Analytic torsion of locally symmetric spaces and cohomology of arithmetic groups

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1. Analytic torsion

Analytic torsion is an analytic tool to study torsion in the cohomolgy of arithmetic groups.

General set up:

- (X, g) a compact Riemannian manifold of dimension n.
- $\rho\colon \pi_1(X) \to \mathrm{GL}(V)$ finite-dimensional representation.
- $E_\rho \to X$ associated flat vector bundle.
- h Hermitean fibre metric in E_ρ.

Let

$$\Delta_p(\rho)\colon \Lambda^p(X, E_\rho) \to \Lambda^p(X, E_\rho)$$

be the Laplace operator on E_ρ-valued p-forms.

- $\Delta_p(\rho)$ elliptic, self-adjoint, non-negative.
- $\Delta_p(\rho)$: $0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty$ spectrum of $\Delta_p(\rho)$.

Let

$$\zeta_p(s;\rho) := \sum_{j=1}^{\infty} \lambda_j^{-s}, \quad \mathrm{Re}(s) > n/2,$$

be the zeta function of $\Delta_p(\rho)$. $\zeta_p(s;\rho)$ admits meromorphic extension to $\mathbb{C}$, holomorphic at $s=0$. Put

$$\det \Delta_p(\rho) = \exp\left(-\frac{d}{ds}\zeta_p(s;\rho)\Big|_{s=0}\right).$$

Ray-Singer analytic torsion:

$$T_X(\rho) := \prod_{j=1}^{n} \left(\det \Delta_p(\rho)\right)^{(-1)^{p+1}p/2}.$$

- $T_X(\rho)$ depends on the metrics g on X and h in E_ρ.
- If $\dim X$ is odd and $H^*(X;E_\rho)=0$, then $T_X(\rho)$ is independent of g and h.

- $\tau_X(\rho)$ Reidemeister torsion, defined in terms of a triangulation of X. Defined by a formula analogous to Ray-Singer torsion with Hodge Laplacian replaced by combinatorial Laplace operators.

Theorem (Cheeger, M.) For all unitary ρ, one has $T_X(\rho) = \tau_X(\rho)$.
Extensions:

- Mü: ρ unimodular, equality holds.
- Bismut-Zhang: general ρ, equality does not hold in genral.

Relation to cohomology

- $M \subset V_\rho$ $\pi_1(X)$-invariant lattice, $\mathcal{M}$ associated local system of free $\mathbb{Z}$-modules over X.
- Hodge isomorphism $H^p(X; \mathcal{M} \otimes \mathbb{R}) \cong \mathcal{H}^p(X; \mathcal{M} \otimes \mathbb{R})$, introduces inner product in $H^p(X; \mathcal{M} \otimes \mathbb{R})$.

$$R_p(\mathcal{M}) = \mathrm{vol}(H^p(X; \mathcal{M} \otimes \mathbb{R}) / H^p(X; \mathcal{M})_{free})$$

The regulator $R(\mathcal{M})$ is defined as

$$R(\mathcal{M}) := \prod_{p=0}^{n} R_p(\mathcal{M})^{(-1)^p}.$$

Lemma (Cheeger):

$$\tau_X(\rho) = R(\mathcal{M}) \cdot \prod_{p=0}^{n} \left| H^p(X;\mathcal{M})_{tors} \right|^{(-1)^{p+1}}.$$

Especially, if $H^*(X, \mathcal{M} \otimes \mathbb{R}) = 0$, then $H^*(X;\mathcal{M})$ is finite and

$$T_X(\rho) = \prod_{p=1}^{n} (\det \Delta_p(\rho))^{(-1)^{p+1} p/2} = \prod_{p=0}^{n} \left| H^p(X;\mathcal{M}) \right|^{(-1)^{p+1}}.$$

2. Analytic torsion for special classes of non-compact manifolds

i) Manifolds with iterated fibred wedges,

ii) Manifolds with iterated fibred cusps.

Basic examples

i) Manifolds with conical singularities.

$$X = X_0 \cup C(N), \quad \partial X_0 = N$$
$$C(N) = [0,1) \times N, \quad g|_{C(N)} = dr^2 + r^2 g_N.$$

Results by: Dar, ..., Ludwig

ii) Manifolds with cusps.

$$X = X_0 \cup Z(Y), \quad \partial X_0 = Y$$
$$Z(Y)) = [1,\infty) \times Y, \quad g|_{Z(Y)} = \frac{dr^2}{r^2} + r^{-2} g_Y.$$

Results by: Albin-Rochon-Sher, Vertman, Mü.-Rochon

3. Locally symmetric spaces

Special case of fibred corner metrics.

