THE WEYL LAW FOR CONGRUENCE SUBGROUPS AND ARBITRARY K_∞-TYPES

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Dedicated to my teacher Friedrich Güssefeldt

ABSTRACT. Let G be a reductive algebraic group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup. Let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup. We study the asymptotic behavior of the counting functions of the cuspidal and residual spectrum, respectively, of the regular representation of $G(\mathbb{R})$ in $L^2(\Gamma\backslash G(\mathbb{R}))$ of a fixed K_∞-type σ. A conjecture, which is due to Sarnak, states that the counting function of the cuspidal spectrum of type σ satisfies Weyl's law and the residual spectrum is of lower order growth. Using the Arthur trace formula we reduce the conjecture to a problem about L-functions occurring in the constant terms of Eisenstein series. If G satisfies property (L), introduced by Finis and Lapid, we establish the conjecture. This includes classical groups over a number field.

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1. Introduction

Let G be a connected semisimple algebraic group over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup, which we assume to be torsion free. A basic problem in the theory of automorphic forms is the study of the spectral resolution of the regular representation R_Γ of $G(\mathbb{R})$ in $L^2(\Gamma\backslash G(\mathbb{R}))$. Of particular importance is the determination of the structure of the discrete spectrum. Let $L^2_{\text{dis}}(\Gamma\backslash G(\mathbb{R}))$ be the discrete part of $L^2(\Gamma\backslash G(\mathbb{R}))$, i.e., the closure of the span of all irreducible subrepresentations of R_Γ. Denote by $R_{\Gamma,\text{dis}}$ the corresponding restriction of R_Γ. Denote by $\Pi_{\text{dis}}(G(\mathbb{R}))$ the set of isomorphism classes of irreducible unitary representations of $G(\mathbb{R})$, which occur in R_Γ. By definition we have

$$R_{\Gamma,\text{dis}} = \widehat{\bigoplus}_{\pi\in\Pi_{\text{dis}}(G(\mathbb{R}))} m_\Gamma(\pi)\pi, \tag{1.1}$$

where

$$m_\Gamma(\pi) = \dim \operatorname{Hom}_{G(\mathbb{R})}(\pi, R_\Gamma) = \dim \operatorname{Hom}_{G(\mathbb{R})}(\pi, R_{\Gamma,\text{dis}})$$

is the multiplicity with which π occurs in R_Γ. Apart from special cases, as for example discrete series representations, one cannot hope to describe the multiplicity function m_Γ on $\Pi(G(\mathbb{R}))$ explicitly. Therefor it is feasible to study asymptotic questions such as the limit multiplicity problem [FLM2] and the Weyl law, which is the subject of this article.

To begin with we recall that the discrete spectrum decomposes into the cuspidal and the residual spectrum. Let $\mathbf{K}_\infty$ be a maximal compact subgroup of $G(\mathbb{R})$. Let $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the center of the universal enveloping algebra of the complexification of the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$. Recall that a cusp form for Γ is a smooth and right $\mathbf{K}_\infty$-finite function $\phi\colon \Gamma\backslash G(\mathbb{R}) \to \mathbb{C}$ which is a simultaneous eigenfunction of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ and which satisfies

$$\int_{\Gamma\cap N_P(\mathbb{R})\backslash N_P(\mathbb{R})} \phi(nx)dn = 0 \tag{1.2}$$

for all unipotent radicals N_P of proper rational parabolic subgroups P of G. By Langlands' theory of Eisenstein series [La1], cusp forms are the building blocks of the spectral resolution. We note that each cusp form $\phi \in C^\infty(\Gamma\backslash G(\mathbb{R}))$ is rapidly decreasing on $\Gamma\backslash G(\mathbb{R})$ and hence square integrable. Let $L^2_{\text{cus}}(\Gamma\backslash G(\mathbb{R}))$ be the closure of the linear span of all cusp forms. The restriction of the regular representation R_Γ to $L^2_{\text{cus}}(\Gamma\backslash G(\mathbb{R}))$ decomposes discretely and $L^2_{\text{cus}}(\Gamma\backslash G(\mathbb{R}))$ is a subspace of $L^2_{\text{dis}}(\Gamma\backslash G(\mathbb{R}))$. Denote by $L^2_{\text{res}}(\Gamma\backslash G(\mathbb{R}))$ the orthogonal complement of $L^2_{\text{cus}}(\Gamma\backslash G(\mathbb{R}))$ in $L^2_{\text{dis}}(\Gamma\backslash G(\mathbb{R}))$. This is the residual subspace. Let (σ, V_σ) be an irreducible unitary representation of $\mathbf{K}_\infty$. Set

$$L^2(\Gamma\backslash G(\mathbb{R}),\sigma) := (L^2(\Gamma\backslash G(\mathbb{R})) \otimes V_\sigma)^{\mathbf{K}_\infty}. \tag{1.3}$$

Define the subspaces $L^2_{\text{dis}}(\Gamma\backslash G(\mathbb{R}),\sigma)$, $L^2_{\text{cus}}(\Gamma\backslash G(\mathbb{R}),\sigma)$ and $L^2_{\text{res}}(\Gamma\backslash G(\mathbb{R}),\sigma)$ in a similar way. Then $L^2_{\text{cus}}(\Gamma\backslash G(\mathbb{R}),\sigma)$ is the space of cusp forms with fixed $\mathbf{K}_\infty$-type σ. Let $\widetilde{X} := G(\mathbb{R})/\mathbf{K}_\infty$ be the Riemannian symmetric space associated to $G(\mathbb{R})$ and $X = \Gamma\backslash\widetilde{X}$ the corresponding locally symmetric space. Since we assume that Γ is torsion free, X is a manifold. Let $E_\sigma \to \Gamma\backslash X$ be the locally homogeneous vector bundle associated to σ

and let $L^2(X, E_\sigma)$ be the space of square integrable sections of E_σ. There is a canonical isomorphism

$$L^2(\Gamma\backslash G(\mathbb{R}), \sigma) \cong L^2(X, E_\sigma). \tag{1.4}$$

Let $\Omega_{G(\mathbb{R})} \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the Casimir element of $G(\mathbb{R})$. Then $-\Omega_{G(\mathbb{R})} \otimes \mathrm{Id}$ induces a self-adjoint operator Δ_σ in the Hilbert space $L^2(\Gamma\backslash G(\mathbb{R}), \sigma)$ which is bounded from below. With respect to the isomorphism (1.4) we have

$$\Delta_\sigma = (\nabla^\sigma)^* \nabla^\sigma - \lambda_\sigma \,\mathrm{Id}, \tag{1.5}$$

where ∇^σ is the canonical invariant connection in E_σ and λ_σ denotes the Casimir eigenvalue of σ. In particular, if σ_0 is the trivial representation, then $L^2(\Gamma\backslash G(\mathbb{R}), \sigma_0) \cong L^2(X)$ and Δ_{σ_0} equals the Laplacian Δ on X.

The restriction of Δ_σ to the subspace $L^2_{\mathrm{dis}}(\Gamma\backslash G(\mathbb{R}), \sigma)$ has pure point spectrum consisting of eigenvalues $\lambda_0(\sigma) < \lambda_1(\sigma) < \cdots$ of finite multiplicities. Let $\mathcal{E}(\lambda_i(\sigma))$ be the eigenspace corresponding to $\lambda_i(\sigma)$. Then we define the eigenvalue counting function $N_{\Gamma,\mathrm{dis}}(\lambda, \sigma)$, $\lambda \geq 0$, by

$$N_{\Gamma,\mathrm{dis}}(\lambda, \sigma) = \sum_{\lambda_i(\sigma) \leq \lambda} \dim \mathcal{E}(\lambda_i(\sigma)). \tag{1.6}$$

The counting functions $N_{\Gamma,\mathrm{cus}}(\lambda, \sigma)$ and $N_{\Gamma,\mathrm{res}}(\lambda, \sigma)$ of the cuspidal and residual spectrum are defined by considering the restriction of Δ_σ to the cuspidal and residual subspace, respectively. The main goal is to determine the asymptotic behavior of the counting functions as $\lambda \to \infty$. If X is compact, the Weyl law holds. Recall that for a compact Riemannian manifold X of dimension n, the Weyl law states that the number $N_X(\lambda)$ of eigenvalues $\lambda_i \leq \lambda$, counted with multiplicity, of the Laplace operator Δ of X satisfies

$$N_X(\lambda) = \frac{\mathrm{vol}(X)}{(4\pi)^n \Gamma(\frac{n}{2}+1)} \lambda^{n/2} + o(\lambda^{(n/2}) \tag{1.7}$$

as $\lambda \to \infty$. A standard method to prove (1.7) is the heat equation method. Using the wave equation, one gets a more precise version with an estimation of the remainder term:

$$N_X(\lambda) = \frac{\mathrm{vol}(X)}{(4\pi)^n \Gamma(\frac{n}{2}+1)} \lambda^{n/2} + O(\lambda^{(n-1)/2}) \tag{1.8}$$

as $\lambda \to \infty$. This is due to Avakumovic [Av] and Hörmander. Without further assumptions on the Riemannian manifold, the remainder term is optimal [Av]. More generally, one can consider the Bochner-Laplace operator Δ_E for a Hermitian vector bundle $E \to X$ with Hermitian connection. There is a similar formula (1.7) for the eigenvalue counting function $N_X(\lambda, E)$ of Δ_E. The only difference is the rank of E which appears on the right hand side in the leading coefficient.

For non-uniform lattices Γ the self-adjoint operator Δ_σ has a large continuous spectrum so that alomost all eigenvalues of Δ_σ will be embedded in the continuous spectrum which makes it very difficult to study them. A number of results are known for the spherical cuspidal spectrum. The first results concerning the growth of the cuspidal spectrum are due to Selberg [Se1]. He proved that for every congruence subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$, the counting function of the cuspidal spectrum satisfies Weyl's law, i.e., one has

$$N_\Gamma(\lambda) = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^2)}{4\pi}\lambda^2 - \frac{2m}{\pi}\lambda\log\lambda + O(\lambda), \tag{1.9}$$

as $\lambda \to \infty$. This shows that for congruence subgroups eigenvalues exist in abundance. On the other hand, based on their work on the dissolution of cusp forms under deformation of lattices, Phillips and Sarnak [Sa2] conjectured that except for the Teichmüller space of the once punctured torus, the point spectrum of the Laplacian on $\Gamma\backslash\mathbb{H}^2$ for a generic non-uniform lattice Γ in $\mathrm{SL}(2,\mathbb{R})$ is finite and is contained in $[0,1/4)$. In the more general context of manifolds with cusps Colin de Verdère [CV] has shown that under a generic compactly supported conformal deformation of the metric of a non-compact hyperbolic surface of finite area all eigenvalues $\lambda \geq 1/4$ are dissolved.

If $\mathrm{rank}(G) > 1$, the situation is very different. By the results of Margulis, we have rigidity of irreducible lattices and irreducible lattices are arithmetic. One expects that arithmetic groups have a large discrete spectrum. The following conjecture is due to Sarnak [Sa1].

Conjecture 1.1. *Let $\Gamma \subset G(\mathbb{Q})$ be a congruence subgroup. Then for every $\nu \in \Pi(\mathbf{K}_\infty)$, $N_{\Gamma,\mathrm{cus}}(\lambda;\nu)$ satisfies Weyl's law and $N_{\Gamma,\mathrm{res}}(\lambda;\nu)$ is of lower order growth.*

There are some general results concerning the conjecture. Let G be a connected real semisimple Lie group, K a maximal compact subgroup of G, and $\Gamma \subset G$ a torsion free lattice. Let $n = \dim G/K$. Donnelly [Do, Theorem 9.1] has established the following upper bound for the cuspidal spectrum

$$\limsup_{\lambda\to\infty} \frac{N_{\Gamma,\mathrm{cus}}(\lambda;\nu)}{\lambda^{n/2}} \leq \frac{\dim(\nu)\,\mathrm{vol}(\Gamma\backslash G/K)}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)}, \tag{1.10}$$

which holds for every $\nu \in \Pi(K)$. Concerning the residual spectrum, it was proved in [Mu1, Theorem 0.1] that for a general lattice one has

$$N_{\Gamma,\mathrm{res}}(\lambda;\nu) \ll 1 + \lambda^{2n}. \tag{1.11}$$

However, this is not the optimal bound that one expects. In general, one would expect that the residual spectrum is of order $O(\lambda^{n/2})$ and for arithmetic groups of order $O(\lambda^{(n-1)/2})$ as $\lambda \to \infty$.

Conjecture (1.1) has been verified in a number of cases. Most of the results are obtained for the spherical spectrum. The first result in higher rank is due to S.D. Miller [Mil] who established the Weyl law for spherical cusp forms for $\Gamma = \mathrm{SL}(3,\mathbb{Z})$. The author [Mu3] proved it for a principal congruence subgroup $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$. The method of proof follows Selberg's approach and uses the trace formula. Then Lindenstrauss and Venkatesh [LV] proved the Weyl law for spherical cusps forms in great generality, namely for congruence subgroups $\Gamma \subset \mathbf{G}(\mathbb{R})$, where $\mathbf{G}$ is a split adjoint semisimple group over $\mathbb{Q}$. The method is different. It uses Hecke operators to eliminate the contribution of Eisenstein series. For congruence subgroups of $\mathrm{SL}(n,\mathbb{Z})$, E. Lapid and W. Müller [LM] established the Weyl law for the cuspidal spectrum with an estimation of the remainder term. The order of the remainder term is $O(\lambda^{(d-1)/2}(\log\lambda)^{\max(n,3)})$ where $d = \dim \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. The method is also based on the Arthur trace formula as in [Mu3]. However, the argument is simplified and strengthened, which corresponds to the use of the wave equation in the derivation of the Weyl law for a compact Riemannian manifold. Recently T. Finis and E. Lapid [FL2] estimated the remainder term for the cuspidal spectrum of a locally symmetric space $X = \Gamma\backslash\mathbf{G}(\mathbb{R})/\mathbf{K}_\infty$, where $\mathbf{G}$ is a simply connected, simple Chevalley group and Γ a congruence subgroup of $\mathbf{G}(\mathbb{Z})$. The method also uses Hecke operators as in [LV], but in a slightly different way. The estimation they obtain is $O(\lambda^{d-\delta})$, where $d = \dim X$ and $\delta > 0$ some constant which is not further specified. In [FM], T. Finis and J. Matz included Hecke operators. They studied the asymptotic behavior of the traces of Hecke operators for the spherical discrete spectrum.

For the non-spherical case, the Weyl law was proved in [Mu3] for a principal congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$. Recently, A. Maiti [Ma] has generalized the approach of Lindenstrauss and Venkatesh [LV] to establish the Weyl law for cusp forms and arbitrary K_∞-types. As in [LV], the method works for a semi-simple, split, adjoint linear algebraic group over $\mathbb{Q}$. It provides no results for the residual spectrum.

