WEYL'S LAW FOR THE CUSPIDAL SPECTRUM OF SL_n

WERNER MÜLLER

Abstract. Let Γ be a principal congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$ and let σ be an irreducible unitary representation of $\mathrm{SO}(n)$. Let $N^{\Gamma}_{\mathrm{cus}}(\lambda,\sigma)$ be the counting function of the eigenvalues of the Casimir operator acting in the space of cusp forms for Γ which transform under $\mathrm{SO}(n)$ according to σ. In this paper we prove that the counting function $N^{\Gamma}_{\mathrm{cus}}(\lambda,\sigma)$ satisfies Weyl's law. Especially, this implies that there exist infinitely many cusp forms for the full modular group $\mathrm{SL}_n(\mathbb{Z})$.

Contents

Let G be a connected reductive algebraic group over $\mathbb{Q}$ and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. An important problem in the theory of automorphic forms is the question of the existence and the construction of cusp forms for Γ. By Langlands' theory of Eisenstein series [La], cusp forms are the building blocks of the spectral resolution of the regular representation of $G(\mathbb{R})$ in $L^2(\Gamma\backslash G(\mathbb{R}))$. Cusp forms are also fundamental in number theory. Despite of their importance, very little is known about the existence of cusp forms in general. In this paper we will address the question of existence of cusp forms for the group

Date: March 11, 2005.
1991 *Mathematics Subject Classification.* Primary: 22E40; Secondary: 58G25.
Key words and phrases. trace formula, Automorphic forms, spectral decomposition.

$G = \mathrm{SL}_n$. The main purpose of this paper is to prove that cusp forms exist in abundance for congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 2$.

To formulate our main result we need to introduce some notation. For simplicity assume that G is semisimple. Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$ and let $X = G(\mathbb{R})/K_\infty$ be the associated Riemannian symmetric space. Let $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the center of the unviersal enveloping algebra of the complexification of the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$. Recall that a cusp form for Γ in the sense of [La] is a smooth and K_∞-finite function $\phi : \Gamma\backslash G(\mathbb{R}) \to \mathbb{C}$ which is a simultaneous eigenfunction of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ and which satisfies

$$\int_{\Gamma\cap N_P(\mathbb{R})\backslash N_P(\mathbb{R})} \phi(nx)\, dn = 0,$$

for all unipotent radicals N_P of proper rational parabolic subgroups P of G. We note that each cusp form $f \in C^\infty(\Gamma\backslash G(\mathbb{R}))$ is rapidly decreasing on $\Gamma\backslash G(\mathbb{R})$ and hence square integrable. Let $L^2_{\mathrm{cus}}(\Gamma\backslash G(\mathbb{R}))$ be the closure of the linear span of all cusp forms. Let (σ, V_σ) be an irreducible unitary representation of K_∞. Set

$$L^2(\Gamma\backslash G(\mathbb{R}), \sigma) = (L^2(\Gamma\backslash G(\mathbb{R})) \otimes V_\sigma)^{K_\infty}$$

and define $L^2_{\mathrm{cus}}(\Gamma\backslash G(\mathbb{R}), \sigma)$ similarly. Then $L^2_{\mathrm{cus}}(\Gamma\backslash G(\mathbb{R}), \sigma)$ is the space of cusp forms with fixed K_∞-type σ. Let $\Omega_{G(\mathbb{R})} \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the Casimir element of $G(\mathbb{R})$. Then $-\Omega_{G(\mathbb{R})} \otimes \mathrm{Id}$ induces a selfadjoint operator Δ_σ in the Hilbert space $L^2(\Gamma\backslash G(\mathbb{R}), \sigma)$ which is bounded from below. If Γ is torsion free, $L^2(\Gamma\backslash G(\mathbb{R}), \sigma)$ is isomorphic to the space $L^2(\Gamma\backslash X, E_\sigma)$ of square integrable sections of the locally homogeneous vector bundle E_σ associated to σ, and $\Delta_\sigma = (\nabla^\sigma)^*\nabla^\sigma - \lambda_\sigma \mathrm{Id}$, where ∇^σ is the canonical invariant connection and λ_σ the Casimir eigenvalue of σ. This shows that Δ_σ is a second order elliptic differential operator. Especially, if σ_0 is the trivial representation, then $L^2(\Gamma\backslash G(\mathbb{R}), \sigma_0) \cong L^2(\Gamma\backslash X)$ and Δ_{σ_0} equals the Laplacian Δ of X.

The restriction of Δ_σ to the subspace $L^2_{\mathrm{cus}}(\Gamma\backslash G(\mathbb{R}), \sigma)$ has pure point spectrum consisting of eigenvalues $\lambda_0(\sigma) < \lambda_1(\sigma) < \cdots$ of finite multiplicity. We call it the *cuspidal spectrum* of Δ_σ. A convenient way of counting the number of cusp forms for Γ is to use their Casimir eigenvalues. For this purpose we introduce the counting function $N^\Gamma_{\mathrm{cus}}(\lambda, \sigma)$, $\lambda \geq 0$, for the cuspidal spectrum of type σ which is defined as follows. Let $\mathcal{E}(\lambda_i(\sigma))$ be the eigenspace corresponding to the eigenvalue $\lambda_i(\sigma)$. Then

$$N^\Gamma_{\mathrm{cus}}(\lambda, \sigma) = \sum_{\lambda_i(\sigma)\leq\lambda} \dim \mathcal{E}(\lambda_i(\sigma)).$$

For non-uniform lattices Γ the selfadjoint operator Δ_σ has a large continuous spectrum so that almost all of the eigenvalues of Δ_σ will be embedded in the continous spectrum. This makes it very difficult to study the cuspidal spectrum of Δ_σ.

The first results concerning the growth of the cuspidal spectrum are due to Selberg [Se]. Let H be the upper half-plane and let Δ be the hyperbolic Laplacian of H. Let $N^\Gamma_{\mathrm{cus}}(\lambda)$ be the counting function of the cuspidal spectrum of Δ. In this case the cuspidal eigenfunctions of Δ are called *Maass cusp forms*. Using the trace formula, Selberg [Se, p.668] proved that for every congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, the counting function satisfies Weyl's law, i.e.

$$N^{\Gamma}_{\mathrm{cus}}(\lambda) \sim \frac{\mathrm{vol}(\Gamma\backslash H)}{4\pi}\lambda \tag{0.1}$$

as $\lambda \to \infty$. In particular this implies that for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ there exist as many Maass cusp forms as one can expect. On the other hand, it is conjectured by Phillips and Sarnak [PS] that for a non-uniform lattice Γ of $\mathrm{SL}_2(\mathbb{R})$ whose Teichmüller space T is non trivial and different from the Teichmüller space corresponding to the once punctured torus, a generic lattice $\Gamma \in T$ has only finitely many Maass cusp forms. This indicates that the existence of cusp forms is very subtle and may be related to the arithmetic nature of Γ.

Let $d = \dim X$. In has been conjectured in [Sa] that for $\mathrm{rank}(X) > 1$ and Γ an irreducible lattice

$$\limsup_{\lambda\to\infty} \frac{N^{\Gamma}_{\mathrm{cus}}(\lambda)}{\lambda^{d/2}} = \frac{\mathrm{vol}(\Gamma\backslash X)}{(4\pi)^{d/2}\Gamma(d/2+1)}, \tag{0.2}$$

where $\Gamma(s)$ denotes the Gamma function. A lattice Γ for which (0.2) holds is called by Sarnak *essentially cuspidal*. An analogous conjecture was made in [Mu3, p.180] for the counting function $N^{\Gamma}_{\mathrm{dis}}(\lambda,\sigma)$ of the discrete spectrum of any Casimir operator Δ_σ. This conjecture states that for any arithmetic subroup Γ and any K_∞-type σ

$$\limsup_{\lambda\to\infty} \frac{N^{\Gamma}_{\mathrm{dis}}(\lambda,\sigma)}{\lambda^{d/2}} = \dim(\sigma)\frac{\mathrm{vol}(\Gamma\backslash X)}{(4\pi)^{d/2}\Gamma(d/2+1)}. \tag{0.3}$$

Up to now these conjectures have been verified only in a few cases. Besides of Selberg's result, Weyl's law (0.2) has been proved in the following cases: For congruence subgroups of $G = \mathrm{SO}(n,1)$ by Reznikov [Rez], for congruence subgroups of $G = R_{F/\mathbb{Q}}\,\mathrm{SL}_2$, where F is a totally real number field, by Efrat [Ef, p.6], and for $\mathrm{SL}_3(\mathbb{Z})$ by St. Miller [Mil].

In this paper we will prove that each principal congruence subgroup Γ of $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 2$, is essentially cuspidal, i.e. Weyl's law holds for Γ. Actually we prove the corresponding result for all K_∞-types σ. Our main result is the following theorem.

Theorem 0.1. *For $n \geq 2$ let $X_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. Let $d_n = \dim X_n$. For every principal congruence subgroup Γ of $\mathrm{SL}_n(\mathbb{Z})$ and every irreducible unitary representation σ of $\mathrm{SO}(n)$ such that $\sigma|_{Z_\Gamma} = \mathrm{Id}$ we have*

$$N^{\Gamma}_{\mathrm{cus}}(\lambda,\sigma) \sim \dim(\sigma)\frac{\mathrm{vol}(\Gamma\backslash X_n)}{(4\pi)^{d_n/2}\Gamma(d_n/2+1)}\lambda^{d_n/2} \tag{0.4}$$

as $\lambda \to \infty$.

The method that we use is similar to Selberg's method [Se]. In particular, it does not give any estimation of the remainder term. For $n = 2$ a much better estimation of the remainder term exists. Using the full strength of the trace formula, it is possible to get a three-term asymptotic expansion of $N^{\Gamma}_{\mathrm{cus}}(\lambda)$ with remainder term of order $O(\sqrt{\lambda}/\log\lambda)$

[He, Theorem 2.28], [Ve, Theorem 7.3]. The method is based on the study of the Selberg zeta function. It is quite conceivable that the Arthur trace formula can be used to obtain a good estimation of the remainder term for arbitrary n.

Next we reformulate Theorem 0.1 in the adèlic language. Let $G = \mathrm{GL}_n$ regarded as an algebraic group over $\mathbb{Q}$. Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$. Denote by A_G the split component of the center of G and let $A_G(\mathbb{R})^0$ be the component of 1 in $A_G(\mathbb{R})$. Let ξ_0 be the trivial character of $A_G(\mathbb{R})^0$ and denote by $\Pi(G(\mathbb{A}), \xi_0)$ the set of equivalence classes of irreducible unitary representations of $G(\mathbb{A})$ whose central character is trivial on $A_G(\mathbb{R})^0$. Let $L^2_{\mathrm{cus}}(G(\mathbb{Q})A_G(\mathbb{R})^0\backslash G(\mathbb{A}))$ be the subspace of cusp forms in $L^2(G(\mathbb{Q})A_G(\mathbb{R})^0\backslash G(\mathbb{A}))$. Denote by $\Pi_{\mathrm{cus}}(G(\mathbb{A}), \xi_0)$ the subspace of all π in $\Pi(G(\mathbb{A}), \xi_0)$ which are equivalent to a subrepresentation of the regular representation in $L^2_{\mathrm{cus}}(G(\mathbb{Q})A_G(\mathbb{R})^0\backslash G(\mathbb{A}))$. By [Sk] the multiplicity of any $\pi \in \Pi_{\mathrm{cus}}(G(\mathbb{A}), \xi_0)$ in the space of cusp forms $L^2_{\mathrm{cus}}(G(\mathbb{Q})A_G(\mathbb{R})^0\backslash G(\mathbb{A}))$ is one. Let A_f be the ring of finite adèles. Any irreducible unitary representation π of $G(\mathbb{A})$ can be written as $\pi = \pi_\infty \otimes \pi_f$, where π_∞ and π_f are irreducible unitary representations of $G(\mathbb{R})$ and $G(\mathbb{A}_f)$, respectively. Let $\mathcal{H}_{\pi_\infty}$ and $\mathcal{H}_{\pi_f}$ denote the Hilbert space of the representation π_∞ and π_f, respectively. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. Denote by $\mathcal{H}_{\pi_f}^{K_f}$ the subspace of K_f-invariant vectors in $\mathcal{H}_{\pi_f}$. Let $G(\mathbb{R})^1$ be the subgroup of all $g \in G(\mathbb{R})$ with $|\det(g)| = 1$. Given $\pi \in \Pi(G(\mathbb{A}), \xi_0)$, denote by λ_π the Casimir eigenvalue of the restriction of π_∞ to $G(\mathbb{R})^1$. For $\lambda \geq 0$ let $\Pi_{\mathrm{cus}}(G(\mathbb{A}), \xi_0)_\lambda$ be the space of all $\pi \in \Pi_{\mathrm{cus}}(G(\mathbb{A}), \xi_0)$ which satisfy $|\lambda_\pi| \leq \lambda$. Set $\varepsilon_{K_f} = 1$, if $-1 \in K_f$ and $\varepsilon_{K_f} = 0$ otherwise. Then we have

Theorem 0.2. *Let $G = \mathrm{GL}_n$ and let $d_n = \dim \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let (σ, V_σ) be an irreducible unitary representation of $\mathrm{O}(n)$ such that $\sigma(-1) = \mathrm{Id}$ if $-1 \in K_f$. Then*

$$\begin{aligned}\sum_{\pi \in \Pi_{\mathrm{cus}}(G(\mathbb{A}),\xi_0)_\lambda} & \dim\big(\mathcal{H}_{\pi_f}^{K_f}\big) \dim\big(\mathcal{H}_{\pi_\infty} \otimes V_\sigma\big)^{\mathrm{O}(n)} \\ & \sim \dim(\sigma) \frac{\mathrm{vol}(G(\mathbb{Q})A_G(\mathbb{R})^0\backslash G(\mathbb{A})/K_f)}{(4\pi)^{d_n/2}\Gamma(d_n/2+1)} (1 + \varepsilon_{K_f}) \lambda^{d_n/2}\end{aligned} \tag{0.5}$$

as $\lambda \to \infty$.

Here we have used that the multiplicity of any $\pi \in \Pi(G(\mathbb{A}), \xi_0)$ in the space of cusp forms is one.

The asymptotic formula (0.5) may be regarded as the adèlic version of Weyl's law for GL_n. A similar result holds if we replace ξ_0 by any unitary character of $A_G(\mathbb{R})^0$. If we specialize Theorem 0.2 to the congruence subgroup $K(N)$ which defines $\Gamma(N)$, we obtain Theorem 0.1.

Theorem 0.2 will be derived from the Arthur trace formula combined with the heat equation method. The heat equation method is a very convenient way to derive Weyl's law for the counting function of the eigenvalues of the Laplacian on a compact Riemannian manifold [Cha]. It is based on the study of the asymptotic behaviour of the trace of the heat operator.

Our approach is similar. We will use the Arthur trace formula to compute the trace of the heat operator on the discrete spectrum and to determine its asymptotic behaviour as $t \to 0$.

We will now describe our method in more detail. Let $G(\mathbb{A})^1$ be the subgroup of all $g \in G(\mathbb{A})$ satisfying $|\det(g)| = 1$. Then $G(\mathbb{Q})$ is contained in $G(\mathbb{A})^1$ and the noninvariant trace formula of Arthur [A1] is an identity

$$\sum_{\chi \in \mathfrak{X}} J_\chi(f) = \sum_{\mathfrak{o} \in \mathfrak{O}} J_{\mathfrak{o}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1), \tag{0.6}$$

between distributions on $G(\mathbb{A})^1$. The left hand side is the *spectral side* $J_{\text{spec}}(f)$ and the right hand side the *geometric side* $J_{\text{geo}}(f)$ of the trace formula. The distributions J_χ are defined in terms of truncated Eisenstein series. They are parametrized by the set of cuspidal data $\mathfrak{X}$. The distributions $J_{\mathfrak{o}}$ are parametrized by semisimple conjugacy in $G(\mathbb{Q})$ and are closely related to weighted orbital integrals on $G(\mathbb{A})^1$.

For simplicity we consider only the case of the trivial K_∞-type. We choose a certain family of test functions $\widetilde{\phi}_t^1 \in C_c^\infty(G(\mathbb{A})^1)$, depending on $t > 0$, which at the infinite place are given by the heat kernel $h_t \in C^\infty(G(\mathbb{R})^1)$ of the Laplacian on X, multiplied by a certain cutoff function φ_t, and which at the finite places is given by the normalized characteristic function of an open compact subgroup K_f of $G(\mathbb{A}_f)$. Then we evaluate the spectral and the geometric side at $\widetilde{\phi}_t^1$ and study their asymptotic bahaviour as $t \to 0$. Let $\Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)$ be the set of irreducible unitary representations of $G(\mathbb{A})$ which occur discretely in the regular representation of $G(\mathbb{A})$ in $L^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A}))$. Given $\pi \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)$, let $m(\pi)$ denote the multiplicity with which π occurs in $L^2(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A}))$. Let $\mathcal{H}_{\pi_\infty}^{K_\infty}$ be the space of K_∞-invariant vectors in $\mathcal{H}_{\pi_\infty}$. Comparing the asymptotic behaviour of the two sides of the trace formula, we obtain

$$\begin{aligned} &\sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A}), \xi_0)} m(\pi) e^{t\lambda_\pi} \dim(\mathcal{H}_{\pi_f}^{K_f}) \dim(\mathcal{H}_{\pi_\infty}^{K_\infty}) \\ &\qquad\qquad \sim \frac{\operatorname{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1 / K_f)}{(4\pi)^{d_n/2}} (1 + \varepsilon_{K_f}) t^{-d_n/2} \end{aligned} \tag{0.7}$$

as $t \to 0$, where the notation is as in Theorem 0.2. Applying Karamatas theorem [Fe, p.446], we obtain Weyl's law for the discrete spectrum with respect to the trivial K_∞-type. A nontrivial K_∞-type can be treated in the same way. The discrete spectrum is the union of the cuspidal and the residual spectrum. It follows from [MW] combined with Donnelly's estimation of the cuspidal spectrum [Do], that the order of growth of the counting function of the residual spectrum for GL_n is at most $O(\lambda^{(d_n-1)/2})$ as $\lambda \to \infty$. This implies (0.5).

To study the asymptotic behaviour of the geometric side, we use the fine $\mathfrak{o}$-expansion [A10]

$$J_{\text{geo}}(f) = \sum_{M \in \mathcal{L}} \sum_{\gamma \in (M(\mathbb{Q}_S))_{M,S}} a^M(S, \gamma) J_M(f, \gamma), \tag{0.8}$$

which expresses the distribution $J_{\mathrm{geo}}(f)$ in terms of weighted orbital integrals $J_M(\gamma, f)$. Here M runs over the set of Levi subgroups $\mathcal{L}$ containing the Levi component M_0 of the standard minimal parabolic subgroup P_0, S is a finite set of places of $\mathbb{Q}$, and $(M(\mathbb{Q}_S))_{M,S}$ is a certain set of equivalence classes in $M(\mathbb{Q}_S)$. This reduces our problem to the investigation of weighted orbital integrals. The key result is that

$$\lim_{t\to 0} t^{d_n/2} J_M(\widetilde{\phi}_t^1, \gamma) = 0,$$

unless $M = G$ and $\gamma = \pm 1$. The contributions to (0.8) of the terms where $M = G$ and $\gamma = \pm 1$ are easy to determine. Using the behaviour of the heat kernel $h_t(\pm 1)$ as $t \to 0$, it follows that

$$J_{\mathrm{geo}}(\widetilde{\phi}_t^1) \sim \frac{\operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_f)}{(4\pi)^{d/2}}(1+\varepsilon_{K_f})t^{-d_n/2} \tag{0.9}$$

as $t \to 0$.

To deal with the spectral side, we use the results of [MS]. Let $\mathcal{C}^1(G(\mathbb{A})^1)$ denote the space of integrable rapidly decreasing functions on $G(\mathbb{A})^1$ (see [Mu2, §1.3] for its definition). By Theorem 0.1 of [MS], the spectral side is absolutely convergent for all $f \in \mathcal{C}^1(G(\mathbb{A})^1)$. Furthermore, it can be written as a finite linear combination

$$J_{\mathrm{spec}}(f) = \sum_{M\in\mathcal{L}} \sum_{L\in\mathcal{L}(M)} \sum_{P\in\mathcal{P}(M)} \sum_{s\in W^L(\mathfrak{a}_M)_{\mathrm{reg}}} a_{M,s} J^L_{M,P}(f,s).$$

of distributions $J^L_{M,P}(f,s)$, where $\mathcal{L}(M)$ is the set of Levi subgroups containing M, $\mathcal{P}(M)$ denotes the set of parabolic subgroups with Levi component M and $W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ is a certain set of Weyl group elements. Given $M \in \mathcal{L}$, the main ingredients of the distribution $J^L_{M,P}(f,s)$ are generalized logarithmic derivatives of the intertwining operators

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad P, Q \in \mathcal{P}(M),\ \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

acting between the spaces of automorphic forms attached to P and Q, respectively. First of all, Theorem 0.1 of [MS] allows us to replace $\widetilde{\phi}_t^1$ by a similar function $\phi_t^1 \in \mathcal{C}^1(G(\mathbb{A})^1)$ which is given as the product of the heat kernel at the infinite place and the normalized characteristic function of K_f. Consider the distribution where $M = L = G$. Then $s = 1$ and

$$J^G_{G,G}(\phi_t^1) = \sum_{\pi\in\Pi_{\mathrm{dis}}(G(\mathbb{A}),\xi_0)} m(\pi) e^{t\lambda_\pi} \dim(\mathcal{H}_{\pi_f}^{K_f}) \dim(\mathcal{H}_{\pi_\infty}^{K_\infty}). \tag{0.10}$$

This is exactly the left hand side of (0.7). Thus in order to prove (0.7) we need to show that for all proper Levi subgroups M, all $L \in \mathcal{L}(M)$, $P \in \mathcal{P}(M)$ and $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$, we have

$$J^L_{M,P}(\phi_t^1, s) = O(t^{-(d_n-1)/2}) \tag{0.11}$$

as $t \to 0$. This is the key result where we really need that our group is GL_n. It relies on estimations of the logarithmic derivatives of intertwining operators for $\lambda \in i\mathfrak{a}^*_M$. Given $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}), \xi_0)$, let $M_{Q|P}(\pi, \lambda)$ be the restriction of the intertwining operator $M_{Q|P}(\lambda)$

to the subspace $\mathcal{A}^2_\pi(P)$ of automorphic forms of type π. The intertwining operators can be normalized by certain meromorphic functions $r_{Q|P}(\pi,\lambda)$ [A7]. Thus

$$M_{Q|P}(\pi,\lambda) = r_{Q|P}(\pi,\lambda)^{-1} N_{Q|P}(\pi,\lambda),$$

where $N_{Q|P}(\pi,\lambda)$ are the normalized intertwining operators. Using Arthur's theory of (G,M)-families [A5], our problem can be reduced to the estimation of derivatives of $N_{Q|P}(\pi,\lambda)$ and $r_{Q|P}(\pi,\lambda)$ on $i\mathfrak{a}^*_M$. The derivatives of $N_{Q|P}(\pi,\lambda)$ can be estimated using Proposition 0.2 of [MS]. Let $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. Then $\pi = \otimes_i \pi_i$ with $\pi_i \in \Pi_{\mathrm{dis}}(\mathrm{GL}_{n_i}(\mathbb{A})^1)$ and the normalizing factors $r_{Q|P}(\pi,\lambda)$ are given in terms of the Rankin-Selberg L-functions $L(s,\pi_i \times \widetilde{\pi}_j)$ and the corresponding ϵ-factors $\epsilon(s,\pi_i \times \widetilde{\pi}_j)$. So our problem is finally reduced to the estimation of the logarithmic derivative of Rankin-Selberg L-functions on the line $\mathrm{Re}(s) = 1$. Using the available knowledge of the analytic properties of Rankin-Selberg L-functions together with standard methods of analytic number theory, the necessary estimates can be derived.

In the proof of Theorem 0.1 and Theorem 0.2 we have used the following key results which at present are only known for GL_n: 1) The nontrivial bounds of the Langlands parameters of local components of cuspidal automorphic representations [LRS] which are needed in [MS], 2) The description of the residual spectrum given in [MW], 3) The theory of the Rankin-Selberg L-functions [JPS].

The paper is organized as follows. In section 2 we prove some estimations for the heat kernel on a symmetric space. In section 3 we establish some estimates for the growth of the discrete spectrum in general. We are essentially using Donnelly's result [Do] combined with the description of the residual spectrum [MW]. The main purpose of section 4 is to prove estimates for the growth of the number of poles of Rankin-Selberg L-functions in the critical strip. We use these results in section 5 to establish the key estimates for the logarithmic derivatives of normalizing factors. In section 6 we study the asymptotic behaviour of the spectral side $J_{\mathrm{spec}}(\phi^1_t)$. Finally, in section 7 we study the asymptotic behaviour of the geometric side, compare it to the asymptotic behaviour of the spectral side and prove the main results.

Acknowledgment. The author would like to thank W. Hoffmann, D. Ramakrishnan and P. Sarnak for very helpful discussions on parts of this paper. Especially Lemma 7.1 is due to W. Hoffmann.

1. Preliminaries

1.1. Fix a positive integer n and let G be the group GL_n considered as algebraic group over $\mathbb{Q}$. By a parabolic subgroup of G we will always mean a parabolic subgroup which is defined over $\mathbb{Q}$. Let P_0 be the subgroup of upper triangular matrices of G. The Levi subgroup M_0 of P_0 is the group of diagonal matrices in G. A parabolic subgroup P of G is called standard, if $P \supset P_0$. By a Levi subgroup we will mean a subgroup of G which contains M_0 and is the Levi component of a parabolic subgroup of G defined over $\mathbb{Q}$. If $M \subset L$ are Levi subgroups, we denote the set of Levi subgroups of L which contain M

by $\mathcal{L}^L(M)$. Furthermore, let $\mathcal{F}^L(M)$ denote the set of parabolic subgroups of L defined over $\mathbb{Q}$ which contain M, and let $\mathcal{P}^L(M)$ be the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. If $L = G$, we shall denote these sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$. Write $\mathcal{L} = \mathcal{L}(M_0)$. Suppose that $P \in \mathcal{F}^L(M)$. Then

$$P = N_P M_P,$$

where N_P is the unipotent radical of P and M_P is the unique Levi component of P which contains M.

Let $M \in \mathcal{L}$ and denote by A_M the split component of the center of M. Then A_M is defined over $\mathbb{Q}$. Let $X(M)_{\mathbb{Q}}$ be the group of characters of M defined over $\mathbb{Q}$ and set

$$\mathfrak{a}_M = \operatorname{Hom}(X(M)_{\mathbb{Q}}, \mathbb{R}).$$

Then $\mathfrak{a}_M$ is a real vector space whose dimension equals that of A_M. Its dual space is

$$\mathfrak{a}_M^* = X(M)_{\mathbb{Q}} \otimes \mathbb{R}.$$

Let P and Q be groups in $\mathcal{F}(M_0)$ with $P \subset Q$. Then there is a canonical surjection $\mathfrak{a}_P \to \mathfrak{a}_Q$ and a canonical injection $\mathfrak{a}_Q^* \hookrightarrow \mathfrak{a}_P^*$. The kernel of the first map will be denoted by $\mathfrak{a}_P^Q$. Then the dual vector space of $\mathfrak{a}_P^Q$ is $\mathfrak{a}_P^*/\mathfrak{a}_Q^*$.

