1. The wave equation

The wave equation is an important tool to study the relation between spectral theory and geometry on manifolds.

Let $U \subset \mathbb{R}^n$ be an open set and let

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

be the Euclidean Laplace operator. Then the wave equation on U is the following differential equation.

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(x,t) &= f(x,t), \\ u(x,t) &= 0, \quad x \in \partial U,\ t > 0. \\ u(x,0) = &u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x). \end{aligned}$$

Here f, u_0 and u_1 are given functions.

6.1. The wave equation on R^n.

To understand the behavior of the solution of the wave equation we consider first the wave equation on the real line.

On $\mathbb{R}$ we consider the following equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = 0,$$

$$u(x,0) = g(x), \quad \frac{\partial u}{\partial t}(x,0) = h(x),$$

where $g, h \in C^2(\mathbb{R})$.

The first equation can be factored as follows

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0.$$

Put

$$v(x,t) \colon = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t).$$

Then we get

$$\frac{\partial}{\partial t} v(x,t) + \frac{\partial}{\partial x} v(x,t) = 0, \quad x \in \mathbb{R}, \quad t > 0.$$

This is a transport equation with constant coefficients. To solve this equation, we apply the Fourier transform with respect to x. Let

$$\hat{v}(\xi,t) := \int_{\mathbb{R}} e^{-i\xi x} v(x,t)dx.$$

Then we have

$$\frac{d}{dt}\hat{v}(\xi,t) + i\xi\hat{v}(\xi,t) = 0.$$

The solution is given by

$$\hat{v}(\xi,t) = e^{-it\xi}\hat{v}(\xi,0).$$

Hence we have

$$v(x,t) = v(x-t,0).$$

Let $a(x) = v(x,0)$. Then we get

$$\frac{\partial}{\partial t}u(x,t) - \frac{\partial}{\partial x}u(x,t) = a(x-t), \quad x \in \mathbb{R}, \quad t > 0.$$

this is a non-homogeneous transport equation. It can be solved by a similar method. The result is

$$\begin{aligned} u(x,t) &= \int_0^t a(x+(t-s)-s)\, ds + u(x+t,0) \\ &= \frac{1}{2}\int_{x-t}^{x+t} a(y)\, dy + u(x+t,0). \end{aligned}$$

If we use the inatrial conditions

$u(x,0) = g(x)$ and $u_t(x,0) = h(x)$,

we get

$$\begin{aligned} a(x) = v(x,0) &= u_t(x,0) - u_x(x,0) \\ &= h(x) - g'(x) \end{aligned}$$

Inserting this formula for a in the above equation, we obtain the following final form for the solution

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \ , \ x \in \mathbb{R}, \ t > 0. \tag{6.1}$$

From this expression for the solution one can derive the following theorem.

Theorem 6.1. *Assume that $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and define $u(x,t)$ by (6.1). Then the following holds*

1) $u \in C^2(\mathbb{R} \times [0,\infty])$.
2) $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u(x,t) = 0$ *in* $\mathbb{R} \times \mathbb{R}^+$

3)

$$\lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u(x,t) = g(x_0)$$
$$\lim_{\substack{(x,t)\to(x_0,0)\\ t>0}} u_t(x,t) = h(x_0)$$

Assume that $\operatorname{supp} g, \operatorname{supp} h \subset (r-, r)$. Then it follows from (6.1) that $\operatorname{supp} u \subset (-r-t, r+t)$. This means that the wave equation on $\mathbb{R}$ has finite propagation speed. Similar formulars hold for all $n \geq 1$. Namely consider on $\mathbb{R}^n$ the wave equation.

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(x,t) = 0, \quad u(x,0) = g(x),\ u_t(x,0) = h(x).$$

Let $n \geq 3$ be odd. Let $m = (n+1)/2$, $g \in C^{m+1}(\mathbb{R}^n)$ and $h \in C^m(\mathbb{R}^n)$. Let $\gamma_n = 1 \cdot 3 \cdot 5 \cdots (n-2)$. Then the unique solution of the wave equation is given by

$$u(x,t) = \frac{1}{\gamma_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\,\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\int_{\partial B(x,t)} g\, dS\right) + \left(\frac{1}{t}\,\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\int_{\partial B(x,t)} hds\right)\right].$$

This formula also shows that the wave equation satisfies finite propagation speed.

