Spectral theory on locally symmetric spaces of finite volume

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Garwitz, 1981

Göhren-Lebbin, 1988

Garwitz, 1981

1. Automorphic forms, automorphic representations

Basic set up

- G connected semisimple Lie group, finite center, non-compact type. For example: $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{SO}(p,q)$, $p,q \in \mathbb{N}$.
- H Lie group, $\Pi(H)$ equivalence classes of irreducible unitary representations of H.
- $K \subset G$ maximal compact subgroup, Ex.: $\mathrm{SO}(n) \subset \mathrm{SL}(n,\mathbb{R})$.
- $\widetilde{X} = G/K$ Riemannian symmetric space.
- $\Gamma \subset G$ lattice, i.e., discrete subgroup with $\mathrm{vol}(\Gamma\backslash G) < \infty$.
- $X = \Gamma\backslash\widetilde{X}$ locally symmetric space, manifold if Γ is torsion free.

Example: $\mathbb{H}^2 \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, $\Gamma\backslash\mathbb{H}^2$ modular surface.

Automorphic forms

- $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ center of the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}$, $\Omega \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ Casimir element.
- $\phi \in C^{\infty}(\Gamma \backslash G)$ is an automorphic form, if ϕ is K-finite, $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$-finite, and of moderate growth.
- Especially, a joint eigenfunction $\phi \in L^2(\Gamma \backslash G)^K \cong L^2(\Gamma \backslash \widetilde{X})$ of $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ is an automorpic function.
- $P \subset G$ parabolic subgroup, $P = M_P A_P N_P$, M_P reductive, $A_P \cong (\mathbb{R}^+)^k$, N_P nilpotent.
- P is called cuspidal parabolic, if $(\Gamma \cap N_P) \backslash N_P$ is compact.
- $\phi \in C^{\infty}(\Gamma \backslash G)$ automorhic form, ϕ is called cusp form, if

$$\int_{(\Gamma \cap N_P) \backslash N_P} f(ng) dn = 0$$

for all proper cuspidal parabolic subgroups P of G.

Representation theory

- R_Γ right regular representation of G in $L^2(\Gamma\backslash G)$, defined by

$$(R_\Gamma(g)f)(g') = f(g'g), \quad f \in L^2(\Gamma\backslash G).$$

- Langlands' theory of Eisenstein series implies decomposition in invariant subspaces

$$L^2(\Gamma\backslash G) = L^2_{\mathrm{dis}}(\Gamma\backslash G) \oplus L^2_{ac}(\Gamma\backslash G)$$

- $L^2_{\mathrm{dis}}(\Gamma\backslash G)$ maximal invariant subspace, spanned by irreducible subrepresentations,

$$R_{\Gamma,\mathrm{dis}} \cong \widehat{\bigoplus}_{\pi\in\Pi(G)} m_\Gamma(\pi)\pi.$$

- $m_\Gamma(\pi) = \dim \mathrm{Hom}_G(\pi, R_\Gamma) = \dim \mathrm{Hom}_G(\pi, R_{\Gamma,\mathrm{dis}})$.
- $m_\Gamma(\pi) < \infty$ for all $\pi \in \Pi(G)$.

- $L^2_{\mathrm{cus}}(\Gamma\backslash G)$ subspace of cusp forms, $L^2_{\mathrm{cus}}(\Gamma\backslash G) \subset L^2_{\mathrm{dis}}(\Gamma\backslash G)$ and
$$L^2_{\mathrm{dis}}(\Gamma\backslash G) = L^2_{\mathrm{cus}}(\Gamma\backslash G) \oplus L^2_{\mathrm{res}}(\Gamma\backslash G).$$

Main problem: Study multiplicities $m_\Gamma(\pi)$.

- Apart from special cases, one cannot hope to describe $m_\Gamma(\pi)$ explicitly.
- There exist formulas for $m_\Gamma(\pi)$ if π is a discrete series representation.

2. Asymptotic behavior of automorphic spectra

- Study behavior of multiplicity with respect to the growth of various parameters such as the infinitesimal character or/and the level of congruence subgroups.

Examples:

a) Weyl law.

- For $\sigma \in \Pi(K)$ let
$$\Pi(G;\sigma) = \{\pi \in \Pi(G) \colon [\pi|_K : \sigma] > 0\}.$$
- $\lambda_\pi = \pi(\Omega)$ Casimir eigenvalue of $\pi \in \Pi(G)$.
$$N_\Gamma(\lambda;\sigma) = \sum_{\substack{\pi\in\Pi(G;\sigma)\\ |\lambda_\pi|\le\lambda}} m_\Gamma(\pi).$$

Problem: Behavior of $N_\Gamma(\lambda;\sigma)$ as $\lambda \to \infty$.

Equivalent problem:

- $\sigma = 1$, $\widetilde{X} = G/K$, $X = \Gamma\backslash\widetilde{X}$,
$L^2(\Gamma\backslash G)^K \cong L^2(\Gamma\backslash G/K) = L^2(X)$, $\Delta \colon C^\infty(X) \to C^\infty(X)$
Laplace operator.