Let $P \in \mathcal{F}(M_0)$. We shall denote the roots of (P, A_P) by Σ_P, and the simple roots by Δ_P. Note that for GL_n all roots are reduced. They are elements in $X(A_P)_{\mathbb{Q}}$ and are canonically embedded in $\mathfrak{a}_P^*$.

For any $M \in \mathcal{L}$ there exists a partition $(n_1, ..., n_r)$ of n such that

$$M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}.$$

Then $\mathfrak{a}_M^*$ can be canonically identified with $(\mathbb{R}^r)^*$ and the Weyl group $W(\mathfrak{a}_M)$ coincides with the group S_r of permutations of the set $\{1, ..., r\}$.

1.2. Let F be a local field of characteristic zero. If π is an admissible representation of $\mathrm{GL}_m(F)$, we shall denote by $\widetilde{\pi}$ the contragredient representation to π. Let π_i, $i = 1, ..., r$, be irreducible admissible representations of the group $\mathrm{GL}_{n_i}(F)$. Then $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ is an irreducible admissible representation of

$$M(F) = \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F).$$

For $\mathbf{s} \in \mathbb{C}^r$ let $\pi_i[s_i]$ be the representation of $\mathrm{GL}_{n_i}(F)$ which is defined by

$$\pi_i[s_i](g) = |\det(g)|^{s_i} \pi_i(g), \quad g \in \mathrm{GL}_{n_i}(F).$$

Let

$$I_P^G(\pi, \mathbf{s}) = \operatorname{Ind}_{P(F)}^{G(F)}(\pi_1[s_1] \otimes \cdots \otimes \pi_r[s_r])$$

be the induced representation and denote by $\mathcal{H}_P(\pi)$ the Hilbert space of the representation $I_P^G(\pi, \mathbf{s})$. We refer to $\mathbf{s}$ as the continuous parameter of $I_P^G(\pi, \mathbf{s})$. Sometimes we will write $I_P^G(\pi_1[s_1], ..., \pi_r[s_r])$ in place of $I_P^G(\pi, \mathbf{s})$.

1.3. Let $\mathcal{G}$ be a locally compact topological group. Then we denote by $\Pi(\mathcal{G})$ the set of equivalence classes of irreducible unitary representations of $\mathcal{G}$.

1.4. Let $M \in \mathcal{L}$. Denote by $A_M(\mathbb{R})^0$ the component of 1 of $A_M(\mathbb{R})$. Set

$$M_P(\mathbb{A})^1 = \bigcap_{\chi \in X(M)_{\mathbb{Q}}} \ker(|\chi|).$$

This is a closed subgroup of $M(\mathbb{A})$, and $M(\mathbb{A})$ is the direct product of $M(\mathbb{A})^1$ and $A_M(\mathbb{R})^0$. Given a unitary character ξ of $A_M(\mathbb{R})^0$, denote by $L^2(M(\mathbb{Q})\backslash M(\mathbb{A}),\xi)$ the space of all measurable functions ϕ on $M(\mathbb{Q})\backslash M(\mathbb{A})$ such that

$$\phi(xm) = \xi(x)\phi(m), \quad x \in A_M(\mathbb{R})^0,\ m \in M(\mathbb{A}),$$

and ϕ is square integrable on $M(\mathbb{Q})\backslash M(\mathbb{A})^1$. Let $L^2_{\text{dis}}(M(\mathbb{Q})\backslash M(\mathbb{A}),\xi)$ denote the discrete subspace of $L^2(M(\mathbb{Q})\backslash M(\mathbb{A}),\xi)$ and let $L^2_{\text{cus}}(M(\mathbb{Q})\backslash M(\mathbb{A}),\xi)$ be the subspace of cusp forms in $L^2(M(\mathbb{Q})\backslash M(\mathbb{A}),\xi)$. The orthogonal complement of $L^2_{\text{cus}}(M(\mathbb{Q})\backslash M(\mathbb{A}),\xi)$ in the discrete subspace is the resiudal subspace $L^2_{\text{res}}(M(\mathbb{Q})\backslash M(\mathbb{A}),\xi)$. Denote by $\Pi_{\text{dis}}(M(\mathbb{A}),\xi)$, $\Pi_{\text{cus}}(M(\mathbb{A}),\xi)$, and $\Pi_{\text{res}}(M(\mathbb{A}),\xi)$ the subspace of all $\pi \in \Pi(M(\mathbb{A}),\xi)$ which are equivalent to a subrepresentation of the regular representation of $M(\mathbb{A})$ in $L^2(M(\mathbb{Q})\backslash M(\mathbb{A}),\xi)$, $L^2_{\text{cus}}(M(\mathbb{Q})\backslash M(\mathbb{A}),\xi)$, and $L^2_{\text{res}}(M(\mathbb{Q})\backslash M(\mathbb{A}),\xi)$, respectively.

Let $\Pi_{\text{dis}}(M(\mathbb{A})^1)$ be the subspace of all $\pi \in \Pi(M(\mathbb{A})^1)$ which are equivalent to a subrepresentation of the regular representation of $M(\mathbb{A})^1$ in $L^2(M(\mathbb{Q})\backslash M(\mathbb{A})^1)$. We denote by $\Pi_{\text{cus}}(M(\mathbb{A})^1)$ (resp. $\Pi_{\text{res}}(M(\mathbb{A})^1)$) the subspaces of all $\pi \in \Pi_{\text{dis}}(M(\mathbb{A})^1)$ occurring in the cuspidal (resp. residual) subspace $L^2_{\text{cus}}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)$ (resp. $L^2_{\text{res}}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)$).

1.5. Let P be a parabolic subgroup of G. We denote by $\mathcal{A}^2(P)$ the space of square integrable automorphic forms on $N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A})$ (see [Mu2, §1.7]).

Given $\pi \in \Pi_{\text{dis}}(M_P(\mathbb{A}),\xi_0)$, let $\mathcal{A}^2_\pi(P)$ be the subspace of $\mathcal{A}^2(P)$ of automorphic forms of type π [A1, p.925]. Let $\pi \in \Pi(M_P(\mathbb{A})^1)$. We identify π with a representation of $M_P(\mathbb{A})$ which is trivial on $A_P(\mathbb{R})^0$. Hence we can define $\mathcal{A}^2_\pi(P)$ for any $\pi \in \Pi(M_P(\mathbb{A})^1)$. It is a space of square integrable functions on $N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A})$ such that for every $x \in G(\mathbb{A})$, the function

$$\phi_x(m) = \phi(mx), \quad m \in M_P(\mathbb{A}),$$

belongs to the π-isotypical subspace of the regular representation of $M_P(\mathbb{A})$ in the Hilbert space $L^2(A_P(\mathbb{R})^0M_P(\mathbb{Q})\backslash M_P(\mathbb{A}))$.

2. Heat kernel estimates

In this section we shall prove some estimates for the heat kernel of the Bochner-Laplace operator acting on sections of a homogeneous vector bundle over a symmetric space. Let G be a connected semisimple algebraic group defined over $\mathbb{Q}$. Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$ and let (σ, V_σ) be an irreducible unitary representation of K_∞ on a complex vector space V_σ. Let $\widetilde{E}_\sigma = (G(\mathbb{R}) \times V_\sigma)/K_\infty$ be the associated homogeneous vector bundle over $X = G(\mathbb{R})/K_\infty$. We equip $\widetilde{E}_\sigma$ with the $G(\mathbb{R})$–invariant Hermitian fibre metric which is induced by the inner product in V_σ. Let $C^\infty(\widetilde{E}_\sigma)$, $C_c^\infty(\widetilde{E}_\sigma)$ and $L^2(\widetilde{E}_\sigma)$ denote the space of smooth sections, the space of compactly supported smooth sections and the Hilbert space of square integrable sections of $\widetilde{E}_\sigma$, respectively. Then we have

$$C^\infty(\widetilde{E}_\sigma) = (C^\infty(G(\mathbb{R})) \otimes V_\sigma)^{K_\infty}, \quad L^2(\widetilde{E}_\sigma) = (L^2(G(\mathbb{R})) \otimes V_\sigma)^{K_\infty} \tag{2.1}$$

and similarly for $C_c^\infty(\widetilde{E}_\sigma)$. Let $\Omega \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the Casimir element of $G(\mathbb{R})$ and let R be the right regular representation of $G(\mathbb{R})$ on $C^\infty(G(\mathbb{R}))$. Let $\widetilde{\Delta}_\sigma$ be the second order elliptic operator which is induced by $-R(\Omega) \otimes \mathrm{Id}$ in $C^\infty(\widetilde{E}_\sigma)$. Let $\widetilde{\nabla}^\sigma$ be the canonical connection on $\widetilde{E}_\sigma$, and let Ω_K be the Casimir element of K_∞. Let $\lambda_\sigma = \sigma(\Omega_K)$ be the Casimir eigenvalue of σ. Then with respect to the identification (2.1), we have

$$(\widetilde{\nabla}^\sigma)^* \widetilde{\nabla}^\sigma = -R(\Omega) \otimes \mathrm{Id} + \lambda_\sigma \,\mathrm{Id} \tag{2.2}$$

[Mia, Proposition 1.1], and therefore

$$\widetilde{\Delta}_\sigma = (\widetilde{\nabla}^\sigma)^* \widetilde{\nabla}^\sigma - \lambda_\sigma \mathrm{Id}. \tag{2.3}$$

Hence $\widetilde{\Delta}_\sigma \colon C_c^\infty(\widetilde{E}_\sigma) \to L^2(\widetilde{E}_\sigma)$ is essentially selfadjoint and bounded from below. We continue to denote its unique selfadjoint extension by $\widetilde{\Delta}_\sigma$. Let $\exp(-t\widetilde{\Delta}_\sigma)$ be the associated heat semigroup. The heat operator is a smoothing operator on $L^2(\widetilde{E}_\sigma)$ which commutes with the representation of $G(\mathbb{R})$ on $L^2(\widetilde{E}_\sigma)$. Therefore, it is of the form

$$(e^{-t\widetilde{\Delta}_\sigma}\varphi)(g) = \int_{G(\mathbb{R})} H_t^\sigma(g^{-1}g_1)(\varphi(g_1))dg_1, \quad g \in G(\mathbb{R}), \tag{2.4}$$

where $\varphi \in (L^2(G(\mathbb{R})) \otimes V_\sigma)^{K_\infty}$ and $H_t^\sigma \colon G(\mathbb{R}) \to \mathrm{End}(V_\sigma)$ is in $L^2 \cap C^\infty$ and satisfies the covariance property

$$H_t^\sigma(g) = \sigma(k) H_t^\sigma(k^{-1}gk')\sigma(k')^{-1}, \quad \text{for } g \in G(\mathbb{R}),\ k, k' \in K_\infty. \tag{2.5}$$

In order to get estimates for H_t^σ, we proceed as in [BM] and relate H_t^σ to the heat kernel of the Laplace operator of $G(\mathbb{R})$ with respect to a left invariant metric on $G(\mathbb{R})$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G(\mathbb{R})$ and K_∞, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition and let θ be the corresponding Cartan involution. Let $B(Y_1, Y_2)$ be the Killing form of $\mathfrak{g}$. Set $\langle Y_1, Y_2 \rangle = -B(Y_1, \theta Y_2)$, $Y_1, Y_2 \in \mathfrak{g}$. By translation of $\langle \cdot, \cdot \rangle$ we get a left invariant Riemannian metric on $G(\mathbb{R})$. Let $X_1, \cdots, X_p$ be an orthonormal basis for $\mathfrak{p}$ with respect to $B|\mathfrak{p} \times \mathfrak{p}$ and let $Y_1, \cdots, Y_k$ be an orthonormal basis for $\mathfrak{k}$ with respect to $-B|\mathfrak{k} \times \mathfrak{k}$. Then we have

$$\Omega = \sum_{i=1}^p X_i^2 - \sum_{i=1}^k Y_i^2 \quad \text{and} \quad \Omega_K = -\sum_{i=1}^k Y_i^2.$$

Let

$$P = -\Omega + 2\Omega_K = -\sum_{i=1}^p X_i^2 - \sum_{i=1}^k Y_i^2. \tag{2.6}$$

Then $R(P)$ is the Laplace operator Δ_G on $G(\mathbb{R})$ with respect to the left invariant metric defined above. The heat semigroup $e^{-t\Delta_G}$ is represented by a smooth kernel p_t, i.e.

$$\left(e^{-t\Delta_G} f\right)(g) = \int_{G(\mathbb{R})} p_t(g^{-1}g')f(g')dg', \quad f \in L^2(G(\mathbb{R})),\ g \in G(\mathbb{R}), \tag{2.7}$$

where $p_t \in C^\infty(G(\mathbb{R})) \cap L^2(G(\mathbb{R}))$. In fact, p_t belongs to $L^1(G(\mathbb{R}))$ [N] so that (2.7) can be written as

$$e^{-t\Delta_G} = R(p_t).$$

Let

$$Q = \int_{K_\infty} R(k) \otimes \sigma(k)\, dk$$

be the orthogonal projection of $L^2(G(\mathbb{R}))\otimes V_\sigma$ onto its K_∞–invariant subspace $(L^2(G(\mathbb{R}))\otimes V_\sigma)^{K_\infty}$. By (2.6) we have

$$\begin{aligned}\widetilde{\Delta}_\sigma &= -Q(R(\Omega)\otimes \mathrm{Id})Q\\ &= Q(R(P)\otimes \mathrm{Id})Q - 2Q(R(\Omega_K)\otimes \mathrm{Id})Q\\ &= Q(\Delta_G \otimes \mathrm{Id})Q - 2\lambda_\sigma \mathrm{Id}_{L^2(\widetilde{E}_\sigma)}.\end{aligned}$$

Hence, we get

$$e^{-t\widetilde{\Delta}_\sigma} = Q(e^{-t\Delta_G}\otimes \mathrm{Id})Q \cdot e^{t2\lambda_\sigma}$$

which implies that

$$H_t^\sigma(g) = e^{t2\lambda_\sigma}\int_{K_\infty}\int_{K_\infty} p_t(k^{-1}gk')\sigma(kk'^{-1})\, dk\, dk'. \tag{2.8}$$

Let $\mathcal{C}^1(G(\mathbb{R}))$ be Harish-Chandra's space of integrable rapidly decreasing functions on $G(\mathbb{R})$. Then (2.8) can be used to show that

$$H_t^\sigma \in \left(\mathcal{C}^1(G(\mathbb{R}))\otimes \mathrm{End}(V_\sigma)\right)^{K_\infty\times K_\infty} \tag{2.9}$$

[BM, Proposition 2.4].

Now we turn to the estimation of the derivatives of H_t^σ. By (2.8), this problem can be reduced to the estimation of the derivatives of p_t. Let ∇ denote the Levi-Civita connection and $\rho(g,g')$ the geodesic distance of $g,g' \in G(\mathbb{R})$ with respect to the left invariant metric. Then all covariant derivatives of the curvature tensor are bounded and the injectivity radius has a positive lower bound. Let $a = \dim G(\mathbb{R})$, $l \in \mathbb{N}_0$ and $T > 0$. Then it follows from Corollary 8 in [CLY] that there exist $C, c > 0$ such that

$$\parallel \nabla^l p_t(g) \parallel \leq C t^{-(a+l)/2} \exp\left(-\frac{c\rho^2(g,1)}{t}\right) \tag{2.10}$$

for all $0 < t \le T$ and $g \in G(\mathbb{R})$. By (2.8) and (2.10) we get

$$\begin{aligned} \| \nabla^l H_t^\sigma(g) \| &\le e^{2t\lambda_\sigma} \int_{K_\infty} \int_{K_\infty} \| (\nabla^l p_t)(k^{-1} g k') \| \, dk dk' \\ &\le C t^{-(a+l)/2} \int_{K_\infty} \int_{K_\infty} \exp\left(-\frac{c\rho^2(gk, k')}{t} \right) dk dk' \end{aligned} \tag{2.11}$$

for all $0 < t \le T$. Choose the invariant Riemannian metric on X which is defined by the restriction of the Killing form to $T_e X \cong \mathfrak{p}$. Then the canonical projection map $G(\mathbb{R}) \to X$ is a Riemannian submersion. Let $d(x, y)$ denote the geodesic distance on X. Then it follows that

$$\rho(g, e) \ge d(gK_\infty, K_\infty), \quad g \in G(\mathbb{R}).$$

Set $r(g) = d(gK_\infty, K_\infty)$, $g \in G(\mathbb{R})$. Together with (2.11) we get the following result.

Proposition 2.1. *Let $a = \dim G(\mathbb{R})$, $l \in \mathbb{N}_0$ and $T > 0$. There exist $C, c > 0$ such that*

$$\| \nabla^l H_t^\sigma(g) \| \le C t^{-(a+l)/2} \exp\left(-\frac{c r^2(g)}{t} \right) \tag{2.12}$$

for all $0 < t \le T$ and $g \in G(\mathbb{R})$.

We note that the exponent of t on the right hand side of (2.12) is not optimal. Using the method of Donnelly [Do2], this estimate can be improved for $l \le 1$. Indeed by Theorem 3.1 of [Mu1] we have

Proposition 2.2. *Let $n = \dim X$ and $T > 0$. There exist $C, c > 0$ such that*

$$\| \nabla^l H_t^\sigma(g) \| \le C t^{-n/2-l} \exp\left(-\frac{c r^2(g)}{t} \right) \tag{2.13}$$

for all $0 < t \le T$, $0 \le l \le 1$, and $g \in G(\mathbb{R})$.

We also need the asymptotic behaviour of the heat kernel on the diagonal. It is described by the following lemma.

Lemma 2.3. *Let $n = \dim X$ and let $e \in G(\mathbb{R})$ be the identity element. Then*

$$\operatorname{tr} H_t^\sigma(e) = \frac{\dim(\sigma)}{(4\pi)^{n/2}} t^{-n/2} + O(t^{-(n-1)/2})$$

as $t \to 0$.

Proof. Note that for each $x \in X$, the injectivity radius at x is infinite. Hence we can construct a parametrix for the fundamental solution of the heat equation for Δ_σ as in [Do2]. Let $\epsilon > 0$ and set

$$U_\epsilon = \{ (x, y) \in X \times X \mid d(x, y) < \epsilon \}.$$

For any $l \in \mathbb{N}$ we define an approximate fundamental solution $P_l(x,y,t)$ on U_ϵ by the formula

$$P_l(x,y,t) = (4\pi t)^{-n/2} \exp\left(\frac{-d^2(x,y)}{4t}\right) \left(\sum_{i=0}^{l} \Phi_i(x,y) t^i\right),$$

where the $\Phi_i(x,y)$ are smooth sections of $E_\sigma \boxtimes E_\sigma^*$ over $U_\epsilon \times U_\epsilon$ which are constructed recursively as in Theorem 2.26 of [BGV]. In particular, we have

$$\Phi_0(x,x) = \mathrm{Id}_{V_\sigma}, \quad x \in X.$$

Let $\psi \in C^\infty(X \times X)$ be equal to 1 on $U_{\epsilon/4}$ and 0 on $X \times X - U_{\epsilon/2}$. Set

$$Q_l(x,y,t) = \psi(x,y) P_l(x,y,t).$$

If $l > n/2$, then the section Q_l of $E_\sigma \boxtimes E_\sigma^*$ is a parametrix for the heat equation. Since X is a Riemannian symmetric space, we get

$$H_t^\sigma(e) = \mathrm{Id}_{V_\sigma}(4\pi t)^{-n/2} + O(t^{-(n-1)/2})$$

as $t \to 0$. This implies the lemma. □

3. Estimations of the discrete spectrum

In this section we shall establish a number of facts concerning the growth of the discrete spectrum. Let $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$, $r \geq 1$, and let

$$M(\mathbb{R})^1 = M(\mathbb{R}) \cap M(\mathbb{A})^1.$$

Then $M(\mathbb{R}) = M(\mathbb{R})^1 \cdot A_M(\mathbb{R})^0$. Let $K_{M,\infty} \subset M(\mathbb{R})$ be the standard maximal compact subgroup. Then $K_{M,\infty}$ is contained in $M(\mathbb{R})^1$. Let

$$X_M = M(\mathbb{R})^1 / K_{M,\infty}$$

be the associated Riemannian symmetric space. Let $\Gamma_M \subset M(\mathbb{Q})$ be an arithmetic subgroup and let (τ, V_τ) be an irreducible unitary representation of $K_{M,\infty}$ on V_τ. Set

$$C^\infty(\Gamma_M \backslash M(\mathbb{R})^1, \tau) := (C^\infty(\Gamma_M \backslash M(\mathbb{R})^1) \otimes V_\tau)^{K_{M,\infty}}.$$

If Γ_M is torsion free, then $\Gamma_M \backslash X_M$ is a Riemannian manifold and the homogeneous vector bundle $\widetilde{E}_\tau$ over X_M, which is associated to τ, can be pushed down to a vector bundle $E_\tau \to \Gamma_M \backslash X_M$. Then $C^\infty(\Gamma_M \backslash M(\mathbb{R})^1, \tau)$ equals $C^\infty(\Gamma_M \backslash X_M, E_\tau)$, the space of smooth sections of E_τ. Define $C_c^\infty(\Gamma_M \backslash M(\mathbb{R})^1, \tau)$ and $L^2(\Gamma_M \backslash M(\mathbb{R})^1, \tau)$ similarly. Let $\Omega_{M(\mathbb{R})^1}$ be the Casimir element of $M(\mathbb{R})^1$ and let Δ_τ be the operator in $C^\infty(\Gamma_M \backslash M(\mathbb{R})^1, \tau)$ which is induced by $-\Omega_{M(\mathbb{R})^1} \otimes \mathrm{Id}$. As unbounded operator in $L^2(\Gamma_M \backslash M(\mathbb{R})^1, \tau)$ with domain $C_c^\infty(\Gamma_M \backslash M(\mathbb{R})^1, \tau)$, Δ_τ is essentially selfadjoint. Let $L^2_{\mathrm{cus}}(\Gamma_M \backslash M(\mathbb{R})^1, \tau)$ be the subspace of cusp forms of $L^2(\Gamma_M \backslash M(\mathbb{R})^1, \tau)$. Then $L^2_{\mathrm{cus}}(\Gamma_M \backslash M(\mathbb{R})^1, \tau)$ is an invariant subspace of Δ_τ, and Δ_τ has pure point spectrum in this subspace consisting of eigenvalues $\lambda_0 < \lambda_1 < \cdots$ of finite multiplicity. Let $\mathcal{E}(\lambda_i)$ be the eigenspace of λ_i. Set

$$N_{\mathrm{cus}}^{\Gamma_M}(\lambda, \tau) = \sum_{\lambda_i \leq \lambda} \dim \mathcal{E}(\lambda_i).$$

Let $d = \dim X_M$ and let

$$C_d = \frac{1}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}$$

be Weyl's constant, where $\Gamma(s)$ denotes the Gamma function. Then Donnelly [Do, Theorem 9] has established the following basic estimation of the counting function of the cuspidal spectrum.

Theorem 3.1. *For every $\tau \in \Pi(K_{M,\infty})$ we have*

$$\limsup_{\lambda\to\infty} \frac{N_{\text{cus}}^{\Gamma_M}(\lambda,\tau)}{\lambda^{d/2}} \le C_d \dim(\tau) \operatorname{vol}(\Gamma_M\backslash X_M).$$

Actually, Donnelly proved this theorem only for the case of a torsion free discrete group. However, it is easy to extend his result to the general case.

We shall now reformulate this theorem in the representation theoretic context. Let ξ_0 be the trivial character of $A_M(\mathbb{R})^0$ and let $\pi \in \Pi(M(\mathbb{A}),\xi_0)$. Let $m(\pi)$ be the multiplicity with which π occurs in the regular representation of $M(\mathbb{A})$ in $L^2(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))$. Then $\Pi_{\text{dis}}(M(\mathbb{A}),\xi_0)$ consists of all $\pi \in \Pi(M(\mathbb{A}),\xi_0)$ with $m(\pi) > 0$. Write

$$\pi = \pi_\infty \otimes \pi_f,$$

where $\pi_\infty \in \Pi(M(\mathbb{R}))$ and $\pi_f \in \Pi(M(\mathbb{A}_f))$. Denote by $\mathcal{H}_{\pi_\infty}$ (resp. $\mathcal{H}_{\pi_f}$) the Hilbert space of the representation π_∞ (resp. π_f). Let $K_{M,f}$ be an open compact subgroup of $M(\mathbb{A}_f)$ and let $\tau \in \Pi(K_{M,\infty})$. Denote by $\mathcal{H}_{\pi_\infty}(\tau)$ the τ-isotypical subspace of $\mathcal{H}_{\pi_\infty}$ and let $\mathcal{H}_{\pi_f}^{K_{M,f}}$ be the subspace of $K_{M,f}$-invariant vectors in $\mathcal{H}_{\pi_f}$. Denote by λ_π the Casimir eigenvalue of the restriction of π_∞ to $M(\mathbb{R})^1$. Given $\lambda > 0$, let

$$\Pi_{\text{dis}}(M(\mathbb{A}),\xi_0)_\lambda = \{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}),\xi_0) \mid |\lambda_\pi| \le \lambda\}.$$

Define $\Pi_{\text{cus}}(M(\mathbb{A}),\xi_0)_\lambda$ and $\Pi_{\text{res}}(M(\mathbb{A}),\xi_0)_\lambda$ similarly.

Lemma 3.2. *Let $d = \dim X_M$. For every open compact subgroup $K_{M,f}$ of $M(\mathbb{A}_f)$ and every $\tau \in \Pi(K_{M,\infty})$ there exists $C > 0$ such that*

$$\sum_{\pi\in\Pi_{\text{cus}}(M(\mathbb{A}),\xi_0)_\lambda} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_{M,f}})\dim(\mathcal{H}_{\pi_\infty}(\tau)) \le C(1+\lambda^{d/2})$$

for $\lambda \ge 0$.

Proof. Extending the notation of §1.4, we write $\Pi(M(\mathbb{R}),\xi_0)$ for the set of representations in $\Pi(M(\mathbb{R}))$ whose central character is trivial on $A_M(\mathbb{R})^0$. Given $\pi_\infty \in \Pi(M(\mathbb{R}),\xi_0)$, let $m(\pi_\infty)$ be the multiplicity with which π_∞ occurs discretely in the regular representation of $M(\mathbb{R})$ in $L^2(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))^{K_{M,f}}$. Then

$$m(\pi_\infty) = {\sum}'_{\pi'\in\Pi_{\text{cus}}(M(\mathbb{A}),\xi_0)} m(\pi')\dim(\mathcal{H}_{\pi'_f}^{K_{M,f}}), \tag{3.1}$$

where the sum is over all $\pi' \in \Pi_{\text{dis}}(M(\mathbb{A}),\xi_0)$ such that the Archimedean component π'_∞ of π' equals π_∞.