6.2. **Energy methods.**

Energy methods are an important tool to establish finite propagation speed for the wave equation. We illustrate this for the Laplace operator. For $x_0 \in \mathbb{R}^n$, $t_0 > 0$, let

$$C = \Big\{(x,t) \colon 0 \leq t \leq t_0,\ \parallel x - x_0 \parallel \leq t_0 - t\Big\}.$$

Theorem 6.2. *(Finite propagation speed).*

Assume that $u(x,0) = u_t(x,0) \equiv 0$ on $B(x_0,t_0)$. Then $u \equiv 0$ in C.

Proof. Define the energy of the solution by

$$e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} \Big(u_t(x,t)^2 + \parallel \nabla u(x,t) \parallel^2\Big)dx, 0 \leq t \leq t_0. \tag{6.2}$$

Then we have

$$\begin{aligned}\frac{d}{dt}e(t) &= \int_{B(x_0,t_0-t)} (u_t u_{tt} + \langle \nabla u, \nabla u_t\rangle)dx \\ &- \frac{1}{2}\int_{\partial B(x_0,t_0-t)} (u_t^2 + |\nabla u|^2)ds \\ &= \int_{B(x_0,t_0-t)} u_t(u_{tt} - \Delta u)dx \\ &+ \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t \; ds - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \frac{1}{2}\left(u_t^2 \parallel \nabla u \parallel^2\right)dS. \\ &= \int_{\partial B(x_0,t_0-t)} \left(\frac{\partial u}{\partial \nu}u_t - \frac{1}{2}u_t^2 - \frac{1}{2}\parallel \nabla u \parallel^2\right)dS.\end{aligned}$$

Now note that

$$\left|\frac{\partial u}{\partial \nu}u_t\right| \leq |u_t| \cdot \parallel \nabla u \parallel \leq \frac{1}{2}|u_t|^2 + \frac{1}{2}\parallel \nabla u \parallel^2 .$$

This implies that

$$\frac{d}{dt}e(t) \leq 0.$$

Thus $e(t) \leq e(0) = 0$ for $0 \leq t \leq t_0$. By (6.2) it follows that $u_t \equiv 0$ and $\nabla u \equiv 0$ in C. This implies that $u \equiv c$ and therefore, $u = 0$. ☐

6.3. **Gradient and divergence.** As preparation for the study of the wave equation on manifolds we recall some facts about the dicergence and the gradient on a Riemannian manifold.

Let X be a Riemannian manifold. Let $f \in C^\infty(X)$. Then the gradient $\operatorname{grad} f \in C^\infty(TX)$ of f is defined by

$$\langle \operatorname{grad} f(p), Y_p\rangle = Y(f)(p) = df(Y)(p)$$

for all $Y \in C^\infty(TY)$. Let

$$\nabla \colon C^\infty(TX) \to C^\infty(T^x X \otimes TX)$$

be the Levi-Civita connection associated to the Riemannian metric of X. Let $Y \in C^\infty(TX)$. The divergence $\operatorname{div} Y$ of the vector field Y is defined by

$$\operatorname{div} Y(p) = \operatorname{Tr}(\xi \in T_pX \longmapsto \Delta_\xi Y \in T_pX).$$

In local coordinates $\operatorname{grad} f$ and $\div Y$ can described as follows. Let $x_1, ..., x_n$ be local coordinates. Let

$$g = \sum_{i,j=1}^n g_{ij} dx_i \otimes dx_j$$

be the Riemannian metric in these coordinates. Furthermore, let

$$(g^{kl}) = (g_{ij})^{-1}, \quad \bar{g} = \det(g_{ij}).$$

and

$$Y = \sum_{j=1}^{n} f_j \frac{\partial}{\partial x_j}.$$

Then we have

$$\operatorname{grad} f = \sum_{k=1}^{n} \sum_{l=1}^{n} \left(g^{kl} \frac{\partial f}{\partial x_l} \right) \frac{\partial}{\partial x_k}$$

and

$$\operatorname{div} Y = \frac{1}{\sqrt{g}} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{g} f_j \right)$$

Lemma 6.3. *For all $f \in C^\infty(X)$ and $Y \in C^\infty(TX)$ we have* $(\operatorname{grad} f, Y) = -(f, \operatorname{div} Y)$.

