1. THE WAVE EQUATION

The wave equation is an important tool to study the relation between spectral theory and geometry on manifolds.

Let $U \subset \mathbb{R}^n$ be an open set and let

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

be the Euclidean Laplace operator. Then the wave equation on $U$ is the following differential equation.

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(x, t) = f(x, t),$$

$$u(x, t) = 0, \quad x \in \partial U, \quad t > 0.$$

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x).$$

Here $f, u_0$ and $u_1$ are given functions.

6.1. The wave equation on $\mathbb{R}^n$.

To understand the behavior of the solution of the wave equation we consider first the wave equation on the real line.

On $\mathbb{R}$ we consider the following equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u(x, t) = 0,$$

$$u(x, 0) = g(x), \quad \frac{\partial u}{\partial t}(x, 0) = h(x),$$

where $g, h \in C^2(\mathbb{R})$.

The first equation can be factored as follows

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0.$$

Put

$$v(x, t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x, t).$$

Then we get

$$\frac{\partial}{\partial t} v(x, t) + \frac{\partial}{\partial x} v(x, t) = 0, \quad x \in \mathbb{R}, \quad t > 0.$$
This is a transport equation with constant coefficients. To solve this equation, we apply the Fourier transform with respect to $x$. Let

$$\hat{v}(\xi,t) = \int_{\mathbb{R}} e^{-ix\xi} v(x,t) dx.$$ 

Then we have

$$\frac{d}{dt} \hat{v}(\xi,t) + i\xi \hat{v}(\xi,t) = 0.$$ 

The solution is given by

$$\hat{v}(\xi,t) = e^{-it\xi} \hat{v}(\xi,0).$$ 

Hence we have

$$v(x,t) = v(x - t, 0).$$ 

Let $a(x) = v(x, 0)$. Then we get

$$\frac{\partial}{\partial t} u(x,t) - \frac{\partial}{\partial x} u(x,t) = a(x-t), \quad x \in \mathbb{R}, \quad t > 0.$$ 

This is a non-homogeneous transport equation. It can be solved by a similar method. The result is

$$u(x,t) = \int_0^t a(x + (t - s) - s) ds + u(x + t, 0) = \frac{1}{2} \int_{x-t}^{x+t} a(y) dy + u(x + t, 0).$$ 

If we use the initial conditions

$$u(x,0) = g(x)$$

and $u_t(x, 0) = h(x)$,

we get

$$a(x) = v(x, 0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x)$$

Inserting this formula for $a$ in the above equation, we obtain the following final form for the solution

$$(6.1) \quad u(x,t) = \frac{1}{2} [g(x + t) + g(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy \quad x \in \mathbb{R}, \quad t > 0.$$ 

From this expression for the solution one can derive the following theorem.

**Theorem 6.1.** Assume that $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and define $u(x,t)$ by (6.1). Then the following holds

1) $u \in C^2(\mathbb{R} \times [0,\infty])$.

2) $\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2} \right) u(x,t) = 0$ in $\mathbb{R} \times \mathbb{R}^+$.
Assume that supp $g$, supp $h \subset (r-, r)$. Then it follows from (6.1) that supp $u \subset (-r - t, r + t)$. This means that the wave equation on $\mathbb{R}$ has finite propagation speed. Similar formulars hold for all $n \geq 1$. Namely consider on $\mathbb{R}^n$ the wave equation.

$$\frac{\partial^2}{\partial t^2} u(x,t) - \Delta u(x,t) = 0,$$

$$u(x,0) = g(x), \quad u_t(x,0) = h(x).$$

Let $n \geq 3$ be odd. Let $m = (n + 1)/2$, $g \in C^{m+1}(\mathbb{R}^n)$ and $h \in C^m(\mathbb{R}^n)$. Let $\gamma_n = 1 \cdot 3 \cdot 5 \cdots (n - 2)$. Then the unique solution of the wave equation is given by

$$u(x,t) = \frac{1}{\gamma_n} \left[ \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left( t^{n-2} \int_{\partial B(x,t)} g dS \right) 
+ \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-5}{2}} \left( t^{n-2} \int_{\partial B(x,t)} h ds \right) \right].$$

This formula also shows that the wave equation satisfies finite propagation speed.

