Analytic torsion of locally symmetric spaces and cohomology of arithmetic groups

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Göttingen, September 30 - October 4, 2019

1. Analytic torsion

- (X, g) a compact Riemannian manifold, $\dim X = n$.
- $\rho\colon \pi_1(X) \to \mathrm{GL}(V)$ finite-dimensional representation.
- $E_\rho \to X$ associated flat vector bundle.
- h Hermitian fibre metric in E_ρ.

Let

- $\Lambda^p(X, E_\rho) := C^\infty(X, \Lambda^p T^*(X) \otimes E_\rho)$ space of E_ρ-valued p-forms, equipped with inner product defined by g and h.

$$\cdots \to \Lambda^p(X, E_\rho) \xrightarrow{d_p} \Lambda^{p+1}(X, E_\rho) \to \cdots$$

de Rham complex.

$U \subset X$ open, $E_\rho|_U \cong U \times V$. $s_1, \dots, s_r \in C^\infty(U, E_\rho|_U)$ bases of flat sections. $\varphi \in \Lambda^p(U, E_\rho)$. Then $\varphi = \sum_{j=1}^r \varphi_j \otimes s_j$, $\varphi_j \in \Lambda^p(U)$.

$$d\varphi = \sum_{j=1}^r d\varphi_j \otimes s_j.$$

- $d_{p+1}^*\colon \Lambda^{p+1}(X,E_\rho)\to\Lambda^p(X,E_\rho)$ formal adjoint of d_p.
- $$\Delta_p(\rho)=d_{p+1}^*d_p+d_{p-1}d_p^*\colon \Lambda^p(X,E_\rho)\to\Lambda^p(X,E_\rho)$$

  Laplace operator on E_ρ-valued p-forms.

- $\Delta_p(\rho)$ 2nd order, elliptic, self-adjoint, $\Delta_p(\rho)\geq 0$.
- Spectrum of $\Delta_p(\rho)$: Eigenvalues $0\leq\lambda_1\leq\lambda_2\leq\cdots\to\infty$
- $e^{-t\Delta_p(\rho)}$ trace class operator, $\operatorname{Tr}(e^{-t\Delta_p(\rho)})=\sum_{j=1}^\infty e^{-t\lambda_j}$.

Let $h_p(\rho)=\dim\ker\Delta_p(\rho)$ and let

$$\zeta_p(s;\rho):=\sum_{\lambda_j>0}^{\infty}\lambda_j^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty(\operatorname{Tr}(e^{-t\Delta_p(\rho)})-h_p(\rho))t^{s-1}\,dt,$$

$\operatorname{Re}(s)>n/2$, be the zeta function of $\Delta_p(\rho)$.

- $\zeta_p(s;\rho)$ admits meromorphic extension to $\mathbb{C}$, holomorphic at $s=0$.

Put

$$\det\Delta_p(\rho) := \exp\left(-\frac{d}{ds}\zeta_p(s;\rho)\Big|_{s=0}\right).$$

This is the regularized or functional determinant of $\Delta_p(\rho)$.

- $A\in\mathrm{End}(\mathbb{R}^n)$, $A=A^t$, $A>0$. $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n$ eigenvalues of A.
- $\zeta(s,A):=\sum_{j=1}^n\lambda_j^{-s}$, $s\in\mathbb{C}$.

Then

$$\exp\left(-\frac{d}{ds}\zeta(s,A)\big|_{s=0}\right)=\prod_{j=1}^n\lambda_j=\det(A).$$

Ray-Singer analytic torsion:

$$T_X(\rho):=\prod_{j=1}^n\left(\det\Delta_p(\rho)\right)^{(-1)^{p+1}p/2}.$$

- $T_X(\rho)$ depends on the metrics g on X and h in E_ρ.
- If $\dim X$ is odd and $H^*(X; E_\rho) = 0$, then $T_X(\rho)$ is independent of g and h.
- Combinatorial analogue: Reidemeister torsion $\tau_X(\rho)$, defined in terms of C^∞-triangulation $K \cong X$.

Theorem (J. Cheeger, Mü.) For all unitary ρ we have

$$T_X(\rho) = \tau_X(\rho).$$

Extensions:

- $\rho \colon \pi_1(X) \to \mathrm{GL}(V)$ is called unimodular, if $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(X)$.
- Mü.: $T_X(\rho) = \tau_X(\rho)$ for all unimodular ρ.
- Bismut, Zhang: General ρ, $T_X(\rho) = \tau_X(\rho) + \delta$, defect δ is explixitely described.

Relation with cohomology.

- Assume there exists a $\pi_1(X)$-invariant lattice $M \subset V_\rho$.
  Example: $\rho = 1$, $\mathbb{Z} \subset \mathbb{R}$.
- $\mathcal{M}$ associated local system of free $\mathbb{Z}$-modules of finite rank over X, $\mathcal{M} \otimes \mathbb{R} = E_\rho$.
- $H^p(X, \mathcal{M})$ cohomology of X with coefficients in $\mathcal{M}$, finitely generated abelian group. Moreover

$$H^j(\Gamma, M) = H^j(X, \mathcal{M}).$$

- $H^p(X, \mathcal{M})_{\mathrm{tors}}$ torsion subgroup.
- $H^p(X, M)_{\mathrm{free}} = H^p(X, \mathcal{M})/H^p(X, \mathcal{M})_{\mathrm{tors}}$ free part, lattice in $H^p(X, E_\rho)$.
- $\langle \cdot, \cdot \rangle$ inner product in $H^p(X, E_\rho)$ induced by Hodge isomorphism $H^p(X; E_\rho) \cong \mathcal{H}^p(X; E_\rho)$.

$$R_p(\mathcal{M}) := \mathrm{vol}(H^p(X, E_\rho)/H^p(X; \mathcal{M})_{free})$$

The regulator $R(\mathcal{M})$ is defined as

$$R(\mathcal{M}) := \prod_{p=0}^{n} R_p(\mathcal{M})^{(-1)^p}.$$

Corollary.

$$T_X(\rho) = R(\mathcal{M}) \cdot \prod_{p=0}^{n} \left| H^p(X;\mathcal{M})_{tors} \right|^{(-1)^{p+1}}.$$

Especially, if $H^*(X, E_\rho) = 0$, then $H^*(X;\mathcal{M})$ is finite and

$$T_X(\rho) = \prod_{p=1}^{n} (\det \Delta_p(\rho))^{(-1)^{p+1}p/2} = \prod_{p=0}^{n} \left| H^p(X;\mathcal{M}) \right|^{(-1)^{p+1}}.$$

- This was first observed by Cheeger in his paper on analytic torsion in the case of pure torsion groups, Bergeron-Venkatesh extended it.

2. L^2-torsion

J. Lott, V. Mathai, 1992

Recall that

$$\log T_X(\rho) = \frac{1}{2}\sum_{p=1}^{n}(-1)^p p \frac{d}{ds}\zeta_p(s;\rho)\big|_{s=0}.$$

The zeta function can be written as

$$\zeta_p(s;\rho) = \frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}(e^{-t\Delta_p(\rho)}) - b_p(\rho))t^{s-1}dt, \quad \mathrm{Re}(s) > n/2,$$

and $b_p(\rho) := \dim\ker\Delta_p(\rho)$.