General set up

- G semi-simple Lie group, non-compact type, ...
- $K \subset G$ maximal compact subgroup,
- $\widetilde{X} = G/K$ Riemannian symmetric space,
- $\Gamma \subset G$ lattice, i.e., discrete subgroup, $\mathrm{vol}(\Gamma\backslash G) < \infty$.
- $X = \Gamma\backslash\widetilde{X}$ locally symmetric space of finite volume,
- Γ torsion free, then X is a manifold.

Flat vector bundles

- $\tau\colon G \to \mathrm{GL}(V)$ finite dimensional complex representation,
- $\rho := \tau|_\Gamma\colon \Gamma \to \mathrm{GL}(V)$, G semi-simple $\Rightarrow$ ρ unimodular.
- $\sigma := \tau|_K\colon K \to \mathrm{GL}(V)$,
- $\widetilde{E}_\sigma \to \widetilde{X}$ homogeneous vector bundle, $E_\sigma = \Gamma\backslash\widetilde{E}_\sigma \to X$ locally homogeneous vector bundle.

Lemma $E_\rho \cong E_\sigma$ canonically isomorphic, induces canonical metric in E_ρ.

- One can use harmonic analysis to study $\Delta_p(\rho)$.

$$C^\infty(X, E_\rho) \cong (C^\infty(\Gamma\backslash G) \otimes V)^K.$$

$$\Delta_p(\rho) = -R_\Gamma(\Omega) \otimes \mathrm{Id} + \tau(\Omega_K)\,\mathrm{Id}\,.$$

- $\Omega \in Z(\mathfrak{g}_\mathbb{C})$, $\Omega_K \in Z(\mathfrak{k}_\mathbb{C})$ Casimir elements,
- R_Γ right regular representation of G in $C^\infty(\Gamma\backslash G)$.
- $\Delta_p(\rho)$ has non-empty continous spectrum,
- Usual definition of analytic torsion does not work.

4. Regularized analytic torsion

4.1. Regularized trace of heat operators

Idea goes back to Melrose.

Example: $X = \Gamma\backslash\mathbb{H}^2$ hyperbolic surface of finite area,

$$X = X_0 \cup \bigcup_{i=1}^{m} Y_i, \quad Y_i = [a_i, \infty) \times S^1.$$

- $T \gg 0$, X_T obtained by truncation of cusps at level T.
- $\Delta \colon C^\infty(X) \to C^\infty(X)$ hyperbolic Laplace operator.
- $e^{-t\Delta}$, $t > 0$, heat operator, integral operator with kernel $K(x, y, t)$.
- X compact, pre-trace formula: $\mathrm{Tr}(e^{-t\Delta}) = \int_X K(x, x, t)dx$.

If X is not compact, then

$$\int_{X_T} K(x,x,t)dx = \log(T)a(t) + b(t) + O(T^{-1}), \quad \text{as } T \to \infty.$$

Definition: $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta}) := b(t)$, $t > 0$.

- $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ eigenvalues of Δ, $E_k(z,s)$ Eisenstein series, $C(s)$ "scattering matrix", obtained from zero Fourier coefficients of $E_k(z,s)$, $\varphi(s) := \det C(s)$, $s \in \mathbb{C}$.

Lemma:

$$Tr_{\mathrm{reg}}(e^{-t\Delta}) = \sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_R e^{-(1/4+r^2)t}\frac{\varphi'}{\varphi}(1/2+ir)dr + \cdots .$$

- Spectral side of Selberg trace formula.
- Used to define regularized Ray-Singer torsion $T_X^{\mathrm{reg}}(\rho)$ for hyperbolic manifolds X of finite volume.

Approximation of L^2-torsion:

1. Hyperbolic manifolds of finite volume

Theorem (M., Pfaff). $\mathbb{H}^d = \mathrm{SO}_0(d,1)/\mathrm{SO}(d)$, $\Gamma(N) \subset \Gamma_0 = \mathrm{SO}_0(d,1)(\mathbb{Z})$ principal congruence subgroup, $\tau \in \mathrm{Rep}(\mathrm{SO}_0(d,1))$ with $\tau \ncong \tau \circ \theta$, θ Cartan involution, $X_N = \Gamma(N)\backslash\mathbb{H}^d$. Then

$$\lim_{N\to\infty} \frac{\log T^{\mathrm{reg}}_{X_N}(\tau)}{[\Gamma_0 \colon \Gamma(N)]} = t^{(2)}_{\widetilde{X}}(\tau)\,\mathrm{vol}(\Gamma_0\backslash\mathbb{H}^d).$$

- J. Raimbeault treated the 3-dimensional case.