6.2. Energy methods.

Energy methods are an important tool to establish finite propagation speed for the wave equation. We illustrate this for the Laplace operator. For $x_0 \in \mathbb{R}^n$, $t_0 > 0$, let

$$C = \left\{ (x,t) : 0 \leq t \leq t_0, \| x - x_0 \| \leq t_0 - t \right\}.$$

**Theorem 6.2.** (Finite propagation speed).

Assume that $u(x,0) = u_t(x,0) \equiv 0$ on $B(x_0, t_0)$. Then $u \equiv 0$ in $C$.

**Proof.** Define the energy of the solution by

$$e(t) = \frac{1}{2} \int_{B(x_0,t_0-t)} \left( u_t(x,t)^2 + \| \nabla u(x,t) \|^2 \right) dx, 0 \leq t \leq t_0.$$
Then we have
\[
\frac{d}{dt} e(t) = \int_{B(x_0,t_0-t)} (u_t u_{tt} + \langle \nabla u, \nabla u_t \rangle) dx
- \frac{1}{2} \int_{\partial B(x_0,t_0-t)} (u_t^2 + |\nabla u|^2) ds
= \int_{B(x_0,t_0-t)} u_t (u_{tt} - \Delta u) dx
+ \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t ds - \frac{1}{2} \int_{\partial B(x_0,t_0-t)} \frac{1}{2} \left( u_t^2 \| \nabla u \|^2 \right) ds.
\]
Now note that
\[
\left| \frac{\partial u}{\partial \nu} u_t \right| \leq |u_t| \cdot \| \nabla u \| \leq \frac{1}{2} |u_t|^2 + \frac{1}{2} \| \nabla u \|^2.
\]
This implies that
\[
\frac{d}{dt} e(t) \leq 0.
\]
Thus $e(t) \leq e(0) = 0$ for $0 \leq t \leq t_0$. By (6.2) it follows that $u_t \equiv 0$ and $\nabla u \equiv 0$ in $C$. This implies that $u \equiv c$ and therefore, $u = 0$. 

6.3. Gradient and divergence. As preparation for the study of the wave equation on manifolds we recall some facts about the divergence and the gradient on a Riemannian manifold.

Let $X$ be a Riemannian manifold. Let $f \in C^\infty(X)$. Then the gradient $\nabla f \in C^\infty(TX)$ of $f$ is defined by
\[
\langle \nabla f(p), Y_p \rangle = Y(f)(p) = df(Y)(p)
\]
for all $Y \in C^\infty(TY)$. Let
\[
\nabla : C^\infty(TX) \to C^\infty(T^*X \otimes TX)
\]
be the Levi-Civita connection associated to the Riemannian metric of $X$. Let $Y \in C^\infty(TX)$. The divergence $\text{div} Y$ of the vector field $Y$ is defined by
\[
\text{div} Y(p) = \text{Tr}(\xi \in T_pX \mapsto \Delta_\xi Y \in T_pX).
\]
In local coordinates $\nabla f$ and $\nabla Y$ can described as follows. Let $x_1, \ldots, x_n$ be local coordinates. Let
\[
g = \sum_{i,j=1}^n g_{ij} dx_i \otimes dx_j
\]
be the Riemannian metric in these coordinates. Furthermore, let
\[
(g^{ij}) = (g_{ij})^{-1}, \quad \bar{g} = \det(g_{ij}).
\]
and
\[ Y = \sum_{j=1}^{n} f_j \frac{\partial}{\partial x_j}. \]

Then we have
\[
\text{grad } f = \sum_{k=1}^{n} \sum_{l=1}^{n} \left( g^{kl} \frac{\partial f}{\partial x_l} \right) \frac{\partial}{\partial x_k}
\]
and
\[
\text{div } Y = \frac{1}{\sqrt{\bar{g}}} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( \sqrt{g} f_j \right)
\]

**Lemma 6.3.** For all \( f \in C^\infty(X) \) and \( Y \in C^\infty(TX) \) we have \((\text{grad } f, Y) = -(f, \text{div } Y)\).

