Spectral theory of automorphic forms

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Introduction

The theory of automorphic forms has a (relatively) long history dating back to the time of H. Poincaré and F. Klein. Poincaré named them Fuschsian functions. Since then the concept of automorphic forms has been changed considerably and the theory has undergone a tremendous development. Today it is one of the central research areas in mathematics with links to many different fields in mathematics including representation theory, number theory, PDE's, algebraic geometry and differential geometry

The modern theory of auomorphic forms is a repsonse to many different impulses and influences. But so far, the most powerful techniques are the issue, direct or indirect, of the introduction of spectral theory into the subject by Maass and then Selberg.

Chapter 1

General set up and basic facts

1.1 Preliminaries

Let G be a connected real semi-simple Lie group with finite center. Assume that G is of non-compact type. This means that G has no compact factors. Fix a maximal compact subgroup K of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. Let

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$$

be the Cartan-Killing form of $\mathfrak{g}$. Since $\mathfrak{g}$ is semi-simple, B is a non-degenerate bilinear form on $\mathfrak{g}$. Let

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$$

be the Cartan decomposition w.r.t. B. Then $B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite and $B|_{\mathfrak{k}\times\mathfrak{k}}$ negative definite. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Let $A \subset G$ be the connected subgroup with Lie algebra $\mathfrak{a}$. Let

$$G = NAK$$

be the corresponding Cartan decompostion. Recall that N is a nilpotent Lie group. Let $M = Z_K(A)$ be the centralizer of A in K. Put

$$P_0 = MAN.$$

Then P_0 is a minimal parabolic subgroup of G.

Example. Let $G = \mathrm{SL}(n,\mathbb{R})$ and $K = \mathrm{SO}(n)$. Then P_0 is group of upper triangular matrices.

1.2 Symmetric spaces

Let G and K be as above. Then

$$S = G/K$$

a smooth C^∞-manifold which is diffeomorphic to $\mathbb{R}^n$. Given $g \in G$, we denote by $L_g\colon S \to S$ the diffeomorphism defined by

$$L_g(g_1K) = gg_1K, \quad g_1K \in S.$$

Let $x_0 = eK$ and let $T_{x_0}S$ be the tangent space at x_0. There is a canonical isomorphism

$$T_{x_0}S \cong \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}.$$

Using this isomorphism, the restriction of the Killing form B to $\mathfrak{p}$ defines an inner product $\langle\cdot,\cdot\rangle_{x_o}$ in $T_{x_0}S$. Then $\langle\cdot,\cdot\rangle_{x_o}$ satisfies

$$\langle \mathrm{Ad}(k)t_1, \mathrm{Ad}(k)t_2\rangle_{x_o} = \langle t_1, t_2\rangle_{x_0}, \quad k \in K,\ t_1, t_2 \in T_{x_0}S. \tag{1.2.1}$$

Let $x = gK \in S$. Define an inner product in the tangent space T_xS by

$$\langle t_1, t_2\rangle_x := \langle dL_{g^{-1}}(t_1), dL_{g^{-1}}(t_2)\rangle_{x_0}, \quad t_1, t_2 \in T_xS.$$

By (1.2.1) the right hand side is independent of the chosen representative of the coset gK. In this way we get Riemannian metric ds^2 on S, which is G-invariant. This means that G acts on S by isometries. Then (S, ds^2) is a complete Riemannian manifold. The geodesic reflection about any $x \in S$ is a global isometry. So (S, ds^2) is global Riemannian symmetric space. The rank $\mathrm{rk}(S)$of S is defined as the dimension the maximal flat subspace of S. It equals $\dim \mathfrak{a}$.

Examples.

1) Let $G = \mathrm{SL}(2,\mathbb{R})$ and $K = SO(2)$. G acts on the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C}\colon\ \mathrm{Im}(z) > 0\}$$

by fractional linear transformations. The action action is transitive and the stabilzer of $i \in \mathbb{H}$ is $K = SO(2)$. This is a maximal compact subgroup and

$$\mathbb{H} \cong G/K = \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2).$$

The invariant metric on $\mathbb{H}$ defined by the Killing form is the Poinncaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad z = x + iy.$$

2) Let

$$\mathbb{H}^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}\colon x_1^2 - (x_2^2 + \cdots x_{n+1}^2) = 1,\ x_1 > 0\}$$

be the hyperbolic n-space with Riemannian metric given by the restriction of

$$ds^2 = dx_1^2 - (dx_2^2 + \cdots dx_{n+1}^2)$$

to $\mathbb{H}^n$. Then $\mathrm{SO}_0(n,1)$ acts transitively on $\mathbb{H}^n$. The stabilizer of the point $(1, 0, ..., 0)$ is $SO(n)$. This is a maximal compact subgroup of $\mathrm{SO}_0(n,1)$ and

$$\mathbb{H}^n \cong \mathrm{SO}_0(n,1)/\,\mathrm{SO}(n).$$

3) Let

$$S = \{Y \in \mathrm{Mat}(n, \mathbb{R}) \colon Y > 0, \, \det Y = 1\}$$

be the space of positive definite symmetric $n \times n$-matrices with determinant 1. $\mathrm{SL}(n, \mathbb{R})$ acts on S by

$$g \cdot Y := g^t Y g, \quad Y \in S, \, g \in \mathrm{SL}(n, \mathbb{R}),$$

and the stabilizer of the identity matrix I is $\mathrm{SO}(n)$. Thus

$$S \cong \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n).$$

The invariant metric Riemannian metric on S is given by

$$ds^2 = \mathrm{Tr}\left(Y^{-1}(dY)Y^{-1}dY\right).$$

The rank of S is $n-1$.

A differential operator $D \colon C^\infty(S) \to C^\infty(S)$ is called invariant, if it commutes with the left action L_g of G on $C^\infty(S)$, i.e.,

$$D \circ L_g = L_g \circ D, \quad \text{for all } g \in G,$$

where $(L_g f)(x) = f(g^{-1}x)$, $f \in C^\infty(S)$. Let

$$\Delta = -\operatorname{div} \circ \operatorname{grad}$$

be the Laplacian with respect to the invariant metric on S. Then Δ is an invariant differential operator on S.

Let $\mathcal{D}(S)$ be the ring of invariant differential operators on S. The structure of $\mathcal{D}(S)$ is described by the following theorem of Harish-Cahndra. Let $\mathfrak{a}_\mathbb{C} = \mathfrak{a} \otimes \mathbb{C}$ be the complexification of the Lie algebra $\mathfrak{a}$ and let $S(\mathfrak{a}_\mathbb{C})^W$ be the subspace of W-invariant elements in the symmetric algebra $S(\mathfrak{a}_\mathbb{C})$ over $\mathfrak{a}_\mathbb{C}$. Then we have [HC1, Theorem 1, p. 260]

Theorem 1.2.1. *There is a canonical isomorphism*

$$\gamma \colon \mathcal{D}(S) \cong S(\mathfrak{a}_\mathbb{C})^W.$$

It follows that $\mathcal{D}(S)$ is a commutative, finitely generated algebra. The minimal number of generators of $\mathcal{D}(S)$ equals $r = \mathrm{rank}(S)$.

Given $\lambda \in \mathfrak{a}_\mathbb{C}^*$, we extend it to an algebra homorphism $\lambda \colon S(\mathfrak{a}_\mathbb{C}) \to \mathbb{C}$. By Theorem 1.2.1 we get a homorphism $\chi_\lambda \colon \mathcal{D}(S) \to \mathbb{C}$, which is defined by

$$\chi_\lambda(D) = \lambda(\gamma(D)), \quad D \in \mathcal{D}(S).$$

Using $\gamma(D) \in S(\mathfrak{a}_\mathbb{C})^W$, it follows that χ_λ satisfies

$$\chi_{w\lambda} = \chi_\lambda \quad \text{for } w \in W.$$

The converse is also true. If $\chi_{\lambda_1} = \chi_{\lambda_2}$, then there exists $w \in W$ such that $w\lambda_1 = \lambda_2$. Futhermore, we have [Kn, Chapt. VIII, Proposition 8.21]

Lemma 1.2.2. *Every homorphism from $\mathcal{D}(S)$ into $\mathbb{C}$ is of the form χ_λ for some $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.*

Thus we get

$$\widehat{\mathcal{D}(S)} \cong \mathfrak{a}^*_{\mathbb{C}}/W. \tag{1.2.2}$$

1.3 Discrete subgroups

A lattice Γ in G is a discrete subgroup of G such that $\mathrm{vol}(\Gamma\backslash G) < \infty$, where the volume is taken with respect to any Haar measure on G. Let $\Gamma \subset G$ be lattice in G. Then Γ acts properly discontinuously on S. Let

$$X_\Gamma := \Gamma\backslash S = \Gamma\backslash G/K.$$

Then X_Γ is a locally symmetric space. Let $F \subset S$ be a fundamental domain of Γ. Then

$$\mathrm{vol}(X_\Gamma) := \int_F d\mu < \infty,$$

where $d\mu$ is the volume form attached to the metric ds^2. So X_Γ is a locally symmetric space of finite volume. If Γ is torsion free, then X_Γ is a smooth manifold. Since the Riemannian metric ds^2 on S is G-invariant, it induces a canonical Riemannian metric ds^2_Γ on X_Γ. The volume of X_Γ with respect to this metric is finite.

Examples.

1) For $N \in \mathbb{N}$ let

$$\Gamma(N) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z})\colon \gamma \equiv \mathrm{I} \mod N\}. \tag{1.3.1}$$

This is the principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of level N. Especially $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$ is the modular group. The standard fundamental domain $F(1)$ of $\Gamma(1)$ is given by

$$F(1) = \{z \in \mathbb{H}\colon |z| \geq 1,\ |\mathrm{Re}(z)| \leq 1/2\}.$$

It has finite area. In fact, using hyperbolic geometry one has

$$\mathrm{Area}(\Gamma(1)\backslash\mathbb{H}) = \frac{\pi}{3}.$$

Let

$$X(N) = \Gamma(N)\backslash\mathbb{H}.$$

If $N \geq 3$, then $\Gamma(N)$ is torsion free. Then $X(N)$ is a hyperbolic surface of finite area.

2) The hyperbolic 3-space is given by

$$\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2).$$

The Picard modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z}[i])$ is a lattice in $\mathrm{SL}(2,\mathbb{C})$.

3) $\mathrm{SL}(n,\mathbb{Z})$ and congruence subgroups of $\mathrm{SL}(n,\mathbb{Z})$ are lattices in $\mathrm{SL}(n,\mathbb{R})$.

Of particular importance are arithmetic subgroups. Let $\mathbf{G}$ be a connected linear semi-simple albegraic group defined over $\mathbb{Q}$ and let $G=\mathbf{G}(\mathbb{R})$ be the group of real points of $\mathbf{G}$. Let $\mathbf{G}\subset \mathrm{GL}(n)$ be an embedding defined over $\mathbb{Q}$. A subgroup $\Gamma\subset\mathbf{G}(\mathbb{Q})$ is called *arithmetic* if it is commensurable with $\mathbf{G}\cap\mathrm{GL}(n,\mathbb{Z})$.

1.4 Automorphic forms

Let G be a subgroup of finte index in the group of real points of a connected semi-simple algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. By definition, there exists $N\in\mathbb{N}$ such that $G\subset \mathrm{SL}(N,\mathbb{R})$, and it is closed. Let $\| g \|$ be the Hilbert-Schmidt norm of $g\in\mathrm{SL}(N,\mathbb{R})$. Thus

$$\| g \|^2=\mathrm{tr}(g^t\cdot g)=\sum_{i,j} g_{ij}^2.$$

A function $f\in C(G)$ is said to be of *moderate growth* or *slowly increasing*, if there exist $m\in\mathbb{N}$ and $C>0$ such that

$$|f(g)|\le C\,\| g \|^m,\quad g\in G. \tag{1.4.1}$$

Let ν_m be the semi-norm on $C(G)$ defined by

$$\nu_m(f)=\sup\left\{|f(g)|\cdot \| g \|^{-m}\colon g\in G\right\}.$$

Then f is moderate growth, if and only if there exists $m\in\mathbb{N}$ such that $\nu_m(f)<\infty$.

$f\in C^\infty(G)$ is called right K-finite, if the set of right translates $\{R(k)f\colon k\in K\}$ spans a finite-dimensional subspace. Furthermore, f is called $\mathcal{Z}(\mathfrak{g})$-finite, if there exists an ideal of finite co-dimension in $\mathcal{Z}(\mathfrak{g})$ which annihilates f.

Let $\Gamma\subset G$ be a lattice.

Definition 1.4.1. *A function $f\in C^\infty(G)$ is an automorphic form for Γ if it satisfies the following conditions:*

(A1) $f(\gamma g)=f(g)$ for all $\gamma\in\Gamma$ and $g\in G$.

(A2) f is right K-finite.

(A3) f is $\mathcal{Z}(\mathfrak{g})$-finite.

(A4) f is of moderate growth.

An automorphic form f is called cusp form, if it also satisfies

A5) For all proper Γ-cuspidal parabolic subgroups P of G one has

$$\int_{(\Gamma\cap N_P)\backslash N_P} f(nx)\,dn = 0.$$

We denote the space of automorphic forms for Γ by $\mathcal{A}(\Gamma)$ and the subsapce of cusp forms by $\mathcal{A}_{\text{cus}}(\Gamma)$. Cusp forms have the following important property [HC2, Chapt. I, §4, Lemma 12].

Theorem 1.4.2. *Let $f \in C^\infty(G)$ be a cusp form. Then f is rapidely decreasing on every Siegel domain $\mathfrak{S}$.*

In particular, a cusp form is bounded. Since $\Gamma\backslash G$ has finite volume, it follows that $\mathcal{A}_{\text{cus}}(\Gamma) \subset L^2(\Gamma\backslash G)$. Let $L^2_{\text{cus}}(\Gamma\backslash G)$ be the closure of $\mathcal{A}_{\text{cus}}(\Gamma)$ in $L^2(\Gamma\backslash G)$.

A special type of an automorphic form is a *Maass form.* A Maass form for Γ is a smooth function f on S which satisfies:

1) $f(\gamma x) = f(x)$ for all $\gamma \in \Gamma$ and $x \in S$.
2) There exists $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ such that $Df = \chi_\lambda(D)f$ for all $D \in \mathcal{D}(S)$.
3) f is of moderate growth.

Examples.

1. Classical automorphic forms. Let $G = \mathrm{SL}(2,\mathbb{R})$ and $K = \mathrm{SO}(2)$. For $g \in G$ and $z \in \mathbb{H}$ we define the automorphie factor by

$$j(g,z) = cz + d, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $\Gamma \subset G$ be a lattice. Let $m \in \mathbb{N}$. An automorphic form of weight m on $\mathbb{H}$ is a function $f\colon \mathbb{H} \to \mathbb{C}$ such that

1) f is holomorphic on $\mathbb{H}$.
2) $f(\gamma(z)) = j(\gamma,z)^m f(z)$ for all $\gamma \in \Gamma$.
3) f is regular in all cusps of $\Gamma\backslash\mathbb{H}$.

Let $\tilde f$ be the function on G defined by

$$\tilde f(g) = j(g,i)^{-m} f(g(i)), \quad g \in G.$$

Then $\tilde f \in C^\infty(G)$. It follows from 2) that $\tilde f$ is left Γ-invariant and it satisfies

$$\tilde f(gk) = \chi_m(k)\tilde f(g), \quad k \in K,\ g \in g,$$

where $\chi_m \colon K \to \mathbb{C}^*$ is the character defined by $\chi_m(k(\theta)) = e^{-im\theta}$ for a rotation $k(\theta)$ of angle θ. Thus $\tilde{f}$ satisfies (A2). 1) implies that $\tilde{f}$ is an eigenfunction of the Casimir operator $\Omega \in \mathcal{Z}(\mathfrak{g})$. Since $\mathcal{Z}(\mathfrak{g}) = \mathbb{C}[\Omega]$, $\tilde{f}$ satisfies (A3). For (A4) suppose that ∞ is a cusp ∞. Then Γ contains a translation $z \mapsto z + p$ for some $p \in \mathbb{N}$. Since f is invariant under this translation, f admits a Fourier expansion w.r.t to x of the form

$$f(z) = \sum_{n\in\mathbb{Z}} a_n \exp\left(\frac{2\pi i n z}{p}\right).$$

The regularity condition 3) means that $a_n = 0$ for $n < 0$. This implies that on the "Siegel set"

$$\mathfrak{S}_t = \{z \in \mathbb{H} \colon |\operatorname{Re}(z)| \le c,\, \operatorname{Im}(z) > t\},$$

where $c, t > 0$, we have $|f(z)| \ll y^m$. This implies that $\tilde{f}$ is of moderate growth. Thus $\tilde{f}$ is an automorphic form for Γ.