- $\widetilde{X} \to X$ universal covering of X, $\Gamma := \pi_1(X)$, $d\tilde{\mu}$ Γ-invariant measure on $\widetilde{X}$, $F \subset \widetilde{X}$ fundamental domain for the action of Γ.
- **Assume:** $\widetilde{X}$ is noncompact, i.e., Γ infinite.
- $\widetilde{\Delta}_p(\rho)\colon \Lambda^p(\widetilde{X}, \widetilde{E}_\rho) \to \Lambda^p(\widetilde{X}, \widetilde{E}_\rho)$ lift of $\Delta_p(\rho)$ to $\widetilde{X}$.
- $\widetilde{K}_p^\rho(t,x,y) \in \mathrm{Hom}(\Lambda^p T_y^*(\widetilde{X}) \otimes E_{\rho,y}, \Lambda^p T_x^*(\widetilde{X}) \otimes E_{\rho,x})$ kernel of $e^{-t\widetilde{\Delta}_p(\rho)}$. The von Neumann trace of $e^{-t\widetilde{\Delta}_p(\rho)}$ is given by

$$\mathrm{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\rho)}\right) := \int_F \mathrm{tr}\, \widetilde{K}_p^\rho(t,x,x)dx.$$

- We need to determine the asymptotic behavior of $\mathrm{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\rho)}\right)$ as $t \to +0$ and $t \to \infty$.

a) $t \to +0$. Lott, 1992: Let d be the length of the shortest closed geodesic on X. Put $N = [n/4] + 1$. For any $\varepsilon > 0$ we have

$$\mathrm{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\rho)}\right) - \mathrm{Tr}\left(e^{-t\Delta_p(\rho)}\right) = O(t^{-4N+(1/2)}e^{-(d-\varepsilon)^2/4t}), \quad t \to 0.$$

b) $t \to \infty$. Novikov-Shubin invariants:

Let $b_p^{(2)}(\rho) := \dim_\Gamma \ker \widetilde{\Delta}_p(\rho)$. Define $\widetilde{\alpha}_p(X, \rho) \in [0, \infty]$ by

$$\widetilde{\alpha}_p(\widetilde{X}, \rho) := \sup \left\{ \beta_p : \ \mathrm{Tr}_\Gamma(e^{-t\widetilde{\Delta}_p(\rho)}) - b_p^{(2)}(\rho) = O(t^{-\frac{\beta_p}{2}}) \ \text{ as } \ t \to \infty \right\}.$$

Novikov, Shubin: $\rho = 1$, $\widetilde{\alpha}_p(\widetilde{X})$ are homeomorphism invariants.

Conjecture: $\widetilde{\alpha}_p(\widetilde{X}, \rho) > 0$ for all X, ρ, and $p = 0, \dots, n$.

Assume: $\widetilde{\alpha}_p(\widetilde{X}, \rho) > 0$, $p = 0, \dots, n$.

Let $\widetilde{\Delta}_p(\rho)'$ be the restriction of $\widetilde{\Delta}_p(\rho)$ to $\ker(\widetilde{\Delta}_p(\rho))^\perp$. Then $T_X^{(2)}(\rho) \in \mathbb{R}^+$ is defined by

$$\log T_X^{(2)}(\rho) := \frac{1}{2} \sum_{p=1}^{n} (-1)^p p \left\{ \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^1 \mathrm{Tr}_\Gamma(e^{-t\widetilde{\Delta}_p(\rho)'}) t^{s-1} dt \right) \Bigg|_{s=0} + \int_1^\infty \mathrm{Tr}_\Gamma(e^{-t\widetilde{\Delta}_p(\rho)'}) t^{-1} dt \right\}.$$

If $\widetilde{\alpha}_p(X,\rho)=\infty$, $p=0,\dots,n$, then

$$\log T_X^{(2)}(\rho) := \frac{1}{2}\sum_{p=1}^{n}(-1)^p p \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty Tr_\Gamma(e^{-t\widetilde{\Delta}_p(\rho)'})t^{s-1}dt\right)\Bigg|_{s=0}.$$

3. Locally symmetric spaces

i) Symmetric spaces

- G semisimple real Lie group with finite center of non-compact type
- $K \subset G$ maximal compact subgroup
- $\widetilde{X} = G/K$ Riemannian symmetric space of non-positive curvature, equipped with a G-invariant Riemannian metric g.
- geodesic inversion about any $x \in S$ is a global isometry.

Examples. 1. $\mathbb{H}^n$ hyperbolic n-space

$$\mathbb{H}^n = \left\{(x_1, ..., x_n) \in \mathbb{R}^n \colon x_n > 0\right\} \cong \mathrm{SO}_0(n,1)/\,\mathrm{SO}(n).$$

The invariant metric is given by

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

2. S space of positive definite $n \times n$-matrices of determinant 1.

$$S = \{Y \in \mathrm{Mat}_n(\mathbb{R}) \colon Y = Y^*, \ Y > 0, \ \det Y = 1\}$$
$$\cong \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$$

- Invariant metric: $\quad ds^2 = \mathrm{Tr}(Y^{-1} dY \cdot Y^{-1} dY)$.
- $G = \mathrm{SL}(n, \mathbb{R})$ acts on S by $Y \mapsto g^t Y g$, $g \in G$.

ii) Locally symmetric spaces

- $\Gamma \subset G$ a lattice, i.e., Γ is a discrete subgroup of G and $\mathrm{vol}(\Gamma \backslash G) < \infty$
- Γ acts properly discontinuously on $\widetilde{X}$.
- $X = \Gamma \backslash \widetilde{X} = \Gamma \backslash G / K$ locally symmetric space

Example:

- $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2) \cong \mathbb{H}^2 = \{z \in \mathbb{C}\colon\ \mathrm{Im}(z) > 0\}$, $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by fractional linear transformations:

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

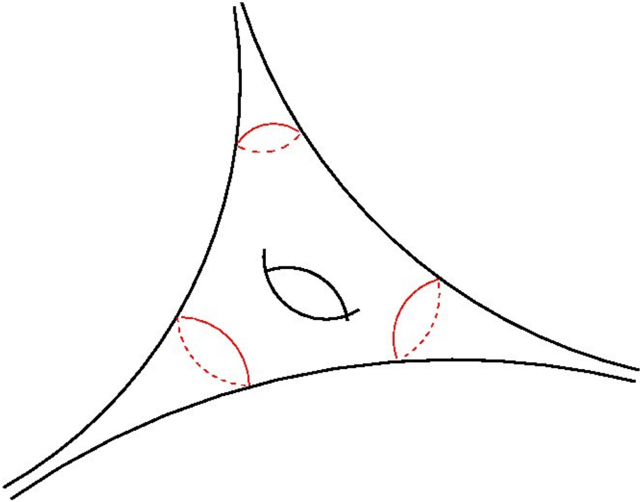
- $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ discrete subgroup, torsion free, $\mathrm{vol}(\Gamma\backslash\,\mathrm{SL}(2,\mathbb{R})) < \infty$.
- $X = \Gamma\backslash\mathbb{H}^2$ is a hyperbolic surface of finite area.

