# Spectral theory on locally symmetric spaces and automorphic L-functions

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# Introduction

Analysis on loc. sym. spaces ⟺ Number theory

Automorphic forms

The Langlands program establishes a deep connection between harmonic analysis and number theory. At the heart of Langlands' program is the general notion of an "automorphic representation" $\pi$ and its $L$-function $L(s,\pi)$. These notions are both defined via group theory and the theory of harmonic analysis. The conjectures of Langlands amount (roughly) to the assertion that all zeta-functions arising in number theory are but special realizations of automorphic $L$-functions $L(s,\pi)$.

## a) The Riemann zeta function

**Riemann:** Use of analytic methods to study the distribution of primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1.$$

**Properties:**

- **Analytic continuation:** $\zeta(s)$ admits a meromorphic extenion to $\mathbb{C}$ with a simple pole at $s = 1$.
- **Functional equation:** Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then

$$\xi(s) = \xi(1 - s).$$

- **Euler product:** $P$ set of prime numbers.

$$\zeta(s) = \prod_{p \in P} \left(1 - p^{-s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

- Non-trivial zeros are contained in the strip $0 < \mathrm{Re}(s) < 1$.
- Location of zeros is of great significance for the distribution of prime numbers

**Hadamard, de la Vallée-Poussin:** $\zeta(s) \neq 0$ for $\mathrm{Re}(s) = 1$.
$\Rightarrow$ prime number theorem
Let $\mathcal{P} \subset \mathbb{N}$ be the prime numbers, $\pi(x) = \#\{p\colon p \in \mathcal{P},\ \ p \leq x\}$.
Then the **Prime number theorem** states

$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty.$$

**Riemann hypothesis:** All non-trivial zeros of $\zeta(s)$ are on the line $\mathrm{Re}(s) = 1/2$. This implies the best possible estimation of the remainder term

$$\pi(x) = Li(x) + O(\sqrt{x}\log x), \quad x \to \infty.$$

## b) Dirichlet $L$-series

$\chi\colon \mathbb{Z} \to \mathbb{C}^\times$ Dirichlet character.

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \mathrm{Re}(s) > 1.$$

Has similar properties as the Riemann zeta function: Analytic continuation, functional equation, Euler product.

- **Dirichlet:** $L(s;\chi)$ entire for $\chi \neq \chi_0$ and $L(s,\chi) \neq 0$ on $\mathrm{Re}(s) = 1$.

This implies

- **Dirichlet prime number theorem:** Let $a, k \in \mathbb{N}$, $(a,k) = 1$. Then the sequence $a_n = a + nk$, $n \in \mathbb{N}$, contains infinitely many prime numbers.

### c) Dedekind zeta function of a number field

$F/\mathbb{Q}$ number field, Ex.: $\mathbb{Q}(\sqrt{D})$, $\mathbb{Q}(\sqrt{-D})$, $D \in \mathbb{N}$ square free.
$\mathcal{O}_F \subset F$ ring of algebraic integers.

$$\zeta_F(s) := \sum_{\mathfrak{a} \subseteq \mathcal{O}_F} N(\mathfrak{a})^{-s}, \quad \mathrm{Re}(s) > 1,$$

where $\mathfrak{a}$ runs over all ideals in $\mathcal{O}_F$.

- Analytic continuation to $\mathbb{C}$, functional equation, Euler product.

### d) $L$-function of an elliptic curve

$E$ elliptic curve over $\mathbb{Q}$, given in Weierstrass normal form by

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q}, \ \Delta := -4a^3 - 27b^2 \neq 0.$$

Complex points:

$$E(\mathbb{C}) = \{(x,y) \in \mathbb{C}^2 \colon y^2 = x^3 + ax + b\} \cup \{\infty\}.$$

- $E(\mathbb{C})$ is a 2-dimensional torus: $E(\mathbb{C}) = \mathbb{C}^2/\Lambda$, $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$, where $v_1, v_2 \in \mathbb{C}$ is a basis of $\mathbb{R}^2$.
- $p$ prime, $\bar{E}_p$ reduction mod $p$ of $E$, curve over $\mathbb{F}_p$.
- For almost all $p$, $\bar{E}_p$ is an elliptic curve, $E$ has good reduction at $p$.
- If $E$ has good reduction at $p$, let $c_p = \#\bar{E}_p(\mathbb{F}_p)$, $a_p = 1 + p - c_p$. $$L(s,E) = \prod_p (1 - a_p p^{-s} + \varepsilon(p) p^{1-2s})^{-1}, \quad \mathrm{Re}(s) > 3/2,$$

where $\varepsilon(p) = 1$, if $E$ has good reduction at $p$ and $\varepsilon(p) = 0$ otherwise, and $a_p$ is defined differently.

