A SPECTRAL INTERPRETATION OF THE ZEROS OF THE CONSTANT TERM OF CERTAIN EISENSTEIN SERIES

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ABSTRACT. In this paper we consider the constant term $\varphi_K(y,s)$ of the non-normalized Eisenstein series attached to $\mathrm{PSL}(2,\mathcal{O}_K)$, where K is either $\mathbb{Q}$ or an imaginary quadratic field of class number one. The main purpose of this paper is to show that for every $a \geq 1$ the zeros of the Dirichlet series $\varphi_K(a,s)$ admit a spectral interpretation in terms of eigenvalues of a natural self-adjoint operator Δ_a. This implies that, except for at most two real zeros, all zeros of $\varphi_K(a,s)$ are on the critical line, and all zeros are simple. For $K=\mathbb{Q}$ this is due to Lagarias and Suzuki [LS] and Ki [Ki].

0. INTRODUCTION

Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ be the modular group and $\mathbb{H}$ the upper half-plane. We consider the (completed) nonholomorphic Eisenstein series $E^*(z,s)$ for the modular group which is given for $z \in \mathbb{H}$ and $\mathrm{Re}(s) > 1$ by

$$E^*(z,s) := \pi^{-s}\Gamma(s)\left(\frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{y^s}{|mz+n|^{2s}}\right). \tag{0.1}$$

It is well known that for fixed z, $E^*(z,s)$ admits a meromorphic continuation to the whole s-plane, and satisfies the functional equation

$$E^*(z,s) = E^*(z,1-s).$$

Its only singularities are simple poles at $s=1$ and $s=0$. As a function of z, it is invariant under Γ

$$E^*(\gamma z,s) = E^*(z,s), \quad \gamma \in \Gamma.$$

In particular $E^*(z,s)$ is invariant under $z \mapsto z+1$ and so it has a Fourier expansion

$$E^*(x+iy,s) = \sum_{n\in\mathbb{Z}} a_n(y,s)e^{2\pi i x}.$$

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The zeroth Fourier coefficient $\varphi_0(y,s)$ is given by

$$\varphi_0(y,s) = \Lambda(2s)y^s + \Lambda(2s-1)y^{1-s},$$

where $\Lambda(s)$ is the completed zeta function

$$\Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Let $a > 0$. The zeros of $\varphi_0(a,s)$ have been studied by various authors. Hejhal [He, Proposition 5.3(f)] proved that for all $a \geq 1$, the complex zeros of $\varphi_0(a,s)$ are on the critical line $\mathrm{Re}(s) = 1/2$. Lagarias and Suzuki [LS] reproved this result and also determined the occurrence of real zeros. Ki [Ki] proved that all complex zeros are simple. Putting these results together, we have following theorem.

Theorem 0.1. *For each $a \geq 1$ all complex zeros of $\varphi_0(y,s)$ are simple and lie on the critical line $\mathrm{Re}(s) = 1/2$. Moreover there is a critical value $a^* = 4\pi e^{-\gamma} = 7.055...$ such that the following holds:*

1) *For $1 \leq a \leq a^*$ all zeros are on the critical line.*
2) *For $a > a^*$ there are exactly two zeros off the critical line. These are real simple zeros $\rho_a, 1-\rho_a$ with $\rho_a \in (1/2,1)$. The zero ρ_a is a nondecreasing function of a and $\lim_{a\to\infty}\rho_a = 1$.*

The first aim of this paper is to point out that there is actually a spectral interpretation of the zeros of $\varphi_0(a,s)$, which gives another proof of this theorem. For $a > 0$ let Δ_a be the cut-off Laplacian Δ_a introduced by Lax and Phillips [LP]. It acts in the subspace $\mathcal{H}_a \subset L^2(\Gamma\backslash\mathbb{H})$ of all $f \in L^2(\Gamma\backslash\mathbb{H})$ satisfying $\int_0^1 f(x+iy)\,dx = 0$ for almost all $y \geq a$. The cut-off Laplacian Δ_a is a nonnegative self-adjoint operator with pure point spectrum. The spectrum has been studied by Colin de Verdiere [CV]. Let

$$c(s) = \frac{\Lambda(2s-1)}{\Lambda(2s)}, \quad s \in \mathbb{C}. \tag{0.2}$$

The following theorem is an immediate consequence of [CV, Théorème 5].

Theorem 0.2. *For every $a > 0$, the spectrum of Δ_a is the union of the cuspidal eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ of Δ and a sequence of eigenvalues*

$$0 < \mu_0(a) < \mu_1(a) < \cdots$$

with the following properties:

1) *Each eigenvalue $\mu_j(a)$ is a decreasing function of a.*
2) *If $a \geq 1$, each eigenvalue $\mu_j(a)$ has multiplicity 1. Moreover $\lim_{a\to\infty}\mu_0(a) = 0$ and if $j \geq 1$, then $\mu_j(a) \geq 1/4$ and $\lim_{a\to\infty}\lambda_j(a) = 1/4$.*
3) *Let $a \geq 1$. Then the map $s \mapsto s(1-s)$ is a bijection between the zeros $\rho \neq 1/2$ of $\varphi_0(a,s)$ and the eigenvalues $\mu_j(a) \neq 1/4$ of Δ_a.*
4) *$1/2$ is a zero of $\varphi_0(a,s)$ for all $a > 0$. There is j with $\mu_j(a) = 1/4$ if and only if $c'(1/2) = -2\log a$.*

This theorem implies that for $a \geq 1$ there is at most one zero of $\varphi_0(a,s)$ which is off the line $\mathrm{Re}(s) = 1/2$. The simplicity of the zeros follows from the consideration of the corresponding eigenfunctions of Δ_a.

The main purpose of this article is to extend these results to the constant term of other Eisenstein series. In the present paper we will consider the Eisenstein series attached to $\mathrm{PSL}(2, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ of class number one, D being a square free positive integer. These are exactly the fields $\mathbb{Q}(\sqrt{-D})$ with $D = 1, 2, 3, 7, 11, 19, 43, 67, 163$ [St]. Let d_K be the discriminant of K and let $\zeta_K(s)$ be the Dedekind zeta function of K. Let

$$\Lambda_K(s) = \left(\frac{\sqrt{|d_K|}}{2\pi}\right)^s \Gamma(s)\zeta_K(s) \tag{0.3}$$

be the completed zeta function. Then $\Lambda_K(s)$ satisfies the functional equation $\Lambda_K(s) = \Lambda_K(1-s)$. For $a > 0$ let

$$\varphi_K(a,s) := a^s\Lambda_K(s) + a^{2-s}\Lambda_K(s-1). \tag{0.4}$$

Note that $\varphi_K(a,s)$ is a Dirichlet series which satisfies the functional equation $\varphi_K(a,s) = \varphi_K(a, 2-s)$. Let $\xi_K(s) = s(s-1)\Lambda_K(s)$. Then $\xi_K(s)$ is entire. Put

$$a_K^* := \exp\left(1 + \frac{\xi_K'(1)}{\xi_K(1)}\right).$$

Then our main result is the following theorem.