Proof. Using a partition of unity, the proof can be reduced to the case where $\operatorname{supp} f$ is contained in a coordinate chart U. Then

$$\begin{aligned}
(\operatorname{grad} f, Y) &= \int_U \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(g^{kl} \frac{\partial f}{\partial x_l} \right) g_{kj} f_j \sqrt{g} dx = \int_U \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} f_j \sqrt{g} dx \\
&= - \int_U f \frac{1}{\sqrt{g}} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(f_j \sqrt{g} \right) \sqrt{g} dx = - \int_U f \operatorname{div} Y d\mu(x) \\
&= -(f, \operatorname{div} Y).
\end{aligned}$$

□

The Riemannian metric defines an isomorphism.

$$\phi\colon TX \cong T^\star X.$$

It induces an isomorphism

$$\phi\colon C^\infty(TX) \cong \Lambda^1(X).$$

Lemma 6.4. (1) *For all $f \in C^\infty(X)$ we have*

$$\phi(\operatorname{grad} f) = df.$$

(2) *For all $Y \in C^\infty(TX)$ we have*

$$-\operatorname{div}(Y) = d^\star(\phi(Y)).$$

6.4. **Symmetric hyperbolic systems.** Let X be a Riemannian manifold and $E \to X$ a Hermitian vector bundle over X. Denote by $(\cdot,\cdot)$ the inner product in $C_c^\infty(E)$ induced by the Riemannian metric and the fibre metric in E. Let

$$D \colon C^\infty(E) \to C^\infty(E)$$

be an elliptic differential operator of order 1. Assume that D is formally self-adjoint.

Example: The basic example is the Dirac operator $D \colon C^\infty(S) \to C^\infty(S)$ on a spin manifold.

Let $\pi \colon T^\star X \to X$ the cotangent bundle. Let

$$\sigma_D \colon \pi^* E \to \pi^* E$$

be the principal symbol of D. We recall its definition. Let $x \in X$, $\xi \in T_x^* X$, and $e \in E_x$. We choose $f \in C^\infty(X)$ with

$$f(x) = 0, \qquad df(x) = \xi,$$

and $\varphi \in C^\infty(E)$ with $\varphi(x) = e$. Then

$$\sigma_D(x,\xi)(e) = D(F\varphi)(x).$$

Lemma 6.5. *For any $f \in C^\infty(X)$ and $\varphi \in C^\infty(E)$ we have*

$$D(f\varphi) = \sigma_D(df)(\varphi) + f d\varphi. \tag{6.3}$$

Note that

$$\sigma_D(x,\xi)^t = -\sigma_d(x,\xi).$$

Definition 6.6. *For $\Omega \subset X$ let*

$$c(\Omega) \colon = \sup \left\{ \parallel \sigma_D(x,\xi) \parallel : \xi \in T_x^\star, \ \parallel \xi \parallel, \ x \in \Omega \right\}.$$

$c(\Omega)$ is called the propagation speed of D on Ω.

Now we consider the wave equation

$$\frac{\partial u}{\partial t} = iDu, \quad u(x,0) = u_0(x), \tag{6.4}$$

where $u_0 \in C^\infty(E)$.

Proposition 6.7. *Let $x_0 \in X$ and suppose that $B_r(x_0)$ is a geodesic coordinate system. Let $c \colon = c(B_r(x_0))$. Let $u \in C^\infty([-T,T], C^\infty(E))$ be a solution of*

$$\frac{\partial u}{\partial t} = iD_u.$$

Then we have

$$\parallel u(t) \parallel_{B_{r-ct}(x_0)} \leq \parallel u(0) \parallel_{B_r(x_0)}$$

for $0 \leq t < r/c$.