Lemma (Kuga): $-R(\Omega) = \Delta$.

- $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ eigenvalues of Δ in $L^2(X)$.
- $N_\Gamma(\lambda) = \#\{j \colon \lambda_j \leq \lambda\}$

Problem: Study behavior of $N_\Gamma(\lambda)$ as $\lambda \to \infty$.

b) Limit multiplicity problem.

- Let $\Pi(G)$ be equipped with the Fell topology.

Define a measure on $\Pi(G)$ by

$$\mu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi \in \Pi(G)} m_\Gamma(\pi)\delta_\pi,$$

where δ_π is the delta distribution.

Problem: Study the behavior of μ_Γ as $\mathrm{vol}(\Gamma\backslash G) \to \infty$.

c) Distribution of Hecke eigenvalues

Put $\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})$, $S_k(\Gamma(1))$ space of cusp forms w.r.t. $\Gamma(1)$ of weight k.

$$T_n \colon S_k(\Gamma(1)) \to S_k(\Gamma(1))$$

the n-th Hecke operator.

- $\mathcal{S}_k$ the set of all normalized Hecke eigenforms $f \in S_k(\Gamma(1))$.

Then

$$T_n f = a_f(n) f, \quad f \in \mathcal{S}_k.$$

Put $\lambda_f(n) = n^{(1-k)/2} a_f(n)$.
Deligne: $\lambda_f(p) \in [-2, 2]$ for p prime.

Conjecture (Serre): For each $h \in C([-2, 2])$

$$\frac{1}{\pi(x)} \sum_{p \leq x} h(\lambda_f(p)) \to \frac{1}{2\pi} \int_{-2}^{2} h(t) \sqrt{4 - t^2}\, dt, \quad x \to \infty.$$

Sato-Tate conjecture for modular forms.

Theorem (H. Nagoshi, 2006): Suppose that $k = k(x)$ satisfies $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. Then for every $h \in C([-2,2])$, we have

$$\frac{1}{\pi(x)\# S_k} \sum_{\substack{p \leq x \\ f \in S_k}} h(\lambda_f(p)) \to \frac{1}{2\pi} \int_{-2}^{2} h(t)\sqrt{4-t^2}\, dt, \quad x \to \infty.$$

d) Analytic torsion

Analytic torsion is a more sophisticated spectral invariant which recently found interesting applications to the study of the growth of torsion in the cohomology of arithmetic groups. It is defined as follows.

- Let (X, g) be a compact Riemannian manifold of dimension n, $\rho\colon \pi_1(X) \to \mathrm{GL}(V)$ a finite dimensional representation of the fundamental group, $E_\rho \to X$ flat vector bundle, associated to ρ, h fibre metric in E_ρ.

- $\Delta_p(\rho)\colon \Lambda^p(X, E_\rho) \to \Lambda^p(X, E_\rho)$ Laplace operator on E_ρ-valued p-forms, defined with respect to g and h.
- $\Delta_p(\rho)$ is essentially self-adjoint. Its spectrum consisits of eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty$ of finite multiplicities.
- Let
$$\zeta_p(s;\rho) := \sum_{\lambda_j > 0} \lambda_j^{-s}, \quad \mathrm{Re}(s) > n/2,$$
zeta function of $\Delta_p(\rho)$.
- $\zeta_p(s;\rho)$ admits meromorphic extension to $s \in \mathbb{C}$, holomorphic at $s = 0$.
-
$$\det \Delta_p(\rho) := \exp\left(-\frac{d}{ds}\zeta_p(s;\rho)\Big|_{s=0}\right)$$
regularized determinant of $\Delta_p(\rho)$.

- The Ray-Singer analytic torsion $T_X(\rho) \in \mathbb{R}^+$ is defined by

$$T_X(\rho) := \prod_{q=1}^{n} \left[\det \Delta_q(\rho)\right]^{(-1)^{q+1} q/2}.$$

- $\pi_1(X) = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_j \supset \cdots \supset \{e\}$, $\cap \Gamma_j = \{e\}$, tower of normal subgroups of finite index.
- $X_j := \Gamma_j \backslash \widetilde{X}$ finite covering of X, $E_j \to X_j$ flat vector bundle associated to $\rho|_{\Gamma_j}$.

Problem: Study behavior of $\frac{\log T_{X_j}(\rho_j)}{\mathrm{vol}(X_j)}$ as $j \to \infty$.

e) Families of automorphic forms (Sarnak)

2. Conjectures and results

a) Wely law

i) Γ cocompact. $n = \dim X$. Then the following Weyl law holds

$$N_\Gamma(\lambda;\sigma) = \frac{\dim(\sigma)\,\mathrm{vol}(\Gamma\backslash G)}{(4\pi)^{n/2}\Gamma(n/2+1)}\lambda^{n/2} + O(\lambda^{(n-1)/2}), \quad \lambda \to \infty.$$

- $\sigma = 1$: Avakumović, Hörmander: general elliptic operators.
- σ arbitrary: P. Ramacher.

ii) $\Gamma\backslash G$ non-compact:

Problem: Nonempty continuous spectrum.