**Proof.** Using a partition of unity, the proof can be reduced to the case where \( \text{supp } f \) is contained in a coordinate chart \( U \). Then
\[
(\text{grad } f, Y) = \int_U \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \left( g^{kl} \frac{\partial f}{\partial x_l} \right) g_{kj} f_j \sqrt{\bar{g}} dx = \int_U \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} f_j \sqrt{\bar{g}} dx
\]
\[
= - \int_U f \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( f_j \sqrt{\bar{g}} \right) \sqrt{\bar{g}} dx = - \int_U \text{div } Y d\mu(x)
\]
\[
= -(f, \text{div } Y).
\]

\(\square\)

The Riemannian metric defines an isomorphism.
\[ \phi: TX \cong T^*X. \]

It induces an isomorphism
\[ \phi: C^\infty(TX) \cong \Lambda^1(X). \]

**Lemma 6.4.**

1. For all \( f \in C^\infty(X) \) we have
   \[ \phi(\text{grad } f) = df. \]
2. For all \( Y \in C^\infty(TX) \) we have
   \[ -\text{div}(Y) = d^*(\phi(Y)). \]
6.4. Symmetric hyperbolic systems. Let $X$ be a Riemannian manifold and $E \to X$ a Hermitian vector bundle over $X$. Denote by $(\cdot, \cdot)$ the inner product in $C^\infty_c(E)$ induced by the Riemannian metric and the fibre metric in $E$. Let
\[ D : C^\infty(E) \to C^\infty(E) \]
be an elliptic differential operator of order 1. Assume that $D$ is formally self-adjoint.

Example: The basic example is the Dirac operator $D : C^\infty(S) \to C^\infty(S)$ on a spin manifold.

Let $\pi : T^* X \to X$ the cotangent bundle. Let $\sigma_D : \pi^* E \to \pi^* E$ be the principal symbol of $D$. We recall its definition. Let $x \in X$, $\xi \in T^*_x X$, and $e \in E_x$. We choose $f \in C^\infty(X)$ with $f(x) = 0$, $df(x) = \xi$, and $\varphi \in C^\infty(E)$ with $\varphi(x) = e$. Then
\[ \sigma_D(x, \xi)(e) = D(F\varphi)(x). \]

Lemma 6.5. For any $f \in C^\infty(X)$ and $\varphi \in C^\infty(E)$ we have
\[ D(f\varphi) = \sigma_D(df)(\varphi) + f d\varphi. \]

Note that
\[ \sigma_D(x, \xi)^t = -\sigma_d(x, \xi). \]

Definition 6.6. For $\Omega \subset X$ let
\[ c(\Omega) := \sup \left\{ \| \sigma_D(x, \xi) \| : \xi \in T^*_x, \| \xi \|, x \in \Omega \right\}. \]
$c(\Omega)$ is called the propagation speed of $D$ on $\Omega$.

Now we consider the wave equation
\[ \frac{\partial u}{\partial t} = iDu, \quad u(x, 0) = u_0(x), \]
where $u_0 \in C^\infty(E)$.

Proposition 6.7. Let $x_0 \in X$ and suppose that $B_r(x_0)$ is a geodesic coordinate system. Let $c : = c(B_r(x_0))$. Let $u \in C^\infty([-T, T], C^\infty(E))$ be a solution of
\[ \frac{\partial u}{\partial t} = iDu. \]
Then we have
\[ \| u(t) \|_{B_r} \leq \| u(0) \|_{B_r(x_0)} \]
for $0 \leq t < r/c$. 
Proof. We define a smooth vector field $Y_t$ on $X$ by

$$Y_t(f)(x) = -i \langle u(x, t), \sigma_D(df_x, x)(u(x, t)) \rangle_x$$

Let $f \in C^\infty_c(X)$. Then by Lemma 6.4 we have

$$\int_X \text{div} Y_t(x) \bar{f}(x) dx = (\text{div} Y_t, f) = -(Y_t, \text{grad} f) = -Y_t(f).$$