Concerning the residual spectrum, there is the general upper bound (1.11), which, however, is not the expected optimal one. For $\mathrm{rank}(G) = 1$, the residual spectrum is known to be finite. For $\mathrm{GL}(n)$ the residual spectrum has been determined by Mœglin and Waldspurger [MW]. This has been used in [Mu3, Proposition 3.6] to prove that in this case the residual spectrum is of lower order growth.,

The main goal of the present paper is to prove Conjecture (1.1) for a certain class of reductive groups including classical groups over a number field. We use the Arthur trace formula to reduce the proof of the conjecture to a problem about automorphic L-functions occurring in the constant terms of Eisenstein series. This problem can be dealt with if the reductive group G satisfies property (L), which was introduced by Finis and Lapid in [FL1, Definition 3.4]. Let G be a reductive group over $\mathbb{Q}$. As usual, let $G(\mathbb{R})^1$ denote the intersection of the kernels of the homorphisms $|\chi|\colon G(\mathbb{R}) \to \mathbb{R}^{>0}$, where χ ranges over the $\mathbb{Q}$-rational characters of G. Then our main result is the following theorem. vvv

Theorem 1.2. *Let G_0 be a connected reductive algebraic group over a number field F which satisfies property (L). Let $G = \mathrm{Res}_{F/\mathbb{Q}}(G_0/F)$. Let $\mathbf{K}_\infty \subset G(\mathbb{R})^1$ be a maximal compact subgroup and let $n = \dim G(\mathbb{R})^1/\mathbf{K}_\infty$. Let $\Gamma \subset G(\mathbb{Q})$ be a torsion free congruence subgroup. Then for every $\nu \in \Pi(\mathbf{K}_\infty)$ we have*

$$N_{\Gamma,\mathrm{cus}}(\lambda;\nu) \sim \frac{\dim(\nu) vol(\Gamma\backslash G(\mathbb{R})^1/\mathbf{K}_\infty)}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)}\lambda^{n/2}, \quad \lambda \to \infty. \tag{1.12}$$

and

$$N_{\Gamma,\mathrm{res}}(\lambda;\nu) \ll 1 + \lambda^{(n-1)/2}, \quad \lambda > 0. \tag{1.13}$$

Thus in order to establish the Weyl law and the estimation of the residual spectrum for every $\mathbf{K}_\infty$-type, we are reduced to the verification that G_0 satisfies property (L). For $\mathrm{GL}(n)$

the relevant L-functions are the Rankin-Selberg L-functions, which are known to satisfy the pertinent properties. Using Arthur's work on functoriality from classical groups to $\mathrm{GL}(n)$, T. Finis and E. Lapid [FL1, Theorem 3.11] proved that quasi-split classical groups over a number field F satisfy property (L). Moreover, they also proved that inner forms of $\mathrm{GL}(n)$ and the exceptional group G_2 over a number field F satisfy property (L). In fact, one expects that property (L) holds for all reductive groups. Currently, we only know [FL1, Theorem 3.11]. Together with Theorem 1.2 this leads to the following corollary.

Corollary 1.3. *Let F be a number field and let G_0 be one of the following groups over F:*

(1) $\mathrm{GL}(n)$ and its inner forms.
(2) Quasi-split classical groups.
(3) The exceptional group G_2.

Let $G = \mathrm{Res}_{F/\mathbb{Q}}(G_0/F)$. Let $\Gamma \subset G(\mathbb{Q})$ be a congruence subgroup and $\nu \in \Pi(\mathbf{K}_\infty)$. Then (1.12) *and* (1.13) *hold.*

Our approach to prove Theorem 1.2 is a generalization of the heat equation method to the non-compact setting. The basic tool is the Arthur trace formula. This requires to pass to the adelic setting. We will work with reductive groups over a number field F. However, for the rest of the introduction we will assume that $F = \mathbb{Q}$. So let G be a connected reductive group defined over $\mathbb{Q}$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. Let $G(\mathbb{A})^1 := \cap_\chi \ker|\chi|$, where χ runs over the rational characters of G. Denote by T_G the split component of the center of G and let A_G be the component of the identity of $T_G(\mathbb{R})$. Then

$$G(\mathbb{A}) = A_G \times G(\mathbb{A})^1.$$

We replace $\Gamma\backslash G(\mathbb{R})^1$ by the adelic quotient $A_G G(\mathbb{Q})\backslash G(\mathbb{A})/K_f = G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_f$, where $K_f \subset G(\mathbb{A}_f)$ is an open compact subgroup. Let $\Pi(G(\mathbb{A}))$ (resp. $\Pi(G(\mathbb{A})^1)$) be the set of equivalence classes of irreducible unitary representations of $G(\mathbb{A})$ (resp. $G(\mathbb{A})^1$). We identify a representation of $G(\mathbb{A})^1$ with a representation of $G(\mathbb{A})$, which is trivial on A_G. Let $L^2_{\mathrm{dis}}(A_G G(\mathbb{Q})\backslash G(\mathbb{A}))$ be the closure of the span of all irreducible subrepresentations of the regular representation R of $G(\mathbb{A})$ in $L^2(A_G G(\mathbb{Q})\backslash G(\mathbb{A}))$. Denote by $\Pi_{\mathrm{dis}}(G(\mathbb{A}))$ the subspace of all $\pi \in \Pi(G(\mathbb{A}))$ which are equivalent to a subrepresentation of the regular representation of $G(\mathbb{A})$ in $L^2(A_G G(\mathbb{Q})\backslash G(\mathbb{A}))$. Note that this is a countable set. Denote by R_{dis} the restriction of R to $L^2_{\mathrm{dis}}(A_G G(\mathbb{Q})\backslash G(\mathbb{A}))$. Then

$$R_{\mathrm{dis}} \cong \widehat{\bigoplus}_{\pi\in\Pi_{\mathrm{dis}}(G(\mathbb{A}))} m(\pi)\pi, \tag{1.14}$$

where

$$m(\pi) = \dim \mathrm{Hom}(\pi, L^2(A_G G(\mathbb{Q})\backslash G(\mathbb{A})) \tag{1.15}$$

is the multiplicity with which π occurs in $L^2(A_G G(\mathbb{Q})\backslash G(\mathbb{A}))$. Any $\pi \in \Pi(G(\mathbb{A}))$ can be written as $\pi = \pi_\infty \otimes \pi_f$, where π_∞ and π_f are irreducible unitary representations of $G(\mathbb{R})$ and $G(\mathbb{A}_f)$, respectively. Let $\mathcal{H}_{\pi_\infty}$ and $\mathcal{H}_{\pi_f}$ denote the Hilbert space of the representation π_∞ and π_f, respectively. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup. Denote by

$\mathcal{H}_{\pi_f}^{K_f}$ the subspace of K_f-invariant vectors in $\mathcal{H}_{\pi_f}$. Let $G(\mathbb{R})^1 = G(\mathbb{A})^1 \cap G(\mathbb{R})$. Given $\pi \in \Pi(G(\mathbb{A}))$, denote by λ_{π_∞} the Casimir eigenvalue of the restriction of π_∞ to $G(\mathbb{R})^1$. Let $\nu \in \Pi(\mathbf{K}_\infty)$. Then we define the adelic counting function of the discrete spectrum by

$$N_{\text{dis}}^{K_f,\nu}(\lambda) := \sum_{\substack{\pi\in\Pi_{\text{dis}}(G(\mathbb{A}))\\ -\lambda_{\pi_\infty}\le\lambda}} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_f})\dim(\mathcal{H}_{\pi_\infty}\otimes V_\nu)^{K_\infty}. \tag{1.16}$$

In the same way we define the counting functions $N_{\text{cus}}^{K_f,\nu}(\lambda)$ and $N_{\text{res}}^{K_f,\nu}(\lambda)$ of the cuspidal and residual spectrum, respectively. The adelic version of Theorem 1.2 is then

Theorem 1.4. *Let G_0 be a connected reductive algebraic group over a number field F. Assume that G_0 satisfies property (L). Let $G = \operatorname{Res}_{F/\mathbb{Q}}(G_0)$ be the group that is obtained from G_0 by restriction of scalars. Let $\mathbf{K}_\infty$ be a maximal compact subgroup of $G(\mathbb{R})^1$. Let $d := \dim(G(\mathbb{R})^1/\mathbf{K}_\infty)$. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup and let $\nu \in \Pi(\mathbf{K}_\infty)$. Then we have*

$$N_{\text{cus}}^{K_f,\nu}(\lambda) = \frac{\dim(\nu)\operatorname{vol}(A_G G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}\lambda^{d/2} + o(\lambda^{d/2}), \quad \lambda\to\infty, \tag{1.17}$$

and

$$N_{\text{res}}^{K_f,\nu}(\lambda) \ll (1+\lambda^{(n-1)/2}), \quad \lambda > 0. \tag{1.18}$$

To deduce Theorem 1.2 from the adelic version, we recall that there exist finitely many congruence subgroups $\Gamma_i \subset G(\mathbb{Q})$, $i = 1, ..., m$, such that

$$A_G G(\mathbb{Q})\backslash G(\mathbb{A})/K_f = \bigsqcup_{i=1}^{m} \Gamma_i\backslash G(\mathbb{R})^1. \tag{1.19}$$

(see sect. 3). Denote by $N_{\Gamma_i,\text{cus}}(\lambda,\nu)$ the counting function for the cuspidal spectrum $L^2(\Gamma_i\backslash G(\mathbb{R})^1)\otimes V_\nu)^{K_\infty}$. Then it follows that

$$N_{\text{cus}}^{K_f,\nu}(\lambda) = \sum_{i=1}^{m} N_{\Gamma_i,\text{cus}}(\lambda,\nu). \tag{1.20}$$

This is used to derive Theorem 1.2 from Theorem 1.4.

To prove Theorem 1.4 we start with the estimation of the residual counting function, which is needed to establish the Weyl law. For this purpose we use Langlands' description of the residual spectrum in terms of iterated residues of Eisenstein series [La1, Ch. 7], [MW, V.3.13]. Using the Maass-Selberg relations, the problem is finally reduced to the estimation of the number of real poles of the normalizing factors of intertwining operators, which appear in the constant terms of Eisenstein series. To obtain the appropriate bounds, we need that G satisfies property (L) which was introduced by Finis and Lapid [FL1, Definition 3.4]. In this way we get (1.18).

To prove the Weyl law, we use the Arthur trace formula. We will work with groups over a number field F. However, in order to explain the method we will simply assume that $F=\mathbb{Q}$. We proceed as in [Mu3]. We choose test functions $\phi_t^\nu \in C_c^\infty(G(\mathbb{A})^1)$, $t>0$, which at the infinite place are obtained from the heat kernel H_t^ν of the Bochner-Laplace operator $\widetilde{\Delta}_\nu$ on the symmetric space $\widetilde{X}=G(\mathbb{R})^1/\mathbf{K}_\infty$ and which at the finite places is given by the normalized characteristic function of K_f (see (8.11) for the precise definition). Then we insert ϕ_t^ν into the spectral side J_{spec} of the trace formula and study the asymptotic behavior of $J_{\text{spec}}(\phi_t^\nu)$ as $t\to 0$. The spectral side is a sum of distributions $J_{\text{spec},M}$ associated to conjugacy classes of Levi subgroups M of G. For $M=G$ we have

$$J_{\text{spec},G}(\phi_t^\nu)=\sum_{\pi\in\Pi_{\text{dis}}(G(\mathbb{A}))} m(\pi)e^{t\lambda_{\pi_\infty}}\dim(\mathcal{H}_{\pi_f}^{K_f})\dim(\mathcal{H}_{\pi_\infty}\otimes V_\nu)^{K_\infty}, \tag{1.21}$$

which is the contribution of the discrete spectrum to the spectral side. For $M\neq G$, the main ingredients of the distributions $J_{\text{spec},M}$ are logarithmic derivatives of intertwining operators. The intertwining operators can be normalized by certain meromorphic functions. Then the logarithmic derivatives of the intertwining operators are expressed in terms of logarithmic derivatives of the normalizing factors and logarithmic derivatives of the local normalized intertwining operators. In fact, we only need to control integrals of logarithmic derivatives which simplifies the problem. To deal with the integrals of logarithmic derivatives of the normalizing factors, we use property (TWN+) [FL1, Definition 3.3]. By [FL1, Proposition 3.8], property (TWN+) is a consequence of property (L) [FL1, Definition 3.4], which we assume to be satisfied by G. To deal with the local intertwining operators, we follow essentially the approach used in [FLM2]. The final result is Theorem 8.5, which states that if G satisfies property (L), then

$$J_{\text{spec}}(\phi_t^\nu)=J_{\text{spec},G}(\phi_t^\nu)+O(t^{-(d-1)/2}) \tag{1.22}$$

as $t\to 0$.

Next we come to the geometric side $J_{\text{geom}}(\phi_t^\nu)$. Its asymptotic behavior as $t\to 0$ has been determined in [MM2, Theorem 1.1]. We will briefly recall the main steps of the proof and determine the leading coefficient. By the trace formula we have $J_{\text{spec}}(\phi_t^\nu)=J_{\text{geom}}(\phi_t^\nu)$, which together with (1.21) and (1.22) leads to

$$\begin{aligned}\sum_{\pi\in\Pi_{\text{dis}}(G(\mathbb{A}))} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_f})\dim\left(\mathcal{H}_{\pi_\infty}\otimes V_\nu\right)^{K_\infty}\, e^{t\lambda_{\pi_\infty}}\\ =\frac{\dim(\nu)\,\operatorname{vol}(X(K_f))}{(4\pi)^{d/2}}t^{-d/2}+O(t^{-(d-1)/2})\end{aligned} \tag{1.23}$$

as $t\to 0$. Applying Karamata's theorem, we obtain the adelic Weyl law (1.17).

2. Preliminaries

We will mostly use the notation of [FLM1]. Let G be a reductive algebraic group defined over a number field F. We fix a minimal parabolic subgroup P_0 of G defined over F and a Levi decomposition $P_0=M_0U_0$, both defined over F. Let T_0 be the F-split component of the center of M_0. Let $\mathcal{F}$ be the set of parabolic subgroups of G which contain M_0 and are defined over F. Let $\mathcal{L}$ be the set of subgroups of G which contain M_0 and are Levi components of groups in $\mathcal{F}$. For any $P \in \mathcal{F}$ we write

$$P = M_P N_P,$$

where N_P is the unipotent radical of P and M_P belongs to $\mathcal{L}$.

Let $M \in \mathcal{L}$. Denote by T_M the F-split component of the center of M. Put $T_P = T_{M_P}$. With our previous notation, we have $T_0 = T_{M_0}$. Let $L \in \mathcal{L}$ and assume that L contains M. Then L is a reductive group defined over F and M is a Levi subgroup of L. We shall denote the set of Levi subgroups of L which contain M by $\mathcal{L}^L(M)$. We also write $\mathcal{F}^L(M)$ for the set of parabolic subgroups of L, defined over F, which contain M, and $\mathcal{P}^L(M)$ for the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. Each of these three sets is finite. If $L = G$, we shall usually denote these sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$.

Let $W_0 = N_{G(F)}(T_0)/M_0$ be the Weyl group of (G, T_0), where $N_{G(F)}(H)$ denotes the normalizer of H in $G(F)$. For any $s \in W_0$ we choose a representative $w_s \in G(F)$. Note that W_0 acts on $\mathcal{L}$ by $sM = w_s M w_s^{-1}$. For $M \in \mathcal{L}$ let $W(M) = N_{G(F)}(M)/M$, which can be identified with a subgroup of W_0.