Example: $X = \Gamma\backslash\mathbb{H}^2$ hyperbolic surface of finite area, non-compact.

- Continuous spectrum of Δ: $[1/4, \infty)$.
- possible eigenvalues of Δ: $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \cdots$,
- Eigenvalues $\lambda \ge 1/4$ are embedded into the continuous spectrum, unstable under perturbations.

Arithmetic groups.

Let $\mathbf{G} \subset \mathrm{GL}_n$ be a semisimple algebraic group over $\mathbb{Q}$. $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is called arithmetic, if Γ is commensurable to $\mathbf{G}(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$.

Examples:

1) Principal congruence subgroup

$$\Gamma(N) := \left\{ \gamma \in \mathrm{SL}(2,\mathbb{Z}) \colon \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod (N) \right\}$$

2) $F = \mathbb{Q}(\sqrt{-D})$, $D \in \mathbb{N}$, square free, $\mathcal{O}_F$ ring of integers of F, $\mathrm{SL}(2,\mathcal{O}_F) \subset \mathrm{SL}(2,\mathbb{C})$ Bianchi group, $\mathfrak{a} \subset \mathcal{O}_F$ ideal, $\Gamma(\mathfrak{a}) \subset \mathrm{SL}(2,\mathcal{O}_F)$ principal congruence subgroup.

Theorem (Margulis): Let $\mathrm{rank}_{\mathbb{R}}\, G > 1$ and $\Gamma \subset G$ an irreducible lattice. Then Γ is arithmetic.

Theorem (Serre): Let $n \geq 3$. Every arithmetic subgroup $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$ contains a principal congruence subgroup.

Conjecture 1. Weyl law holds for congruence subgroups or, more generally, arithmetic groups.

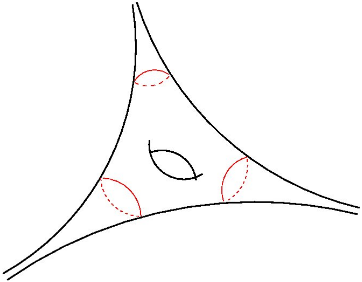
Conjecture 2 (Phillips, Sarnak, 1986). Except for the Teichmüller space of the once punctured torus, for a generic $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$, the Laplace operator has only finitely many eigenvalues. They are all contained in $[0,1/4)$.

Reason: Eigenvalues, embedded in the continuous spectrum, are unstable with respect to perturbations.

Results.

a) The real rank one case. Selberg 1954,

- $X = \Gamma\backslash\mathbb{H}^2$, $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ lattice.
- X is a hyperbolic surface of finite area.



Theorem 1: *The resolvent* $R_X(s) = (\Delta - s(1-s))^{-1}$, *defined for* $\mathrm{Re}(s) > 1/2$, $s \neq \bar{s}$, *extends to a meromorphic family of bounded operator*

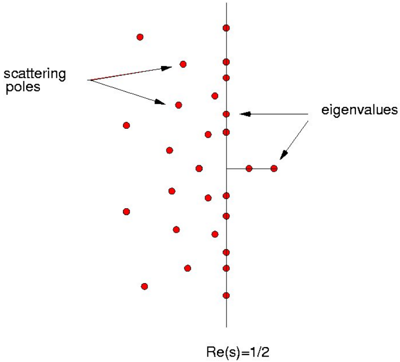
$$R_X(s) : L^2_{\mathrm{cpt}}(X) \to H^2_{\mathrm{loc}}(X)$$

with poles of finite rank.

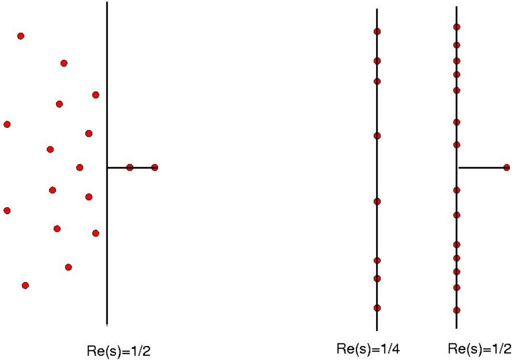
Faddejev, Colin de Verdiere, Mü., Zworski-Guillopé,

Methods: Resolvent equation, Lax-Phillips cut-off Laplacian, weighted L^2-spaces, Fredholm theory.

- The poles of $R_X(s)$ are called resonances. The poles in $(1/2, 1] \cup (1/2 + i\mathbb{R})$ correspond to eigenvalues. Scattering resonances are poles of $R_X(s)$ with $\mathrm{Re}(s) < 1/2$.



- $s \in (1/2, 1] \cup (1/2 + i\mathbb{R})$, then $\lambda = s(1-s)$ is an eigenvalue.
- The poles are distributed in a strip of the form $-c < \mathrm{Re}(s) \leq 1$.