Theorem 0.3. *Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of class number one. Then for each $a \geq 1$ all complex zeros of $\varphi_K(a,s)$ are simple and lie on the line* $\mathrm{Re}(s) = 1$. *Moreover*

1) *For $a > \max\{a_K^*, 1\}$ there are exactly two zeros off the critical line. These are simple zeros $\rho_a, 2-\rho_a$ with $\rho_a \in (1,2)$. The zero ρ_a is a nondecreasing function of a, and $\lim_{a\to\infty} \rho_a = 2$.*
2) *If $1 \leq a_K^*$ and $1 \leq a \leq a_K^*$, all zeros of $\varphi_K(a,s)$ are on the critical line.*

To prove Theorem 0.3, we extend Colin de Verdiere's Theorem [CV, Théorème 5] to our setting. Let $\mathcal{O}_K$ be the ring of integers of K and let $\Gamma = \mathrm{PSL}(2, \mathcal{O}_K)$. Then Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ which acts properly discontinuously on the 3-dimensional hyperbolic space $\mathbb{H}^3$. The quotient $\Gamma\backslash\mathbb{H}^3$ has finite volume and a single cups at ∞. The function $\varphi_K(y,s)$ appears to be the zeroth Fourier coefficient of the modified Eisenstein series attached to the cusp $\kappa = \infty$. Let Δ_a be the corresponding cut-off Laplacian on $\Gamma\backslash\mathbb{H}^3$. Then we generalize Theorem 0.2 to this setting. As above, this leads to a spectral interpretation of the zeros of $\varphi_K(a,s)$, and we deduce Theorem 0.3 from this spectral interpretation.

The constant a^*_K can be computed using the Kronecker limit formula [La, p. 273] and the Chowla-Selberg formula [SC, (2), p.110]. For example, we get

$$a^*_{\mathbb{Q}(i)} = \frac{4\pi^2 e^{2+\gamma}}{\Gamma(1/4)^4} \approx 3.00681, \quad a^*_{\mathbb{Q}(\sqrt{-2})} = \frac{8\pi^2 e^{2+\gamma}}{\left(\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)\right)^2} \approx 3.2581.$$

Thus $a^*_{\mathbb{Q}(\sqrt{-D})} > 1$ for $D = 1, 2$ and therefore, in the range $1 \le a \le a^*_{\mathbb{Q}(\sqrt{-D})}$ all zeros of $\varphi_{\mathbb{Q}(\sqrt{-D})}(a, s)$ are on the line $\mathrm{Re}(s) = 1$.

The method used to prove Theorem 0.3 can be extended so that an arbitrary number field K of class number one can be treated. The underlying global Riemannian symmetric space is $X = \mathbb{H}^{r_1} \times (\mathbb{H}^3)^{r_2}$, where r_1 is the number of real places and r_2 the number of pairs of complex conjugate places of K. The structure of $\Gamma\backslash X$ is slightly more complicated, however the proof of the corresponding statement is completely analogous. More generally, it seems to be possible to prove similar results for the constant terms of rank one cuspidal Eisenstein series attached to $\mathrm{PSL}(n, \mathbb{Z})$.

1. The cut-off Laplacian

The cut-off Laplacian can be defined for every rank one locally symmetric space. In the present paper we are dealing only with hyperbolic manifolds of dimension 2 and 3. So we discuss only the case of a hyperbolic 3-manifold. The general case is similar.

Let

$$\mathbb{H}^3 = \{(x_1, x_2, y) \in \mathbb{R}^3 \colon y > 0\}$$

be the hyperbolic 3-space with its hyperbolic metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dy^2}{y^2}.$$

The hyperbolic Laplacian Δ is given by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial}{\partial y}.$$

If we regard $\mathbb{H}^3$ as the set of all quaternions $z + yj$ with $z \in \mathbb{C}$ and $y > 0$, then $G = \mathrm{PSL}(2, \mathbb{C})$ is the group of all orientation preserving isometries of $\mathbb{H}^3$. It acts by linear fractional transformations, i.e., for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$,

$$\gamma(w) = (aw + b)(cw + d)^{-1}, \quad w \in \mathbb{H}^3.$$

Let

$$B(\mathbb{C}) := \left\{\begin{pmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{C}^*,\ z \in \mathbb{C}\right\}/\{\pm \mathrm{Id}\}, \quad N(\mathbb{C}) := \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}\right\}.$$

Let $\Gamma \subset G$ be a discrete subgroup of finite co-volume. Let $\kappa_1 = \infty, \kappa_2, ..., \kappa_m \in \mathbb{P}^1(\mathbb{C})$ be a complete set of Γ-inequivalent cusps, and let Γ_i be the stabilizer of κ_i in Γ. Choose $\sigma_i \in G$ such that $\sigma_i(\kappa_i) = \infty$, $i = 1, ..., m$. Then

$$\sigma_i \Gamma_i \sigma_i^{-1} \cap N(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in L_i \right\}, \tag{1.1}$$

where L_i is a lattice in $\mathbb{C}$ [EGM, Theorem 2.1.8]. Note that for $D \neq 1, 3$ the intersection actually coincides with $\sigma_i \Gamma_i \sigma_i^{-1}$. Choose closed fundamental sets $\mathcal{P}_i$ for the action of $\sigma_i \Gamma_i \sigma_i^{-1}$ on $\mathbb{P}^1(\mathbb{C}) \setminus \{\infty\} = \mathbb{C}$. For $T > 0$ define

$$\tilde{\mathcal{F}}_i(T) := \left\{ (z, y) \in \mathbb{H}^3 \colon z \in \mathcal{P}_i, \;\; y \geq T \right\}.$$

Let $\mathcal{F}_i(T) = \sigma_i^{-1}(\tilde{\mathcal{F}}_i(T))$. There exists $T_i > 0$ such that any two points in $\mathcal{F}_i(T_i)$ are Γ-equivalent if and only if they are Γ_i-equivalent. For such a choice of $T_1, ..., T_m$ there exists a compact subset $\mathcal{F}_0 \subset \mathbb{H}^3$ such that

$$\mathcal{F} := \mathcal{F}_0 \cup \mathcal{F}_1(T_1) \cup \cdots \cup \mathcal{F}_m(T_m) \tag{1.2}$$

is a fundamental domain for Γ [EGM, Proposition 2.3.9]. Let

$$b := \max\{T_1, ..., T_m\}. \tag{1.3}$$

By $C_c^\infty(\Gamma \backslash \mathbb{H}^3)$ we denote the space of Γ-invariant C^∞-functions on $\mathbb{H}^3$ with compact support in $\mathcal{F}$. For $f \in C_c^\infty(\Gamma \backslash \mathbb{H}^3)$ let

$$\| f \|^2 = \int_{\mathcal{F}} |f(x_1, x_2, y)|^2 \, \frac{dx_1 dx_2 dy}{y^3},$$

and let $L^2(\Gamma \backslash \mathbb{H}^3)$ be the completion of $C_c^\infty(\Gamma \backslash \mathbb{H}^3)$ with respect to this norm. Similarly let

$$\| df \|^2 = \int_{\mathcal{F}} df \wedge * \overline{df} = \int_{\mathcal{F}} \| df(x_1, x_2, y) \|^2 \; \frac{dx_1 dx_2 dy}{y}.$$