2. Maass forms.

Let Δ be the Laplace operator on the upper half-plane $\mathbb{H}$ w.r.t. the Poincaré metric. It is given by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \quad z = x + iy. \tag{1.4.2}$$

Since $\operatorname{rank}(\mathbb{H}) = 1$, it follows that $\mathcal{D}(\mathbb{H}) = \mathbb{C}[\Delta]$. Then a Maass form for $\Gamma(1)$ is a smooth function f on $\mathbb{H}$, which satisfies

1) $f(\gamma(z)) = f(z)$, $\gamma \in \Gamma(1)$.

2) There exists $\lambda \in \mathbb{C}$ such that $\Delta f = \lambda f$.

3) There exists $N \in \mathbb{N}$ such that $|f(x+iy)| \ll y^N$ for $y \ge 1$.

Furthermore f is a cusp form, if

$$\int_0^1 f(x+iy)\,dx = 0 \quad \text{for}\, y > 0.$$

Let $f \in L^2(\Gamma(1)\backslash\mathbb{H})$ and assume that $\Delta f = \lambda f$. Since Δ is an essentially self-adjoint operator in L^2, it follows that $\lambda \ge 0$. For $\Gamma(1)$ it is known that any non-zero eigenvalue λ satisfies $\lambda > 1/4$. Write $\lambda = \frac{1}{4} + r^2$, $r > 0$. Since f is invariant under $z \mapsto z + 1$, it has a Fourier expansion of the form

$$f(x+iy) = \sum_{n\neq 0} a_n \sqrt{y} K_{ir}(2\pi|n|y) e^{2\pi n x},$$

where $K_\nu(y)$ is the modified Bessel function which may be defined by

$$K_\nu(y) = \int_0^\infty e^{-y\cosh(t)} \cosh(\nu t)\, dt.$$

In particular, f is a cusp form. Furthermore, it is known that K_ν satisfies $K_\nu(y) \ll e^{-cy}$ for some $c > 0$ and $y \geq 1$. Hence we get $|f(x+iy)| \ll e^{-cy}$, $y \geq 1$. This shows that every square integrable eigenfunction of Δ is a Maass form.

Another example of a Maass form is the non-holomorphic Eisenstein series. Let $\Gamma = \Gamma(1)$ and let Γ_∞ denote the stabilizer of ∞. Then the Eisenstein series attached to the cusp ∞ is defined by

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma(z))^s = \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}}, \quad \operatorname{Re}(s) > 1. \tag{1.4.3}$$

By definition, $E(\cdot, s)$ is invariant under $\Gamma(1)$. Furthermore, it follows from (1.4.2) that $\Delta y^s = s(1-s)y^s$. Since Δ commutes with the action of $\Gamma(1)$, it follows that the Eisenstein series satisfies

$$\Delta_z E(z,s) = s(1-s)E(z,s).$$

Using (2.3.3), it is easy to verify that $E(z,s)$ is of moderate growth. Thus it is a Maass form which is not square integrable.

1.5 Geometric interpretation of automorphic forms

Let $\sigma\colon K \to \mathrm{GL}(V)$ be a finite-dimensional complex representation.

1.6 Spectral decomposition

Let $S = G/K$ be Riemannian symmetric space and $\mathcal{D}(S)$ the algebra of invariant differential operators of S. Let $\Gamma \subset G$ be a lattice. Assume that Γ is torsion free. Let $f \in C^\infty(\Gamma\backslash S)$ be a Maass cusp form for Γ. Then f is a square integrable joint eigenfunction of $\mathcal{D}(S)$. Therefore the study of Maass automorphic forms is intimately connected with the study of the spectral resolution of the algebra $\mathcal{D}(S)$ acting in $L^2(\Gamma\backslash S)$.

Langlands' theory of Eisenstein series [La1] provides a decomposition

$$L^2(\Gamma\backslash S) = L^2_{\mathrm{dis}}(\Gamma\backslash S) \oplus L^2_{ac}(\Gamma\backslash S),$$

where $L^2_{\mathrm{dis}}(\Gamma\backslash S)$ and $L^2_{ac}(\Gamma\backslash S)$ are the subspaces conrresponding to the point spectrum and the absolutely continuous spectrum, respectively. $L^2_{ac}(\Gamma\backslash S)$ is described in terms of Eisenstein series and

$$L^2_{\mathrm{dis}}(\Gamma\backslash S) = \bigoplus_{i\in I} \mathbb{C} f_i,$$

where $\{f_i\}_{i\in I}$ is an orthonormal basis of $L_{\text{dis}}(\Gamma\backslash S)$ consisting of joint eigenfunctions of $\mathcal{D}(S)$. Each f_i, $i \in I$, is a square integrable Maass automorphic form. The space of cusp forms $L^2_{\text{cus}}(\Gamma\backslash S)$ is contained in $L^2_{\text{dis}}(\Gamma\backslash S)$. Let $L^2_{\text{res}}(\Gamma\backslash S)$ be the orthogonal complement of $L^2_{\text{cus}}(\Gamma\backslash S)$ in $L^2_{\text{dis}}(\Gamma\backslash S)$. This is the so called residual subspace corresponding to the residual spectrum. By Langlands [La1], $L^2_{\text{res}}(\Gamma\backslash S)$ is spanned by iterated residues of Eisenstein series. Thus we have the following decomposition of the point spectrum

$$L^2_{\text{dis}}(\Gamma\backslash S) = L^2_{\text{cus}}(\Gamma\backslash S) \oplus L^2_{\text{res}}(\Gamma\backslash S).$$

By Langlands [La1], the cusp forms are the building blocks of the spectral resolution. Recall that by (1.2.2) each joint eigenfunction f of $\mathcal{D}(S)$ determines a character $\chi_\lambda\colon \mathcal{D}(S) \to \mathbb{C}$, where $\lambda \in \mathfrak{a}^*_{\mathbb{C}}/W$. Given $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, let

$$\mathcal{E}(\lambda) = \left\{ f \in L^2(\Gamma\backslash S) \cap C^\infty(\Gamma\backslash S)\colon Df = \chi_\lambda(D)f, \text{ for all } D \in \mathcal{D}(S) \right\}.$$

Let $m(\lambda) = \dim \mathcal{E}(\lambda)$. Then the discrete spectrum of $\mathcal{D}(S)$ is defined as

$$\Lambda_{\text{dis}}(\Gamma) = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}}/W \colon m(\lambda) > 0\}.$$

Then we get an orthogonal decomposition

$$L^2_{\text{dis}}(\Gamma\backslash S) = \bigoplus_{\lambda\in\Lambda_{\text{dis}}(\Gamma)} \mathcal{E}(\lambda).$$

Similarly we define the cuspidal spectrum $\Lambda_{\text{cus}}(\Gamma)$ and the residual spectrum $\Lambda_{\text{res}}(\Gamma)$ by replacing $L^2(\Gamma\backslash S)$ by $L^2_{\text{cus}}(\Gamma\backslash S)$ and $L^2_{\text{res}}(\Gamma\backslash S)$, respectively. Thus we have orthogonal decompositions

$$L^2_{\text{cus}}(\Gamma\backslash S) = \bigoplus_{\lambda\in\Lambda_{\text{cus}}(\Gamma)} \mathcal{E}_{\text{cus}}(\lambda), \quad \text{and} \quad L^2_{\text{res}}(\Gamma\backslash S) = \bigoplus_{\lambda\in\Lambda_{\text{res}}(\Gamma)} \mathcal{E}_{\text{res}}(\lambda).$$

If we choose a fundamental domain for W in $\mathfrak{a}^*_{\mathbb{C}}$, we may regard the spectra as subsets of $\mathfrak{a}^*_{\mathbb{C}}$.

If $\text{rank}(S) = 1$, then $\mathcal{D}(S) = \mathbb{C}[\Delta]$. Then the problem is reduced to the study of the spectral resolution of the Laplace oeprator Δ in $L^2(\Gamma\backslash S)$. This case will be discussed in more detail in the next chapter. If $\text{rank}(S) > 1$, then the spectra are multidimensional. Nevertheless, the spectral decomposition of the Laplace operator alone plays a central role. This is so because the closure of the Laplace operator in L^2 is self-adjoint. If $\mathcal{E}$ is a finite-dimensional eigenspace of Δ, we get a representation of $\mathcal{D}(S)$ in $\mathcal{E}$ by a commutative algebra of normal operators. So this algebra can be diagonalized.

More generally, we may consider vector valued automorphic forms. Let $\tilde{E}_\sigma \to S = G/K$ be a homogeneous vector bundle associated to a finite-dimensional representation of K. Let $E_\sigma = \Gamma\backslash\tilde{E}_\sigma$ be the associated locally homogeneous vector bundle over $\Gamma\backslash S$. The center

$\mathcal{Z}(\mathfrak{g})$ of the universal envoloping algebra of $\mathfrak{g}_{\mathbb{C}}$ acts in $L^2(\Gamma\backslash S, E_\sigma)$ and we may study its spectral resolution.

The representation theoretic viewpoint leads to a unified treatment. Consider the right regular representation R of G in $L^2(\Gamma\backslash G)$,which is defined by

$$(R(g)f)(g') = f(g'g), \quad f \in L^2(\Gamma\backslash G),\, g \in G.$$

This is a unitary representation and we may decompose it into irreducible representations of G. Let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of G, equipped with the Fell topology. Again by Langlands [La1] there is an orthogonal decomposition into two invariant subspaces of R:

$$L^2(\Gamma\backslash G) = L^2_{\text{dis}}(\Gamma\backslash G) \oplus L^2_{ac}(\Gamma\backslash G),$$

where $L^2_{\text{dis}}(\Gamma\backslash G)$ is the closure of the span of all irreducible subrepresentations of R. Let R_{dis} denote the restrictions of R to $L^2_{\text{dis}}(\Gamma\backslash G)$. Then

$$R_{\text{dis}} = \bigoplus_{\pi\in\hat{G}} m_\Gamma(\pi)\pi,$$

with finite multiplicities $m_\Gamma(\pi)$.

1.7 Basic problems

Here are some of the basic questions related to the spectral decomposition.

1) Existence of cusp forms and the asymptotic distribution of the spectrum (Weyl's law).
2) Location of the spectrum (Ramanujan - Selberg conjectures, Arthur conjectures).
3) Description of the residual spectrum.
4) Bahavior of the joint eigenfunctions as the eigenvalues of the Laplacian tend to ∞. (semiclassical limit).
5) Size of eigenfunctions, equidistribution of mass.
6) Langlands functoriality principle (relation between spectra for different groups).

Why study these problems?

There is a number of reasons why one wants to study these problems.

1. Automorphic L-functions - Langlands program.

This is the most important reason to study the spectral decomposition. Let $\mathbf{G}$ be a reductive algebraic group over $\mathbb{Q}$ (or a number field F). Let ${}^L\mathbf{G}$ be the Langlands dual group. For each cuspidal automorphic representation π of $\mathbf{G}(\mathbb{A})$ and each finite-dimensional representation of ${}^L\mathbf{G}$ one can form an L-function $L(s, \pi, \rho)$. All L-functions from arithmetic (Artin, Hasse-Weil zeta function) are expected to be special cases of automorphic L-functions.

Example: Let $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ and $F \subset \mathbb{H}$ the standard fundamental domain. Let $f \in C^\infty(\mathbb{H})$ be a square integrable Γ-automorphic form with eigenvalue $\lambda = 1/4 + r^2$, $r \geq 0$, i.e., f satisfies

$$f(\gamma z) = f(z),\ \gamma \in \Gamma, \quad \Delta f = (1/4 + r^2) f, \quad \int_F |f(z)|^2 dA(z) < \infty.$$

In addition assume that f is symmetric w.r.t. to the reflection $x + iy \mapsto -x + iy$.

Since f satisfies $f(z+1) = f(z)$ it admits the following Fourier expansion w.r.t. to x:

$$f(x+iy) = \sum_{n=1}^{\infty} a_n y^{1/2} K_{ir}(2\pi n y) \cos(2\pi n x),$$

where

$$K_\nu(y) = \int_0^\infty e^{-y \cosh t} \cosh(\nu t)\, dt$$

is the modified Bessel function. Let

$$L(s, f) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \mathrm{Re}(s) > 1.$$

The modularity of f implies that $L(s, f)$ admits a meromorphic extension to $\mathbb{C}$, and satisfies a functional equation. Let

$$\Lambda(s, f) = \pi^{-s} \Gamma\left(\frac{s + ir}{2}\right) \Gamma\left(\frac{s - ir}{2}\right) L(s, f).$$

Then the **functional equation** is $\Lambda(s) = \Lambda(1 - s)$.

- $L(s, f)$ is an example of an **automorphic L-function**.
- This construction can be generalized to automorphic forms w.r.t. other semisimple (or reductive) groups.

Basic problem: Establish analytic continuation and functional equation in the general case

Langlands' functoriallity principle: Provides relation between auotmorphic forms on different groups by relating the corresponding L-functions.

Basic conjecture: All L-functions occurring in number theory and algebraic geometry are automorphic L-functions.

- **Leads to deep connections between harmonic analysis and number theory.**

Example. A. Wiles, proof of the **Shimura-Taniyama conjecture:** The L-function of an elliptic curve is automorphic.

Langlands program: This theorem holds in much greater generality. There is a conjectured correspondence

$$\begin{Bmatrix} irreducible \;\; n-dim. \\ repr's \;\; of \;\; \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \end{Bmatrix} \to \begin{Bmatrix} automorphic \;\; forms \\ of \;\; \mathrm{GL}(n) \end{Bmatrix}$$

2. Mathematical physics. If the curvature of S is strictly negative, then the geodesic flow on $X = \Gamma\backslash S$ is known to be chaotic. The Laplacian is the Hamiltonian of the quantization of this Hamiltonian system. So it is a model for questions from "Quantum chaos". This is the only known such mechanical system for which some of the basic questions of quantum chaos can be answered.

- $\Gamma\backslash\mathbb{H}$ surfaces negative curvature, geodesic flow is ergodic.
- $\Gamma\backslash\mathbb{H}$ models for quantun chaos
- L^p-estimates for eigenfunctions, "random wave conjecture"
- "Quantum unique ergodicity"

Let $X(N) = \Gamma(N)\backslash\mathbb{H}$. Let $\Delta\phi_j = \lambda_j\phi_j$, $\{\phi_j\}_{j\in\mathbb{N}}$ an orthonormal basis of $L^2_{pp}(X(N))$. Existence: see Theorem 5.

L^∞-conjecture: Fix $K \subset X(N)$ compact. For $\varepsilon > 0$

$$\| \phi_j|_K \|_\infty \ll_\varepsilon \lambda_j^\varepsilon, \quad j \in \mathbb{N}.$$

- Implies Lindeloef hypothesis for $\zeta(s)$, and also for $L(s, \phi_j)$.

Seger, Sogge: L^∞ bounds on general compact surfaces.

$$\| \phi_j \|_\infty \ll \lambda_j^{1/4}.$$

Theorem 1.7.1. *Let ϕ_j on $X(N)$.*

$$\| \phi_j \|_\infty \ll \lambda_j^{5/24}.$$

Let

$$\mu_j = |\phi_j(z)|^2 \, dA(z).$$

μ_j is a probability measure on $X(N)$.

Quantum unique ergodicity conjecture:

$$\mu_j \to \frac{1}{\text{Area}(X(N))} dA(z), \quad j \to \infty.$$

The existence of infinitely many L^2 eigenfunctions of Δ on $X(N)$ is essential for these conjectures.

- One of the basic tools to study the above problems is the trace formula of Selberg and Arthur.

Chapter 2

Spectral decomposition I. The rank one case.

In this chapter we summarize the basic facts about the spectral decomposition for the group $\mathrm{SL}(2,\mathbb{R})$. The case of an arbitrary group G of real rank one is analogous.