The figure on the left hand side shows the expected distribution of the resonances for a generic Γ. The figure on the right shows the distribution of resonances for the modular surface $SL(2,\mathbb{Z})\backslash\mathbb{H}^2$, under the assumption of the Riemann hypothesis. Except for the pole at $s = 1$, the scattering resonances are on the line $\mathrm{Re}(s) = 1/4$ and the poles corresponding to eigenvalues are on the line $\mathrm{Re}(s) = 1/2$.

Put

$$N_\Gamma(\lambda) = \#\left\{j \colon \lambda_j \le \lambda^2\right\}.$$

Let $N_{\mathrm{scr}}(\lambda)$ be the number of scattering resonances, counted with their order, in the circle of radius λ.

Theorem (Selberg): We have

$$2N_\Gamma(\lambda) + N_{\mathrm{scr}}(\lambda) \sim \frac{\mathrm{Area}(X)}{2\pi}\lambda^2, \quad \lambda \to \infty.$$

For $\mathrm{SL}(2,\mathbb{Z})$ one can determine the scattering resonances:

$$\begin{aligned}\{\rho \in \mathbb{C} \colon \rho \text{ pole of } R_X(s),\ \mathrm{Re}(s) < 1/2\}\\ = \{\rho \in \mathbb{C} \colon \zeta(2\rho) = 0,\ 0 < \mathrm{Re}(\rho) < 1/2\},\end{aligned}$$

$\zeta(s)$ Riemann zeta function. Let $N(T)$ be the number of zeros of $\zeta(s)$ with $0 < \mathrm{Im}(s) < T$, $0 < \mathrm{Re}(s) < 1$. Then

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \quad T \to \infty.$$

- Similar for $\Gamma(N) \subset \mathrm{SL}(2,\mathbb{Z})$. Scattering resonances for $\Gamma(N)\backslash\mathbb{H}$ correspond to zeros of Dirichlet L-functions $L(s,\chi)$ for Dirichlet characters χ mod m with $m|N$.

Corollary (Selberg):

$$N_{\Gamma(N)}(\lambda) \sim \frac{\mathrm{Area}(X)}{4\pi}\lambda^2, \quad \lambda \to \infty.$$

b) Higher rank The Weyl law holds in the following cases:

i) $\sigma = 1$, no estimation of the remainder term.

- S. Miller: $X = \mathrm{SL}(3,\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$.
- E. Lindenstrauss, A. Venkatesh: G of ajoint type, $\Gamma \subset G$ congruence subgroup.

ii) $\sigma \in \Pi(K)$ arbitrary, no estimation of the remainder term.

- Mü.: $\Gamma(N) \subset \mathrm{SL}(n,\mathbb{Z})$ principal congruence subgroup of level N, $X = \Gamma(N)\backslash \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

Matz, Mü. (Work in progress): Let G be a quasi-split classical group, an inner form of $\mathrm{SL}(n)$, or the split exceptional group G_2. Then the Weyl law holds for all congruence subgroups $\Gamma \subset G(\mathbb{Q})$ and all $\sigma \in \Pi(K)$.

iii) $\sigma = 1$, estimation of the remainder term.
Let

$$S_n = \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n), \quad n \geq 2,$$

and let $\Gamma \subset \mathrm{SL}(n,\mathbb{R})$ be a congruence subgroup. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of Δ_Γ in $L^2(\Gamma\backslash S_n)$.

$$N_\Gamma(\lambda) = \#\{j \colon \lambda_j \leq \lambda^2\}.$$

Theorem 2 (Lapid, Mü.): Let $d = \dim S_n$, $N \geq 3$. Then

$$N_{\Gamma(N)}(\lambda) = \frac{\mathrm{Vol}(\Gamma(N)\backslash S_n)}{(4\pi)^{d/2}\boldsymbol{\Gamma}(d/2+1)}\lambda^d + O(\lambda^{d-1}(\log\lambda)^{\max(n,3)})$$

as $\lambda \to \infty$.

Theorem 3 (Finis, Lapid) Let G be a simply connected, simple Chevalley group. Then there exists $\delta > 0$ such that for any congruence subgroup Γ of $G(\mathbb{Z})$ one has

$$N_{X,\mathrm{cus}}(\lambda) = \frac{\mathrm{Vol}(X)}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}\lambda^d + O(\lambda^{d-\delta}), \quad \lambda \to \infty,$$

where $X = \Gamma\backslash G(\mathbb{R})/K$ and $d = \dim X$.

- The method is based on the use of Hecke operators as in the work of Lindenstrauss and Venkatesh, but in a slightly different way.

Multidimensional version

- $G = \mathrm{SL}(n, \mathbb{R})$, $G = NAK$ Iwasawa decomposition,
- $\mathfrak{a} = \mathrm{Lie}(A)$, $W = W(G, A)$.
- $\mathcal{D}(S_n)$ ring of invariant differential operators on S_n.