Let $\Pi_{\mathrm{cus}}(M(\mathbb{R}),\xi_0)$ be the subset of all $\pi_\infty \in \Pi(M(\mathbb{R}),\xi_0)$ which are equivalent to an irreducible subrepresentation of the regular representation of $M(\mathbb{R})$ in the Hilbert space $L^2_{\mathrm{cus}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))^{K_{M,f}}$. Given $\pi_\infty \in \Pi_{\mathrm{cus}}(M(\mathbb{R}),\xi_0)$, denote by λ_{π_∞} the Casimir eigenvalue of the restriction of π_∞ to $M(\mathbb{R})^1$. For $\lambda \geq 0$, let

$$\Pi_{\mathrm{cus}}(M(\mathbb{R}),\xi_0)_\lambda = \{\pi_\infty \in \Pi_{\mathrm{cus}}(M(\mathbb{R}),\xi_0) \mid |\lambda_{\pi_\infty}| \leq \lambda\}.$$

Then by (3.1), it suffices to show that for each $\tau \in \Pi(K_{M,\infty})$ there exists $C > 0$ such that

$$\sum_{\pi_\infty \in \Pi_{\mathrm{cus}}(M(\mathbb{R}),\xi_0)_\lambda} m(\pi_\infty)\dim(\mathcal{H}_{\pi_\infty}(\tau)) \leq C(1+\lambda^{d/2}).$$

To deal with this problem recall that there exist arithmetic subgroups $\Gamma_{M,i} \subset M(\mathbb{R})$, $i = 1,\dots,l$, such that

$$M(\mathbb{Q})\backslash M(\mathbb{A})/K_{M,f} \cong \bigsqcup_{i=1}^{l}(\Gamma_{M,i}\backslash M(\mathbb{R}))$$

(cf. [Mu1, section 9]). Hence

$$L^2(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))^{K_{M,f}} \cong \bigoplus_{i=1}^{l} L^2(A_M(\mathbb{R})^0\Gamma_{M,i}\backslash M(\mathbb{R})) \tag{3.2}$$

as $M(\mathbb{R})$-modules. For each i, $i = 1,\dots,l$, and $\pi_\infty \in \Pi(M(\mathbb{R}))$ let $m_{\Gamma_{M,i}}(\pi_\infty)$ be the multiplicity with which π_∞ occurs discretely in the regular representation of $M(\mathbb{R})$ in $L^2(A_M(\mathbb{R})^0\Gamma_{M,i}\backslash M(\mathbb{R}))$. Then $m(\pi_\infty) = \sum_{i=1}^{l} m_{\Gamma_{M,i}}(\pi_\infty)$ and

$$\begin{aligned}\sum_{\pi_\infty \in \Pi_{\mathrm{cus}}(M(\mathbb{R}),\xi_0)_\lambda} & m(\pi_\infty)\dim(\mathcal{H}_{\pi_\infty}(\tau)) \\ &= \sum_{i=1}^{l}\sum_{\pi_\infty \in \Pi_{\mathrm{cus}}(M(\mathbb{R}),\xi_0)_\lambda} m_{\Gamma_{M,i}}(\pi_\infty)\dim(\mathcal{H}_{\pi_\infty}(\tau)).\end{aligned}$$

The interior sum can be interpreted as follows. Fix i and set $\Gamma_M := \Gamma_{M,i}$. Let $\lambda_1 < \lambda_2 < \cdots$ be the eigenvalues of Δ_τ in the space of cusp forms $L^2_{\mathrm{cus}}(\Gamma_M\backslash M(\mathbb{R})^1,\tau)$ and let $\mathcal{E}(\lambda_i)$ be the eigenspace of λ_i. By Frobenius reciprocity it follows that

$$\dim \mathcal{E}(\lambda_i) = \sum_{-\lambda_{\pi_\infty}=\lambda_i} m_{\Gamma_M}(\pi_\infty),$$

where the sum is over all $\pi_\infty \in \Pi_{\mathrm{cus}}(M(\mathbb{R}),\xi_0)$ such that the Casimir eigenvalue λ_{π_∞} equals $-\lambda_i$. Hence we obtain

$$\sum_{\pi_\infty \in \Pi_{\mathrm{cus}}(M(\mathbb{R}),\xi_0)_\lambda} m_{\Gamma_M}(\pi_\infty)\dim(\mathcal{H}_{\pi_\infty}(\tau)) = N^{\Gamma_M}_{\mathrm{cus}}(\lambda,\tau).$$

Combined with Theorem 3.1 the desired estimation follows. $\square$

Next we consider the residual spectrum.

Lemma 3.3. *Let $d = \dim X_M$. For every open compact subgroup $K_{M,f}$ of $M(\mathbb{A}_f)$ and every $\tau \in \Pi(K_{M,\infty})$ there exists $C > 0$ such that*

$$\sum_{\pi \in \Pi_{\mathrm{res}}(M(\mathbb{A}),\xi_0)_\lambda} m(\pi) \dim(\mathcal{H}_{\pi_f}^{K_{M,f}}) \dim(\mathcal{H}_{\pi_\infty}(\tau)) \leq C(1 + \lambda^{(d-1)/2})$$

for $\lambda \geq 0$.

Proof. We can assume that $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. Let $K_{M,f}$ be an open compact subgroup of $M(\mathbb{A}_f)$. There exist open compact subgroups $K_{i,f}$ of $\mathrm{GL}_{n_i}(\mathbb{A}_f)$ such that $K_{1,f} \times \cdots \times K_{r,f} \subset K_{M,f}$. Thus we can replace $K_{M,f}$ by $K_{1,f} \times \cdots \times K_{r,f}$. Next observe that $K_{M,\infty} = \mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_r)$ and therefore, τ is given as $\tau = \tau_1 \otimes \cdots \otimes \tau_r$, where each τ_i is an irreducible unitary representation of $\mathrm{O}(n_i)$. Finally note that every $\pi \in \Pi(M(\mathbb{A}), \xi_0)$ is of the form $\pi = \pi_1 \otimes \cdots \otimes \pi_r$. Hence we get $m(\pi) = \prod_{i=1}^r m(\pi_i)$ and

$$\dim\big(\mathcal{H}_{\pi_f}^{K_{M,f}}\big) = \prod_{i=1}^r \dim\big(\mathcal{H}_{\pi_{i,f}}^{K_{i,f}}\big), \quad \dim\big(\mathcal{H}_{\pi_\infty}(\tau)\big) = \prod_{i=1}^r \dim\big(\mathcal{H}_{\pi_{i,\infty}}(\tau_i)\big).$$

This implies immediately that it suffices to consider a single factor.

With the analogous notation the proof of the proposition is reduced to the following problem. For $m \in \mathbb{N}$ set $X_m = \mathrm{SL}_m(\mathbb{R})/\,\mathrm{SO}(m)$ and $d_m = \dim X_m$. Then we need to show that for every open compact subgroup $K_{m,f}$ of $\mathrm{GL}_m(\mathbb{A}_f)$ and every $\tau \in \Pi(\mathrm{O}(m))$ there exists $C > 0$ sucht that

$$\sum_{\pi \in \Pi_{res}(\mathrm{GL}_m(\mathbb{A}),\xi_0)_\lambda} m(\pi) \dim(\mathcal{H}_{\pi_f}^{K_{m,f}}) \dim(\mathcal{H}_{\pi_\infty}(\tau)) \leq C(1 + \lambda^{(d_m-1)/2})$$

for $\lambda \geq 0$. To deal with this problem recall the description of the residual spectrum of GL_m by Mœglin and Waldspurger [MW]. Let $\pi \in \Pi_{\mathrm{res}}(\mathrm{GL}_m(\mathbb{A}))$ and suppose that π is trivial on $A_{\mathrm{GL}_m}(\mathbb{R})^0$. There exist $k|m$, a standard parabolic subgroup P of GL_m of type $(l, \dots, l)$, $l = m/k$, and a cuspidal automorphic representation ρ of GL_l which is trivial on $A_{\mathrm{GL}_l}(\mathbb{R})^0$, such that π is equivalent to the unique irreducible quotient $J(\rho)$ of the induced representation

$$I_{P(\mathbb{A})}^{\mathrm{GL}_m(\mathbb{A})}(\rho[(k-1)/2] \otimes \cdots \otimes \rho[(1-k)/2]).$$

Here $\rho[s]$ denotes the representation $g \longmapsto \rho(g)|\det g|^s$, $s \in \mathbb{C}$. At the Archimedean place, the corresponding induced representation

$$I_P^{\mathrm{GL}_m}(\rho_\infty, k) := I_{P(\mathbb{R})}^{\mathrm{GL}_m(\mathbb{R})}(\rho_\infty[(k-1)/2] \otimes \cdots \otimes \rho_\infty[(1-k)/2])$$

has also a unique irreducible quotient $J(\rho_\infty)$. Comparing the definitions, we get $J(\rho)_\infty = J(\rho_\infty)$. Hence the Casimir eigenvalue of $\pi_\infty = J(\rho)_\infty$ equals the Casimir eigenvalue of $J(\rho_\infty)$ which in turn coincides with the Casimir eigenvalue of the induced representation $I_P^{\mathrm{GL}_m}(\rho_\infty, k)$. Let λ_ρ be the Casimir eigenvalue of ρ_∞. Then it follows that there exists

$C > 0$ such that $|\lambda_\pi - k\lambda_\rho| \leq C$ for all $\pi \in \Pi_{\mathrm{res}}(\mathrm{GL}_m(\mathbb{A}), \xi_0)$. Using the main theorem of [MW, p.606] it follows that it suffices to fix $l|m$, $l < m$, and to estimate

$$\sum_{\rho \in \Pi_{\mathrm{cus}}(\mathrm{GL}_l(\mathbb{A}), \xi_0)_\lambda} m(\rho) \dim\big(\mathcal{H}_{J(\rho)_f}^{K_{m,f}}\big) \dim\big(\mathcal{H}_{J(\rho)_\infty}(\tau)\big). \tag{3.3}$$

First note that by [Sk], we have $m(\rho) = 1$ for all $\rho \in \Pi_{\mathrm{cus}}(\mathrm{GL}_l(\mathbb{A}), \xi_0)$. So it remains to estimate the dimensions. We begin with the infinite place. Observe that $\dim(\mathcal{H}_{J(\rho)_\infty}(\tau)) = \dim(\tau)[J(\rho_\infty)|_{\mathrm{O}(m)} : \tau]$. Thus in order to estimate $\dim(\mathcal{H}_{J(\rho)_\infty}(\tau))$ it suffices to estimate the multiplicity $[J(\rho_\infty)|_{\mathrm{O}(m)} : \tau]$. Since $J(\rho_\infty)$ is an irreducible quotient of $I_P^{\mathrm{GL}_m}(\rho_\infty, k)$, we have

$$[J(\rho_\infty)|_{\mathrm{O}(m)} : \tau] \leq [I_P^{\mathrm{GL}_m}(\rho_\infty, k)|_{\mathrm{O}(m)} : \tau].$$

Let $K_{l,\infty} = \mathrm{O}(l) \times \cdots \times \mathrm{O}(l)$. Using Frobenius reciprocity as in [Kn, p.208], we obtain

$$\begin{aligned}&[I_P^{\mathrm{GL}_m}(\rho_\infty, k)|_{\mathrm{O}(m)} : \tau] \\ &\qquad = \sum_{\omega \in \Pi(K_{l,\infty})} [(\rho_\infty \otimes \cdots \otimes \rho_\infty)|_{K_{l,\infty}} : \omega] \cdot [\tau|_{K_{l,\infty}} : \omega].\end{aligned}$$

Finally note that $\omega = \omega_1 \otimes \cdots \otimes \omega_k$ with $\omega_i \in \Pi(\mathrm{O}(l))$. Therefore we have

$$[(\rho_\infty \otimes \cdots \otimes \rho_\infty)|_{K_{l,\infty}} : \omega] = \prod_{i=1}^{k} [\rho_\infty|_{\mathrm{O}(l)} : \omega_i].$$

At the finite places we proceed in an analogous way. This implies that there exist an open compact subgroup $K_{l,f}$ of $\mathrm{GL}_l(\mathbb{A}_f)$ and $\omega_1, ..., \omega_p \in \Pi(\mathrm{O}(l))$ such that (3.3) is bounded from above by a constant times

$$\sum_{i=1}^{p} \left(\sum_{\rho \in \Pi_{\mathrm{cus}}(\mathrm{GL}_l(\mathbb{A}), \xi_0)_\lambda} m(\rho) \dim\big(\mathcal{H}_{\rho_f}^{K_{l,f}}\big) \dim\big(\mathcal{H}_{\rho_\infty}(\omega_i)\big) \right)^k.$$

By Lemma 3.2 this term is bounded by a constant times $(1+\lambda^{d_l/2})^k$, where $d_l = l(l+1)/2-1$. Since $m = k \cdot l$ and $k > 1$, we have

$$d_l k = \frac{l(l+1)k}{2} - k \leq \frac{m(m+1)}{2} - 2 = d_m - 1.$$

This proves the desired estimation in the case of $M = \mathrm{GL}_m$, and as explained above, this suffices to prove the lemma. $\square$

Combining Lemma 3.2 and Lemma 3.3, we obtain

Proposition 3.4. *Let $d = \dim X_M$. For every open compact subgroup $K_{M,f}$ of $M(\mathbb{A}_f)$ and every $\tau \in \Pi(K_{M,\infty})$ there exists $C > 0$ such that*

$$\sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}), \xi_0)_\lambda} m(\pi) \dim(\mathcal{H}_{\pi_f}^{K_{M,f}}) \cdot \dim(\mathcal{H}_{\pi_\infty}(\tau)) \leq C(1 + \lambda^{d/2})$$

for $\lambda \geq 0$.

Next we restate Proposition 3.4 in terms of dimensions of spaces of automorphic forms. Let $P \in \mathcal{P}(M)$ and let $\mathcal{A}^2(P)$ be the space of square integrable automorphic forms on $N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A})$. Given $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}),\xi_0)$ let $\mathcal{A}^2_\pi(P)$ be the subspace of $\mathcal{A}^2(P)$ of automorphic forms of type π [A1, p.925]. Let K_∞ be the standard maximal compact subgroup of $G(\mathbb{R})$. Given an open compact subgroup K_f of $G(\mathbb{A}_f)$ and $\sigma \in \Pi(K_\infty)$, let $\mathcal{A}_\pi(P)_{K_f}$ denote the subspace of K_f-invariant automorphic forms in $\mathcal{A}^2_\pi(P)$ and let $\mathcal{A}^2_\pi(P)_{K_f,\sigma}$ be the σ-isotypical subspace of $\mathcal{A}^2_\pi(P)_{K_f}$.

Proposition 3.5. *Let $d = \dim X_M$. For every open compact subgroup K_f of $G(\mathbb{A}_f)$ and every $\sigma \in \Pi(K_\infty)$ there exists $C > 0$ such that*

$$\sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}),\xi_0)_\lambda} \dim \mathcal{A}^2_\pi(P)_{K_f,\sigma} \le C(1+\lambda^{d/2})$$

for $\lambda \ge 0$.

Proof. Let $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}),\xi_0)$. Let $\mathcal{H}_P(\pi)$ be the Hilbert space of the induced representation $I^{G(\mathbb{A})}_{P(\mathbb{A})}(\pi)$. There is a canonical isomorphism

$$j_P : \mathcal{H}_P(\pi) \otimes \mathrm{Hom}_{M(\mathbb{A})}(\pi, I^{M(\mathbb{A})}_{M(\mathbb{Q})A_M(\mathbb{R})^0}(\xi_0)) \to \overline{\mathcal{A}}^2_\pi(P), \tag{3.4}$$

which intertwines the induced representations. Let $\pi = \pi_\infty \otimes \pi_f$. Let $\mathcal{H}_P(\pi_\infty)$ (resp. $\mathcal{H}_P(\pi_f)$) be the Hilbert space of the induced representation $I^{G(\mathbb{R})}_{P(\mathbb{R})}(\pi_\infty)$) (resp. $I^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}(\pi_f)$). Denote by $\mathcal{H}_P(\pi_\infty)_\sigma$ the σ-isotypical subspace of $\mathcal{H}_P(\pi_\infty)$ and by $\mathcal{H}_P(\pi_f)^{K_f}$ the subspace of K_f-invariant vectors of $\mathcal{H}_P(\pi_f)$. Then it follows from (3.4) that

$$\dim \mathcal{A}^2_\pi(P)_{K_f,\sigma} = m(\pi)\dim(\mathcal{H}_P(\pi_f)^{K_f})\dim(\mathcal{H}_P(\pi_\infty)_\sigma). \tag{3.5}$$

Using Frobenius reciprocity as in [Kn, p.208] we get

$$[I^{G(\mathbb{R})}_{P(\mathbb{R})}(\pi_\infty)|_{K_\infty} : \sigma] = \sum_{\tau \in \Pi(K_{M,\infty})} [\pi_\infty|_{K_{M,\infty}} : \tau]\cdot[\sigma|_{K_{M,\infty}} : \tau].$$

Hence we get

$$\dim(\mathcal{H}_P(\pi_\infty)_\sigma) \le \dim(\sigma) \sum_{\tau \in \Pi(K_{M,\infty})} \dim(\mathcal{H}_{\pi_\infty}(\tau))[\sigma|_{K_{M,\infty}} : \tau]. \tag{3.6}$$

Next we consider $\pi_f = \otimes_{p<\infty}\pi_p$. Replacing K_f by a subgroup of finite index if necessary, we can assume that $K_f = \Pi_{p<\infty}K_p$. For any $p < \infty$, denote by $\mathcal{H}_P(\pi_p)$ the Hilbert space of the induced representation $I^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)}(\pi_p)$. Let $\mathcal{H}_P(\pi_p)^{K_p}$ be the subspace of K_p-invariant vectors. Then $\dim \mathcal{H}_P(\pi_p)^{K_p} = 1$ for alomost all p and

$$\mathcal{H}_P(\pi_f)^{K_f} \cong \bigotimes_{p<\infty} \mathcal{H}_P(\pi_p)^{K_p}.$$

Furthermore

$$
\begin{aligned}
I_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\pi_p)^{K_p} &= \left(I_{P(\mathbb{Z}_p)}^{G(\mathbb{Z}_p)}(\pi_p)\right)^{K_p} \\
&\hookrightarrow \bigoplus_{G(\mathbb{Z}_p)/K_p} I_{K_p\cap P}^{K_p}(\pi_p)^{K_p} \\
&\cong \bigoplus_{G(Z_p)/K_p} \pi_p^{K_p\cap P}.
\end{aligned}
\tag{3.7}
$$

Let $K_{M,f} = K_f \cap M(\mathbb{A}_f)$. Using (3.5)–(3.7), it follows that in order to prove the proposition, it suffices to fix $\tau \in \Pi(K_{M,\infty})$ and to estimate

$$\sum_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}),\xi_0)_\lambda} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_{M,f}})\dim(\mathcal{H}_{\pi_\infty}(\tau)).$$

The proof is now completed applying Proposition 3.4. □

Finally we consider the analogous statement of Lemma 3.3 at the Archimedean place. For simplicity we consider only the case $M = G$. Let K_∞ be the standard maximal compact subgroup of $G(\mathbb{R})$. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup and $\sigma \in \Pi(K_\infty)$. Then the discrete subspace $L^2_{\mathrm{dis}}(\Gamma\backslash G(\mathbb{R})^1, \sigma)$ of Δ_σ decomposes as

$$L^2_{\mathrm{dis}}(\Gamma\backslash G(\mathbb{R})^1,\sigma) = L^2_{\mathrm{cus}}(\Gamma\backslash G(\mathbb{R})^1,\sigma) \oplus L^2_{\mathrm{res}}(\Gamma\backslash G(\mathbb{R}),\sigma),$$

where $L^2_{\mathrm{res}}(\Gamma\backslash G(\mathbb{R})^1,\sigma)$ is the subspace which corresponds to the residual spectrum of Δ_σ. Let

$$L^2_{\mathrm{res}}(\Gamma\backslash G(\mathbb{R})^1,\sigma) = \bigoplus_i \mathcal{E}_{\mathrm{res}}(\lambda_i)$$

be the decomposition into eigenspaces of Δ_σ. For $\lambda \geq 0$ set

$$N^\Gamma_{\mathrm{res}}(\lambda,\sigma) = \sum_{\lambda_i\leq\lambda} \dim \mathcal{E}_{\mathrm{res}}(\lambda_i).$$

Proposition 3.6. *Let $d = G(\mathbb{R})^1/K_\infty$. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. For every $\sigma \in \Pi(K_\infty)$ there exists $C > 0$ such that*

$$N^\Gamma_{\mathrm{res}}(\lambda,\sigma) \leq C(1+\lambda^{(d-1)/2})$$

for $\lambda \geq 0$.

Proof. First assume that $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$. Let $\Gamma(N) \subset \Gamma$ be a congruence subgroup. Then

$$N^\Gamma_{\mathrm{res}}(\lambda,\sigma) \leq N^{\Gamma(N)}_{\mathrm{res}}(\lambda,\sigma). \tag{3.8}$$

Let

$$N = \Pi_p p^{r_p}, \quad r_p \geq 0.$$

Set

$$K_p(N) = \{k \in \mathrm{GL}_n(\mathbb{Z}_p) \mid k \equiv 1 \bmod p^{r_p}\mathbb{Z}_p\}$$

and

$$K(N) = \Pi_{p<\infty} K_p(N). \tag{3.9}$$

Then $K(N)$ is an open compact subgroup of $G(\mathbb{A}_f)$ and

$$A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A})/K(N) \cong \bigsqcup_{(\mathbb{Z}/N\mathbb{Z})^*} (\Gamma(N)\backslash \mathrm{SL}_n(\mathbb{R})) \tag{3.10}$$

(cf. [A9]). Hence

$$L^2_{\mathrm{res}}(A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))^{K(N)} \cong \bigoplus_{(\mathbb{Z}/N\mathbb{Z})^*} L^2_{\mathrm{res}}(\Gamma(N)\backslash \mathrm{SL}_n(\mathbb{R}))$$

as $\mathrm{SL}_n(\mathbb{R})$-modules. Put $M = G$ in Lemma 3.3. Then

$$\sum_{\pi\in\Pi_{res}(G(\mathbb{A}),\xi_0)_\lambda} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_f})\dim(\mathcal{H}_{\pi_\infty}(\sigma)) = \varphi(N) N_{\mathrm{res}}^{\Gamma(N)}(\lambda,\sigma),$$

where $\varphi(N) = \#[(\mathbb{Z}/N\mathbb{Z})^*]$. Hence by Lemma 3.3 it follows that there exists $C > 0$ such that

$$N_{\mathrm{res}}^{\Gamma(N)}(\lambda,\sigma) \leq C(1+\lambda^{(d-1)/2}).$$

This proves the proposition for $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$. Since an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ is commensurable with $G(\mathbb{Z})$, the general case can be easily reduced to this one.

$\square$

4. Rankin-Selberg L-functions

The main purpose of this section is to prove estimates for the number of zeros of Rankin-Selberg L-functions. We shall consider the Rankin-Selberg L-functions over an arbitrary number field, although in the present paper we shall use them only in the case of $\mathbb{Q}$. We begin with the description of the local L-facors.

Let F be a local field of characteristic zero. Recall that any irreducible admissible representation of $\mathrm{GL}_m(F)$ is given as a Langlands quotient: There exist a standard parabolic subgroup P of type $(m_1,...,m_r)$, discrete series representations δ_i of $\mathrm{GL}_{m_i}(F)$ and complex numbers $s_1,...,s_r$ satisfying $\mathrm{Re}(s_1) \geq \mathrm{Re}(s_2) \geq \cdots \geq \mathrm{Re}(s_r)$ such that

$$\pi = J_P^{\mathrm{GL}_m}(\delta_1[s_1]\otimes\cdots\otimes\delta_r[s_r]), \tag{4.1}$$

where the representation on the right is the unique irreducible quotient of the induced representation $I_P^{\mathrm{GL}_m}(\delta_1[s_1]\otimes\cdots\otimes\delta_r[s_r])$ [MW, I.2]. Furthermore any irreducible generic representation π of $\mathrm{GL}_m(F)$ is equivalent to a fully induced representation $I_P^{\mathrm{GL}_m}(\delta_1[s_1]\otimes\cdots\otimes\delta_r[s_r])$. If π is generic and unitary, it follows from the classification of the unitary dual of $\mathrm{GL}_m(F)$ that the parameters s_i satisfy

$$|\mathrm{Re}(s_i)| < 1/2, \quad i = 1,...,r. \tag{4.2}$$

Suppose that π is given as Langlands quotient of the form (4.1). Then the L-function satisfies

$$L(s,\pi) = \prod_j L(s+s_j,\delta_j) \tag{4.3}$$

[J]. Furthermore, suppose that π_1 and π_2 are irreducible admissible representations of $G_1 = \mathrm{GL}_{m_1}(\mathbb{R})$ and $G_2 = \mathrm{GL}_{m_2}(\mathbb{R})$, respectively. Let

$$\pi_i \cong J_{P_i}^{\mathrm{GL}_{n_i}}(\tau_{i1}[s_{i1}],\dots,\tau_{ir_i}[s_{ir_i}])$$

be the Langlands parametrizations of π_i, $i=1,2$. Then it follows from the multiplicativity of the local Rankin-Selberg L-factors [JPS, (9.4)], [Sh6] that

$$L(s,\pi_1\times\pi_2) = \prod_{i=1}^{r_1}\prod_{j=1}^{r_2} L(s+s_{1i}+s_{2j},\tau_{1i}\times\tau_{2j}). \tag{4.4}$$

This reduces the description of the local L-factors to the square-integrable case. Now we distinguish three cases according to the type of the field.

1. *F non-Archimedean*

Let $\mathcal{O}_F$ denote the ring of integers of F and $\mathfrak{P}$ the maximal ideal of $\mathcal{O}_F$. Set $q = \mathcal{O}_F/\mathfrak{P}$. The square-integrable case can be further reduced to the supercuspidal one. Finally for supercuspidal representations the L-factor is given by an elementary polynomial in q^{-s}. For details see [JPS] (see also [MS]). If we put together all steps of the reduction, we get the following result. Let π_1 and π_2 be irreducible admissible representations of $\mathrm{GL}_{n_1}(F)$ and $\mathrm{GL}_{n_2}(F)$, resprectively. Then there is a polynomial $P_{\pi_1,\pi_2}(x)$ of degree at most $n_1\cdot n_2$ with $P_{\pi_1,\pi_2}(0)=1$ such that

$$L(s,\pi_1\times\pi_2) = P_{\pi_1,\pi_2}(q^{-s})^{-1}.$$

In the special case where π_1 and π_2 are unitary and generic the L-factor has the following special form.

