has pure point spectrum in $L^2_-(\Gamma\backslash H)$. Correponds to Dirichlet problem on one half F_+ of the fundamental domain. One can use the same methods as in the case of compact surfaces.

$$N_-^\Gamma(\lambda) \sim \frac{\text{Area}(F_+)}{4\pi}\lambda, \quad \lambda \to \infty.$$

b) General case.

• G connected and simply connected algebraic group over $\mathbb{Q}$.

• Use simple trace formula which avoids continuous spectrum.

• Adèlic framework: $G(\mathbb{A}) = \prod'_{p\leq\infty} G(\mathbb{Q}_p)$

$$K_{\text{fin}} = \prod_{p<\infty} K_p, \quad K_p \subset G(\mathbb{Q}_p)$$

$K_{\text{fin}} \subset G(\mathbb{A}_{\text{fin}})$ decomposable open compact subgroup,

$$\Gamma = K_{\text{fin}} \cap G(\mathbb{Q}).$$

Strong approximation:

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\text{fin}} \cong \Gamma\backslash G(\mathbb{R}).$$

• S finite set of primes, $|S| \geq 2$.

$$L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), S) \subset L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

spanned by cusp form orthogonal to 1 and on which

$$G_S = \prod_{p \in S} G(\mathbb{Q}_p)$$

acts by the Steinberg representation. Put

$$H^{\Gamma,S}_{\text{cusp}} := L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), S)^{K_{\text{fin}}}.$$

Let $K = K_\infty K_{\text{fin}}$. Set

$$H^{\Gamma}_{\text{cusp}}(\sigma, S) := \left(L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), S) \otimes V_\sigma\right)^K.$$

• $\Lambda_{\text{cusp}}(\sigma, S)$ spectrum of $-\rho_\infty(\Omega) \otimes \text{Id}$ in $H^{\Gamma}_{\text{cusp}}(\sigma, S)$.

• $h_t \in C^\infty(G(\mathbb{R}))$ kernel of $e^{-t\Delta_\sigma}$.

• ρ_∞ regular representation of $G(\mathbb{R})$ in H^{Γ}_{cusp}.

$$\text{Tr}\left(\rho_\infty(h_t) \upharpoonright H^{\Gamma,S}_{\text{cusp}}\right) = \sum_{\lambda \in \Lambda_{\text{cusp}}(\sigma,S)} m(\lambda) e^{-t\lambda}.$$

• Apply simple version of Arthur's trace formula to compute the trace and to determina the asymptotic bahaviour as $t \to 0+$.

3. Simple trace formula

a) Adèlic version

Let

$$e_{\text{fin}} = \frac{1}{\text{Vol}(K_{\text{fin}})}\chi_{K_{\text{fin}}}.$$

For $f_\infty \in C_c^\infty(G(\mathbb{R}))$:

$$\text{Tr}\Big(\rho_\infty(f_\infty) \upharpoonright H^{\Gamma,S}_{\text{cusp}}\Big)$$

$$= \text{Tr}\left(\rho_\infty(f_\infty \otimes e_{\text{fin}}) \upharpoonright L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), S)\right).$$

- ρ regular representation of $G(\mathbb{A})$.

b) Steinberg representation

- $G_p = G(\mathbb{Q}_p)$, $P_p \subset G_p$ minimal parabolic.

$$\pi_{\text{St}} \subset \text{Ind}_{P_p}^{G_p}(1)$$

unique irreducible subrepresentation.

• $f_p \in C_c^\infty(G_p)$ pseudo-coefficient of Steinberg representation, if

$$\mathrm{Tr}\,(\pi_p(f_p)) = 0, \quad \text{unless} \quad \pi_p = \begin{cases} \pi_{\mathrm{St}}; \\ 1. \end{cases}$$

$$\mathrm{Tr}(\pi_{\mathrm{St}}(f_p)) = 1, \quad \mathrm{Tr}(\mathbf{1}_p(f_p)) = (-1)^q$$

for some integer q.

Existence of pseudo-coefficients: Kazdan, Kottwitz, Euler-Poincaré functions

• Kottwitz: $\mathcal{O}_\gamma(f_p) = 0$, for $\gamma \in G(\mathbb{Q})$ non elliptic.

• π_S Steinberg representation of G_S.