Proof. We define a smooth vector field Y_t on X by

$$Y_t(f)(x) = -i\langle u(x,t), \sigma_D(df_x, x)(u(x,t))\rangle_x$$

Let $f \in C_c^\infty(X)$. Then by Lemma 6.4 we have

$$\int_X \operatorname{div} Y_t(x)\bar{f}(x)dx = (\operatorname{div} Y_t, f) = -(Y_t, \operatorname{grad} f) = -Y_t(f).$$

By definition of $Y_t(f)$ and (6.3) we get

$$\begin{aligned}\int_X \operatorname{div} Y_t(x)\bar{f}(x)dx &= i\big(u(t), D(fu(t)) - fDu(t)\big)\\ &= i\big(Du(t), fu(t)\big) - i\big(u(t), fDu(t)\big)\\ &= i\int_X \Big(\langle du(x,t), u(x,t)\rangle_x - \langle u(x,t), Du(x,t)\rangle_x\Big)\bar{f}(x)dx.\end{aligned}$$

Since this equality holds for every $f \in C_c^\infty(X)$, it follows that

$$\operatorname{div} Y_t(x) = i\langle Du(x,t), u(x,t)\rangle_x - i\langle u(x,t), Du(x,t)\rangle_x. \tag{6.5}$$

Now

$$\begin{aligned}&\frac{d}{dt}\int_{B_{r-ct}(x_0)} \| u(x,t) \|^2 \, dx\\ &\quad= \int_{B_{r-ct}(x_0)} \Big(\big\langle \frac{\partial}{\partial t}u(x,t), u(x,t)\big\rangle_x + \big\langle u(x,t), \frac{\partial}{\partial t}u(x,t)\big\rangle_x\Big)dx\\ &\quad\quad - c\int_{\partial B_{r-ct}(x_0)} \| u(x,t) \|_x^2 \, dS(x)\\ &\quad= i\int_{B_{r-ct}(x_0)} \Big(\big\langle Du(x,t), u(x,t)\big\rangle_x - \big\langle u(x,t), Du(x,t)\big\rangle_x\Big)dx\\ &\quad\quad - c\int_{\partial B_{r-ct}(x_0)} \| u(x,t) \|_x^2 \, dS(x).\end{aligned}$$

For the last equality we used that $u(x,t)$ satisfies the wave equation (6.6). Using (6.5) and the divergence theorem it follows that

$$\begin{aligned}\frac{d}{dt}\int_{B_{r-ct}(x_0)} \| u(x,t) \|_x^2 \, dx &= \int_{B_{r-ct}(x_0)} \operatorname{div} Y_t(x) \cdot \overline{u(x,t)}dx\\ &\quad - c\int_{\partial B_{r-ct}(x_0)} \| u(x,t) \|_x^2 \, dS(x)\\ &= \int_{\partial B_{r-ct}(x_0)} \langle Y_t(x), \nu(x)\rangle_x dS(x)\\ &\quad - c\int_{\partial B_{r-ct}(x_0)} \| u(x,t) \|_x^2 \, dS(x),\end{aligned}$$

where $\nu(x)$ denotes the exterior unit normal vector field. Now observe that by the definition of $c = c(B_r(x_0))$

$$|\langle Y_t(x), \nu(x)\rangle_x| = |\langle u(x,t), \sigma_D(\nu(x), x)(u(x,t))\rangle_x| \leq c \parallel u(x,t) \parallel^2 .$$

This implies that

$$\frac{d}{dt}\int_{B_{r-ct}(x_0)} \|u(x,t)\|_x^2 dx \leq 0.$$

Thus we obtain

$$\|u(t)\|_{B_{r-ct}(x_0)} \leq \|u(0)\|_{B_r}(x_0),$$

which concludes the proof. □

Let $c = c(B_r(x_0))$ and let

$$C = \{(t,x) \colon t \geq 0, \quad d(x, x_0) \leq r - ct\}$$

Corollary 6.8. *Let $u \in C^\infty([-T,T], C^\infty(E))$ be a solution of the equation*

$$\frac{\partial u}{\partial t} = iDu$$

on C. Suppose that $u(0) = 0$ on $B_r(x_0)$. Then $u = 0$ on C.