2.1 Hyperbolic surfaces of finite area

Let $G = \mathrm{SL}(2,\mathbb{R})$ and $K = \mathrm{SO}(2)$. Then $\mathbb{H} = G/K$. Let $\Gamma \subset G$ be a discrete subgroup of finite co-volume. Let $F \subset \mathbb{H}$ be a fundamental domain for Γ. If Γ is torsion free, then $X_\Gamma = \Gamma\backslash\mathbb{H}$ is a hyperbolic surface of finite area. It has the following structure. There is a decomposition

$$X_\Gamma = M \cup Y_1 \cup \cdots \cup Y_m, \tag{2.1.1}$$

where M is a compact surface with boundary and

$$Y_i \cong [a_i,\infty) \times S^1, \quad i = 1,...,m,\, a_i > 0.$$

The metric on Y_k is given by

$$\frac{dx_k^2 + dy_k^2}{y_k^2}, \quad (y_k, x_k) \in [a_k,\infty) \times S^1.$$

2.2 The hyperbolic Laplace operator

Then $\mathbb{H} = G/K$. Let $\Gamma \subset G$ be a discrete subgroup of finte co-volume. Let $F \subset \mathbb{H}$ be a fundamental domain for Γ. Let

$$C^\infty(\Gamma\backslash\mathbb{H}) = \{f \in C^\infty(\mathbb{H})\colon f(\gamma(z)) = f(z), \quad \gamma \in \Gamma\},$$

By $C_c^\infty(\Gamma\backslash\mathbb{H})$ we denote the subspace of all $f \in C^\infty(\Gamma\backslash\mathbb{H})$ such that supp $f \cap F$ is compact. If Γ is torsion free, then $X_\Gamma = \Gamma\backslash\mathbb{H}$ is a hyperbolic surface of finte area. Then $C_c^\infty(\Gamma\backslash\mathbb{H})$ is the space of smooth functions on X_Γ and $C_c^\infty(\Gamma\backslash\mathbb{H})$ the subspace of compactly supported smooth functions on X_Γ. For $f_1, f_2 \in C_c^\infty(\Gamma\backslash\mathbb{H})$ let

$$\langle f_1, f_2\rangle = \int_F f_1(z)\overline{f_2(z)}\,\frac{dx\,dy}{y^2},$$

and let $L^2(\Gamma\backslash\mathbb{H})$ be the completion of $C_c^\infty(\Gamma\backslash\mathbb{H})$ with respect to this inner product. Similarly, let

$$\| df \|^2 = \int_F df \wedge *\overline{df} = \int_F \| df(z) \|^2 \,\frac{dx\,dy}{y^2}.$$

Let $\Delta = d^*d$ be the Laplacian of $\mathbb{H}$ w.r.t. the Poincaré metric. Recall that

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \quad z = x + iy. \tag{2.2.1}$$

It commutes with the action of G on $\mathbb{H}$. Therefore it descends to an operator in $C^\infty(\Gamma\backslash\mathbb{H})$. Then Δ, regarded as linear operator

$$\Delta\colon C_c^\infty(\Gamma\backslash\mathbb{H}) \to L^2(\Gamma\backslash\mathbb{H}),$$

is a symmetric, non-negative operator L^2, i.e., it satisfies

$$\langle \Delta f_1, f_2\rangle = \langle f_1, \Delta f_2\rangle, \quad f_1, f_2 \in C_c^\infty(\Gamma\backslash\mathbb{H}),$$

and

$$\langle \Delta f, f\rangle \geq 0, \quad f \in C_c^\infty(\Gamma\backslash\mathbb{H}).$$

If Γ is torsion free, X_Γ is complete Riemannian manifold. It follows that Δ is essentailly self-adjoint [Ch]. This means that the closure $\bar\Delta$ of Δ in $L^2(X_\Gamma)$ is a self-adjoint operator. This is also true for an arbitrary lattice. Therefore we can talk about the spectral decomposition of $\bar\Delta$. The following basic result about the spectrum of $\bar\Delta$ is due to Roelcke [Roe].

Proposition 2.2.1. *The spectrum of $\bar\Delta$ is the union of a pure point spectrum $\sigma_{pp}(\bar\Delta)$ and an absolutely continuous spectrum $\sigma_{ac}(\bar\Delta)$.*
1) The pure point spectrum consists of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \cdots$ of finite multiplicities with no finite points of accumulation.
2) The absolutely continuous spectrum equals $[1/4, \infty)$ with multiplicity equal to the number of cusps of $\Gamma\backslash\mathbb{H}$.

Of particular interest is the point spectrum. The eigenfunctions corresponding to the eigenvalues λ_i are Maass forms. Let $\{f_i\}_{i\in I}$ be an orthonormal system of eigenfunctions with eigenvalues λ_i. Put

$$L^2_{\rm dis}(\Gamma\backslash\mathbb{H}) = \bigoplus_i \mathbb{C} f_i.$$

If $\Gamma\backslash\mathbb{H}$ is non-compact, then the continuous spectrum of $\bar{\Delta}$ equals $[1/4, \infty)$. Then the only obvious eigenvalue of Δ is $\lambda = 0$. The existence of eigenvalues $\neq 0$ is by no means obvious - see section ?.

Let $L^2_{ac}(\Gamma\backslash\mathbb{H})$ be the orthogonal complement of $L^2_{\mathrm{dis}}(\Gamma\backslash\mathbb{H})$. This is the absolutely continuous subspace. It can be explicitely described in terms of Eisenstein series.

2.3 Eisenstein series

Let $a_1, ..., a_m \in \mathbb{R} \cup \{\infty\}$ be representatives of the Γ-conjugacy classes of parabolic fixed points of Γ. The a_i's are called *cusps*. For each a_i let Γ_{a_i} be the stabilizer of a_i in Γ. Choose $\sigma_i \in \mathrm{SL}(2, \mathbb{R})$ such that

$$\sigma_i(\infty) = a_i, \quad \sigma_i^{-1}\Gamma_{a_i}\sigma_i = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}. \tag{2.3.1}$$

Then the Eisenstein series $E_i(z, s)$ associated to the cusp a_i is defined as

$$E_i(z, s) = \sum_{\gamma \in \Gamma_{a_i}\backslash\Gamma} \mathrm{Im}(\sigma_i^{-1}\gamma z)^s, \quad \mathrm{Re}(s) > 1. \tag{2.3.2}$$

The series converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > 1$ and it satisfies the following properties.

1) $E_i(\gamma z, s) = E_i(z, s)$ for all $\gamma \in \Gamma$.
2) As a function of s, $E_i(z, s)$ admits a meromorphic continuation to $\mathbb{C}$ which is regular on the line $\mathrm{Re}(s) = 1/2$.
3) $E_i(z, s)$ is a smooth function of z and satisfies $\Delta_z E_i(z, s) = s(1 - s)E_i(z, s)$.

As example consider the modular group $\Gamma(1)$ which has a single cusp ∞. The Eisenstein series attached to ∞ is the well-known series

$$E(z, s) = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n)=1}} \frac{y^s}{|mz + n|^{2s}}. \tag{2.3.3}$$

In the general case, the Eisenstein series were first studied by Selberg [Se1].

• *The important property of the Eisenstein series is the existence of the meromorphic continuation to $s \in \mathbb{C}$ and the functional equations satisfied by the Eisenstein series.*

For the Eisenstein series (2.3.3) this is a classical result of analytic number theory. In this case one can use the Mellin transform and the Poisson summation formula to establish the analytic continuation and the functional equation. There exist different methods to deal with the general case. One of them is *geometric scattering theory*.

2.4 Fourier expansion of Eisenstein series

Let σ_i, $i = 1, ..., m$, be defined by (2.3.1). Let $f \in C(\Gamma\backslash\mathbb{H})$. Put $f_i(z) = f(\sigma_i(z))$. Then it follows from the definition of σ_i that f_i satisfies $f_i(z+1) = f_i(z)$. Therefore it can be expanded in a Fourier series with respect to x. This is the Fourier expansion of f at the cusp a_l.

The Fourier expansion of $E_k(z,s)$ at a_l has the following form:

$$E_k(\sigma_l(z), s) = \delta_{kl} y^s + C_{kl}(s) y^{1-s} + \sum_{n \neq 0} \varphi_{kl}(n,s) \sqrt{y} K_{s-1/2}(2\pi |n| y) e^{2\pi n x}$$

with certain meromorphic functions $C_{kl}(s)$ and $\varphi_{kl}(n,s)$ [Iw, Theorem 3.4]. Using that $K_\nu(y) \ll e^{-cy}$ as $y \to \infty$, it follows that

$$E_k(\sigma_l(z), s) = \delta_{kl} y^s + C_{kl}(s) y^{1-s} + O\left(e^{-c_1 y}\right) \tag{2.4.1}$$

as $y \to \infty$. Put

$$C(s) = \left(C_{kl}(s)\right)_{k,l=1}^m . \tag{2.4.2}$$

This is the so called *scattering matrix*. As we will see later, its analytic properties are important for the existence of cusp forms. For a general group Γ there is very little we can say about $C(s)$. On the hand, for the principle cingruence subgroup $\Gamma(N)$ the entries of $C(s)$ can be expressed in terms of known functions from analytic number theory. For $\mathrm{SL}(2,\mathbb{Z})$ the scattering matric is just a function $c(s)$ which is given by

$$c(s) = \sqrt{\pi} \frac{\mathbf{\Gamma}(s-1/2)}{\mathbf{\Gamma}(s)} \frac{\zeta(2s-1)}{\zeta(2s)}, \tag{2.4.3}$$

where $\mathbf{\Gamma}(s)$ is the Gamma function and $\zeta(s)$ is the Riemann zeta-function [Iw, (3.24)]. For $\Gamma(N)$ the entries of $C(s)$ have been computed by Huxley [Hu]. In particular, the determinant is given by

$$\det C(s) = (-1)^l A^{1-2s} \left(\frac{\mathbf{\Gamma}(1-s)}{\mathbf{\Gamma}(s)}\right)^k \prod_\chi \frac{L(2-2s, \bar{\chi})}{L(2s, \chi)}, \tag{2.4.4}$$

where $k, l \in \mathbb{Z}$, $A > 0$, the product runs over Dirichlet characters χ to some modulus dividing N and $L(s,\chi)$ is the Dirichlet L-function with character χ.

2.5 Analytic continuation of Eisenstein series

To simplify notation, we assume that $\Gamma\backslash\mathbb{H}$ has a single cusp. An example is the modular group $\mathrm{SL}(2,\mathbb{Z})$. Then $\Gamma\backslash\mathbb{H}$ has a decomposition of the form

$$\Gamma\backslash\mathbb{H} = M \cup Y,$$

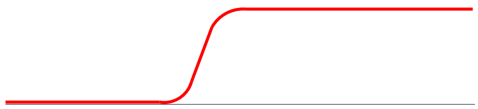
where M is a compact surface with boundary and $Y = [b, \infty) \times S^1$. In this case there is a single Eisenstein series $E(z, s)$. There exists different approaches to achieve the analytic continuation of Eisenstein series [Se2], [Co], [Mu1]. The latter two are based on the following result.

Lemma 2.5.1. *For* $\mathrm{Re}(s) > 1/2$, $s \neq \bar{s}$, $E(z, s)$ *is the unique solution of the equation* $\Delta u(s) = s(1-s)u(s)$ *such that for* $(y, x) \in Y$ *we have*

$$u((y,x),s) = y^s + \psi((y,x),s), \quad \text{where } \psi(s) \in L^2(Y). \tag{2.5.1}$$

Proof. Let $u_1(s)$ and $u_2(s)$ be any two solutions such that (2.5.1) holds. Put $v(s) = u_1(s) - u_2(s)$. Then $v(s)$ is a square integrable eigenfunction of Δ with eigenvalue $\lambda = s(1-s) \notin \mathbb{R}$. Since $\bar{\Delta}$ is self-adjoint, it follows that $v(s) = 0$. □

Let $f \in C^\infty(\mathbb{R})$ such that $f(u) = 1$ for $u \geq b+2$, and $f(u) = 0$ for $u \leq b$.



Set

$$\varphi(z,s) = f(y)y^s \quad \text{and} \quad \psi(s) = (\Delta - s(1-s))(\varphi(s)).$$

Then $\psi(\cdot, s) \in C_c^\infty(\Gamma\backslash\mathbb{H})$, and the Lemma implies that

$$E(z,s) = f(y)y^s - (\Delta - s(1-s))^{-1}(\psi(s))(z) \tag{2.5.2}$$

for $\mathrm{Re}(s) > 1/2$, $s \neq \bar{s}$. The analytic continuation of $E(z,s)$ will follow from the analytic continuation of the resolvent. Let $L^2_{\mathrm{cpt}}(\Gamma\backslash\mathbb{H})$ the subspace of all $f \in L^2(\Gamma\backslash\mathbb{H})$ with compact support. Let $H^2_{\mathrm{loc}}(\Gamma\backslash\mathbb{H})$ be the space of distributions on $\Gamma\backslash\mathbb{H}$ which are locally in the Sobolev space H^2. Then we have [Mu1]

Theorem 2.5.2. *The resolvent* $R(s) = (\Delta - s(1-s))^{-1}$, *defined for* $\mathrm{Re}(s) > 1/2$, $s \neq \bar{s}$, *extends to a meromorphic family of bounded operators*

$$R(s) : L^2_{\mathrm{cpt}}(X) \to H^2_{\mathrm{loc}}(X)$$

with poles of finite rank.

Since $\psi(s) \in L^2_{\mathrm{cpt}}(\Gamma\backslash\mathbb{H})$ for all $s \in \mathbb{C}$, the theorem can be applied to the right hand side of (2.5.2) and yields the analytic continuation of $E(z,s)$. This method worsk equally well for a group Γ with several cusps.

A similar approach has been used by Colin de Verdiere [Co]. It this based on the cut-off Laplacian of Lax and Phillips. Let $H^1(\Gamma\backslash\mathbb{H})$ be the completion of $C_c^\infty(\Gamma\backslash\mathbb{H})$ with respect to the norm

$$\| f \|^2_{H^1} := \| f \|^2 + \| df \|^2 .$$

For $a > b+3$ let

$$H^1_a(\Gamma\backslash\mathbb{H}) = \left\{ f \in H^1(\Gamma\backslash\mathbb{H})\colon \int_{S^1} f(y,x)\,dx = 0 \quad \text{for}\, y \geq a \right\}.$$

Let $q_a\colon H^1_a(\Gamma\backslash\mathbb{H}) \to \mathbb{R}$ be the quadratic form defined by

$$q_a(f) := \| \, df \, \|^2, \quad f \in H^1_a(\Gamma\backslash\mathbb{H}).$$

This is closed quadratic form and therefore there exists a unique self-adjoint operator Δ_a which represents q_a. This is the cut-off Laplacian. The important point about Δ_a is the following result.

Lemma 2.5.3. *Δ_a has pure point spectrum and $R_a(s) = (\Delta_a - s(1-s))^{-1}$ is a meromorphic function of $s \in \mathbb{C}$, with values in the bounded operators on $L^2(\Gamma\backslash\mathbb{H})$.*

If follows from the description of the domain of Δ_a and the definition of $\psi(s)$ that $\psi(s) \in \operatorname{dom}(\Delta_a)$. Put

$$F(z,s) = f(y)y^s - (\Delta_a - s(1-s))^{-1}(\psi(s))(z), \quad s \in \mathbb{C}. \tag{2.5.3}$$

By Lemma 2.5.3, $F(z,s)$ is a meromorphic function in $s \in \mathbb{C}$. Moreover, it is smooth outside $\{a\} \times S^1$. Let

$$F_0(y,s) = \int_{S^1} F((y,x),s)\,dx, \quad y \in [b,\infty).$$

Then it follows from the definition that F_0 has the following form

$$F_0(y,s) = \begin{cases} A(s)y^s + B(s)y^{1-s}, & b \leq y \leq a; \\ y^s, & y > a. \end{cases}$$

Let χ_a denote the characteristic function of $[a,\infty) \times S^1$ in $\Gamma\backslash\mathbb{H}$. Put

$$\tilde{F}(z,s) = F(z,s) - \chi_a(y^s - (A(s)y^s + B(s)y^{1-s})).$$

Then $\tilde{F}(z,s)$ is smooth and satisfies $\Delta\tilde{F}(z,s) = s(1-s)\tilde{F}(z,s)$. Moreover $\tilde{F}(z,s) - \chi_a(A(s)y^s + B(s)y^{1-s})$ is square integrable. Hence by Lemma 2.5.1 we get

$$E(z,s) = A(s)^{-1}\tilde{F}(z,s), \quad \operatorname{Re}(s) > 1/2,\ s \neq \bar{s}.$$

Now the right hand side provides the analytic continuation of the Esienstein series. This method works in genral. Then $A(s)$ and $B(s)$ become $m \times m$ -matrices. It follows that the scattering matrix $C(s)$ is given by

$$C(s) = B(s) \cdot A(s)^{-1}.$$

Moreover, it follows from the analytic continuation that the scattering matrix and the Eisenstein series satisfy the following functional equations:

$$\begin{aligned} &C(s)C(1-s) = \mathrm{Id}, \\ &E_k(z,s) = \sum_{j=1}^{m} C_{kl}(s)E_l(z,1-s). \end{aligned} \tag{2.5.4}$$

2.6 The spectral thoerem

Let Γ be non-compact lattice. Let m be the number of cusps of $\Gamma\backslash\mathbb{H}$. The Eisenstein series $E_k(z,s)$, $k=1,...,m$, are holomorphic on $\mathrm{Re}(s)=1/2$. For each k, $k=1,...,m$, we define the Eisenstein transform

$$E_k\colon C_c^\infty(\mathbb{R}^+)\to L^2(\Gamma\backslash\mathbb{H})$$

by

$$E_k(f)(z)=\frac{1}{4\pi}\int_0^\infty f(r)E_k(z,1/2+ir)\,dr.$$

The fact that $E_k(f)$ is square integrable follows from (2.4.1).