For $N \in \mathbb{N}$ let

$$\Gamma(N) = \left\{\gamma \in \mathrm{SL}(2,\mathbb{Z})\colon \gamma \equiv \mathrm{Id} \mod N\right\}.$$

- principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of level N.
- $\Gamma(N)\backslash\mathbb{H}^2$ hyperbolic surface, non-compact, $\mathrm{Area}(\Gamma(N)\backslash\mathbb{H}^2) < \infty$.

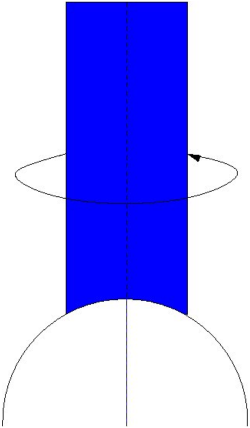
Surface of genus 2



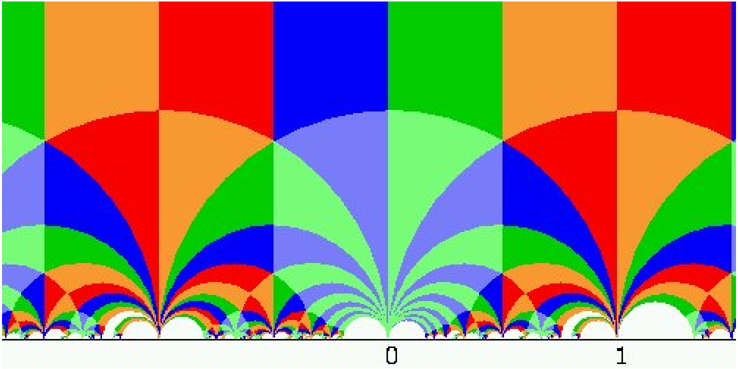
Surface of genus 1 with 3 cusps

Fundamental domain

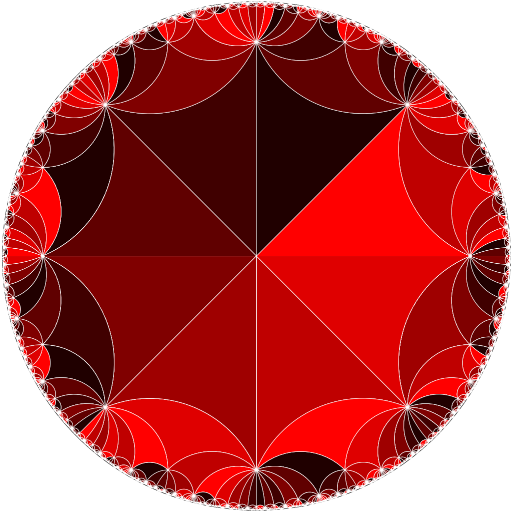
A discrete group can be visualized by its fundamental domain.



The standard fundamental domain of the modular group $\mathrm{SL}(2, \mathbb{Z})$.



Tessellation of the hyperbolic plane by the fundamental domain of $SL(2, \mathbb{Z})$



Tessellation of the Poincaré disc by the fundamental domain of a Coxeter group (courtesy of H. Koch, Bonn)

Flat bundles

- $\tau\colon G \to \mathrm{GL}(V)$ representation, $\dim_{\mathbb{C}} V < \infty$.
- $\rho := \tau|_{\Gamma}\colon \Gamma \to \mathrm{GL}(V)$, unimodular representation.
- $E_\rho = \Gamma\backslash(\widetilde{X} \times V) \to X$ associated flat vector bundel over X.
- $\sigma := \tau|_K\colon K \to \mathrm{GL}(V)$.
- K acts on $G \times V$ from the right by $(g, v) \cdot k = (gk, \sigma(k)^{-1} v)$.
- $\widetilde{E} := (G \times V)/K$ homogeneous vector bundle over $\widetilde{X} = G/K$associated to σ.

Lemma (Matsushima-Murakami) There is a canonical isomorphism

$$\Gamma\backslash\widetilde{E} \cong E_\rho.$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition, where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{k} = \mathrm{Lie}(K)$.

Matsushima-Murakami: There exists inner product $\langle \cdot, \cdot \rangle$ in V such that

$$\langle \tau(Y)u, v\rangle = -\langle u, \tau(Y)v\rangle, \quad Y \in \mathfrak{k},\ u, v \in V;$$
$$\langle \tau(Y)u, v\rangle = \langle u, \tau(Y)v\rangle, \quad Y \in \mathfrak{p},\ u, v \in V.$$

- Induces G-invariant metric in $\widetilde{E}$ and by the Lemma a metric in $\Gamma\backslash\widetilde{E} \cong E_\rho$.
- $\Lambda^p(X, E_\rho) \cong (C^\infty(\Gamma\backslash G) \otimes \Lambda^p\mathfrak{p} \otimes V^*)^K$, where K acts on $\Lambda^p\mathfrak{p} \otimes V^*$ by $\Lambda^p \operatorname{Ad}_{\mathfrak{p}} \otimes\tilde{\sigma}$.
- R_Γ right regular representation of G in $C^\infty(\Gamma\backslash G)$, $\Omega \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ Casimir element.

Lemma (Kuga) With respect to the isomorphism above, we have

$$\Delta_p(\rho) = -R_\Gamma(\Omega) + \tau(\Omega).$$

By the Lemma of Kuga, harmonic analysis can be used to study the spectrum of $\Delta_p(\rho)$.

4. Approximation of L^2-torsion

Lück: Approximation of L^2-invariants.

- X compact Riemannian manifold, $\Gamma := \pi_1(X, x_0)$.
- Assume that Γ is residual finite: $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_j \supset \cdots \supset \{e\}$, $\cap_j \Gamma_j = \{e\}$, tower of normal subgroups of finite index.
- $X_j := \Gamma_j \backslash \widetilde{X}$, $X_j \to X$ finite covering.
- $\rho\colon \Gamma \to \mathrm{GL}(V)$ finite dimensional representation.
- $T_{X_j}(\rho)$ analytic torsion with respect to $\rho_j := \rho|_{\Gamma_j}$.
- $T_X^{(2)}(\rho)$ L^2-torsion.

Conjecture. (Lück,...)

$$\lim_{j \to \infty} \frac{\log T_{X_j}(\rho)}{[\Gamma : \Gamma_j]} = \log T_X^{(2)}(\rho).$$

Known results:

Case of locally symmetric spaces

- $\widetilde{X} = G/K$, $\Gamma \subset G$ co-compact, torsion free lattice, $X = \Gamma \backslash \widetilde{X}$ compact locally symmetric manifold.
- $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_j \supset \cdots \supset \{e\}$ tower of normal subgroups of finite index, $X_j = \Gamma_j \backslash \widetilde{X}$ finite covering of X.
- $\tau \in \mathrm{Rep}(G)$, $E_j \to X_j$ flat vector bundle attached to $\rho_j = \tau|_{\Gamma_j}$.
- $\theta\colon G \to G$ Cartan involution, $\tau_\theta := \tau \circ \theta$.
- $\Delta_{X_j,p}(\rho_j)\colon \Lambda^p(X_j, E_j) \to \Lambda^p(X_j, E_j)$ Laplacian on E_j-valued p-forms on X_j.