Let $U = B_r(x_0) \subset X$ be a normal coordinate chart and let

$$\phi \colon E|_U \cong U \times \mathbb{C}^N$$

be a trivialization of $E|_U$. Let $D|_U$ be restriction of D to $C^\infty(U, E|_U)$. Then

$$\frac{\partial}{\partial t} u = iD|_U(u), \quad u(0,x) = u_0(x)$$

is a hyperbolic system of order 1 in $\mathbb{R}^n$. The usual theory for such systems implies existence of solutions with smooth initial conditions. In this way we get

Proposition 6.9. *For every $x_0 \in X$ there exists $r > 0$ such that for $u_0 \in C^\infty(B_r(x_0, E))$ there exists a unique solution of*

$$\frac{\partial u}{\partial t} = iD(u), \quad u(0,x) = u_0(x)$$

on

$$C_0 = \{(x,t) \colon t \geq 0, \quad d(x, s_0) \leq r - ct\},$$

where $c = c(B_r(x_0))$.

The next proposition extends the above result to a larger region.

Proposition 6.10. *Let $S = B_R(x_0)$ be a compact ball in X. Let $c \colon c(S)$ and*

$$C_0 = \{(x,t) \colon t \geq 0, \; d(x, x_0) \leq r - ct\}$$

Let $u_0 \in C^\infty(S, E)$. Then there is in C_0 a unique smooth solution of the equation

$$\frac{\partial u}{\partial t} = iD(u), \quad u(0) = u_0.$$

Proof. Since S is compact, there exists $r > 0$ such for all $y \in S$ the injectivity radius $i(y) \geq r$. Thus for all $y \in S$, $B_r(y)$ is a normal coordinate chart. It follows from Proposition (6.9) that for all $y \in B_{R-r}(x_0)$ and $u_0 \in C^\infty(B_r(y), E)$, the wave equation

$$\frac{\partial u}{\partial t} = iD(u), \quad u(0) = u_0,$$

has a unique C^∞-solution on the truncated cone

$$C_y = \{(x,t)\colon d(x,y) \leq r - ct, 0 \leq t \leq r/2c\}.$$

By uniqueness, solutions agree on $C_y \cap C_z$. Therefore, we obtain a solution on the truncated cone

$$\{(x,t)\colon d(x,x_0) \leq R - ct, \;\; 0 \leq t \leq r/2c\}.$$

The solution u at time $t = r/2c$ serves as initial condition on $B(x_0, R - r/2)$. If we repeat the above argument, we get a solution on the truncated cone

$$\{(x,t)\colon d(x,x_0) \leq R - ct, \; r/2c \leq t \leq r/c\}$$

and therefore, a solution on

$$\{(x,t)\colon \; d(x,x_0) \leq R - ct, \; 0 \leq t \leq r/c\}.$$

After a finite number of steps, we obtain a smooth solution on the cone with base S. $\square$

Let $x_0 \in X$. Put

$$c(r)\colon = c(B_r(x_0)), \quad r > 0.$$

Theorem 6.11. *Let X be a complete Riemannian manifold. Suppose that*

$$\int_0^\infty \frac{dr}{c(r)} = \infty.$$

1) *Uniqueness: Suppose that $u \in C^\infty(X, E)$ is a solution of*

$$\frac{\partial u}{\partial t} = iDu$$

on $[0,T] \times X$ with $u(0) = 0$. Then $u \equiv 0$.

2) *Existence: Let $u_0 \in C^\infty(X, E)$. Then the wave equation*

$$\frac{\partial u}{\partial t} = iDu \quad , \quad u(0) = u_0, \tag{6.6}$$

has a unique solution on $\mathbb{R} \times X$. Moreover, for fixed t, $u(\cdot, t)$ has compact support.