Proposition 2.6.1. *For $f_1,f_2\in C_c^\infty(\Gamma\backslash\mathbb{H})$ and $k,l=1,...,m$ we have*

$$\langle E_kf_1,E_lf_2\rangle=\frac{1}{2\pi}\langle f_1,f_2\rangle.$$

The proof follows from the inner product formula for truncated Eisenstein series. By Proposition 2.6.1, the Eisenstein transform extends to an isometry $L^2(\mathbb{R}^+,\frac{dr}{2\pi})$ into $L^2(\Gamma\backslash\mathbb{H})$. Define

$$E\colon\bigoplus_{k=1}^m L^2(\mathbb{R}^+,\frac{dr}{2\pi})\to L^2(\Gamma\backslash\mathbb{H})$$

by

$$E(f_1,...,f_m)=\sum_{k=1}^m E_k(f_k).$$

Recall that $L^2_{ac}(\Gamma\backslash\mathbb{H})$ is defined as the orthogona complement in $L^2(\Gamma\backslash\mathbb{H})$ to $L^2_{\mathrm{dis}}(\Gamma\backslash\mathbb{H})$ - the subspace spanned by the eigenfunctions.

Proposition 2.6.2. *E is an isometry of $\oplus_{k=1}^m L^2(\mathbb{R}^+,\frac{dr}{2\pi})$ onto $L^2_{ac}(\Gamma\backslash\mathbb{H})$.*

The adjoint map

$$E^*\colon L^2(\Gamma\backslash\mathbb{H})\to\bigoplus_{k=1}^m L^2(\mathbb{R}^+,\frac{dr}{2\pi})$$

is given by

$$E^*(\varphi)(r)=\left(\langle\varphi,E_1(\cdot,1/2+ir)\rangle,...,\langle\varphi,E_m(\cdot,1/2+ir)\rangle\right).$$

$P_{ac}=E\circ E^*$ is the orthogonal projection of $L^2(\Gamma\backslash\mathbb{H})$ onto $L^2_{ac}(\Gamma\backslash\mathbb{H})$.

Theorem 2.6.3. *Let $\{f_i\}_{i\in I}$ be an orthonormal basis of $L^2_{\mathrm{dis}}(\Gamma\backslash\mathbb{H})$ consisting of eigenfunctions of Δ. Let $\varphi\in C_c^\infty(\Gamma\backslash\mathbb{H})$. Then φ has the following spectral expansion*

$$\varphi(z)=\sum_{i\in I}\langle\varphi,f_i\rangle+\sum_{k=1}^m\frac{1}{4\pi}\int_{\mathbb{R}}\langle\varphi,E_k(\cdot,1/2+ir)\rangle E_k(z,1/2+ir)\,dr.$$

It converges in the C^∞-topology.

Chapter 3

Spectral decomposition II. The higher rank case.

The theory of Eisenstein series is the basic tool to study the spectral resolution of the regular representation. For a general resuctive group, the theory of Eisenstein series has been developed by Langlands [La1]. By the fundamental results of Margulis, irreducible lattices in higher rank groups are arithmetic. Therefore, it is convenient to pass to the adelic framework.

3.1 Eisenstein series

Let G be a reducive algebraic group which is defined over $\mathbb{Q}$. More generally, we may consider G defined over a number field F. All parabolic subgroups of G will be assumed to be defined over $\mathbb{Q}$. Let P_0 be a fixed minimal parabolic subgroup, and let M_0 be a fixed Levi component of P_0, defined over $\mathbb{Q}$. If P is a standard parabolic subgroup of G, we shall write N_P for the unipotent radical of P and M_P for the Levi component of P which contains M_0. Let $X(M)_{\mathbb{Q}}$ be the obelian group of maps from M to GL_1 defined over $\mathbb{Q}$. If $m \in M(\mathbb{A})$ and $\chi \in X(M)_{\mathbb{Q}}$, the value of χ at m, m^χ is an idèle, so it has an absolute value. Define a map H_M from $M(\mathbb{A})$ to the real vector space

$$\mathfrak{a}_P = \mathrm{Hom}(X(M)_{\mathbb{Q}}, \mathbb{R}) \tag{3.1.1}$$

by

$$e^{<\chi, H_M(x)>} = |m^\chi|, \chi \in X(M)_{\mathbb{Q}}, m \in M(\mathbb{A}).$$

Let A_P be the split component of the center of M, and let Z_M be the connected component of 1 in $A_P(\mathbb{R})$. Then $M(\mathbb{A})$ is the direct product of the kernel of H_M and Z_M. Finally, let $K = \prod_v K_v$ be a maximal compact subgroup of $G(\mathbb{A})$, admissible relative to M_0.

If $G = GL_n$, we can take

$$P_0 = \left\{ \begin{pmatrix} \star & \cdots & \star \\ 0 & \ddots & \vdots \\ & & \star \end{pmatrix} \right\}, \quad M_0 = \left\{ \begin{pmatrix} \star & & 0 \\ & \ddots & \\ 0 & & \star \end{pmatrix} \right\}$$

$$P = \left\{ \begin{pmatrix} g_1 & & \star \\ & \ddots & \\ 0 & & g_r \end{pmatrix} : g_i \in GL_{n_i} \right\}, \quad M = \left\{ \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_r \end{pmatrix} : g_i \in GL_{n_i} \right\},$$

where $n = n_1 + \cdots + n_r$. Then $X(M)_{\mathbb{Q}}$ is the group of maps

$$\chi_\nu \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_r \end{pmatrix} = \prod_{i=1}^{r} (\det\ m_i)^{\nu_i},$$

where $\nu = (\nu_1, ...\nu_r)$ ranges over $\mathbb{Z}^r$. The space $\mathfrak{a}_P$ is isomorphic to $\mathbb{R}^r$ and

$$\langle \chi_\nu, H_M(m) \rangle = \sum_{i=1}^{r} \nu_i \log|\det m_i|, \quad \nu \in \mathbb{Z}^r.$$

Finally, we can take

$$K = \mathrm{O}_n(\mathbb{R}) \times \prod_p \mathrm{GL}_n(\mathbb{Z}_p).$$

If x is an element in $G(\mathbb{A})$, we can write

$$x = nmk, \ n \in N(\mathbb{A}), \ m \in M(\mathbb{A}), \ k \in K.$$

Define

$$H_P(x) = H_M(m).$$

If $\rho = \rho_P \in \mathfrak{a}_P^\star$ is the half sum (with multiplicities) of the roots of (P, A_P) then

$$\delta(p) = e^{2\rho(H_P(p))}, \quad p \in P(\mathbb{A}),$$

is the modular function of $P(\mathbb{A})$. Finally let $\Delta_P \subset X(M)_{\mathbb{Q}} \subset \mathfrak{a}_P^*$ be the simple roots of (P, A_P). Every $\alpha \in \Delta_P$ is the restriction to $\mathfrak{a}_P$ of a unique simple root β in Δ_{P_0}. Let $\alpha^\vee$ be the projection of the co-root $\beta^\vee$ onto $\mathfrak{a}_P$.

Let $R_{M,\mathrm{dis}}$ be the subrepresentation of the regular representation of $M(\mathbb{A})$ on the Hilbert space $L^2(Z_M \cdot M(\mathbb{Q})\backslash M(\mathbb{A}))$ that decomposes discretely. It acts on a closed invariant subspace $L^2_{\mathrm{dis}}(Z_M \cdot M(\mathbb{Q})\backslash M(\mathbb{A}))$ of $L^2(Z_M \cdot M(\mathbb{Q})\backslash M(\mathbb{A}))$.

If δ is any representation of $M(\mathbb{A})$ and λ belongs to $\mathfrak{a}_{P,\mathbb{C}}^\star$, set

$$\delta_\lambda(m) = \delta(m) e^{\lambda(H_M(m))}, \quad m \in M(\mathbb{A}).$$

Then $R_{M,\mathrm{dis},\lambda}$ is a representation of $M(\mathbb{A}) \cong P(\mathbb{A})/N(\mathbb{A})$ which we can lift to $P(\mathbb{A})$. Let

$$I_P(\lambda) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(R_{M,\mathrm{dis},\lambda})$$

be the induced representation. It acts on the Hilbert space of complex valued functions ϕ on $N(\mathbb{A})Z_M M(\mathbb{Q})\backslash M(\mathbb{A})$ such that

(i) the function $m \mapsto \phi(mx)$, $m \in M(\mathbb{A})$, belongs to $L^2_{\mathrm{dis}}(Z_M M(\mathbb{Q})\backslash M(\mathbb{A}))$ for each $x \in G(\mathbb{A})$.

(ii) $||\phi||^2 = \int_K \int_{Z_M M(\mathbb{Q})\backslash M(\mathbb{A})} |\phi(mk)|^2 dm\, dk < \infty$.

If λ is purely imaginary, $I_P(\lambda)$ is unitary.

There are intertwining operators between these induced representations. Let W be the restricted Weyl group of G. It acts on $A_0 = A_{P_0}$ and also on $\mathfrak{a} = \mathfrak{a}_{P_0}$. If P and P' are standard parabolic subgroups, $\mathfrak{a}_P$ and $\mathfrak{a}_{P'}$ are both contained in $\mathfrak{a}_0$. Let $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ be the set of distinct isomorphisms from $\mathfrak{a}_P$ onto $\mathfrak{a}'_P$, obtained by restricting elements in W to $\mathfrak{a}_P$. The groups P and P' are said to be associated if $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ is not empty.

If $G = GL_n$, W is isomorphic to the symmetric group S_n, by

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda_{\sigma(1)} & & 0 \\ & \ddots & \\ 0 & & \lambda_{\sigma(n)} \end{pmatrix}, \quad \sigma \in S_n.$$

The groups P and P' are associated if and only if the corresponding partitions are such that $r = r'$ and $(n'_1, ..., n'_r) = (n_{\tau(r)})$ for some $\tau \in S_r$.

For each $s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ let w_s be a representative of s in the normalizer of A_0 in $G(\mathbb{Q})$. Define

$$(M(s,\lambda)\phi)(x) = \int\limits_{N'(\mathbb{A}) \cap w_s N(\mathbb{A}) w_s^{-1} \backslash N(\mathbb{A})} \phi(w_s^{-1} n x) e^{(\lambda+\rho)(H_P(w_s^{-1} n x))} e^{-(s\lambda+\rho')(H_{P'}(x))}\, dn,$$

for $\phi \in \mathcal{H}_P$, $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ and $\rho' = \rho_{P'}$. Let $\mathcal{H}^0_P \subset \mathcal{H}_P$ be the subspace of functions which are right K-finite and $Z(\mathfrak{m}_P)$-finite. This is a dense subspace of $\mathcal{H}_P$.

Lemma 3.1.1. *Let $\phi \in \mathcal{H}^0_P$ and $\lambda \in \mathfrak{a}^\star_{P,\mathbb{C}}$ with*

$$(\mathrm{Re}(\lambda) - \rho)(\alpha^\vee) > 0, \quad \alpha \in \Delta_P.$$

Then the integral defining $M(s,\lambda)\phi$ converges absolutely. For λ in this range, $M(s,\lambda)$ is an analytic function with values in $\mathrm{Hom}(\mathcal{H}^0_P, \mathcal{H}^0_P)$*, which intertwines $I_P(\lambda)$ and $I_{P'}(s\lambda)$.*

Next define

$$E(x,\phi,\lambda)=\sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})}\phi(\gamma x)e^{(\lambda+\rho)(H_P(\gamma x))}$$

for $\phi\in\mathcal{H}_P$, $x\in G(\mathbb{A})$ and $\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}$.

Lemma 3.1.2. *If $\phi\in\mathcal{H}^0_P$ and*

$$(\mathrm{Re}(\lambda)-\rho)(\alpha^v)>0,\quad \alpha\in\Delta_P,$$

then the series converges absolutely. It defines an analytic function in this range.

(see [La1]).

We can now state the fundamental theorem of Eisenstein series.

Theorem 3.1.3. (a) *Suppose that $\phi\in\mathcal{H}^0_P$. Then $E(x,\phi,\lambda)$ and $M(s,\lambda)\phi$ can be analytically contiued as meromorphic functions to $\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}$. On $i\mathfrak{a}^\star_P$, $E(x,\phi,\lambda)$ is regular and $M(s,\lambda)$ is unitary. If $t\in W(\mathfrak{a}_{P'},\mathfrak{a}_{P''})$ the following functional equations hold:*

(i) $E(x,M(s,\lambda)\phi,s\lambda)=E(s,\phi,\lambda)$

(ii) $M(ts,\lambda)\phi=M(t,s\lambda)M(s,\lambda)\phi,$

(b) *Let $\mathcal{P}$ be an equivalence class of associated standard parabolic subgroups. Let $\hat{L_\mathcal{P}}$ be the set of collections $\{F_P\colon P\in\mathcal{P}\}$ of measurable functions*

$$F_P\colon i\mathfrak{a}^*_P\to\mathcal{H}_P$$

such that

(i) $F_{P'}(s\lambda)=M(s,\lambda)F_P(\lambda),\ s\in W(\mathfrak{a}_P,\mathfrak{a}'_P).$

(ii) $\parallel F\parallel^2=\sum_{P\in\mathcal{P}}\int_{i\mathfrak{a}^*_P}\parallel F_P(\lambda)\parallel^2\ d\lambda<\infty.$

Then the map which sends F to the function

$$\sum_{P\in\mathcal{P}}\int_{i\mathfrak{a}^*_P}E\left(x,F_P(\lambda),\lambda\right)\ d\lambda,$$

defined for F in a certain dense subspace of $\hat{L_P}$, extends to a unitary map from $\hat{L_P}$ onto a closed $G(\mathbb{A})$-invariant subspace $L^2_\mathcal{P}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Moreover there is an orthogonal decomposition

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))=\bigoplus_{\mathcal{P}}L^2_\mathcal{P}(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

The proof of this theorem is given in [La1, Chapt. 7].

The theorem implies that the regular representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ decomposes as the direct integral over all (P, λ), where P is a standard parabolic subgroup and λ belongs to the positive chamber in $i\mathfrak{a}_P^\star$, of the representations $I_P(\lambda)$.

These representations were obtained by induction from the discrete spectrum of $M(\mathbb{A})$. The discrete spectrum of $G(\mathbb{A})$ corresponds to the case that $P = G$.

Chapter 4

The Selberg trace formula

4.1 Uniform lattices

Let G be a connected real semi-simple Lie group with finite center of non-compact type and let $\Gamma \subset G$ be uniform lattice. Let R_Γ be the right regular representation of G in $L^2(\Gamma\backslash G)$. Then we have the following result of Gelfand, Graev, and Piateski-Shapiro.

Theorem 4.1.1. R_Γ *decomposes discretely:*

$$R_\Gamma = \bigoplus_{\pi \in \hat{G}} m_\Gamma(\pi)\pi.$$

Let $f \in C_c^\infty(G)$. Define

$$R_\Gamma(f) = \int_G f(g) R_\Gamma(g)\, dg.$$

Then $R_\Gamma(f)$ is an integral operator

$$(R_\Gamma(f)\varphi)(g) = \int_{\Gamma\backslash G} K_f(g, g')\, dg', \quad \varphi \in L^2(\Gamma\backslash G)$$

with kernel

$$K(g, g') = \sum_{\gamma \in \Gamma} f(g^{-1}\gamma g').$$

Since $\Gamma\backslash G$ is compact, $R_\Gamma(f)$ is a *trace class operator* and

$$\operatorname{Tr} R_\Gamma(f) = \int_{\Gamma\backslash G} K(g, g)\, dg = \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma} f(g^{-1}\gamma g)\, dg.$$

- break the sum over γ into conjugacy classes $\{\gamma\}$ of Γ.

Let Γ_γ and G_γ be the centralizer of γ in Γ and G, respectively. The contribution of a conjugacy class $\{\gamma\}$ is

$$\int_{\Gamma_\gamma\backslash G} f(g^{-1}\gamma g)\, d\dot{g} = \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) I(\gamma, f),$$

where $I(\gamma, f)$ is the *orbital integral*

$$I(\gamma, f) = \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\, d\dot{g}, \quad f \in C_c^\infty(G).$$

Thus we get

$$\mathrm{Tr}\, R_\Gamma(f) = \sum_{\{\gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) I(\gamma, f).$$

For $\pi \in \hat{G}$ let

$$\Theta_\pi(f) = \mathrm{Tr}\, \pi(f), \quad f \in C_c^\infty(G),$$

be the character of π. By the result of Gelfand, Graev, and Piatetski-Shapiro, we get

$$\mathrm{Tr}\, R_\Gamma(f) = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\Theta_\pi(f).$$

Comparing the two expressions, we obtain

Trace formula (1. version):

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\Theta_\pi(f) = \sum_{\{\gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) I(\gamma, f). \tag{4.1.1}$$

spectral side = geometric side

- $I(\gamma, f)$ and $\mathrm{Tr}\, \pi(f)$ are invariant distributions on G, i.e., invarinat under $f \to f^g$, where $f^g(g') = f(g^{-1}g'g)$.

- Fourier inversion formula can be used to express $I(\gamma, f)$ in terms of characters.

To make the trace formula useful, one has to understand the distributions $I(\gamma, f)$ and $\mathrm{Tr}\, \pi(f)$ and to express them in differential geometric terms. This is possible if G has split rank 1.