Set

$$C(K_S) = \dim\left(\mathcal{H}_{\pi_S}^{K_S}\right), \quad e_S = \frac{1}{\mathrm{Vol}(K_S)}\chi_{K_S}.$$

$$\mathrm{Tr}\pi_S(e_S) = C(K_S)\mathrm{Tr}\pi_S(f_S)$$

• Replace e_S by f_S.

c) Assume: S contains two different finite primes.

Set

$$f = f_\infty \otimes f_S \otimes e_{\mathrm{fin},S}.$$

• Then f is cuspidal at two places in the sense of Arthur.

Then Arthur's trace formula

$$\sum_{\chi \in \mathfrak{X}} I_\chi(f) = \sum_{\mathfrak{o} \in \mathcal{O}} I_{\mathfrak{o}}(f)$$

is reduced to

$$\mathrm{Tr}\Big(\rho(f) \restriction L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), S)\Big) + 1(f)$$

$$= \sum_{\gamma \in \mathfrak{G}_e} a(\gamma)\mathcal{O}_\gamma(f).$$

• $\mathfrak{G}_e$ set of representatives of conjugacy classes of semisimple elliptic elements in $G(\mathbb{Q})$.

$$\mathcal{O}_\gamma(f) = \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} f(g^{-1}\gamma g)\, dg.$$

Final formula

$$\mathrm{Tr}\Big(\rho_\infty(f_\infty) \restriction H_{\mathrm{cusp}}^{\Gamma,S}\Big) = C(K_S)\left(\sum_{\gamma\in\mathfrak{G}_e} a(\gamma)\mathcal{O}_\gamma(f) - 1(f)\right).$$

- **Advantage:** avoids all difficulties due to the continous spectrum.

- **Disadvantage:** $C(K_S) \neq 0$, only if Γ is a deep enough congruence subgroup.

4) Application to the heat kernel

a) The heat kernel

- $\sigma : K_\infty \to \mathrm{GL}(V_\sigma)$ irreducible unitary representation.

- $E_\sigma \to X = G(\mathbb{R})/K_\infty$ associated homogeneous vector bundle.

$$\Delta_\sigma\colon C^\infty(X, E_\sigma) \to C^\infty(X, E_\sigma)$$

elliptic differential operator induced by $-\Omega \otimes \mathrm{Id}$.

$$\Delta_\sigma\colon C_c^\infty(X, E_\sigma) \to L^2(X, E_\sigma)$$

essentially self-adjoint.

- $e^{-t\Delta_\sigma}$ smoothing operator.

$$\left(e^{-t\Delta_\sigma}\varphi\right)(g) = \int_{G(\mathbb{R})} H_t(g^{-1}g_1)\varphi(g_1)\, dg_1,$$

- $H_t \in \left(\mathcal{C}^1(G(\mathbb{R})) \otimes \mathrm{End}(V_\sigma)\right)^{K_\infty \times K_\infty}$.

Set

$$h_t(g) := \mathrm{tr}\, H_t(g), \quad g \in G(\mathbb{R}),\ t \geq 0.$$

- π irreducible unitary representation of $G(\mathbb{R})$

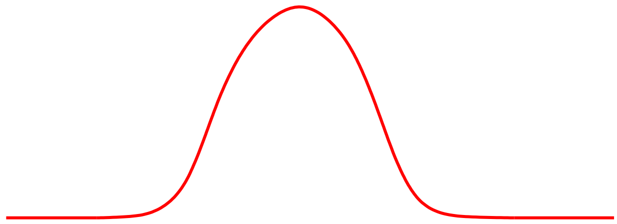
$$\mathrm{Tr}\,\pi(h_t) = e^{-t\pi(\Omega)} \dim \mathrm{Hom}_{K_\infty}(\mathcal{H}(\pi), V_\sigma^*).$$

b) Modified heat kernel

- $h_t \notin C_c^\infty(G(\mathbb{R}))$ needs modification.

Let

$$\varphi \in C_c^\infty(\mathbb{R}), \quad \varphi(u) = \begin{cases} 1, & |u| < 1/2; \\ 0, & |u| > 1. \end{cases}$$



Let $d(x, y)$ be the geodesic distance of $x, y \in X$. Set

$$\varphi_t(g) := \varphi\left(d^2(gx_0, x_0)/\sqrt{t}\right), \quad g \in G(\mathbb{R}),\ t > 0.$$

The modified heat kernel is defined by

$$\tilde{h}_t(g) := \varphi_t(g) h_t(g), \quad g \in G(\mathbb{R}),\ t > 0.$$

Proposition.

$$\left| \text{Tr}\left(\rho_\infty(h_t)\right) - \text{Tr}\left(\rho_\infty(\tilde{h}_t)\right) \right| \leq C e^{-c/\sqrt{t}}$$

for $0 < t \leq 1$.