Let $X(M)_F$ be the group of characters of M which are defined over F. Put

$$\mathfrak{a}_M := \operatorname{Hom}(X(M)_F, \mathbb{R}). \tag{2.1}$$

This is a real vector space whose dimension equals that of T_M. Its dual space is

$$\mathfrak{a}_M^* = X(M)_F \otimes \mathbb{R}.$$

We shall write,

$$\mathfrak{a}_P = \mathfrak{a}_{M_P} \quad \text{and} \quad \mathfrak{a}_0 = \mathfrak{a}_{M_0}. \tag{2.2}$$

For any $L \in \mathcal{L}(M)$ we identify $\mathfrak{a}_L^*$ with a subspace of $\mathfrak{a}_M^*$. We denote by $\mathfrak{a}_M^L$ the annihilator of $\mathfrak{a}_L^*$ in $\mathfrak{a}_M$. Then $r = \dim \mathfrak{a}_0^G$ is the semisimple rank of G. We set

$$\mathcal{L}_1(M) = \{L \in \mathcal{L}(M) : \dim \mathfrak{a}_M^L = 1\} \tag{2.3}$$

and

$$\mathcal{F}_1(M) = \bigcup_{L \in \mathcal{L}_1(M)} \mathcal{P}(L). \tag{2.4}$$

Let $\Sigma_P \subset \mathfrak{a}_P^*$ be the set of reduced roots of T_P on the Lie algebra $\mathfrak{n}_P$ of N_P. Let Δ_P be the subset of simple roots of P, which is a basis for $(\mathfrak{a}_P^G)^*$. Denote by Σ_M the set of reduced roots of T_M on the Lie algebra of G. For any $\alpha \in \Sigma_M$ we denote by $\alpha^\vee \in \mathfrak{a}_M$ the corresponding co–root. Let P_1 and P_2 be parabolic subgroups with $P_1 \subset P_2$. Then $\mathfrak{a}_{P_2}^*$ is embedded into $\mathfrak{a}_{P_1}^*$, while $\mathfrak{a}_{P_2}$ is a natural quotient vector space of $\mathfrak{a}_{P_1}$. The group $M_{P_2} \cap P_1$ is a parabolic subgroup of M_{P_2}. Let $\Delta_{P_1}^{P_2}$ denote the set of simple roots of $(M_{P_2} \cap P_1, T_{P_1})$. It is a subset of Δ_{P_1}. For a parabolic subgroup P with $P_0 \subset P$ we write $\Delta_0^P := \Delta_{P_0}^P$.

Let $\mathbb{A}$ be the ring of adeles of F, $\mathbb{A}_f$ the ring of finite adeles and $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. We fix a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v = \mathbf{K}_\infty \cdot \mathbf{K}_f$ of $G(\mathbb{A}) = G(F_\infty) \cdot G(\mathbb{A}_f)$. We assume that the maximal compact subgroup $\mathbf{K} \subset G(\mathbb{A})$ is admissible with respect to M_0 [Ar6, § 1]. Let $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ be the homomorphism given by

$$e^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_{\mathbb{A}} = \prod_v |\chi(m_v)|_v \tag{2.5}$$

for any $\chi \in X(M)_F$ and denote by $M(\mathbb{A})^1 \subset M(\mathbb{A})$ the kernel of H_M.

Let $G_1 = \operatorname{Res}_{F/\mathbb{Q}}(G)$ be the group over $\mathbb{Q}$ obtained from G by restriction of scalars [We]. Similar for any $M \in \mathcal{L}$ let $M_1 := \operatorname{Res}_{F/\mathbb{Q}}(M)$. Let T_{M_1} be the $\mathbb{Q}$-split component of the center of M_1. For $M \in \mathcal{L}$ let A_M denote the connected component of the identity of $T_{M_1}(\mathbb{R})$, which is viewed as a subgroup of $T_M(\mathbb{A}_F)$ via the diagonal embedding of $\mathbb{R}$ into F_∞. Note that it follows from the properties of the restriction of scalars that $M_1(\mathbb{A}_\mathbb{Q})^1 \cong M(\mathbb{A}_F)^1$. Thus we have

$$M(\mathbb{A}_F) = A_M \times M(\mathbb{A}_F)^1.$$

Let $L^2_{\text{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ be the discrete part of $L^2(A_M M(F)\backslash M(\mathbb{A}))$, i.e., the closure of the sum of all irreducible subrepresentations of the regular representation of $M(\mathbb{A})$. We denote by $\Pi_{\text{disc}}(M(\mathbb{A}))$ the countable set of equivalence classes of irreducible unitary representations of $M(\mathbb{A})$ which occur in the decomposition of the discrete subspace $L^2_{\text{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ into irreducible representations. Let $L^2_{\text{cus}}(A_M M(F)\backslash M(\mathbb{A}))$ be the subspace of cusp forms. Denote by $\Pi_{\text{cus}}(M(\mathbb{A}))$ the set of equivalence classes of irreducible unitary representations of $M(\mathbb{A})$ which occur in the decomposition of the space of cusp forms $L^2_{\text{cus}}(A_M M(F)\backslash M(\mathbb{A}))$ into irreducible representations.

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G(F_\infty)$ and $\mathbf{K}_\infty$, respectively. Let θ be the Cartan involution of $G(F_\infty)$ with respect to $\mathbf{K}_\infty$. It induces a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We fix an invariant bi-linear form B on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. This choice defines a Casimir operator Ω on $G(F_\infty)$, and we denote the Casimir eigenvalue of any $\pi \in \Pi(G(F_\infty))$ by λ_π. Similarly, we obtain a Casimir operator $\Omega_{\mathbf{K}_\infty}$ on $\mathbf{K}_\infty$ and write λ_τ for the Casimir eigenvalue of a representation $\tau \in \Pi(\mathbf{K}_\infty)$ (cf. [BG, § 2.3]). The form B induces a Euclidean scalar product $(X, Y) = -B(X, \theta(Y))$ on $\mathfrak{g}$ and all its subspaces. For $\tau \in \Pi(\mathbf{K}_\infty)$ we define $\|\tau\|$ as in [CD, § 2.2]. Note that the restriction of the scalar product $(\cdot,\cdot)$ on $\mathfrak{g}$ to $\mathfrak{a}_0$ gives $\mathfrak{a}_0$ the structure of a Euclidean space. In particular, this fixes Haar measures on the spaces $\mathfrak{a}_M^L$ and their duals $(\mathfrak{a}_M^L)^*$. We follow Arthur in the corresponding normalization of Haar measures on the groups $M(\mathbb{A})$ ([Ar1, § 1]).

Let H be a topological group. We will denote by $\Pi(H)$ the set of equivalence classes of irreducible unitary representations of H.

Next we introduce the space $\mathcal{C}(G(\mathbb{A})^1)$ of Schwartz functions. For any compact open subgroup K_f of $G(\mathbb{A}_f)$ the space $G(\mathbb{A})^1/K_f$ is the countable disjoint union of copies of $G(F_\infty)^1 = G(F_\infty) \cap G(\mathbb{A})^1$ and therefore, it is a differentiable manifold. Any element $X \in \mathcal{U}(\mathfrak{g}^1_\infty)$ of the universal enveloping algebra of the Lie algebra $\mathfrak{g}^1_\infty$ of $G(F_\infty)^1$ defines a left invariant differential operator $f \mapsto f * X$ on $G(\mathbb{A})^1/K_f$. Let $\mathcal{C}(G(\mathbb{A})^1; K_f)$ be the space of smooth right K_f-invariant functions on $G(\mathbb{A})^1$ which belong, together with all their derivatives, to $L^1(G(\mathbb{A})^1)$. The space $\mathcal{C}(G(\mathbb{A})^1; K_f)$ becomes a Fréchet space under the seminorms

$$\|f * X\|_{L^1(G(\mathbb{A})^1)}, \quad X \in \mathcal{U}(\mathfrak{g}^1_\infty).$$

Denote by $\mathcal{C}(G(\mathbb{A})^1)$ the union of the spaces $\mathcal{C}(G(\mathbb{A})^1; K_f)$ as K_f varies over the compact open subgroups of $G(\mathbb{A}_f)$ and endow $\mathcal{C}(G(\mathbb{A})^1)$ with the inductive limit topology.

3. Arithmetic manifolds

In this section we introduce the adelic description of the locally symmetric spaces we will work with. We also explain the relation to the usual set up.

Let G be a reductive algebraic group over a number field F. Fix a faithful F-rational representation $\rho\colon G \to \mathrm{GL}(V)$ and an $\mathcal{O}_F$-lattice Λ in the representation space V such that the stabilizer of $\hat{\Lambda} := \hat{\mathcal{O}}_F \otimes \Lambda \subset \mathbb{A}_f \otimes V$ in $G(\mathbb{A}_f)$ is the group $\mathbf{K}_f$. Since the maximal compact subgroups of $\mathrm{GL}(\mathbb{A}_f \otimes V)$ are precisely the stabilizers of lattices, it is easy to see that such a lattice exists. For any non-zero ideal $\mathfrak{n}$ of $\mathcal{O}_F$, let

$$\mathbf{K}(\mathfrak{n}) = \mathbf{K}_G(\mathfrak{n}) = \{g \in G(\mathbb{A}_f)\colon \rho(g)v \equiv v \pmod{\mathfrak{n}\hat{\Lambda}},\ v \in \hat{\Lambda}\}$$

be the principal congruence subgroup of level $\mathfrak{n}$. Note that $\mathbf{K}(\mathfrak{n})$ is a factorizable normal subgroup of $\mathbf{K}_f$. Moreover, the groups $\mathbf{K}(\mathfrak{n})$ form a neighborhood base of the identity element in $G(\mathbb{A}_f)$, i.e., every compact open subgroup $K_f \subset G(\mathbb{A}_f)$ contains a $K(\mathfrak{n})$ for some ideal $\mathfrak{n}$. We denote by $N(\mathfrak{n}) := [\mathfrak{o}_F\colon \mathfrak{n}]$ the ideal norm of $\mathfrak{n}$.

A subgroup $\Gamma \subset G(F)$ is a congruence subgroup if it contains a a finite-index subgroup of the form $\Gamma(\mathfrak{n}) := G(F) \cap \mathbf{K}(\mathfrak{n})$ for some ideal $\mathfrak{n}$. This definition of a congruence subgroup is independent of the choice of a faithful representation, i.e., it is intrinsic to the F-group G. Let $K_f \subset G(\mathbb{A}_f)$ be a compact open subgroup. Then there exists an ideal $\mathfrak{n}$ of $\mathcal{O}_F$ such that $\mathbf{K}(\mathfrak{n}) \subset K_f$. Let $\Gamma_{K_f} := G(F) \cap K_f$. Then $\Gamma(\mathfrak{n}) \subset \Gamma_{K_f}$ is a finite index subgroup. Thus Γ_{K_f} is a congruence subgroup of $G(F)$.

By [Bo1, § 5.6] the double coset space $G(F)\backslash G(\mathbb{A})/G(F_\infty)K_f$ is finite. Let $x_1 = 1, x_2, \dots, x_l$ be a set of representatives in $G(\mathbb{A}_f)$ of the double cosets. Then the groups

$$\Gamma_i := \left(G(F_\infty) \times x_i K_f x_i^{-1}\right) \cap G(F), \quad 1 \le i \le l, \tag{3.1}$$

are arithmetic subgroups of $G(F_\infty)$ and the action of $G(F_\infty)$ on the space of double cosets $A_G G(F)\backslash G(\mathbb{A})/K_f$ induces the following decomposition into $G(F_\infty)$-orbits:

$$A_G G(F)\backslash G(\mathbb{A})/K_f \cong \bigsqcup_{i=1}^{l} \left(\Gamma_i\backslash G(F_\infty)^1\right), \tag{3.2}$$

where $G(F_\infty)^1 = G(F_\infty)/A_G$. Given a function f on $G(\mathbb{A})$, let f_i be the function on $G(F_\infty)$ which is defined by $g \mapsto f(x_i \cdot g)$, $g \in G(F_\infty)$. Then the map $f \mapsto (f_i)_{i=1}^l$ yields an isomorphism of $G(F_\infty)$-modules

$$L^2(A_G G(F)\backslash G(\mathbb{A}))^{K_f} \cong \bigoplus_{i=1}^{l} L^2(\Gamma_i\backslash G(F_\infty)^1) \tag{3.3}$$

[BJ, 4.3]. We note that, in general, $l > 1$. However, if G is semisimple, simply connected, and without any F-simple factors H for which $H(F_\infty)$ is compact, then by strong approximation we have

$$G(F)\backslash G(\mathbb{A})/K_f \cong \Gamma\backslash G(F_\infty),$$

where $\Gamma = (G(F_\infty)\times K_f)\cap G(F)$. In particular this is the case for $G = \mathrm{SL}(n)$. Since (3.3) is an isomorphism of $G(F_\infty)$-modules, it holds also for the discrete spectrum, i.e., we have an isomorphism of $G(F_\infty)$-modules

$$L^2_{\mathrm{dis}}(A_G G(F)\backslash G(\mathbb{A}))^{K_f} \cong \bigoplus_{i=1}^{l} L^2_{\mathrm{dis}}(\Gamma_i\backslash G(F_\infty)^1). \tag{3.4}$$

Let $P \in \mathcal{L}$ with Levi decomposition $P = M \ltimes N$. Let $f \in L^2(A_G G(F)\backslash G(\mathbb{A}))$ correspond to $(f_i)_{i=1}^l \in \oplus_{i=1}^l L^2(\Gamma_i\backslash G(F_\infty)^1)$ as above. As explained in [BJ, Sect. 4.4], f is a cusp function if and only if each f_i, $i = 1,\dots,l$, is a cusp function. Hence (3.3) induces an isomorphism

$$L^2_{\mathrm{cus}}(A_G G(F)\backslash G(\mathbb{A}))^{K_f} \cong \bigoplus_{i=1}^{l} L^2_{\mathrm{cus}}(\Gamma_i\backslash G(F_\infty)^1) \tag{3.5}$$

of $G(F_\infty)$-modules. Since $L^2_{\mathrm{res}}(\cdot)$ is the orthogonal complement of $L^2_{\mathrm{cus}}(\cdot)$ in $L^2(\cdot)$, it follows from (3.4) and (3.5), that we also have an isomorphism of $G(F_\infty)$-modules

$$L^2_{\mathrm{res}}(A_G G(F)\backslash G(\mathbb{A}))^{K_f} \cong \bigoplus_{i=1}^{l} L^2_{\mathrm{res}}(\Gamma_i\backslash G(F_\infty)^1). \tag{3.6}$$

Given $\tau \in \Pi(G(F_\infty))$, let $m(\tau)$ be the multiplicity with which the representation τ occurs in $L^2_{\mathrm{dis}}(A_G G(F)\backslash G(\mathbb{A}))^{K_f}$. Then

$$m(\tau) = \sum_{\substack{\pi\in\Pi_{\mathrm{dis}}(G(\mathbb{A}))\\ \tau=\pi_\infty}} m(\pi)\dim(\mathcal{H}_{\pi_f'}^{K_f}), \tag{3.7}$$

where $\pi = \pi_\infty \otimes \pi_f$. Similarly, let $m_{\Gamma_i}(\tau)$ be the multiplicity with which τ occurs in $L^2_{\mathrm{dis}}(\Gamma_i\backslash G(F_\infty)^1)$. Since (3.4) is an isomorphism of $G(F_\infty)^1$-modules, it follows that

$$\sum_{\substack{\pi\in\Pi_{\mathrm{dis}}(G(\mathbb{A}))\\ \pi_\infty=\tau}} m(\pi)\dim(\mathcal{H}_{\pi_f'}^{K_f}) = \sum_{j=1}^{l} m_{\Gamma_j}(\tau). \tag{3.8}$$

Let $\mathbf{K}_\infty \subset G(F_\infty)^1$ be a maximal compact subgroup. Let

$$\widetilde{X} := G(F_\infty)^1/\mathbf{K}_\infty \tag{3.9}$$

be the associated global Riemannian symmetric space. Given an open compact subgroup $K_f \subset G(\mathbb{A}_f)$, we define the arithmetic manifold $X(K_f)$ by

$$X(K_f) := G(F)\backslash(\widetilde{X} \times G(\mathbb{A}_f)/K_f). \tag{3.10}$$

By (3.2) we have

$$X(K_f) = \bigsqcup_{i=1}^{l} \left(\Gamma_i\backslash\widetilde{X}\right), \tag{3.11}$$

where each component $\Gamma_i\backslash\widetilde{X}$ is a locally symmetric space. We will assume that K_f is neat. Then $X(K_f)$ is a locally symmetric manifold of finite volume.