4.2 The rank one case

Let $G = KAN$ be the Iwasawa decomposition of G. We assume that $\mathrm{rank}_{\mathbb{R}}(G) = 1$. This means that $\dim A = 1$. Let M be the centralizer of A in K. Put $P = MAN$. This is the standard parabolic subgroup of G. Since G has split rank 1, every proper parabolic subgroup of G is conjugate to P. Let α be the unique simple root of $(\mathfrak{g}, \mathfrak{a})$. Let $H \in \mathfrak{a}$ be such that $\alpha(H) = 1$. Then $\mathfrak{a} = \mathbb{R}H$. For $t \in R$ we set $a_t = \exp(tH)$ and $\log a_t = t$.

For $(\sigma, V_\sigma) \in \hat{M}$ and $\lambda \in \mathbb{R}$ let $\pi_{\sigma,\lambda}$ be the unitarily induced representation from P to G acting in the Hilbert space $\mathcal{H}_\sigma$ of measurable functions $f\colon K \to V_\sigma$ which satisfy

$$f(km) = \sigma(m)^{-1} f(k), \quad \| f \|^2 = \int_K \| f(k) \|^2 \; dk < \infty.$$

If $f \in \mathcal{H}_\sigma$ let $f_\lambda(k \exp(tH) n) = e^{-(i\lambda+\rho)t} f(k)$, $k \in K$, $t \in \mathbb{R}$, $n \in N$. Then

$$(\pi_{\sigma,\lambda}(g) f)(k) = f_\lambda(g^{-1}k).$$

Let $\Theta_{\sigma,\lambda}$ denote the character of $\pi_{\sigma,\lambda}$.

If $\gamma \in \Gamma$, $\gamma \neq e$, then there exists $g \in G$ such that $g\gamma g^{-1} \in MA^+$. Thus there are $m_\gamma \in M$ and $a_\gamma \in A^+$ such that $g\gamma g^{-1} = m_\gamma a_\gamma$. By [Wa, Lemma 6.6], a_γ depends only on γ and m_γ is determined by γ up to conjugacy in M. Let

$$l(\gamma) = \log a_\gamma.$$

Then $l(\gamma)$ is the length of the unique closed geodesic of $\Gamma\backslash S$ determined by $\{\gamma\}_\Gamma$. Futhermore, by the above remark

$$D(\gamma) := e^{-l(\gamma)\rho} \Big| \det \left(\mathrm{Ad}(m_\gamma a_\gamma)|_{\mathfrak{n}} - \mathrm{Id} \right) \Big| \tag{4.2.1}$$

is well defined. Let

$$u(\gamma) = \mathrm{vol}(G_{m_\gamma a_\gamma}/A).$$

Let $h \in C_c^\infty(G)$ be K-finite. Then by [Wa, pp. 177-178] (correcting a misprint) we have

$$\int_{G_\gamma\backslash G} h(g\gamma g^{-1}) \, d\dot{g} = \frac{1}{2\pi} \frac{1}{u(\gamma) D(\gamma)} \sum_{\sigma \in \widehat{M}} \overline{\mathrm{tr}\, \sigma(\gamma)} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(h) \cdot e^{-il(\gamma)\lambda} \, d\lambda. \tag{4.2.2}$$

Since h is K-finite, $\Theta_{\sigma,\lambda}(h) \neq 0$ only for finitely many σ. Thus the sum over $\sigma \in \widehat{M}$ is finite. The volume factors in (4.1.1) are computed as follows. Since G has rank one, Γ_γ is infinite cyclic [DKV, Proposition 5.16]. Thus there is $\gamma_0 \in \Gamma_\gamma$ such that γ_0 generates Γ_γ and $\gamma = \gamma_0^{n(\gamma)}$ for some integer $n(\gamma) \geq 1$. Then

$$\frac{\mathrm{vol}(\Gamma_\gamma \backslash G_\gamma)}{u(\gamma)} = l(\gamma_0). \tag{4.2.3}$$

Inserting (4.2.2) and (4.2.3) into (4.1.1), we get the following form of the trace formula in the rank one case.

Proposition 4.2.1. *Let $f \in C_c^\infty(G)$. Then*

$$\sum_{\pi\in\widehat{G}} m_\Gamma(\pi)\Theta_\pi(f) = \mathrm{vol}(\Gamma\backslash S)f(e) + \sum_{\{\gamma\}_\Gamma\neq e} \frac{1}{2\pi}\frac{l(\gamma_0)}{D(\gamma)} \sum_{\sigma\in\widehat{M}} \overline{\operatorname{tr}\sigma(\gamma)} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(f)\cdot e^{-il(\gamma)\lambda}\, d\lambda. \tag{4.2.4}$$

The right hand side is still not in an explicite form. First of all we can use the Plancherel formula [Kn] to express $f(e)$ in terms of charaters. In this way we are reduced to the computation of the characters Θ_π, $\pi \in \widehat{G}$, evaluated on f.

We consider this problem in the special case of a bi-K-invariant function. To this end we recall some facts about the spherical Fourier transform. [He1]. Let

$$C_c^\infty(G//K) = \{f \in C_c^\infty(G)\colon f(k_1 g k_2) = f(g),\ k_1, k_2 \in K\}$$

be the space of smooth, bi-K-invariant functions on G. Let ϕ_λ, $\lambda \in \mathfrak{a}_\mathbb{C}^*$, be the spherical function given by Harish-Chandra's formula

$$\phi_\lambda(g) = \int_K e^{\langle\lambda+\rho, H(gk)\rangle}\, dk. \tag{4.2.5}$$

Let $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)$ be the topological algebra of Paley-Wiener functions on $\mathfrak{a}_\mathbb{C}^*$ (with the usual product). Recall that

$$\mathcal{P}(\mathfrak{a}_\mathbb{C}^*) = \bigcup_{R>0} \mathcal{P}^R(\mathfrak{a}_\mathbb{C}^*),$$

with the inductive limit topology where $\mathcal{P}^R(\mathfrak{a}_\mathbb{C}^*)$ is the Frechet space of entire functions f on $\mathfrak{a}_\mathbb{C}^*$ such that for every $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|f(\lambda)| \leq C_N(1+\|\lambda\|)^{-N} e^{R\|\mathrm{Re}\,\lambda\|}, \quad \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

Let $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)^W$ be the subalgebra of W-invariant Paley-Wiener functions on $\mathfrak{a}_\mathbb{C}^*$. Then the Harich-Chandra transform

$$(\mathcal{H}f)(\lambda) = \int_G f(g)\phi_\lambda(g)\, dg, \quad f \in C_c^\infty(G//K),\ \lambda \in \mathfrak{a}_\mathbb{C}^*, \tag{4.2.6}$$

defines an isomorphism of topological algebras

$$\mathcal{H}\colon C_c^\infty(G//K) \to \mathcal{P}(\mathfrak{a}_\mathbb{C}^*)^W.$$

Put $\tilde{f}(\lambda) = \mathcal{H}(f)(\lambda)$. In inverse spherical transform is given by

$$f(g) = \frac{1}{|W|}\int_{\mathfrak{a}^*} \tilde{f}(\lambda)\phi_{-\lambda}(g)\beta(\lambda)\, d\lambda, \tag{4.2.7}$$

where $\beta(\lambda) = |c(\lambda)c(\rho)^{-1}|^{-2}$ is the Plancherel measure on $\mathfrak{a}^*$ and the c-function is given by the Gindikin-Karpelevic formula [He1].

Let $f \in C_c^\infty(G//K)$. Then it follows that $\Theta_\pi(f) = 0$, unless $\mathcal{H}_\pi$ contains a non-zero K-invariant vector. Let $\mathcal{H}_\pi^K$ be the subspace of $\mathcal{H}_\pi$ of $v \in \mathcal{H}_\pi$ such that $\pi(k)v = v$ for all $k \in K$. Then $\dim \mathcal{H}_\pi^K \leq 1$ and by Froebnius reciprocity it follows that $\mathcal{H}_{\sigma,\lambda}^K \neq 0$, only if $\sigma = 1$ (the trivial representation). Denote $\pi_{1,\lambda}$ by π_λ. Let $v \in \mathcal{H}_\lambda^K$, $\| v \| = 1$. Then it follows that

$$\Theta_\lambda(f) = \operatorname{Tr} \pi_\lambda(f) = \int_G \langle \pi_\lambda(g)v, v \rangle \, f(g) \, dg.$$

Moreover

$$\phi_\lambda(g) = \langle \pi_\lambda(g)v, v \rangle$$

is the spherical function. Thus we get

$$\Theta_\lambda(f) = \int_G \phi_\lambda(g) f(g) \, dg = \mathcal{H}(f)(\lambda), \quad \lambda \in \mathbb{R}. \tag{4.2.8}$$

Put $h(\lambda) = \mathcal{H}(f)(\lambda)$. Using (4.2.7) to express $f(e)$ in terms of $\mathcal{H}(f)$ and (4.2.8), the right hand side of (4.2.4)becomes

$$\operatorname{vol}(\Gamma\backslash S) \int_{\mathbb{R}} h(\lambda)\beta(\lambda) \, d\lambda + \sum_{\{\gamma\}_\Gamma \neq e} \frac{l(\gamma_0)}{D(\gamma)} \hat{h}(\ell(\gamma)).$$

To describe the left hand side, we observe that

$$L^2(\Gamma\backslash G)^K = L^2(\Gamma\backslash S).$$

Let

$$\widehat{G}(1) = \left\{ \pi \in \widehat{G} \colon \mathcal{H}_\pi^K = 1 \right\}$$

Then we get

$$L^2(\Gamma\backslash S) = \oplus_{\pi \in \widehat{G}(1)} m_\Gamma(\pi) \mathcal{H}_\pi^K.$$

Let $\Omega \in \mathcal{Z}(\mathfrak{g})$ be the Casimir element. By Schur's lemma Ω acts in the subspace of smooth vectors $\mathcal{H}_\pi^\infty$ of $\mathcal{H}_\pi$ by a scalar μ_π. Let $\pi \in \widehat{G}(1)$ and $v \in \mathcal{H}_\pi^K$. Then $v \in \mathcal{H}_\pi^\infty$ and $\Omega v = \mu_\pi v$. Assume that $m(\pi) > 0$. Then v corresponds to a function $\varphi \in C^\infty(\Gamma\backslash S)$ and by Kuga's lemma we have $\Delta\varphi = -\lambda_\pi \varphi$. Finally note that

$$\mu_{\pi_\lambda} = \lambda^2 + |\rho|^2.$$

The same is true for the complementary spectrum. Summarizing we obtain

Theorem 4.2.2. *Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the spectrum of Δ. Let $m(\lambda_j)$ be the multiplicity of λ_j. Let $h \in \mathcal{S}(\mathbb{R})$ be even and such that $\hat{h} \in C_c^\infty(\mathbb{R})$. Then*

$$\sum_{j=0}^\infty h\left(\sqrt{\lambda_j - |\rho|^2}\right) = \operatorname{vol}(\Gamma\backslash S) \int_{\mathbb{R}} h(\lambda)\beta(\lambda) \, d\lambda + \sum_{\{\gamma\}_\Gamma \neq e} \frac{l(\gamma_0)}{D(\gamma)} \hat{h}(\ell(\gamma)). \tag{4.2.9}$$

Now let us specialize to $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2)$. Write the eigenvalues of Δ as

$$\lambda_j = \frac{1}{4} + r_j^2, \quad r_j \in \mathbb{R} \cup i[-1/2, 1/2].$$

The Plancherel measure is given by $\beta(\lambda) = r\tanh(\pi r)$. Then Theorem 4.2.2 gives

Selberg's trace formula (K-invariant form):

$$\sum_j m(\lambda_j)h(r_j) = \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{2\pi}\int_{\mathbb{R}} h(r)r\tanh(\pi r)\,dr + \sum_{\{\gamma\}_\Gamma \neq e} \frac{\ell(\gamma_0)}{\sinh(\ell(\gamma)/2)}\widehat{h}(\ell(\gamma)).$$

4.3 Non-uniform lattices

We now assume that $\Gamma\backslash G$ is a non-uniform lattice. This means that $\Gamma\backslash G$ non-compact, but $\mathrm{vol}(\Gamma\backslash G) < \infty$. Then the major differences to the co-compact case are

- R_Γ does not decompose discretely.
- $R_\Gamma(f)$ is not trace class

Langlands's theory of Eisenstein series provides a decomposition into invariant subspaces

$$L^2(\Gamma\backslash G) = L^2_{\mathrm{dis}}(\Gamma\backslash G) \oplus L^2_{ac}(\Gamma\backslash G),$$

where $L^2_{\mathrm{dis}}(\Gamma\backslash G)$ is the maximal invariant subspace, in which R_Γ decomposes discretely. Let R_Γ^{dis} and R_Γ^{ac} denote the right regular representation of G in $L^2_{\mathrm{dis}}(\Gamma\backslash G)$ and $L^2_{\mathrm{ac}}(\Gamma\backslash G)$, respectively. Then

$$R_\Gamma^{\mathrm{dis}} = \hat{\bigoplus}_{\pi\in\widehat{G}} m_\Gamma(\pi)\pi$$

with finite multiplicities $m_\Gamma(\pi)$. The absolutely continuous subspace is described in terms of Eisenstein series. Let $L^2_{\mathrm{cus}}(\Gamma\backslash G)$ be the closed subspace of $L^2(\Gamma\backslash G)$ which is spaned by cusp forms. Then $L^2_{\mathrm{cus}}(\Gamma\backslash G)$ is contained in $L^2_{\mathrm{dis}}(\Gamma\backslash G)$ and we have decomposition

$$L^2_{\mathrm{dis}}(\Gamma\backslash G) = L^2_{\mathrm{cus}}(\Gamma\backslash G) \oplus L^2_{\mathrm{res}}(\Gamma\backslash G),$$

where $L^2_{\mathrm{res}}(\Gamma\backslash G)$ the residual subspace which according to Langlands [La1] is spanned by iterated residues of Eisenstein series.

The fist result we need is that the integral operator $R_\Gamma(f)$ is trace class in $L^2_{\mathrm{dis}}(\Gamma\backslash G)$.

Let $\mathcal{C}^1(G)$ be Harish-Chandra's Schwartz space of integrable rapidely decreasing functions on G (see [Mu3, Section 1] for the definition). For $f \in \mathcal{C}^1(G)$. Since $f \in L^1(G)$, we can define

$$R_\Gamma^{\mathrm{dis}}(f) = \int_G f(g)R_\Gamma^{\mathrm{dis}}(g)\,dg.$$

The first main result is

Theorem 4.3.1. *For every $f \in \mathcal{C}^1(G)$, the operator $R_\Gamma^{\rm dis}(f)$ is a trace class operator.*

For K-finite functions $f \in \mathcal{C}^1(G)$ this was proved in [Mu2]. The extension to the general case was proved by Ji [Ji] and the author [Mu3], independently. The proof of Theorem 4.3.1 follows from the estimation of the eigenvalue counting function for Bochner-Laplace operatorson the locally symmetric space $\Gamma\backslash S$. Let $\sigma\colon K \to \mathrm{GL}(V_\sigma)$ be an irreducible unitary representation of K. Let

$$\tilde{E}_\sigma = (G \times V_\sigma)/K \to G/K$$

be the associated homogeneous vextor bundle and let $E_\sigma = \Gamma\backslash\tilde{E}_\sigma$ be the corresponding locally homogeneous vector bundle over $\Gamma\backslash S$. Let ∇^σ be the push-down of the connection on the homogeneous vector bundle $\tilde{E}_\sigma$. Let

$$\Delta_\sigma = (\nabla^\sigma)^*\nabla^\sigma$$

be the Bochner-Laplace operator. This is a second order elliptic differential operator acting in $C^\infty(\Gamma\backslash S, E_\sigma)$. Regarded as operator

$$\Delta_\sigma\colon C_c^\infty(\Gamma\backslash S, E_\sigma) \to L^2(\Gamma\backslash S, E_\sigma)$$

it is essentially self-adjoint and nonnegative. Let $\bar{\Delta}_\sigma$ be the unique self-adjoint extension of Δ_σ and let

$$0 \le \lambda_1 < \lambda_2 < \cdots$$

be the eigenvalues of $\bar{\Delta}_\sigma$. Let $m(\lambda_j)$ be the multiplicity of λ_j. Let

$$N_\Gamma^{\rm dis}(\lambda,\sigma) = \sum_{\lambda_j \le \lambda} m(\lambda_j).$$

be the counting function of eigenvalues. The following theorem was proved in [Mu2].