Let $H^1(\Gamma \backslash \mathbb{H}^3)$ denote the completion of $C_c^\infty(\Gamma \backslash \mathbb{H}^3)$ with respect to the norm

$$\| f \|_1^2 := \| f \|^2 + \| df \|^2 .$$

Denote by $|\mathcal{P}_i|$ the Euclidean area of the fundamental domain $\mathcal{P}_i \subset \mathbb{C}$ of $\sigma_i \Gamma_i \sigma_i^{-1}$. For $f \in L^2(\Gamma \backslash \mathbb{H}^3)$ let

$$f_{j,0}(y) = \frac{1}{|\mathcal{P}_i|} \int_{\mathcal{P}_i} f(\sigma_i^{-1}(x_1, x_2, y)) \, dx_1 dx_2$$

be the zeroth coefficient of the Fourier expansion of f at the cusp κ_j. Note that for $f \in H^1(\Gamma \backslash \mathbb{H}^3)$, each $f_{j,0}$ belongs to $H^1(\mathbb{R}^+)$ and therefore, each $f_{j,0}$ is a continuous function on $\mathbb{R}^+$. For $a > 0$ let

$$H_a^1(\Gamma \backslash \mathbb{H}^3) := \left\{ f \in H^1(\Gamma \backslash \mathbb{H}^3) \colon f_{j,0}(y) = 0 \;\; \text{f}or\ y \geq a, \; j = 1, ..., m \right\}.$$

Then $H_a^1(\Gamma \backslash \mathbb{H}^3)$ is a closed subspace of $H^1(\Gamma \backslash \mathbb{H}^3)$. Hence the quadratic form

$$q_a(f) = \| df \|^2, \quad f \in H_a^1(\Gamma \backslash \mathbb{H}^3),$$

is closed. Let Δ_a denote the self-adjoint operator which represents the quadratic form q_a. It acts in the Hilbert space $\mathcal{H}_a$ which is the closure of $H_a^1(\Gamma \backslash \mathbb{H}^3)$ in $L^2(\Gamma \backslash \mathbb{H}^3)$. By definition, Δ_a is nonnegative. Its domain can be described as follows. Let $\psi_{j,a} \in \mathcal{D}'(\Gamma\backslash\mathbb{H}^3)$ be defined by

$$\psi_{j,a}(f) := f_{j,0}(a), \quad f \in C_c^\infty(\Gamma\backslash\mathbb{H}^3).$$

Let $b > 0$ be defined by (1.3).

Lemma 1.1. *Let $a \geq b$. Then the domain of Δ_a consists of all $f \in H_a^1(\Gamma\backslash\mathbb{H}^3)$ for which there exist $C_1, ..., C_m \in \mathbb{C}$ such that*

$$\Delta f - \sum_{j=1}^{m} C_j \psi_{j,a} \in L^2(\Gamma\backslash\mathbb{H}^3). \tag{1.4}$$

The proof of the lemma is analogous to the proof of Théorèm 1 in [CV]. Let $f \in \mathrm{dom}(\Delta_a)$. By the lemma, there exist $C_1, ..., C_m \in \mathbb{C}$ such that (1.4) holds. Then $\Delta_a f$ is given by

$$\Delta_a f = \Delta f - \sum_{j=1}^{m} C_j \psi_{j,a}. \tag{1.5}$$

Furthermore, we have

Lemma 1.2. Δ_a *has a compact resolvent.*

Proof. The proof is a simple extension of the argument in [LP, p.206]. □

So the spectrum of Δ_a consists of a sequence of eigenvalues $0 \leq \lambda_1(a) \leq \lambda_2(a) \leq \cdots \to \infty$ with finite multiplicities. To describe the eigenvalues and eigenfunctions of Δ_a more explicitely, we need to consider the Eisenstein series. Let $\sigma_j \in G$, $j = 1, ..., m$, be as above. For each cusp κ_i, the Eisenstein series $E_i(w, s)$ attached to κ_i is defined to be

$$E_i(w, s) := \sum_{\gamma \in \Gamma_i\backslash\Gamma} y(\sigma_i \gamma w)^s, \quad \mathrm{Re}(s) > 2,$$

where $y(\sigma_i\gamma w)$ denotes the y-component of $\sigma_i\gamma w$. Selberg has shown that $E_i(w, s)$ can be meromorphically continued in s to the whole complex plane $\mathbb{C}$, and it is an automorphic eigenfunction of Δ with

$$\Delta E_i(w, s) = s(2 - s)E_i(w, s).$$

Since $E_i(w, s)$ is Γ-invariant, it is invariant under the stabilizer Γ_j of the cusp κ_j and therefore, admits a Fourier expansion at κ_j. The constant Fourier coefficient is given by

$$\delta_{ij} y^s + C_{ij}(s) y^{2-s}. \tag{1.6}$$

Put

$$C(s) := \left(C_{ij}(s)\right)_{i,j=1,\ldots,m}.$$

This is the so called "scattering matrix". The scattering matrix and the Eisenstein series satisfy the following system of functional equations.

$$\begin{aligned} &C(s)C(2-s) = \mathrm{Id}, \\ &E_i(w,s) = \sum_{j=1}^{m} C_{ij}(s)E_j(w,2-s), \quad i=1,...,m. \end{aligned} \tag{1.7}$$

Now recall that a square integrable eigenfunction f of Δ is called cuspidal, if the zeroth Fourier coefficient $f_{j,0}$ of f at the cusp κ_j vanishes for all $j=1,...,m$. Denote by $S(\lambda;\Gamma)$ the space of cuspidal eigenfunctions of Δ with eigenvalue λ. A function $f \in C^0(\Gamma\backslash\mathbb{H}^3)$ is called of moderate growth, if there exists $R \in \mathbb{R}^+$ such that

$$f(\sigma_i^{-1}(x_1,x_2,y)) = O(y^R), \quad for \ \ (x_1,x_2,y) \in \mathcal{F}_i(b), \ i=1,...,m.$$

It follows from the Fourier expansion in the cusps [EGM, Section 3.3.3] that a cuspidal eigenfunction f of Δ is rapidly decreasing in each cusp. Therefore for $\psi \in \mathcal{E}(\lambda,\Gamma)$ and $\varphi \in S(\lambda,\Gamma)$, the inner product $\langle\psi,\varphi\rangle$ is well defined. For $\lambda \in \mathbb{C}$ let

$$\mathcal{E}(\lambda;\Gamma) := \left\{\psi \in L^2_{\mathrm{loc}}(\Gamma\backslash\mathbb{H}^3)\colon \Delta\psi = \lambda\psi, \ \psi \ \textit{ is of moderate growth}, \ \psi \perp S(\lambda;\Gamma)\right\}. \tag{1.8}$$

Lemma 1.3. *Let m be the number of cusps of $\Gamma\backslash\mathbb{H}^3$. Then*

1) *For every $\lambda \in \mathbb{C}$ we have $\dim\mathcal{E}(\lambda;\Gamma) \leq m$.*
2) *Suppose that $\lambda = s(2-s)$, where $\mathrm{Re}(s) \geq 1$, $s \neq 1$, and s is not a pole of any Eisenstein series. Then $E_1(w,s),...,E_m(w,s)$ is a basis of $\mathcal{E}(\lambda;\Gamma)$.*