By definition of $Y_t(f)$ and (6.3) we get

$$\int_X \text{div} Y_t(x) \bar{f}(x) dx = i \langle Du(t), fu(t) \rangle - f Du(t)$$

Since this equality holds for every $f \in C^\infty_c(X)$, it follows that

(6.5) $\text{div} Y_t(x) = i \langle Du(x, t), u(x, t) \rangle_x - i \langle u(x, t), Du(x, t) \rangle_x.$

Now

$$\frac{d}{dt} \int_{B_{r-ct}(x_0)} \| u(x, t) \|^2 dx$$

$$= \int_{B_{r-ct}(x_0)} \left( \left( \frac{\partial}{\partial t} u(x, t), u(x, t) \right)_x + \langle u(x, t), \frac{\partial}{\partial t} u(x, t) \rangle_x \right) dx$$

$$- c \int_{\partial B_{r-ct}(x_0)} \| u(x, t) \|^2 dS(x)$$

$$= i \int_{B_{r-ct}(x_0)} \left( \langle Du(x, t), u(x, t) \rangle_x - \langle u(x, t), Du(x, t) \rangle_x \right) dx$$

$$- c \int_{\partial B_{r-ct}(x_0)} \| u(x, t) \|^2 dS(x).$$

For the last equality we used that $u(x, t)$ satisfies the wave equation (6.6). Using (6.5) and

the divergence theorem it follows that

$$\frac{d}{dt} \int_{B_{r-ct}(x_0)} \| u(x, t) \|^2 dx = \int_{B_{r-ct}(x_0)} \text{div} Y_t(x) \cdot \overline{u(x, t)} dx$$

$$- c \int_{\partial B_{r-ct}(x_0)} \| u(x, t) \|^2 dS(x)$$

$$= \int_{\partial B_{r-ct}(x_0)} \langle Y_t(x), \nu(x) \rangle dS(x)$$

$$- c \int_{\partial B_{r-ct}(x_0)} \| u(x, t) \|^2 dS(x),$$
where \( \nu(x) \) denotes the exterior unit normal vector field. Now observe that by the definition of \( c = c(B_r(x_0)) \)
\[
|\langle Y_t(x), \nu(x) \rangle_x| = |\langle u(x, t), \sigma_D(\nu(x), x)(u(x, t)) \rangle_x| \leq c \| u(x, t) \|^2.
\]
This implies that
\[
\frac{d}{dt} \int_{B_{r-ct}} \| u(x, t) \|^2 dx \leq 0.
\]
Thus we obtain
\[
\| u(t) \|_{B_{r-ct}} \leq \| u(0) \|_{B_r(x_0)},
\]
which concludes the proof. \( \square \)

Let \( c = c(B_r(x_0)) \) and let
\[
C = \{(t, x) : t \geq 0, \ d(x, x_0) \leq r - ct\}
\]

**Corollary 6.8.** Let \( u \in C^\infty([-T, T], C^\infty(E)) \) be a solution of the equation
\[
\frac{\partial u}{\partial t} = iD u
\]
on \( C \). Suppose that \( u(0) = 0 \) on \( B_r(x_0) \). Then \( u = 0 \) on \( C \).

Let \( U = B_r(x_0) \subset X \) be a normal coordinate chart and let
\[
\phi : E|_U \cong U \times \mathbb{C}^N
\]
be a trivialization of \( E|_U \). Let \( D|_U \) be restriction of \( D \) to \( C^\infty(U, E|_U) \). Then
\[
\frac{\partial}{\partial t} u = iD|_U(u), \quad u(0, x) = u_0(x)
\]
is a hyperbolic system of order 1 in \( \mathbb{R}^n \). The usual theory for such systems implies existence of solutions with smooth initial conditions. In this way we get

**Proposition 6.9.** For every \( x_0 \in X \) there exists \( r > 0 \) such that for \( u_0 \in C^\infty(B_r(x_0, E)) \) there exists a unique solution of
\[
\frac{\partial u}{\partial t} = iD(u), \quad u(0, x) = u_0(x)
\]
on
\[
C_0 = \{(x, t) : t \geq 0, \ d(x, s_0) \leq r - ct\},
\]
where \( c = c(B_r(x_0)) \).