Theorem 4.3.2. *Let $d = \dim S$. There exists $C > 0$ such that*

$$N_\Gamma^{\rm dis}(\lambda,\sigma) \le C(1+\lambda)^{2d}, \quad \lambda \ge 0. \tag{4.3.1}$$

Now observe that

$$L^2_{\rm dis}(\Gamma\backslash S, E_\sigma) \cong (L^2_{\rm dis}(\Gamma\backslash G) \otimes V_\sigma)^K = \hat{\bigoplus}_{\pi \in \widehat{G}} m_\Gamma(\pi)(\mathcal{H}_\pi \otimes V_\sigma)^K.$$

For $\pi \in \widehat{G}$ denote by λ_π the eigenvalue of the Casimir operator acting in $\mathcal{H}_\pi^\infty$. Then Theorem 4.3.2 implies that there exists $N \in \mathbb{N}$ such that

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \dim(\mathcal{H}_\pi \otimes V_\sigma)^K (1+\lambda)^{-N} < \infty.$$

This estimation implies Theorem 4.3.1 for all K-finite $f \in \mathcal{C}^1(G)$. To prove it in general one needs to control the dependence on σ of the constant C in Theorem 4.3.2. The refined estimation is proved in [Mu3, Theorem 0.2].

The exponent $2d$ in 4.3.1 is not optimal. One expects it to be $d/2$. This is supported by the following result of Donnelly [Do].

Theorem 4.3.3. *Let* $d = \dim S$. *Let* $N_\Gamma^{\text{cus}}(\lambda, \sigma)$ *the counting function for the cuspidal spectrum of* $\bar{\Delta}_\sigma$. *Then*

$$\limsup_{\lambda \to \infty} \frac{N_\Gamma^{\text{cus}}(\lambda, \sigma)}{\lambda^{d/2}} \leq \frac{\dim(V_\sigma)\,\text{vol}(\Gamma\backslash S)}{(4\pi)^{d/2}\mathbf{\Gamma}(d/2+1)}.$$

The main issue of [Mu2] was to estimate the growth of the residual spectrum. Let $N_\Gamma^{\text{res}}(\lambda, \sigma)$ be the counting function of the residual spectrum. Then we have

$$N_\Gamma^{\text{res}}(\lambda, \sigma) \leq C(1+\lambda)^{2d}, \quad \lambda \geq 0. \tag{4.3.2}$$

The proof is based on the following methods and results:

1) The description of the the residual eigenfunctions as iterated residues of Eisenstein series [La1].

2) Donnelly's result [Do] applied to the cuspidal spectrum for Levi components of proper Γ-cuspidal parabolic subgroups.

3) Extension of Colin de Verdier's method [Co] for the analytic continuation of cuspidal Eisenstein series attached to maximal parabolic subgroups.

Moeglin and Waldspurger [MW] gave a precise description of the residual spectrum for $\text{GL}(n)$. Using these results and Donnelly's estimation of th cuspidal spectrum, one gets

Theorem 4.3.4. *Let* $G = \text{SL}(n, \mathbb{R})$ *and* $\Gamma \subset \text{SL}(n, \mathbb{Z})$ *a congruence subgroup. Then we have*

$$N_\Gamma^{\text{res}}(\lambda, \sigma) \leq C(1+\lambda)^{d/2-1}, \quad \lambda \geq 0.$$

One expects that this holds in general.

Conjecture 1. Let $S = G/K$, $\Gamma \subset G$ a lattice and $\sigma \in \widehat{K}$. Let $d = \dim S$. There exists $C > 0$ such that

$$N_\Gamma^{\text{res}}(\lambda, \sigma) \leq C(1+\lambda)^{d/2-1}, \quad \lambda \geq 0.$$

Let $f \in \mathcal{C}^1(G)$. By Theorem 4.3.1 it follows as in the co-compact case that

$$\text{Tr}\, R_\Gamma^{\text{dis}}(f) = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\Theta_\pi(f).$$

Now $R_\Gamma^{\text{dis}}(f)$ is an integral operator with kernel $K_d(g, g')$ and a pre-trace formula is

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\Theta_\pi(f) = \int_{\Gamma\backslash G} K_d(g, g)\, dg. \tag{4.3.3}$$

Let $K_c(g,g')$ be the kernel of $R_\Gamma^{ac}(f)$. For simplicity assume that $f \in C_c^\infty(G)$ is K-finite. Using the spectral resolution, $K_c(g,g')$ it can be described in terms of Eisenstein series. Then

$$K_d(g,g') = K(g,g') - K_c(g,g')$$

and we get

$$\sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\Theta_\pi(f) = \int_{\Gamma\backslash G} (K(g,g) - K_c(g,g))\, dg. \tag{4.3.4}$$

The main goal of the trace formula is to compute the right hand side. For higher rank groups this is not possible within this framework. An appropriate replacement is the Arthur trace formula.

4.4 The rank one case II: non-uniform lattices

If $\operatorname{rank}(G) = 1$ there is a generalization of the Selberg trace formula to non-uniform lattices. For simplicity we consider the case $G = \mathrm{SL}(2,\mathbb{R})$ and we assume that f is bi-K-invariant, where $K = \mathrm{SO}(2)$. Then we are dealing with the trace formula for a hyperbolic surface $X = \Gamma\backslash\mathbb{H}$ of finite area with m cusps and $R_\Gamma(f)$ equals $h(\Delta)$, where h is the spherical Fourier transform of f. Let $K(z,z')$ be the kernel of the integral operator $h\left((\Delta - 1/4)^{1/2}\right)$. It is given by

$$K(z,z') = \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g'),$$

where $z = gK$, $z' = g'K$. Since f is bi-K-invariant, the right hand side is independent of the choice of the representatives of the cosets. Let $E_k(z,s)$, $k = 1,\dots,m$, be the Eisenstein series attached to the cusps of $\Gamma\backslash\mathbb{H}$. By Proposition 2.6.2 it follows that the continuous part$K_c(z,z')$ of the kernel $K(z,z')$ is given by

$$K_c(z,z') = \frac{1}{2\pi}\sum_{k=1}^{m}\int_{\mathbb{R}} h(r)E_k(z,1/2+ir)E_k(z',1/2-ir)\, dr. \tag{4.4.1}$$

The surface X has the form (2.1.1). Let $T > \max\{a_1,\dots,a_m\}$. Let

$$X_T = M \cup ([a_1,T]\times S^1) \cup \cdots \cup ([a_m,T]\times S^1)$$

be the surface obtained from X by truncating the cusps at level T. Then

$$\lambda_j = 1/4 + r_j^2, \quad r_j \in \mathbb{R}\cup i[-1/2,1/2],\ j \in I,$$

be the eigenvalues of Δ. Then the pre-trace formula (4.3.4) gives

$$\sum_j h(r_j) = \int_X (K(z,z) - K_c(z,z))\, dz = \lim_{T\to\infty}\left\{\int_{X_T} K(z,z)\, dz - \int_{X_T} K_c(z,z)\, dz\right\}.$$

By (4.4.1) we get

$$\int_{X_T} K_c(z,z)\,dz = \frac{1}{2\pi}\sum_{k=1}^{m}\int_{\mathbb{R}} h(r)\left(\int_{X_T} |E_k(z,1/2+ir)|^2\,dz\right)dr. \tag{4.4.2}$$

Let Λ^T be the trucation operator. The truncated Eisenstein $\Lambda^T E_k(z,s)$ are square integrable and

$$\int_{X_T} |E_k(z,1/2+ir)|^2\,dz = \| \Lambda^T E_k(z,1/2+ir) \|^2 + O\left(e^{-cT}\right)$$

as $T \to \infty$. The L^2 norm of the truncated Eisenstein series is given in terms of the constant term. The integral $\int_{X_T} K(z,z)\,dz$ can be computed as in the compact case. Each conjugacy class of Γ makes a contribution. The difference to the compact case is that we have now also parabolic conjugacy classes to deal with.

Let $C(s)$ be the scattering matrix(2.4.2). Put

$$\phi(s) = \det C(s).$$

Then we get the following **Selberg trace formula** (see [Se2]):

Theorem 4.4.1. *Let $h \in \mathcal{S}(\mathbb{R})$ be even with $\hat{h} \in C_c^\infty(\mathbb{R})$. Then*

$$\begin{aligned}\sum_j h(r_j) - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{\phi'}{\phi}(1/2+ir)\,dr + \frac{1}{4}\operatorname{Tr}(\mathrm{Id} - C(1/2))h(0)&\\ = \frac{\operatorname{Area}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{\mathbb{R}} h(r) r\tanh(\pi r)\,dr&\\ + \sum_{\{R\}}\sum_{0<l<n}\frac{1}{m\sin\left(\frac{\pi l}{n}\right)}\int_{\mathbb{R}}\frac{h(r)e^{-\frac{\pi l}{n}r}}{1+e^{-2\pi r}}\,dr&\\ + \sum_{\{\gamma\}}\sum_{k=1}^{\infty}\frac{\ell(\gamma)}{e^{k\ell(\gamma)/2}-e^{-k\ell(\gamma)/2}}\widehat{h}(k\ell(\gamma))&\\ - \frac{m}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr + \frac{m}{4}h(0) - m\ln 2\,\widehat{h}(0).&\end{aligned} \tag{4.4.3}$$

Here $\{R\}$ runs over the primitive elliptic classes, and $n = n(R)$ is the order of the primitive elliptic class R. Similarly $\{\gamma\}$ runs over the primitive hyperbolic conjugacy classes.

Remark 4.4.2. The main new ingredient in the trace formula is the integral involving the logarithmic derivative of the determinant of the scattering matrix. It is due to the presence of the continuous spectrum.

Remark 4.4.3. The contribution of the parabolic conjugacy classes to the trace formula is the integral $\frac{m}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr$.

Remark 4.4.4. The Selberg trace formula can be extended to lattices of rank one. This includes, for example, Hilbert modular groups.

Chapter 5

The Arthur trace formula

The trace formula has become a central tool in the modern theory of automorphic forms. Arthur, driven by Langlands' functoriality conjectures, developed the trace formula an arbirtrary reductive group over a number field F.

Let G be a connected reductive algebraic group over $\mathbb{Q}$. Let $\mathbb{A}$ be the ring of adels of $\mathbb{Q}$. Let $X(G)_{\mathbb{Q}}$ denote the group of characters of G defined over $\mathbb{Q}$. Let $G(\mathbb{A})^1$ be the intersection of the kernels of the maps

$$x \in G(\mathbb{A}) \mapsto |\xi(x)|, \quad \xi \in X(G)_{\mathbb{Q}}.$$

Then $G(\mathbb{Q})$ is contained in $G(\mathbb{A})^1$. The **noninvariant** trace formula of Arthur is an identity

$$\sum_{\chi \in \mathfrak{X}} J_\chi(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f), \quad f \in G(\mathbb{A})^1, \tag{5.0.1}$$

between distributions on $G(\mathbb{A})^1$. The left hand side is the spectral expansion $J_{\text{spec}}(f)$ and the rigt hand side the (coarse) geometric expansion $J_{geom}(f)$ of the trace formula. The distributions J_χ are defined in terms of truncated Eisenstein series. They are parametrized by the set $\mathfrak{X}$ of Weyl group orbits of pairs (M_B, π_B), where M is the Levi component of a standard parabolic subgroup B and π_B is an irreducible cuspidal automorphic representation of $M_B(\mathbb{A})^1$. The distributions $J_{\mathfrak{o}}$ are parametrized by semisimple conjugacy classes in $G(\mathbb{Q})$ and are closely related to weighted orbital integrals on $G(\mathbb{A})^1$.

5.1 The spectral expansion

The distributions J_χ are derived from the constant terms of Eisenstein series and generalize the integral

$$\frac{1}{4\pi} \int_{\mathbb{R}} \widehat{h}(r) \frac{\phi'}{\phi}(1/2 + ir)\, dr,$$

which appears in the Selberg trace formula for $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$.

To describe the geometric expansion in more detail, we need to recall some notation from [A3], [A4]. We shall fix a minimal parabolic subgroup P_0 of G and a Levi component M_0 of P_0, defined over $\mathbb{Q}$. A parabolic subgroup will mean a parabolic subgroup of G, defined over $\mathbb{Q}$, which contains P_0. Suppose that P is such a subgroup. We shall write N_P for the unipotent radical of P, and M_P for the unique Levi component of P which contains M_0. If $M \subset L$ are Levi subgroups, we denote the set of Levi subgroups of L which contain M by $\mathcal{L}^L(M)$. Furthermore, let $\mathcal{F}^L(M)$ denote the set of parabolic subgroups of L defined over $\mathbb{Q}$ which contain M, and let $\mathcal{P}^L(M)$ be the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. If $L = G$, we shall denote these sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$. Write $\mathcal{L} = \mathcal{L}(M_0)$. Suppose that $P \in \mathcal{F}^L(M)$. Then

$$P = N_P M_P.$$

Let

$$\mathfrak{a}_P = \mathrm{Hom}(X(M_P)_{\mathbb{Q}}, \mathbb{R}).$$

Denote by $\mathcal{A}^2(P)$ the space of square integrable automorphic forms on $N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})$. This is the space of smooth functions

$$\phi\colon N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$$

which satisfy the following conditions:

i) The span of the set of functions

$$x \mapsto (z\phi)(xk), \quad x \in G(\mathbb{A}),$$

indexed by $k \in K$ and $z \in Z(\mathfrak{g}_{\mathbb{C}})$, is finite dimensional.

ii)

$$\| \phi \|^2 = \int_K \int_{M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1} |\phi(mk)|^2 \, dm \, dk < \infty.$$

Let $M \in \mathcal{L}$ and $P, Q \in \mathcal{P}(M)$. Let $\mathcal{A}^2(P)$ and $\mathcal{A}^2(Q)$ be the corresponding spaces of automorphic functions. For $s \in W(\mathfrak{a}_P)$ let

$$M_{Q|P}(s,\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}},$$

be the intertwining operator. Set

$$M_{Q|P}(\lambda) := M_{Q|P}(1,\lambda).$$

Generalized logarithmic derivatives of $M_{Q|P}(\lambda)$ are the main ingredients of the spectral side. For $M \in \mathcal{L}$, $L \in \mathcal{L}(M)$, $P \in \mathcal{P}(M)$ let

$$\mathfrak{M}_L(P,\lambda) = \lim_{\Lambda\to 0}\left(\sum_{Q_1\in\mathcal{P}(L)} \operatorname{vol}(\mathfrak{a}^G_{Q_1}/\mathbb{Z}(\Delta^\vee_{Q_1}))M_{Q|P}(\lambda)^{-1}\frac{M_{Q|P}(\lambda+\Lambda)}{\prod_{\alpha\in\Delta_{Q_1}}\Lambda(\alpha^\vee)}\right),$$

where λ and Λ are constrained to lie in $i\mathfrak{a}^*_L$, and for each $Q_1 \in \mathcal{P}(L)$, Q is a group in $\mathcal{P}(M_P)$ which is contained in Q_1. It follows from Arthur's theory of (G,M)-families [A3], [A4] that the limit exists. Then $\mathfrak{M}_L(P,\lambda)$ is an unbounded operator which acts on the Hilbert space $\overline{\mathcal{A}}^2(P)$. For $\pi \in \Pi(M(\mathbb{A})^1)$ let $\mathcal{A}^2_\pi(P)$ be the subspace of $\mathcal{A}^2(P)$ determined by π. Let $\rho_\pi(P,\lambda)$ be the induced representation of $G(\mathbb{A})$ in $\overline{\mathcal{A}}^2_\pi(P)$. Let $W^L(\mathfrak{a}_M)_{\text{reg}}$ be the set of elements $s \in W(\mathfrak{a}_M)$ such that $\{H \in \mathfrak{a}_M \mid sH = H\} = \mathfrak{a}_L$. Let $\mathcal{C}^1(G(\mathbb{A})^1)$ be Harisch-Chandra's Schwartz space of integrable, rapidly decreasing functions on $G(\mathbb{A})^1$. For any function $f \in \mathcal{C}^1(G(\mathbb{A})^1)$ and $s \in W^L(\mathfrak{a}_M)_{\text{reg}}$ set

$$J^L_{M,P}(f,s) = \sum_{\pi\in\Pi_{\text{dis}}(M(\mathbb{A})^1)}\int_{i\mathfrak{a}^*_L/i\mathfrak{a}^*_G}\operatorname{tr}(\mathfrak{M}_L(P,\lambda)M_{P|P}(s,0)\rho_\pi(P,\lambda,f))\,d\lambda. \tag{5.1.1}$$

Concerning the convergence of the integral-sum, the following result is proved in [FLM1, Teorem 1], [FLM2, Corollary 2].