Proof. We first establish uniqueness. Let $R > 0$. Put $S_R = B_R(x_0)$. We shall show that $u(T)$ vanishes on S_R. Of course, we also have that $u(T')$ vanishes for $T' < T$. Let

$$t_R = c(R+1)^{-1}, R > 0.$$

By Proposition (6.10), the wave equation (6.6) on

$$\{(x,t)\colon d(x,x_0) \leq R - c(R+1)t, 0 \leq t \leq t_R\}$$

with initial condition $u_0 \in C^\infty(S_{R+1}, E)$ has a unique solution. Hence $u(T)$ on S_R is determined by $u(T - t_R)$ on S_{R+1}. By the same argument $u(T - t_R)$ is determined by $u(T - t_R - t_{R+1})$ on S_{R+2}. Since the sets S_R are compact, this may be continued indefinitely. Now we have

$$\sum_{n=0}^{\infty} t_{R+n} = \infty, \tag{6.7}$$

because $c(R)$ is monoton and

$$\int_1^\infty \frac{dr}{c(r)} = \infty$$

by assumption. Hence there exists $N \in \mathbb{N}$ such that

$$T' \colon = t_R + t_{R+1} + \cdots + t_{R+N} < T$$

and $T' + t_{R+N+1} > T$. From our considerations above follows that $u(T)|_{S_R}$ is determined by $u(T - T')|_{S_{R+N}}$. Let

$$\varepsilon = (R + N + 1)(T - T').$$

Then it follows as above that $u(T - T')|_{S_{R+N}}$ is determined by $u(0)|_{S_{R+N+\varepsilon'}}$. Note that $T - T' \leq t_{R+N+1}$. But $u(0) = 0$. Hence $u(T - T')|_{S_{R+N}} = 0$. This implies that $u(T)|_{S_R} = 0$.

Next we establish existence. The argument is like the uniqueness proof run in reverse. Let $u_0 \in C_c^\infty(X, E)$. Let $R > 0$ such that $\operatorname{supp} u_0 \subset S_R$. Since X is complete, S_R is compact. By Proposition (6.10) there exists a C^∞-solution of (6.6) in

$$C = \{(x, t) \colon 0 \leq t \leq t_{R+2} \ , d(x, x_0) \leq R + 3 - c(R + 3)t\}$$

Moreover, it follows from the uniqueness part of Proposition (6.10) that the solution vanishes outside S_{R+1}. So we can extend it by 0 to a global solution on X which exists for time $0 \leq t \leq t_{R+2}$. Now we iterate this process. The solution at time $t = t_{R+2}$ is supported in S_{R+1}. By the above argument, it extends to a solution for $0 \leq t \leq t_{R+2} + t_{R+3}$ with support in S_{R+2}. Using again (6.7), we can extend the solution to any time t and for fixed t, $u(t)$ has compact support. □

For each t define a map

$$U_t \colon C_c^\infty(X, E) \to C^\infty(X, E)$$

by

$$U_t(u_0) = u(t),$$

where u is the unique solution of

$$\frac{\partial u}{\partial t} = iDu, \quad u(0) = u_0.$$

Corollary 6.12. *Under the assumptions of Theorem* (6.11), $\{U_t\}$ *is a one-parameter group. Moreover, if* $w \in C_c^\infty(X, E)$, *we have*

$$\frac{d}{dt}(U_t(u_0), w) = (iDU_t(u_0), w).$$

Finally $DU_t(u_0) = U_t(Du_0)$.

Proof. Let $u(t){:}\ = U_t(u_0)$. Then $u(t) \in C^\infty(X,E)$ and

$$\frac{\partial u}{\partial t} = iDu.$$

Since $w \in C_c^\infty(X,E)$, we can differentiate unter the integral which gives

$$\frac{d}{dt}(U_t(u_0), w) = \left(\frac{\partial u}{\partial t}, w\right) = (iDU_t(u_0), w)$$

Moreover, $U_{s+t} = U_s \circ U_t$ and $DU_t = U_t D$ follows from uniqueness. □

Note that U_t is unitary. Indeed we have

$$\begin{aligned}\frac{d}{dt} \parallel u(t) \parallel^2 &= \Big(\frac{du}{dt}(t), u(t)\Big) + \Big(u(t), \frac{du}{dt}(t)\Big) \\ &= \Big(iDu(t), u(t)\Big) + \Big(u(t), iDu(t)\Big) \\ &= \Big(iDu(t), u(t)\Big) - \Big(iD(t), u(t)\Big) = 0.\end{aligned}$$

6.5. **Essential self-adjointness.** In this section we apply the results obtained in the previous section to establish the essential self-adjointness of geometric operators.