Proof. The proof is analogous to the proof of Satz 11 in [Ma, p.171]. Let $\phi,\psi \in \mathcal{E}(\lambda;\Gamma)$ and $\lambda = s(2-s)$. The zeroth Fourier coefficient of ϕ, ψ at the j-th cusp is given by

$$\phi_{0,j}(y) = a_j y^s + b_j y^{2-s}, \quad \psi_{0,j}(y) = c_j y^s + d_j y^{2-s}, \ s \neq 1,$$

and for $s=1$ by

$$\phi_{0,j}(y) = a_j y + b_j y\log y, \quad \psi_{0,j}(y) = c_j y + d_j y\log y.$$

Let $\chi_{j,[a,\infty)}$ denote the characteristic function of $\Gamma_i\backslash\mathcal{F}_i(a)$ in $\Gamma\backslash\mathbb{H}^3$. Let

$$\phi_a = \phi - \sum_{j=1}^{m} \chi_{j,[a,\infty)}\phi_{0,j}, \quad \psi_a = \psi - \sum_{j=1}^{m} \chi_{j,[a,\infty)}\psi_{0,j}$$

be the truncations of ϕ and ψ, respectively, at level $a \geq b$. Let $s \neq 1$. Integrating by parts, we get

$$\begin{aligned} 0 = \int_{\Gamma\backslash\mathbb{H}^3} (\phi_a\Delta\psi_a - \psi_a\Delta\phi_a)\, dvol &= \sum_{j=1}^{m}\left(\psi_{0,j}(y)\frac{\partial\phi_{0,j}}{\partial y}(y) - \phi_{0,j}(y)\frac{\partial\psi_{0,j}}{\partial y}(y)\right)\Bigg|_{y=a} \\ &= (2-2s)\sum_{j=1}^{m}(a_jd_j - b_jc_j). \end{aligned} \tag{1.9}$$

For $s=1$ the right hand side equals $\sum_{j=1}^{m}(a_jd_j - b_jc_j)$.

An element of $\mathcal{E}(\lambda)$ is uniquely determined by its zeroth Fourier coefficients. Define $i\colon \mathcal{E}(\lambda) \to \mathbb{C}^{2m}$ by $i(\phi) := (a_1, b_1, ..., a_m, b_m)$, where a_j, b_j are the 0-th Fourier coefficients of ϕ at the j-th cusp. Then i is an embedding. Let

$$I(x, y) := \sum_{j=1}^{m} (x_{2j-1}y_{2j} - x_{2j}y_{2j-1})$$

be the standard symplectic form on $\mathbb{C}^{2m}$. Then by (1.9) and the corresponding statement for $s = 1$, $\mathcal{E}(\lambda)$ is an isotropic subspace for I. Hence $\dim \mathcal{E}(\lambda) \leq m$.

If $s \neq 1$ and s is not a pole of $E_j(w, s)$, $j = 1, ..., m$, the Eisenstein series have linearly independent 0-th Fourier coefficients given by (1.6). This implies that $E_1(w, s), ..., E_m(w, s)$ are linearly independent and hence, form a basis of $\mathcal{E}(\lambda, \Gamma)$, where $\lambda = s(2 - s)$. $\square$

We are now ready to describe the spectrum of Δ_a. Let $0 = \Lambda_0 < \Lambda_1 < \cdots < \Lambda_N < 1$ be the eigenvalues of Δ with non-cuspidal eigenfunctions. Then Λ_j has the form $\Lambda_j = \sigma_j(2 - \sigma_j)$ with $\sigma_j \in (1, 2]$ and σ_j is a simple pole of some Eisenstein series $E_k(w, s)$. The corresponding residue is an eigenfunction of Δ with eigenvalue Λ_j [EGM, Proposition 6.2.2]. The following theorem is an extension of Theorem 5 in [CV] to the 3-dimensional case.

Theorem 1.4. *The spectrum of Δ_a is the union of the cuspidal eigenvalues $(\lambda_i)_i$ of Δ and a sequence $0 < \mu_0(a) \leq \mu_1(a) \leq \cdots$ with the following properties:*

1) *Each $\mu_j(a)$ is a decreasing function of a,*
2) *Let $a \geq b$. Then each $\mu_j(a)$ has multiplicity $\leq m$, and the map $s \mapsto s(2 - s)$ is a bijection between the eigenvalues $\mu_j(a) \notin \{1, \Lambda_0, ..., \Lambda_N\}$ and the zeros of $\rho \in \mathbb{C} \setminus \{1, \sigma_0, ..., \sigma_N\}$ of $\varphi(s) := \det(C(s) + a^{2s-2}\,\mathrm{Id})$.*

Proof. Let $f \in L^2(\Gamma\backslash\mathbb{H}^3)$ be a cuspidal eigenfunction of Δ. Then f belongs to $H^1_a(\Gamma\backslash\mathbb{H}^3)$ and Δf is square integrable. Hence by (1.4) and (1.5), $f \in \mathrm{dom}(\Delta_a)$ and $\Delta_a f = \Delta f$. Thus all cuspidal eigenfunctions of Δ are also eigenfunctions of Δ_a with the same eigenvalues.

For 1) observe that for $a \leq a'$ we have

$$H^1_a(\Gamma\backslash\mathbb{H}^3) \subset H^1_{a'}(\Gamma\backslash\mathbb{H}^3).$$

Since Δ_a is a non-negative self-adjoint operator with compact resolvent, 1) follows from the mini-max characterization of its eigenvalues.

2) Let Φ be an eigenfunction of Δ_a with eigenvalue $\mu_j(a) = s_j(2 - s_j)$ and suppose that $\Phi \perp S(\mu_j(a); \Gamma)$. Let $\Phi_{i,0}$ be the zeroth Fourier coefficient of Φ at the cusp κ_i. There exists $1 \leq j \leq m$ such that $\Phi_{j,0} \not\equiv 0$. Let $\hat{\Phi}_{i,0}$ denote the analytic continuation of $\Phi_{i,0}$ from $(0, a)$ to $\mathbb{R}^+$. Put

$$\hat{\Phi}(w) := \Phi(w) + \sum_{i=1}^{m} \chi_{i,[a,\infty)} \hat{\Phi}_{i,0}.$$

Then $\hat{\Phi} \in C^\infty(\Gamma\backslash\mathbb{H}^3)$ and $\Delta\hat{\Phi} = \mu_j(a)\hat{\Phi}$. Moreover $\hat{\Phi}$ is of moderate growth and $\hat{\Phi} \perp S(\mu_j(a), \Gamma)$. Therefore $\hat{\Phi} \in \mathcal{E}(\mu_j(a); \Gamma)$. By Lemma 1.3, 1), it follows that the multiplicity of $\mu_j(a)$ is bounded by m.

Next suppose that $\mu_j(a) = s_j(2 - s_j) \notin \{1, \Lambda_0, ..., \Lambda_N\}$. We can assume that $\mathrm{Re}(s_j) \geq 1$. As explained above, s_j is not a pole of any Eisenstein series $E_k(w, s)$, $k = 1, ..., m$. Therefore, by Lemma 1.3 there exist $c_1, ..., c_m \in \mathbb{C}$ such that

$$\hat{\Phi}(w) = \sum_{i=1}^{m} c_i E_i(w, s_j).$$

By construction, the zeroth Fourier coefficient $\hat{\Phi}_{l,0}(y)$ of $\hat{\Phi}(w)$ at κ_l vanishes at $y = a$ for all $l = 1, ..., m$. By (1.6) this implies

$$\sum_{i=1}^{m} c_i \left(\delta_{il} a^{s_j} + C_{il}(s_j) a^{2-s_j}\right) = 0, \quad l = 1, ..., m.$$

Hence $\det(C(s_j) + a^{2s_j-2}\,\mathrm{Id}) = 0$.