The next proposition extends the above result to a larger region.

**Proposition 6.10.** Let \( S = B_R(x_0) \) be a compact ball in \( X \). Let \( c : c(S) \) and
\[
C_0 = \{(x, t) : t \geq 0, \ d(x, x_0) \leq r - ct\}
\]
Let \( u_0 \in C^\infty(S, E) \). Then there is in \( C_0 \) a unique smooth solution of the equation
\[
\frac{\partial u}{\partial t} = iD(u), \quad u(0) = u_0.
\]
Proof. Since $S$ is compact, there exists $r > 0$ such for all $y \in S$ the injectivity radius $i(y) \geq r$. Thus for all $y \in S$, $B_r(y)$ is a normal coordinate chart. It follows from Proposition (6.9) that for all $y \in B_{R-r}(x_0)$ and $u_0 \in C^\infty(B_r(y), E)$, the wave equation
\[
\frac{\partial u}{\partial t} = iD(u), \quad u(0) = u_0,
\]
has a unique $C^\infty$-solution on the truncated cone
\[
C_y = \{(x,t): d(x,y) \leq r - ct, 0 \leq t \leq r/2c\}.
\]
By uniqueness, solutions agree on $C_y \cap C_z$. Therefore, we obtain a solution on the truncated cone
\[
\{(x,t): d(x,x_0) \leq R - ct, 0 \leq t \leq r/2c\}.
\]
The solution at time $t = r/2c$ serves as initial condition on $B(x_0, R - r/2)$. If we repeat the above argument, we get a solution on the truncated cone
\[
\{(x,t): d(x,x_0) \leq R - ct, r/2c \leq t \leq r/c\}
\]
and therefore, a solution on
\[
\{(x,t): d(x,x_0) \leq R - ct, 0 \leq t \leq r/c\}.
\]
After a finite number of steps, we obtain a smooth solution on the cone with base $S$. □

Let $x_0 \in X$. Put
\[
c(r) := c(B_r(x_0)), \quad r > 0.
\]

Theorem 6.11. Let $X$ be a complete Riemannian manifold. Suppose that
\[
\int_0^\infty \frac{d r}{c(r)} = \infty.
\]
1) Uniqueness: Suppose that $u \in C^\infty(X, E)$ is a solution of
\[
\frac{\partial u}{\partial t} = iD u
\]
on $[0, T] \times X$ with $u(0) = 0$. Then $u \equiv 0$.
2) Existence: Let $u_0 \in C^\infty(X, E)$. Then the wave equation
\[
(6.6) \quad \frac{\partial u}{\partial t} = iD u, \quad u(0) = u_0,
\]
has a unique solution on $\mathbb{R} \times X$. Moreover, for fixed $t$, $u(\cdot, t)$ has compact support.

Proof. We first establish uniqueness. Let $R > 0$. Put $S_R = B_R(x_0)$. We shall show that $u(T)$ vanishes on $S_R$. Of course, we also have that $u(T')$ vanishes for $T' < T$. Let
\[
t_R = c(R + 1)^{-1}, \quad R > 0.
\]
By Proposition (6.10), the wave equation (6.6) on
\[
\{(x,t): d(x,x_0) \leq R - c(R + 1)t, 0 \leq t \leq t_R\}
\]
with initial condition \( u_0 \in C^\infty(S_{R+1}, E) \) has a unique solution. Hence \( u(T) \) on \( S_R \) is determined by \( u(T - t_R) \) on \( S_{R+1} \). By the same argument \( u(T - t_R) \) is determined by \( u(T - t_R - t_{R+1}) \) on \( S_{R+2} \). Since the sets \( S_R \) are compact, this may be continued indefinitely. Now we have