Set

$$f_t = \tilde{h}_t \otimes e_{\mathsf{fin},S} \otimes f_S,$$

where f_S is a pseudo-coefficient of the Steinberg representation.

The simple trace formula combined with the proposition yields

$$\mathrm{Tr}\left(\rho_\infty(h_t) \restriction H^{\Gamma,S}_{\mathsf{cusp}}\right) = C(K_S) \sum_{\gamma \in \mathfrak{G}_e} a(\gamma)\mathcal{O}_\gamma(f_t) + O(1)$$

as $t \to 0+$.

c) Geometric side

We have

$$\mathcal{O}_\gamma(f_t) = \mathcal{O}_\gamma(\tilde{h}_t)\mathcal{O}_\gamma(e_{\mathsf{fin},S})\mathcal{O}_\gamma(f_S).$$

where

$$\mathcal{O}_\gamma(\tilde{h}_t) = \int_{G(\mathbb{R})_\gamma \backslash G(\mathbb{R})} \tilde{h}_t(g_\infty^{-1}\gamma g_\infty)\, dg_\infty.$$

• $\mathrm{supp}\tilde{h}_t \to K_\infty$ as $t \to 0+$.

For $h \in G(\mathbb{R})$ let

$$C_h = \{ghg^{-1} \mid g \in G(\mathbb{R})\}.$$

Let $\gamma \notin Z_G(\mathbb{R})$. Then

$$C_\gamma \cap K_\infty \subset C_\gamma$$

is a proper submanifold. By dominated convergence:

$$t^{d/2}\mathcal{O}_\gamma(\tilde{h}_t) \longrightarrow 0, \quad t \to 0.$$

Assume: $z \in Z_G(\mathbb{R})$. Then

$$\mathcal{O}_\gamma(\tilde{h}_t) = h_t(z)$$

and

$$t^{d/2}h_t(z) \longrightarrow \frac{\mathrm{tr}\sigma(z)}{(4\pi)^{d/2}\mathrm{vol}(K_\infty)}, \quad t \to 0.$$

Assume: $\sigma \restriction Z_\Gamma = \mathrm{Id}$.

Set $d_S = f_S(z)$ and

$$C_S(\Gamma) = C(K_S)d_S\mathrm{Vol}(K_S).$$

Then

$$\lim_{t\to 0} t^{d/2}\mathrm{Tr}\Big(\rho_\infty(h_t) \restriction H^{\Gamma,S}\Big)$$

$$= C_S(\Gamma)\frac{\dim(\sigma)\mathrm{Vol}(\Gamma\backslash X)}{(4\pi)^{d/2}}.$$

Let $\lambda_\pi = \pi(\Omega)$ be the Casimir eigenvalue of π. Then

$$\mathrm{Tr}\Big(\rho_\infty(h_t) \restriction H^{\Gamma,S}_{\mathrm{cusp}}\Big)$$

$$= \sum_{\pi\in\Pi(G(\mathbb{R}))} e^{t\lambda_\pi} m_\Gamma(\pi,S)\dim \mathrm{Hom}_{K_\infty}(\mathcal{H}(\pi),V_\sigma^*).$$

Set

$$N^\Gamma_{\mathrm{cusp}}(T,\sigma,S)$$

$$= \sum_{|\lambda_\pi|\le T} m_\Gamma(\pi,S)\dim \mathrm{Hom}_{K_\infty}(\mathcal{H}(\pi),V_\sigma^*).$$

Then Karamata's theorem implies

$$\lim_{T\to\infty}\frac{N^\Gamma_{\mathrm{cusp}}(T,\sigma,S)}{T^{d/2}} = \dim(\sigma)C_S(\Gamma)C_\Gamma.$$

- $C_S(\Gamma) \neq 0$ if and only if $C(K_S) \neq 0$.

- $C(K_S) \neq 0$, if the Steinberg representation contains a non zero K_S-invariant vector. This is the case if $K_p \subset I_P$, a minimal parahoric subgroup for $p \in S$.