Now assume that $\det(C(s_j) + a^{2s_j-2}\,\mathrm{Id}) = 0$, $\mathrm{Re}(s_j) \geq 1$, and $s_j \notin \{1, \sigma_0, ..., \sigma_N\}$. Then s_j is not a pole of any Eisenstein series. Let $0 \neq \psi \in \mathbb{C}^m$ such that

$$C(s_j)\psi = -a^{2s_j-2}\psi. \tag{1.10}$$

Let $\psi = (c_1, ..., c_m)$ and set

$$\hat{\Phi}(w) := \sum_{k=1}^{m} c_k E_k(w, s_j).$$

By (1.6) and (1.10), the 0-th Fourier coefficient of $\hat{\Phi}(w)$ at the cusp κ_l is given by

$$\sum_{k=1}^{m} c_k \left(\delta_{kl} y^{s_j} + C_{kl}(s_j) y^{2-s_j}\right) = c_l \left(y^{s_j} - a^{2s_j-2} y^{2-s_j}\right). \tag{1.11}$$

Therefore, the 0-th Fourier coefficient $\hat{\Phi}_{l,0}(y)$ vanishes at $y = a$ for all $l = 1, ..., m$. Put

$$\Phi(w) = \hat{\Phi}(w) - \sum_{i=1}^{m} \chi_{i,[a,\infty)} \hat{\Phi}_{l,0}(y).$$

Then $\Phi \in H^1_a(\Gamma\backslash\mathbb{H})$ and it follows from the description of the domain of Δ_a that $\Phi \in \mathrm{dom}(\Delta_a)$ and $\Delta_a\Phi = s_j(2 - s_j)\Phi$. □

Now assume that $\Gamma\backslash\mathbb{H}^3$ has a single cusp $\kappa = \infty$. Then there is a single Eisenstein series $E(w, s)$, which is given by

$$E(w, s) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} y(\gamma w)^s, \quad \mathrm{Re}(s) > 2.$$

The zeroth Fourier coefficient of $E(w, s)$ at ∞ equals

$$\varphi_0(y, s) := y^s + c(s) y^{2-s}, \tag{1.12}$$

where $c(s)$ is a meromorphic function of $s \in \mathbb{C}$. The following facts are well known [EGM, pp.243-245] . The poles of $E(w, s)$ in the half-plane $\mathrm{Re}(s) > 1$ are contained in the interval $(1, 2]$ and are all simple. The residue of $E(w, s)$ at a pole $\sigma \in (1, 2]$ is a square integrable eigenfunction of Δ with eigenvalue $\sigma(2 - \sigma)$, which is non-cuspidal. Moreover all non-cuspidal eigenfunctions of Δ are obtained in this way. Thus the non-cuspidal eigenvalues have multiplicity one. So in this case we get the following refinement of Theorem 1.4.

Theorem 1.5. *Assume that $\Gamma\backslash\mathbb{H}^3$ has a single cusp. Then the spectrum of Δ_a is the union of the cuspidal eigenvalues $(\lambda_i)_i$ of Δ and a sequence $0 < \mu_0(a) < \mu_1(a) < \cdots$ with the following properties:*

1) *Each $\mu_j(a)$ is a decreasing function of a;*
2) *Let $a \geq b$. Then each $\mu_j(a)$ has multiplicity 1 and the map $s \mapsto s(2 - s)$ is a bijection between the zeros $\rho \neq 1$ of $\varphi_0(a, s)$ and the eigenvalues $\mu_j(a) \neq 1$ of Δ_a.*
3) *There is j with $\mu_j(a) = 1$ if and only if $c(1) = -1$ and $c'(1) = -2\log a$.*
4) *Let $0 = \Lambda_0 < \Lambda_1 < \cdots < \Lambda_N < 1$ be the eigenvalues of Δ with non-cuspidal eigenfunctions. If $a \geq b$, then*
$$0 = \Lambda_0 < \mu_0(a) < \Lambda_1 < \mu_1(a) < \cdots < \Lambda_N < \mu_N(a).$$
Moreover,

$$\lim_{a\to\infty} \mu_j(a) = \begin{cases} \Lambda_j & , \ 0 \leq j \leq N; \\ 1 & , \ j > N. \end{cases} \tag{1.13}$$

Proof. 1) follows from Theorem 1.4. If $s \notin \{1, \sigma_0, ..., \sigma_N\}$, then 2) follows also from 2) of Theorem 1.4. Now suppose that s_0 is a pole of $E(w, s)$ in the half-plane $\mathrm{Re}(s) \geq 1$. Then the residue ψ of $E(w, s)$ at s_0 is an eigenfunction of Δ with non-vanishing 0-th Fourier coefficient. Moreover $\psi \perp S(\lambda, \Gamma)$. Therefore $\psi \in \mathcal{E}(\lambda, \Gamma)$, where $\lambda = s_0(2 - s_0)$. By Lemma 1.3 it follows that $\dim \mathcal{E}(\lambda, \Gamma) = 1$. Moreover the constant term $\psi_0(y)$ of ψ has the form $\psi_0(y) = cy^{2-s_0}$. Especially, it never vanishes on $\mathbb{R}^+$. On the other hand, if Φ were an eigenfunction of Δ_a with eigenvalue λ, then the corresponding eigenfunction $\hat{\Phi} \in \mathcal{E}(\lambda, \Gamma)$, constructed in the proof of Theorem 1.4 would have a constant term which vanishes at $y = a$. Hence the eigenvalues Λ_j can not be eigenvalues of Δ_a. This proves 2)

For 3) we note that $c(1)^2 = 1$ by the functional equation (1.7). Thus $c(1) = \pm 1$. If $c(1) = -1$, then $E(w, 1) \equiv 0$. Put $\psi(w) := \frac{d}{ds} E(w, s)\big|_{s=1}$. Then $\Delta\psi = \psi$ and the zeroth Fourier coefficient of ψ is given by
$$\psi_0(y) = (2\log y + c'(1))y.$$
Suppose that $c'(1) = -2\log a$. Put $\hat{\psi}_a(w) := \psi(w) - \chi_{[a,\infty)}\psi_0(y(w))$. Then $\hat{\psi}_a$ is in the domain of Δ_a and $\Delta_a\hat{\psi}_a = \hat{\psi}_a$.