7. Finite volume case

- $\Gamma(N) \subset \mathrm{SL}(n;\mathbb{Z})$ principal congruence subgroup,
- F imaginary quadratic field, $\mathcal{O}_F$ ring of integers, $\mathfrak{a} \subset \mathcal{O}_F$ ideal, $\Gamma(\mathfrak{a}) \subset \mathrm{SL}(2,\mathcal{O}_F)$ principal congruence subgroups.
- finite co-volume, not co-compact.

Goal: Extension of the results about growth of torsion to the finite volume case.

Conjecture (Bergeron): $(\Gamma_i)_{i\in\mathbb{N}}$ sequence of distinct finite index subgroups of $\mathrm{SL}_n(\mathbb{Z})$. Then

$$\lim_{i\to\infty} \frac{\log |H_q(\Gamma_i;\mathbb{Z})_{\mathrm{tors}}|}{[\mathrm{SL}_n(\mathbb{Z})\colon \Gamma_i]} = \alpha(n,q)$$

where $\alpha(n,q) = 0$, unless $n = 3$ and $q = 2$, or $n = 4$ and $q = 4$.

Moreover

$$\alpha(3,2) = \mathrm{vol}(\mathrm{SL}_3(\mathbb{Z})\backslash\widetilde{X}_3) t^{(2)}_{\widetilde{X}_3}, \quad \alpha(4,4) = \mathrm{vol}(\mathrm{SL}_4(\mathbb{Z})\backslash\widetilde{X}_4) t^{(2)}_{\widetilde{X}_4},$$

where $\widetilde{X}_n = \mathrm{SL}_n(\mathbb{R})/\,\mathrm{SO}(n)$.

Problems:

1. $\Delta_p(\rho)$ has continous spectrum, therefore $e^{-t\Delta_p(\rho)}$ is not trace class. We need to define a regularized trace of $e^{-t\Delta_p(\rho)}$, which leads to a regularized version $T_X^{\mathrm{reg}}(\rho)$ of the Ray-Singer torsion.
2. Approximation of L^2-torsion:

$$\lim_{N\to\infty} \frac{\log T^{\mathrm{reg}}_{X_N}(\rho)}{[\Gamma : \Gamma_N]} = t^{(2)}_{\widetilde{X}}(\rho)\,\mathrm{vol}(X).$$

3. $\overline{X}$ Borel-Serre compactification of $X = \Gamma\backslash\widetilde{X}$. Relation between $T_X^{\mathrm{reg}}(\rho)$ and Reidemeister torsion of $\overline{X}$.

Regularized trace of heat operators.

Example: $X = \Gamma\backslash\mathbb{H}^2$ hyperbolic surface of finite area,

$$X = X_0 \cup \bigcup_{i=1}^{m} Y_i, \quad Y_i = [a_i, \infty) \times S^1.$$

- $T \gg 0$, X_T obtained by truncation of cusps at level T.
- $\Delta\colon C^\infty(X) \to C^\infty(X)$ hyperbolic Laplace operator.
- $e^{-t\Delta}$, $t > 0$, heat operator, integral operator with kernel $K(x, y, t)$.
- X compact, pre-trace formula: $\mathrm{Tr}(e^{-t\Delta}) = \int_X K(x, x, t)dx$.

If X is not compact, then

$$\int_{X_T} K(x,x,t)dx = \log(T)a(t) + b(t) + O(T^{-1}), \quad \text{as } T \to \infty.$$

Definition: $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta}) := b(t)$, $t > 0$. Hadamard regularization.

Example: $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, $X = \Gamma\backslash\mathbb{H}^2$.

- $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ eigenvalues of Δ.
- $E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma(z))^s$, $\mathrm{Re}(s) > 1$ Eisenstein series,

$$E(x+iy,s) = y^s + c(s)y^{1-s} + O(e^{-cy}), \quad y \to \infty.$$

- $c(s)$ scattering matrix.

Theorem:

$$Tr_{\mathrm{reg}}(e^{-t\Delta}) = \sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_R e^{-(1/4+r^2)t}\frac{c'}{c}(1/2+ir)dr + \cdots.$$

- Spectral side of Selberg trace formula.
- Used to define regularized Ray-Singer torsion $T_X^{\mathrm{reg}}(\rho)$ for hyperbolic manifolds X of finite volume.