Let $\nu \in \Pi(\mathbf{K}_\infty)$. Let $\widetilde{E}_\nu \to \widetilde{X}$ be the homogeneous vector bundle associated to ν. Denote by $C^\infty(\widetilde{X}, \widetilde{E}_\nu)$ the space of smooth sections of $\widetilde{E}_\nu$. Let

(3.12)

$$C^\infty(G(F_\infty)^1, \nu) := \{f : G(F_\infty)^1 \to V_\nu\colon f \in C^\infty,\ f(gk) = \nu(k^{-1})f(g),\ \forall g \in G(F_\infty)^1,\ \forall k \in \mathbf{K}_\infty\}.$$

Let $L^2(G(F_\infty)^1, \nu)$ be the corresponding L^2-space. There is a canonical isomorphism

$$\widetilde{A}\colon C^\infty(\widetilde{X}, \widetilde{E}_\nu) \cong C^\infty(G(F_\infty)^1, \nu), \tag{3.13}$$

(see [Mia, p. 4]). $\widetilde{A}$ extends to an isometry of the corresponding L^2-spaces.

Over each component of $X(K_f)$, $\widetilde{E}_\sigma$ induces a locally homogeneous Hermitian vector bundle $E_{i,\sigma} \to \Gamma_i\backslash\widetilde{X}$. Let

$$E_\sigma := \bigsqcup_{i=1}^{l} E_{i,\sigma}.$$

Then E_σ is a vector bundle over $X(K_f)$ which is locally homogeneous. Let $L^2(X(K_f), E_\sigma)$ be the space of square integrable sections of E_σ.

4. Eisenstein series and intertwining operators

In this section we recall some basic facts about Eisenstein series and intertwining operators, which are the main ingredients of the spectral side of the Arthur trace formula.

Let $M \in \mathcal{L}$ and $P \in \mathcal{P}(M)$ with $P = M \ltimes N_P$. Recall that we denote by $\Sigma_P \subset \mathfrak{a}_P^*$ the set of reduced roots of T_M on the Lie algebra $\mathfrak{n}_P$ of N_P. Let Δ_P be the subset of simple roots of P, which is a basis for $(\mathfrak{a}_P^G)^*$. Write $\mathfrak{a}_{P,+}^*$ for the closure of the Weyl chamber of P, i.e.

$$\mathfrak{a}_{P,+}^* = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee\rangle \geq 0 \text{ for all } \alpha \in \Sigma_P\} = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee\rangle \geq 0 \text{ for all } \alpha \in \Delta_P\}.$$

Denote by δ_P the modulus function of $P(\mathbb{A})$. Let $\bar{\mathcal{A}}^2(P)$ be the Hilbert space completion of

$$\{\phi \in C^\infty(M(F)U_P(\mathbb{A})\backslash G(\mathbb{A})) : \delta_P^{-\frac{1}{2}}\phi(\cdot x) \in L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A})),\ \forall x \in G(\mathbb{A})\}$$

with respect to the inner product

$$(\phi_1, \phi_2) = \int_{A_M M(F) N_P(\mathbb{A})\backslash G(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}\, dg.$$

Let $\alpha \in \Sigma_M$. We say that two parabolic subgroups $P, Q \in \mathcal{P}(M)$ are *adjacent* along α, and write $P|^\alpha Q$, if $\Sigma_P \cap -\Sigma_Q = \{\alpha\}$. Alternatively, P and Q are adjacent if the group $\langle P, Q\rangle$ generated by P and Q belongs to $\mathcal{F}_1(M)$ (see (2.4) for its definition). Any $R \in \mathcal{F}_1(\mathcal{M})$ is of the form $\langle P, Q\rangle$, where P, Q are the elements of $\mathcal{P}(M)$ contained in R. We have $P|^\alpha Q$ with $\alpha^\vee \in \Sigma_P^\vee \cap \mathfrak{a}_M^R$. Interchanging P and Q changes α to $-\alpha$.

For any $P \in \mathcal{P}(M)$ let $H_P\colon G(\mathbb{A}) \to \mathfrak{a}_P$ be the extension of H_M to a left $N_P(\mathbb{A})$-and right $\mathbf{K}$-invariant map. Denote by $\mathcal{A}^2(P)$ the dense subspace of $\bar{\mathcal{A}}^2(P)$ consisting of its $\mathbf{K}$- and $\mathfrak{Z}$-finite vectors, where $\mathfrak{Z}$ is the center of the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}$. That is, $\mathcal{A}^2(P)$ is the space of automorphic forms ϕ on $N_P(\mathbb{A})M(F)\backslash G(\mathbb{A})$ such that $\delta_P^{-\frac{1}{2}}\phi(\cdot k)$ is a square-integrable automorphic form on $A_M M(F)\backslash M(\mathbb{A})$ for all $k \in \mathbf{K}$. Let $\rho(P, \lambda)$, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, be the induced representation of $G(\mathbb{A})$ on $\bar{\mathcal{A}}^2(P)$ given by

$$(\rho(P, \lambda, y)\phi)(x) = \phi(xy)e^{\langle \lambda, H_P(xy) - H_P(x)\rangle}.$$

It is isomorphic to the induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\left(L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A})) \otimes e^{\langle \lambda, H_M(\cdot)\rangle}\right).$$

For $\phi \in \mathcal{A}^2(P)$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, the associated Eisenstein series is defined by

$$E(g, \phi, \lambda) := \sum_{\gamma \in P(F)\backslash G(F)} \phi(\gamma g)e^{(\lambda + \rho_P)(H_P(\gamma g))}. \tag{4.1}$$

The series converges absolutely and locally uniformly in g and λ for $\mathrm{Re}(\lambda)$ sufficiently regular in the positive Weyl chamber of $\mathfrak{a}_P^*$ ([MW, II.1.5]. By Langlands [La1] the Eisenstein series can be continued analytically to a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. Its singularities lie along hypersurfaces defined by root equations.

Let $M, M_1 \in \mathcal{L}$. Let $W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ be the set of isomorphisms from $\mathfrak{a}_M$ onto $\mathfrak{a}_{M_1}$ obtained by restricting elements in W_0, the Weyl group of (G, T_0), to $\mathfrak{a}_M$. Each $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ has a representative w_s in $G(F)$. Given $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$, $P \in \mathcal{P}(M)$ and $P_1 \in \mathcal{P}(M_1)$, let

$$M_{P_1|P}(s, \lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(P_1), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}, \tag{4.2}$$

be the standard *intertwining operator* [Ar9, § 1], which is the meromorphic continuation in λ of the integral

$$\begin{aligned}[M_{P_1|P}(s,\lambda)\phi](x) \\ = \int_{N_{P_1}(\mathbb{A})\cap w_s N_P(\mathbb{A}) w_s^{-1}\backslash N_{P_1}(\mathbb{A})} \phi(w_s^{-1}nx) e^{(\lambda+\rho_P)(H_P(w_s^{-1}nx))} e^{-(s\lambda+\rho_{P_1})(H_{P_1}(x))}\, dn,\end{aligned} \tag{4.3}$$

for $\phi \in \mathcal{A}^2(P)$, $x \in G(\mathbb{A})$. Let $M = M_1$. Then for $Q, P \in \mathcal{P}(M)$ and $1 \in W(\mathfrak{a}_M)$ the identity element, we put

$$M_{Q|P}(\lambda) := M_{Q|P}(1,\lambda). \tag{4.4}$$

Recall that $L^2_{\text{dis}}(A_M M(F)\backslash M(\mathbb{A}))$ decomposes as the completed direct sum of its π-isotopic components for $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$. We have a corresponding decomposition of $\bar{\mathcal{A}}^2(P)$ as a direct sum of Hilbert spaces $\hat{\oplus}_{\pi\in\Pi_{\text{dis}}(M(\mathbb{A}))}\bar{\mathcal{A}}^2_\pi(P)$ and the corresponding algebraic sum decomposition

$$\mathcal{A}^2(P) = \bigoplus_{\pi\in\Pi_{\text{dis}}(M(\mathbb{A}))} \mathcal{A}^2_\pi(P). \tag{4.5}$$

We further decompose $\mathcal{A}^2_\pi(P)$ according to the action of $\mathbf{K}_\infty$ into isotypic subspaces

$$\mathcal{A}^2_\pi(P) = \bigoplus_{\nu\in\Pi(\mathbf{K}_\infty)} \mathcal{A}^2_\pi(P)^\nu. \tag{4.6}$$

Furthermore, for an open compact subgroup $K_f \subset G(\mathbb{A}_f)$ let $\mathcal{A}^2_\pi(P)^{K_f}$ be the subspace of K_f-invariant functions in $\mathcal{A}^2_\pi(P)$ and for $\nu \in \Pi(\mathbf{K}_\infty)$ we let $\mathcal{A}^2_\pi(P)^{K_f,\nu}$ be the ν-isotypic subspace of $\mathcal{A}^2_\pi(P)^{K_f}$.

Given $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$, let $(\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi), \mathcal{H}_P(\pi))$ be the induced representation. Let $\mathcal{H}^0_P(\pi)$ be the subspace of $\mathcal{H}_P(\pi)$, consisting of all $\phi \in \mathcal{H}_P(\pi)$ which are right $\mathbf{K}$-finite and right $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$-finite. There is a canonical isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_\mathbb{C}, \mathbf{K}_\infty)$-modules

$$j_P : \text{Hom}(\pi, L^2(A_M M(F)\backslash M(\mathbb{A}))) \otimes \mathcal{H}^0_P(\pi) \to \mathcal{A}^2_\pi(P). \tag{4.7}$$

If we fix a unitary structure on π and endow $\text{Hom}(\pi, L^2(A_M M(F)\backslash M(\mathbb{A})))$ with the inner product $(A,B) = B^*A$ (which is a scalar operator on the space of π), the isomorphism j_P becomes an isometry. Let

$$M_{Q|P}(\pi,\lambda) := M_{Q|P}(\lambda)|_{\mathcal{A}^2_\pi(P)} \tag{4.8}$$

be the restriction of $M_{Q|P}(\lambda)$ to the subspace $\mathcal{A}^2_\pi(P)$. Suppose that $P|^\alpha Q$. The operator $M_{Q|P}(\pi,z) := M_{Q|P}(\pi, z\varpi)$, where $\varpi \in \mathfrak{a}^\star_M$ is such that $\langle\varpi,\alpha^\vee\rangle = 1$, admits a normalization by a global factor $n_\alpha(\pi,z)$ which is a meromorphic function in $z \in \mathbb{C}$. We may write

$$M_{Q|P}(\pi,z) \circ j_P = n_\alpha(\pi,z) \cdot j_Q \circ (\text{Id} \otimes R_{Q|P}(\pi,z)) \tag{4.9}$$

where $R_{Q|P}(\pi,z) = \otimes_v R_{Q|P}(\pi_v,z)$ is the product of the locally defined normalized intertwining operators and $\pi = \otimes_v \pi_v$ [Ar9, § 6], (cf. [Mu2, (2.17)]). In many cases, the normalizing factors can be expressed in terms automorphic L-functions [Sh1], [Sh2].

For any $P, Q \in \mathcal{P}(M)$ there exists a sequence of parabolic subgroups $P_0, ..., P_k$ and roots $\alpha_1, ..., \alpha_k \in \Sigma_M$ such that $P = P_0$, $Q = P_k$, and $P_{i-1}|^{\alpha_i} P_i$ for $i = 1, ..., k$. By the product rule for intertwining operators we have

$$M_{Q|P}(\pi, \lambda) = M_{P_k|P_{k-1}}(\pi, \lambda) \circ M_{P_{k-1}|P_{k-2}}(\pi, \lambda) \circ \cdots \circ M_{P_1|P_0}(\pi, \lambda). \tag{4.10}$$

Thus the study of the operators $M_{Q|P}(\pi, \lambda)$ is reduced to the case where $Q, P \in \mathcal{P}(M)$ are adjacent along some root $\alpha \in \Sigma_M$. Let

$$n_{Q|P}(\pi, \lambda) := \prod_{\alpha \in \Sigma_P \cap \Sigma_{\bar{Q}}} n_\alpha(\pi, \lambda(\alpha^\vee)) \tag{4.11}$$

The product is a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. Then $M_{Q|P}(\pi, \lambda)$ is normalized by $n_{Q|P}(\pi, \lambda)$, i.e.,

$$M_{Q|P}(\pi, \lambda) \circ j_P = n_{Q|P}(\pi, \lambda) \cdot j_Q \circ (\operatorname{Id} \otimes R_{Q|P}(\pi, \lambda)). \tag{4.12}$$

Recall that $\pi = \otimes_v \pi_v$, where $\pi_v \in \Pi(M(F_v))$. If $\mathcal{H}_P(\pi_v)$ is the Hilbert space of the induced representation $\operatorname{Ind}_{P(F_v)}^{G(F_v)}(\pi_v)$, then one has

$$\mathcal{H}_P(\pi) \cong \bigotimes_v \mathcal{H}_P(\pi_v)$$

and, with respect to this isomorphism, it follows that $R_{Q|P}(\pi, \lambda)$ is the product of the corresponding local normalized intertwining operators

$$R_{Q|P}(\pi, \lambda) = \otimes_v R_{Q|P}(\pi_v, \lambda) \tag{4.13}$$

[Ar4], [Ar9, § 6], [Mu2, § 2].

5. Normalizing factors

In this section we consider the global normalizing factors of intertwining operators. The goal is to estimate the number of singular hyperplanes of normalizing factors which intersect a given compact set. The normalizing factors can be expressed in terms of L-functions. To begin with we recall some basic facts about L-functions. As above, we assume that G is a reductive group over a number field F. Recall that $A_G = T_{G_1}(\mathbb{R})^0$, where T_{G_1} is the $\mathbb{Q}$-split part of the connected component of the center of $G_1 = \operatorname{Res}_{F/\mathbb{Q}}(G)$, viewed as a subgroup of $T_{G_1}(\mathbb{A}_\mathbb{Q})$ and hence of $G(\mathbb{A}_F)$.

Recall that we denote by $\Pi_{\text{dis}}(G(\mathbb{A}))$ the set of equivalence classes of automorphic representations of $G(\mathbb{A})$ which occur in the discrete spectrum of $L^2(A_G G(F)\backslash G(\mathbb{A}))$. For any $\pi = \otimes_v \pi_v \in \Pi_{\text{dis}}(G(\mathbb{A}))$ let $S(\pi)$ be the finite set of places of F containing all archimedean places and such that for each finite place $v \in S(\pi)$ at least one of the following conditions holds:

1. v is archimedean.
2. $F/\mathbb{Q}$ is ramified at v.
3. G is ramified at v, i.e., either G is not quasi-split over F_v or G does not split over an unramified extension of F_v.

(4) For every hyperspecial maximal compact subgroup K_v of $G(F_v)$, π_v does not have a nonzero vector which is invariant under K_v.

Let S_∞ denote the set of archimedean places of F and let $S_f(\pi)$ denote the set of non-archimedean places in $S(\pi)$. Thus $S(\pi) = S_\infty \cup S_f(\pi)$. For any $v \in S_f(\pi)$ let q_v denote the order of the residue field of F_v. Let $S_{\mathbb{Q},f}(\pi)$ be the set of rational primes which lie below the primes in $S_f(\pi)$. Also set $S_{\mathbb{Q}}(\pi) := \{\infty\} \cup S_{\mathbb{Q},f}(\pi\}$.