Proposition (Bergeron/Venkatesh): Assume that $\tau_\theta \ncong \tau$. There exists $c > 0$ such that

$$\mathrm{Spec}(\Delta_{X_j,p}(\rho_j)) \subset [c, \infty)$$

for all $j \in \mathbb{N}$ and $p = 0, ..., n$.

- Such a τ is called strongly acyclic.

Example: For $p, q \in \mathbb{N}_0$ let $\rho_{p,q}\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(V_{p,q})$ be defined by

$$\rho_{p,q} := \mathrm{Sym}^p(\mathbb{C}^2) \otimes \overline{\mathrm{Sym}^q(\mathbb{C}^2)}$$

Then $\rho_{p,q}$ is strongly acyclic if and only if $p \neq q$.

Theorem (Bergeron-Venkatesh) Assume that $\tau \not\cong \tau_\theta$. Then

$$\lim_{j\to\infty} \frac{\log T_{X_j}(\rho_j)}{[\Gamma : \Gamma_j]} = \mathrm{vol}(X) t^{(2)}_{\widetilde{X}}(\tau),$$

where $\mathrm{vol}(X) t^{(2)}_{\widetilde{X}}(\tau)$ is the L^2-torsion.

Sketch of the proof.

1) Under the assumption $\tau_\theta \not\cong \tau$ there exists $c > 0$ s.th. $\mathrm{Spec}(\Delta_{X_j,p}(\rho_j)) \subset [c, \infty)$ for all $j \in \mathbb{N}$ and $p = 0, \ldots, n$.

Then

$$-\log\det\Delta_{X_j,p}(\rho)=\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}\left(e^{-t\Delta_{X_j,p}(\rho)}\right)t^{s-1}\,dt\right)\Big|_{s=0}.$$

2) Let $A>0$. Then

$$\begin{aligned}-\log\det\Delta_{X_j,p}(\rho)=&\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^A \operatorname{Tr}\left(e^{-t\Delta_{X_j,p}(\rho)}\right)t^{s-1}\,dt\right)\Big|_{s=0}\\&+\int_A^\infty t^{-1}\operatorname{Tr}\left(e^{-t\Delta_{X_j,p}(\rho)}\right)\,dt.\end{aligned}$$

Existence of a uniform spectral gap implies: For $\varepsilon>0$ there exists $A>0$ s.th.

$$\frac{1}{[\Gamma:\Gamma_j]}\int_A^\infty t^{-1}\operatorname{Tr}\left(e^{-t\Delta_{X_j,p}(\rho)}\right)\,dt\le\varepsilon$$

for all $j\in\mathbb{N}$.

To deal with the first term, we apply the Selberg trace formula.

- Let $p = 0$, $\tau = 1$, $X = \Gamma\backslash\widetilde{X}$.
- $\Delta\colon C^\infty(X) \to C^\infty(X)$ Laplace operator.
- $K(t, x, y)$ kernel of $e^{-t\Delta}$.
- $\widetilde{\Delta}$ Laplace operator of $\widetilde{X}$, $k(t, x, y)$ kernel of $e^{-t\widetilde{\Delta}}$.

$$K(t, x, y) = \sum_{\gamma \in \Gamma} k(t, x, \gamma(y)).$$

$$\operatorname{Tr}\left(e^{-t\Delta}\right) = \int_X K(t, x, x)\, dx = \int_X \sum_{\gamma \in \Gamma} k(t, x, \gamma(x))\, dx.$$

Moreover, there exists $h_t \in C^\infty(\mathbb{R}^+)$ such that

$$k(t, x, y) = h_t(d(x, y)).$$

- $\ell(X)$ length of shortest closed geodesic of X.

There exist $C, c > 0$, depending only on (G, ρ, A) such that

$$\frac{1}{\mathrm{vol}(X)}\left| \int_X \sum_{\gamma \in \Gamma - \{1\}} k(t, x, \gamma(x))\, dx \right| \leq C t^{-(n+1)} e^{-(\ell(X) - ct)^2/5t}$$

for $0 < t \leq A$. Moreover

$$\int_X k(t, x, x)\, dx = h_t(0)\, \mathrm{vol}(X).$$

This implies

$$\frac{\log T_{X_j}(\rho)}{[\Gamma : \Gamma_j]} \to t^{(2)}_{\widetilde{X}}(\tau)\, \mathrm{vol}(X).$$

as $j \to \infty$.

4. Arithmetic groups

- $\mathbf{G}$ connected semisimple (reductive) algebraic group over $\mathbb{Q}$,
- $\mathbf{G} \subset \mathrm{GL}_n$ fixed embedding,
- $G = \mathbf{G}(\mathbb{R})$,
- $\Gamma \subset \mathbf{G}(\mathbb{Q})$ arithmetic subgroup, Γ is commensurable with $\mathbf{G}(\mathbb{Z}) := \mathbf{G}(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$.

Examples:
1) For $N \in N$. Let $\Gamma(N) \subset \mathrm{SL}(2,\mathbb{Z})$ be the principal congruence sungroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \colon a, d \equiv 1 \mod N, b, c \equiv 0 \mod N \right\}$$

2) $F = \mathbb{Q}(\sqrt{-D})$, $D > 0$, square-free, $\mathcal{O}_D$ ring of integers of F, $\Gamma(D) = \mathrm{SL}(2, \mathcal{O}_D)$ Biachi-group. Discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$. Let $\mathfrak{a} \subset \mathcal{O}_D$ be a nonzero ideal. Congruence subgroup of level $\mathfrak{a}$:

$$\Gamma(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(D) \colon a - 1 \in \mathfrak{a},\ d - 1 \in \mathfrak{a};\ b, c \in \mathfrak{a} \right\}.$$

- $\rho \colon G \to \mathrm{GL}(V)$ representation, $M \subset V$ Γ-invariant lattice.
- $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_j \supset \cdots \supset \{1\}$ tower of congruence subgroups, $X_j = \Gamma_j \backslash \widetilde{X}$.

The Γ-module M is called strongly acyclic if the spectrum of $\Delta_{X_j,p}(\rho)$ is uniformly bounded away from 0 for all p and j.

- strongly acyclic Γ-modules exist (Bergeron-Venkatesh)

Example: Let $\Gamma \subset \mathrm{SL}(2, \mathbb{Z}[i])$ be a congruence subgroup. For $p, q \in \mathbb{N}$,

$$M_{p,q} := \mathrm{Sym}^p(\mathbb{Z}[i]^2) \otimes \overline{\mathrm{Sym}^q(Z[i]^2)}$$

is strongly acyclic if and only if $p \neq q$.

Let

$$\delta(G) = \mathrm{rank}_{\mathbb{C}}\, G - \mathrm{rank}_{\mathbb{C}}\, K$$

be the fundamental rank.