- $L(s,E)$ admits analytic continuation to $\mathbb{C}$ and satisfies a functional equation.

## e) Automorphic $L$-functions

Maaß forms: $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$, $F \subset \mathbb{H}^2$ fundamental domain. Let

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \quad z = x + iy,$$

be the hyperbolic Laplace operator on $\mathbb{H}^2$. Let $f \in C^\infty(\mathbb{H}^2)$ be such that

- $\Delta f = \lambda f$, $\lambda = 1/4 + r^2$, $r \geq 0$, $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$, and $\int_F |f(z)|^2 d\mu(z) < \infty$.
- Assume that $f(z) = f(-\bar{z})$, $z \in \mathbb{H}^2$.

Thus $f$ is a square integrable eigenfunction of $\Delta$ on $\Gamma\backslash\mathbb{H}^2$. Since $f$ satisfies $f(z+1) = f(z)$, it has the following Fourier expansion with resepct to $x$:

$$f(x+iy) = \sum_{n=1}^{\infty} a_n y^{1/2} K_{ir}(2\pi n y)\cos(2\pi n x),$$

where $K_\nu(y)$ is the modified Bessel function. Put

$$L(s,f) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \mathrm{Re}(s) > 3/2.$$

Using that $f$ satisfies $f(z) = f(-1/z)$, it follows that $L(s,f)$ admits a meromorphic extension to $\mathbb{C}$, and satisfies a functional equation. Put

$$\Lambda(s,f) = \pi^{-s}\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)L(s,f).$$

Then the functional equation is $\Lambda(s) = \Lambda(1-s)$.

Hecke operators:

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right), \quad n \in \mathbb{N}.$$

- $[T_n, T_m] = 0$, $[T_n, \Delta] = 0$, $m, n \in \mathbb{N}$, selfadjoint in $L^2(\Gamma\backslash\mathbb{H})$.

- Maass forms can be simultaneously diagonalized.

$$T_n f = \lambda_f(n) f, \quad n \in \mathbb{N}.$$

It follows that

$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p L_p(s,f), \quad \mathrm{Re}(s) > 1,$$

where

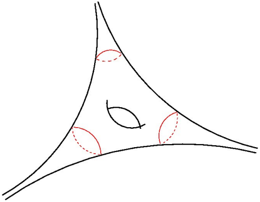
$$L_p(s,f) = (1 - \lambda_f(p) p^{-s} + p^{-2s})^{-1}.$$

- $L(s,f)$ is an example of an **automorphic L-function**.
- This construction can be generalized to automorphic forms w.r.t. other semisimple (or reductive) groups.

## 2. Spectral theory for hyperbolic surfaces

The existence of $L^2$-automorphic forms is intimately related with the structure of the continuous spectrum.

- $\mathbb{H}^2$ upper half plane, $\mathbb{H}^2 \cong \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$.
- $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ discrete, torsion free subgroup with $\mathrm{vol}(\Gamma\backslash\,\mathrm{SL}(2,\mathbb{R})) < \infty$.
- $X := \Gamma\backslash\mathbb{H}^2$ hyperbolic surface of finite area.



- $X = X_0 \cup Y_1 \cup \cdots \cup Y_m$, $X_0$ compact surface with boundary
- $Y_i \cong [a_i, \infty) \times S^1$ cusp, metric on $Y_i$ has the form

$$h|_{Y_i} = \frac{dy_i^2 + dx_i^2}{y_i^2}, \quad (y_i, x_i) \in Y_i.$$

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad z = x + iy.$$

Hyperbolic Laplace operator.

- $\Delta \colon C^\infty(X) \to C^\infty(X)$ is essentially self-adjoint in $L^2(X)$.