Theorem 5.1.1. *For every $f \in \mathcal{C}^1(G(\mathbb{A})^1)$, the integral-series* (5.1.1) *is absolutely convergent with respect to the trace norm.*

For $G = \mathrm{GL}(n)$ this theorem was proved in [MS]. The statement of the theorem means the following. Let $\|\cdot\|_{\pi,1}$ denote the trace norm in $\overline{\mathcal{A}}^2_\pi(P)$. Then

$$\sum_{\pi\in\Pi_{\text{dis}}(M(\mathbb{A})^1)}\int_{i\mathfrak{a}^*_L/i\mathfrak{a}^*_G}\|\,\mathfrak{M}_L(P,\lambda)M_{P|P}(s,0)\rho_\pi(P,\lambda,f)\,\|_{\pi,1}\;d\lambda < \infty.$$

For $M \in \mathcal{L}$ and $s \in W^L(\mathfrak{a}_M)_{\text{reg}}$ set

$$a_{M,s} = |\mathcal{P}(M)|^{-1}|W^M_0||W_0|^{-1}|\det(s-1)_{\mathfrak{a}^L_M}|^{-1}.$$

Then for any f in $\mathcal{C}^1(G(\mathbb{A})^1)$, the spectral side $J_{\text{spec}}(f)$ of the Arthur trace formula is given by

$$J_{\text{spec}}(f) = \sum_{M\in\mathcal{L}}\sum_{L\in\mathcal{L}(M)}\sum_{P\in\mathcal{P}(M)}\sum_{s\in W^L(\mathfrak{a}_M)_{\text{reg}}} a_{M,s}J^L_{M,P}(f,s). \tag{5.1.2}$$

Note that all sums in this expression are finite.

To illustrate the theorem, we discuss two special cases. For $M = G$ and $s = 1$ we have

$$J^G_{G,G}(f,1) = \operatorname{Tr} R^{\text{dis}}_\Gamma(f).$$

The fact that $R_\Gamma^{\mathrm{dis}}(f)$ is trace class was proved in [Mu5].

Next consider the case that $L = M$ and $\dim \mathfrak{a}_M^G = 1$. Then $P \in \mathcal{P}(M)$ is a maximal parabolic subgroup.Let $\bar{P}$ denote the parabolic subgroup opposite to P. If α is the unique root in Δ_P, let $\tilde{\omega}$ be the element in $(\mathfrak{a}_M^G)^*$ such that $\tilde{\omega}(\alpha^\vee) = 1$, and set

$$\lambda = z\tilde{\omega}, \quad z \in \mathbb{C}.$$

The intertwining operator $M_{\bar{P}|P}(\lambda)$ may be regarded as a function of the complex variable z. Then the following integral-series

$$\sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} \int_{\mathbb{R}} \operatorname{tr}\left(M_{\bar{P}|P}(ir,\pi)^{-1} \frac{d}{dz} M_{\bar{P}|P}(ir,\pi)\rho(P,ir,f)\right) dr \tag{5.1.3}$$

is part of the spectral side. To study the convergence of this integral-series, we make use of the factorization of the global intertwining operator. Let $\pi = \otimes_v \pi_v$, $\phi \in \mathcal{A}_\pi^2(P)$, $\phi = \otimes_v \phi_v$. S a finite set of places, containing ∞, such that ϕ_v is fixed under $G(\mathbb{Z}_p)$ for $p \notin S$. There exist finite-dimensional representations $r_1, ..., r_m$ of ${}^LM$ such that

$$M_{\overline{P}|P}(z,\pi)\phi = \bigotimes_{v\in S} M_{\overline{P}|P}(z,\pi_v)\phi_v \otimes \bigotimes_{v \notin S} \widetilde{\phi}_v \cdot \prod_{j=1}^m \frac{L_S(jz,\pi,\widetilde{r}_j)}{L_S(1+jz,\pi,\widetilde{r}_j)}, \tag{5.1.4}$$

where

$$L_S(s,\pi,r) = \prod_{v\notin S} L(s,\pi_v,r_v), \quad \mathrm{Re}(s) \gg 0,$$

is the partial automorphic L-function attached to π and r. Using (5.1.4), (5.1.2) can be written as a finite sum of similar expressions, involving either logarithmic derivatives of partial L-functions or logarithmic derivatives of local intertwining operators.

- This reduces the problem to the estimation of the number of zeros of $L_S(s,\pi,\widetilde{r}_j)$ in a circle of radius T as $T \to \infty$. One needs to control the constants, appearing in the estimations, in terms of π.

Remark 5.1.2. (5.1.2) explicates Arthur's fine spectral expansion which was previously only known to be conditionally convergent.

5.2 The fine geometric expansion

In this section we describe the fine expansion of the geometric side of the trace formula. The coarse $\mathfrak{o}$-expansion of $J_{\mathrm{geo}}(f)$ is a sum of distributions

$$J_{\mathrm{geo}}(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1),$$

which are parametrized by the set $\mathcal{O}$ of conjugacy classes of semisimple elements in $G(\mathbb{Q})$. The distributions $J_{\mathfrak{o}}(f)$ is the value at $T = 0$ of the polynomial $J_{\mathfrak{o}}^T(f)$ defined in [A1]. The fine $\mathfrak{o}$-expansion of the spectral side [A10] expresses the distributions $J_{\mathfrak{o}}(f)$ in terms of weighted orbital integrals $J_M(\gamma, f)$.

Let S be a finite set of places of $\mathbb{Q}$ containing ∞. Set

$$\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v, \quad \text{and} \quad G(\mathbb{Q}_S) = \prod_{v \in S} G(\mathbb{Q}_v).$$

Let $M \in \mathcal{L}$ and $\gamma \in M(\mathbb{Q}_S)$. The general weighted orbital integrals $J_M(\gamma, f)$ defined in [A11] are distributions on $G(\mathbb{Q}_S)$. If γ is such that $M_\gamma = G_\gamma$, then, as the name suggests, $J_M(\gamma, f)$ is given by an integral of the form

$$J_M(\gamma, f) = \left|D(\gamma)\right|^{1/2} \int_{G_\gamma(\mathbb{Q}_S)\backslash G(\mathbb{Q}_S)} f(x^{-1}\gamma x) v_M(x)\, dx,$$

where $D(\gamma)$ is the discriminant of γ [A11, p. 231] and $v_M(x)$ is the volume of the convex hull of the set

$$\{H_P(x) \colon P \in \mathcal{P}(M)\}.$$

For general γ the definition is more complicated. In this case, $J_M(\gamma, f)$ is obtained as a limit of a linear combination of integrals as above. For more details we refer to [A11].

Set

$$G(\mathbb{Q}_S)^1 = G(\mathbb{Q}_S) \cap G(\mathbb{A})^1.$$

Suppose that ω is a compact neighborhood of 1 in $G(\mathbb{A})^1$. There is a finite set S of valuations of $\mathbb{Q}$, which contains the Archimedean place, such that ω is the product of a compact neighborhood of 1 in $G(\mathbb{Q}_S)^1$ with $\prod_{v \notin S} K_v$. Let S_ω^0 be the minimal such set. Let $C_\omega^\infty(G(\mathbb{A})^1)$ denote the space of functions in $C_c^\infty(G(\mathbb{A})^1)$ which are supported on ω. For any finite set $S \supset S_\omega^0$ set

$$C_\omega^\infty(G(\mathbb{Q}_S)^1) = C_\omega^\infty(G(\mathbb{A})^1) \cap C_c^\infty(G(\mathbb{Q}_S)^1).$$

Let us recall the notion of (M, S)-equivalence [A10, p.205]. For any $\gamma \in M(\mathbb{Q})$ denote by γ_s (resp. γ_u) the semisimple (resp. unipotent) Jordan component of γ. Then two elements γ and γ' in $M(\mathbb{Q})$ are called (M, S)-equivalent if there exists $\delta \in M(\mathbb{Q})$ with the following two properties.

(i) γ_s is also the semisimple Jordan component of $\delta^{-1}\gamma'\delta$.

(ii) γ_u and $(\delta^{-1}\gamma'\delta)_u$, regarded as unipotent elements in $M_{\gamma_s}(\mathbb{Q}_S)$, are $M_{\gamma_s}(\mathbb{Q}_S)$-conjugate.

Denote by $(M(\mathbb{Q}))_{M,S}$ the set of (M,S)-equivalence classes in $M(\mathbb{Q})$. Note that (M,S)-equivalent elements γ and γ' in $M(\mathbb{Q})$ are, in particular, $M(\mathbb{Q}_S)$-conjugate. Given $\gamma \in M(\mathbb{Q})$, let

$$J_M(\gamma, f), \quad f \in C_c^\infty(G(\mathbb{Q}_S)^1),$$

be the weighted orbital integral associated to M and γ [A11]. We observe that $J_M(\gamma, f)$ depends only on the $M(\mathbb{Q}_S)$-orbit of γ. Then by Theorem 9.1 of [A10] there exists a finite set $S_\omega \supset S_\omega^0$ of valuations of $\mathbb{Q}$ such that for all $S \supset S_\omega$ and any $f \in C_\omega^\infty(G(\mathbb{Q}_S)^1)$, we have

$$J_{\text{geo}}(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(\mathbb{Q}))_{M,S}} a^M(S,\gamma) J_M(\gamma, f). \tag{5.2.1}$$

This is the fine $\mathfrak{o}$-expansion of the geometric side of the trace formula. The interior sum is finite.

5.3 Problems related to the trace formula

In this section we list some problems connected with the trace formula, which need to be solved in order to make the formula suitable for applications to spectral problems.

1. In the trace formula (4.4.3), the orbital integrals on the right hand side of (5.2.1) have been replaced by their Fourier transform, which makes the formula more suitable for applications. Therefore, an important problem in the higher rank case is to study the Fourier inversion of the weighted orbital integrals. The constants $a^M(S,\gamma)$ are also not known explicitely. It is important to study these constants.

2. By Theorem 5.1.1 the spectral side is absolutely convergent for all $f \in \mathcal{C}^1(G(\mathbb{A})^1)$. On the hand, the geometric side is only known to converge for $f \in C_c^\infty(G(\mathbb{A})^1)$.

Chapter 6

Some applications of the trace formula

A key issue in the work of Arthur is the comparison of trace formulas of two different groups. Besides functoriality, the trace formula admits striking applications to the Langlands correspondence and to the arithmetic of Shimura varieties. We are mainly interested in the application of the trace formula to spectral problems.

6.1 Weyl's law and the existence of cusp forms

Selberg used the trace formula to prove Weyl's law for congrunce sugroups of $\mathrm{SL}(2, \mathbb{Z})$, which shows that for such groups there exist cusp forms in abundance. In this section we will recall some results about the Weyl law on compact Riemannian manifolds and then discuss discuss it in the context of automorphic forms.

6.1.1 Compact Riemannian manifolds

Let M be a smooth, compact Riemannian manifold of dimension n with smooth boundary ∂M (which may be empty). Let

$$\Delta = -\operatorname{div} \circ \operatorname{grad} = d^* d$$

be the Laplace-Beltrami operator associated with the metric g of M. We consider the Dirichlet eigenvalue problem

$$\Delta\phi = \lambda\phi, \quad \phi\big|_{\partial M} = 0. \tag{6.1.1}$$

As is well known, (6.1.1) has a discrete set of solutions

$$0 \leq \lambda_0 \leq \lambda_2 \leq \cdots \to \infty$$

whose only accumulation point is at infinity and each eigenvalue occurs with finite multiplicity. The corresponding eigenfunctions ϕ_i can be chosen such that $\{\phi_i\}_{i\in\mathbb{N}_0}$ is an orthonormal basis of $L^2(M)$. A fundamental problem in analysis on manifolds is to study the distribution of the eigenvalues of Δ and their relation to the geometric and topological structure of the underlying manifold. One of the first results in this context is Weyl's law for the asymptotic behavior of the eigenvalue counting function. For $\lambda \geq 0$ let

$$N(\lambda) = \#\{j\colon \lambda_j \leq \lambda^2\}$$

be the counting function, where eigenvalues are counted with multiplicities. Then the Weyl law states

$$N(\lambda) = \frac{\operatorname{vol}(M)}{(4\pi)^{n/2}\Gamma\left(\frac{n}{2}+1\right)}\lambda^n + o(\lambda^n), \quad \lambda \to \infty. \tag{6.1.2}$$

This was first proved by Weyl [We1] for a bounded domain $\Omega \subset \mathbb{R}^3$. Written in a slightly different form it is known in physics as the Rayleigh-Jeans law. Raleigh [Ra] derived it for a cube. Garding [Ga] proved Weyl's law for a general elliptic operator on a domain in $\mathbb{R}^n$. For a closed Riemannian manifold (6.1.2) was proved by Minakshisundaram and Pleijel [MP].

Formula (6.1.2) does not say very much about the finer structure of the eigenvalue distribution. The basic question is the estimation of the remainder term

$$R(\lambda) := N(\lambda) - \frac{\operatorname{vol}(M)}{(4\pi)^{n/2}\Gamma\left(\frac{n}{2}+1\right)}\lambda^n.$$

That this is a deep problem shows the following example. Consider the flat 2-dimensional torus $T = \mathbb{R}^2/(2\pi\mathbb{Z})^2$. Then the eigenvalues of the flat Laplacian are $\lambda_{m,n} := m^2 + n^2$, $m, n \in \mathbb{Z}$ and the counting function equals

$$N(\lambda) = \#\{(m,n) \in \mathbb{Z}^2\colon m^2 + n^2 \leq \lambda^2\}.$$

Thus $N(\lambda)$ is the number of lattice points in the circle of radius λ. An elementary packing argument, attributed to Gauss, gives

$$N(\lambda) = \pi\lambda^2 + O(\lambda),$$

and the circle problem is to find the best exponent μ such that

$$N(\lambda) = \pi\lambda^2 + O_\varepsilon\left(\lambda^{\mu+\varepsilon}\right), \quad \forall\varepsilon > 0.$$

The conjecture of Hardy is $\mu = 1/2$. The first nontrivial result is due to Sierpinski who showed that one can take $\mu = 2/3$. Currently the best known result is $\mu = 131/208 \approx 0.629$ which is due to Huxley. Levitan [?] has shown that for a domain in $\mathbb{R}^n$ the remainder term is of order $O(\lambda^{n-1})$.

For a closed Riemannian manifold, Avakumović [Av] proved the Weyl estimate with optimal remainder term:

$$N(\lambda) = \frac{\operatorname{vol}(M)}{(4\pi)^{n/2}\Gamma\left(\frac{n}{2}+1\right)}\lambda^n + O(\lambda^{n-1}), \quad \lambda \to \infty. \tag{6.1.3}$$

This result was extended to more general, and higher order operators by Hörmander [Ho]. As shown by Avakumović the bound $O(\lambda^{n-1})$ of the remainder term is optimal for the sphere. On the other hand, under certain assumption on the geodesic flow, the estimate can be slightly improved. Let S^*M be the unit cotangent bundle and let Φ_t be the geodesic flow. Suppose that the set of $(x,\xi) \in S^*M$ such that Φ_t has a contact of infinite order with the identity at (x,ξ) for some $t \neq 0$, has measure zero in S^*M. Then Duistermaat and Guillemin [DG] proved that the remainder term satisfies $R(\lambda) = o(\lambda^{n-1})$. This is a slight improvement over (6.1.3).

In [We3] Weyl formulated a conjecture which claims the existence of a second term in the asymptotic expansion for a bounded domain $\Omega \subset \mathbb{R}^3$, namely he predicted that

$$N(\lambda) = \frac{\operatorname{vol}(\Omega)}{6\pi^2}\lambda^{3/2} - \frac{\operatorname{vol}(\partial\Omega)}{16\pi}\lambda + o(\lambda).$$

This was proved for manifolds with boundary under a certain condition on the periodic billiard trajectories, by Ivrii [Iv] and Melrose [Me].

6.1.2 Hyperbolic surfaces of finite area

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be a lattice such that $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ is noncompact. Assume that Γ is torsion free. Then $X = \Gamma\backslash\mathbb{H}$ is noncompact, hyperbolic surface of finite area.

Since $\Gamma\backslash\mathbb{H}$ is not compact, it is not clear that there exist any eigenvalues $\lambda > 0$. By Proposition 2.2.1 the continuous spectrum of $\bar{\Delta}$ equals $[1/4,\infty)$. Thus all eigenvalues $\lambda \geq 1/4$ are embedded in the continuous spectrum. It is well-known in mathematical physics, that embedded eigenvalues are unstable under perturbations and therefore, are difficult to study.

One of the basic tools to study the cuspidal spectrum is the Selberg trace formula [Se2]. Let $0 = \lambda_0 < \lambda_1 \leq \cdots$ be the eigenvalues of Δ, $C(s)$ scattering matrix, and $\phi(s) = \det C(s)$. Put

$$N_\Gamma(\lambda) = \#\{j \colon \lambda_j \leq \lambda^2\}, \quad M_\Gamma(\lambda) = -\frac{1}{4\pi}\int_{-\lambda}^{\lambda} \frac{\phi'}{\phi}(1/2+ir)\,dr.$$

Using his trace formula [Se2], Selberg established the following version of Weyl's law.