Let W_F be the Weil group of F and let ${}^LG$ be the Langlands L-group of G [Bo2]. Let $r\colon {}^LG \to \mathrm{GL}(N,\mathbb{C})$ be a continuous and W_F-semisimple N-dimensional complex representation of ${}^LG$. For any $\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}))$ and any place v of F with $v \notin S(\pi)$ let $t_{\pi_v} \in^L G$ be the Hecke-Frobenius parameter of π_v. Then the local L-function $L_v(s,\pi,r)$ is defined by

$$L_v(s,\pi,r) := \det\left(\mathrm{Id} - r(t_{\pi_v})q_v^{-s}\right)^{-1}. \tag{5.1}$$

Since π is unitary, the $|r(t_{\pi_v})|$ are bounded by q_v^c, where c depends only on G and r, [Bo2], [La2]. Therefore, for $S \supset S(\pi)$ the partial L-function

$$L^S(s,\pi,r) := \prod_{v\notin S} L_v(s,\pi,r) \tag{5.2}$$

converges absolutely and uniformly on compact subsets of $\mathrm{Re}(s) > c+1$. One of the goals of the Langlands program is to show that each of these L-functions admits a meromorphic extension to the entire complex plane and satisfies a functional equation. This is far from being proved. In [FL1, Definition 2.1], Finis and Lapid formulated a precise version of the expected functional equation. According to this definition, (G,r) has property **(FE)**, if for any $\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}_F)$ the partial L-function $L^{S(\pi)}(s,\pi,r)$ admits a meromorphic continuation to $\mathbb{C}$ with a functional equation of the form

$$L^{S(\pi)}(s,\pi,r) = \left(\prod_{p\in S_{\mathbb{Q}}(\pi)} \gamma_p(s,\pi,r))\right) L^{S(\pi)}(1-s,\pi,r^\vee), \tag{5.3}$$

where for each $p \in S_{\mathbb{Q},f}(\pi)$, $\gamma_p(s,\pi,r) = R_p(p^{-s})$ for some rational function R_p and

$$\gamma_\infty(s,\pi,r) = C_\infty \prod_{i=1}^m \frac{\Gamma_{\mathbb{R}}(1-s+\alpha_i^\vee)}{\Gamma_{\mathbb{R}}(s+\alpha_i)} \tag{5.4}$$

for certain parameters $\alpha_1,...,\alpha_m,\alpha_1^\vee,...,\alpha_m^\vee \in \mathbb{C}$ and a constant C_∞. By [FL1, Lemma 2.2] the parameters $\alpha_1^\vee,...,\alpha_m^\vee$ are determined by $\alpha_1,...,\alpha_m$. Moreover, the integer m is uniquely determined. The parameters $\alpha_1,...,\alpha_m$ are said to be *reduced*, if $\alpha_i+\alpha_j$ is not a negative odd integer for any $1 \le i,j \le m$. By [FL1, Lemma 2.2] one may choose the parameters $\alpha_1,...,\alpha_m$ to be reduced. Assuming that this is satisfied, Finis and Lapid introduce the reduced L-factor at the Archimedean place by

$$L_\infty^{\mathrm{red}}(s,\pi,r) := \prod_{i=1}^m \Gamma_{\mathbb{R}}(s+\alpha_i). \tag{5.5}$$

Now let $p \in S_{\mathbb{Q},f}(\pi)$. Then by [FL1, (2.7)], $\gamma_p(s,\pi,r)$ can be written in a unique way as

$$\gamma_p(s,\pi,r) = c_p p^{\left(\frac{1}{2}-s\right)\mathfrak{e}_p(\pi,r)} P_p(p^{-s})/\bar{P}_p(p^{s-1}), \tag{5.6}$$

where $c_p \in \mathbb{C}^*$, $\mathfrak{e}_p(\pi,r) \in \mathbb{Z}$, and P_p is a polynomial with $P_p(0) = 1$ such that no zeros α and β of P_p satisfy $\alpha\bar{\beta} = p^{-1}$. Then Finis and Lapid define the reduced L-factor at p by

$$L_p^{\mathrm{red}}(s,\pi,r) := P_p(p^{-s})^{-1}, \quad p \in S_{\mathbb{Q},f}(\pi), \tag{5.7}$$

and introduce the *reduced completed L-function* by

$$L^{\mathrm{red}}(s,\pi,r) := \Big(\prod_{p\in S_{\mathbb{Q}}(\pi)} L_p^{\mathrm{red}}(s,\pi,r)\Big) L^{S(\pi)}(s,\pi,r). \tag{5.8}$$

There is also a corresponding reduced epsilon factor $\epsilon^{\mathrm{red}}(s,\pi,r)$, which is defined by

$$\epsilon^{\mathrm{red}}(s,\pi,r) = c_\infty \prod_{p\in S_{\mathbb{Q},f}(\pi)} c_p p^{\left(\frac{1}{2}-s\right)\mathfrak{e}_p(\pi,r)} = \mathfrak{n}(\pi,r)^{\frac{1}{2}-s} \prod_{p\in S_{\mathbb{Q}}(\pi)} c_p, \tag{5.9}$$

where

$$\mathfrak{n}(\pi,r) = \prod_{p\in S_{\mathbb{Q},f}(\pi)} p^{\mathfrak{e}_p(\pi,r)}. \tag{5.10}$$

Then the functional equation (5.3) becomes

$$L^{\mathrm{red}}(s,\pi,r) = \epsilon^{\mathrm{red}}(s,\pi,r) L^{\mathrm{red}}(1-s,\pi,r^\vee). \tag{5.11}$$

In [FL1, Definition 2.4] a stronger version of property **(FE)** is introduced. The pair (G,r) is said to satisfy property **(FE+)**, if it satisfies **(FE)** and in addition some uniformity conditions for γ_∞ and P_p are fulfilled. For the precise statement see [FL1, Definition 2.4].

The normalizing factors are described in [FL1, Sect. 3]. To recall the description, we need to introduce some notation. Let $M \in \mathcal{L}$ and $\alpha \in \Sigma_M$. Let $\tilde{M}_\alpha$ be the Levi subgroup of M of co-rank one, defined in [FL1, p. 254], together with the map $p^{\mathrm{sc}}\colon \tilde{M}_\alpha^{\mathrm{sc}} \to \tilde{M}_\alpha$, which is also defined in [FL1, p. 254]. Furthermore, let U_α be the unipotent subgroup of G corresponding to α. Thus the eigenvalues of T_M acting on the Lie algebra of U_α are positive integer multiples of α. The adjoint action of ${}^LM$ on $\mathrm{Lie}({}^LU_\alpha)$ factors through the composed homomorphism ${}^LM \to^L \tilde{M}_\alpha$. The contragredient of the adjoint representation of ${}^L\tilde{M}_\alpha$ on $\mathrm{Lie}({}^LU_\alpha)$ is decomposed as $\oplus_{j=1}^l r_j$ into irreducible representations r_j.

By T. Finis and E. Lapid [FL1, Definition 3.4], G satisfies property (L), if for any standard Levi subgroup M, any $\alpha \in \Sigma_M$, and any irreducible constituent $r = r_j$ as above, the pair $(\tilde{M}_\alpha, r)$ satisfies properties **(FE+)** [FL1, Definition 2.4] and the conductor condition **(CC)** [FL1, Definition 2.9].

Assume that G satisfies property **(L)**. Then one can describe the normalizing factors in terms of L-functions. Let $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$ and let $n_\alpha(\pi,s)$ be the normalizing factor as in (4.9). First note that $n_\alpha(\pi,s)$ satisfies the functional equation

$$n_\alpha(\pi,s)\overline{n_\alpha(\pi,-\bar{s})} = 1. \tag{5.12}$$

Next recall that for $\mathrm{Re}(s) \gg 0$, $n_\alpha(\pi, s)$ factorizes as

$$n_\alpha(\pi, s) = \prod_v n_{\alpha,v}(\pi_v, s). \tag{5.13}$$

By [FL1, Lemma 2.13] there exist $\sigma \in \Pi_{\mathrm{dis}}(\tilde{M}_\alpha(\mathbb{A}))$ and a character χ of $\tilde{M}_\alpha(\mathbb{A})$, which is trivial on $\tilde{M}_\alpha(F)p^{\mathrm{sc}}(\tilde{M}_\alpha^{\mathrm{sc}})$, such that $\sigma\chi$ is a subrepresentation of $\pi|_{\tilde{M}_\alpha(\mathbb{A})}$. Let

$$m(\sigma, s) := \prod_{j=1}^{l} \frac{\epsilon^{\mathrm{red}}(1, \sigma, r_j) L^{\mathrm{red}}(js, \sigma, r_j)}{L^{\mathrm{red}}(js+1, \sigma, r_j)}. \tag{5.14}$$

Then at the end of the proof of Proposition 3.8 in [FL1, p. 259] it has been shown that there exists $C > 0$ such that

$$n_\alpha(\pi, s) = C \cdot m(\sigma, s) \cdot \prod_{j=1}^{l} \prod_{p \in S_{\mathbb{Q}}(\sigma)} \frac{L_p^{\mathrm{red}}(js+1, \sigma, r_j)}{L_p^{\mathrm{red}}(js, \sigma, r_j)} \prod_{v \in S(\sigma)} n_{\alpha,v}(\pi, s). \tag{5.15}$$

We use this formula to estimate the number of poles of $n_\alpha(\pi, s)$. First we consider the two finite products. For this purpose we need to estimate the cardinality of $S(\sigma)$ and $S_{\mathbb{Q}}(\sigma)$. Recall that the level of σ is defined as $\mathrm{level}(\sigma) = N(\mathfrak{n})$, where $\mathfrak{n}$ is the largest ideal of $\mathcal{O}_F$ such that $\sigma^{\mathbf{K}(\mathfrak{n}) \cap \tilde{M}_\alpha} \neq 0$. We obviously have

$$|S(\sigma)|, |S_{\mathbb{Q}}(\sigma)| \leq C(1 + \log \mathrm{level}(\sigma))$$

for some constant $C > 0$ which is independent of σ. Furthermore, recall that by [FL1, Sec. 2.3], $\mathrm{level}(\pi; p^{\mathrm{sc}})$ is defined as $\mathrm{level}(\pi; p^{\mathrm{sc}}) = N(\mathfrak{n})$, where $\mathfrak{n}$ is the largest ideal in $\mathcal{O}_F$ such that $\pi^{\mathbf{K}(\mathfrak{n}) \cap p^{\mathrm{sc}}(\tilde{M}_\alpha^{\mathrm{sc}})} \neq 0$.

By [FL1, Lemma 2.13] there exists $N_1 \in \mathbb{N}$, which depends only on p^{sc} and G, such that for every $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$ the corresponding representation $\sigma \in \Pi_{\mathrm{dis}}(\tilde{M}_\alpha(\mathbb{A}))$ is such that $\mathrm{level}(\sigma)$ divides $N_1 \,\mathrm{level}(\pi; p^{\mathrm{sc}})$. Thus there exists $C > 0$ such that

$$|S_{\mathbb{Q}}(\sigma)|, |S(\sigma)| \leq C \log \mathrm{level}(\pi; p^{\mathrm{sc}}) \tag{5.16}$$

for all π and σ which are related as above.

Now consider the last product on the right hand side of (5.15). Let $v \in S_f(\sigma)$. By [FL1, p. 255, (1)], $n_{\alpha,v}(\pi, s)$ is a rational function in $X = q_v^{-s}$, whose degree is bounded in terms of G only and which is regular and non-zero at $X = 0$. Hence the poles of $n_{\alpha,v}(\pi, s)$ form a finite union of arithmetic progressions with imaginary difference and the number of progressions is bounded by a constant that depends only on G.

If $v \in S_\infty(\sigma)$, then by [FL1, p. 255, (2)] we have

$$n_{\alpha,v}(\pi, s) = c_v \prod_{i=1}^{N_v} \frac{\Gamma_{\mathbb{R}}(j_i s + \alpha_i)}{\Gamma_{\mathbb{R}}(j_i s + \alpha_i + 1)}, \tag{5.17}$$

where $c_v \neq 0$, $\alpha_1, \ldots, \alpha_{N_v} \in \mathbb{C}$ and the integers $N_v \geq 1$ and $j_i \geq 1$, $i = 1, \ldots, N_v$ are bounded in terms of G only. Recall that $\Gamma(z)$ has no zeros and the poles are simple and occur at the negative integers. Let $R > 0$. It follow that the number of poles of $n_{\alpha,v}(\pi, s)$ in a fixed half-strip $|\operatorname{Im}(s)| \le R$, $\operatorname{Re}(s) \ge -R$ is bounded by a constant independent of π. Thus by (5.16) it follows that for every $R > 0$ there exists $C_1 > 0$, which is independent of π, such that the number of poles in the half-strip $|\operatorname{Im}(s)| \le R$, $\operatorname{Re}(s) \ge -R$, counted with their order, of the last product is bounded by $C_1 \log \operatorname{level}(\pi; p^{\mathrm{sc}})$.

Next we deal with the product over $S_{\mathbb{Q}}(\sigma)$, Let $p \in S_{\mathbb{Q},f}(\sigma)$. By (5.7), there exist a polynomial $P_p(x;\sigma, r_j)$ such that $L_p^{\mathrm{red}}(s,\sigma,r_j) = P_p(p^{-s};\sigma,r_j)^{-1}$. By definition, (G,r_j) satisfies property **(FE+)** [FL1, Definition 2.4]. By (2) of this definition, the degree of $P_p(x;\sigma,r_j)$ is bounded in terms of (G,r_j) only. Thus the poles of $L_p(s,\sigma,r_j)$ form a finite union of arithmetic progressions with imaginary difference and the number of progressions is bounded by a constant that depends only on (G,r_j). Hence for every $R > 0$ there exists $C > 0$, which depends only on (G,r_j), such that the number of poles of $L_p(s,\sigma,r_j)$ in the strip $|\operatorname{Im}(s)| \le R$ is bounded by C. For $p = \infty$ we use (5.5). By [FL1, Definition 2.4, (3)], there exists $\beta \in \mathbb{R}$ which depends only on (G,r_j) such that the reduced parameters α_i satisfy $\operatorname{Re}(\alpha_i) \ge -\beta$, $i = 1,\dots,l$, and $\gamma_\infty(s,\sigma,r_j)$ has no zeros in $\operatorname{Re}(s) > \beta$. So it follows as above, that the number of poles of $L_p^{\mathrm{red}}(js+1,\sigma,r_j)/L_p^{\mathrm{red}}(js,\sigma,r_j)$ in the half-strip $|\operatorname{Im}(s)| \le R$, $\operatorname{Re}(s) \ge -R$, counted with their order, is bounded by a constant independent of π. Using (5.16) it follows that for each $R > 0$ there exists $C_2 > 0$ such that the number of poles of the product over $S_{\mathbb{Q}}(\sigma)$, counted with their order, in the half-strip $|\operatorname{Im}(s)| \le R$, $\operatorname{Re}(s) \ge -R$, is bounded by $C_2 \log \operatorname{level}(\pi; p^{\mathrm{sc}})$.