For the other direction suppose that $\hat{\psi}$ is an eigenfunction of Δ_a with eigenvalue 1 and $\hat{\psi} \perp S(1, \Gamma)$. Let $\hat{\psi}_0(y)$ be the 0-th Fourier coefficient of $\hat{\psi}$. Extend $\hat{\psi}_0(y)$ in the obvious way from (b,a) to a smooth function ψ_0 on (b,∞). Set

$$\psi := \hat{\psi} + \chi_[a,\infty)\psi_0.$$

Then ψ is smooth, of moderate growth, satisfies $\Delta\psi = \psi$ and $\psi \perp S(1,\Gamma)$. Thus $\psi \in \mathcal{E}(1,\Gamma)$. Therefore by Lemma 1.3 it follows that $\dim \mathcal{E}(1,\Gamma) = 1$. On the other hand, we have $0 \neq \frac{d}{ds}E(s,w)\big|_{s=1} \in \mathcal{E}(1,\Gamma)$. Therefore it follows that there exists $c \neq 0$ such that $\psi(w) = c\frac{d}{ds}E(w,s)\big|_{s=1}$. Comparing the constant terms, it follows that $c'(1) = -2\log a$.

4) follows from the mini-max principle and the fact that $\cup_{a\geq b} H^1_a(\Gamma\backslash\mathbb{H}^3)$ is dense in $H^1(\Gamma\backslash\mathbb{H}^3)$. □

2. Hyperbolic surfaces of finite area

In this section we prove Theorem 0.2 and deduce Theorem 0.1 from it.

Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ be the modular group. Then $\Gamma\backslash\mathbb{H}$ has a single cusp $\kappa = \infty$. As fundamental domain we take the standard domain

$$F := \{z \in \mathbb{H} \colon |\operatorname{Re}(z)| < 1/2,\ |z| > 1\}.$$

So we can take $b = 1$. The Eisenstein series attached to the cusp ∞ is given by

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^s = \sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}}, \quad \operatorname{Re}(s) > 1.$$

The constant term of $E(z,s)$ equals

$$y^s + c(s)y^{1-s}, \tag{2.1}$$

where $c(s)$ is the meromorphic function defined by (0.2).

Proof of Theorem 0.2: Since the the zeros of the completed zeta function $\Lambda(s)$ are all contained in the strip $0 < \operatorname{Re}(s) < 1$, it follows that $\Lambda(2s-1)$ and $\Lambda(2s)$ have no common zeros. This implies that the zeros of

$$\varphi_0(a,s) = \Lambda(2s)a^s + \Lambda(2s-1)a^{1-s}$$

coincide with the zeros of $a^s + c(s)a^{1-s}$. Now note that $\Lambda(s)$ has poles of order 1 at $s = 1, 0$. The residue at $s = 1$ is 1 and the residue at $s = 0$ is -1. Hence

$$c(1/2) = \lim_{s\to 1/2} \frac{\Lambda(2s-1)}{\Lambda(2s)} = -1. \tag{2.2}$$

Now the statements 1), 3) and 4) follow immediately from [CV, Théorèm 5]. For 2) we use that by Roelcke [Ro], the smallest positive eigenvalue λ_1 of Δ on $\Gamma\backslash\mathbb{H}$ satisfies $\lambda_1 > 1/4$. Then 2) follows from Théorèm 5, (iii), of [CV].

Proof of Theorem 0.1: Let $a \geq 1$ and let $\rho \neq 1/2$ be a zero of $\varphi_0(a,s)$. Then by Theorem 0.2, 3), $\rho(1-\rho)$ is an eigenvalue of Δ_a. Hence $\rho(1-\rho)$ is a non-negative real number. If $\rho(1-\rho) > 1/4$, then ρ is a complex zero with $\operatorname{Re}(\rho) = 1/2$. By Theorem 0.2, 2), there exist at most two zeros, ρ_a with $0 < \rho_a(1-\rho_a) < 1/4$. Thus ρ_a and $1-\rho_a$ are zeros and we may assume that $\rho_a \in (1/2, 1)$. Let $a < a'$. Then by 1), we have

$$-(\rho_a - 1/2)^2 = \rho_a(1-\rho_a) - 1/4 \geq \rho_{a'}(1-\rho_{a'}) - 1/4 = -(\rho_{a'} - 1/2)^2.$$

Thus $\rho_a \leq \rho_{a^*}$ and by Theorem 0.2, 2), it follows that $\lim_{a\to\infty} \rho_a = 1$. This proves the second part of 2).

Now consider the eigenvalue 1/4. Let

$$\xi(s) = s(s-1)\Lambda(s).$$

Then we get

$$c'(1/2) = -\frac{c'(1/2)}{c(1/2)} = -4 \lim_{s\to 0}\left(\frac{\Lambda'(s)}{\Lambda(s)} + \frac{1}{s}\right) = -4\left(1 + \frac{\xi'(0)}{\xi(0)}\right).$$

By [Da, pp.80-81] it follows that

$$2\left(1 + \frac{\xi'(0)}{\xi(0)}\right) = -\gamma + \log 4\pi,$$

where γ denotes Euler's constant. Let $a^* := 4\pi e^{-\gamma}$. Then we get

$$c'(1/2) = -2\log a^*. \tag{2.3}$$

By Theorem 0.2, 4), 1/4 is an eigenvalue of Δ_a if and only if $a = a^*$. Since all eigenvalues $\mu_i(a)$ of Δ_a are non-increasing functions of a and have multiplicity one, it follows that $\mu_0(a^*) = 1/4$, $\mu_0(a) < 1/4$, if $a > a^*$, and $\mu_0(a) > 1/4$, if $1 \leq a < a^*$. Since $\mu_0(a) = \rho_a(1-\rho_a)$, this proves 1) and the first part of 2).

It remains to show that all zeros $\rho \neq 1/4$ are simple. Let ρ be a zeros of $\varphi_0(a,s)$ of multiplicity ≥ 2. Then ρ is also a zero of $1 + c(s)a^{1-2s}$ of multiplicity ≥ 2. Thus

$$0 = \frac{d}{ds}\left(1 + c(s)a^{1-2s}\right)\big|_{s=\rho} = a^{1-2s}\left(c'(\rho) - 2c(\rho)\log a\right).$$

and hence, we get

$$2\log a - \frac{c'}{c}(\rho) = 0. \tag{2.4}$$

Let $\chi_{[a,\infty)}$ be the characteristic function of the interval $[a,\infty)$ in $\mathbb{R}^+$. For $a \geq 1$ let

$$\hat{E}(z,s) := E(z,s) - \chi_{[a,\infty)}(y^s + c(s)y^{1-s})$$

be the truncated Eisenstein series. Then $\hat{E}(z,s)$ is square integrable and its norm is given by the Maass-Selberg relation. For $r \in \mathbb{R}$, $r \neq 0$, we have [Se, (7.42')]

$$\| \hat{E}(1/2+ir) \|^2 = 2\log a - \frac{c'}{c}(1/2+ir) + \frac{1}{r}\operatorname{Im}\left(c(1/2+ir)a^{-2ir}\right). \tag{2.5}$$

Let $\rho = 1/2 + ir$, $r \neq 0$, be a zero of $1 + c(s)a^{1-2s}$ of multiplicity ≥ 2. Then

$$c(1/2+ir) = -a^{2ir}.$$

Thus $\operatorname{Im}(c(1/2+ir)a^{-2ir})=0$. This together with (2.4) implies that $\hat{E}(1/2+ir)=0$. On the other hand, the zeroth Fourier coefficient is not identically zero. Hence we arrive at a contradiction.