Harish-Chandra: $\mathcal{D}(S_n) \cong S(\mathfrak{a}_{\mathbb{C}})^W$.
Thus, if

$$\chi \colon \mathcal{D}(S_n) = S(\mathfrak{a}_{\mathbb{C}}^*)^W \to \mathbb{C}$$

is a character, then

$$\chi = \chi_\lambda \leftrightarrow \lambda \in \mathfrak{a}_{\mathbb{C}}^*/W.$$

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let

$$\mathcal{E}_{\mathrm{cus}}(\lambda) = \left\{\varphi \in L^2_{\mathrm{cus}}(\Gamma \backslash S_n) \colon D\varphi = \chi_\lambda(D)\varphi,\ D \in \mathcal{D}(S_n)\right\}$$

Let $m_{\mathrm{cus}}(\lambda) = \dim \mathcal{E}_{\mathrm{cus}}(\lambda)$. Then the cuspidal spectrum is defined as

$$\Lambda_{\mathrm{cus}}(\Gamma) = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W \colon m_{\mathrm{cus}}(\lambda) > 0\}.$$

- $\Lambda_{\text{cus}}(\Gamma) \cap i\mathfrak{a}^*/W$ is the tempered spectrum
- $\Lambda_{\text{cus}}(\Gamma) - (\Lambda_{\text{cus}}(\Gamma) \cap i\mathfrak{a}^*/W)$ the complementary spectrum.

Theorem (Lapid, Mü, 2007): Let $S_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ and $d_n = \dim S_n$, $\Omega \subset i\mathfrak{a}^*$ a bounded open subset with piecewise C^2 boundary, $\beta(\lambda)$ be the Plancherel measure. Then as $t \to \infty$

$$\sum_{\lambda \in \Lambda_{\text{cus}}(\Gamma(N)), \lambda \in t\Omega} m(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash S_n)}{|W|} \int_{t\Omega} \beta(\lambda)\, d\lambda + O\left(t^{d_n - 1} (\log t)^{\max(n,3)}\right)$$

and

$$\sum_{\substack{\lambda \in \Lambda_{\text{cus}}(\Gamma(N)) \\ \lambda \in B_t(0) \backslash i\mathfrak{a}^*}} m(\lambda) = O\left(t^{d_n - 2}\right).$$

Duistermaat, Kolk, Varadarajan, 1979: This results holds for G arbitrary, and $\Gamma \subset G$ a uniform lattice.

b) Limit multiplicities

- μ_{PL} Plancherel measure on $\Pi(G)$, support of μ_{PL} is the tempered dual $\Pi_{\mathrm{temp}}(G)$.
- Up to a closed subset of Plancherel measure zero, the topological space $\Pi_{\mathrm{temp}}(G)$ is homeomorphic to a countable union of Eucledian spaces of bounded dimension.
- Under this homeomorphism, the Plancherel density is given by a continuous function.
- A relatively quasi-compact subset of $\Pi(G)$ is called bounded.
- $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_j \supset \cdots$ tower of normal subgroups of finite index, $\cap_j \Gamma_j = \{e\}$.

$$\mu_j = \frac{1}{\mathrm{vol}(\Gamma_j \backslash G)} \sum_{\pi \in \Pi(G)} m_{\Gamma_j} \delta_\pi .$$

Conjecture: $\mu_j \to \mu_{PL}$ as $j \to \infty$.

Results:

a) $\Gamma \subset G$ uniform lattice.
De George-Wallach, Delorme: Answer affirmative.

b) $\Gamma \subset G$ non-uniform lattice.
Savin: $\pi \in \Pi_{\mathrm{dis}}(G)$ (discrete series), $d(\pi)$ formal degree.

$$\lim_{j\to\infty} \mu_j(\{\pi\}) = d(\pi).$$

Theorem (Finis, Lapid, Mü.), 2014: Let $G = \mathrm{SL}(n,\mathbb{R})$ and $\Gamma_n(N) \subset \mathrm{SL}(n,\mathbb{Z})$ the principal congruence subgroup of level N. Let $\mu_N := \mu_{\Gamma_n(N)}$.

1) For every Jordan measurable set $A \subset \Pi_{\mathrm{temp}}(G)$ we have

$$\mu_N(A) \to \mu_{PL}(A), \quad N \to \infty.$$

2) For every bounded subset $A \subset \Pi(G) \setminus \Pi_{\mathrm{temp}}(G)$ we have

$$\mu_N(A) \to 0, \quad N \to \infty.$$

d) Analytic torsion

- (X, g) compact Riemannian manifold, $\rho\colon \pi_1(X) \to \mathrm{GL}(V)$ finite dimensional representation, $E_\rho \to X$ flat vector bundle associated to ρ, h fibre metric in E_ρ.
- Let $T_X^{(2)}(\rho)$ be the L^2-torsion, introduced by John Lott and Mathai Varghese.
- Let $\pi_1(X) = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_j \supset \cdots \supset \{e\}$, $\cap \Gamma_j = \{e\}$, be a tower of normal subgroups of finite index.
- $X_j := \Gamma_j \backslash \widetilde{X}$ finite covering of X, $E_j \to X_j$ flat vector bundle associated to $\rho|_{\Gamma_j}$.