\[
(6.7) \quad \sum_{n=0}^{\infty} t_{R+n} = \infty,
\]

because \( c(R) \) is monoton and

\[
\int_1^{\infty} \frac{dr}{c(r)} = \infty
\]

by assumption. Hence there exists \( N \in \mathbb{N} \) such that \( T' = t_R + t_{R+1} + \cdots + t_{R+N} < T \) and \( T' + t_{R+N+1} > T \). From our considerations above follows that \( u(T)|_{S_R} \) is determined by \( u(T - T')|_{S_{R+N}} \). Let

\[
\varepsilon = (R + N + 1)(T - T').
\]

Then it follows as above that \( u(T - T')|_{S_{R+N}} \) is determined by \( u(0)|_{S_{R+N+1}} \). Note that \( T - T' \leq t_{R+N+1} \). But \( u(0) = 0 \). Hence \( u(T - T')|_{S_{R+N}} = 0 \). This implies that \( u(T)|_{S_R} = 0 \).

Next we establish existence. The argument is like the uniqueness proof run in reverse. Let \( u_0 \in C^\infty_c(X, E) \). Let \( R > 0 \) such that \( \text{supp} \ u_0 \subset S_R \). Since \( X \) is complete, \( S_R \) is compact. By Proposition (6.10) there exists a \( C^\infty \)-solution of (6.6) in

\[
C = \{(x, t) : 0 \leq t \leq t_{R+2}, d(x, x_0) \leq R + 3 - c(R + 3)t\}
\]

Moreover, it follows from the uniqueness part of Proposition (6.10) that the solution vanishes outside \( S_{R+1} \). So we can extend it by 0 to a global solution on \( X \) which exists for time \( 0 \leq t \leq t_{R+2} \). Now we iterate this process. The solution at time \( t = t_{R+2} \) is supported in \( S_{R+1} \). By the above argument, it extends to a solution for \( 0 \leq t \leq t_{R+2} + t_{R+3} \) with support in \( S_{R+2} \). Using again (6.7), we can extend the solution to any time \( t \) and for fixed \( t \), \( u(t) \) has compact support. \( \square \)

For each \( t \) define a map

\[
U_t : C^\infty_c(X, E) \to C^\infty(X, E)
\]

by

\[
U_t(u_0) = u(t),
\]

where \( u \) is the unique solution of

\[
\frac{\partial u}{\partial t} = iDu, \quad u(0) = u_0.
\]

**Corollary 6.12.** Under the assumptions of Theorem (6.11), \( \{U_t\} \) is a one-parameter group. Moreover, if \( w \in C^\infty_c(X, E) \), we have

\[
\frac{d}{dt}(U_t(u_0), w) = (iDU_t(u_0), w).
\]
Finally \( DU_t(u_0) = U_t(Du_0) \).

**Proof.** Let \( u(t) : = U_t(u_0) \). Then \( u(t) \in C^\infty(X, E) \) and
\[
\frac{\partial u}{\partial t} = iDu.
\]

Since \( w \in C^\infty_c(X, E) \), we can differentiate under the integral which gives
\[
\frac{d}{dt}(U_t(u_0), w) = \left( \frac{\partial u}{\partial t}, w \right) = (iDU_t(u_0), w)
\]
Moreover, \( U_{s+t} = U_s \circ U_t \) and \( DU_t = U_tD \) follows from uniqueness. \( \square \)

Note that \( U_t \) is unitary. Indeed we have
\[
\frac{d}{dt} \| u(t) \|^2 = \left( \frac{du}{dt}(t), u(t) \right) + \left( u(t), \frac{du}{dt}(t) \right)
= \left( iDu(t), u(t) \right) + \left( u(t), iDu(t) \right)
= \left( iDu(t), u(t) \right) - \left( iD(t), u(t) \right) = 0.
\]

6.5. **Essential self-adjointness.** In this section we apply the results obtained in the previous section to establish the essential self-adjointness of geometric operators.