The case of real zeros is treated similarly. Let $\sigma \in (1/2,1)$ be a zero of $1+c(s)a^{1-2s}$ of multiplicity ≥ 2. In this case the Maass-Selberg relation is

$$\| \hat{E}(\sigma) \|^2 = \frac{a^{2\sigma-1} - |c(\sigma)|^2 a^{1-2\sigma}}{2\sigma-1} + 2c(\sigma)\log a - c'(\sigma) \tag{2.6}$$

[Se, (7.42")]. Since σ is a zero, we have $|c(\sigma)| = a^{2\sigma-1}$. This together with (2.4) implies $\hat{E}(\sigma)=0$, which again leads to a contradiction.

Finally note that by (2.2) and (2.3), $1/2$ is a double zero of $\varphi_0(a^*,s)$. This completes the proof of Theorem 0.1. □

We note that Théorèm 5 of [CV] has an extension to hyperbolic surfaces with several cusps similar to Theorem 1.4. In the case of m cusps the scattering matrix $C(s)$ is an $m\times m$ matrix. In this case we study the zeros of the function

$$\phi(a,s) := \det(a^s + a^{1-s}C(s)). \tag{2.7}$$

For a congruence subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{Z})$ the entries of $C(s)$ consist of ratios of Dirichlet L-series [Hu]. If we multiply (2.7) by the common denominator, we get a certain linear combination of products of Dirichlet L-functions. As example, consider the case of the Hecke group $\Gamma=\Gamma_0(n)$, where $n=p_1\cdots p_r$ is a product of pairwise distinct prime numbers p_i. Then the number of cusps of $\Gamma_0(n)\backslash\mathbb{H}$ is $m=2^r$. The scattering matrix can be described in the following way. For a prime p let

$$N_p(s) := \frac{1}{p^{2s}-1}\begin{pmatrix} p-1 & p^s-p^{1-s} \\ p^s-p^{1-s} & p-1\end{pmatrix}$$

Then

$$C(s) = \frac{\Lambda(2s-1)}{\Lambda(2s)}\left(N_{p_1}(s)\otimes\cdots\otimes N_{p_r}(s)\right)$$

[He]. Suppose that $n=p$ is a prime number. Then we get

$$\begin{aligned}
&\det\left(a^s\,\mathrm{Id} + a^{1-s}C(s)\right)\\
&= a^{2s} + 2a\frac{p-1}{p^{2s}-1}\cdot\frac{\Lambda(2s-1)}{\Lambda(2s)} + a^{2-2s}\frac{p^{2-2s}-1}{p^{2s}-1}\cdot\frac{\Lambda(2s-1)^2}{\Lambda(2s)^2}\\
&= \frac{\prod_{k=1}^2\left(a^s(p^s+(-1)^k)\Lambda(2s) + a^{1-s}(p^{1-s}+(-1)^k)\Lambda(2s-1)\right)}{(p^{2s}-1)\Lambda(2s)^2}.
\end{aligned} \tag{2.8}$$

Note that the numerator and the denominator have no common zeros. Therefore the zeros of the function $\phi(a,s)$ coincide with the zeros of the numerator. Let $a\geq 1$. Then it follows as above that the map $s\mapsto s(1-s)$ induces a bijection between the zeros $\rho\neq 1/2$ of $\phi(a,s)$ and the eigenvalues $\mu_j(a)\neq 1/4$ of Δ_a. This implies that for $a\geq 1$ the complex zeros of

$$a^s(p^s\pm 1)\Lambda(2s) + a^{1-s}(p^{1-s}\pm 1)\Lambda(2s-1)$$

lie all on the line $\mathrm{Re}(s) = 1/2$. As for the real zeros, we observe that $\Gamma_0(p)$ has no residual eigenvalues other than 0 [Iw, Theorem 11.3]. By the mini-max principle it follows that $\mu_1(a) \geq 1/4$. Therefore, there exist at most two real zeros, ρ_a and $1-\rho_a$, with $\rho_a \in (1/2, 1)$. Moreover $\lim_{a\to\infty} \rho_a = 1$.

3. Imaginary quadratic fields

In this section we derive Theorem 0.3 from Theorem 1.5.

Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field, where D is a square free positive integer, and let $\mathcal{O}_K$ be the ring of integers in K. Let $\Gamma_K = \mathrm{PSL}(2, \mathcal{O}_K)$. Then $\Gamma_K \subset \mathrm{PSL}(2, \mathbb{C})$ is a discrete subgroup of finite co-volume. The number of cusps of $\Gamma_K\backslash\mathbb{H}^3$ equals the class number h_K of K. Assume that $h_K = 1$. This is the case if and only if $D = 1, 2, 3, 7, 11, 19, 43, 67, 163$ [St]. Then $\Gamma_K\backslash\mathbb{H}^3$ has a single cusps which we choose to be at $\kappa = \infty$. A fundamental domain $\mathcal{F}_K$ of Γ_K can be constructed as in [EGM, Section 7.3], [Sw, pp. 17-19]. Let F_K be a fundamental domain for the group

$$\mathrm{PSL}(2, \mathcal{O}_K)_\infty = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \lambda \end{pmatrix} \in \mathrm{PSL}(2, \mathcal{O}_K) \right\}.$$

If $\mu, \lambda \in \mathcal{O}_K$ generate $\mathcal{O}_K$ and $\mu \neq 0$, let

$$H(\mu, \lambda) := \{(z, r) \in \mathbb{H}^3 \colon |\mu z + \lambda|^2 + |\mu|^2 r^2 \geq 1\}.$$

This is the set of points $w \in \mathbb{H}^3$ which lie above or on the hemisphere $S(\mu, \lambda)$ in $\mathbb{H}^3$ which is the set of all (z, r) such that $|\mu z + \lambda|^2 + |\mu|^2 r^2 = 1$. This hemisphere has center $(\lambda/\mu, 0)$ and radius $1/|\mu|$. Let

$$B_K := \bigcap H(\mu, \lambda),$$

where the intersection is taken over all $\mu, \lambda \in \mathcal{O}_K$ as above. Then a fundamental domain $\mathcal{F}_K$ of Γ_K is given by

$$\mathcal{F}_K = \{(z, r) \in B_K \colon z \in F_K\}$$

[EGM, Theorem 7.3.4], [Sw, Section 3]. Thus $\mathcal{F}_K$ is the complement in $F_K \times \mathbb{R}^+$ of Euclidean semi-balls with center at $(\lambda/\mu, 0)$ and radius $1/|\mu|$, $\mu \neq 0$. Since $\mu \in \mathcal{O}_K$, it follows that $|\mu| \geq 1$ and therefore, we have

$$F_K \times [1, \infty) \subset \mathcal{F}_K. \tag{3.1}$$

For $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$, the fundamental domain is a hyperbolic pyramid with one vertex at ∞.

Let d_K be the discriminant of K and let $\zeta_K(s)$ be the Dedekind zeta function of K. Let

$$\Lambda_K(s) = \left(\frac{\sqrt{|d_K|}}{2\pi} \right)^s \Gamma(s) \zeta_K(s)$$

be the completed zeta function. Then the constant term of the Eisenstein series $E(w,s)$ is given by

$$E_0(y,s) = y^s + \frac{\Lambda_K(s-1)}{\Lambda_K(s)} y^{2-s} \tag{3.2}$$

[EGM, Section 8.3], [ES], and therefore, the scattering matrix $c_K(s)$ equals

$$c_K(s) = \frac{\Lambda_K(s-1)}{\Lambda_K(s)}. \tag{3.3}$$

The completed zeta function satisfies the functional equation $\Lambda_K(s) = \Lambda_K(1-s)$. All its zeros are contained in the strip $0 < \mathrm{Re}(s) < 1$. This implies that $\Lambda_K(s-1)$ and $\Lambda_K(s)$ have no common zeros. Therefore for $a > 0$, the zeros of

$$\varphi_K(a,s) = a^s \Lambda_K(s-1) + a^{2-s}\Lambda_K(s)$$

coincide with the zeros of $E_0(a,s)$.