Recall that for a strongly acyclic Γ-module we have

$$T_X(\rho) = \prod_{p=1}^{n} (\det \Delta_p(\rho))^{(-1)^{p+1}p/2} = \prod_{p=0}^{n} \left| H^p(X; \mathcal{M}) \right|^{(-1)^{p+1}}.$$

Combined with the approximation of the L^2-torsion, it follows that

Theorem (Bergeron-Venkatesh): Let M be strongly acyclic. Then

$$\lim_{N\to\infty} \sum_{p=0}^{n} (-1)^{p+1} \frac{\log |H^p(\Gamma_N; M)|}{[\Gamma : \Gamma_N]} = \mathrm{vol}(X) t^{(2)}_{\widetilde{X}}(M).$$

If $\delta(G) = 1$, we have $t^{(2)}_{\widetilde{X}}(M) \neq 0$. Then $\dim \widetilde{X}$ is odd. It follows that

$$\liminf_{N} \sum_{p} \frac{\log |H^p(\Gamma_N; M)|}{[\Gamma : \Gamma_N]} \geq C_{G,M} \,\mathrm{vol}(X),$$

where p is taken over integers with the same parity as $\frac{\dim X - 1}{2}$ and $C_{G,M} > 0$.

Example:

- $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$, $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ co-compact arithmetic subgroup, derived from quaternion algebra.
- Each even symmetric power $\mathrm{Sym}^{2k}(\mathbb{C}^2)$ contains a Γ-invariant lattice M_{2k} and is strongly acyclic.

Corollary (Bergeron, Venkatesh):
Let $\Gamma_N \subset \Gamma$ be a decreasing sequence of congruence subgroups with $\cap_N \Gamma_N = \{1\}$. Then there is $C_k > 0$ such that

$$\lim_{N\to\infty} \frac{\log |H_1(\Gamma_N, M_{2k})|}{[\Gamma : \Gamma_N]} = C_k \,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3).$$

Conjecture (B-V): Let $\Gamma_N \subset \Gamma$ be a decreasing sequence of congruence subgroups with $\cap_N \Gamma_N = \{1\}$. The limit of

$$\frac{\log |H_j(\Gamma_N; M)_{\text{tors}}|}{[\Gamma : \Gamma_N]}$$

always exists. It vanishes unless $\delta(G) = 1$ and $j = \frac{\dim(\widetilde{X})-1}{2}$. In the latter case it equals $C_{G,M}\,\mathrm{vol}(\Gamma\backslash\widetilde{X})$ with $C_{G,M} > 0$.

This conjecture can be considered as predicting three different types of behavior:

1. If $\delta(G) = 0$, then there is little torsion, but $H_j(\Gamma_N, M \otimes \mathbb{Q})$ is large.
2. If $\delta(G) = 1$, then there is a lot of torsion, but $H_j(\Gamma_N, M \otimes \mathbb{Q})$ is small.
3. If $\delta(G) \geq 2$, there is "relatively little" torsion or characteristic zero homology.

Motivation: Mod p-Langlands program. Torsion classes which are eigenclasses of Hecke operators are expected to correspond to Galois representations over finite fields.

- $F/\mathbb{Q}$ a number field, $\mathbb{A}_F = \prod'_\nu F_\nu$ adeles of F,
- $K \subset \mathrm{GL}_n(\mathbb{A}_{F,f})$ compact open subgroup,
- $K_\infty \subset \mathrm{GL}_n(F \otimes_\mathbb{Q} \mathbb{R})$ a maximal compactsubgroup,
- $\widetilde{X} = \mathrm{GL}_n(F \otimes_\mathbb{Q} \mathbb{R})/\mathbb{R}_{>0}K_\infty$.

Let

$$X_K := \mathrm{GL}_n(F)\backslash(\widetilde{X} \times \mathrm{GL}_n(\mathbb{A}_{F,f})/K).$$

Conjecture (Ash-Serre): For any system of Hecke eigenvalues appearing in $H^i(X_K, \mathbb{F}_p)$, there is a continuous semisimple representation $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ such that Frobenius and Hecke eigenvalues match up.

Theorem (P. Scholze)
Conjecture true for F totally real or CM field.

Some problems

1)

- Remove the assumption $\tau \ncong \tau_\theta$. In particular, consider the trivial representation.
- Let $\rho = 1$. There are no uniform spectral gaps for Δ_p.
- Let $X = \Gamma\backslash\mathbb{H}^n$, $n = 2d+1$. Then $\sigma(\widetilde{\Delta}^d) = [0,\infty)$.

Sufficient condition for convergence (Bergeron, Sengün, Venkatesh): Let $\lambda_i^{(j)}$ be the eigenvalues of $\Delta_{X_j,d}$. For every $\varepsilon > 0$ there exists $c > 0$ such that

$$\lim_{j\to\infty} \frac{1}{\operatorname{vol}(X_j)} \sum_{0<\lambda_i^{(j)}\le c} |\log(\lambda_i^{(j)})| \le \varepsilon.$$

Let $\lambda_1^{(j)}$ be the first positive eigenvalue of $\Delta_{X_j,d}$. Then this will follow if there exists $c > 0$ such that

$$\frac{1}{\lambda_1^{(j)}} = O(\mathrm{vol}(X_j)^c), \quad j \in \mathbb{N}.$$

- Let $\lambda_1(X)$ be the first positive eigenvalue of Δ_0.
- R. Schoen: Exists $C(n) > 0$ such that

$$\lambda_1(X) \geq \frac{C(n)}{\mathrm{vol}(X)^2}$$

  for all X.

2)
Estimation of $\mathrm{vol}(H_j(X,\mathbb{R})/H_j(X,Z)_{\mathrm{free}})$.

Sequences of representations

Now we fix Γ and vary the representation.

- $X = \Gamma\backslash\mathbb{H}^3$, compact oriented hyperbolic 3-manifold, defined by arithmetic group.

Theorem (Marshall, Mü., 2012)

For every choice of a Γ-invariant lattice M_{2k} in $Sym^{2k}(\mathbb{C}^2)$ one has

$$\lim_{k\to\infty} \frac{\log|H_1(\Gamma; M_{2k})|}{k^2} = \frac{2}{\pi}\operatorname{vol}(\Gamma\backslash\mathbb{H}^3).$$

Furthermore,

$$\log|H_2(\Gamma; M_{2k})| \ll k\log k$$

uniformly for all choices of lattices M_{2k}.

Higher dimensions:

- Bismut-Ma-Zhang, Mü.-Pfaff, 2014

Example: $\widetilde{X} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, $X = \Gamma\backslash\widetilde{X}$.

- $\widetilde{X}_d$ compact dual of $\widetilde{X}$.
- ω_i, $i = 1, 2$, fundamental weights of $\mathfrak{sl}(3,\mathbb{C})$.
- $\rho_i(m)$ irreducible representations with heighest weight $m\omega_i$.
- $\Gamma \subset \mathrm{SL}(3,\mathbb{R})$ co-compact arithmetic subgroup, derived from a 9-dimensional division algebra over $\mathbb{Q}$.
- $M_{i,m} \subset V_{\rho_i(m)}$, $i = 1, 2$, $m \in \mathbb{N}$, Γ-invariant lattice.

Theorem (M.-Pfaff, 2014)

$$\liminf_m \sum_{j=0}^{2} \frac{\log |H^{2j+1}(\Gamma; M_{i,m})_{\mathrm{tors}}|}{m^3} \geq \frac{2\pi}{9\,\mathrm{vol}(\widetilde{X}_d)} \mathrm{vol}(X).$$

Conjecture

$$\lim_{m\to\infty} \frac{\log |H^3(\Gamma; M_{i,m})_{\mathrm{tors}}|}{m^3} = \frac{2\pi}{9\,\mathrm{vol}(\widetilde{X}_d)} \mathrm{vol}(X).$$

$$\log |H^j(\Gamma; M_{i,m})_{\mathrm{tors}}| = o(m^3), \quad j \neq 3.$$

II. The non-compact case

Many arithmetic groups like $\mathrm{SL}(2,\mathbb{Z}[i]) \subset \mathrm{SL}(2,\mathbb{C})$ or $\mathrm{SL}(n,\mathbb{Z}) \subset \mathrm{SL}(n,\mathbb{R})$ are not co-compact. Extension of the results in the compact case to these groups is very desirable.