Lemma 4.1. *Let π_1 and π_2 be irreducible unitary generic representations of $\mathrm{GL}_{n_1}(F)$ and $\mathrm{GL}_{n_2}(F)$, respectively. There exist complex numbers a_i, $i=1,...,n_1\cdot n_2$, with $|a_i|<q$ such that*

$$L(s,\pi_1\times\pi_2) = \prod_{i=1}^{n_1\cdot n_2}(1-a_iq^{-s})^{-1}. \tag{4.5}$$

Proof. Let δ_1 and δ_2 be square-integrable representations of $\mathrm{GL}_{d_1}(F)$ and $\mathrm{GL}_{d_2}(F)$, respectively. As explained above there is a polynomial $P_{\delta_1,\delta_2}(x)$ of degree at most $d_1\cdot d_2$ with $P_{\delta_1,\delta_2}(0)=1$ such that

$$L(s,\delta_1\times\delta_2) = P_{\delta_1,\delta_2}(q^{-s})^{-1}.$$

By (6) of [JPS], p. 445, $L(s,\delta_1\times\delta_2)$ is holomorphic in the half-plane $\mathrm{Re}(s)>0$. Hence $P_{\delta_1,\delta_2}(x)$ has no zeros in the unit disc. Thus there exist complex numbers b_i with $|b_i|<1$ such that

$$L(s,\delta_1\times\delta_2)=\prod_{i=1}^{d_1\cdot d_2}(1-b_iq^{-s})^{-1}. \tag{4.6}$$

Now let π_1 and π_2 be unitary and generic. Then $L(s,\pi_1\times\pi_2)$ can be written as a product of the form (4.4) and by (4.2) the parameters s_{ij} satisfy $|\operatorname{Re}(s_{ij})|<1/2$, $i=1,2$, $j=1,...,r_i$. Combined with (4.6) the lemma follows. □

If F is Archimedean the L-factors are defined in terms of the L-factors attached to semisimple representations of the Weyl group W_F by means of the Langlands correspondence [La1]. The structure of the L-factors are described, for example, in [MS, §3]. We briefly recall the result.

2. $F=\mathbb{R}$.

First note that $\mathrm{GL}_m(\mathbb{R})$ does not have square-integrable representations if $m\geq 3$. To describe the principal L-factors in the remaining cases $d=1$ and $d=2$, we define Gamma factors by

$$\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right),\quad \Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma(s). \tag{4.7}$$

In the case $d=1$, the unitary representations of $\mathrm{GL}_1(\mathbb{R})=\mathbb{R}^\times$ are of the form $\psi_{\epsilon,t}(x)=\operatorname{sign}^\epsilon(x)|x|^t$ with $\epsilon\in\{0,1\}$ and $t\in i\mathbb{R}$. Then

$$L(s,\psi_{\epsilon,t})=\Gamma_{\mathbb{R}}(s+t+\epsilon).$$

For $k\in\mathbb{Z}$ let D_k be the k-th discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with the same infinitesimal character as the k-dimensional representation. Then the unitary square-integrable representations of $\mathrm{GL}_2(\mathbb{R})$ are unitary twists of D_k, $k\in\mathbb{Z}$, for which the L-factor is given by

$$L(s,D_k)=\Gamma_{\mathbb{C}}(s+|k|/2).$$

Let $\psi_\epsilon=\operatorname{sign}^\epsilon$, $\epsilon\in\{0,1\}$. Then up to twists by unramified characters the following list describes the Rankin-Selberg L-factors in the square-integrable case:

$$\begin{aligned} L(s,D_{k_1}\times D_{k_2})&=\Gamma_{\mathbb{C}}(s+|k_1-k_2|/2)\cdot\Gamma_{\mathbb{C}}(s+|k_1+k_2|/2)\\ L(s,D_k\times\psi_\epsilon)&=L(s,\psi_\epsilon\times D_k)=\Gamma_{\mathbb{C}}(s+|k|/2)\\ L(s,\psi_{\epsilon_1}\times\psi_{\epsilon_2})&=\Gamma_{\mathbb{R}}((s+\epsilon_{1,2})),\end{aligned} \tag{4.8}$$

where $0\leq\epsilon_{1,2}\leq 1$ with $\epsilon_{1,2}\equiv\epsilon_1+\epsilon_2 \mod 2$.

3. $F=\mathbb{C}$.

There exist square-integrable representations of $\mathrm{GL}_k(\mathbb{C})$ only if $k=1$. For $r\in\mathbb{Z}$ let χ_r be the character of $\mathrm{GL}_1(\mathbb{C})=\mathbb{C}^\times$ which is given by $\chi(z)=(z/\overline{z})^r$, $z\in\mathbb{C}^*$. Then

$$L(s,\chi_r)=\Gamma_{\mathbb{C}}(s+|r_i|/2). \tag{4.9}$$

If χ_{r_1} and χ_{r_2} are two characters as above, then we have

$$L(s, \chi_{r_1} \times \chi_{r_2}) = \Gamma_{\mathbb{C}}(s + |r_1 + r_2|/2).$$

Up to twists by unramified characters, these are all possibilities for the L-factors in the square-integrable case.

To summarize we obtain the following description of the local L-factors in the complex case. Let π be an irreducible unitary representation of $\mathrm{GL}_m(\mathbb{C})$. It is given by a Langlands quotient of the form

$$\pi = J_B^{\mathrm{GL}_m}(\chi_1[s_1] \otimes \cdots \otimes \chi_m[s_m]),$$

where B is the standard Borel subgroup of GL_m and the χ_i's are characters of $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ which are defined by $\chi(z) = (z/\overline{z})^{r_i}$, $r_i \in \mathbb{Z}$, $i = 1, \dots, m$. Then

$$L(s, \pi) = \prod_{i=1}^{m} \Gamma_{\mathbb{C}}(s + s_i + |r_i|/2). \tag{4.10}$$

Let π_1 and π_2 be irreducible unitary representations of $\mathrm{GL}_{m_1}(\mathbb{C})$ and $\mathrm{GL}_{m_2}(\mathbb{C})$, respectively. Let $B_i \subset \mathrm{GL}_{m_i}$ be the standard Borel subgroup. There exist characters χ_{ij} of $\mathbb{C}^\times$ of the form $\chi_{ij}(z) = (z/\overline{z})^{r_{ij}}$, $r_{ij} \in \mathbb{Z}$, and complex numbers s_{ij}, $i = 1, \dots, m_1$, $j = 1, \dots, m_2$, satisfying

$$\mathrm{Re}(s_{i1}) \geq \cdots \geq \mathrm{Re}(s_{im_i}), \quad |\,\mathrm{Re}(s_{ij}| < 1/2,$$

such that

$$\pi_i = J_{B_i}^{\mathrm{GL}_{m_i}}(\chi_{i1}[s_{i1}] \otimes \cdots \otimes \chi_{im_i}[s_{im_i}]), \quad i = 1, 2. \tag{4.11}$$

Then the Rankin-Selberg L-factor is given by

$$L(s, \pi_1 \times \pi_2) = \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} \Gamma_{\mathbb{C}}(s + s_{1i} + s_{2j} + |r_{1i} + r_{2j}|/2). \tag{4.12}$$

If $F = \mathbb{R}$, the L-factors have a similar form.

The description of the L-factors in the Archimedean case can be unified in the following way. By the duplication formula of the Gamma function we have

$$\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1). \tag{4.13}$$

Let F be Archimedean. Set $e_F = 1$, if $F = \mathbb{R}$, and $e_F = 2$, if $F = \mathbb{C}$. Let $\pi \in \Pi(\mathrm{GL}_m(F))$. Then it follows from (4.13) and the definition of the L-factors, that there exist complex numbers $\mu_j(\pi)$, $j = 1, \dots, me_F$, such that

$$L(s, \pi) = \prod_{j=1}^{me_F} \Gamma_{\mathbb{R}}(s + \mu_j(\pi)). \tag{4.14}$$

The numbers $\mu_j(\pi)$ are determined by the Langlands parameters of π. Similarly, if $\pi_i \in \Pi(\mathrm{GL}_{m_i}(F))$, $i = 1, 2$, it follows from the description of the Rankin-Selberg L-factors that there exist complex numbers $\mu_{j,k}(\pi_1 \times \pi_2)$ such that

$$L(s, \pi_1 \times \pi_2) = \prod_{j,k} \Gamma_{\mathbb{R}}(s + \mu_{j,k}(\pi_1 \times \pi_2)). \tag{4.15}$$

Lemma 4.2. *Let F be Archimedean. There exists $C > 0$ such that*

$$\sum_{j,k} |\mu_{j,k}(\pi_1 \times \pi_2)|^2 \leq C \left(\sum_i |\mu_i(\pi_1)|^2 + \sum_j |\mu_j(\pi_2)|^2 \right)$$

for all generic $\pi_i \in \Pi(\mathrm{GL}_{m_i}(F))$, $i = 1, 2$.

Proof. First consider the case $F = \mathbb{C}$. Let π_1 and π_2 be irreducible unitary generic representations of $\mathrm{GL}_{m_1}(\mathbb{C})$ and $\mathrm{GL}_{m_2}(\mathbb{C})$, respectively. Write π_i as Langlands quotient of the form (4.11). Using (4.10) and (4.12) together with (4.13), it follows that it suffices to prove that there exist $C > 0$ such that

$$\sum_{j,k} |s_{1j} + s_{2k} + |r_{1j} + r_{2k}|/2|^2 \leq C \sum_{i,j} |s_{ij} + |r_{ij}|/2|^2$$

for all generic $\pi_i \in \Pi(\mathrm{GL}_{m_i}(\mathbb{C}))$, $i = 1, 2$. This follows immediately, if we use the fact that the parameters s_{ij} satisfy $|\operatorname{Re}(s_{ij})| < 1/2$ and the r_{ij}'s are integers.

The proof in the case $F = \mathbb{R}$ is essentially the same. We only have to check the different possible cases for the L-factors as listed above. ☐

Next we consider the global Rankin-Selberg L-functions. Let E be a number filed and let $\mathbb{A}_E$ be the ring of adèles of E. Given $m \in \mathbb{N}$, let $\Pi_{\mathrm{dis}}(\mathrm{GL}_m(\mathbb{A}_E))$ and $\Pi_{\mathrm{cus}}(\mathrm{GL}_m(\mathbb{A}_E))$ be defined in the same way as in the case of $\mathbb{Q}$ (see §1.4). Recall that the Rankin-Selberg L-function attached to a pair of automorphic representations π_1 of $\mathrm{GL}_{m_1}(\mathbb{A}_E)$ and π_2 of $\mathrm{GL}_{m_2}(\mathbb{A}_E)$ is defined by the Euler product

$$L(s, \pi_1 \times \pi_2) = \prod_v L(s, \pi_{1,v} \times \pi_{2,v}), \tag{4.16}$$

where v runs over all places of E. The Euler product is known to converge in a certain half-plane $\operatorname{Re}(s) > c$. Suppose that π_1 and π_2 are unitary cuspidal automorphic representations of $\mathrm{GL}_{m_1}(\mathbb{A}_E)$ and $\mathrm{GL}_{m_2}(\mathbb{A}_E)$, respectively. Then $L(s, \pi_1 \times \pi_2)$ has the following basic properties:

i) The Euler product $L(s, \pi_1 \times \pi_2)$ converges absolutely for all s in the half-plane $\operatorname{Re}(s) > 1$.
ii) $L(s, \pi_1 \times \pi_2)$ admits a meromorphic continuation to the entire complex plane with at most simple poles at 0 and 1.
iii) $L(s, \pi_1 \times \pi_2)$ is of order one and is bounded in vertical strips outside of the poles.

iv) It satisfies a functional equation of the form

$$L(s,\pi_1\times\pi_2)=\epsilon(s,\pi_1\times\pi_2)L(1-s,\widetilde{\pi}_1\times\widetilde{\pi}_2) \tag{4.17}$$

with

$$\epsilon(s,\pi_1\times\pi_2)=W(\pi_1\times\pi_2)(D_E^{m_1m_2}N(\pi_1\times\pi_2))^{1/2-s}, \tag{4.18}$$

where D_E is the discriminant of E, $W(\pi_1\times\pi_2)$ is a complex number of absolute value 1 and $N(\pi_1\times\pi_2)\in\mathbb{N}$.

The absolute convergence of the Euler product (4.16) in the half-plane $\mathrm{Re}(s)>1$ was proved in [JS1]. The functional equation is established in [Sh1, Theorem 4.1] combined with [Sh3, Prop. 3.1] and [Sh3, Theorem 1]. See also [Sh5] for the general case. The location of the poles has been determined in the appendix of [MW]. Property iii) was proved in [RS, p.280].

Now let $\pi_1\in\Pi_{\mathrm{dis}}(\mathrm{GL}_{m_1}(\mathbb{A}_E))$ and $\pi_2\in\Pi_{\mathrm{dis}}(\mathrm{GL}_{m_2}(\mathbb{A}_E))$. Using the description of the residual spectrum for GL_n [MW], $L(s,\pi_1\times\pi_2)$ can be expressed in terms of Rankin-Selberg L-functions attached to cuspidal automorphic representations. Indeed, by [MW] there exist $k_i\in\mathbb{N}$ with $k_i|m_i$, parabolic subgroups P_i of $G_i=\mathrm{GL}_{m_i}$ of type $(d_i,...,d_i)$, $d_i=m_i/k_i$, and unitary cuspidal automorphic representations δ_i of $\mathrm{GL}_{d_i}(\mathbb{A}_E)$ such that

$$\pi_i=J_{P_i}^{G_i}(\delta_i[(k_i-1)/2]\otimes\cdots\otimes\delta_i[(1-k_i)/2]), \tag{4.19}$$

where the right hand side denotes the unique irreducible quotient of the induced representation $I_{P_i}^{G_i}(\delta_i[(k_i-1)/2]\otimes\cdots\otimes\delta_i[(1-k_i)/2])$. Set $k=k_1+k_2-2$. Then it follows from (4.4) that

$$L(s,\pi_1\times\pi_2)=\prod_{i=0}^{k_1-1}\prod_{j=0}^{k_2-1}L(s+k/2-i-j,\delta_1\times\delta_2). \tag{4.20}$$

Using this equality and i)–iv) above, we deduce immediately the corresponding properties satisfied by $L(s,\pi_1\times\pi_2)$. Especially, $L(s,\pi_1\times\pi_2)$ satisfies a functional equation of the form (4.17) with an ϵ-factor similar to (4.18).

We shall now investigate the logarithmic derivatives of the Rankin-Selberg L-functions. First we need to introduce some notation. Let $\pi_i\in\Pi_{\mathrm{dis}}(\mathrm{GL}_{m_i}(\mathbb{A}_E))$, $i=1,2$. For each Archimedean place w of E let $\mu_{j,k}(\pi_{1,w}\times\pi_{2,w})$ $j=1,\dots,r_w$, $k=1,\dots,h_w$, be the parameters attached to $(\pi_{1,w},\pi_{2,w})$ by means of (4.15). Set

$$c(\pi_1\times\pi_2)=\sum_{w|\infty}\sum_{j,k}|\mu_{j,k}(\pi_{1,w}\times\pi_{2,w})|. \tag{4.21}$$

Let $N(\pi_1\times\pi_2)$ be the integer that is determined by the ϵ-factor as in (4.18). Set

$$\nu(\pi_1\times\pi_2)=D_E^{m_1m_2}N(\pi_1\times\pi_2)(2+c(\pi_1\times\pi_2)). \tag{4.22}$$

We call $\nu(\pi_1\times\pi_2)$ the level of (π_1,π_2). Given $\pi\in\Pi(\mathrm{GL}_m(\mathbb{A}_E))$, set

$$\pi_\infty=\otimes_{v|\infty}\pi_v,\quad \pi_f=\otimes_{v<\infty}\pi_v.$$

Lemma 4.3. *For every $\epsilon > 0$ there exists $C > 0$ such that*

$$\left| \frac{L'(s, \pi_{1,f} \times \pi_{2,f})}{L(s, \pi_{1,f} \times \pi_{2,f})} \right| \le C$$

for all s in the half-plane $\operatorname{Re}(s) \ge 2 + \epsilon$ and all $\pi_i \in \Pi_{\mathrm{cus}}(\mathrm{GL}_{m_i}(\mathbb{A}_E))$, $i = 1, 2$.

Proof. Let $\pi_i \in \Pi_{\mathrm{cus}}(\mathrm{GL}_{m_i}(\mathbb{A}_E))$, $i = 1, 2$, and let $v < \infty$. By [Sk], $\pi_{1,v}$ and $\pi_{2,v}$ are unitary generic representations. Hence by Lemma 4.1 there exist complex numbers $a_i(v)$, $i = 1, \dots, m_1 \cdot m_2$, with

$$|a_i(v)| < N(v) \tag{4.23}$$

such that

$$L(s, \pi_{1,v} \times \pi_{2,v}) = \prod_{i=1}^{m_1 \cdot m_2} (1 - a_i(v) N(v)^{-s})^{-1}.$$

Suppose that $\operatorname{Re}(s) > 1$. By (4.23) we have $|a_i(v)/N(v)^s| < 1$. Hence, taking the logarithmic derivative, we get

$$\begin{aligned} \frac{L'(s, \pi_{1,v} \times \pi_{2,v})}{L(s, \pi_{1,v} \times \pi_{2,v})} &= -\sum_i \frac{a_i(v) \log N(v)}{N(v)^s (1 - a_i(v) N(v)^{-s})} \\ &= -\log N(v) \sum_i \sum_{k=1}^{\infty} \frac{a_i(v)^k}{N(v)^{sk}}. \end{aligned}$$

Suppose that $\sigma = \operatorname{Re}(s) > 1$. Then by (4.23) we get

$$\left| \frac{L'(s, \pi_{1,v} \times \pi_{2,v})}{L(s, \pi_{1,v} \times \pi_{2,v})} \right| \le m_1 m_2 \sum_{k=1}^{\infty} \frac{\log N(v)}{N(v)^{(\sigma - 1)k}}.$$

Let $\zeta_E(s)$ be the Dedekind zeta function of E. Let $\epsilon > 0$ and set $\sigma = 2 + \epsilon$. Then for $\operatorname{Re}(s) \ge \sigma$ we get

$$\left| \frac{L'(s, \pi_{1,f} \times \pi_{2,f})}{L(s, \pi_{1,f} \times \pi_{2,f})} \right| \le m_1 m_2 \left| \frac{\zeta_E'(\sigma - 1)}{\zeta_E(\sigma - 1)} \right|.$$

□

Lemma 4.4. *For every $\epsilon > 0$ there exists $C > 0$ such that*

$$\left| \frac{L'(s, \pi_{1,\infty} \times \pi_{2,\infty})}{L(s, \pi_{1,\infty} \times \pi_{2,\infty})} \right| \le C(1 + \log(|s| + c(\pi_1 \times \pi_2)))$$

for all s with $\operatorname{Re}(s) \ge 1 + \epsilon$ and all $\pi_i \in \Pi_{\mathrm{cus}}(\mathrm{GL}_{m_i}(\mathbb{A}_E))$, $i = 1, 2$.

Proof. Let $w | \infty$. By (4.15) we have

$$L(s, \pi_{1,w} \times \pi_{2,w}) = \prod_{j,k} \Gamma_{\mathbb{R}}(s + \mu_{j,k}(\pi_{1,w} \times \pi_{2,w})). \tag{4.24}$$

Since $\pi_{1,w}$ and $\pi_{2,w}$ are unitary and generic, the complex numbers $\mu_{j,k}(\pi_{1,w} \times \pi_{2,w})$ satisfy

$$\operatorname{Re}(\mu_{j,k}(\pi_{1,w} \times \pi_{2,w})) > -1. \tag{4.25}$$

Now recall that by Stirlings formula

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(|s|^{-1})$$

is valid as $|s| \to \infty$, in the angle $-\pi+\delta < \arg s < \pi - \delta$, for any fixed $\delta > 0$. Hence

$$\frac{\Gamma'_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s)}(s) = -\frac{1}{2}\log\pi + \log s + O(|s|^{-1}) \tag{4.26}$$

holds in the same range of s. Let $\epsilon > 0$. Using (4.24), (4.25) and (4.26), it follows that there exists $C > 0$ such that

$$\left|\frac{L'(s,\pi_{1,w}\times\pi_{2,w})}{L(s,\pi_{1,w}\times\pi_{2,w})}\right| \le C + \sum_{j,k}\log(|s| + |\mu_{j,k}(\pi_{1,w}\times\pi_{2,w})|).$$

for all $w|\infty$, all s with $\mathrm{Re}(s) \ge 1+\epsilon$ and all $\pi_i \in \Pi_{\mathrm{cus}}(\mathrm{GL}_m(\mathbb{A}_E))$, $i = 1,2$. This implies the lemma. □

Let $\pi_i \in \Pi_{\mathrm{dis}}(\mathrm{GL}_{m_i}(\mathbb{A}_E))$, $i = 1,2$, and $T > 0$, be given. Denote by $N(T;\pi_1,\pi_2)$ the number of zeros of $L(s,\pi_1\times\pi_2)$, counted with multiplicity, which are contained in the disc of radius T centered at 0.

Proposition 4.5. *There exists $C > 0$ such that*

$$N(T;\pi_1,\pi_2) \le CT\log(T+\nu(\pi_1\times\pi_2))$$

for all $T \ge 1$ and all $\pi_i \in \Pi_{\mathrm{dis}}(\mathrm{GL}_m(\mathbb{A}_E))$, $i = 1,2$.

Proof. By (4.20) we can assume that π_1 and π_2 are unitary cuspidal automorphic representations. Set

$$\Lambda(s) = s^a(1-s)^a\left(D_E^{m_1m_2}N(\pi_1\times\pi_2)\right)^{s/2}L(s,\pi_1\times\pi_2), \tag{4.27}$$

where a denotes the order of the pole of $L(s,\pi_1\times\pi_2)$ at $s=1$. Note that a can be at most 1. Since π_i is unitary, we have $\widetilde{\pi}_i = \overline{\pi_i}$, $i = 1,2$. Hence by (4.17), it follows that $\Lambda(s)$ satisfies the functional equation

$$\Lambda(s) = W(\pi_1\times\pi_2)\left(D_E^{m_1m_2}N(\pi_1\times\pi_2)\right)^{1/2}\overline{\Lambda(1-\overline{s})}. \tag{4.28}$$

Since $L(s,\pi_1\times\pi_2)$ is of order one, $\Lambda(s)$ is an entire function of order one and hence, it admits a representation as a Weierstraß product of the form

$$\Lambda(s) = e^{A+Bs}\prod_\rho(1-s/\rho)e^{s/\rho},$$

where $A, B \in \mathbb{C}$ and the product runs over the set of zeros of $\Lambda(s)$. We note that for $s = \sigma + iT$

$$\mathrm{Re}\sum_\rho\frac{1}{s-\rho} = \sum_\rho\frac{\sigma-\mathrm{Re}(\rho)}{(\sigma-\mathrm{Re}(\rho))^2+(\mathrm{Im}(\rho)-T)^2} \tag{4.29}$$

and this series is convergent since $\Lambda(s)$ is of oder one. Taking the real part of the logarithmic derivative of $\Lambda(s)$, and applying the functional equation (4.28) to the right hand side, we get

$$\begin{aligned}\operatorname{Re}(B)+\operatorname{Re}\sum\frac{1}{\rho}+\operatorname{Re}\sum_{\rho}\frac{1}{s-\rho}=&-\operatorname{Re}(\overline{B})-\operatorname{Re}\sum\frac{1}{\rho}\\&+\operatorname{Re}\sum_{\rho}\frac{1}{s-(1-\overline{\rho})}.\end{aligned}$$

Now observe that by (4.28), ρ is a zero of $\Lambda(s)$ if and only if $1-\overline{\rho}$ is a zero of $\Lambda(s)$. Hence the two sums involving s are equal, as they run over the same set of zeros. It follows that

$$\operatorname{Re}(B)=-\operatorname{Re}(\sum_{\rho}\frac{1}{\rho}). \tag{4.30}$$

Together with (4.27) this leads to

$$\begin{aligned}\operatorname{Re}\sum_{\rho}\frac{1}{s-\rho}=\operatorname{Re}\frac{\Lambda'(s)}{\Lambda(s)}=\frac{a}{s}+\frac{a}{s-1}+\frac{1}{2}\log(D_E^{m_1m_2}N(\pi_1\times\pi_2))\\+\frac{L'(s,\pi_{1,\infty}\times\pi_{2,\infty})}{L(s,\pi_{1,\infty}\times\pi_{2,\infty})}+\frac{L'(s,\pi_{1,f}\times\pi_{2,f})}{L(s,\pi_{1,f}\times\pi_{2,f})}.\end{aligned}$$

Let $\epsilon>0$, and set $\sigma=2+\epsilon$. By Lemma 4.3, Lemma 4.4 and (4.29) it follows that there exists $C>0$ such that

$$\begin{aligned}\sum_{\rho}\frac{\sigma-\operatorname{Re}(\rho)}{(\sigma-\operatorname{Re}(\rho))^2+(\operatorname{Im}(\rho)-T)^2}&\le\frac{1}{2}\log(D_E^{m_1m_2}N(\pi_1\times\pi_2))\\&\quad+C(1+\log(|T|+c(\pi_1\times\pi_2))\\&\le C_1\log(|T|+\nu(\pi_1\times\pi_2))\end{aligned}\tag{4.31}$$

for all $T\in\mathbb{R}$ and $\pi_i\in\Pi_{\text{cus}}(\mathrm{GL}_{m_i}(\mathbb{A}_E))$, $i=1,2$. Let $T>0$. Then it follows from (4.31) that

$$\begin{aligned}N(T+1;\pi_1,\pi_2)-&N(T;\pi_1,\pi_2)\\&\le 2(3+\epsilon)\sum_{\rho}\frac{\sigma-\operatorname{Re}(\rho)}{(\sigma-\operatorname{Re}(\rho))^2+(\operatorname{Im}(\rho)-T)^2}\\&\le C\log(T+\nu(\pi_1\times\pi_2))\end{aligned}$$

for all $\pi_i\in\Pi_{\text{cus}}(\mathrm{GL}_{m_i}(\mathbb{A}_E))$, $i=1,2$. This implies the proposition. □

5. Normalizing factors

In this section we consider the global normalizing factors of intertwining operators. Our main purpose is to estimate certain integrals involving the logarithmic derivatives of the normalizing factors. The behaviour of these integrals is crucial for the estimation of the spectral side. From now on we assume that the ground field is $\mathbb{Q}$. Denote by $\mathbb{A}$ the ring of adèles of $\mathbb{Q}$.