So it remains to consider $m(\sigma,s)$. Let $r := r_j$ for some j and let

$$\Lambda(s,\sigma,r) := \mathfrak{n}(\sigma,r)^{s/2} L^{\mathrm{red}}(s,\sigma,r), \tag{5.18}$$

where $\mathfrak{n}(\sigma,r)$ is defined by (5.10) Then, using functional equation (5.11) and the definition of the epsilon factor by (5.9), it follows that $\Lambda(s,\sigma,r)$ satisfies

$$\Lambda(s,\sigma,r) = \epsilon^{\mathrm{red}}(\frac{1}{2},\sigma,r)\overline{\Lambda(1-\bar{s},\sigma,r)}. \tag{5.19}$$

By (5.9) and (5.18) we get

$$\frac{\Lambda(s,\sigma,r)}{\overline{\Lambda(-\bar{s},\sigma,r)}} = \frac{\epsilon^{\mathrm{red}}(1,\sigma,r) L^{\mathrm{red}}(s,\sigma,r)}{L^{\mathrm{red}}(s+1,\sigma,r)}. \tag{5.20}$$

Thus by the definition (5.14) it follows that

$$m(\sigma,s) = \prod_{j=1}^{l} \frac{\Lambda(js,\sigma,r_j)}{\overline{\Lambda(-j\bar{s},\sigma,r_j)}}. \tag{5.21}$$

As explained in the proof of [FL1, Proposition 2.6], $\Lambda(s,\sigma,r)$ is the quotient of two holomorphic functions of order one. Therefore $\Lambda(s,\sigma,r)$ admits a Hadamard factorization

$$\Lambda(s,\sigma,r) = e^{a+bs} s^{n(0)} \prod_{\rho\neq 0} \left[(1-s/\rho)e^{s/\rho}\right]^{n(\rho)}, \tag{5.22}$$

where $a,b \in \mathbb{C}$, the product ranges over the zeros and poles of $\Lambda(s)$ different from 0, and $n(\rho)$ is the order of the function $\Lambda(s)$ at $s = \rho$. In the poof of [FL1, Proposition 2.6] it was shown that the conditions of property **(FE+)**, [FL1, Definition 2.4], together with the functional equation (5.19) imply that there exists $A \geq 1$, depending only on G and r, such that all zeros and poles of $\Lambda(s, \sigma, r)$ lie in the strip $1 - A \leq \mathrm{Re}(s) \leq A$. Moreover, by [FL1, (2.12)] we have for $T \geq 0$

$$\sum_{\rho\colon |\operatorname{Im}(\rho)-T|<2} |n(\rho)| \ll \log \mathfrak{n}(\sigma, r) + \sum_{p \in S_{\mathbb{Q},f}(\sigma)} \log p + \log \mathfrak{c}_\infty(\sigma, r) + \log(1 + |T|) + 1, \tag{5.23}$$

where $\mathfrak{c}_\infty(\pi, r)$ is the archimedean conductor defined by [FL1, (2.6)] and $\mathfrak{n}(\sigma, r)$ the finite conductor (5.10). Since we assume that G satisfies property **(L)**, $(\tilde{M}_\alpha, r_j)$ satisfies property (CC). Let $\Lambda(\pi_\infty; p^{\mathrm{sc}})$ be defined by [FL1, (2.18)]. Then by [FL1, (2.14), (2.15)] and [FL1, Lemma 2.13] we obtain

$$\sum_{\rho\colon |\operatorname{Im}(\rho)-T|<2} |n(\rho)| \ll \log(|T| + 2) + \log \operatorname{level}(\pi; p^{\mathrm{sc}}) + \log \Lambda(\pi_\infty; p^{\mathrm{sc}}). \tag{5.24}$$

We combine (5.24) with the results above concerning the other factors occurring in (5.15). Note that by the functional equation (5.12), the poles of $n_\alpha(\pi, s)$ are contained in a strip $|\operatorname{Re}(s)| \leq C$ for some $C > 0$. We can summarize our results as follows. Denote by $\Sigma_\alpha(\pi)$ the poles of $n_\alpha(\pi, s)$. Given $\rho \in \Sigma_\alpha(\pi)$, denote by $n(\rho)$ its order. Then combined with the results above concerning the other factors occurring in (5.15), we obtain the following proposition.

Proposition 5.1. *Assume that G satisfies property (L). Let $M \in \mathcal{L}$, $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$, and $\alpha \in \Sigma_M$. Let $\Sigma_\alpha(\pi)$ be the set of poles of $n_\alpha(\pi, s)$ and for any $\rho \in \Sigma_\alpha(\pi)$ denote by $n(\rho)$ the order of the pole ρ. Then for every $R > 0$ there exist $C > 0$ such that*

$$\sum_{\rho \in \Sigma_\alpha(\pi), |\operatorname{Im}(\rho)|<R} |n(\rho)| \leq C(1 + \log \operatorname{level}(\pi; p^{\mathrm{sc}}) + \log \Lambda(\pi_\infty; p^{\mathrm{sc}})). \tag{5.25}$$

Let $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$. Let $W_P(\pi_\infty)$ be the set of minimal K_∞-types of $\operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty)$. Then $W_P(\pi_\infty)$ is a non empty finite subset of $\Pi(K_\infty)$. Let λ_{π_∞} be the Casimir eigenvalue of π_∞ and for each $\tau \in \Pi(K_\infty)$, let λ_τ be the Casimir eigenvalue of τ. Put

$$\Lambda_{\pi_\infty} := \min_{\tau \in W_P(\pi_\infty)} \sqrt{\lambda_{\pi_\infty}^2 + \lambda_\tau^2}. \tag{5.26}$$

Then by [FLM2, (10)] one has

$$\Lambda(\pi_\infty; p^{\mathrm{sc}}) \ll_G 1 + \Lambda_{\pi_\infty}^2. \tag{5.27}$$

Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. Put

$$\Pi(M(\mathbb{A}); K_f) := \{\pi \in \Pi(M(\mathbb{A})\colon \pi_f^{K_f \cap M(\mathbb{A}_f)} \neq 0\}. \tag{5.28}$$

Furthermore, given $\nu \in \Pi(\mathbf{K}_\infty)$, let

$$\Pi(M(\mathbb{A}); K_f, \nu) = \{\pi \in \Pi(M(\mathbb{A}); K_f)\colon [\operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty)|_{K_\infty} : \nu] > 0\} \tag{5.29}$$

and put

$$\Pi_{\mathrm{dis}}(M(\mathbb{A});K_f,\nu)=\Pi_{\mathrm{dis}}(M(\mathbb{A}))\cap\Pi(M(\mathbb{A});K_f,\nu). \tag{5.30}$$

Now recall the definition of level$(\pi;p^{\mathrm{sc}})$ [FL1, Sec. 2.3]. It follows that there exists $C>0$ such that all for $\pi\in\Pi(M(\mathbb{A});K_f)$ we have level$(\pi;p^{\mathrm{sc}})\le C$. Furthermore, by (5.26) there exists $C_1>0$ such that for all $\pi\in\Pi(M(\mathbb{A});K_f,\nu)$ one has $\Lambda^2_{\pi_\infty}\le C_1(1+\lambda^2_{\pi_\infty})$, and it follows from (5.27) that in this case $\Lambda(\pi_\infty;p^{\mathrm{sc}})\ll_G(1+\lambda^2_{\pi_\infty})$. In this way we get

Corollary 5.2. *Assume that G satisfies property (L). Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and $\nu\in\Pi(K_\infty)$. Let $M\in\mathcal{L}$ and $\alpha\in\Sigma_M$. Let the notation be as above. For every $R>0$ there exist $C>0$ such that*

$$\sum_{\rho\in\Sigma_\alpha(\pi),|\operatorname{Im}(\rho)|<R}|n(\rho)|\le C(1+\log(1+\lambda^2_{\pi_\infty})) \tag{5.31}$$

for all $\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A});K_f,\nu)$.

6. Logarithmic derivatives of local intertwining operators

In this section we prove some auxiliary results for local intertwining operators. To begin with we recall some facts concerning local intertwining operators and normalizing factors. Let $M\in\mathcal{L}$ and $P,Q\in\mathcal{P}(M)$. Let v be a place of F. If v is finite, let K_v be an open compact subgroup of $G(F_v)$ and if $v\in S_\infty$, let K_v be a maximal compact subgroup of $G(F_v)$. Let $\pi_v\in\Pi(M(F_v))$. Given $\lambda\in\mathfrak{a}^*_{M,\mathbb{C}}$, let $(I^G_P(\pi_v,\lambda),\mathcal{H}_P(\pi_v))$ denote the induced representation. Let $\mathcal{H}^0_P(\pi_v)\subset\mathcal{H}_P(\pi_v)$ be the subspace of K_v-finite functions. Let

$$J_{Q|P}(\pi_v,\lambda)\colon\mathcal{H}^0_P(\pi_v)\to\mathcal{H}^0_Q(\pi_v)$$

be the local intertwining operator between the induced representations $I^G_P(\pi_v,\lambda)$ and $I^G_Q(\pi_v,\lambda)$ [Sh1]. It is proved in [Ar4], [CLL, Lecture 15] that there exist scalar valued meromorphic functions $r_{Q|P}(\pi_v,\lambda)$ of $\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}$ such that the normalized intertwining operators

$$R_{Q|P}(\pi_v,\lambda)=r_{Q|P}(\pi_v,\lambda)^{-1}J_{Q|P}(\pi_v,\lambda) \tag{6.1}$$

satisfy the conditions $(R_1)-(R_8)$ of Theorem 2.1 of [Ar4]. We recall some facts about the local normalizing factors. First assume that v is a finite valuation of F with $q_v\in\mathbb{N}$ the cardinality of the residue field of F_v. Furthermore assume that $\dim(\mathfrak{a}_M/\mathfrak{a}_G)=1$ and π_v is square integrable. Let $P\in\mathcal{P}(M)$ and let α be the unique simple root of (P,T_M). Then Langlands [CLL, Lecture 15] has shown that there exists a rational function $V_P(\pi_v,z)$ of one variable such that

$$r_{\overline{P}|P}(\pi_v,\lambda)=V_P(\pi_v,q_v^{-\lambda(\widetilde{\alpha})}), \tag{6.2}$$

where $\widetilde{\alpha}\in\mathfrak{a}_M$ is uniquely determined by α. For the construction of V_P see also [Mu2, Sect. 3]. In this reference, only the case $\mathbb{Q}_v$ has been discussed. However, the case F_v can be dealt with in exactly the same way. We need the following lemma.

Lemma 6.1. *Let $M \in \mathcal{L}$ be such that* $\dim(\mathfrak{a}_M/\mathfrak{a}_G) = 1$. *There exists $C > 0$ such that for all $P \in \mathcal{P}(M)$ and all $\pi \in \Pi(M(F_v))$ the number of zeros of the rational function $V_P(\pi, z)$ is less than or equal to C.*

For the proof see [MM2, Lemma 10.1]. Again, the proof has been carried out for $\mathbb{Q}_v$. It extends to F_v without any changes.

The main goal of this section is to estimate the logarithmic derivatives of the normalized intertwining operators $R_{Q|P}(\pi, \lambda)$. For $G = \mathrm{GL}(n)$ such estimates were derived in [MS, Proposition 0.2]. The proof depends on a weak version of the Ramanujan conjecture, which is not available in general. Therefore we will establish only an integrated version of it, which however, is sufficient for our purpose. For $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$ denote by $\mathcal{H}_P(\pi)$ the Hilbert space of the induced representation $I_P^G(\pi, \lambda)$. Furthermore, for an open compact subgroup $K_f \subset G(\mathbb{A}_f)$ and $\nu \in \Pi(\mathbf{K}_\infty)$, denote by $\mathcal{H}_P(\pi)^{K_f}$ the subspace of vectors, which are invariant under K_f and let $\mathcal{H}_P(\pi)^{K_f,\nu}$ denote the ν-isotypical subspace of $\mathcal{H}_P(\pi)^{K_f}$. Let $P, Q \in \mathcal{P}(M)$ be adjacent parabolic subgroups. Then $R_{Q|P}(\pi, \lambda)$ depends on a single variable $s \in \mathbb{C}$ and we will write

$$R'_{Q|P}(\pi, s_0) := \frac{d}{ds} R_{Q|P}(\pi, s)\big|_{s=s_0}$$

for any regular $s_0 \in \mathbb{C}$.

Proposition 6.2. *Let $M \in \mathcal{L}$, and let $P, Q \in \mathcal{P}(M)$ be adjacent parabolic subgroups. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup and let $\nu \in \Pi(\mathbf{K}_\infty)$. Then there exists $C > 0$ such that*

$$\int_{\mathbb{R}} \left\| R_{Q|P}(\pi, iu)^{-1} R'_{Q|P}(\pi, iu)\big|_{\mathcal{H}_P(\pi)^{K_f,\nu}} \right\| e^{-tu^2} du \le C t^{-1/2} \tag{6.3}$$

for all $0 < t \le 1$ and $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$ with $\mathcal{H}_P(\pi)^{K_f,\nu} \neq 0$.

Proof. We may assume that K_f is factorisable, i.e., $K_f = \prod_v K_v$. Let S be the finite set of finite places such that K_v is not hyperspecial. Since P and Q are adjacent, by standard properties of normalized intertwining operators [Ar4, Theorem 2.1] we may assume that P is a maximal parabolic subgroup and $Q = \overline{P}$, the opposite parabolic subgroup to P. By [Ar4, Theorem 2.1, (R8)], $R_{\overline{P}|P}(\pi_v, s)^{K_v}$ is independent of s if v is finite and $v \notin S$. Thus we have

$$\begin{aligned} R_{\overline{P}|P}(\pi, s)^{-1} R'_{\overline{P}|P}(\pi, s)\big|_{H_P(\pi)^{K_f,\nu}} &= R_{\overline{P}|P}(\pi_\infty, s)^{-1} R'_{\overline{P}|P}(\pi_\infty, s)\big|_{\mathcal{H}_P(\pi_\infty)^\nu} \\ &+ \sum_{v \in S} R_{\overline{P}|P}(\pi_v, s)^{-1} R'_{\overline{P}|P}(\pi_v, s)\big|_{\mathcal{H}_P(\pi_v)^{K_v}} \end{aligned} \tag{6.4}$$

This reduces our problem to the operators at the local places. We distinguish between the archimedean and the non-archimedean case.