Conjecture (Lück): The limit of $\log T_{X_j}(\rho_j)/[\Gamma\colon \Gamma_j]$ as $j \to \infty$ exists and

$$\lim_{j\to\infty} \frac{\log T_{X_j}(\rho_j)}{[\Gamma\colon \Gamma_j]} = \log T_X^{(2)}(\rho).$$

Results

- $\widetilde{X} = G/K$ Riemannian symmetric space of non-positive curvature, $\Gamma \subset G$ co-compact, torsion free lattice, $X = \Gamma\backslash\widetilde{X}$.
- $\tau\colon G \to \mathrm{GL}(V)$ finite dimensional representation, $\rho = \tau|_\Gamma$, $E_\rho \to X$ flat vector bundle associated to ρ.
- $\widetilde{E} \to \widetilde{X}$ homogeneous vector bundle associated to $\sigma := \tau|_K\colon K \to \mathrm{GL}(V)$. Then there is a canonical isomorphism
$$\Gamma\backslash\widetilde{E} \cong E_\rho.$$
- $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{k} = \mathrm{Lie}(K)$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition.
- $\langle\cdot,\cdot\rangle$ inner product in V, skew-symmetric with respect to $\mathfrak{k}$, symmetric with respect to $\mathfrak{p}$.
- $\langle\cdot,\cdot\rangle$ induces G-invariant metric in $\widetilde{E}$ and therefore, fibre metric in $E_\rho \cong \Gamma\backslash\widetilde{E}$.

Theorem (Bergeron/Venkatesh). Let $(\Gamma_j)_{j\in\mathbb{N}}$ be a family of normal subgroups of finite index of Γ. Let $X_j = \Gamma_j \backslash \widetilde{X}$. Let $\tau \in \mathrm{Rep}(G)$ and $E_j \to X_j$ the flat vector bundle associated to $\rho_j = \tau|_{\Gamma_j}$, equipped with the metric defined by the inner product $\langle \cdot, \cdot \rangle$ in V. Let $\theta\colon G \to G$ be the Cartan involution. Assume that $\tau \circ \theta \not\cong \tau$ and $\mathrm{vol}(X_j) \to \infty$. Then

$$\lim_{j\to\infty} \frac{\log T_{X_j}(\rho_j)}{[\Gamma : \Gamma_j]} = \log T_X^{(2)}(\tau).$$

- This theorem has interesting applications to the study of the growth of torsion in the cohomology of arithmetic groups.

4. The Arthur-Selberg trace formula.

The trace formula is the main tool to study spectral problems for locally symmetric spaces.

i) $\mathrm{rank}_{\mathbb{R}}\, G = 1$.

- π unitary representation of G, $f \in C_c^\infty(G)$,

$$\pi(f) := \int_G f(g)\pi(g)dg.$$

- $\Gamma \subset G$ lattice, $R_{\Gamma,\mathrm{dis}}$ right regular representation of G in

$$L^2_{\mathrm{dis}}(\Gamma\backslash G) = \widehat{\bigoplus}_{\pi\in\Pi(G)} m_\Gamma(\pi)H_\pi.$$

Let Γ be cocompact and $f \in C_c^\infty(G)$ K-finite. Then $R_\Gamma(f)$ is a trace class operator, let $C(\Gamma)$ denote the Γ-conjugacy classes.

$$\operatorname{Tr} R_\Gamma(f) = \sum_{\pi\in\Pi(G)} m_\Gamma(\pi)\operatorname{tr}\pi(f)$$

$$= \sum_{[\gamma]\in C(\Gamma)} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)dg.$$

- G_γ and Γ_γ centralizer of γ in G resp. Γ.
- $J_G(f,\gamma) = \int_{G_\gamma \backslash G} f(g^{-1}\gamma g)dg$ orbital integral.
- $J_G(f,\gamma)$ invariant distribution on G, can be studied by Harish-Chandra's Fourier inversion formula.

Example

- $X = \Gamma \backslash \mathbb{H}^2$ a compact hyperbolic surface.
- $\Delta\colon C^\infty(X) \to C^\infty(X)$ Laplace operator. Let $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ the eigenvalues of Δ.
- $\widetilde{\Delta}\colon C^\infty(\mathbb{H}^2) \to C^\infty(\mathbb{H}^2)$ Laplace operator on the universal covering of X.
- $k(t,x,y)$ kernel of $e^{-t\widetilde{\Delta}}$. There exists $h_t(\cdot) \in C^\infty(\mathbb{R}^+)$ such that
$$k(t,x,y) = h_t(d(x,y)).$$
- h_t may be regarded as C^∞-function on $\mathrm{SL}(2,\mathbb{R})$ which is $\mathrm{SO}(2)$-bi-invariant.