We begin with an abstract result.

**Lemma 6.13.** Let \( T \) be a symmetric operator in a Hilbert space \( H \) with dense domain \( D \subset H \). Suppose that \( T(D) \subseteq D \). Furthermore suppose that there is a one-parameter group \( U_t \) of unitary operators on \( H \) such that
\[
U_t(D) \subseteq D, \quad U_t T = TU_t \text{ on } D
\]
and
\[
\frac{d}{dt} U_t(u) = iTU_t(u)
\]
for \( u \in D \). Then every power of \( T \) is essentially self-adjoint.

**Proof.** Let \( n \in \mathbb{N} \) and \( A : = T^n \). Then
\[
A : D \to H
\]
is symmetric. To show that \( A \) is essentially self-adjoint, it suffices to verify that
\[
(A \pm i \text{Id})(D) = H
\]
Let \( \psi \in H \) and suppose that
\[
((A \pm i)(\varphi), \psi) = 0, \ \forall \varphi \in D.
\]
Then it follows that \( \psi \in D(A^*) \) and
\[
A^*\psi = \mp i\psi.
\]
We consider the case where $A^*\psi = i\psi$. For $u \in D$ define 
$$ f(t) = (U_t(u), \psi), \ t \in \mathbb{R}. $$
Since $U_t$ is unitary, $f(t)$ is bounded. Furthermore we have 
$$ \frac{d^n}{dt^n} f(t) = (iT^n U_t(u), \psi) = (i^n U_t(u), A^* \psi) $$
$$ = -i^{n+1} (U_t(u), \psi) = -i^{n+1} f(t). $$
Thus $f(t)$ satisfies the ordinary linear differential equation. 

(6.8) 
$$ \frac{d^n}{dt^n} f(t) = -i^{n+1} f(t). $$
Let $\alpha_j, j = 1, ..., n$, be the different roots of the equation 
$$ z^n = -i^{n+1}. $$
Then $e^{\alpha_j t}, j = 1, ..., n$, is a basis for the space of solutions of (6.8). Therefore $f(t)$ can be written as 
$$ f(t) = \sum_{j=1}^{n} c_j e^{\alpha_j t} $$
for some constants $c_j \in \mathbb{C}$. Now observe that $\text{Re}(\alpha_j) \neq 0$ for $j = 1, ..., n$. Since $f$ is bounded this implies that $f \equiv 0$. Hence we get 
$$ (u, \psi) = f(0) = 0, \ u \in D. $$
Since $D \subset H$ is dense, it follows that $\psi = 0$. The case $A^*\psi = -i\psi$ can be treated in the same way. \[ \square \]

We can now state the main result about essential self-adjointness.

**Theorem 6.14.** Let $X$ be a complete Riemannian manifold and $E \to X$ a Hermitian vector bundle over $X$. Let 
$$ D : C^\infty(X, E) \to C^\infty(X, E) $$
be an elliptic differential operator of order 1 which is formally self-adjoint. Assume that 
$$ \int_1^\infty \frac{dr}{c(r)} = \infty. $$
Let $T : C^\infty_c(X, E) \to L^2(X, E)$ be the operator which is defined by $D$. Then every power of $T$ is essentially self-adjoint.

**Proof.** Let 
$$ U_t : C^\infty_c(X, E) \to C^\infty_c(X, E) $$
be the 1-parameter group, defined by Corollary 6.12. For each $t \in \mathbb{R}$ we have 
$$ \|U_t(u)\| = \|u\|, \ u \in C^\infty_c(X, E). $$
Indeed, for \( u, w \in C_c^\infty(X, E) \), we have

\[
\frac{d}{dt}(U_t(u), U_t(w)) = (iDU_t(u), U_t(w)) + (U_t(u), iDU_t(w))
\]

\[
((iD - iD)U_t(u), U_t(w)) = 0.
\]

Hence \( U_t \) extends by continuity to a one-parameter family

\[ U_t : L^2(X, e) \to L^2(X, E) \]

of unitary operators. The assumptions of Lemma 6.13 are satisfied. This implies the theorem. \( \square \)