Let $M \in \mathcal{L}$. Then there exists a partition $(n_1, \dots, n_r)$ of n such that

$$M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}.$$

Let $Q, P \in \mathcal{P}(M)$. Let v be a place of $\mathbb{Q}$ and let $\pi_v \in \Pi(M(\mathbb{Q}_v))$. Associated to P, Q and π_v is the local intertwining operator

$$J_{Q|P}(\pi_v, \lambda), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

between the induced representations $I_P(\pi_{v,\lambda})$ and $I_Q(\pi_{v,\lambda})$, which is defined by an integral over $N_Q(\mathbb{Q}_v) \cap N_{\overline{P}}(\mathbb{Q}_v)$, and hence depends upon a choice of Haar measure on this group. By [A7] there exist meromorphic functions $r_{Q|P}(\pi_v, \lambda)$, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, such that the normalized local intertwining operators

$$R_{Q|P}(\pi_v, \lambda) = r_{Q|P}(\pi_v, \lambda)^{-1} J_{Q|P}(\pi_v, \lambda)$$

satisfy the conditions of Theorem 2.1 of [A7]. There is a general construction of normalizing factors which works for any group [A7], [CLL]. For GL_n, however, the intertwining operators can be normalized by L-functions [A7, §4], [AC, p.87]. The normalizing factors are given as

$$r_{Q|P}(\pi_v, \lambda) = \prod_{\alpha \in \Sigma_P \cap \Sigma_{\overline{Q}}} r_\alpha(\pi_v, \lambda(\check{\alpha})), \tag{5.1}$$

where $r_\alpha(\pi_v, s)$ is a meromorphic function of one complex variable and Σ_P (resp. $\Sigma_{\overline{Q}}$) denotes the roots of (P, A_M) (resp. $(\overline{Q}, A_M)$). Thus to define the normalizing factors, it is enough to define the functions $r_\alpha(\pi_v, s)$ for any root α of (G, A_M) and any $\pi_v \in \Pi(M(\mathbb{Q}_v))$. To this end note that π_v is equivalent to a representation $\pi_{1,v} \otimes \cdots \otimes \pi_{r,v}$ with $\pi_{i,v} \in \Pi(\mathrm{GL}_{n_i}(\mathbb{Q}_v))$ and the root α corresponds to an ordered pair (i, j) of distinct integers between 1 and r. Fix a nontrivial additive character ψ_v of $\mathbb{Q}_v$. Let $L(s, \pi_{i,v} \times \widetilde{\pi}_{j,v})$ and $\epsilon(s, \pi_{i,v} \times \widetilde{\pi}_{j,v}, \psi_v)$ be the Rankin-Selberg L-function and the ϵ-factor attached to $(\pi_{i,v}, \widetilde{\pi}_{j,v})$ and ψ_v. Set

$$r_\alpha(\pi_v, s) = \frac{L(s, \pi_{i,v} \times \widetilde{\pi}_{j,v})}{L(1+s, \pi_{i,v} \times \widetilde{\pi}_{j,v}) \epsilon(s, \pi_{i,v} \times \widetilde{\pi}_{j,v}, \psi_v)}. \tag{5.2}$$

It follows from Theorem 6.1 of [Sh1] that there are Haar measures on the group $N_Q(\mathbb{Q}_v) \cap N_{\overline{P}}(\mathbb{Q}_v)$, depending on ψ_v, such that the normalizing factors (5.1) have all the right properties (see [A7, §4], [AC, p.87]). Now suppose that $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$. Then the global normalizing factor $r_{Q|P}(\pi, \lambda)$ is defined by the infinite product

$$r_{Q|P}(\pi, \lambda) = \Pi_v r_{Q|P}(\pi_v, \lambda),$$

which converges in a certain chamber. By (5.1) it follows that there exist meromorphic functions $r_\alpha(\pi, s)$ of one complex variable such that

$$r_{Q|P}(\pi,\lambda) = \prod_{\alpha\in\sum_P\cap\sum_{\overline{Q}}} r_\alpha(\pi,\lambda(\check{\alpha})). \tag{5.3}$$

Let $\pi = \pi_1 \otimes \cdots \otimes \pi_r$. If α corresponds to (i,j) then by (5.2) we have

$$r_\alpha(\pi,s) = \frac{L(s,\pi_i\times\widetilde{\pi}_j)}{L(1+s,\pi_i\times\widetilde{\pi}_j)\epsilon(s,\pi_i\times\widetilde{\pi}_j)}, \tag{5.4}$$

where $L(s,\pi_i\times\widetilde{\pi}_j)$ and $\epsilon(s,\pi_i\times\widetilde{\pi}_j)$ are the global L-function and the ϵ-factor, respectively, considered in the previous section.

The main goal of this section is to study the multidimensional logarithmic derivatives of the normalizing factors that occur on the spectral side of the trace formula [A4]. By (5.3) this problem is reduced to the investigation of the logarithmic derivatives of the analytic functions $r_\alpha(\pi, s)$. Furthermore, by (5.4) each $r_\alpha(\pi, s)$ may be regarded as the normalizing factor attached to a standard maximal parabolic subgroup in GL_m with $m \le n$. So let $m_1, m_2 \in \mathbb{N}$ with $m_1 + m_2 \le n$. Given $\pi_i \in \Pi_{\mathrm{dis}}(\mathrm{GL}_{m_i}(\mathbb{A}))$, $i = 1, 2$, set

$$r(\pi_1\otimes\pi_2,s) = \frac{L(s,\pi_1\times\widetilde{\pi}_2)}{L(1+s,\pi_1\times\widetilde{\pi}_2)\epsilon(s,\pi_1\times\widetilde{\pi}_2)}. \tag{5.5}$$

We shall now study the logarithmic derivatives of these functions. For this purpose we need some preparation. Suppose that π_i, $i = 1, 2$, is given in the form (4.20) and assume that $k_1 \le k_2$. Set $k = k_1 + k_2 - 2$. For $j = 0, \dots, k$ let the integers a_j be defined by

$$a_i = \begin{cases} i+1 & : \quad i \le k_1 - 1; \\ k_1 & : \quad k_1 - 1 \le i \le k_2 - 1; \\ k-i+1 & : \quad i \ge k_2 - 1. \end{cases} \tag{5.6}$$

Note that $a_i = a_{k-i}$, $i = 0, ..., k$. It follows from (4.20) that

$$L(s,\pi_1\times\widetilde{\pi}_2) = \prod_{i=0}^{k} L(s+k/2-i,\delta_1\times\widetilde{\delta}_2)^{a_i}. \tag{5.7}$$

Define a polynomial of one variable x by

$$p(x) = \prod_{i=0}^{k} \left((x+k/2-i)(1-x-k/2+i)\right)^{a_i}.$$

Then $p(x)$ has real coeefficients and satisfies $p(x) = p(1-x)$. Let a be the order of the pole of $L(s,\delta_1\times\widetilde{\delta}_2)$ at $s = 1$. Note that $a \le 1$. Set

$$\Lambda(s) = p(s)^a N(\pi_1\times\widetilde{\pi}_2)^{s/2} L(s,\pi_1\times\widetilde{\pi}_2). \tag{5.8}$$

Then $\Lambda(s)$ satisfies the functional equation

$$\Lambda(s) = W(\pi_1 \times \widetilde{\pi}_2) N(\pi_1 \times \widetilde{\pi}_2)^{1/2} \overline{\Lambda(1-\overline{s})}. \tag{5.9}$$

Furthermore $\Lambda(s)$ is an entire function of order 1. Therefore it can be written as Weierstrass product of the form

$$\Lambda(s) = e^{A+Bs} \prod_\rho (1 - s/\rho) e^{s/\rho}$$

with $A, B \in \mathbb{C}$ and ρ runs over the zeros of $\Lambda(s)$. Taking the logarithmic derivative and applying the functional equation (5.9) to the right hand side, we get

$$\begin{aligned}\left(\frac{\Lambda(s)}{\Lambda(s+1)}\right)' \cdot \frac{\Lambda(s+1)}{\Lambda(s)} &= \frac{\Lambda'(s)}{\Lambda(s)} + \frac{\overline{\Lambda'(-\overline{s})}}{\overline{\Lambda(-\overline{s})}} \\ &= 2\operatorname{Re}(B) + 2\operatorname{Re}\sum_\rho \frac{1}{\rho} + \sum_\rho \left\{ \frac{1}{s-\rho} - \frac{1}{s+\overline{\rho}} \right\}.\end{aligned}$$

By (4.30) it follows that the first two terms on the right hand side cancel and hence we get

$$\left(\frac{\Lambda(s)}{\Lambda(s+1)}\right)' \cdot \frac{\Lambda(s+1)}{\Lambda(s)} = 2\sum_\rho \frac{\operatorname{Re}(\rho)}{(s-\rho)(s+\overline{\rho})}. \tag{5.10}$$

Therefore, combining (4.18), (5.5) and (5.8), we obtain

$$\begin{aligned}\frac{r'(\pi_1 \otimes \pi_2, s)}{r(\pi_1 \otimes \pi_2, s)} =& \log N(\pi_1 \times \widetilde{\pi}_2) \\ &+ a\left\{ \frac{p'(s+1)}{p(s+1)} - \frac{p'(s)}{p(s)} \right\} + 2\sum_\rho \frac{\operatorname{Re}(\rho)}{(s-\rho)(s+\overline{\rho})}.\end{aligned}$$

In particular, if $s = i\lambda$, $\lambda \in \mathbb{R}$, then it follows from the definition of $p(s)$ that

$$\begin{aligned}\frac{r'(\pi_1 \otimes \pi_2, i\lambda)}{r(\pi_1 \otimes \pi_2, i\lambda)} =& \log N(\pi_1 \times \widetilde{\pi}_2) \\ &+ 2a \sum_{i=0}^{k} \left\{ \frac{a_i(k/2 - i + 1)}{\lambda^2 + (k/2 - i + 1)^2} - \frac{a_i(k/2 - i - 1)}{\lambda^2 + (k/2 - i - 1)^2} \right\} \\ &+ 2\sum_\rho \frac{\operatorname{Re}(\rho)}{\operatorname{Re}(\rho)^2 + (\operatorname{Im}(\rho) - \lambda)^2}.\end{aligned}$$

Proposition 5.1. *There exists $C > 0$ such that*

$$\int_{-T}^{T} \left| \frac{r'(\pi_1 \otimes \pi_2, i\lambda)}{r(\pi_1 \otimes \pi_2, i\lambda)} \right| d\lambda \leq CT \log(T + \nu(\pi_1 \times \widetilde{\pi}_2))$$

for all $T > 0$ and $\pi_i \in \Pi_{\mathrm{dis}}(\mathrm{GL}_{m_i}(\mathbb{A}))$, $i = 1, 2$.

Proof. By the above formula it suffices to estimate the integral

$$\int_{-T}^{T}\sum_{\rho}\frac{|\operatorname{Re}(\rho)|}{\operatorname{Re}(\rho)^2+(\operatorname{Im}(\rho)-\lambda)^2}\,d\lambda.$$

We split the series as follows

$$\sum_{\rho}=\sum_{|\operatorname{Im}(\rho)|\le T+1}+\sum_{|\operatorname{Im}(\rho)|>T+1}.$$

To estimate the integral of the first sum, observe that for all $\beta\in\mathbb{R}^+$ and $\gamma\in\mathbb{R}$ we have

$$\int_{-T}^{T}\frac{\beta}{\beta^2+(\gamma-\lambda)^2}\,d\lambda\le\int_{-\infty}^{\infty}\frac{d\lambda}{1+\lambda^2}=\pi.$$

Hence by Proposition 4.5 we get

$$\begin{aligned}\int_{-T}^{T}\sum_{|\operatorname{Im}(\rho)|\le T+1}\frac{|\operatorname{Re}(\rho)|}{\operatorname{Re}(\rho)^2+(\operatorname{Im}(\rho)-\lambda)^2}\,d\lambda&\le\pi N(T+1,\pi_1,\widetilde{\pi}_2)\\&\le CT\log(T+\nu(\pi_1\times\widetilde{\pi}_2)).\end{aligned}$$

It remains to consider the integral of the second sum. Observe that by (5.7) the zeros ρ of $\Lambda(s)$ satisfiy $|\operatorname{Re}(\rho)|\le k/2+1$. Set

$$\sigma=k+3,\quad C=2(k+2)^2.$$

Then the following inequality holds for all $\lambda\in\mathbb{R}$ with $|\lambda|\le T$, all $\beta\in\mathbb{R}^\times$ with $|\beta|\le k/2+1$ and all $\gamma\in\mathbb{R}$ with $|\gamma|>T+1$:

$$\frac{|\beta|}{\beta^2+(\gamma-\lambda)^2}\le C\left\{\frac{\sigma-\beta}{(\sigma-\beta)^2+(\gamma-T)^2}+\frac{\sigma-\beta}{(\sigma-\beta)^2+(\gamma+T)^2}\right\}.$$

Thus we get

$$\begin{aligned}&\sum_{|\operatorname{Im}(\rho)|>T+1}\frac{|\operatorname{Re}(\rho)|}{\operatorname{Re}(\rho)^2+(\operatorname{Im}(\rho)-\lambda)^2}\\&\qquad\le C\Bigg\{\sum_{\rho}\frac{\sigma-\operatorname{Re}(\rho)}{(\sigma-\operatorname{Re}(\rho))^2+(\operatorname{Im}(\rho)-T)^2}\\&\qquad\qquad+\sum_{\rho}\frac{\sigma-\operatorname{Re}(\rho)}{(\sigma-\operatorname{Re}(\rho))^2+(\operatorname{Im}(\rho)+T)^2}\Bigg\}.\end{aligned}$$

Combining (5.7) and (4.31), it follows that for $\sigma=k+3$ there exists $C_1>0$ such that

$$\sum_{\rho}\frac{\sigma-\operatorname{Re}(\rho)}{(\sigma-\operatorname{Re}(\rho))^2+(\operatorname{Im}(\rho)-T)^2}\le C_1\log(|T|+\nu(\pi_1\times\widetilde{\pi}_2))$$

for all $T \in \mathbb{R}$ and $\pi_i \in \Pi_{\mathrm{dis}}(\mathrm{GL}_{m_i}(\mathbb{A}))$, $i = 1, 2$. Combining these observation we get

$$\int_{-T}^{T} \sum_{|\operatorname{Im}(\rho)|>T+1} \frac{|\operatorname{Re}(\rho)|}{\operatorname{Re}(\rho)^2 + (\operatorname{Im}(\rho) - \lambda)^2} \leq CT \log(T + \nu(\pi_1 \times \widetilde{\pi}_2)).$$

This completes the proof of the proposition. □

The next proposition will be important for the determination of the asymptotic behaviour of the spectral side.

Proposition 5.2. *There exists $C > 0$ such that*

$$\int_{-\infty}^{\infty} |r'(\pi_1 \otimes \pi_2, i\lambda) r(\pi_1 \otimes \pi_2, i\lambda)^{-1}| e^{-t\lambda^2} \, d\lambda$$
$$\leq C \log(1 + \nu(\pi_1 \times \widetilde{\pi}_2)) \frac{1 + |\log t|}{\sqrt{t}}$$

for all $0 < t \leq 1$ and $\pi_i \in \Pi_{\mathrm{dis}}(\mathrm{GL}_{m_i}(\mathbb{A}))$, $i = 1, 2$.

Proof. By Proposition 5.1 it follows that we have

$$\int_0^{\lambda} |r'(\pi_1 \otimes \pi_2, iu) r(\pi_1 \otimes \pi_2, iu)^{-1}| du \leq C\lambda^2$$

as $|\lambda| \to \infty$. Hence, using integration by parts, it follows that the integral on the left hand side of the claimed inequality equals

$$2t \int_{-\infty}^{\infty} \int_0^{\lambda} |r'(\pi_1 \otimes \pi_2, iu) r(\pi_1 \otimes \pi_2, iu)^{-1}| du \, \lambda e^{-t\lambda^2} \, d\lambda.$$

Applying Proposition 5.1 we get

$$\int_{-\infty}^{\infty} |r'(\pi_1 \otimes \pi_2, i\lambda) r(\pi_1 \otimes \pi_2, i\lambda)^{-1}| e^{-t\lambda^2} \, d\lambda$$
$$\leq Ct \int_{-\infty}^{\infty} \log\left(|\lambda| + \nu(\pi_1 \times \widetilde{\pi}_2)\right) \lambda^2 e^{-t\lambda^2} d\lambda$$
$$\leq C_1 \log(1 + \nu(\pi_1 \times \widetilde{\pi}_2)) \frac{1 + |\log t|}{\sqrt{t}}$$

for all $0 < t \leq 1$ and $\pi_i \in \Pi_{\mathrm{dis}}(\mathrm{GL}_{m_i}(\mathbb{A}))$, $i = 1, 2$. □

Let $M \in \mathcal{L}$ and let $Q, P \in \mathcal{P}(M)$. Our next goal is to estimate the corresponding integrals involving the generalized logarithmic derivatives of the global normalizing factors $r_{Q|P}(\pi, \lambda)$. For this purpose we will use the notion of a (G, M) family introduced by Arthur in Section 6 of [A5]. For the convenience of the reader we recall the definition of a (G, M) family and explain some of its properties.

For each $P \in \mathcal{P}(M)$, let $c_P(\lambda)$ be a smooth function on $i\mathfrak{a}_M^*$. Then the set

$$\{c_P(\lambda) \mid P \in \mathcal{P}(M)\}$$

is called a (G,M) family if the following holds: Let $P, P' \in \mathcal{P}(M)$ be adjacent parabolic groups and suppose that λ belongs to the hyperplane spanned by the common wall of the chambers of P and P'. Then

$$c_P(\lambda) = c_{P'}(\lambda).$$

Let

$$\theta_P(\lambda) = \operatorname{vol}\left(\mathfrak{a}_P^G/\mathbb{Z}(\Delta_P^\vee)\right)^{-1} \prod_{\alpha\in\Delta_P} \lambda(\alpha^\vee), \quad \lambda \in i\mathfrak{a}_P^*, \tag{5.11}$$

where $\mathbb{Z}(\Delta_P^\vee)$ is the lattice in $\mathfrak{a}_P^G$ generated by the co-roots

$$\{\alpha^\vee \mid \alpha \in \Delta_P\}.$$

Let $\{c_P(\lambda)\}$ be a (G,M) family. Then by Lemma 6.2 of [A5], the function

$$c_M(\lambda) = \sum_{P\in\mathcal{P}(M)} c_P(\lambda)\theta_P(\lambda)^{-1} \tag{5.12}$$

extends to a smooth function on $i\mathfrak{a}_M^*$. The value of $c_M(\lambda)$ at $\lambda = 0$ is of particular importance in connection with the spectral side of the trace formula. It can be computed as follows. Let $p = \dim(A_M/A_G)$. Set $\lambda = t\Lambda$, $t \in \mathbb{R}$, $\Lambda \in \mathfrak{a}_M^*$, and let t tend to 0. Then

$$c_M(0) = \frac{1}{p!} \sum_{P\in\mathcal{P}(M)} \left(\lim_{t\to 0}\left(\frac{d}{dt}\right)^p c_P(t\Lambda)\right)\theta_P(\Lambda)^{-1} \tag{5.13}$$

[A5, (6.5)]. This expression is of course independent of Λ.

For any (G,M) family $\{c_P(\lambda) \mid P \in \mathcal{P}(M)\}$ and any $L \in \mathcal{L}(M)$ there is associated a natural (G,L) family which is defined as follows. Let $Q \in \mathcal{P}(L)$ and suppose that $P \subset Q$. The function

$$\lambda \in i\mathfrak{a}_L^* \mapsto c_P(\lambda)$$

depends only on Q. We will denote it by $c_Q(\lambda)$. Then

$$\{c_Q(\lambda) \mid Q \in \mathcal{P}(L)\}$$

is a (G,L) family. We write

$$c_L(\lambda) = \sum_{Q\in\mathcal{P}(L)} c_Q(\lambda)\theta_Q(\lambda)^{-1}$$

for the corresponding function (5.12).

Let $Q \in \mathcal{P}(L)$ be fixed. If $R \in \mathcal{P}^L(M)$, then $Q(R)$ is the unique group in $\mathcal{P}(M)$ such that $Q(R) \subset Q$ and $Q(R) \cap L = R$. Let c_R^Q be the function on $i\mathfrak{a}_M^*$ which is defined by

$$c_R^Q(\lambda) = c_{Q(R)}(\lambda).$$

Then $\{c_R^Q(\lambda) \mid R \in \mathcal{P}^L(M)\}$ is an (L,M) family. Let $c_M^Q(\lambda)$ be the function (5.12) associated to this (L,M) family.

We consider now special (G, M) families defined by the global normalizing factors. Fix $P \in \mathcal{P}(M)$, $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$ and $\lambda \in i\mathfrak{a}_M^*$. Define

$$\nu_Q(P,\pi,\lambda,\Lambda) := r_{Q|P}(\pi,\lambda)^{-1} r_{Q|P}(\pi,\lambda+\Lambda), \quad Q \in \mathcal{P}(M). \tag{5.14}$$

This set of functions is a (G, M) family [A4, p.1317]. It is of a special form. By (5.3) we have

$$\nu_Q(P,\pi,\lambda,\Lambda) = \prod_{\alpha\in\Sigma_Q\cap\Sigma_{\overline{P}}} r_\alpha(\pi,\lambda(\alpha^\vee))^{-1} r_\alpha(\pi,\lambda(\alpha^\vee)+\Lambda(\alpha^\vee)).$$

Suppose that $L \in \mathcal{L}(M)$, $L_1 \in \mathcal{L}(L)$ and $S \in \mathcal{P}(L_1)$. Let

$$\{\nu_{Q_1}^S(P,\pi,\lambda,\Lambda) \mid Q_1 \in \mathcal{P}^{L_1}(L)\}$$

be the associated (L_1, L) family and let $\nu_L^S(P,\pi,\lambda,\Lambda)$ be the function (5.12) defined by this family. Set

$$\nu_L^S(P,\pi,\lambda) := \nu_L^S(P,\pi,\lambda,0).$$

If α is any root in $\Sigma(G, A_M)$, let $\alpha_L^\vee$ denote the projection of $\alpha^\vee$ onto $\mathfrak{a}_L$. If F is a subset of $\Sigma(G, A_M)$, let $F_L^\vee$ be the disjoint union of all the vectors $\alpha_L^\vee$, $\alpha \in F$. Then by Proposition 7.5 of [A4] we have

$$\begin{aligned}\nu_L^S(P,\pi,\lambda) = &\sum_F \mathrm{vol}(\mathfrak{a}_L^{L_1}/\mathbb{Z}(F_L^\vee)) \\ &\cdot\left(\prod_{\alpha\in F} r_\alpha(\pi,\lambda(\alpha^\vee))^{-1} r'_\alpha(\pi,\lambda(\alpha^\vee))\right),\end{aligned} \tag{5.15}$$

where F runs over all subsets of $\Sigma(L_1, A_M)$ such that $F_L^\vee$ is a basis of $\mathfrak{a}_L^{L_1}$. Let $t > 0$. Then by (5.15) we get

$$\begin{aligned}\int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*} |\nu_L^S(P,\pi,\lambda)| e^{-t\|\lambda\|^2}\, d\lambda \le &\sum_F \mathrm{vol}(\mathfrak{a}_L^{L_1}/\mathbb{Z}(F_L^\vee)) \\ &\cdot \int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*} \prod_{\alpha\in F} \left| r_\alpha \pi, \lambda(\alpha\vee))^{-1} r_\alpha\prime(\pi,\lambda(\alpha\vee))\right| e^{-t\|\lambda\|^2}\, d\lambda.\end{aligned}$$

Fix any subset F of $\Sigma(L_1, A_M)$ such that $F_L^\vee$ is a basis of $\mathfrak{a}_L^{L_1}$. Let

$$\{\tilde\omega_\alpha \mid \alpha \in F\}$$

be the basis of $(\mathfrak{a}_L^{L_1})^*$ which is dual to $F_L^\vee$. We can write $\lambda \in i\mathfrak{a}_L^*/i\mathfrak{a}_G^*$ as

$$\lambda = \sum_{\alpha\in F} z_\alpha\tilde\omega_\alpha + \lambda_1, \quad z_\alpha \in i\mathbb{R}, \quad \lambda_1 \in i\mathfrak{a}_{L_1}^*/i\mathfrak{a}_G^*.$$

Observe that $\lambda(\alpha^\vee) = z_\alpha$. Let $l_1 = \dim(A_{L_1}/A_G)$. Then there exists $C > 0$, independent of π, such that for all $t > 0$ we have

$$\begin{aligned}(5.16)\qquad &\int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*} \prod_{\alpha\in F} \left|r_\alpha(\pi,\lambda(\alpha^\vee))^{-1} r_\alpha'(\pi,\lambda(\alpha^\vee))\right| e^{-t\|\lambda\|^2}\, d\lambda \\ &\qquad \le C t^{-l_1/2} \prod_{\alpha\in F} \int_{i\mathbb{R}} \left|r_\alpha(\pi,z_\alpha)^{-1} r_\alpha'(\pi,z_\alpha)\right| e^{-tz_\alpha^2}\, dz_\alpha.\end{aligned}$$

Suppose that $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. Then $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ with $\pi_i \in \Pi_{\mathrm{dis}}(\mathrm{GL}_{n_i}(\mathbb{A}))$. Now recall that a given root $\alpha \in \Sigma(G, A_M)$ corresponds to an ordered pair (i, j) of distinct integers i and j between 1 and r. Then it folows from (5.4) and (5.5) that $r_\alpha(\pi, s) = r(\pi_i \otimes \pi_j, s)$. Let $l = \dim(A_L/A_G)$ and $k = \dim(A_L/A_{L_1})$. Then by Proposition 5.2 and (5.16) it follows that there exists $C > 0$

$$\begin{aligned}(5.17)\qquad &\int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*} |\nu_L^S(P,\pi,\lambda)| e^{-t\|\lambda\|^2}\, d\lambda \\ &\qquad \le C \prod_{i,j} \log(1+\nu(\pi_i \times \widetilde{\pi}_j)) \frac{(1+|\log t|)^k}{t^{l/2}}\end{aligned}$$

for all $0 < t \le 1$ and all $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$.

Next we shall estimate the numbers $\nu(\pi_i \times \widetilde{\pi}_j)$. For $\pi_\infty \in \Pi(\mathrm{GL}_m(\mathbb{R}))$, let the complex numbers $\mu_j(\pi_\infty)$, $j = 1, ..., m$, be defined by (4.14) and set

$$c(\pi_\infty) = \left(\sum_{j=1}^m |\mu_j(\pi_\infty)|^2\right)^{1/2}.$$

Given an open compact subgroup K_f of $\mathrm{GL}_m(\mathbb{A}_f)$, set

$$\Pi(\mathrm{GL}_m(\mathbb{A}))_{K_f} := \{\pi \in \Pi(\mathrm{GL}_m(\mathbb{A})) \mid \mathcal{H}_{\pi_f}^{K_f} \neq 0\},$$

where $\pi = \pi_\infty \otimes \pi_f$ and $\mathcal{H}_{\pi_f}$ denotes the Hilbert space of the representation π_f.