Theorem (Selberg, Roelcke, 1954):

1) $\mathrm{Spec}_{pp}(\Delta)\colon 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, eigenvalues of finite multiplicities,
2) If $\Gamma \backslash \mathbb{H}^2$ is non-compact, $\mathrm{Spec}_{ac}(\Delta) = [1/4, \infty)$,
3) $L^2_{ac}(\Gamma \backslash \mathbb{H}^2)$ can be described in terms of Eisenstein series.

Eisenstein series Example $\Gamma = \mathrm{SL}(2, \mathbb{Z})$:

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma(z))^s = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{\mathrm{Im}(z)^s}{|mz + n|^{2s}}, \quad \mathrm{Re}(s) > 1.$$

$$\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)=1}} \frac{1}{|mz+n|^{2s}}$$

Epstein zeta function

- **Analytic number theory:** $E(z,s)$ admits meromorphic extension to $\mathbb{C}$, holomorphic for $\mathrm{Re}(s)=1/2$.
- $\Delta E(z,s) = s(1-s)E(z,s)$
- $r\in\mathbb{R}\mapsto E(z,1/2+ir)$ generalized eigenfunction.

The map

$$f\in C_c^\infty(\mathbb{R}^+)\mapsto \frac{1}{2\pi}\int_0^\infty f(r)E(z,1/2+ir)dr$$

extends to an isometry $E\colon L^2(\mathbb{R}^+)\to L^2_{ac}(\Gamma\backslash\mathbb{H}^2)$.

# Scattering matrix

**Fourier expansion** of $E(z,s)$:

$$E(x+iy,s) = y^s + C(s)y^{1-s} + O\left(e^{-cy}\right)$$

as $y \to \infty$. Sommerfeld radiation condition

- $y^{1/2+ir}$ incoming plane wave, $y^{1/2-ir}$ outgoing plane wave, $E(z,1/2+ir)$ the distorted plane wave.
- $S(r) = C(1/2+ir)$ scattering matrix,
- $C(s)$ analytic continuation of the scattering matrix,

General hyperbolic surface: $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ discrete, torsion free, $\mathrm{vol}(\Gamma \backslash \mathrm{SL}(2,\mathbb{R})) < \infty$, $X = \Gamma \backslash \mathrm{SL}(2,\mathbb{R})$.

- $E_k(z,s)$, $k = 1, ..., m$, Eisenstein series, attached to the $k$-th cusp.

**Fourier expansion:** Cusp $Y_l \cong [a_l, \infty) \times S^1$, $(y_l, x_l) \in Y_l$.

$$E_k((y_l,x_l),s) = \delta_{k,l} y_l^s + C_{kl}(s) y_l^{1-s} + O(e^{-cy_l}), \quad y_l \to \infty.$$

Scattering matrix: $C(s) = (C_{kl}(s))_{k,l=1}^m$.

- $E_k(z,s)$, $k = 1, \dots, m$, and $C(s)$ are meromorphic functions of $s \in \mathbb{C}$, holomorphic on $\mathrm{Re}(s) = 1/2$.
- $z \mapsto E_k(z, 1/2 + ir)$ are generalized eigenfunctions.

The analytic continuation of $E_k(z,s)$, $k = 1, \dots, m$, follows from

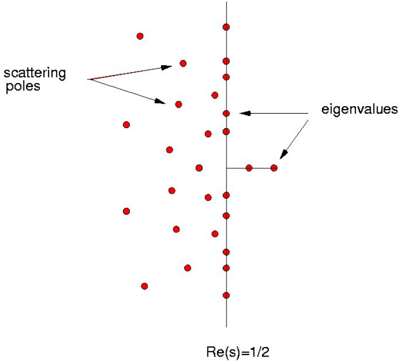
Theorem 1: *The resolvent* $R_X(s) = (\Delta - s(1-s))^{-1}$, *defined for* $\mathrm{Re}(s) > 1/2$, $s \neq \bar{s}$, *extends to a meromorphic family of bounded operator*

$$R_X(s) : L^2_{\mathrm{cpt}}(X) \to H^2_{\mathrm{loc}}(X)$$

*with poles of finite rank.*

Faddejev, Colin de Verdiere, Mü., Zworski-Guillopé,

**Definition:** The poles of $R_X(s)$ are called Resonances. Scattering resonances := poles of $C(s)$ = poles of $R_X(s)$ with $\mathrm{Re}(s) < 1/2$.