We begin with an abstract result.

Lemma 6.13. *Let T be a symmetric operator in a Hilbert space $\mathcal{H}$ with dense domain $\mathcal{D} \subset \mathcal{H}$. Suppose that $T(\mathcal{D}) \subseteq \mathcal{D})$. Furthermore suppose that there is a one-parameter group U_t of unitary operators on $\mathcal{H}$ such that*

$$U_t(\mathcal{D}) \subseteq \mathcal{D}, \quad \mathrm{U}_t T = TU_t \text{ on } \mathcal{D}$$

and

$$\frac{d}{dt}U_t(u) = iTU_t(u)$$

for $u \in \mathcal{D}$. Then every power of T is essentially self-adoint.

Proof. Let $n \in \mathbb{N}$ and $A{:}\ = T^n$. Then

$$A\colon \mathcal{D} \to \mathcal{H}$$

is symmetric. To show that A is essentially self-adjoint, it suffices to verify that

$$\overline{(A \pm i\,\mathrm{Id})(\mathcal{D})} = \mathcal{H}$$

Let $\psi \in \mathcal{H}$ and suppose that

$$\big((A \pm i)(\varphi), \psi\big) = 0,\ \forall \varphi \in \mathcal{D}.$$

Then it follows that $\psi \in \mathcal{D}(A^*)$ and

$$A^*\psi = \mp i\psi.$$

We consider the case where $A^*\psi = i\psi$. For $u \in \mathcal{D}$ define

$$f(t) = (U_t(u), \psi),\ t \in \mathbb{R}.$$

Since U_t is unitary, $f(t)$ is bounded. Furthermore we have

$$\begin{aligned}\frac{d^n}{dt^n} f(t) &= (iT^n U_t(u), \psi) = (iAU_t(u), \psi) = (i^n U_t(u), A^*\psi) \\ &= -i^{n+1}(U_t(u), \psi) = -i^{n+1} f(t).\end{aligned}$$

Thus $f(t)$ satisfies the ordinary linear differential equation.

$$\frac{d^n f}{dt^n}(t) = -i^{n+1} f(t). \tag{6.8}$$

Let α_j, $j = 1, ..., n$, be the different roots of the equation

$$z^n = -i^{n+1}.$$

Then $e^{\alpha_j t}$, $j = 1, ..., n$, is a basis for the space of solutions of (6.8). Therefore $f(t)$ can be written as

$$f(t) = \sum_{j=1}^{n} c_j e^{\alpha_j t}$$

for some constants $c_j \in \mathbb{C}$. Now observe that $\mathrm{Re}(\alpha_j) \neq 0$ for $j = 1, ..., n$. Since f is bounded this implies that $f \equiv 0$. Hence we get

$$(u, \psi) = f(0) = 0,\ u \in \mathcal{D}.$$

Since $\mathcal{D} \subset \mathcal{H}$ is dense, it follows that $\psi = 0$. The case $A^*\psi = -i\psi$ can be treated in the same way. $\square$

We can now state the main result about essential self-adjointness.

Theorem 6.14. *Let X be a complete Riemannian manifold and $E \to X$ a Hermitian vector bundle over X. Let*

$$D\colon C^\infty(X, E) \to C^\infty(X, E)$$

be an elliptic differential operator of order 1 which is formally self-adjoint. Assume that

$$\int_1^\infty \frac{dr}{c(r)} = \infty.$$

Let $T\colon C_c^\infty(X, E) \to L^2(X, E)$ be the operator which is defined by D. Then every power of T is essentially self-adjoint.

Proof. Let

$$U_t\colon C_c^\infty(X, E) \to C_c^\infty(X, E)$$

be the 1-parameter group, defined by Corollary 6.12. For each $t \in \mathbb{R}$ we have

$$\|U_t(u)\| = \|u\|,\ u \in C_c^\infty(X, E).$$

Indeed, for $u, w \in C_c^\infty(X, E)$, we have

$$\begin{aligned}\frac{d}{dt}(U_t(u), U_t(w)) &= (iDU_t(u), U_t(w)) + (U_t(u), iDU_t(w))\\ &((iD - iD)U_t(u), U_t(w)) = 0.\end{aligned}$$

Hence U_t extends by continuity to a one-parameter family

$$U_t \colon L^2(X, e) \to L^2(X, E)$$

of unitary operators. The assumptions of Lemma 6.13 are satisfied. This implies the theorem. □

6.6. **Applications.** Now we are ready to apply the results of the previous section to geometric situations.