Then the STF, applied to the heat operator $e^{-t\Delta}$, gives the following equality

$$\operatorname{Tr} R_\Gamma(h_t) = \sum_{j=0}^{\infty} e^{-t\lambda_j} = \frac{\operatorname{vol}(X)}{4\pi} \int_{\mathbb{R}} e^{-(1/4+r^2)t} r \tanh(\pi r) dr$$

$$+ \sum_{\{\gamma\}\neq\{e\}} \frac{\ell(\gamma_0)}{2\sinh(\ell(\gamma)/2)} \frac{e^{-\frac{t}{4}-\frac{\ell(\gamma)^2}{4t}}}{\sqrt{4\pi t}}.$$

This implies

$$\sum_{j=0}^{\infty} e^{-t\lambda_j} \sim \frac{\operatorname{vol}(X)}{4\pi} t^{-1}, \quad t \to 0,$$

and by Karamata's theorem we obtain Weyl's law

$$N_\Gamma(\lambda) = \frac{\operatorname{vol}(X)}{4\pi}\lambda + o(\lambda),$$

as $\lambda \to \infty$.

b) Γ not uniform.

$$\sum_{\pi \in \Pi_{\mathrm{dis}}(G)} m_\Gamma(\pi) \operatorname{tr} \pi(f) + \text{contr. of cont. spectrum}$$

$$= \operatorname{vol}(\Gamma \backslash G) f(e) + \text{weighted orbital integrals}$$

Example: $G = \mathrm{SL}(2, \mathbb{R})$, $\Gamma = \mathrm{SL}(2, \mathbb{Z})$.

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \operatorname{Im}(\gamma z)^s = \sum_{(m,n)=1} \frac{y^s}{|mz + n|^{2s}}, \quad \operatorname{Re}(s) > 1, \; z = x + iy,$$

Eisenstein series.

- $E(z, s)$ admits meromorphic extension to $s \in \mathbb{C}$, holomorphic on $\operatorname{Re}(s) = 1/2$.
- $E(\cdot, s) \in C^\infty(\Gamma \backslash \mathbb{H}^2)$ and $\Delta_z E(z, s) = s(1 - s) E(z, s)$.
- $r \in \mathbb{R} \mapsto E(z, 1/2 + ir)$ is a generalized eigenfunction.

- Fourier expansion: $E(x+iy,s) = y^s + c(s)y^{1-s} + \sum_{n\neq 0}\cdots$.
- $c(s)$ scattering matrix.
- Contribution of the continuous spectrum given by

$$\int_{\mathbb{R}} f(r)\frac{c'}{c}(1/2+ir)dr.$$

- $G = KAN$ Iwasawa decomposition, $\log\colon N \to \mathfrak{n}$.
- Weighted orbital integral:

$$\int_N f(n)\log\|\log n\|dn.$$

The basic idea in the application of the trace formula is to show that the additional terms are negligible.

Example: Weyl law, $\Gamma = SL(2,\mathbb{Z})$, $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ eigenvalues of Δ on $\Gamma\backslash\mathbb{H}^2$. Selberg trace formula applied to the heat operator $e^{-t\Delta}$ gives

$$\sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-(1/4+r^2)t}\frac{c'}{c}(1/2+ir)dr + \cdots$$
$$= \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{\mathbb{R}} e^{-(1/4+r^2)t} r\tanh(\pi r)dr + \cdots .$$

- study behavior as $t \to 0+$.

One has

$$c(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}$$

The zeta function satisfies $|\zeta'(1+it)| \ll (\log|t|)^6$ for $|t| \ge 2$. This implies that

$$\sum_j e^{-t\lambda_j} \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}t^{-1}, \quad t \to 0+ .$$

Tauberian theorem implies the Weyl law.

Limit multiplicities

- $C^\infty_{c,\mathrm{fin}}(G)$ bi-K-finite functions in $C_c^\infty(G)$.
- $f \in C^\infty_{c,\mathrm{fin}}(G)$, $\pi \in \Pi(G)$, $\hat{f}(\pi) := \operatorname{tr} \pi(f)$.
- $\pi \in \Pi(G) \mapsto \hat{f}(\pi)$ Fourier transform.

Sauvageot's density principle: Assume that for all $f \in C^\infty_{c,\mathrm{fin}}(G)$ one has $\mu_j(\hat{f}) \to \mu_{PL}(\hat{f})$ as $j \to \infty$. Then $\mu_j \to \mu_{PL}$, $j \to \infty$, in the weak sense.

Now

$$\operatorname{vol}(\Gamma_j \backslash G)\mu_j(\hat{f}) = \operatorname{Tr} R_{\Gamma,\mathrm{dis}}(f), \quad \mu_{PL}(\hat{f}) = f(e).$$

STF implies

$$\mu_j(\hat{f}) + \frac{1}{\operatorname{vol}(\Gamma_j \backslash G)}(\text{contr. of cont. spec}) = \mu_{PL}(\hat{f}) + \frac{1}{\operatorname{vol}(\Gamma_j \backslash G)}(\text{sum of weighted orbital int.})$$

5. Automorphic L-functions.

Automorphic L-functions are the generalizations of Dirichlet L-functions. They appear in the constant terms of Eisenstein series. Logarithmic derivatives of automorphic L-functions are the main ingredients of the contribution of the continuous spectrum to the spectral side of the Arthur trace formula.