The poles are distributed in a strip of the form $-c < \mathrm{Re}(s) \leq 1$.

# Distribution of resonances

Put

$$N_\Gamma(\lambda) = \#\{j \colon \lambda_j \le \lambda^2\}, \quad \phi(s) := \det C(s).$$

Theorem 3 (Selberg): As $\lambda \to \infty$, we have

$$N_\Gamma(\lambda) - \frac{1}{4\pi}\int_{-\lambda}^{\lambda} \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right)\, dr \sim \frac{\mathrm{Area}(X)}{4\pi}\lambda^2.$$

**proof:** Selberg trace formula applied to the heat operator gives

$$\sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-(1/4+r^2)t}\frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right)\, dr \sim \frac{\mathrm{Area}(X)}{4\pi}t^{-1}$$

as $t \to 0+$. Let

$$M_\Gamma(\lambda) = -\frac{1}{4\pi}\int_{-\lambda}^{\lambda} \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right)\, dr.$$

Lemma: $M_\Gamma(\lambda)$ is monotonic increasing for $\lambda \gg 0$.
Karamata's theorem implies Theorem 3.

- A more sophisticated use of the trace formula gives an estimation of the remainder term of order $O(\lambda \log \lambda)$. This is essentially Hörmander's method.

$$N_\Gamma(\lambda)+M_\Gamma(\lambda) = \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H}^2)}{2\pi}\lambda^2 + c\lambda\log(\lambda) + O(\lambda), \quad \lambda \to \infty.$$

- Let $N_{scre}(\lambda)$ be the number of poles of $\det C(s)$ in the circle of radius $\lambda$ around the origin, counted with multiplicities.

Theorem(Selberg): As $\lambda \to \infty$

$$M_\Gamma(\lambda) = N_{scre}(\lambda) + O(\lambda).$$

- In general, $N_\Gamma(\lambda)$ and $M_\Gamma(\lambda)$ can not be separated.

# Arithmetic groups

In general, we know very little about the analytic properties of the scattering matrix. For the principal congruence subgroup

$$\Gamma(N) := \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) \colon \gamma \equiv \mathrm{Id} \mod N\},$$

however, the entries of the scattering matrix can be expressed in terms of known functions of analytic number theory.

Theorem (Huxley): For $\Gamma(N)$ we have

$$\det C(s) = (-1)^l A^{1-2s} \left( \frac{\Gamma(1-s)}{\Gamma(s)} \right)^k \prod_{\chi} \frac{L(2-2s,\bar{\chi})}{L(2s,\chi)},$$

where $k, l \in \mathbb{Z}$, $A > 0$, $\chi$ Dirichlet character mod $k$, $k|N$, $L(s,\chi)$ Dirichlet $L$-function with character $\chi$.

# Arithmetic groups

Especially, for $N = 1$ we have

$$C(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)},$$

where $\zeta(s)$ denotes the Riemann zeta function.

Thus for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ we get

$$\left\{\text{scattering resonances}\right\} = \left\{\frac{1}{2}\rho \colon \zeta(\rho) = 0,\ 0 < \mathrm{Re}(\rho) < 1\right\}.$$

A similar result holds for $\Gamma(N)$. By standard facts of analytic number theory, we get

$$M_\Gamma(\lambda) = O(\lambda \log \lambda).$$

# Arithmetic groups

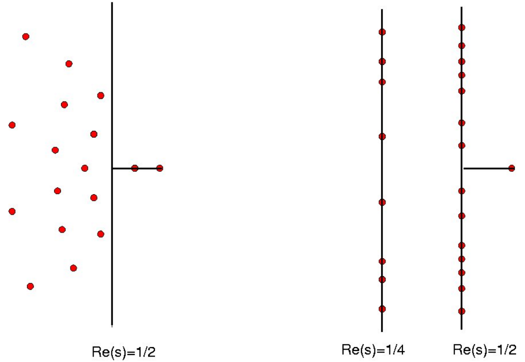
Theorem 5 (Selberg, 1956):

$$N_{\Gamma(N)}(\lambda) = \frac{\text{Area}(X(N))}{4\pi}\lambda^2 + O(\lambda \log \lambda), \quad \lambda \to \infty.$$

Thus for $\Gamma(N)$, $L^2$-eigenfunctions of $\Delta$ with eigenvalue $\lambda \geq 1/4$ ( $=$ Maass automorphic cusp forms) exist in abundance.