Problems.

- If $\Gamma\backslash G/K$ is not compact, but has finite volume, then the Laplace operators have non-empty continuous spectrum
- The zeta function can not be defined in the usual way.
- Regularization of the trace of the heat operator is necessary.

$$\zeta_{reg}(s;\rho) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\mathrm{Tr}_{reg}\left(e^{-t\Delta_p(\rho)}\right) t^{s-1} - b_p(\rho)\right)\, dt.$$

- The equality $T_X(\rho) = \tau_X(\rho)$ is not known.

We start with the case $\operatorname{rank}_{\mathbb{R}} G = 1$.

1. Hyperbolic manifolds of finite volume

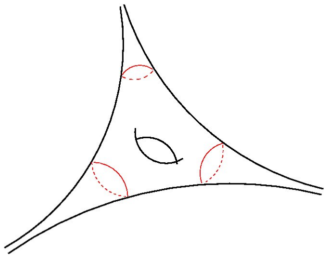
- $\mathbb{H}^n = \mathrm{SO}^0(n,1)/\mathrm{SO}(n)$ hyperbolic n-space, $\Gamma \subset \mathrm{SO}^0(n,1)$ torsion free lattice.
- $X = \Gamma\backslash\mathbb{H}^n$ hyperbolic manifold of finite volume. X is a manifold with finitely many cusp ends

$$X = X_0 \cup Z_1 \cup \cdots \cup Z_m,$$

where X_0 is a compact manifold with boundary and

$$Z_j \cong [a_j, \infty) \times T_j, \quad g|_{Z_j} \cong y_j^{-2}(dy_j^2 + dx_j^2),$$

Here $T_j = \mathbb{R}^{n-1}/\Lambda_j$ is a (n-1)-dimensional torus and dx_j^2 the flat metric on T_j.



Spectral decomposition of the Laplacian

- $X = \Gamma\backslash\mathbb{H}^2$, $\Delta = d^*d\colon C^\infty(X) \to C^\infty(X)$ Laplace operator.
- $\Delta\colon C_c^\infty(X) \to L^2(X)$ is essentially self-adjoint.
- $\sigma(\Delta)$ spectrum of $\overline{\Delta}$.

Theorem $\sigma(\Delta) = \sigma_{pp}(\Delta) \cup \sigma_{ac}(\Delta)$.

1) $\sigma_{pp}(\Delta)\colon \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$.
2) If $m \geq 1$, then $\sigma_{ac}(\Delta) = [1/4, \infty)$.

- The continuous spectrum is described by Eisenstein series attached to the cusps.

Let m be the number of cusps of X. For all $1 \leq k \leq m$ and $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, there exists a unique function $E_k(z, s)$, which is C^∞ in $z \in X$ and holomorphic in s, such that

1) $\Delta E_k(z, s) = s(1 - s)E_k(z, s)$,
2) For all $1 \leq l \leq m$ and $(y_l, x_l) \in Y_l$

$$E_k((y_l, x_l), s) = \delta_{kl} y_l^s + O(1), \quad y_l \to \infty.$$

Further properties:

- $E_k(z, s)$ is given as a series in $\mathrm{Re}(s) > 1$.
- $E_k(z, s)$ has meromorphic extension to $s \in \mathbb{C}$.
- $E_k(z, s)$ is holomorphic on $\mathrm{Re}(s) = 1/2$.

Example: $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. the surface $\Gamma\backslash\mathbb{H}^2$ has a single cusp at $i\infty$. The series

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s = \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}}, \quad \mathrm{Re}(s) > 1.$$

converges for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$. This is the Eisenstein series for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$.

The functions $r \in \mathbb{R} \mapsto E_k(z, 1/2 + ir)$, $k = 1, \ldots, m$, form a complete system of generalized eigenfunctions. Let $\{f_i\}_{i\in I}$ be an orthonormal basis of $L^2_{pp}(X)$ with $\Delta f_i = \lambda_i f_i$. Let $\varphi \in C_c^\infty(X)$. Then φ has the following spectral expansion

$$\varphi(z) = \sum_{i\in I} \langle \varphi, f_i\rangle f_i(z) + \sum_{k=1}^{m} \frac{1}{4\pi} \int_{\mathbb{R}} \langle \varphi, E_k(\cdot, 1/2+ir)\rangle E_k(z, 1/2+ir)dr.$$

Fourier expansion in the cusps

We expand the Eisenstein series $E_k(z,s)$ in the cusp $Y_l = [a,\infty) \times S^1$ in a Fourier series with respect to $x \in S^1$. For $1 \le k, l \le m$ and $(y_l, x_l) \in Y_l$ we have

$$E_k((y_l,x_l),s) = \delta_{kl} y_l^s + C_{kl}(s) y_l^{1-s} + O(e^{-cy_l}), \quad y_l \to \infty.$$

This corresponds to the Sommerfeld radiation condition for potential scattering on $\mathbb{R}$.
Let

$$C(s) := (C_{kl}(s)).$$

$C(s)$ is called scattering matrix. It is a meromorphic function of $s \in \mathbb{C}$. The Eisenstein series and the scattering matrix satisfy the following functional equations

$$C(s)C(1-s) = \mathrm{Id},$$

$$E_k(z,s) = \sum_{l=1}^{m} C_{kl}(s) E_l(z,1-s), \quad k = 1,\dots,m.$$

Selberg trace formula

Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of Δ. Write

$$\lambda_j = 1/4 + r_j^2, \quad r_j \in i(0, 1/2] \cup \mathbb{R}.$$

Put $\phi(s) = \det C(s)$. Let $g \in C_c^\infty(\mathbb{R})$ be even and let $f = \hat{g}$. Then the Selberg trace formula is the follwoing equality

$$\sum_j f(r_j) - \frac{1}{4\pi} \int_{-\infty}^{\infty} f(r) \frac{\phi'}{\phi}(1/2 + ir)\, dr + \frac{1}{4}\phi(1/2) f(0)$$

$$= \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{4\pi} \int_{\mathbb{R}} f(r) r \tanh(\pi r)\, dr + \sum_{\{\gamma\} \neq e} \frac{\ell(\gamma_0)}{2 \sinh(\ell(\gamma)/2)} g(\ell(\gamma))$$

$$- \frac{m}{2\pi} \int_{-\infty}^{\infty} f(r) \frac{\Gamma'}{\Gamma}(1 + ir) dr + \frac{m}{4} f(0) - m \ln 2\, g(0).$$

Here $\{\gamma\}$ runs over the hyperbolic conjugacy classes and $\ell(\gamma)$ is the length of the corresponding closed geodesic.