Higher rank case, Approximation of L^2-torsion: Joint work with Jasmin Matz.

- Pass to adelic framework.
- $\boldsymbol{G}$ semisimple algebraic group over $\mathbb{Q}$, $K_f \subset \boldsymbol{G}(\mathbb{A}_f)$ open compact subgroup, $\widetilde{X} = \boldsymbol{G}(\mathbb{R})/K_\infty$,
- adelic locally symmetric space $X(K_f) = \boldsymbol{G}(\mathbb{Q})\backslash(\widetilde{X} \times G(\mathbb{A}_f))/K_f$.
- There exist lattices $\Gamma_i \subset \boldsymbol{G}(\mathbb{R})$, $i = 1, ..., m$, such that

$$X(K_f) = \Gamma_1\backslash\widetilde{X} \sqcup \cdots \sqcup \Gamma_m\backslash\widetilde{X}.$$

- $\boldsymbol{G}$ simply connected, $\mathbb{Q}$-simple, then by strong approximation

$$X(K_f) = \Gamma\backslash\widetilde{X}, \quad \Gamma = (\boldsymbol{G}(\mathbb{R}) \times K_f) \cap \boldsymbol{G}(\mathbb{Q}).$$

- We use Arthur's trace formula to define the regularized trace of the heat operators.
- h_t kernel of $e^{-t\tilde{\Delta}}$ on $\widetilde{X} = \boldsymbol{G}(\mathbb{R})/K_\infty$, $\phi_t := h_t \otimes 1_{K_f}$, $\phi_t \in \mathcal{C}(\boldsymbol{G}(\mathbb{A}); K_f)$.
- Let $J_{\mathrm{spec}}(\phi_t)$ be the spectral side of the Arthur trace formula.
- We have

$$J_{\mathrm{spec}}(\phi_t) = \sum_{[M]} J_{\mathrm{spec},M}(\phi_t),$$

  where $[M]$ runs over conjugacy classes of Levi subroups of G.
- For $M = G$ we have

$$J_{\mathrm{spec},G}(\phi_t) = \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}))} m(\pi) \dim \left(\mathcal{H}_{\pi_f}^{K_f} \right) \Theta_{\pi_\infty}(h_t).$$

- If M is a proper Levi subgroup, the main ingredients of $J_{\mathrm{spec},M}(\phi_t)$ are logarithmic derivatives of intertwining operators.

We define the regularized trace of the heat operator by

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta}\right) := J_{\mathrm{spec}}(\phi_t).$$

- For $X = \Gamma\backslash\mathbb{H}^2$ it coincides with the previous definition.

If $\rho \in \mathrm{Rep}(G)$ is not self-conjugate, the regularized analytic torsion is defined by

$$\log T_{X(K_f)}^{reg}(\rho) = \frac{1}{2}\sum_{q=1}^{n}(-1)^q q \left(\mathrm{FP}_{s=0}\,\frac{\zeta_p(s;\rho)}{s}\right),$$

$$\zeta_p(s;\rho) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_q(\rho)}\right)t^{s-1}\,dt.$$

Remark. It is a theorem that the integral converges for $\mathrm{Re}(s) > n/2$ and admits a meromorphic extension to $\mathbb{C}$.

Let $\widetilde{X}_n = \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$, $\Gamma(N) \subset \mathrm{SL}(n,\mathbb{Z})$ principal congruence subgroup, $X_n(N) = \Gamma(N)\backslash\widetilde{X}_n$, $\tau \in \mathrm{Rep}(\mathrm{SL}(n,\mathbb{R}))$, $\tau_\theta = \tau \circ \theta$, where θ is the Cartan involution.

Theorem (Matz, M.): Let $\tau \in \mathrm{Rep}(SL(n,\mathbb{R}))$. Assume $\tau \not\cong \tau_\theta$. Then for $n \geq 2$ we have

$$\lim_{N\to\infty} \frac{\log T^{\mathrm{reg}}_{X_n(N)}(\tau)}{\mathrm{vol}(X_n(N))} = t^{(2)}_{\widetilde{X}_n}(\tau).$$

Moreover, if $n > 4$, then $t^{(2)}_{\widetilde{X}_n}(\tau) = 0$, and if $n = 3, 4$, then $t^{(2)}_{\widetilde{X}_n}(\tau) > 0$.

8. Connection with Reidemeister torsion and cohomology

- Joint work with Fréderic Rochon.

i) Hyperbolic manifolds of finite volume.