Conjecture (Phillips, Sarnak, 1986): Except for the Teichmüller space of the once punctured torus, a generic $\Gamma$ has only a finite number of eigenvalues.

- Thus for generic $\Gamma$ the scattering resonances are expected to dominate in the counting function.
- What are the special properties of $\Gamma(N)$ that imply the existence of embedded eigenvalues ?
  The existence of Hecke operators



The left figure shows the expected distribution of resonances for a generic surfaces. The figure on the right shows the distribution of resonances for the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$, under the assumption of the Riemann hypothesis. Except for the pole at $s = 1$, the scattering resonances are on the line $\mathrm{Re}(s) = 1/4$ and the poles corresponding to eigenvalues are on the line $\mathrm{Re}(s) = 1/2$.

# 3. Higher rank

## General framework

- $G$ semisimple real Lie group, non-compact type, finite center.
- $K \subset G$ maximal compact subgroup.
- $\widetilde{X} = G/K$ Riemannian symmetric space of non-positive curvature, equipped with $G$-invariant metric.
- $\Gamma \subset G$ lattice, i.e., a discrete subgroup of $G$ with $\mathrm{vol}(\Gamma\backslash G) < \infty$.
- $\Gamma$ acts properly discontinously on $\widetilde{X}$.
- $X = \Gamma\backslash\widetilde{X} = \Gamma\backslash G/K$ locally symmetric space.

**Example:** $G = \mathrm{SL}(n, \mathbb{R})$, $\widetilde{X} = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, $X = \Gamma(N)\backslash\widetilde{X}$, where

$$\Gamma(N) = \{\gamma \in \mathrm{SL}(n, \mathbb{Z}) \colon \gamma \equiv \mathrm{Id} \mod N\}.$$

## Automorphic forms

- $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ center of the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}$.
- $\phi \in C^\infty(\Gamma \backslash G)$ is an automorphic form, if $\phi$ is $K$-finite, $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$-finite, and of moderate growth.

**Example.** $\phi \in L^2(\Gamma \backslash G)^K \cong L^2(\Gamma \backslash \widetilde{X})$, joint eigenfunction of $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$.

- $G = \mathrm{SL}(2, \mathbb{R})$: Unified treatment of classical and Maass automorphic forms.

## Cusp forms

- $P \subset G$ parabolic subgroup, $P = M_P A_P N_P$, $M_P$ reductive, $A_P \cong (\mathbb{R}^+)^k$, $N_P$ unipotent radical.
- A standard parabolic subgroup $P$ of $\mathrm{SL}(n, \mathbb{R})$ is a stabilizer of a flag
$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_m \subset \mathbb{R}^n.$$

# Cusp forms

- $P$ is called cuspidal parabolic, if $(\Gamma \cap N_P)\backslash N_P$ is compact.
- $\phi \in C^\infty(\Gamma\backslash G)$ automorhic form, $\phi$ is called cusp form, if

  $$\int_{(\Gamma\cap N_P)\backslash N_P} f(ng)dn = 0$$

  for all proper cuspidal parabolic subgroups $P \subset G$.

# Representation theory

- $R_\Gamma$ right regular representation of $G$ in $L^2(\Gamma\backslash G)$:

$$(R_\Gamma(g)f)(g') = f(g'g), \quad f \in L^2(\Gamma\backslash G).$$

- Theory of Eisenstein series implies decomposition in invariant subspaces

$$L^2(\Gamma\backslash G) = L^2_{\mathrm{dis}}(\Gamma\backslash G) \oplus L^2_{ac}(\Gamma\backslash G)$$

- $L^2_{\mathrm{dis}}(\Gamma\backslash G)$ maximal invariant subspace, spanned by irreducible subrepresentations,

$$R_{\Gamma,\mathrm{dis}} \cong \widehat{\bigoplus}_{\pi\in\Pi(G)} m_\Gamma(\pi)\pi.$$

- $m_\Gamma(\pi) = \dim \operatorname{Hom}_G(\pi, R_\Gamma) = \dim \operatorname{Hom}_G(\pi, R_{\Gamma,\mathrm{dis}})$.
- $m_\Gamma(\pi) < \infty$ for all $\pi \in \Pi(G)$.