Case 1: $v<\infty$. Define $A_v\colon \mathbb{C}\to \operatorname{End}(\mathcal{H}_P(\pi_v)^{K_v})$ by

$$A_v(q_v^{-s}):=R_{\overline{P}|P}(\pi_v,s)\big|_{\mathcal{H}_P(\pi_v)^{K_v}}.$$

This is a meromorphic function with values in the space of endomorphisms of a finite dimensional vector space. It has the following properties. By the unitarity of $R_{\overline{P}|P}(\pi_v,iu)$, $u\in\mathbb{R}$, it follows that $A_v(z)$ is holomorphic for $z\in S^1$ and satisfies $\|A_v(z)\|\le 1$, $|z|=1$. By [Ar4, Theorem 2.1], the matrix coefficients of $A_v(z)$ are rational functions. Recall that the operators $R_{\overline{P}|P}(\pi_v,iu)$ are unitary. As in [FLM2, (14)] we get

(6.5)

$$\begin{aligned}\int_{\mathbb{R}}\Big\|R_{\overline{P}|P}(\pi_v,iu)^{-1}R'_{\overline{P}|P}(\pi_v,iu)\big|_{\mathcal{H}_P(\pi_v)^{K_v}}\Big\|e^{-tu^2}du&=\int_{\mathbb{R}}\left\|R'_{\overline{P}|P}(\pi_v,iu)\Big|_{\mathcal{H}_P(\pi_v)^{K_v}}\right\|e^{-tu^2}du\\ &\le 2\sum_{n=0}^{\infty}\exp\left(-t\frac{4\pi^2n^2}{(\log q_v)^2}\right)\int_0^{\frac{2\pi}{\log q_v}}\left\|R'_{\overline{P}|P}(\pi_v,iu)\Big|_{\mathcal{H}_P(\pi_v)^{K_v}}\right\|du\\ &\le 2\left(1+\int_0^\infty\exp\left(-t\frac{4\pi^2x^2}{(\log q_v)^2}\right)dx\right)\int_0^{\frac{2\pi}{\log q_v}}\left\|R'_{\overline{P}|P}(\pi_v,iu)\Big|_{\mathcal{H}_P(\pi_v)^{K_v}}\right\|du\\ &=\left(2+\frac{\log q_v}{\pi}\cdot t^{-1/2}\right)\int_{S^1}\|A_v'(z)\|\,|dz|.\end{aligned}$$

As explained above, A_v satisfies the assumptions of [FLM2, Corollary 5.18]. Denote by $z_1,\dots,z_m\in\mathbb{C}\setminus S^1$ be the poles of $A_v(z)$. Then $(z-z_1)\cdots(z-z_m)A_v(z)$ is a polynomial of degree n with coefficients in $\operatorname{End}(\mathcal{H}_P(\pi_v)^{K_v})$ and by [FLM2, Corollary 5.18] we get

$$\|A_v'(z)\|\le\max\left(\max(n-m,0)+\sum_{j\colon |z_j|>1}\frac{|z_j|^2-1}{|z_j-z|^2},\ \sum_{j\colon|z_j|<1}\frac{1-|z_j|^2}{|z_j-z|^2}\right),\quad z\in S^1. \tag{6.6}$$

Now observe that

$$\frac{1}{2\pi}\int_{S^1}\frac{1-|z_0|^2}{|z-z_0|^2}|dz|=1.$$

$z_0\in\mathbb{C}$, $|z_0|<1$. This follows from the fact that the integrant is the Poisson kernel and so the integral is the unique harmonic function on the unit disc which is equal to 1 on the boundary. This is the constant function 1. Hence by (6.6) we get

$$\int_{S^1}\|A_v'(z)\|\,|dz|\le 2\pi\max(m,n). \tag{6.7}$$

Next we estimate m and n. First consider m. Let $J_{\overline{P}|P}(\pi_v,s)$ be the usual intertwining operator so that

$$R_{\overline{P}|P}(\pi_v,s)=r_{\overline{P}|P}(\pi_v,s)^{-1}J_{\overline{P}|P}(\pi_v,s),$$

where $r_{\overline{P}|P}(\pi_v,s)$ is the normalizing factor [Ar4]. By [Sh1, Theorem 2.2.2] there exists a polynomial $p(z)$ with $p(0)=1$ whose degree is bounded independently of π_v, such that $p(q_v^{-s})J_{\overline{P}|P}(\pi_v,s)$ is holomorphic on $\mathbb{C}$. To deal with the normalizing factor we use (6.2) together with Lemma 6.1 to count the number of poles of $r_{\overline{P}|P}(\pi_v, s)^{-1}$. This leads to a bound for m which depends only on G. To estimate n we fix an open compact subgroup K_v of $G(F_v)$. Our goal is now to estimate the order at ∞ of any matrix coefficient of $R_{\overline{P}|P}(\pi_v, s)$ regarded as a function of $z = q_v^{-s}$. Write π_v as Langlands quotient $\pi_v = J_R^M(\delta_v, \mu)$ where R is a parabolic subgroup of M, δ_v a square integrable representation of $M_R(F_v)$ and $\mu \in (\mathfrak{a}_R^*/\mathfrak{a}_M^*)_{\mathbb{C}}$ with $\mathrm{Re}(\mu)$ in the chamber attached to R. Then by [Ar4, p. 30] we have

$$R_{\overline{P}|P}(\pi_v, s) = R_{\overline{P(R)}|P(R)}(\delta_v, s + \mu)$$

with respect to the identifications described in [Ar4, p. 30]. Here s is identified with a point in $(\mathfrak{a}_R^*/\mathfrak{a}_G^*)_{\mathbb{C}}$ with respect to the canonical embedding $\mathfrak{a}_M^* \subset \mathfrak{a}_G^*$. Using again the factorization of normalized intertwining operators we reduce the problem to the case of a square-integrable representation δ_v. Moreover δ_v has to satisfy $[I_P^G(\delta_v, s)\big|_{K_v} : \mathbf{1}] \geq 1$. By [Si, Lemma 1] we have

$$[I_P^G(\delta_v, s)\big|_{K_v} : \mathbf{1}] \geq 1 \Leftrightarrow [\delta_v\big|_{K_v \cap M(F_v)} : \mathbf{1}] \geq 1 \tag{6.8}$$

Let $\Pi_2(M(F_v))$ be the space of square-integrable representations of $M(F_v)$. This space has a manifold structure [HC1], [Si]. By [HC1, Theorem 10] the set of square-integrable representations $\Pi_2(M(F_v), K_v)$ of $M(F_v)$ with $[\delta_v\big|_{K_v \cap M(F_v)} : \mathbf{1}] \geq 1$ is a compact subset of $\Pi_2(M(F_v))$. Under the canonical action of $i\mathfrak{a}_M$, the set $\Pi_2(M(F_v), K_v)$ decomposes into a finite number of orbits. For $\mu \in i\mathfrak{a}_M$ and $\delta_v \in \Pi_2(M(F_v), K_v)$, let $(\delta_v)_\mu \in \Pi_2(M(F_v), K_v)$ be the result of the canonical action. Then it follows that

$$R_{\overline{P}|P}((\delta_v)_\mu, \lambda) = R_{\overline{P}|P}(\delta_v, \lambda + \mu).$$

In this way our problem is finally reduced to the consideration of the matrix coefficients of $R_{\overline{P}|P}(\pi_v, s)\big|_{K_v}$ for a finite number of representations π_v. This implies that n is bounded by a constant which is independent of π_v. Together with (6.7) it follows that for each finite place v of F and each open compact subgroup K_v of $G(F_v)$ there exists $C_v > 0$ such that

$$\int_{\mathbb{R}} \left\| R_{\overline{P}|P}(\pi_v, iu)^{-1} R'_{\overline{P}|P}(\pi_v, iu)\big|_{\mathcal{H}_P(\pi_v)^{K_v}} \right\| e^{-tu^2} du \leq C_v t^{-1/2} \tag{6.9}$$

for all $0 < t \leq 1$ and $\pi_v \in \Pi(M(F_v))$ with $I_P^G(\pi_v)\big|_{\mathcal{H}_P(\pi_v)^{K_v}} \neq 0$.

Case 2: $v = \infty$. To begin with we need a modification of [FLM2, Lemma 5.19].

Lemma 6.3. *Let $z_j \in \mathbb{C} \setminus i\mathbb{R}$, $j = 1, ..., m$, and let $b(z) = (z - z_1) \cdots (z - z_m)$. Suppose that $A \colon \mathbb{C} \setminus \{z_1, ..., z_m\} \to V$ is such that $\|A(z)\| \leq 1$ for all $z \in i\mathbb{R}$ and $b(z)A(z)$ is a polynomial in $z \in \mathbb{C}$ (necessarily of degree $\leq m$) with coefficients in V. Then*

$$\int_{\mathbb{R}} \|A'(iu)\| e^{-tu^2} du \leq 2\pi m$$

for all $0 \leq t$.

Proof. For any $w\in\mathbb{C}$, let $\phi_w(z):=\frac{z+\bar w}{z-w}$, and set

$$\phi_>(z)=\sum_{j:\operatorname{Re}(z_j)>0}\phi_{z_j}(z),\quad \phi_<(z)=\sum_{j:\operatorname{Re}(z_j)<0}\phi_{z_j}(z).$$

Applying [BE, Theorem 4], it follows that

$$\|A'(z)\|\le\max\{|\phi'_>(z)|,|\phi'_<(z)|\}\le|\phi'_>(z)|+|\phi'_<(z)|,\quad z\in i\mathbb{R}. \tag{6.10}$$

Now observe that for $w=x+iy\in\mathbb{C}\setminus i\mathbb{R}$, one has $|\phi'_w(z)|=\frac{2|x|}{|z-w|^2}=\frac{2|x|}{x^2+(u-y)^2}$ for $z=iu$, $u\in\mathbb{R}$. So we get

$$\begin{aligned}\int_{\mathbb{R}}|\phi'_w(iu)|e^{-tu^2}du&\le\int_{\mathbb{R}}\frac{2|x|}{x^2+(u-y)^2}du=\frac{2}{|x|}\int_{\mathbb{R}}\frac{1}{1+(\frac{u}{|x|}-\frac{y}{|x|})^2}du\\&=2\int_{\mathbb{R}}\frac{1}{1+(u-\frac{y}{|x|})^2}du=2\int_{\mathbb{R}}\frac{du}{1+u^2}=2\pi.\end{aligned}$$

Together with (6.10) the lemma follows. □

As above let $M\in\mathcal{L}$ with $\dim(\mathfrak{a}_M/\mathfrak{a}_G)=1$ and $P\in\mathcal{P}(M)$. Let $\pi_\infty\in\Pi(M(F_\infty))$ and $\nu\in\Pi(\mathbf{K}_\infty)$. As explained in [MS, Appendix], there exist $w_1,...,w_r\in\mathbb{C}$ and $m\in\mathbb{N}$ such that the poles of $R_{\overline{P}|P}(\pi_\infty,s)\big|_{\mathcal{H}_P(\pi_\infty)^\nu}$ are contained in $\cup_{j=1}^r\{w_j-k\colon k=1,...,m\}$. Moreover, by [MS, Proposition A.2] there exists $c>0$ which depends only on G, such that

$$r\le c,\quad m\le c(1+\|\nu\|). \tag{6.11}$$

Let $A\colon\mathbb{C}\to\mathcal{H}_P(\pi_\infty)^\nu$ be defined by

$$A(z):=R_{\overline{P}|P}(\pi_\infty,z)\big|_{\mathcal{H}_P(\pi_\infty)^\nu}$$

and let $b(z)=\prod_{j=1}^r\prod_{k=1}^m(z-w_j+k)$. Then it follows from (R_6) of [Ar4, Theorem 2.1] that $b(z)A(z)$ is a polynomial function. Moreover, by unitarity of $R_{\overline{P}|P}(\pi_\infty,it)$, $t\in\mathbb{R}$, we have $\|A(it)\|=1$. Thus $A(z)$ satisfies the assumptions of [FLM2, Lemma 5.19]. Thus by Lemma 6.3 and (6.11) we get

$$\begin{aligned}\int_{\mathbb{R}}\Big\|R_{\overline{P}|P}(\pi_\infty,iu)^{-1}R'_{\overline{P}|P}(\pi_\infty,iu)\big|_{\mathcal{H}_P(\pi_\infty)^\nu}\Big\|e^{-tu^2}du&=\int_{\mathbb{R}}\|A'(iu)\|e^{-tu^2}du\\&\le2\pi r\cdot m\le2\pi c^2(1+\|\nu\|).\end{aligned} \tag{6.12}$$

Combining (6.4), (6.9) and (6.12), the proposition follows.

Remark 6.4. *For $G=\mathrm{GL}_n$ it is proved in* [MS, Proposition 0.2] *that the corresponding bounds hold for the derivatives of the local intertwining operators itself. This follows from a weak version of the Ramanujan conjecture, which implies that the poles of the local intertwining operators are uniformly bounded away from the imaginary axis. For the integrated derivatives the distance of the poles from the imaginary axis does not matter.*

□

7. The residual spectrum

The goal of this section is to estimate the growth of the counting function of the residual spectrum. To this end we recall the construction of the residual spectrum. By Langlands [La1, Ch. 7], [MW, V.3.13], $L^2_{\mathrm{res}}(G(F)\backslash G(\mathbb{A})^1)$ is spanned by *iterated residues* of cuspidal Eisenstein series. Let us briefly recall this construction.

Let $P = M \ltimes N$ be a F-rational parabolic subgroup of G. If $\alpha \in \Sigma_P$, denote by $\alpha^\vee$ the co-root associated to α. Given $\alpha \in \Sigma_P$ and $c \in \mathbb{R}$, we set

$$H(\alpha, c) := \{\Lambda \in \mathfrak{a}^*_{\mathbb{C}} \colon \Lambda(\alpha^\vee) = c\}.$$

An affine subspace $\mathcal{H} \subset \mathfrak{a}^*_{\mathbb{C}}$ is called admissible, if $\mathcal{H}$ is the intersection of such hyperplanes. Suppose that $\mathcal{H}_1 \supset \mathcal{H}_2$ are two admissible affine subspaces of $\mathfrak{a}^*_{\mathbb{C}}$ and $\mathcal{H}_2$ is of co-dimension one in $\mathcal{H}_1$. Let $\Phi(\Lambda)$ be a meromorphic function on $\mathcal{H}_1$ whose singularities lie along hyperplanes which are admissible as subspaces of $\mathfrak{a}^*_{\mathbb{C}}$. Choose a real unit vector Λ_0 in $\mathcal{H}_1$ which is normal to $\mathcal{H}_2$. Let $\delta > 0$ be such that $\Phi(\Lambda + z\Lambda_0)$ has no singularities in the punctured disc $0 < |z| < 2\delta$. Then we can define a meromorphic function $\operatorname{Res}_{\mathcal{H}_2} \Phi$ on $\mathcal{H}_2$ by

$$\operatorname{Res}_{\mathcal{H}_2} \Phi(\Lambda) := \frac{\delta}{2\pi i} \int_0^1 \Phi(\Lambda + \delta e^{2\pi i\vartheta}\Lambda_0) d(e^{2\pi i \vartheta}),$$

The singularities of $\operatorname{Res}_{\mathcal{H}_2} \Phi$ lie on the intersections with $\mathcal{H}_2$ of the singular hyperplanes of Φ different from $\mathcal{H}_2$. Now consider a complete flag

$$\mathfrak{a}^*_{\mathbb{C}} = \mathcal{H}_p \supset \mathcal{H}_{p-1} \supset \cdots \supset \mathcal{H}_1 \supset \mathcal{H}_0 = \{\Lambda_0\}$$

of affine admissible subspaces of $\mathfrak{a}^*_{\mathbb{C}}$ and let $\Lambda_i \in \mathcal{H}_i$ be a real unit vector which is normal to $\mathcal{H}_{i-1}$, $i = 1, \dots, p$. We call $\mathcal{F} = \{\mathcal{H}_i, \Lambda_i\}$ an admissible flag. Let Φ be a meromorphic function on $\mathfrak{a}^*_{\mathbb{C}}$ whose singularities lie along admissible hyperplanes of $\mathfrak{a}^*_{\mathbb{C}}$. Then we define Φ_i inductively by

$$\Phi_p = \Phi,\ \Phi_i = \operatorname{Res}_{\mathcal{H}_i} \Phi_{i+1}, \quad i = 0, \dots, p-1.$$

Set

$$\operatorname{Res}_{\mathcal{F}} \Phi := \Phi_0.$$

This is the iterated residue of Φ at Λ_0.