- $G = \mathrm{SO}_0(d,1)$, $K = \mathrm{SO}(d)$, $\mathbb{H}^d = G/K$ hyperbolic d-space.
- $\Gamma \subset G$ torsion free lattice, $X = \Gamma\backslash\mathbb{H}^d$ hyperbolic manifold of finite volume.
- $\rho\colon G \to \mathrm{GL}(V)$ irreducible finite-dimensional representation.
- $E = \mathbb{H}^d \times_{\rho|_\Gamma} V$ associated flat vector bundle.
- $\overline{X}$ Borel-Serre compactification of X, $\overline{X}$ compact manifold with boundary $Z = \partial\overline{X} = \sqcup_{i=1}^m T_i$, T_i torus.
- The Reidemeister torsion $\tau(\overline{X}, E, \mu_X)$ depends on a basis of the cohomology $H^*(X; E)$.

Harder: $H^*(X;E) = H^*_{(2)}(X;E) \oplus H^*_{\mathrm{Eis}}(X;E)$.

- $H^*_{\mathrm{Eis}}(X;E)$ Eisenstein cohomology, induced from $H^*(\partial\overline{X};E)$. Eisenstein cohomology classes are represented by differential forms which are Eisenstein series, evaluated at special points.
- If $\rho \not\cong \rho_\theta$, then $H^*_{(2)}(X;E) = 0$.
- Let μ_Z be an orthonormal basis of $H^*(Z;E)$. The theory of Eisenstein series gives rise to a basis μ_X of $H^*_{\mathrm{Eis}}(X;E)$.
- $\tau^{\mathrm{Eis}}(X;E)$ Reidemeister torsion with respect to μ_X.

Theorem (M., Rochon) Assume that $\rho \not\cong \rho_\theta$. Then

$$\log T_X^{\mathrm{reg}}(E) = \log \tau^{Eis}(X;E) - \frac{1}{2}\log \tau(Z;E) - \kappa_\Gamma^\rho c_\rho,$$

where $c_\rho \in \mathbb{R}$ is an explicit constant depending on ρ and κ_Γ^ρ is the number of connected components of Z on which the cohomology with values in E is non-trivial.

Method: Degeneration of closed manifolds.

- $M = \overline{X} \cup_{\partial \overline{X}} \overline{X}$ double of $\overline{X}$, $\{g_\varepsilon\}_{\varepsilon>0}$, family of Riemannian metrics on M, degenerating to the hyperbolic metric on each copy of X inside of M as $\varepsilon \to 0$.
- Similar for the Hermitian metric h_E on E. We introduce a family of Hermitian metrics $\{h_\varepsilon\}_{\varepsilon>0}$ on the double of E on M, degenerating to h_E on each copy of X as $\varepsilon \to 0$.

Application to torsion in the cohomology

Example:

Let $a_1, \ldots, a_d \in \mathbb{N}$ and let q be the quadratic form defined by

$$q(x_1, \ldots, x_d, x_{d+1}) = a_1 x_1^2 + \cdots + a_d x_d^2 - x_{d+1}^2.$$

- Let $\boldsymbol{G}$ be the algebraic group defined by q. Then $\boldsymbol{G}$ is defined over $\mathbb{Q}$ and $G := \boldsymbol{G}(\mathbb{R}) \cong \mathrm{SO}(d,1)$.
- $\Gamma := \boldsymbol{G}(\mathbb{Z})$ is a lattice in G. For $d \geq 4$, $\boldsymbol{G}(\mathbb{Z})$ is not co-compact. $X = \Gamma \backslash \mathbb{H}^d$.
- $\rho\colon \boldsymbol{G} \to \mathrm{GL}(V)$ rational representation, $\Lambda \subset V$ Γ-invariant lattice. $\Gamma(N) \subset \Gamma$ principal congruence subgroup, $X_j := \Gamma(j) \backslash \mathbb{H}^d$.

Theorem (M., Rochon): Let Λ be strongly L^2-acyclic. Let $E_j = \mathbb{H}^d \times_{\rho|_{\Gamma(j)}} \Lambda$ be the bundle of free $\mathbb{Z}$-modules over X_j. Assume that $E_j \cong E_j^*$. Let $d = 2n+1$. Then we have

$$\liminf_{j\to\infty} \frac{\sum_{q \text{ even}} \log |H^q(X_j; E_j)_{\mathrm{tors}}|}{[\Gamma : \Gamma(j)]} \geq (-1)^n t^{(2)}_{\mathbb{H}^d}(\rho)\, \mathrm{vol}(\Gamma \backslash \mathbb{H}^d) > 0.$$