Truncation of X:

Let $Y > \max\{c_1, ..., c_m\}$. Put

$$X(T) := X_0 \cup ([c_1, Y] \times T_1) \cup \cdots \cup ([c_m, Y] \times T_m).$$

$X(Y)$ is a compact manifold with boundary. Let $K_p(t, x, y)$ be the kernel of $e^{-t\Delta_p(\rho)}$, i.e.,
$K_p(t, x, y) \in \mathrm{Hom}((\Lambda^p T_y^*(X) \otimes E_{\rho,y}, \Lambda^p T_x^*(X) \otimes E_{\rho,x})$ and

$$(e^{-t\Delta_p(\rho)}\phi)(x) = \int_X K_p(t, x, y)(\phi(y))dy.$$

Using the spectral decomposition of $\Delta_p(\rho)$, we get

Proposition There exist $a(t)$ and $b(t)$ such that

$$\int_{X(Y)} \operatorname{tr} K_p(t, x, x)dx = a(t)\log Y + b(t) + O(Y^{-1})$$

as $Y \to \infty$.

Definition. $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\rho)}) := b(t),\ t > 0.$

- $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\rho)})$ equals the spectral side of the Selberg trace formula applied to the kernel of $e^{-t\widetilde{\Delta}_p(\rho)}$.

Example:

$X = \Gamma\backslash\mathbb{H}^2$, $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$ eigenvalues of Δ_0, $C(s)$, scattering matrix, associated to continuous spectrum, meromorphic function on $\mathbb{C}$ with values in $\mathrm{End}(\mathbb{C}^m)$, $\phi(s) := \det C(s)$. Then

$$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_0}) = \sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-(1/4+r^2)t}\frac{\phi'}{\phi}(\frac{1}{2}+ir)dr + \frac{e^{-t/4}}{4}\phi(1/2).$$

Theorem. There exist an asymptotic expansion

$$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\rho)}) \sim \sum_{j=0}^{\infty} a_j t^{j-n/2} + \sum_{k=0}^{\infty} b_k t^{k-1/2} \log t$$

as $t \to 0$.

Proof: There exist different methods. 1. microlocal analysis, b-calculus of Melrose, 2. Selberg trace formula.

Example:

$X = \Gamma\backslash\mathbb{H}^2$, $p = 0$, $\rho = 1$. If we apply the trace formula, we get

$$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\rho)}) = \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{4\pi} \int_{\mathbb{R}} e^{-(1/4+r^2)t} r \tanh(\pi r)\, dr$$
$$+ \sum_{\{\gamma\}\neq e} \frac{\ell(\gamma_0)}{2\sinh(\ell(\gamma)/2)} \frac{e^{-\ell(\gamma)^2/4t}}{\sqrt{4\pi t}}$$
$$- \frac{m}{2\pi} \int_{-\infty}^{\infty} e^{-(1/4+r^2)t} \frac{\Gamma'}{\Gamma}(1+ir)dr + \frac{m}{4} e^{-t/4} - m \ln 2 \frac{e^{-t/4}}{\sqrt{4\pi t}}.$$

General case: $X = \Gamma\backslash\mathbb{H}^n$, $p = 0, ..., n$, $\rho = \tau|_\Gamma$, $\tau \in \mathrm{Rep}(G)$. $e^{-t\widetilde{\Delta}_p(\rho)}$ is a convolution operator, its kernel is given by

$$H_t^p\colon G \to \mathrm{End}(\Lambda^p\mathfrak{p}^* \otimes V_\tau),$$

which transforms under K according to $\tau|_K$. Let

$$h_t^p := \mathrm{tr} \circ H_t^p.$$

Then $h_t^p \in \mathcal{C}^1(G)$. Let $P = MAN$ be standard parabolic subgroup of G. The unipotent contribution to $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\rho)})$ is given by

$$C_1(\Gamma)T_1(h_t^p) + C_2(\Gamma)T_2(h_t^p),$$

where T_1 and T_2 are distributions which are defined by

$$T_1(f) = \int_N f(n)dn, \quad T_2(f) = \int_N f(n)\log\|\log(n)\|dn.$$

Here $\log\colon N \to \mathfrak{n}$.

- $T_2(f)$ is a weighted orbital integral.
- $T_2(f)$ is a non-invariant distribution.
- To analyze the bahavior of $T_2(h_t^p)$ as $T \to 0$ we use standard estimates of the heat kernel on $\mathbb{H}^n$ together with the method of stationary phase approximation.

For the large time behavior we have
Proposition. Let $\tau \ncong \tau_\theta$. Then there exists $c > 0$ such that

$$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\rho)}) = O(e^{-ct})$$

as $t \to \infty$.
The proof uses the spectral side of the STF.

$$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\rho)}) = \sum_j e^{-t\lambda_j} + \text{contribution of Eisenstein series.}$$

Example:

$X = \Gamma\backslash\mathbb{H}^2$, $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$ eigenvalues of Δ_0, $C(s)$, scattering matrix, associated to continuous spectrum, meromorphic function on $\mathbb{C}$ with values in $\mathrm{End}(\mathbb{C}^m)$, $\phi(s) := \det C(s)$. Then

$$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_0}) = \sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-(1/4+r^2)t}\frac{\phi'}{\phi}(\frac{1}{2}+ir)dr + \frac{e^{-t/4}}{4}\phi(1/2).$$

It follows that

$$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_0}) = 1 + O(e^{-ct}).$$

Thus we can define $T_X(\rho)$ as in the compact case by

$$\log T_X(\rho) := \sum_{p=1}^{n}(-1)^p p\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\rho)})t^{s-1}dt\right)\bigg|_{s=0}.$$

Approximation of L^2-torsion in the finite volume case

Let

- $\Gamma_0 \subset \mathrm{SO}_0(n,1)$ lattice, $\Gamma_j \subset \Gamma_0$, $j \in \mathbb{N}$, normal subgroups of finite index, $\Gamma_j \supset \Gamma_{j+1}$, $\cap_j \Gamma_j = \{1\}$, torsion free, cusp uniform.
- $X_j = \Gamma_j \backslash \mathbb{H}^n$, $j \in \mathbb{N}_0$.
- $\tau \in \mathrm{Rep}(\mathrm{SO}_0(n,1))$, $\tau \not\cong \tau_\theta$.

Theorem (Mü, Pfaff, 2014) Under the above assumptions we have

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{[\Gamma : \Gamma_i]} = t^{(2)}_{\widetilde{X}}(\tau)\,\mathrm{vol}(X_0).$$

Reidemeister torsion

- $X = \Gamma \backslash \mathbb{H}^n$, $\mathrm{vol}(X) < \infty$, noncompact.
- $\bar{X}$ Borel-Serre compactification of X. $\bar{X}$ is a compact manifold with boundary whose interior is X.
- $K \cong \bar{X}$ C^∞-triangulation.
- $\tau \in \mathrm{Rep}(\mathrm{SO}_0(n,1))$, $\tau \ncong \tau_\theta$, $\rho = \tau|_\Gamma$.

For such representations $H^*(X, E_\rho) \neq 0$. Therefore we need a volume form in $H^*(X, E_\rho) \neq 0$. Harmonic forms with absolute or relative boundary conditions would be the wrong choice.

Harder: Eisenstein cohomology.