- $L^2_{\mathrm{cus}}(\Gamma\backslash G) \subset L^2_{\mathrm{dis}}(\Gamma\backslash G)$ subspace of cusps forms.

$$L^2_{\mathrm{dis}}(\Gamma\backslash G) = L^2_{\mathrm{cus}}(\Gamma\backslash G) \oplus L^2_{\mathrm{res}}(\Gamma\backslash G)$$

- $L^2_{\mathrm{res}}(\Gamma\backslash G)$ residual subspace, spanned by iterated residues of Eisenstein series.

$$R_{\Gamma,\mathrm{cus}} \cong \widehat{\bigoplus}_{\pi\in\Pi(G)} m_{\Gamma,\mathrm{cus}}(\pi)\pi.$$

Main problem: Study of the multiplicities $m_\Gamma(\pi)$.

- Apart from special cases, one cannot hope to describe $m_\Gamma(\pi)$ explicitly.
- There exist formulas for $m_\Gamma(\pi)$ if $\pi$ is a discrete series representation.

## Further problems

- Existence of automorphic forms.
- Selberg conjecture: $\lambda_1(N)$ first positve eigenvalue of $\Delta$ on $\Gamma(N)\backslash\mathbb{H}^2$. For all $N \in \mathbb{N}$: $\lambda_1(N) \geq 1/4$.
- Ramanujan conjecture for $\mathrm{GL}(n)$.
- Generalized Ramanujan conjecture.
- Structure of the residual spectrum.
- Mœglin, Waldspurger: Determination of the residual spectrum for $G = \mathrm{GL}(n)$. What can be said about the residual spectrum for general $G$?

## 3. Asymptotic behavior of automorphic spectra

- Study behavior of multiplicities with respect to the growth of various parameters such as the infinitesimal character or/and the level of congruence subgroups.

**Examples:**

### a) Weyl law.

- For $\sigma \in \Pi(K)$ let
$$\Pi(G;\sigma) = \left\{\pi \in \Pi(G) \colon [\pi|_K : \sigma] > 0\right\}.$$
- $\lambda_\pi = \pi(\Omega)$ Casimir eigenvalue of $\pi \in \Pi(G)$.
$$N_\Gamma(\lambda;\sigma) = \sum_{\substack{\pi \in \Pi(G;\sigma) \\ |\lambda_\pi| \le \lambda}} m_\Gamma(\pi).$$

**Problem:** Behavior of $N_\Gamma(\lambda;\sigma)$ as $\lambda \to \infty$.

### Equivalent problem:

- $\sigma = 1$, $\widetilde{X} = G/K$, $X = \Gamma\backslash\widetilde{X}$,
$L^2(\Gamma\backslash G)^K \cong L^2(\Gamma\backslash G/K) = L^2(X)$, $\Delta \colon C^\infty(X) \to C^\infty(X)$
Laplace operator.

Lemma (Kuga): $-R(\Omega) = \Delta$.

- $0 < \lambda_1 \le \lambda_2 \le \cdots$ eigenvalues of $\Delta$ in $L^2(X)$.
- $N_\Gamma(\lambda) = \#\{j \colon \lambda_j \le \lambda\}$

**Problem:** Study behavior of $N_\Gamma(\lambda)$ as $\lambda \to \infty$.

b) Limit multiplicity problem.

- Let $\Pi(G)$ be equipped with the Fell topology.

Define a measure on $\Pi(G)$ by

$$\mu_\Gamma = \frac{1}{\operatorname{vol}(\Gamma\backslash G)} \sum_{\pi \in \Pi(G)} m_\Gamma(\pi)\delta_\pi,$$

where $\delta_\pi$ is the delta distribution.

**Problem:** Study the behavior of $\mu_\Gamma$ as $\operatorname{vol}(\Gamma\backslash G) \to \infty$.

- Distribution of Hecke eigenvalues
- Sato-Tate conjecture for modular forms.
- Analytic torsion, Approximation of $L^2$-invariants
- Families of automorphic forms and low lying zeros of automorphic $L$-functions (Sarnak).