Lemma 5.3. *Let $K_{f,i} \subset \mathrm{GL}_{m_i}(\mathbb{A}_f)$, $i = 1, 2$, be two open compact subgroups. There exists $C > 0$ such that*

$$\nu(\pi_1 \times \pi_2) \le C(1 + c(\pi_{1,\infty}) + c(\pi_{2,\infty}))$$

for all $\pi_i \in \Pi(\mathrm{GL}_{m_i}(\mathbb{A}))_{K_{f,i}}$, $i = 1, 2$.

Proof. First consider $c(\pi_1 \times \pi_2)$ which is defined by (4.21). It follows from Lemma 4.2 that there exists $C > 0$ such that

$$c(\pi_1 \times \pi_2) \le C(c(\pi_{1,\infty}) + c(\pi_{2,\infty}))$$

for all $\pi_i \in \Pi(\mathrm{GL}_{m_i}(\mathbb{A}))$, $i = 1, 2$. It remains to estimate $N(\pi_1 \times \pi_2)$. For this we first observe that, as the epsilon factor is a product of local epsilon factors, we can factor

$N(\pi_1 \times \pi_2)$ as

$$N(\pi_1 \times \pi_2) = \prod_p N(\pi_{1,p} \times \pi_{2,p}),$$

where p runs over the finite places of $\mathbb{Q}$. This is a finite product. In fact, if p is unramified for both π_1 and π_2, then $N(\pi_{1,p} \times \pi_{2,p}) = 1$. Moreover there is an integer $f(\pi_{1,p} \times \pi_{2,p})$ such that

$$N(\pi_{1,p} \times \pi_{2,p}) = p^{f(\pi_{1,p} \times \pi_{2,p})}$$

(see e.g. [MS]). Since we fix the ramification, there is a finite set S of finite places of $\mathbb{Q}$, such that

$$N(\pi_1 \times \pi_2) = \prod_{p \in S} p^{f(\pi_{1,p} \times \pi_{2,p})}$$

for all $\pi_i \in \Pi(\mathrm{GL}_{m_i}(\mathbb{A}))_{K_{f,i}}$, $i = 1, 2$. This reduces our problem to the estimation of $f(\pi_{1,p} \times \pi_{2,p})$. Let $f(\pi_{i,p})$ be the conductor of $\pi_{i,p}$, $i = 1, 2$. Then $f(\pi_{i,p}) \geq 0$ and by Theorem 1 of [BH] and Corollary (6.5) of [BHK] we have

$$0 \leq f(\pi_{1,p} \times \pi_{2,p}) \leq m_1 f(\pi_{1,p}) + m_2 f(\pi_{2,p}). \tag{5.18}$$

Let $m \in \mathbb{N}$ and let K_p be an open compact subgroup of $\mathrm{GL}_m(\mathbb{Q}_p)$. By Lemma 2.2 of [MS] there exists $C_p > 0$ such that $f(\pi_p) \leq C_p$ for all $\pi_p \in \Pi(\mathrm{GL}_m(\mathbb{Q}_p))$ with $\pi_p^{K_p} \neq 0$. Together with (5.18) this implies that there exists $C > 0$ such that

$$N(\pi_1 \times \pi_2) \leq C$$

for all $\pi_i \in \Pi(\mathrm{GL}_{m_i}(\mathbb{A}))_{K_{f,i}}$, $i = 1, 2$. This completes the proof of the lemma. □

We continue with the estimation of $c(\pi_\infty)$. Given $\pi_\infty \in \Pi(\mathrm{GL}_m(\mathbb{R}), \xi_0)$, let λ_{π_∞} be the Casimir eigenvalue of the restriciton of π_∞ to $\mathrm{GL}_m(\mathbb{R})^1$. Furthermore for $\sigma \in \Pi(\mathrm{O}(m))$ let λ_σ denote the Casimir eigenvalue of σ. We note that if $[\pi_\infty|_{\mathrm{O}(m)} : \sigma] > 0$, then $-\lambda_{\pi_\infty} + \lambda_\sigma \geq 0$ [DH, Lemma 2.6] .

Lemma 5.4. *There exists $C > 0$ such that*

$$c(\pi_\infty) \leq C(1 - \lambda_{\pi_\infty} + \lambda_\sigma)^{1/2}$$

for all $\pi_\infty \in \Pi(\mathrm{GL}_m(\mathbb{R}), \xi_0)$ and $\sigma \in \Pi(\mathrm{O}(m))$ with $[\pi_\infty|_{\mathrm{O}(m)} : \sigma] > 0$.

Proof. Write π_∞ as Langlands quotient $\pi_\infty = J_R^{\mathrm{GL}_m}(\tau, \mathbf{s})$, where τ is a discrete series representation of $M_R(\mathbb{R})$ and the parameters $s_1, ..., s_r \in \mathbb{C}$ satisfy $\mathrm{Re}(s_1) \geq \mathrm{Re}(s_2) \geq \cdots \geq \mathrm{Re}(s_r)$. We may assume that the central character of τ is trivial on $A_R(\mathbb{R})^0$ and hance, we can regard τ as a discrete series representation of $M_R(\mathbb{R})^1$. Let $\mathfrak{m}_R^1$ denote the Lie algebra of $M_R(\mathbb{R})^1$. Note that $\mathfrak{m}_R^1$ is the direct sum of a finite number of copies of $\mathfrak{sl}(2, \mathbb{R})$. Let $\mathfrak{t} \subset \mathfrak{m}_R^1$ be the standard compact Cartan subalgebra equipped with the canonical norm. Then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}_R$ is a Cartan subalgebra of $\mathfrak{gl}_m(\mathbb{R})$. Let $\Lambda_\tau \in i\mathfrak{t}^*$ be the Harish-Cahndra parameter of τ. It follows from the definition of the parameters $\mu_j(\pi_\infty)$ in terms of the Langlands parameters that there exists $C > 0$ such that

$$c(\pi_\infty)^2 \leq C(\| \Lambda_\tau \|^2 + \| \mathbf{s} \|^2)$$

for all $\pi_\infty \in \Pi(\mathrm{GL}_m(\mathbb{R}),\xi_0)$. Let $\gamma : Z(\mathfrak{gl}_m(\mathbb{C})) \to I(\mathfrak{h}_\mathbb{C})$ be the Harish-Chandra homomorphism. By Proposition 8.22 of [Kn] the infinitesimal character χ of the induced representation $I_R^{\mathrm{GL}_m}(\tau,\mathbf{s})$ with respect to $\mathfrak{h}$ is given by $\chi(Z) = (\Lambda_\tau + \mathbf{s})(\gamma(Z))$, $Z \in Z(\mathfrak{gl}_m(\mathbb{C}))$. Since π_∞ is an irreducible quotient of $I_R^{\mathrm{GL}_m}(\tau,\mathbf{s})$, it has the same infinitesimal character. Let $H_1,...,H_r$ be an orthonormal basis of $\mathfrak{a}_R$ and $H_{r+1},...,H_m$ an orthonormal basis of $\mathfrak{t}$. Then

$$\gamma(\Omega) = \sum_{i=1}^r H_i^2 - \sum_{j=r+1}^m H_j^2 - \parallel \rho \parallel^2$$

[Wa1, p.168]. Hence, the Casimir eigenvalue λ_π of π_∞ is given by

$$\lambda_{\pi_\infty} = (\Lambda_\tau + \mathbf{s})(\gamma(\Omega)) = \sum_{i=1}^r s_i^2 + \parallel \Lambda_\tau \parallel^2 - \parallel \rho \parallel^2 .$$

Since π_∞ is unitary, it follows from Theorem 3.3 of Chapter XI of [BW] that there exists $C > 0$, independent of π_∞, such that $\parallel \mathrm{Re}(\mathbf{s}) \parallel \leq C$. Hence there exists $C_1 > 0$ such that

$$\parallel \Lambda_\tau \parallel^2 + \parallel \mathbf{s} \parallel^2 \leq C_1 - \lambda_{\pi_\infty} + \parallel \Lambda_\tau \parallel^2$$

for all $\pi_\infty \in \Pi(\mathrm{GL}_m(\mathbb{R}),\xi_0)$. Now let $\sigma \in \Pi(\mathrm{O}(m))$ and suppose that $\pi_\infty \in \Pi(\mathrm{GL}_m(\mathbb{R}),\xi_0)$ is such that $[\pi_\infty|_{\mathrm{O}(m)} : \sigma] > 0$. Since σ occurs in π_∞, it also occurs in $I_R^{\mathrm{GL}_m}(\tau,\mathbf{s})$. Using Frobenius reciprocity as in [Kn, p.208], it follows that there exists $\omega \in \Pi(\mathrm{O}(m) \cap M_R(\mathbb{R}))$ such that

$$[\tau|_{\mathrm{O}(m)\cap M_R(\mathbb{R})} : \omega] > 0 \quad \text{and} \quad [\sigma|_{\mathrm{O}(m)\cap M_R(\mathbb{R})} : \omega] > 0.$$

Let λ_σ and λ_ω denote the Casimir eigenvalues of σ and ω, respectively. By [Mu2, (5.15)], the second inequality implies $\lambda_\omega \leq \lambda_\sigma$. On the other hand, by [Wa2, p.398], the first inequality implies

$$\parallel \Lambda_\tau \parallel^2 \leq \lambda_\omega + \parallel \rho_R \parallel^2 .$$

Combining our estimations the lemma follows. □

Now let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. Set

$$K_{M,f} = K_f \cap M(\mathbb{A}_f).$$

Then $K_{M,f}$ is an open compact subgroup of $M(\mathbb{A}_f)$. There exist open compact subgroups $K_{f,i}$ of $\mathrm{GL}_{n_i}(\mathbb{A}_f)$, $i = 1,...,r$, such that $K_{f,1} \times \cdots \times K_{f,r}$ is a subgroup of finite index of K_f. Set

$$\Pi(M(\mathbb{A}),\xi_0)_{K_f} = \{\pi \in \Pi(M(\mathbb{A}),\xi_0) \,\big|\, \mathcal{H}_{\pi_f}^{K_{M,f}} \neq \{0\}\},$$

where $\pi = \pi_\infty \otimes \pi_f$. Let $\pi \in \Pi(M(\mathbb{A}),\xi_0)_{K_f}$. Then $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ and π_i belongs to $\Pi(\mathrm{GL}_{n_i}(\mathbb{A}),\xi_0)_{K_{f,i}}$ and by Lemma 5.3 it follows that there exists $C > 0$ such that

$$\prod_{i,j} \log(1 + \nu(\pi_i \times \widetilde{\pi}_j)) \leq C \prod_{i,j} \log(2 + c(\pi_{i,\infty}) + c(\pi_{j,\infty})) \tag{5.19}$$

for all $\pi = \pi_1 \otimes \cdots \otimes \pi_r \in \Pi(M(\mathbb{A}),\xi_0)_{K_f}$.

Let $K_{M,\infty} = \mathrm{O}(n_1) \times \cdots \times \mathrm{O}(n_r)$ be the standard maximal compact subgroup of $M(\mathbb{R})$. Let $\sigma \in \Pi(\mathrm{O}(n))$. For $\pi \in \Pi(M(\mathbb{A}), \xi_0)$ set

$$[\pi_\infty : \sigma] = \sum_{\tau \in \Pi(K_{M,\infty})} [\pi_\infty|_{K_{M,\infty}} : \tau][\sigma|_{K_{M,\infty}} : \tau].$$

Put

$$\Pi(M(\mathbb{A}), \xi_0)_{K_f,\sigma} = \{\pi \in \Pi(M(\mathbb{A}), \xi_0)_{K_f} \mid [\pi_\infty : \sigma] > 0\}$$

and

$$\Pi_{\mathrm{dis}}(M(\mathbb{A}), \xi_0)_{K_f,\sigma} = \Pi_{\mathrm{dis}}(M(\mathbb{A}), \xi_0) \cap \Pi(M(\mathbb{A}), \xi_0)_{K_f,\sigma}.$$

Suppose that $\pi \in \Pi(M(\mathbb{A}), \xi_0)_{K_f,\sigma}$. Let $\tau \in \Pi(K_{M,\infty})$ be such that $[\sigma|_{K_{M,\infty}} : \tau] > 0$ and $[\pi_\infty|_{K_{M,\infty}} : \tau] > 0$.

Let λ_{π_∞} and λ_τ denote the Casimir eigenvalues of the restriction of π_∞ to $M(\mathbb{R})^1$ and of τ, respectively. Note that $\lambda_{\pi_\infty} = \sum_i \lambda_{\pi_{i,\infty}}$ and $\lambda_\tau = \sum_i \lambda_{\tau_i}$, where $\tau = \otimes_i \tau_i$. Then it follows from (5.19) and Lemma 5.4 that there exists $C > 0$ such that

$$\begin{aligned}\prod_{i,j} \log(1 + \nu(\pi_i \times \widetilde{\pi}_j)) &\le C \prod_{i,j} \log(2 - \lambda_{\pi_{i,\infty}} + \lambda_{\tau_i} - \lambda_{\pi_{j,\infty}} + \lambda_{\tau_j}) \\ &\le C\big(\log(2 - \lambda_{\pi_\infty} + \lambda_\tau)\big)^{r^2}\end{aligned}$$

for all $\pi \in \Pi(M(\mathbb{A}), \xi_0)_{K_f,\sigma}$. Since there are only finitely many τ that occur in $\sigma|_{K_{M,\infty}}$, we get

$$\prod_{i,j} \log(1 + \nu(\pi_i \times \widetilde{\pi}_j)) \le C_1\big(\log(2 + |\lambda_{\pi_\infty}|)\big)^{r^2} \tag{5.20}$$

for all $\pi \in \Pi(M(\mathbb{A}), \xi_0)_{K_f,\sigma}$. Combining (5.17)–(5.20) we obtain

Proposition 5.5. *Let $M \in \mathcal{L}$, $L \in \mathcal{L}(M)$ and $P \in \mathcal{P}(M)$. Let $l = \dim(A_L/A_G)$. Let K_f be an open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$ and let $\sigma \in \Pi(\mathrm{O}(n))$. There exists $C > 0$ such that*

$$\int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*} |\nu_L^S(P, \pi, \lambda)| e^{-t\|\lambda\|^2}\, d\lambda \le C\, (\log(2 + |\lambda_{\pi_\infty}|))^{n^2}\, \frac{(1 + |\log t|)^l}{t^{l/2}}$$

for all $0 < t \le 1$ and $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}), \xi_0)_{K_f,\sigma}$.

6. The spectral side

We shall use the noninvariant trace formula of Arthur [A1], [A2], applied to the heat kernel, to determine the growth of the discrete spectrum. To begin with, we explain the general structure of the spectral side of the Arthur trace formula. The spectral side is a sum of distributions

$$\sum_{\chi \in \mathfrak{X}} J_\chi(f), \quad \chi \in C_0^\infty(G(\mathbb{A})^1).$$

Here $\mathfrak{X}$ is the set of cuspidal datas which consists of Weyl group orbits of pairs (M_B, ρ_B), where M_B is the Levi component of a parabolic subgroup and ρ_B is a cuspidal automorphic representation of $M_B(\mathbb{A})$. The distributions J_χ are described by Theorem 8.2 of [A4]. Let $\mathcal{C}^1(G(\mathbb{A})^1)$ be the space of integrable rapidly decreasing functions on $G(\mathbb{A})^1$ [MS, §1.3]. In [MS, Theorem 0.1] it has been proved that the spectral side of the trace formula for GL_n is absolutely convergent for all $f \in \mathcal{C}^1(G(\mathbb{A})^1)$. In this case the expression of the spectral side simplifies.

To describe this in more detail, we need to introduce some notation. Let $M \in \mathcal{L}$ and $P, Q \in \mathcal{P}(M)$. Let $\mathcal{A}^2(P)$ and $\mathcal{A}^2(Q)$ be the corresponding spaces of automorphic functions (see §1.5). Let $W(\mathfrak{a}_P, \mathfrak{a}_Q)$ be the set of all linear isomorphisms from $\mathfrak{a}_P$ to $\mathfrak{a}_Q$ which are restrictions of elements of the Weyl group $W(A_0)$. The theory of Eisenstein series associates to each $s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$ an intertwining operator

$$M_{Q|P}(s, \lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}},$$

which, for $\mathrm{Re}(\lambda)$ in a certain chamber, can be defined by an absolutely convergent integral and admits an analytic continuation to a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ [La]. Set

$$M_{Q|P}(\lambda) := M_{Q|P}(1, \lambda).$$

Fix $P \in \mathcal{P}(M)$ and $\lambda \in i\mathfrak{a}^*_M$. For $Q \in \mathcal{P}(M)$ and $\Lambda \in i\mathfrak{a}^*_M$ define

$$\mathfrak{M}_Q(P, \lambda, \Lambda) = M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda).$$

Then

$$\{\mathfrak{M}_Q(P, \lambda, \Lambda) \mid \Lambda \in i\mathfrak{a}^*_M,\ Q \in \mathcal{P}(M)\} \tag{6.1}$$

is a (G, M) family with values in the space of operators on $\mathcal{A}^2(P)$ [A4, p.1310]. Let $L \in \mathcal{L}(M)$. Then, as explained in the previous section, the (G, M) family (6.1) has an associated (G, L) family

$$\{\mathfrak{M}_{Q_1}(P, \lambda, \Lambda) \mid \Lambda \in i\mathfrak{a}^*_L,\ Q_1 \in \mathcal{P}(L)\}$$

and

$$\mathfrak{M}_L(P, \lambda, \Lambda) = \sum_{Q_1 \in \mathcal{P}(L)} \mathfrak{M}_{Q_1}(P, \lambda, \Lambda) \theta_{Q_1}(\Lambda)^{-1}$$

extends to a smooth function on $i\mathfrak{a}^*_L$. Put

$$\mathfrak{M}_L(P, \lambda) = \mathfrak{M}_L(P, \lambda, 0).$$

This operator depends only on the intertwining operators. It equals

$$\mathfrak{M}_L(P, \lambda) = \lim_{\Lambda \to 0} \left(\sum_{Q_1 \in \mathcal{P}(L)} \mathrm{vol}(\mathfrak{a}^G_{Q_1}/\mathbb{Z}(\Delta^\vee_{Q_1})) M_{Q|P}(\lambda)^{-1} \frac{M_{Q|P}(\lambda + \Lambda)}{\prod_{\alpha \in \Delta_{Q_1}} \Lambda(\alpha^\vee)} \right),$$

where λ and Λ are constrained to lie in $i\mathfrak{a}^*_L$, and for each $Q_1 \in \mathcal{P}(L)$, Q is a group in $\mathcal{P}(M_P)$ which is contained in Q_1. Then $\mathfrak{M}_L(P, \lambda)$ is an unbounded operator which acts on the Hilbert space $\overline{\mathcal{A}}^2(P)$. For $\pi \in \Pi(M(\mathbb{A})^1)$ let $\mathcal{A}^2_\pi(P)$ be the subspace of $\mathcal{A}^2(P)$ determined by π (see §1.5). Let $\rho_\pi(P, \lambda)$ be the induced representation of $G(\mathbb{A})$ in $\overline{\mathcal{A}}^2_\pi(P)$.

Let $W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ be the set of elements $s \in W(\mathfrak{a}_M)$ such that $\{H \in \mathfrak{a}_M \mid sH = H\} = \mathfrak{a}_L$. For any function $f \in \mathcal{C}^1(G(\mathbb{A})^1)$ and $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ set

$$J^L_{M,P}(f,s) = \sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} \int_{i\mathfrak{a}^*_L/i\mathfrak{a}^*_G} \mathrm{tr}(\mathfrak{M}_L(P,\lambda) M_{P|P}(s,0)\rho_\pi(P,\lambda,f))\, d\lambda. \tag{6.2}$$

By Theorem 0.1 of [MS] this integral-series is absolutely convergent with respect to the trace norm. Furthermore for $M \in \mathcal{L}$ and $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ set

$$a_{M,s} = |\mathcal{P}(M)|^{-1} |W_0^M| |W_0|^{-1} |\det(s-1)_{\mathfrak{a}^L_M}|^{-1}.$$

Then for any f in $\mathcal{C}^1(G(\mathbb{A})^1)$, the spectral side $J_{\mathrm{spec}}(f)$ of the Arthur trace formula is given by

$$J_{\mathrm{spec}}(f) = \sum_{M \in \mathcal{L}} \sum_{L \in \mathcal{L}(M)} \sum_{P \in \mathcal{P}(M)} \sum_{s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}} a_{M,s} J^L_{M,P}(f,s). \tag{6.3}$$

Note that all sums in this expression are finite.

We shall now evaluate the spectral side at a function ϕ_t, $t > 0$, which is given in terms of the heat kernel of a Bochner-Laplace operator. Then our main purpose is to determine the behaviour of $J_{\mathrm{spec}}(\phi_t)$ as $t \to 0$.

Let $G(\mathbb{R})^1 = G(\mathbb{A})^1 \cap G(\mathbb{R})$. By definition $G(\mathbb{R})^1$ consisits of all $g \in G(\mathbb{R})$ with $|\det(g)| = 1$. Hence $G(\mathbb{R})^1$ is semisimple and

$$G(\mathbb{R}) = G(\mathbb{R})^1 \cdot A_G(\mathbb{R})^0.$$

Let

$$X = G(\mathbb{R})^1 / K_\infty$$

be the associated Riemannian symmetric space. Given $\sigma \in \Pi(K_\infty)$, let $\widetilde{E}_\sigma \to X$ be the associated homogeneous vector bundle. Let $\Omega_{G(\mathbb{R})^1}$ be the Casimir element of $G(\mathbb{R})^1$ and let $\widetilde{\Delta}_\sigma$ be the operator in $L^2(\widetilde{E}_\sigma)$ which is induced by $-R(\Omega_{G(\mathbb{R})^1}) \otimes \mathrm{Id}$. Let

$$H^\sigma_t \in \left(\mathcal{C}^1(G(\mathbb{R})^1) \otimes \mathrm{End}(V_\sigma)\right)^{K_\infty \times K_\infty} \tag{6.4}$$

be the kernel of the heat operator $e^{-t\widetilde{\Delta}_\sigma}$ where $\mathcal{C}^1(G(\mathbb{R}))$ is Harish-Chandra's space of integrable rapidly decreasing functions. Set

$$h^\sigma_t = \mathrm{tr}\, H^\sigma_t.$$

We extend h^σ_t to a function on $G(\mathbb{R})$ by

$$h^\sigma_t(g \cdot z) = h^\sigma_t(g), \quad g \in G(\mathbb{R})^1,\ z \in A_G(\mathbb{R})^0.$$

Then h^σ_t satisfies

$$h^\sigma_t(gz) = h^\sigma_t(g), \quad g \in G(\mathbb{R}),\ z \in A_G(\mathbb{R})^0.$$

Let χ_σ be the character of σ. Then h^σ_t also satisfies

$$h^\sigma_t = \chi_\sigma * h^\sigma_t * \overline{\chi_\sigma}.$$

Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let $\mathbf{1}_{K_f}$ be the charcteristic function of K_f in $G(\mathbb{A}_f)$. Set

$$\chi_{K_f}=\operatorname{vol}(K_f)^{-1}\mathbf{1}_{K_f}.$$

Define the function ϕ_t on $G(\mathbb{A})$ by

$$\phi_t(g)=h_t^\sigma(g_\infty)\chi_{K_f}(g_f) \tag{6.5}$$

for any point

$$g=g_\infty g_f,\quad g_\infty\in G(\mathbb{R}),\ g_f\in G(\mathbb{A}_f),$$

in $G(\mathbb{A})$. Then ϕ_t satisfies $\phi_t(gz)=\phi_t(g)$ for $z\in A_G(\mathbb{R})^0$, $g\in G(\mathbb{A})$. It follows from (6.4) and the definition of $\mathcal{C}^1(G(\mathbb{A})^1)$ that the restriction ϕ_t^1 of ϕ_t to $G(\mathbb{A})^1$ belongs to $\mathcal{C}^1(G(\mathbb{A})^1)$.

Let π be any unitary representation of $G(\mathbb{A})$ which is trivial on $A_G(\mathbb{R})^0$. Then we can define

$$\pi(\phi_t)=\int_{G(\mathbb{A})/A_G(\mathbb{R})^0}\phi_t(g)\pi(g)\,dg.$$

Suppose that $\pi=\pi_\infty\otimes\pi_f$, where π_∞ and π_f are unitary representations of $G(\mathbb{R})$ and $G(\mathbb{A}_f)$, respectively. Then π_∞ is trivial on $A_G(\mathbb{R})^0$. So we can set

$$\pi_\infty(\phi_t)=\int_{G(\mathbb{R})/A_G(\mathbb{R})^0}\pi_\infty(g_\infty)h_t^\sigma(g_\infty)\,dg_\infty.$$

Let Π_{K_f} denote the orthogonal projection of the Hilbert space $\mathcal{H}_{\pi_f}$ of π_f onto the subspace $\mathcal{H}_{\pi_f}^{K_f}$ of K_f-invariant vectors. Then

$$\pi(\phi_t)=\pi_\infty(h_t^\sigma)\otimes\Pi_{K_f}.$$

Now let $\pi\in\Pi(M(\mathbb{A})^1)$. We identify π with a representation of $M(\mathbb{A})$ which is trivial on $A_M(\mathbb{R})^0$. Let $I_P^G(\pi_\lambda)$, $\lambda\in\mathfrak{a}^*_{M,\mathbb{C}}$, be the induced representation of $G(\mathbb{A})$. Let $\pi=\pi_\infty\otimes\pi_f$. Then

$$I_P^G(\pi_\lambda)=I_P^G(\pi_{\infty,\lambda})\otimes I_P^G(\pi_{f,\lambda}).$$

Let $\mathcal{H}_P(\pi_\infty)_\sigma$ denote the σ-isotypical subspace of the Hilbert space $\mathcal{H}_P(\pi_\infty)$ of the induced representation. Then $\mathcal{H}_P(\pi_\infty)_\sigma$ is an invariant subspace of $I_P^G(\pi_{\infty,\lambda},h_t^\sigma)$. Let λ_π be the Casimir eigenvalue of the restriction of π_∞ to $M(\mathbb{R})^1$. By Proposition 8.22 of [Kn] it follows that

$$I_P^G(\pi_{\infty,\lambda},h_t)\restriction\mathcal{H}_P(\pi_\infty)_\sigma=e^{-t(-\lambda_\pi+\|\lambda\|^2)}\operatorname{Id}.$$

Now observe that there is a canonical isomorphism

$$j_P:\mathcal{H}_P(\pi)\otimes\operatorname{Hom}_{M(\mathbb{A})}\left(\pi,I^{M(\mathbb{A})}_{M(\mathbb{Q})A_M(\mathbb{R})^0}(\xi_0)\right)\to\overline{\mathcal{A}}^2_\pi(P),$$

which intertwines the induced representations. Let $\Pi_{K_f,\sigma}$ denote the orthogonal projection of $\overline{\mathcal{A}}^2_\pi(P)$ onto $\mathcal{A}^2_\pi(P)_{K_f,\sigma}$. Then it follows that

$$\rho_\pi(P,\lambda,\phi_t)=e^{-(-\lambda_\pi+\|\lambda\|^2)}\Pi_{K_f,\sigma}. \tag{6.6}$$

Suppose that $\lambda \in (\mathfrak{a}_P^G)^*_{\mathbb{C}}$. Then $\rho_\pi(P,\lambda,g)$ is trivial on $A_G(\mathbb{R})^0$. This implies $\rho_\pi(P,\lambda,\phi_t) = \rho_\pi(P,\lambda,\phi_t^1)$, where ϕ_t^1 is the restriciton of ϕ_t to $G(\mathbb{A})^1$. Together with (6.6) we get

$$\begin{aligned} J^L_{M,P}(\phi_t^1,s) = &\sum_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} e^{t\lambda_\pi} \\ &\cdot \int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*} e^{-t\|\lambda\|^2}\,\mathrm{tr}(\mathfrak{M}_L(P,\lambda)M_{P|P}(s,0)\Pi_{K_f,\sigma})\,d\lambda. \end{aligned} \tag{6.7}$$

To study this integral-series, we introduce the normalized intertwining operators

$$N_{Q|P}(\pi,\lambda) := r_{Q|P}(\pi,\lambda)^{-1}M_{Q|P}(\pi,\lambda), \quad \lambda\in\mathfrak{a}^*_{M,\mathbb{C}}, \tag{6.8}$$

where $r_{Q|P}(\pi,\lambda)$ are the global normalizing factors considered in the previous section. Let $P\in\mathcal{P}(M)$ and $\lambda\in i\mathfrak{a}_M^*$ be fixed. For $Q\in\mathcal{P}(M)$ and $\Lambda\in i\mathfrak{a}_M^*$ define

$$\mathfrak{N}_Q(P,\pi,\lambda,\Lambda) = N_{Q|P}(\pi,\lambda)^{-1}N_{Q|P}(\pi,\lambda+\Lambda), \tag{6.9}$$

Then as functions of $\Lambda\in i\mathfrak{a}_M^*$,

$$\{\mathfrak{N}_Q(P,\pi,\lambda,\Lambda) \mid Q\in\mathcal{P}(M)\}$$

is a (G,M) family. The verification is the same as in the case of the unnormalized intertwining operators [A4, p.1310]. For $L\in\mathcal{L}(M)$, let

$$\{\mathfrak{N}_{Q_1}(P,\pi,\lambda,\Lambda) \mid \Lambda\in i\mathfrak{a}_L^*,\ Q_1\in\mathcal{P}(L)\}$$

be the associated (G,L) family.