Theorem 6.1.1. *As $\lambda \to \infty$, we have*

$$N_\Gamma(\lambda) + M_\Gamma(\lambda) = \frac{\operatorname{Area}(\Gamma\backslash\mathbb{H})}{4\pi}\lambda^2 + O(\lambda\log\lambda). \tag{6.1.4}$$

Without an estimation of the remainder term, this theorem can be proved as follows. Using the cut-off Laplacian of Lax-Phillips [CV] one can deduce the following elementary bounds

$$N_\Gamma(\lambda) \ll \lambda^2, \quad M_\Gamma(\lambda) \ll \lambda^2, \quad \lambda \geq 1. \tag{6.1.5}$$

These bounds imply that the the trace formula (4.4.3) holds for a larger class of functions. In particular, it can be applied to the heat kernel $k_t(u)$. Its spherical Fourier transform equals $h_t(r) = e^{-t(1/4+r^2)}$, $t > 0$. If we insert h_t into the trace formula we get the following asymptotic expansion as $t \to 0$.

$$\begin{aligned} \sum_j e^{-t\lambda_j} - \frac{1}{4\pi} \int_{\mathbb{R}} e^{-t(1/4+r^2)} \frac{\phi'}{\phi}(1/2 + ir)\, dr \\ = \frac{\operatorname{Area}(\Gamma\backslash\mathbb{H})}{4\pi t} + \frac{a \log t}{\sqrt{t}} + \frac{b}{\sqrt{t}} + O(1) \end{aligned} \tag{6.1.6}$$

for certain constants $a, b \in \mathbb{R}$. Using [Se2, (8.8), (8.9)] it follows that the winding number $M_\Gamma(\lambda)$ is monotonic increasing for $r \gg 0$. Therefore we can apply a Tauberian theorem to (6.1.6) and we get the Weyl law.

A more sophisticated use of the trace formula gives an estimation of the remainder term [Mu6]. The first step is to estimate the number of eigenvalues in an interval. This is **Hörmander's method**.

In general, $N_\Gamma(\lambda)$ and $M_\Gamma(\lambda)$ can not be separated and (6.1.5) gives no information about the asymptotic bahavior of $N_\Gamma(\lambda)$. However, for the principal congruence subgroup $\Gamma(N)$, the entries of the scattering matrix can be expressed in terms of known functions of analytic number theory. Huxley [Hu] has shown that for $\Gamma(N)$ we have

$$\phi(s) = (-1)^l A^{1-2s} \left(\frac{\boldsymbol{\Gamma}(1-s)}{\boldsymbol{\Gamma}(s)} \right)^k \prod_\chi \frac{L(2-2s, \bar{\chi})}{L(2s, \chi)},$$

where $k, l \in \mathbb{Z}$, $A > 0$, χ Dirichlet character mod k, $k|N$, $L(s, \chi)$ Dirichlet L-function with character χ. Especially, for $N = 1$ we have

$$\phi(s) = \sqrt{\pi} \frac{\boldsymbol{\Gamma}(s - 1/2)\zeta(2s-1)}{\boldsymbol{\Gamma}(s)\zeta(2s)},$$

where $\zeta(s)$ denotes the Riemann zeta function.

Thus for $\Gamma(N)$ we get

$$\left| \frac{\phi'}{\phi}(1/2 + ir) \right| \ll \log^k(|r| + 1), \quad r \in \mathbb{R},$$

and therefore

$$M_{\Gamma(N)}(\lambda) = O(\lambda \log \lambda).$$

If we combine this estimation with (6.1.4), we get

Theorem 6.1.2.

$$N_{\Gamma(N)}(\lambda) = \frac{\operatorname{Area}(\Gamma(N)\backslash\mathbb{H})}{4\pi}\lambda^2 + O(\lambda\log\lambda), \quad \lambda \to \infty.$$

Thus for $\Gamma(N)$, L^2-eigenfunctions of Δ with eigenvalue $\lambda \geq 1/4$ (= Maass automorphic cusp forms) exist in abundance. So far, no eigenfunction with eigenvalue $\lambda > 0$ for $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$ has been constructed explicitly.

6.1.3 Higher rank

In this section we consider an arbitrary locally symmetric space $\Gamma\backslash S$ defined by an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$, where $\mathbf{G}$ is a semi-simple algebraic group over $\mathbb{Q}$ with finite center, $G = \mathbf{G}(\mathbb{R})$ and $S = G/K$. The basic example will be $\mathbf{G} = \mathrm{SL}(n)$ and $\Gamma = \Gamma(N)$, the principal congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$ of level N which consists of all $\gamma \in \mathrm{SL}(n,\mathbb{Z})$ such that $\gamma \equiv \mathrm{Id} \mod N$. Let Δ be the Laplacian of $\Gamma\backslash S$, and let $\bar{\Delta}$ be the closure of Δ in L^2. Then $\bar{\Delta}$ is a non-negative self-adjoint operator in $L^2(\Gamma\backslash S)$. The properties of its spectral resolution can be derived from the known structure of the spectral resolution of the regular representation R_Γ of G on $L^2(\Gamma\backslash G)$ [La1], [BG]. In this way we get the following generalization of Proposition 2.2.1.

Proposition 6.1.3. *The spectrum of $\bar{\Delta}$ is the union of a point spectrum $\sigma_{pp}(\bar{\Delta})$ and an absolutely continuous spectrum $\sigma_{ac}(\bar{\Delta})$.*
1) The point spectrum consists of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \cdots$ of finite multiplicities with no finite point of accumulation.
2) The absolutely continuous spectrum equals $[b,\infty)$ for some $b > 0$.

The theory of Eisenstein series [La1] provides a complete set of generalized eigenfunctions for Δ. The corresponding wave packets span the absolutely continuous subspace $L^2_{\mathrm{ac}}(\Gamma\backslash S)$. This allows us to determine the constant b explicitly in terms of the root structure. The statement about the point spectrum was proved in [BG, Theorem 5.5].

Let $N_\Gamma^{\mathrm{cus}}(\lambda)$ and $N_\Gamma^{\mathrm{res}}(\lambda)$ be the counting function of the eigenvalues with eigenfunctions belonging to the cuspidal and residual subspace, respectively.

In [Sa2] Sarnak made the following conjecture.

Conjecture. If $\operatorname{rank}(S) > 1$, each irreducible lattice Γ in G is essentially cuspidal in the sense that Weyl's law holds for $N_\Gamma^{\mathrm{cus}}(\lambda)$, i.e.,

$$N_\Gamma^{\mathrm{cus}}(\lambda) \sim \frac{\operatorname{vol}(\Gamma\backslash S)}{(4\pi)^{n/2}\Gamma\left(\frac{n}{2}+1\right)}\lambda^n$$

as $\lambda \to \infty$, where $n = \dim S$.

This conjecture has now been established in quite generality. A. Reznikov proved it for congruence groups in a group G of real rank one, S. Miller [Mi] proved it for $\mathbf{G} = \mathrm{SL}(3)$ and $\Gamma = \mathrm{SL}(3,\mathbb{Z})$, the author [Mu3] established it for $\mathbf{G} = \mathrm{SL}(n)$ and a congruence group Γ. The method of [Mu3] is an extension of the heat equation method described in the previous section for the case of the upper half-plane. More recently, Lindenstrauss and Venkatesh [LV] proved the following result.

Theorem 6.1.4. *Let* $\mathbf{G}$ *be a split adjoint semi-simple group over* $\mathbb{Q}$ *and let* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *be a congruence subgroup. Let* $n = \dim S$*. Then*

$$N_\Gamma^{\mathrm{cus}}(\lambda) \sim \frac{\mathrm{vol}(\Gamma\backslash S)}{(4\pi)^{n/2}\mathbf{\Gamma}\left(\frac{n}{2}+1\right)}\lambda^n, \quad \lambda \to \infty.$$

The method is based on the construction of convolution operators with pure cuspidal image. It avoids the delicate estimates of the contributions of the Eisenstein series to the trace formula. This proves existence of many cusp forms for these groups.

The next problem is to estimate the remainder term. For $\mathbf{G} = \mathrm{SL}(n)$, this problem has been studied by E. Lapid and the author in [LM]. Actually, we consider not only the cuspidal spectrum of the Laplacian, but the cuspidal spectrum of the whole algebra of invariant differential operators.

As $\mathcal{D}(S)$ preserves the space of cusp forms, we can proceed as in the compact case and decompose $L^2_{\mathrm{cus}}(\Gamma\backslash S)$ into joint eigenspaces of $\mathcal{D}(S)$. Given $\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W$, let

$$\mathcal{E}_{\mathrm{cus}}(\lambda) = \left\{\varphi \in L^2_{\mathrm{cus}}(\Gamma\backslash S) \colon D\varphi = \chi_\lambda(D)\varphi\right\}$$

be the associated eigenspace. Each eigenspace is finite-dimensional. Let $m(\lambda) = \mathcal{E}_{\mathrm{cus}}(\lambda)$. Define the cuspidal spectrum $\Lambda_{\mathrm{cus}}(\Gamma)$ to be

$$\Lambda_{\mathrm{cus}}(\Gamma) = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W \colon m(\lambda) > 0\}.$$

Then we have an orthogonal direct sum decomposition

$$L^2_{\mathrm{cus}}(\Gamma\backslash S) = \bigoplus_{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma)} \mathcal{E}_{\mathrm{cus}}(\lambda).$$

In [LM] we established the following extension of main results of [DKV] to congruence quotients of $S = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

Theorem 6.1.5. *Let* $d = \dim S$*. Let* $\Omega \subset \mathfrak{a}^*$ *be a bounded domain with piecewise smooth boundary. Let* $\beta(\lambda)$ *be the Plancherel measure. Then for* $N \geq 3$ *we have*

$$\sum_{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma(N)), \lambda \in it\Omega} m(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash S)}{|W|}\int_{t\Omega} \beta(i\lambda)\, d\lambda + O\left(t^{d-1}(\log t)^{\max(n,3)}\right), \tag{6.1.7}$$

as $t \to \infty$*, and*

$$\sum_{\substack{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma(N)) \\ \lambda \in B_t(0)\backslash i\mathfrak{a}^*}} m(\lambda) = O\left(t^{d-2}\right), \quad t \to \infty. \tag{6.1.8}$$

If we apply (6.1.7) and (6.1.8) to the unit ball in $\mathfrak{a}^*$, we get the following corollary.

Corollary 6.1.6. *Let* $\mathbf{G} = \mathrm{SL}(n)$ *and let* $\Gamma(N)$ *be the principal congruence subgroup of* $\mathrm{SL}(n,\mathbb{Z})$ *of level* N*. Let* $S = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ *and* $d = \dim S$*. Then for* $N \geq 3$ *we have*

$$N^{\mathrm{cus}}_{\Gamma(N)}(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash S)}{(4\pi)^{d/2}\mathbf{\Gamma}\left(\frac{d}{2}+1\right)}\lambda^d + O\left(\lambda^{d-1}(\log\lambda)^{\max(n,3)}\right), \quad \lambda\to\infty.$$

The condition $N \geq 3$ is imposed for technical reasons. It guarantees that the principal congruence subgroup $\Gamma(N)$ is neat in the sense of Borel, and in particular, has no torsion. This simplifies the analysis by eliminating the contributions of the non-unipotent conjugacy classes in the trace formula.

Note that $\Lambda_{\mathrm{cus}}(\Gamma(N)) \cap i\mathfrak{a}^*$ is the cuspidal tempered spherical spectrum. The Ramanujan conjecture [Sa3] for $\mathrm{GL}(n)$ at the Archimedean place states that

$$\Lambda_{\mathrm{cus}}(\Gamma(N)) \subset i\mathfrak{a}^*$$

so that (6.1.8) is empty, if the Ramanujan conjecture is true. However, the Ramanujan conjecture is far from being proved. Moreover, it is known to be false for other groups $\mathbf{G}$ and (6.1.8) is what one can expect in general.

The method to prove Theorem 6.1.5 is an extension of the method of [DKV] combined with Arthur's trace formula.

6.2 Limit multiplicities

A problem which is closely related to the Weyl law is that of limit multiplicities. Let G be a semisimple Lie group and $\Gamma \subset G$ a lattice. Let

$$R^{\Gamma}_{\mathrm{cus}} = \widehat{\bigoplus}_{\pi\in\hat{G}} m_{\mathrm{cus}}(\Gamma,\pi)\pi$$

be the decomposition of the regular representation in the space of cups of cusp forms. Define a measure on $\hat{G}$ by

$$\mu_{\Gamma,\mathrm{cus}}(S) = \frac{1}{\mathrm{vol}(\Gamma\backslash G)}\sum_{\pi\in S} m_{\mathrm{cus}}(\Gamma,\pi)$$

for any open, relatively compact subset $S \subset \hat{G}$.

Conjecture. *Let*

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n \supset \cdots$$

be a tower of normal subgroups of finite index with $\cap_{j=1}^{\infty}\Gamma_j = \{e\}$*. Let* μ_{PL} *be the Plancherel measure on* $\hat{G}$*. Let* $S \subset \hat{G}$ *be an open, relatively compct subset which is regular for the Plancherel measure (i.e.* $\mu_{PL}(\bar{S}) = \mu_{PL}(S)$*). Then*

$$\lim_{j\to\infty} \mu_{\Gamma_j,\mathrm{cus}}(S) = \mu_{PL}(S).$$

In the co-compact case, this conjecture has an affirmative answer by the work of De-George-Wallach [GW1], [GW2], Delorme [De] and Sauvageot. The basic tool is the Selberg trace formula for compact quotients.

In the non-compact case a partial result is known [Sv]. Namely

$$\lim_{j\to\infty} \frac{m_{\mathrm{cus}}(\Gamma_j, \pi)}{\mathrm{vol}(\Gamma_j\backslash G)} = d(\pi),$$

where $d(\pi)$ is the formal degree of π, which is 0, if π is not a dicrete series representation. The proof does not use the trace formula. A weaker result was obtained by Clozel. The proof is based on the trace formula.

The main tool to deal with the conjecture in the non-compact case is again the Arthur trace formula. It seems to be possible to extend the methods developed in [Mu4] to deal with the case of $\mathrm{GL}(n)$. The problem is to control for a congrunce subgroup $\Gamma(N)$ of $\mathrm{SL}(n,\mathbb{Z})$ the constants appearing in the estimations in terms of the level N.

6.3 Low lying zeros of L-functions

First we recall the density conjecture from [ILS]. Let $\mathcal{F}$ be a family of automorphic forms (needs to be specified). To any $f \in \mathcal{F}$ there is an associated L-function

$$L(s,f) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}.$$

Assume that the completed L-function $\Lambda(s,f) = L_\infty(s,f)L(s,f)$ is entire and satisfies a functional equation of the type

$$\Lambda(s,f) = \varepsilon L(1-s,f),$$

where $\varepsilon = \pm 1$. Also assume that the Riemann hypothesis holds for each $L(s,f)$ with $f \in \mathcal{F}$. Denote the nontrivial zeros of $L(s,f)$ by

$$\rho_f = \frac{1}{2} + i\gamma_f.$$

Let c_f denote the analytic conductor of f. Let $\phi \in \mathcal{S}(\mathbb{R})$ be even and assume that $\hat{\phi}$ has compact support. Define the density of low lying zeros by

$$D(f;\phi) = \sum_{\gamma_f} \phi\left(\gamma_f \log c_f\right).$$

For $Q \in \mathbb{R}^+$ denote by $\mathcal{F}(Q)$ one of the following sets

$$\{f \in \mathcal{F} \colon c_f = Q\}, \quad \{f \in \mathcal{F} \colon c_f \leq Q\}$$

and consider the average

$$E(\mathcal{F}(Q); \phi) = \frac{1}{|\mathcal{F}(Q)|} \sum_{f \in \mathcal{F}(Q)} D(f; \phi).$$

Assume that $\mathcal{F}$ has plenty of independent forms so that $|\mathcal{F}(Q)| \to \infty$ as $Q \to \infty$. Then the problem is to understand the limit of $E(\mathcal{F}(Q); \phi)$ as $Q \to \infty$. By the Katz-Sarnak philosophy there should exist a density $W(\mathcal{F})$, which is explicitely given, such that

$$\lim_{Q \to \infty} E(\mathcal{F}(Q), \phi) = \int_{\mathbb{R}} \phi(x) W(\mathcal{F})(x) \, dx,$$

More precisely, $W(\mathcal{F})$ should be one of 4 families according to the type of symmetry of the family (unitary, symplectic, odd or even orthogonal). For details see [ILS, p. 57]. By the results of Iwaniec, Luo, and Sarnak [ILS], families of modular forms for $GL(2)$ obey the expected laws (under the GRH).

In order to deal with families of automorphic forms of GL(N), we will need to know the "Weyl law for Hecke operators", i.e, something like

$$\operatorname{Tr} T_n(\mathcal{F}(Q)) = \delta_n(\mathcal{F}) |\mathcal{F}(\mathbb{Q})| + O\left(|\mathcal{F}(Q)|^{1-\varepsilon} |n^k\right) \tag{6.3.1}$$

for some $\varepsilon > 0$ and $k \in \mathbb{N}$. The $\delta_n(\mathcal{F})$, which could vanish for many n's, governs the distribution of low lying zeros in the family.

It is conceivable that the methods of [LM] can be extended to prove (6.3.1). This, however, requires a better knowledge of the geometric side of the Arthur trace formula, in particular, some progress on the problems mentioned in section 5.3.

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