## 4. Conjectures and results

### a) Wely law

**i) $\Gamma$ cocompact.** $n = \dim X$. Then the following Weyl law holds

$$N_\Gamma(\lambda;\sigma) = \frac{\dim(\sigma)\,\mathrm{vol}(X)}{(4\pi)^{n/2}\Gamma(n/2+1)}\lambda^{n/2} + O(\lambda^{(n-1)/2}), \quad \lambda \to \infty.$$

- $\sigma = 1$: Avakumović, Hörmander: general elliptic operators.
- **Method:** Wave equation method, Selberg trace formula in the locally symmetric case.

**ii) $\Gamma\backslash G$ non-compact:**

**Problem:** Nonempty continuous spectrum.

- Langlands: Continuous spectrum is described by Eisenstein series.

# b) Higher rank.

The Weyl law holds in the following cases:

**i)** $\sigma = 1$, no estimation of the remainder term.

- S. Miller: $X = \mathrm{SL}(3,\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$.
- E. Lindenstrauss, A. Venkatesh: $\mathbf{G}$ split adjoint semisimple group over $\mathbb{Q}$, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ congruence subgroup, $G = \mathbf{G}(\mathbb{R})$, $X = \Gamma\backslash G/K$. **Method:** Use of Hecke operators.

**ii)** $\sigma \in \Pi(K)$ arbitrary, no estimation of the remainder term.

In this case, the Weyl law is the following statement:

$$N_\Gamma(\lambda;\sigma) \sim \frac{\dim(\sigma)\operatorname{vol}(X)}{(4\pi)^{n/2}\Gamma(n/2+1)}\lambda^{n/2}, \quad \lambda \to \infty.$$

Theorem, Mü.: Let $\mathbf{G}$ be one of the following groups over $\mathbb{Q}$:

1. $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$ and their inner forms.
2. Quasi-split classical groups.
3. The exceptionel group $\mathrm{G}_2$.

Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a congruence subgroup and $\sigma \in \Pi(K)$. Then the Weyl law holds for $N_\Gamma(\lambda, \sigma)$.

**Methods:** Heat equation method.

- $\sigma \in \Pi(K)$, $\tilde{E}_\sigma \to \tilde{X}$ homogeneous vector bundle associated to $\sigma$, $E_\sigma := \Gamma\backslash\tilde{E}_\sigma \to X$ locally homogeneous vector bundle, $\Delta_\sigma := (\nabla^\sigma)^*\nabla^\sigma$ Bochner-Laplace operator acting in $C^\infty(X, E_\sigma)$, $0 \leq \lambda_1(\sigma) \leq \lambda_2(\sigma) \leq \cdots$ eigenvalues of $\Delta_\sigma$.
- Use Arthur trace formula, applied to the heat operator, to show that

$$\sum_j e^{-t\lambda_j(\sigma)} + \text{contr. of cont. spectrum}$$

$$\sim \frac{\dim(\sigma)\,\mathrm{vol}(X)}{(4\pi)^{n/2}} t^{-n/2}$$

as $t \to 0$.

- Main ingredient of the proof: Study of the automorphic $L$-functions that appear in the constant terms of Eisenstein series. $L(\pi, s, r)$, $\pi \in \Pi_{\mathrm{cus}}(\mathbf{M}(\mathbb{A}))$, $\mathbf{M}$ Levi subgroup of $\mathbf{G}$, $r\colon {}^L\mathbf{M} \to \mathrm{GL}(N, \mathbb{C})$ representation of the Langlands dual group.
- $\pi = \otimes_v \pi_v$, $\pi_v \in \Pi(\mathbf{M}(\mathbb{Q}_v))$, $S$ finite set of places, $\infty \in S$.
- $p \notin S$, $\{t_{\pi_p}\} \subset {}^L\mathbf{M}$ semisimple conjugacy class associated to $\pi_p$. Partial $L$-function:

$$L^S(\pi, s, r) := \prod_{p \notin S} \det(\mathrm{Id} - r(t_{\pi_p})p^{-s})^{-1}.$$