Let $\mathfrak{M}_{Q_1}(P,\pi,\lambda,\Lambda)$ be the restriction of $\mathfrak{M}_{Q_1}(P,\lambda,\Lambda)$ to $\overline{\mathcal{A}}^2_\pi(P)$. Then by (6.8) and (5.14) it follows that

$$\mathfrak{M}_{Q_1}(P,\pi,\lambda,\Lambda) = \mathfrak{N}_{Q_1}(P,\pi,\lambda,\Lambda)\nu_{Q_1}(P,\pi,\lambda,\Lambda) \tag{6.10}$$

for all $\Lambda\in i\mathfrak{a}_L^*$ and all $Q_1\in\mathcal{P}(L)$.

For $Q\supset P$ let $\hat{L}_P^Q\subset\mathfrak{a}_P^Q$ be the lattice generated by $\{\tilde{\omega}^\vee \mid \tilde{\omega}\in\hat{\Delta}_P^Q\}$. Define

$$\hat{\theta}_P^Q(\lambda) = \mathrm{vol}(\mathfrak{a}_P^Q/\hat{L}_P^Q)^{-1}\prod_{\tilde{\omega}\in\hat{\Delta}_P^Q}\lambda(\tilde{\omega}^\vee).$$

For $S\in\mathcal{F}(L)$ put

$$\begin{aligned} &\mathfrak{N}'_S(P,\pi,\lambda) \\ &\quad= \lim_{\Lambda\to 0}\sum_{\{R|R\supset S\}}(-1)^{\dim(A_S/A_R)}\hat{\theta}_S^R(\Lambda)^{-1}\mathfrak{N}_R(P,\pi,\lambda,\Lambda)\theta_R(\Lambda)^{-1}. \end{aligned} \tag{6.11}$$

Let $\mathfrak{M}_L(P,\pi,\lambda)$ be the restriction of $\mathfrak{M}_L(P,\lambda)$ to $\overline{\mathcal{A}}^2_\pi(P)$. Then by (6.10) and Lemma 6.3 of [A5] we get

$$\mathfrak{M}_L(P,\pi,\lambda) = \sum_{S\in\mathcal{F}(L)}\mathfrak{N}'_S(P,\pi,\lambda)\nu_L^S(P,\pi,\lambda). \tag{6.12}$$

Let $\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}$ denote the restriction of $\mathfrak{N}'_S(P,\pi,\lambda)$ to $\mathcal{A}^2_\pi(P)_{K_f,\sigma}$. Then by (6.7) we get

$$\begin{aligned} J^L_{M,P}(\phi^1_t,s) = &\sum_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} e^{t\lambda_\pi} \\ &\cdot \sum_{S\in\mathcal{F}(L)} \int_{i\mathfrak{a}^*_L/\mathfrak{a}^*_G} e^{-t\|\lambda\|^2}\nu^S_L(P,\pi,\lambda)\operatorname{tr}(M_{P|P}(s,0)\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma})\,d\lambda. \end{aligned} \tag{6.13}$$

Next we shall estimate the norm of $\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}$. For a given place v of $\mathbb{Q}$ let $J_{Q|P}(\pi_v,\lambda)$ be the intertwining operator between the induced representations $I^G_P(\pi_{v,\lambda})$ and $I^G_Q(\pi_{v,\lambda})$. Let

$$R_{Q|P}(\pi_v,\lambda) = r_{Q|P}(\pi_v,\lambda)^{-1}J_{Q|P}(\pi_v,\lambda),\quad \lambda\in\mathfrak{a}^*_{M,\mathbb{C}},$$

be the normalized local intertwining operator. These operators satisfy the conditions $(R_1)-(R_8)$ of Theorem 2.1 of [A7]. Assume that $K_f=\prod_{p<\infty}K_p$. For any place v denote by $\mathcal{H}_P(\pi_v)$ the Hilbert space of the induced representation $I^G_P(\pi_v)$. If $p<\infty$ let $R_{Q|P}(\pi_p,\lambda)_{K_p}$ be the restriction of $R_{Q|P}(\pi_p,\lambda)$ to the subspace of K_p-invariant vectors $\mathcal{H}_P(\pi_p)^{K_p}$ in $\mathcal{H}_P(\pi_p)$. Let $R_{Q|P}(\pi_\infty,\lambda)_\sigma$ denote the restriction of $R_{Q|P}(\pi_\infty,\lambda)$ to the σ-isotypical subspace of $I^G_P(\pi_\infty)$ in $\mathcal{H}_P(\pi_\infty)$. It was proved in [Mu2, (6.24)] that there exist a finite set of places S_0, including the Archimedean one, and constants $C>0$ and $q\in\mathbb{N}$, such that

$$\begin{aligned}\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}\| \le C\Bigg(&\sum_{p\in S_0\setminus\{\infty\}}\sum_{k=1}^q \|D^k_\lambda R_{Q|P}(\pi_p,\lambda)_{K_p}\| \\ &\sum_{k=1}^q \|D^k_\lambda R_{Q|P}(\pi_\infty,\lambda)_\sigma\|\Bigg)\end{aligned}$$

for all $\lambda\in i\mathfrak{a}^*_M$, $\sigma\in\Pi(K_\infty)$ and $\pi\in\Pi(M(\mathbb{A}))$. By Proposition 0.2 of [MS] it follows that there exists $C>0$ such that

$$\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}\|\le C \tag{6.14}$$

for all $\lambda\in i\mathfrak{a}^*_M$ and $\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A})^1)$. Observe that $M_{P|P}(s,0)$ is unitary. Let $l=\dim(A_L/A_G)$. Using (6.13), (6.14) and Proposition 5.5 it follows that there exists $C>0$ such that

$$\begin{aligned}|J^L_{M,P}(\phi^1_t,s)| \le &\, C\frac{(2+|\log t|)^l}{t^{l/2}} \\ &\cdot\sum_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} \dim\mathcal{A}^2_\pi(P)_{K_f,\sigma}\big(\log(1+|\lambda_\pi|)\big)^{n^2}e^{t\lambda_\pi}\end{aligned} \tag{6.15}$$

for all $0<t\le 1$. The series can be estimated using Proposition 3.5. Let $X_M=M(\mathbb{R})/K'_{M,\infty}$ and let $m=\dim X_M$. It follows from Proposition 3.5 that for every $\epsilon>0$ there exists $C>0$ such that the series is bounded by $Ct^{-m/2-\epsilon}$ for $0<t\le 1$. Together with (6.15) we obtain the following proposition.

Proposition 6.1. *Let $m = \dim X_M$ and $l = \dim A_L/A_G$. For every $\epsilon > 0$ there exists $C > 0$ such that*

$$|J^L_{M,P}(\phi^1_t, s)| \le C t^{-(m+l)/2-\epsilon}$$

for all $0 < t \le 1$.

Now we distinguish two cases. First assume that $M = G$. Then $L = P = G$ and $s = 1$. Let R^1_{dis} be the restriciton of the regular representation R^1 of $G(\mathbb{A})^1$ in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ to the discrete subspace. Then $J^G_{G,G}(\phi^1_t, 1) = \operatorname{Tr} R^1_{\mathrm{dis}}(\phi^1_t)$. Let R_{dis} be the regular representation of $G(\mathbb{A})$ in $L^2_{\mathrm{dis}}(A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))$. Then the operator $R_{\mathrm{dis}}(\phi_t)$ is isomorphic to $R^1_{\mathrm{dis}}(\phi^1_t)$. Thus

$$J^G_{G,G}(\phi^1_t, 1) = \operatorname{Tr} R_{\mathrm{dis}}(\phi_t).$$

Given $\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}), \xi_0)$, let $m(\pi)$ denote the multiplicity with which π occurs in the regular representation of $G(\mathbb{A})$ in $L^2(\mathbb{A}_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))$. Then using Corollary 2.2 in [BM] we get

$$\begin{aligned} &J^G_{G,G}(\phi^1_t, 1) \\ &\quad = \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}),\xi_0)} m(\pi) \dim\big(\mathcal{H}^{K_f}_{\pi_f}\big) \dim\big(\mathcal{H}_{\pi_\infty} \otimes V_\sigma\big)^{\mathrm{O}(n)} e^{t\lambda_\pi}. \end{aligned} \tag{6.16}$$

Now assume that $M \neq G$ is a proper Levi subgroup. Let $P = MN$. Let $X = G(\mathbb{R})^1/K_\infty$. Then

$$X \cong X_M \times A_M(\mathbb{R})^0/A_G(\mathbb{R})^0 \times N(\mathbb{R}).$$

Since $l = \dim A_L/A_G \le \dim A_M/A_G$, it follows that $m + l \le \dim X - 1$. Thus together with Proposition 6.1 we get

Theorem 6.2. *Let $d = \dim X$. For every open compact subgroup K_f of $G(\mathbb{A}_f)$ and every $\sigma \in \Pi(\mathrm{O}(n))$ the spectral side of the trace formula, evaluated at ϕ^1_t, satisfies*

$$\begin{aligned} &J_{\mathrm{spec}}(\phi^1_t) \\ &\quad = \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}),\xi_0)} m(\pi) \dim\big(\mathcal{H}^{K_f}_{\pi_f}\big) \dim\big(\mathcal{H}_{\pi_\infty} \otimes V_\sigma\big)^{\mathrm{O}(n)} e^{t\lambda_\pi} \\ &\qquad + O(t^{-(d-1)/2}) \end{aligned} \tag{6.17}$$

as $t \to 0^+$.

This theorem can be restated in a slightly different way as follows. There exist arithmetic subgroups $\Gamma_i \subset G(\mathbb{R})$, $i = 1, ..., m$, such that

$$A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A})/K_f \cong \bigsqcup_{i=1}^m (\Gamma_i\backslash G(\mathbb{R})^1)$$

(cf. [Mu1, section 9]). Let $\Delta_{\sigma,i}$ be the operator induced by the negative of the Casimir operator in $C^\infty(\Gamma_i\backslash G(\mathbb{R})^1, \sigma)$, $i = 1, ..., m$. Let

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$$

be the L^2-eigenvalues of $\Delta_\sigma = \oplus_{i=1}^m \Delta_{\sigma,i}$, where each eigenvalue is counted with its multiplicity. Let $d = \dim X$. If we proceed in the same way as in the proof of Lemma 3.2, then it follows that (6.17) is equivalent to

$$J_{\mathrm{spec}}(\phi_t^1) = \sum_i e^{-t\lambda_i} + O(t^{-(d-1)/2}) \tag{6.18}$$

as $t \to 0^+$.

Let $\Gamma(N) \subset \mathrm{SL}_n(\mathbb{Z})$ be the principal congruence subgroup of level N. Let $\mu_0 \leq \mu_1 \leq \cdots$ be the eigenvalues, counted with multiplicity, of Δ_σ acting in $L^2(\Gamma(N)\backslash \mathrm{SL}_n(\mathbb{R}), \sigma)$. Then it follows from (6.18) and (3.10) that

$$J_{\mathrm{spec}}(\phi_t^1) = \varphi(N) \sum_i e^{-t\mu_i} + O(t^{-(d-1)/2}) \tag{6.19}$$

as $t \to 0^+$.

Our next purpose is to study J_{spec} as a functional on the Schwartz space. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let $\sigma \in \Pi(K_\infty)$. Denote by $\mathcal{C}^1(G(\mathbb{A})^1; K_f, \sigma)$ the set of all $h \in \mathcal{C}^1(G(\mathbb{A})^1)$ which are bi-invariant under K_f and transform under K_∞ according to σ. Let Δ_G be the Laplace operator of $G(\mathbb{R})^1$. Then we have

Proposition 6.3. *For every open compact subgroup K_f of $G(\mathbb{A}_f)$ and every $\sigma \in \Pi(K_\infty)$ there exist $C > 0$ and $k \in \mathbb{N}$ such that*

$$|J_{\mathrm{spec}}(f)| \leq C \parallel (\mathrm{Id} + \Delta_G)^k f \parallel_{L^1(G(\mathbb{A})^1)}$$

for all $f \in \mathcal{C}^1(G(\mathbb{A})^1; K_f, \sigma)$.

Proof. This follows essentially from the proof of Theorem 0.2 in [Mu2] combined with Proposition 0.2 of [MS]. We include some details. Let $M \in \mathcal{L}$, $L \in \mathcal{L}(M)$ and $P \in \mathcal{P}(M)$. By (6.3) it suffices to estimate $J_{M,P}^L(f, s)$. Since $M_{P|P}(s, 0)$ is unitary, it follows from (6.2) that

$$|J_{M,P}^L(f, s)| \leq \sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} \int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*} \parallel \mathfrak{M}_L(P, \lambda) \rho_\pi(P, \lambda, f) \parallel_1 d\lambda,$$

where $\parallel \cdot \parallel_1$ denotes the trace norm for operators in the Hilbert space $\overline{\mathcal{A}}_\pi^2(P)$. Using (6.12) it follows that the right hand side is bounded by

$$\sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \parallel \mathfrak{N}_S'(P, \pi, \lambda) \rho_\pi(P, \lambda, f) \parallel_1 |\nu_L^S(P, \pi, \lambda)| \, d\lambda$$

The function $\nu_L^S(P, \pi, \lambda)$ can be estimated by Theorem 5.4 of [Mu2]. This reduces our problem to the estimation of the trace norm of the operator $\mathfrak{N}_S'(P, \pi, \lambda) \rho_\pi(P, \lambda, f)$. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let $\sigma \in \Pi(K_\infty)$. Denote by $\Pi_{K_f,\sigma}$ the orthogonal projection of the Hilbert space $\overline{\mathcal{A}}_\pi^2(P)$ onto the finite-dimensional subspace $\mathcal{A}_\pi^2(P)_{K_f,\sigma}$. Let $f \in \mathcal{C}^1(G(\mathbb{A})^1; K_f, \sigma)$. Then

$$\rho_\pi(P, \lambda, f) = \Pi_{K_f,\sigma} \circ \rho_\pi(P, \lambda, f) \circ \Pi_{K_f,\sigma}$$

for all $\pi \in \Pi(M(A)^1)$. Let

$$D = \mathrm{Id} + \Delta_G.$$

For any $k \in \mathbb{N}$ let $\rho_\pi(P, \lambda, D^{2k})_{K_f,\sigma}$ denote the restriction of the operator $\rho_\pi(P, \lambda, D^{2k})$ to the subspace $\mathcal{A}^2_\pi(P)_{K_f,\sigma}$. Then we get

$$\begin{aligned} &\| \mathfrak{N}'_S(P, \pi, \lambda)\rho_\pi(P, \lambda, f) \|_1 \\ &\le \| \mathfrak{N}'_S(P, \pi, \lambda)_{K_f,\sigma} \| \cdot \| \rho_\pi(P, \lambda, D^{2k})^{-1}_{K_f,\sigma} \|_1 \\ &\qquad \cdot \| \rho_\pi(P, \lambda, D^{2k} f) \|, \end{aligned} \tag{6.20}$$

By (6.9) of [Mu2] we get

$$\| \rho_\pi(P, \lambda, D^{2k})^{-1}_{K_f,\sigma} \| \le C \frac{\dim \mathcal{A}^2_\pi(P)_{K_f,\sigma}}{(1 + \| \lambda \|^2 + \lambda_\pi^2)^k}, \tag{6.21}$$

and since $\rho_\pi(P, \lambda)$ is unitary, we have

$$\| \rho_\pi(P, \lambda, D^{2k} f) \| \le \| D^{2k} f \|_{L^1(G(\mathbb{A})^1)} . \tag{6.22}$$

Together with (6.14) it follows that there exists $C > 0$ such that

$$\begin{aligned} &\| \mathfrak{N}'_S(P, \pi, \lambda)\rho_\pi(P, \lambda, f) \|_1 \\ &\le C \| D^{2k} f \|_{L^1(G(\mathbb{A})^1)} (1 + \| \lambda \|)^{-k/2} \frac{\dim \mathcal{A}^2_\pi(P)_{K_f,\sigma}}{(1 + \lambda_\pi^2)^{k/2}} \end{aligned} \tag{6.23}$$

for all $\lambda \in i\mathfrak{a}^*_M$ and $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)$. Let $d = \dim G(\mathbb{R})^1/K_\infty$. By Theorem 5.4 of [Mu2] there exists $k_0 \in \mathbb{N}$ such that for $k \ge k_0$ we have

$$\int_{i\mathfrak{a}^*_L/\mathfrak{a}^*_G} |\nu^S_L(P, \pi, \lambda)| (1 + \| \lambda \|^2)^{-k/2} \, d\lambda \le C_k (1 + \lambda_\pi^2)^{8d^2} \tag{6.24}$$

for all $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)$ with $\mathcal{A}^2_\pi(P)_{K_f,\sigma} \ne 0$. Furthermore, by Proposition 3.4 we have

$$\sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} \frac{\dim \mathcal{A}^2_\pi(P)_{K_f,\sigma}}{(1 + \lambda_\pi^2)^{k/2}} < \infty \tag{6.25}$$

for $k > m/2 + 1$, where $m = \dim M(\mathbb{R})^1/K_{M,\infty}$. Combining (6.23)- (6.25), it follows that for each $k > m/2 + 16d^2 + 1$ there exists $C_k > 0$ such that

$$\begin{aligned} \sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} \int_{i\mathfrak{a}^*_L/i\mathfrak{a}^*_G} &\| \mathfrak{N}'_S(P, \pi, \lambda)\rho_\pi(P, \lambda, f) \|_1 \, |\nu^S_L(P, \pi, \lambda)| \, d\lambda \\ &\le C_k \| D^{2k} f \|_{L^1(G(\mathbb{A})^1)} . \end{aligned}$$

This completes the proof. □

Now we return to the function ϕ_t defined by (6.5). It follows from the definition that the restriction ϕ_t^1 of ϕ_t belongs to $\mathcal{C}^1(G(\mathbb{A})^1, K_f, \sigma)$. We shall now modify ϕ_t in the following way. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\varphi(u) = 1$, if $|u| \le 1/2$, and $\varphi(u) = 0$, if $|u| \ge 1$. Let $d(x, y)$ denote the geodesic distance of $x, y \in X$ and set

$$r(g_\infty) := d(g_\infty K_\infty, K_\infty).$$

Given $t > 0$, let $\varphi_t \in C_0^\infty(G(\mathbb{R})^1)$ be defined by

$$\varphi_t(g_\infty) = \varphi(r^2(g_\infty)/t^{1/2}).$$

Then $\operatorname{supp} \varphi_t$ is contained in the set $\{g_\infty \in G(\mathbb{R})^1 \mid r(g_\infty) < t^{1/4}\}$. Extend φ_t to $G(\mathbb{R})$ by

$$\varphi_t(g_\infty z) = \varphi_t(g_\infty), \quad g_\infty \in G(\mathbb{R})^1,\ z \in A_G(\mathbb{R})^0,$$

and then to a function on $G(\mathbb{A})$ by multiplying φ_t by the characteristic function of K_f. Put

$$\widetilde{\phi}_t(g) = \varphi_t(g)\phi_t(g), \quad g \in G(\mathbb{A}). \tag{6.26}$$

Then the restriction $\widetilde{\phi}_t^1$ of $\widetilde{\phi}_t$ to $G(\mathbb{A})^1$ belongs to $C_c^\infty(G(\mathbb{A})^1)$.

Proposition 6.4. *There exist $C, c > 0$ such that*

$$|J_{\text{spec}}(\phi_t^1) - J_{\text{spec}}(\widetilde{\phi}_t^1)| \le C e^{-c/\sqrt{t}}$$

for $0 < t \le 1$.

Proof. Let $\psi_t = \phi_t - \widetilde{\phi}_t$ and $f_t = 1 - \varphi_t$. Let ψ_t^1 denote the restriction of ψ_t to $G(\mathbb{A})^1$. Then by Proposition 6.3 there exists $k \in \mathbb{N}$ such that

$$|J_{\text{spec}}(\phi_t^1) - J_{\text{spec}}(\widetilde{\phi}_t^1)| = |J_{\text{spec}}(\psi_t^1)| \le C_k \parallel (\operatorname{Id} + \Delta_G)^k \psi_t^1 \parallel_{L^1(G(\mathbb{A})^1)} .$$

In order to estimate the L^1-norm of ψ_t^1, recall that by definition

$$\psi_t(g_\infty g_f) = f_t(g_\infty) h_t^\sigma(g_\infty) \chi_{K_f}(g_f).$$

Hence

$$\parallel (\operatorname{Id} + \Delta_G)^k \psi_t^1 \parallel_{L^1(G(\mathbb{A})^1)} = \parallel (\operatorname{Id} + \Delta_G)^k (f_t h_t^\sigma) \parallel_{L^1(G(\mathbb{R})^1)} .$$

Let $\mathfrak{g}(\mathbb{R})^1$ be the Lie algebra of $G(\mathbb{R})^1$ and let $X_1, ..., X_a$ be an orthonormal basis of $\mathfrak{g}(\mathbb{R})^1$. Then $\Delta_G = -\sum_i X_i^2$. Denote by ∇ the canonical connection on $G(\mathbb{R})^1$. Then it follows that there exists $C > 0$ such that

$$|(\operatorname{Id} + \Delta_G)^k f(g)| \le C \sum_{l=0}^{2k} \parallel \nabla^l f(g) \parallel, \quad g \in G(\mathbb{R})^1,$$

for all $f \in C^\infty(G(\mathbb{R})^1)$. By Proposition 2.1 there exist constants $C, c > 0$ such that

$$\parallel \nabla^j h_t^\sigma(g) \parallel \le C t^{-(a+j)/2} e^{-cr^2(g)/t}, \quad g \in G(\mathbb{R})^1, \tag{6.27}$$

for $j \le 2k$ and $0 < t \le 1$. Let χ_t be the characteristic function of the set $\mathbb{R} - (-t^{1/4}, t^{1/4})$. Recall that $f_t(g) = (1-\varphi)(r^2(g)/t^{1/2})$ and $(1-\varphi)(u)$ is constant for $|u| \ge 1$. This implies that there exist constants $C, c > 0$ such that

$$\parallel \nabla^j f_t(g) \parallel \le C t^{-k} \chi_t(r(g)), \quad g \in G(\mathbb{R})^1, \tag{6.28}$$

for $j \leq 2k$ and $0 < t \leq 1$. Combining (6.27) and (6.28) we obtain

$$\begin{aligned}\sum_{l=0}^{2k} \parallel \nabla^l (f_t h_t^\sigma)(g) \parallel &\leq C_1 t^{-a/2-2k} \chi_t(r(g)) e^{-cr^2(g)/t} \\ &\leq C_2 e^{-c_1/\sqrt{t}} e^{-c_1 r^2(g)}\end{aligned}$$

for all $g \in G(\mathbb{R})^1$ and $0 < t \leq 1$. Finally note that for every $c > 0$, $e^{-cr^2(g)}$ is an integrable function on $G(\mathbb{R})^1$. This finishes the proof. $\square$

7. Proof of the main theorem

In this section we evaluate the geometric side of the trace formula at the function $\widetilde{\phi}_t^1$ and investigate its asymptotic behaviour as $t \to 0$. Then we compare the geometric and the spectral side and prove our main theorem.

Let us briefly recall the structur of the geometric side J_{geo} of the trace formula [A1]. The coarse $\mathfrak{o}$-expansion of $J_{\text{geo}}(f)$ is a sum of distributions

$$J_{\text{geo}}(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1),$$

which are parametrized by the set $\mathcal{O}$ of conjugacy classes of semisimple elements in $G(\mathbb{Q})$. The distributions $J_{\mathfrak{o}}(f)$ are defined in [A1]. We shall use the fine $\mathfrak{o}$-expansion of the spectral side [A10] which expresses the distributions $J_{\mathfrak{o}}(f)$ in terms of weighted orbital integrals $J_M(\gamma, f)$. To describe the fine $\mathfrak{o}$-expansion we have to introduce some notation. Suppose that S is a finite set of valuations of $\mathbb{Q}$. Set

$$G(\mathbb{Q}_S)^1 = G(\mathbb{Q}_S) \cap G(\mathbb{A})^1,$$

where

$$\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v.$$

Suppose that ω is a compact neighborhood of 1 in $G(\mathbb{A})^1$. There is a finite set S of valuations of $\mathbb{Q}$, which contains the Archimedean place, such that ω is the product of a compact neighborhood of 1 in $G(\mathbb{Q}_S)^1$ with $\prod_{v \notin S} K_v$. Let S_ω^0 be the minimal such set. Let $C_\omega^\infty(G(\mathbb{A})^1)$ denote the space of functions in $C_c^\infty(G(\mathbb{A})^1)$ which are supported on ω. For any finite set $S \supset S_\omega^0$ set

$$C_\omega^\infty(G(\mathbb{Q}_S)^1) = C_\omega^\infty(G(\mathbb{A})^1) \cap C_c^\infty(G(\mathbb{Q}_S)^1).$$

Let us recall the notion of (M,S)-equivalence [A10, p.205]. For any $\gamma \in M(\mathbb{Q})$ denote by γ_s (resp. γ_u) the semisimple (resp. unipotent) Jordan component of γ. Then two elements γ and γ' in $M(\mathbb{Q})$ are called (M,S)-equivalent if there exists $\delta \in M(\mathbb{Q})$ with the following two properties.