We can now apply Theorem 1.5. By (3.1) we can take $b = 1$. Let $a \geq 1$ and let $\rho \neq 1$ be a zero of $\psi_K(a,s)$. Then by Theorem 1.5, 2), $\rho(2-\rho)$ is an eigenvalue of Δ_a. This implies that if $\rho(2-\rho) \geq 1$, then $\rho = 1 + ir$ with $r \in \mathbb{R}$, $r \neq 0$. If $\rho(2-\rho) < 1$, then ρ is real. Thus all complex zeros of $\varphi_K(a,s)$ are on the line $\mathrm{Re}(s) = 1$.

For the real zeros we need to consider the non-cuspidal eigenvalues of Δ. By the spectral resolution of the Laplacian, these eigenvalues are in one-to-one correspondence with the poles of the Eisenstein series $E(w,s)$ in the interval $(1,2]$ [EGM, Proposition 6.2.2]. On the other hand, using the Maass-Selberg relations, it follows that the poles of $E(w,s)$ in $(1,2]$ coincide with the poles of the scattering matrix $c_K(s)$ in $(1,2]$. Since $\Lambda_K(s)$ has no zeros in $\mathrm{Re}(s) > 1$ and the only pole in $\mathrm{Re}(s) > 0$ is a simple pole at $s = 1$, it follows from (3.3) that the only pole of $c_K(s)$ in $\mathrm{Re}(s) > 1$ is a simple pole at $s = 2$ which corresponds to the eigenvalue 0. This shows that Δ has no non-cuspidal eigenvalues in the interval $(0,1)$. By Theorem 1.5, 4), it follows that $\varphi_K(a,s)$ has at most two real zeros ρ_a and $2-\rho_a$ with $\rho_a \in (1,2)$. Let $a < a'$. Then by Theorem 1.5, 1), we have

$$-(\rho_a - 1)^2 = \rho_a(2-\rho_a) - 1 \geq \rho_{a'}(2-\rho_{a'}) - 1 = -(\rho_{a'} - 1)^2.$$

This implies $\rho_a \leq \rho_{a'}$, and by 4) it follows that $\lim_{a\to\infty} \rho_a = 1$.

Next we consider the eigenvalue 1. Recall that $\Lambda_K(s)$ has simple poles at $s = 0$ and $s = 1$. Using the functional equation, we get $\mathrm{R}es_{s=1}\Lambda_K(s) = -\mathrm{R}es_{s=0}\Lambda_K(s)$. This implies $c_K(1) = -1$. By Theorem 1.5, 3), 1 occurs as an eigenvalue $\mu_j(a)$ of Δ_a if and only if $c_K'(1) = -2\log a$. Let

$$\xi_K(s) = s(s-1)\Lambda_K(s).$$

Then

$$c_K'(1) = -\frac{c_K'(1)}{c_K(1)} = \lim_{s\to 1}\left(\frac{\Lambda_K'(s)}{\Lambda_K(s)} - \frac{\Lambda_K'(s-1)}{\Lambda_K(s-1)}\right) = -2\left(1 + \frac{\xi_K'(0)}{\xi_K(0)}\right).$$

Let

$$a_K^* := \exp\left(1 + \frac{\xi_K'(0)}{\xi_K(0)}\right).$$

Then by the above, there is j such that $\mu_j(a) = 1$ if and only if $a = a_K^*$. Assume that $a_K^* \geq 1$. Using that the eigenvalues $\mu_j(a)$ of Δ_a are non-increasing and have multiplicity one, we get $\mu_0(a_K^*) = 1$, $\mu_0(a) < 1$, if $a > a_K^*$, and $\mu_0(a) > 1$, if $a < a_K^*$. If $a_K^* < 1$, then $\mu_0(a) < 1$ for all $1 \leq a$. Using that $\mu_0(a) = \rho_a(2-\rho_a)$, we get 1) and 2) of Theorem 0.3. The simplicity of the zeros $\rho \neq 1$ follows again from the Maass-Selberg relations [EGM, p.270] in the same way as in the case of $\mathrm{PSL}(2,\mathbb{Z})$. □

The constant a_K^* in Theorem 0.3 can be computed as follows. Let w_K be the order of the group of units of $\mathcal{O}_K$. Let $1, \omega$ be a basis for $\mathcal{O}_K$ as a $\mathbb{Z}$ module. Since K has class number one, we have

$$\zeta_K(s) = \frac{1}{w_K} {\sum}'_{m,n} \frac{1}{|m+n\omega|^{2s}},$$

the sum being taken for all integers $(m,n) \neq (0,0)$. Using the Kronecker limit formula [La, p. 273], we get for $K = \mathbb{Q}(\sqrt{-D})$

$$a_K^* = \frac{e^{2+\gamma}}{2\pi\sqrt{D}} \begin{cases} \frac{1}{2|\eta(i\sqrt{D})|^4}, & D \equiv 2,3 \mod 4 \\ \frac{1}{\left|\eta\left(\frac{1+i\sqrt{D}}{2}\right)\right|^4}, & D \equiv 1 \mod 4, \end{cases} \tag{3.4}$$

where γ denotes Euler's constant and $\eta(z)$ is the Dedekind eta function. The values of the eta function can be computed by the Chowla-Selberg formula [SC, (2), p.110]. Let $\Delta(z) = \eta(z)^{24}$. Let $D \equiv 2,3 \mod 4$ and assume that $h(D) = 1$. Then the Chowla-Selberg formula gives

$$\Delta(\sqrt{D}i) = \frac{1}{(8\pi D)^6} \left\{ \prod_{m=1}^{4D} \Gamma\left(\frac{m}{4D}\right)^{\left(\frac{-4D}{m}\right)} \right\}^{3w_K}. \tag{3.5}$$

So for $D = 1$ we get

$$\Delta(i) = \frac{1}{(8\pi)^6} \left(\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \right)^{12} = \frac{\Gamma\left(\frac{1}{4}\right)^{24}}{2^6(2\pi)^{18}},$$

where in the right equality we used the duplication formula of the Gamma function. This implies

$$a^*_{\mathbb{Q}(i)} = \frac{4\pi^2 e^{2+\gamma}}{\Gamma(1/4)^4} \approx 3.0068.$$

Similarly for $D = 2$ we get

$$\Delta(\sqrt{2}i) = \frac{1}{(2\pi)^6 8^6} \left(\frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)} \right)^6 = \frac{\left(\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)\right)^{12}}{2^{15}(2\pi)^{18}},$$

and therefore

$$a^*_{\mathbb{Q}(\sqrt{-2})} = \frac{8\pi^2 e^{2+\gamma}}{\left(\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)\right)^2} \approx 3.2581.$$

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