6.6. Applications. Now we are ready to apply the results of the previous section to geometric situations.

Let \( X \) be a complete Riemannian manifold. Let

\[ D = d + d^* : \Lambda^*(X) \to \Lambda^*(X). \]

Then \( D \) is formally self-adjoint. To determine its principal symbol, fix \( p \in X, \xi \in T_p^*X \) and \( v \in \Lambda^*T_p^*X \). Let \( f \in C^\infty(X) \) and \( \varphi \in \Lambda^*(X) \) be such that \( f(p) = 0, df_p = \xi \) and \( \varphi(p) = v \).

Then we have

\[
\sigma_d(p, \xi)v = D(f\varphi)(p) = (d + d^*)(f\varphi)(p) = df_p \wedge \varphi(p) - *(df \wedge *\varphi)(p) = \xi \wedge v - i_\xi(v),
\]

where \( i_\xi : \Lambda^*T_p^*X \to \Lambda^*T_p^*X \) denotes interior multiplication by \( \xi \).

This implies that

\[
\| \sigma_D(x, \xi) \| = \| \xi \|.
\]

Hence we have \( c(x) = 1 \), i.e. \( D \) has unit propagation speed. By Theorem 6.14 it follows that for all \( n \in \mathbb{N} \), the operator

\[
(d + d^*)^n : \Lambda^*_n(X) \to L^2\Lambda^*(X)
\]

is essentially self-adjoint. Now recall that the laplace operator \( \Delta \) is given by

\[ \Delta = (d + d^*)^2. \]

Thus it follows that for all \( n \in \mathbb{N} \),

\[ \Delta^n : \Lambda^*_n(X) \to L^2\Lambda^*(X) \]

is essentially self-adjoint. The Laplace operator preserves \( \Lambda^p(X) \) for every \( p \). Therefore

\[ \Delta^n : \Lambda^p(X) \to L^2\Lambda^p(X) \]

is essentially self-adjoint for all \( p = 0, ..., n \).

Next we consider a complex manifold equipped with a Hermitian metric, so that \( X \), equipped with the associated Riemannian metric is complete. Let \( E \to X \) be a holomorphic Hermitian vector bundle over \( X \). Then we define the space of \( (p, q) \)-forms with values in \( E \) as the space of \( C^\infty \)-sections of \( \Lambda^pT^{(1,0)}(X) \otimes \Lambda^qT^{(0,1)}(X) \otimes E \). The operator

\[ \bar{\partial} : \Lambda^{p,q}(X, E) \to \Lambda^{p,q+1}(X, E) \]
is uniquely defined by demanding that
\[ \bar{\partial}(\omega \otimes \varphi) = (\bar{\partial}\omega) \otimes \varphi \]
for every \( \omega \in \Lambda^{p,q}(X) \) and every holomorphic section of \( E \). In this way we get the Dolbeault complex
\[
\cdots \overset{\bar{\partial}}{\longrightarrow} \Lambda^{p,q}(X, E) \overset{\bar{\partial}}{\longrightarrow} \Lambda^{p,q+1}(X, E) \longrightarrow \cdots
\]
Let \( D = (\bar{\partial} + \bar{\partial}^*) \). Then we have
\[ \sigma_D(x, \xi)(\omega \otimes \varphi) = (\pi(\xi) \wedge \omega - i\pi(\xi)(\omega)) \otimes \varphi \]
where \( \pi: T^*_x X \otimes \mathbb{C} \to T_x^{*(0,1)} X \) is the canonical projection. It follows that
\[ \| \sigma_D(x, \xi) \| = \| \pi(\xi) \| \]
and hence, \( c(x) = 1/\sqrt{2} \). By Theorem 6.14,
\[ (\bar{\partial} + \bar{\partial}^*)^n: \Lambda^{p,*}_e(X, E) \to L^2 \Lambda^{p,*}(X, E) \]
is essentially self-adjoint for all \( p \) and all \( n \in \mathbb{N} \).

References