(i) γ_s is also the semisimple Jordan component of $\delta^{-1}\gamma'\delta$.

(ii) γ_u and $(\delta^{-1}\gamma'\delta)_u$, regarded as unipotent elements in $M_{\gamma_s}(\mathbb{Q}_S)$, are $M_{\gamma_s}(\mathbb{Q}_S)$-conjugate.

Denote by $(M(\mathbb{Q}))_{M,S}$ the set of (M,S)-equivalence classes in $M(\mathbb{Q})$. Note that (M,S)-equivalent elements γ and γ' in $M(\mathbb{Q})$ are, in particular, $M(\mathbb{Q}_S)$-conjugate. Given $\gamma \in M(\mathbb{Q})$, let

$$J_M(\gamma, f), \quad f \in C_c^\infty(G(\mathbb{Q}_S)^1),$$

be the weighted orbital integral associated to M and γ [A11]. We observe that $J_M(\gamma,f)$ depends only on the $M(\mathbb{Q}_S)$-orbit of γ. Then by Theorem 9.1 of [A10] there exists a finite set $S_\omega \supset S_\omega^0$ of valuations of $\mathbb{Q}$ such that for all $S \supset S_\omega$ and any $f \in C_\omega^\infty(G(\mathbb{Q}_S)^1)$, we have

$$J_{\mathrm{geo}}(f) = \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma\in(M(\mathbb{Q}))_{M,S}} a^M(S,\gamma) J_M(\gamma,f). \tag{7.1}$$

This is the fine $\mathfrak{o}$-expansion of the geometric side of the trace formula. The interior sum is finite.

Recall that the restriction $\widetilde{\phi}_t^1$ of $\widetilde{\phi}_t$ to $G(\mathbb{A})^1$ belongs to $C_c^\infty(G(\mathbb{A})^1)$ and hence, J_{geo} can be evaluated at $\widetilde{\phi}_t^1$. By construction of $\widetilde{\phi}_t^1$ there exists a compact neighborhood ω of 1 in $G(\mathbb{A})^1$ and a finite set $S \supset S_\omega$ of valuations of $\mathbb{Q}$ such that

$$\widetilde{\phi}_t^1 \in C_\omega^\infty(G(\mathbb{Q}_S)^1), \quad 0 < t \le 1.$$

Hence we can apply (7.1) to evaluate $J_{\mathrm{geo}}(\widetilde{\phi}_t^1)$. In this way our problem is reduced to the investigation of the weighted orbital integrals $J_M(\gamma, \widetilde{\phi}_t^1)$. Actually for $\gamma \in M(\mathbb{Q})$ we may replace $\widetilde{\phi}_t^1$ by $\widetilde{\phi}_t$.

To begin with we establish some auxiliary results. Given $h \in G(\mathbb{R})$, let

$$C_h = \{g^{-1}hg \mid g \in G(\mathbb{R})\}$$

be the conjugacy class of h in $G(\mathbb{R})$.

Lemma 7.1. *Let $k \in K_\infty$. Then $C_k \cap K_\infty$ is the K_∞-conjugacy class of k.*

Proof. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G(\mathbb{R})$ and K_∞, respectively. Let θ be a Cartan involution of $\mathfrak{g}$ with fixed point set $\mathfrak{k}$ and let $\mathfrak{p}$ be the (-1)-eigenspace of θ. Then the map

$$(k', X) \in K_\infty \times \mathfrak{p} \longmapsto k' \exp(X) \in G(\mathbb{R})$$

is an analytic isomorphism of analytic manifolds. If $k_1 \in K_\infty$, then k_1 is a θ-invariant semisimple element. Therefore, its centralizer G_{k_1} is a reductive subgroup and the restriction of θ to G_{k_1} is a Cartan involution. Thus the restriction of the above Cartan decomposition to the centralizer of k_1 yields a Cartan decomposition of $G_{k_1}(\mathbb{R})$. Let $g \in G(\mathbb{R})$ such that $g^{-1}kg \in K_\infty$. Write $g = k'\exp(X)$ with $k' \in K_\infty$ and $X \in \mathfrak{p}$. Since $g^{-1}kg$ is θ-invariant, we get

$$\exp(-X){k'}^{-1}kk'\exp(X) = \exp(X){k'}^{-1}kk'\exp(-X).$$

Hence $\exp(2X) \in G_{k'^{-1}kk'}(\mathbb{R})$. From the Cartan decomposition of the latter group we conclude that $\exp(2X) = \exp(Y)$ for some $Y \in \mathfrak{p}_{k'^{-1}kk'}$, and hence $X \in \mathfrak{p}_{k'^{-1}kk'}$. This implies that $g^{-1}kg = k'^{-1}kk'$. $\square$

It follows from Lemma 7.1 that $C_k \cap K_\infty$ is a submanifold of C_k.

Lemma 7.2. *Let $k \in K_\infty - \{\pm 1\}$. Then $C_k \cap K_\infty$ is a proper submanifold of C_k.*

Proof. Let the notation be as in the previous lemma. First note that the tangent space of C_k at k is given by

$$T_k C_k \cong (\mathrm{Ad}(k) - \mathrm{Id})(\mathfrak{g}).$$

Furthermore

$$\mathrm{Ad}(k)(\mathfrak{k}) \subset \mathfrak{k}, \quad \mathrm{Ad}(k)(\mathfrak{p}) \subset \mathfrak{p}.$$

Hence we get

$$T_k(C_k \cap K_\infty) = T_k C_k \cap \mathfrak{k} = (\mathrm{Ad}(k) - \mathrm{Id})(\mathfrak{k}),$$

and so the normal space N_k to $C_k \cap K_\infty$ in C_k at k is given by

$$N_k \cong (\mathrm{Ad}(k) - \mathrm{Id})(\mathfrak{p}).$$

Suppose that $\mathrm{Ad}(k) = \mathrm{Id}$ on $\mathfrak{p}$. Since $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$, it follows that $\mathrm{Ad}(k) = \mathrm{Id}$ on $\mathfrak{g}$. Hence k belongs to the center of G^0, which implies that $k = \pm 1$. Thus if $k \neq \pm 1$, we have $\dim N_k > 0$. $\square$

Next we recall the notion of an induced space of orbits [A11, p.255]. Given an element $\gamma \in M(\mathbb{Q}_S)$, let γ^G be the union of those conjugacy classes in $G(\mathbb{Q}_S)$ which for any $P \in \mathcal{P}(M)$ intersect $\gamma N_P(\mathbb{Q}_S)$ in an open set. There are only finitely many such conjugacy classes.

Proposition 7.3. *Let $d = \dim G(\mathbb{R})^1/K_\infty$. Let $M \in \mathcal{L}$ and $\gamma \in M(\mathbb{Q})$. Then*

$$\lim_{t \to 0} t^{d/2} J_M(\gamma, \widetilde{\phi}_t) = 0$$

if either $M \neq G$, or $M = G$ and $\gamma \neq \pm 1$.

Proof. By Corollary 6.2 of [A11] the distribution $J_M(\gamma, \widetilde{\phi}_t)$ is given by the integral of $\widetilde{\phi}_t$ over γ^G with respect to a measure $d\mu$ on γ^G which is absolutely continuous with respect to the invariant measure class. Thus $J_M(\gamma, \widetilde{\phi}_t)$ is equal to a finite sum of integrals of the form

$$\int_{G_{\gamma n}(\mathbb{Q}_S)\backslash G(\mathbb{Q}_S)} \widetilde{\phi}_t(g^{-1}\gamma n g)\, d\mu(g),$$

where $n \in N_P(\mathbb{Q}_S)$ for some $P \in \mathcal{P}(M)$. Now recall that by (6.5) and (6.26), $\widetilde{\phi}_t(g)$ is the product of $\varphi_t(g_\infty)h_t^\sigma(g_\infty)$ with $\chi_{K_f}(g_f)$ for any $g = g_\infty g_f$. Hence our problem is reduced to the investigation of the integral

$$\int_{G_{\gamma n_\infty}(\mathbb{R})\backslash G(\mathbb{R})} (\varphi_t h_t^\sigma)(g_\infty^{-1}\gamma n_\infty g_\infty)\; d\mu(g_\infty).$$

Furthermore, by Proposition 2.1 there exists $C > 0$ such that

$$|h_t^\sigma(g_\infty)| \le C t^{-d/2}, \quad 0 < t \le 1.$$

Hence it suffices to show that

$$\lim_{t\to 0} \int_{G_{\gamma n_\infty}(\mathbb{R})\backslash G(\mathbb{R})} \varphi_t(g_\infty^{-1}\gamma n_\infty g_\infty)\, d\mu(g_\infty) = 0 \tag{7.2}$$

if either $M \neq G$, or $M = G$ and $\gamma \neq \pm 1$.

By definition of γ^G, the conjugacy class of γn in $G(\mathbb{Q}_S)$ has to intersect $\gamma N_P(\mathbb{Q}_S)$ in an open subset. This implies that $\gamma n_\infty \neq \pm 1$, if either $M \neq G$, or $M = G$ and $\gamma \neq \pm 1$. Then it follows from Lemma 7.2 that $C_{\gamma n_\infty} \cap K_\infty$ is a proper submanifold of $C_{\gamma n_\infty}$. The measure $d\mu(g_\infty)$ is of the form $f(g_\infty) dg_\infty$ for some smooth function f on $G(\mathbb{R})$. Hence being a proper submanifold, $C_{\gamma n_\infty} \cap K_\infty$ is a subset of $C_{\gamma n_\infty}$ with measure zero with respect to $d\mu$. Next observe that

$$\int_{G_{\gamma n_\infty}(\mathbb{R})\backslash G(\mathbb{R})} \varphi_t(g_\infty^{-1}\gamma n_\infty g_\infty)\, |f(g_\infty)|\, dg_\infty < \infty.$$

Since $\operatorname{supp}\varphi_{t'} \subset \operatorname{supp}\varphi_t$ for $t' < t$, and $0 \le \varphi_t \le 1$ for all $t > 0$, it follows that there exists $C > 0$ such that

$$\left| \int_{G_{\gamma n_\infty}(\mathbb{R})\backslash G(\mathbb{R})} \varphi_t(g_\infty^{-1}\gamma n_\infty g_\infty)\, d\mu(g_\infty) \right| \le C$$

for all $0 < t \le 1$. Furthermore by definition of φ_t we have

$$\lim_{t\to 0} \varphi_t(x) = 0$$

for all $x \in C_{\gamma n_\infty} - (C_{\gamma n_\infty} \cap K_\infty)$. Since $C_{\gamma n_\infty} \cap K_\infty$ has measure zero with respect to $d\mu$, (7.2) follows by the dominated convergence theorem. $\square$

We can now state the main result of this section.

Theorem 7.4. *Let $d = \dim G(\mathbb{R})^1/K_\infty$, let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let $\sigma \in \Pi(\mathrm{O}(n))$ such that $\sigma(-1) = \mathrm{Id}$ if $-1 \in K_f$. Then*

$$\lim_{t\to 0} t^{d/2} J_{\mathrm{geo}}(\widetilde{\phi}_t^1) = \frac{\dim(\sigma)}{(4\pi)^{d/2}} \operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_f)\big(1 + \mathbf{1}_{K_f}(-1)\big).$$

Proof. By (7.1) and Proposition 7.3 if follows that

$$\lim_{t\to 0} t^{d/2} J_{\mathrm{geo}}(\widetilde{\phi}_t^1) = \lim_{t\to 0} t^{d/2}(a^G(S,1)\widetilde{\phi}_t^1(1) + a^G(S,-1)\widetilde{\phi}_t^1(-1)).$$

By Theorem 8.2 of [A10] we have

$$a^G(S, \pm 1) = \operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1).$$

Furthermore

$$\widetilde{\phi}_t^1(\pm 1) = h_t^\sigma(\pm 1)\chi_{K_f}(\pm 1).$$

Since σ satisfies $\sigma(-1) = \mathrm{Id}$, if $-1 \in K_f$, it follows from (2.5) that $h_t^\sigma(-1) = h_t^\sigma(1)$. Finally, by Lemma 2.3 we have

$$h_t^\sigma(\pm 1) = \frac{\dim(\sigma)}{(4\pi)^{d/2}} t^{-d/2} + O(t^{-(d-1)/2})$$

as $t \to 0$. Combined with $\chi_{K_f}(\pm 1) = \mathbf{1}_{K_f}(\pm 1) \operatorname{vol}(K_f)^{-1}$, the theorem follows. □

We shall now use the trace formula to prove the main results of this paper. Recall that the coarse trace formula is the identity

$$J_{\mathrm{spec}}(f) = J_{\mathrm{geo}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1),$$

between distributions on $G(\mathbb{A})^1$ [A1]. Applied to $\widetilde{\phi}_t^1$ we get the equality

$$J_{\mathrm{spec}}(\widetilde{\phi}_t^1) = J_{\mathrm{geo}}(\widetilde{\phi}_t^1), \quad t > 0.$$

Put $\varepsilon_{K_f} = 1$, if $-1 \in K_f$ and $\varepsilon_{K_f} = 0$ otherwise. Combining Theorem 6.2, Proposition 6.4 and Theorem 7.4, we obtain

$$\begin{aligned} \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}),\xi_0)} & m(\pi) \dim\big(\mathcal{H}_{\pi_f}^{K_f}\big) \dim\big(\mathcal{H}_{\pi_\infty} \otimes V_\sigma\big)^{\mathrm{O}(n)} e^{t\lambda_\pi} \\ & \sim \frac{\dim(\sigma)}{(4\pi)^{d/2}} \operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_f)(1+\varepsilon_{K_f}) t^{-d/2} \end{aligned} \tag{7.3}$$

as $t \to 0$. Applying Karamat's theorem [Fe, p.446], we obtain

$$\begin{aligned} \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}),\xi_0)} & m(\pi) \dim\big(\mathcal{H}_{\pi_f}^{K_f}\big) \dim\big(\mathcal{H}_{\pi_\infty} \otimes V_\sigma\big)^{\mathrm{O}(n)} \\ & \sim \dim(\sigma) \frac{\operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_f)}{(4\pi)^{d/2}\Gamma(d/2+1)} (1+\varepsilon_{K_f}) \lambda^{d/2} \end{aligned} \tag{7.4}$$

as $\lambda \to \infty$. By Lemma 3.3 it follows that this asymptotic formula continues to hold if we replace the sum over $\Pi_{\mathrm{dis}}(G(\mathbb{A}), \xi_0)$ by the sum over $\Pi_{\mathrm{cus}}(G(\mathbb{A}), \xi_0)$. Finally note that by [Sk] we have $m(\pi) = 1$ for all $\pi \in \Pi_{\mathrm{cus}}(G(\mathbb{A}), \xi_0)$. This completes the proof of Theorem 0.2.

Now suppose that K_f is the congruence subgroup $K(N)$ and $\Gamma(N) \subset \mathrm{SL}_n(\mathbb{Z})$ the principal congruence subgroup of level N. Then by (3.10) we have

$$\operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K(N)) = \varphi(N) \operatorname{vol}(\Gamma(N)\backslash \mathrm{SL}_n(\mathbb{R})).$$

Furthermore, $\varepsilon_{K(N)} = 1$ if and only if $-1 \in \Gamma(N)$. If -1 is contained in $\Gamma(N)$, then the fibre of the canonical map

$$\Gamma(N)\backslash \mathrm{SL}_n(\mathbb{R}) \to \Gamma(N)\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$$

is equal to $\mathrm{SO}(n)/\{\pm 1\}$. Otherwise the fibre is equal to $\mathrm{SO}(n)$. We normalize the Haar measure on $\mathrm{SL}_n(\mathbb{R})$ so that $\operatorname{vol}(\mathrm{SO}(n)) = 1$. Then in either case we have

$$\operatorname{vol}(\Gamma(N)\backslash \mathrm{SL}_n(\mathbb{R}))(1+\varepsilon_{K(N)}) = \operatorname{vol}(\Gamma(N)\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)).$$

Let $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ and let $\lambda_0 \le \lambda_1 \le \cdots$ be the eigenvalues, counted with multiplicity, of the Bochner-Laplace operator Δ_σ acting in $L^2(\Gamma(N)\backslash \mathrm{SL}_n(\mathbb{R}), \sigma)$.

Combining (6.18), Proposition 6.4, Theorem 7.4 and the above observations, we get

$$\sum_i e^{-t\lambda_i} = \dim(\sigma)\frac{\mathrm{vol}(\Gamma(N)\backslash X)}{(4\pi)^{d/2}} t^{-d/2} + o(t^{-d/2})$$

as $t \to 0$. Using again Karamata's theorem [Fe, p.446], we get

$$N_{\mathrm{dis}}^{\Gamma(N)}(\lambda, \sigma) = \dim(\sigma)\frac{\mathrm{vol}(\Gamma(N)\backslash X)}{(4\pi)^{d/2}\Gamma(d/2+1)}\lambda^{d/2} + o(\lambda^{d/2})$$

as $\lambda \to \infty$. By Proposition 3.6 it follows that the same asymptotic formula holds if we replace $N_{\mathrm{dis}}^{\Gamma(N)}(\lambda, \sigma)$ by $N_{\mathrm{cus}}^{\Gamma(N)}(\lambda, \sigma)$. This is exactly the statement of Theorem 0.1.

References

[A1] J. Arthur, *A trace formula for reductive groups I: terms associated to classes in $G(\mathbb{Q})$,* Duke. Math. J. **45** (1978), 911–952.

[A2] J. Arthur, *A trace formula for reductive groups II: applications of a truncation operator*, Comp. Math. **40** (1980), 87–121.

[A3] J. Arthur, *On a family of distributions obtained from Eisenstein series I: Applications of the Paley-Wiener theorem*, Amer. J. Math. **104** (1982), 1243–1288.

[A4] J. Arthur, *On a family of distributions obtained from Eisenstein series II: Explicit formulas*, Amer. J. Math. **104** (1982), 1289–1336.

[A5] J. Arthur, *The trace formula in invariant form,* Annals of Math. **114** (1981), 1–74.

[A6] J. Arthur, *The invariant trace formula. II. Global theory*, J. Amer. Math. Society **1** (1988), 501–554.

[A7] J. Arthur, *Intertwining operators and residues. I. Weighted characters*, J. Funct. Analysis **84** (1989), 19–84.

[A8] J. Arthur, *On the Fourier transforms of weighted orbital integrals*, J. reine angew. Math. **452** (1994), 163–217.

[A9] J. Arthur, *Automorphic Representations and Number Theory*, In: 1980 Seminar on Harmonic Analysis, Canadian Math. Soc., Conference Proceedings, Volume 1, AMS, Providence, RI, 1981.

[A10] J. Arthur, *On a family of distributions obtained from orbits*, Canadian J. Math. **38** (1986), 179–214.

[A11] J. Arthur, *The local behaviour of weighted orbital integrals*, Duke Math. J. **56** (1988), 223–293.

[AC] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals Math. Studies **120**, Princeton Univ. Press, Princeton, NJ, 1989.

[BM] D. Barbasch and H. Moscovici, *L^2-index and the Selberg trace formual*, J. Funct. Analysis **53** (1983), 151–201.

[BGV] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer, Berlin, 1996.

[BW] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Annals of Math. Stud. **94**, Princeton Univ. Press, 1980.

[Bu] C.J. Bushnell, *Induced representations of locally profinite groups*, Journal of Algebra **134** (1990), 104–114.

[BH] C.J. Bushnell and G. Henniart, *An upper bound on conductors for pairs*, J. Number Theory **65** (1997), 183–196.

[BHK] C.J. Bushnell, G.M. Henniart, and P.C. Kutzko, *Local Rankin-Selberg convolutions for GL_n: Explicit conductor formula*, J. Amer. Math. Soc. **11** (1998), 703–730.

[Cha] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, New York, 1984.

[CLL] L. Clozel, J.P. Labesse, and R. Langlands, *Morning seminar on the trace formula*, mimeoraphed notes, IAS, Princeton, 1984.

[CLY] S.Y. Cheng, P. Li, and S.-T. Yau, *On the upper estimate of the heat kernel of a complete Riemannian manifold*, Amer. J. Math. **103** (1981), 1021–1063.

[DH] A. Deitmar and W. Hoffmann, *On limit multiplicities for spaces of automorphic forms*, Canad. J. Math. **51** (1999), 952-976.

[Do] H. Donnelly, *On the cuspidal spectrum for finite volume symmetric spaces*, J. Diff. Geom. **17** (1982), 239–253.

[Do2] H. Donnelly, *Asymptotitc expansions for the compact quotients of properly discontinuous group actions*, Illinois J. Math. **23** (1979), 485–496.

[Ef] I. Efrat, *The Selberg trace formula for* $\mathrm{PSL}_2(\mathbb{R})$, Mem. Amer. Math. Soc. **65** (1987), no. 359.

[Fe] W. Feller, *An introduction to probability theory and its applications*, 2nd ed., Vol. II, Wiley, New York, 1971.

[He] D. Hejhal, *The Selberg trace formula for* $\mathrm{PSL}(2,\mathbb{R})$, Vol. 2, Lecture Notes in Math. **1001**, Springer-Verlag, Berlin.

[J] H. Jacquet, *Principal L-functions of the linear group*, Proc. Symp. Pure Math. **33** (1979), 63-86.

[JPS] H. Jacquet, I.I. Piatetski-Shapiro, and J.A. Shalika, *Rankin-Selberg convolutions,* Amer. J. Math. **105** (1983), 367-464.

[JS1] H. Jacquet and J.A. Shalika, *Euler products and classification of automorphic representations I*, Amer. J. Math. **103** (1981), 499-558.

[JS3] H. Jacquet, J.A. Shalika, *The Whittaker models of induced representations*, Pacific J. Math. **109** (1983), 107–120.

[Kn] A.W. Knapp, *Representation theory of semisimple groups*, Princeton University Press, Princeton, New Jersey 1986.

[La] R.P. Langlands, *On the functional equations satisfied by Eisenstein series,* Lecture Notes in Math. **544**, Springer–Verlag, Berlin–Heidelberg–New York, 1976.

[La1] R.P. Langlands, *On the classification of irreducible representations of real algebraic groups*, Mathematical Surveys and Monographs **31**, Amer. Math. Soc. 1989.

[LRS] W. Luo, Z. Rudnick, and P. Sarnak, *On the generalized Ramanujan conjecture for* $\mathrm{GL}(n)$. Proc. Symp. Pure Math. **66**, Vol. II, pp. 301–310, Amer. Math. Soc., Providence, Rhode Island, 1999.

[MW] C. Moeglin et J.-L. Waldspurger, *Le spectre résiduel de GL(n)*, Ann. scient. Éc. Norm. Sup., 4^e série, t. **22** (1989), 605–674.

[Mia] R. Miatello, *The Minakshisundaram-Pleijel coefficients for the vector valued heat kernel on compact locally symmetric spaces of negative curvature*, Trans. Amer. Math. Soc. **260** (1980), 1–33.

[Mil] St. Miller, *On the existence and temperedness of cusp forms for* $\mathrm{SL}_3(\mathbb{Z})$. J. Reine Angew. Math. **533** (2001), 127–169.

[Mu1] W. Müller, *The trace class conjecture in the theory of automorphic forms. II*, Geom. funct. anal. **8** (1998), 315–355.

[Mu2] W. Müller, *On the spectral side of the Arthur trace formula,* Geom. funct. anal. **12** (2002), 669–722.

[Mu3] W. Müller, *Eigenvalue estimates for locally symmetric spaces of finite volume.* Symposium "Partial Differential Equations" (Holzhau, 1988), 179–196, Teubner-Texte Math., 112, Teubner, Leipzig, 1989.

[MS] W. Müller and B. Speh, with appendix by E. Lapid, *Absolute convergence of the spectral side of the Arthur trace formula for* GL_n, Preprint, 2003.

[N] E. Nelson, *Analytic vectors*, Ann. of Math. **70** (1959), 572–615.

[PS] R. Phillips and P. Sarnak, *On cusp forms for cofinite subgroups of* $\mathrm{PSL}(2,\mathbb{R})$, Invent. Math. **80** (1985), 339–364.

[Rez] A. Reznikov, *Eisenstein matrix and existence of cusp forms in rank one symmetric spaces.* Geom. Funct. Anal. **3** (1993), no. 1, 79–105.

[RS] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory.* Duke Math. J. **81** (1996), 269–322.

[Sa] P. Sarnak, *On cusp forms*, in: The Selberg trace formula and related topics, ed. by D. Hejhal et al., Contemporary Math. vol. **53**, Amer. Math. Soc., 1984, pp. 393–407.

[Se] A. Selberg, *Harmonic analysis*, in "Collected Papers", Vol. I, Springer-Verlagf, Berlin-Heidelberg-New York (1989), 626–674.

[Sh1] F. Shahidi, *Fourier transforms of intertwining operators and Plancherel measures for* GL_n, Amer. J. Math **106** (1984), 67–111.

[Sh2] F. Shahidi, *On certain L-functions*, Amer. J. Math. **103** (1981), 297–356.

[Sh3] F. Shahidi, *Local coefficients and normalization of intertwining operators for* GL_n, Comp. Math. **48** (1983), 271–295.

[Sh4] F. Shahidi, *Local coefficients as Artin factors for real groups,* Duke Math. J. **52** (1985), 973–1007.

[Sh5] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; Complementary series for p-adic groups*, Annals Math. **132** (1990), 273–330.

[Sh6] F. Shahidi, *On multiplicativity of local factors*, Israel Math. conf. Proc. Vol. **3** (1990), 279–289.

[Sk] J.A. Shalika, *The multiplicity one theorem for* GL_n, Annals Math. **100** (1974), 171-193.

[Ve] A.B. Venkov, *Spectral theory of automorphic forms and its applications*, Kluwer Academic Pub., Soviet Series **51**, 1990.

[Wa1] G. Warner, *Harmonic analysis on semi-simple Lie groups I*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[Wa2] G. Warner, *Harmonic analysis on semi-simple Lie groups II*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

Universität Bonn, Mathematisches Institut, Beringstrasse 1, D – 53115 Bonn, Germany

E-mail address: `mueller@math.uni-bonn.de`