Let X be a complete Riemannian manifold. Let

$$D = d + d^* \colon \Lambda^*(X) \to \Lambda^*(X).$$

Then D is formally self-adjoint. To determine its principal symbol, fix $p \in X, \xi \in T_p^*X$ and $v \in \Lambda^* T_p^* X$. Let $f \in C^\infty(X)$ and $\varphi \in \Lambda^*(X)$ be such that $f(p) = 0, df_p = \xi$ and $\varphi(p) = v$. Then we have

$$\begin{aligned}\sigma_d(p, \xi)v = D(f\varphi)(p) = (d + d^*)(f\varphi)(p) &= df_p \wedge \varphi(p) - *(df \wedge *\varphi)(p)\\ &= \xi \wedge v - i_\xi(v),\end{aligned}$$

where $i_\xi \colon \Lambda^* T_p^* X \to \Lambda^* T_p^* X$ denotes interior multiplication by ξ. This implies that

$$\| \sigma_D(x, \xi) \| = \| \xi \| .$$

Hence we have $c(x) = 1$, i.e. D has unite propagation speed. By Theorem 6.14 it follows that for all $n \in \mathbb{N}$, the operator

$$(d + d^*)^n \colon \Lambda_c^*(X) \to L^2\Lambda^*(X)$$

is essentially self-adjoint. Now recall that the laplace operator Δ is given by

$$\Delta = (d + d^*)^2.$$

Thus it follows that for all $n \in \mathbb{N}$,

$$\Delta^n \colon \Lambda_c^*(X) \to L^2\Lambda^*(X)$$

is essentially self-adjoint. The Laplace operator preserves $\Lambda^p(X)$ for every p. Therefore

$$\Delta^n \colon \Lambda_c^p(X) \to L^2\Lambda^p(X)$$

is essentially self-adjoint for all $p = 0, ..., n$.

Next we consider a complex manifold equipped with a Hermitian metric, so that X, equipped with the associated Riemannian metric is complete. Let $E \to X$ be a holomorphic Hermitian vector bundle over X. Then we define the space of (p, q)-forms with values in E as the space of C^∞-sections of $\Lambda^p T^{*(1.0)}(X) \otimes \Lambda^q T^{*(0,1)}(X) \otimes E$. The operator

$$\bar{\partial} \colon \Lambda^{p,q}(X, E) \to \Lambda^{p,q+1}(X, E)$$

is uniquely defined by demanding that

$$\bar{\partial}(\omega \otimes \varphi) = (\bar{\partial}\omega) \otimes \varphi$$

for every $\omega \in \Lambda^{p,q}(X)$ and every **holomorphic** section of E. In this way we get the Dolbeault complex

$$\cdots \xrightarrow{\bar{\partial}} \Lambda^{p,q}(X, E) \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(X, E) \longrightarrow \cdots$$

Let $D = (\bar{\partial} + \bar{\partial}^*)$. Then we have

$$\sigma_D(x, \xi)(\omega \otimes \varphi) = (\pi(\xi) \wedge \omega - i_{\pi(\xi)}(\omega)) \otimes \varphi$$

where $\pi\colon T_x^*X \otimes \mathbb{C} \to T_x^{*(0,1)}X$ is the canonical projection. It follows that

$$\| \sigma_D(X, \xi) \| = \| \pi(\xi) \|$$

and hence, $c(x) = 1/\sqrt{2}$. By Theorem 6.14,

$$(\bar{\partial} + \bar{\partial}^*)^n\colon \Lambda_c^{p,*}(X, E) \to L^2\Lambda^{p,*}(X, E)$$

is essentially self-adjoint for all p and all $n \in \mathbb{N}$.

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