Example: $G = \mathrm{SL}(2, \mathbb{R})$, $\Gamma = \mathrm{SL}(2, \mathbb{Z})$

Maass forms:

- $f \in L^2(\Gamma \backslash \mathbb{H})$, $\Delta f = \lambda f$, $\lambda \geq 1/4$, $f(-\bar{z}) = f(z)$.

Hecke operators:

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right), \quad n \in \mathbb{N}.$$

- $[T_n, T_m] = 0$, $[T_n, \Delta] = 0$, $m, n \in \mathbb{N}$, selfadjoint in $L^2(\Gamma \backslash \mathbb{H})$.
- Maass forms can be simultaneously diagonalized.

$$T_n f = \lambda_f(n) f, \quad n \in \mathbb{N}.$$

$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p L_p(s,f), \quad \mathrm{Re}(s) > 1,$$

where

$$L_p(s,f) = (1 - \alpha_{1,f}(p)p^{-s})^{-1}(1 - \alpha_{2,f}(p)p^{-s})^{-1}$$

- $L(s,f)$ is an example of an automorphic L-function, Euler product of degree two.
- $N(T)$ number of zeros of $L(s,f)$ in $|\mathrm{Im}(s)| < T$, $0 < \mathrm{Re}(s) < 1$. Then

$$N(T) \sim \frac{2}{\pi} T \log T, \quad T \to \infty.$$

- Can be generalized to GL_n and other groups G.
- To this end we need to pass to the adelic framework

- p prime number, $\mathbb{Q}_p$ field of p-adic numbers, $\mathbb{Z}_p$ ring of p-adic intergers.

$$\mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{R}) \times \prod_p{}' \, \mathrm{GL}_n(\mathbb{Q}_p)$$

- Restricted direct product. $\mathrm{GL}_n(\mathbb{A})$ consists of all sequences $(g_\infty, g_2, g_3, \cdots, g_p, \cdots)$, $g_p \in \mathrm{GL}_n(\mathbb{Q}_p)$, such that $g_p \in \mathrm{GL}_n(\mathbb{Z}_p)$ for almost all p.
- $K_p := \mathrm{GL}_n(\mathbb{Z}_p)$ maximal compact in $\mathrm{GL}_n(\mathbb{Q}_p)$.
- π irreducible unitary representation of $\mathrm{GL}_n(\mathbb{A})$. Then $\pi = \otimes_v \pi_v$, $\pi_v \in \Pi(\mathrm{GL}_n(\mathbb{Q}_v))$, $v = \infty$ or $v = p$,
- π_p unramified for almost all p, i.e., π_p has non-zero K_p-fixed vector for almost all p.
- $GL_n(\mathbb{Q}) \subset \mathrm{GL}_n(\mathbb{A})$ by diagonal embedding, discrete subgroup.
- $\pi \in \Pi(\mathrm{GL}_n(\mathbb{A}))$ cuspidal automorphic representation, if π occurs in the space of cusp forms $L^2_{\mathrm{cus}}(\mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A}))$.

- π cuspidal automorphic representation
- Automorphic L-function

$$L(s,\pi) = \prod_p L_p(s,\pi), \quad \mathrm{Re}(s) \gg 0.$$

Using the Langlands parameters of π_p, one defines numbers $\alpha_{j,\pi}(p)$ as above. Then

$$L_p(s,\pi) = \prod_{j=1}^{n} (1 - \alpha_{j,\pi}(p) p^{-s})^{-1}.$$

- $L(s,\pi)$ standard automorphic L-function.
- One can form more general L-functions from these basic ones, that is the tensor powers.
- π_i cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$, $i = 1, 2$.
- Rankin-Selberg convolution is given as Euler product

$$L(s,\pi_1 \otimes \pi_2) = \prod_p L(s,\pi_{1,p} \otimes \pi_{2,p}), \quad \mathrm{Re}(s) \gg 0.$$

- For p such that $\pi_{1,p}$ and $\pi_{2,p}$ are unramified, the local L-factor is given by
$$L(s,\pi_{1,p}\otimes\pi_{2,p})=\prod_{j=1}^{n_1}\prod_{k=1}^{n_2}(1-\alpha_{j,\pi_1}(p)\alpha_{k,\pi_2}(p)p^{-s})^{-1}.$$
- $L(s,\pi_1\otimes\pi_2)$ admits meromorphic extension to $\mathbb{C}$, satisfies functional equation.
- Logarithmic derivatives of Rankin-Selberg L-functions are the main ingredients of the contribution of the continuous spectrum to the trace formula for GL_n.
- Using the analytic properties of the Rankin-Selberg L-functions, one can show that for the Weyl law and the limit multiplicity problem for GL_n, the contribution of the continuous spectrum is negligible.
- To deal with the corresponding automorphic L-functions for classical groups (symplectic, orthogonal, unitery) one can use Arthur's work on the endoscopic classification of automorphic representations to reduce the problems to the case of L-functions for GL_n.