Let $H^*_!(X, E_\rho) = \mathrm{Im}(H^*_c(X, E_\rho) \to H^*(X, E_\rho))$. Then

$$H^*(X, E_\rho) = H^*_!(X, E_\rho) \oplus H^*_{Eis}(X, E_\rho),$$

where $H^*_{Eis}(X, E_\rho)$ is spanned by special values of Eisenstein series. These are lifts of harmonic forms on $\partial \bar{X}$ to harmonic forms on X.

Under the assumption $\tau \not\cong \tau_\theta$ we have $H^*_!(X, E_\rho) = 0$. Let $n = 2d + 1$. Then

$$\begin{aligned} H^k(\bar{X}, E_\rho) &= 0, \quad \text{if } k < d, \\ H^k(\bar{X}, E_\rho) &\cong H^k(\partial\bar{X}, E_\rho), \quad \text{if } k > d, \\ H^d(\bar{X}, E_\rho) &\cong H^d(\partial\bar{X}, E_\rho)_+, \end{aligned}$$

where $H^d(\partial\bar{X}, E_\rho) = H^d(\partial\bar{X}, E_\rho)_+ \oplus H^d(\partial\bar{X}, E_\rho)_-$.
The isomorphisms can be described as follows.

- $P_i = M_i N_i$, $i = 1, ..., m$, representatives of the Γ-conjugacy classes of proper paprabolic subgroups of G.
- $T_i = (\Gamma \cap N_i)\backslash N_i$ torus, $\partial\overline{X} = \sqcup_i T_i$.
- For $\phi \in \oplus_i \mathcal{H}^k(T_i, E_\rho)$ let $E(\phi, s)$, $s \in \mathbb{C}$, be the associated Eisenstein series.
- There are special points $\lambda_{\rho,k} \in \mathbb{C}$ such that $E(\phi, -\lambda_{\rho,k})$ is a closed k-form.

- For $d < k$ the map

$$\Phi \in \bigoplus_i H^k(\Gamma \cap N_i \backslash N_i, E_\rho) \mapsto [E(\Phi, -\lambda_{\rho,k})] \in H^k(X, E_\rho)$$

  is an isomorphism with $i_k^*(E(\Phi, -\lambda_{\rho,k})) = \Phi$.
- $\dim H^n(X, E_\rho) = \frac{1}{2} \dim H^n(\partial \overline{X}, E_\rho)$. Moreover, the map

$$\Phi \in \bigoplus_i H^d(\Gamma \cap N_i \backslash N_i, E_\rho)^- \mapsto [E(\Phi, -\lambda_{\rho,n}^-)] \in H^d(X, E_\rho)$$

  is an isomorphism with

$$i_n^*(E(\Phi, -\lambda_{\rho,n})) = \Phi + [C(\sigma_{\rho,n}^-, -\lambda_{\rho,n}^-)\Phi]_+.$$

- Choose orthonormal basis of $H^k(\partial \overline{X}, E_\rho)$, $n < k \leq 2n$, and $H^n(\partial \overline{X}, E_\rho)^-$. Induces basis of $H^*_{Eis}(X, E_\rho)$ and inner product.

Choose an orthonormal bases for $H^k(\partial\bar{X}, E_\rho)$, $k > d$, and $H^d(\partial\bar{X}, E_\rho)_+$. This gives a bases μ_X of $H^*(\bar{X}, E_\rho)$ and a corresponding volume form. Let $\tau_X(\rho)$ be the Reidemeister torsion with respect to this volume form.

Theorem (Mü, Rochon 2018) We have

$$T_X(\rho) = \tau_X(\rho) + \delta(\tau),$$

where $\delta(\tau)$ is a constant that depends only on τ.

Sketch of the proof: The method is based on degeneration as used by Albin, Rochon and Sher in the case of unitary representations ρ of Γ. Let

$$M := \bar{X} \cup_{\partial \bar{X}} \bar{X}$$

be the double of $\bar{X}$. The metric g_X on X is given near $\partial \bar{X}$ by

$$\frac{dx^2}{x^2} + x^2 g_{\partial \bar{X}},$$

where x is boundary defining function and $g_{\partial \bar{X}}$ the flat metric on each component T_j of $\partial \bar{X} = \sqcup_j T_j$. For $\varepsilon > 0$ choose a metric g_ε on M such that on a tubular neighborhood $N \cong (-\delta, \delta) \times \partial \bar{X}$ of $\partial \bar{X}$ in M it is given by

$$\frac{dx^2}{x^2 + \varepsilon^2} + (x^2 + \varepsilon^2) g_{\partial \bar{X}},$$

and g_ε degenerates to the hyperbolic metric on X.

Let $F_\rho := E_\rho \cup E_\rho$ be the double of E_ρ. Choose metric h_ε in F_ρ which degenerates to h in E_ρ. The first step is the following theorem.

Theorem (Mü, Rochon 2018) As $\varepsilon \to 0$, $\log T_M(F_\varepsilon; g_\varepsilon, h_\varepsilon)$ has a polyhomogeneus expansion and the finite part is given by

$$\mathrm{FP}_{\varepsilon=0} \log T_M(F_\varepsilon; g_\varepsilon, h_\varepsilon) = 2 \log T_X(E; g_X, h_E) + \log T(D_b^2),$$

where D_b is a model operator on the cusp.

The proof uses the b-calculus of Melrose and the fact that there are only finitely many eigenvalues that converge to zero.

Reidemeister torsion.

- $Z \subset M$ separating hypersurface, μ_Z basis of $H^*(Z, F)$ constructed as above.
- μ_Z induces basis μ_M of $H^*(M, F)$.

Using the Meyer-Vietoris sequence for Reidemeister torsion, we obtain

Theorem (M.-Rochon). With the above choices of bases in cohomology, we have that the Reidemeister torsions of M and $\overline{X} \cong X(Y)$ are related by

$$\tau(M, F_\varepsilon, \mu_M) = \frac{\tau(X(Y), E, \mu_X)^2}{\tau(Z, F, \mu)}.$$

Furthermore, if n is odd, then $\tau(Z, F, \mu) = 1$ and the formula simplifies to

$$\tau(M, F_\varepsilon, \mu_M) = \tau(X(Y), E, \mu_X)^2.$$

Finally, using the equality of analytic torsion and Reidemeister torsion for compact manifolds, we get

$$2\log\tau(X(Y),E,\mu_X) = \log T(M,E,g_\varepsilon,h_\varepsilon) - \log\left(\prod_q [\mu_M^q|\omega^q]^{(-1)^q}\right),$$

where ω^q is an orthonormal basis of harmonic forms with respect to g_ε and h_ε. To obtain our formula, we take the finite part of each term on the right hand side.

Application to cohomology

Theorem (Rochon, Mü, 2018)

Let $\Gamma \subset \mathrm{SO}_0(n,1)$ be an arithmetic subgroup and $\{\Gamma_j\}$ a decreasing sequence of cusp uniform congruence subgroups such that $\cap_j \Gamma_j = 1$. Let M be a Γ-module with $M^* \cong M$. Then

$$\liminf_{j\to\infty} \frac{\sum_{q \text{ even}} \log |H^q(\Gamma_j, M)_{\mathrm{tors}}|}{[\Gamma : \Gamma_j]} \geq -t^{(2)}_{\mathbb{H}^n}(\rho)\,\mathrm{vol}(X) > 0.$$