Now let $\mathcal{A}^2_{\mathrm{cus}}(P)$ the subspace of functions $\phi \in \mathcal{A}^2(P)$ such that for almost all $x \in G(\mathbb{A})$, the function $\phi_x(m) := \phi(mx)$ on $M(F)\backslash M(\mathbb{A})^1$ lies in the space $L^2_{\mathrm{cus}}(M(F)\backslash M(\mathbb{A})^1)$. For $\pi \in \Pi_{\mathrm{cus}}(M(\mathbb{A})^1)$ let $\mathcal{A}^2_{\mathrm{cus},\pi}(P)$ be the subspace of functions $\phi \in \mathcal{A}^2_{\mathrm{cus}}(P)$ such that each of the functions ϕ_x lies in the subspace $L^2_{\mathrm{cus},\pi}(M(F)\backslash M(A)^1)$ (isotypical subspace). Let $\phi \in \mathcal{A}^2_{\mathrm{cus}}(P)$. As shown by Langlands [La1, §7], the singularities of the Eisenstein series $E(\phi, \lambda)$ lie along hyperplanes of $\mathfrak{a}^*_{\mathbb{C}}$ which are defined by equations of the form $\Lambda(\alpha^\vee) = w$, $w \in \mathbb{C}$, $\alpha \in \Sigma_P$. Let $H(\alpha_i, c_i)$, $i = 0, \dots, p-1$, be a set of singular hyperplanes of $E(\phi, \lambda)$ with $\cap_i H(\alpha_i, c_i) = \{\Lambda_0\}$. Set $\mathcal{H}_i := \cap_{j \geq i} H(\alpha_j, c_j)$, $i = 0, \dots, p-1$, and $\mathcal{H}_P = \mathfrak{a}^*_{\mathbb{C}}$. Choose real unit vectors $\Lambda_i \in \mathcal{H}_i$ normal to $\mathcal{H}_{i-1}$. Then $\mathcal{F} := \{\mathcal{H}_i, \Lambda_i\}$ is an admissible flag.

Furthermore, let $\varphi \in C_c^\infty(\mathfrak{a})$ and let $\hat{\varphi}(\Lambda)$ be its Fourier transform. It is holomorphic on $\mathfrak{a}_{\mathbb{C}}^*$. Put

$$\psi := \operatorname{Res}_{\mathcal{F}}\big[E(\phi,\Lambda)\hat{\varphi}(\Lambda)\big].$$

Note that ψ depends only on the derivatives of $\hat{\varphi}$ at Λ_0. Let $\mathcal{C}(\mathfrak{a}^*)$ be the positive cone in $\mathfrak{a}^*$ spanned by the simple roots of (P,A). If $\Lambda_0 \in \mathcal{C}(\mathfrak{a}^*)$ then ψ is square integrable. Let $\Omega_M \in \mathcal{Z}(\mathfrak{m}_{\mathbb{C}})$ be the Casimir element of $M(F_\infty)$. Assume that $\Omega_M\phi = \mu\phi$. Then it follows that

$$\Omega\psi = (\|\Lambda_0\|^2 - \|\rho_P\|^2 + \mu)\psi. \tag{7.1}$$

[HC2, p. 29]. Moreover $\|\Lambda_0\| \le \|\rho_P\|$. As shown by Langlands [La1, Theorem 7.2], [MW, V.3.13], $L^2_{\text{res}}(G(F)\backslash G(\mathbb{A})^1)$ is spanned by all such ψ, where P runs over the standard parabolic subgroups of G, π over $\Pi_{\text{cus}}(M_P(\mathbb{A})^1)$ and ϕ over a basis of $\mathcal{A}^2_{\text{cus},\pi}(P)$. For all details concerning the description of the discrete and residual spectrum see [MW, Theorem V.3.13, p. 221], and [MW, Corollary, p. 224]. Moreover, the question of positivity is dealt with by [MW, Corollary VI.1.6 (d), p. 255].

Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. Let $L^2_{\text{res}}(A_G G(F)\backslash G(\mathbb{A}))^{K_f}$ be the subspace of K_f-invariant functions of $L^2_{\text{res}}(A_G G(F)\backslash G(\mathbb{A}))$. Moreover, for $\nu \in \Pi(\mathbf{K}_\infty)$ let $L^2_{\text{res}}(A_G G(F)\backslash G(\mathbb{A}))^{K_f,\nu}$ be the ν-isotypical subspace of $L^2_{\text{res}}(A_G G(F)\backslash G(\mathbb{A}))^{K_f}$. Then $L^2_{\text{res}}(A_G G(F)\backslash G(\mathbb{A}))^{K_f,\nu}$ is spanned by residues as above, where for a given pair (P,π), ϕ runs over a basis of $\mathcal{A}^2_{\text{cus},\pi}(P)^{K_f,\sigma}$. Recall that $\mathcal{A}^2_{\text{cus},\pi}(P)^{K_f,\sigma}$ is finite dimensional. So the estimation of the counting function of the residual spectrum is reduced to the following problems:

(1) Estimation of $\dim \mathcal{A}^2_{\text{cus},\pi}(P)^{K_f,\sigma}$ in terms of π, K_f, and σ.
(2) For a given cuspidal Eisenstein series $E(\phi,\Lambda)$, $\phi \in \mathcal{A}^2_{\text{cus},\pi}(P)$, we need to estimate the number of its singular hyperplanes, counted to multiplicity, which are real and intersect a given compact set containing the origin.

We start with (1). Let $\pi \in \Pi_{\text{cus}}(M(\mathbb{A}))$, $\pi = \pi_\infty \otimes \pi_f$. Let $\mathcal{H}_P(\pi_\infty)$ (resp. $\mathcal{H}_P(\pi_f)$) be the Hilbert space of the induced representation $\operatorname{Ind}_{P(F_\infty)}^{G(F_\infty)}(\pi_\infty)$ (resp. $\operatorname{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}$). Denote by $\mathcal{H}_P(\pi_\infty)_\sigma$ the σ-isotypical subspace of $\mathcal{H}_P(\pi_\infty)$ and by $\mathcal{H}_P(\pi_f)^{K_f}$ the subspace of K_f-invariant vectors of $\mathcal{H}_P(\pi_f)$. Let $m(\pi)$ denote the multiplicity with which $\pi \in \Pi_{\text{cus}}(M(\mathbb{A}))$ occurs in $L^2_{\text{cus}}(A_G M(F)\backslash M(\mathbb{A}))$. Then by (4.7) we obtain

$$\dim \mathcal{A}^2_\pi(P)^{K_f,\sigma} = m(\pi)\dim(\mathcal{H}_P(\pi_f)^{K_f})\dim(\mathcal{H}_P(\pi_\infty)_\sigma). \tag{7.2}$$

Using Frobenius reciprocity [Kn, p. 208] we get

$$[\operatorname{Ind}_{P(F_\infty)}^{G(F_\infty)}(\pi_\infty)|_{\mathbf{K}_\infty} : \sigma] = \sum_{\tau\in\Pi(\mathbf{K}_{M,\infty})} [\pi_\infty|_{\mathbf{K}_{M,\infty}} : \tau]\cdot[\sigma|_{\mathbf{K}_{M,\infty}} : \tau].$$

For $\tau \in \Pi(\mathbf{K}_{M,\infty})$ let $\mathcal{H}_{\pi_\infty}(\tau)$ denote the τ-isotypical subspace of $\mathcal{H}_{\pi_\infty}$. Then we obtain

$$\dim(\mathcal{H}_P(\pi_\infty)_\sigma) \le \dim(\sigma) \sum_{\tau\in\Pi(\mathbf{K}_{M,\infty})} \dim(\mathcal{H}_{\pi_\infty}(\tau))\cdot[\sigma|_{\mathbf{K}_{M,\infty}} : \tau]. \tag{7.3}$$

Next we consider $\pi_f = \otimes_{v<\infty}\pi_v$. Replacing K_f by a subgroup of finite index, if necessary, we can assume that $K_f = \prod_{v<\infty} K_v$. For any $v<\infty$, denote by $\mathcal{H}_P(\pi_v)$ the Hilbert space of the induced representation $\operatorname{Ind}_{P(F_v)}^{G(F_v)}(\pi_v)$. Let $\mathcal{H}_P(\pi_v)^{K_v}$ be the subspace of K_v-invariant vectors. Then $\dim \mathcal{H}_P(\pi_v)^{K_v} = 1$ for almost all v and

$$\mathcal{H}_P(\pi_f)^{K_f} \cong \bigotimes_{v<\infty} \mathcal{H}_P(\pi_v)^{K_v}.$$

$$\begin{aligned} \operatorname{Ind}_{P(F_v)}^{G(F_v)}(\pi_v)^{K_v} &= \Big(\operatorname{Ind}_{P(\mathcal{O}_v)}^{G(\mathcal{O}_v)}(\pi_v)\Big)^{K_v} \\ &\hookrightarrow \bigoplus_{G(\mathcal{O}_v)/K_v} \operatorname{Ind}_{K_v\cap P}^{K_v}(\pi_v)^{K_v} \\ &\cong \bigoplus_{G(\mathcal{O}_v)/K_v} \pi_v^{K_v\cap P}. \end{aligned} \tag{7.4}$$

Let $K_{M,f} = K_f \cap M(\mathbb{A}_f)$. For $\sigma \in \Pi(K_\infty)$ let

$$\mathcal{F}_M(\sigma) := \{\tau \in \Pi(K_{M,\infty})\colon [\sigma|_{K_{M,\infty}} : \tau] > 0\}.$$

Combining (7.2) - (7.4), it follows that there exists $C>0$, which depends on σ, such that

$$\dim \mathcal{A}^2_\pi(P)^{K_f,\sigma} \le C m(\pi) \dim(\mathcal{H}_{\pi_f}^{K_{M,f}}) \sum_{\tau\in\mathcal{F}(\sigma)} \dim(\mathcal{H}_{\pi_\infty}(\tau)). \tag{7.5}$$

In order to deal with (2), we use the inner product formula for truncated cuspidal Eisenstein series proved by Langlands [La3, Sect. 9], [Ar2, Lemma 4.2]. We recall the formula. Let $T\in\mathfrak{a}_0^+$ be sufficiently regular. For $\phi\in\mathcal{A}^2_{\text{cus}}(P)$ let $\Lambda^T E(g,\phi,\lambda)$ be the truncated Eisenstein series [Ar2, Sect. 1]. Let $\phi\in\mathcal{A}^2_{\text{cus}}(P)$ and $\phi'\in\mathcal{A}^2_{\text{cus}}(P')$. Then we have the following inner product formula

$$\begin{aligned} &\int_{G(F)\backslash G(\mathbb{A})^1} \Lambda^T E(g,\phi,\lambda)\overline{\Lambda^T E(g,\phi',\lambda')}dg \\ &= \sum_Q\sum_s\sum_{s'} \operatorname{vol}(\mathfrak{a}_Q^G/\mathbb{Z}(\Delta_Q^\vee)) \frac{e^{(s\lambda+s'\lambda')(T)}}{\prod_{\alpha\in\Delta_Q}(s\lambda+s'\bar{\lambda}')(\alpha^\vee)} (M_{Q|P}(s,\lambda)\phi, M_{Q|P'}(s',\lambda')\phi'), \end{aligned} \tag{7.6}$$

where Q runs over all standard parabolic subgroups, $s\in W(\mathfrak{a}_P,\mathfrak{a}_Q)$, and $s'\in W(\mathfrak{a}_{P'},\mathfrak{a}_Q)$, as meromorphic functions of $\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}$ and $\lambda'\in\mathfrak{a}^*_{P',\mathbb{C}}$ [Ar5, Prop. 15.3], [Ar2, Lemma 4.2]. It follows from the inner product formula that in order to settle (2), it suffices to estimate the corresponding number of singular hyperplanes of the intertwining operators $M_{Q|P}(s,\lambda)|_{\mathcal{A}^2_{\text{cus},\pi}(P)}$ for $\pi\in\Pi_{\text{cus}}(M(\mathbb{A}))$. To deal with this problem, we reduce it to the case of $M_{Q|P}(\lambda)|_{\mathcal{A}^2_{\text{cus},\pi}(P)}$, $Q,P\in\mathcal{P}(M)$, $\pi\in\Pi_{\text{cus}}(M(\mathbb{A}))$. Let $M,M_1\in\mathcal{L}$ and let $P\in\mathcal{P}(M)$, $P_1\in\mathcal{P}(M_1)$. Suppose that P and P_1 are associated and let $t\in W(\mathfrak{a}_M,\mathfrak{a}_{M_1})$. Let $w_t\in G(F)$ be a representative of t. Then $M_1 := w_t M w_t^{-1}$ and $tP = w_t P w_t^{-1}$ is a parabolic subgroup which belongs to $\mathcal{P}(M_1)$. The restriction of t to $\mathfrak{a}_M\subset\mathfrak{a}_0$ defines an element in $W(\mathfrak{a}_M,\mathfrak{a}_{tM})$. Let

$$t\colon \mathcal{A}^2(P)\to\mathcal{A}^2(tP)$$

be the linear operator defined by $(t\phi)(x) = \phi(w_t^{-1}x)$, $x \in G(\mathbb{A})$. By [Ar3, Lemma 1.1] there exists $T_0 \in \mathfrak{a}_0$ such that

$$H_{P_0}(w_t^{-1}) = T_0 - t^{-1}T_0.$$

Then by [Ar9, (1.5)] one has

$$M_{P_1|P}(s,\lambda)t^{-1} = M_{P_1|tP}(st^{-1}, t\lambda)e^{(\lambda+\rho_P)(T_0-t^{-1}T_0)} \tag{7.7}$$

for $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$. Thus as far as the singular hyperplanes of $M_{P_1|P}(s,\lambda)$ are concerned, we can assume that P_1 and P have the same Levi component M, that is $P, P_1 \in \mathcal{P}(M)$. Let $t \in W(\mathfrak{a}_M)$. By the functional equation [Ar9, (1.2)] we have

$$M_{P_1|P}(t,\lambda) = M_{P_1|tP}(1, t\lambda)M_{tP|P}(t,\lambda). \tag{7.8}$$

Using (7.7) with $s = t$, we get

$$M_{tP|P}(t,\lambda) = M_{tP|tP}(1, t\lambda)te^{(\lambda+\rho_P)(T_0-t^{-1}T_0)}.$$

Since $M_{tP|tP}(1,\lambda) = \mathrm{Id}$, it follows that

$$M_{P_1|P}(t,\lambda) = M_{P_1|tP}(1, t\lambda)te^{(\lambda+\rho_P)(T_0-t^{-1}T_0)}. \tag{7.9}$$

Thus we are reduced to the consideration of the singular hyperplanes of the intertwining operators $M_{Q|P}(\lambda) = M_{Q|P}(1,\lambda)$. Given $\pi \in \Pi_{\mathrm{cus}}(M_P(\mathbb{A}))$, an open compact subgroup $K_f \subset G(\mathbb{A}_f)$ and $\sigma \in \Pi(\mathbf{K}_\infty)$, we need to estimate the singular hyperplanes, counted to multiplicity, of $M_{Q|P}(\pi,\lambda)^{K_f,\sigma}$, which are real and intersect a fixed compact set. By (4.12) the problem is reduces to the consideration of the normalizing factors $n_{Q|P}(\pi,\lambda)$ and the normalized intertwining operators $R_{Q|P}(\pi,\lambda)$ restricted to $\mathcal{A}^2_{\mathrm{cus},\pi}(P)^{K_f,\sigma}$.

To begin with we consider the normalized intertwining operators (4.13). Let v be a place of F. For $\pi_v \in \Pi(G(F_v))$ let

$$J_{Q|P}(\pi_v,\lambda)\colon \mathcal{H}^0_P(\pi_v) \to \mathcal{H}^0_Q(\pi_v), \quad \lambda \in \mathfrak{a}^*_{Q,\mathbb{C}},$$

be the local intertwining operator and

$$R_{Q|P}(\pi_v,\lambda) := n_{Q|P}(\pi_v,\lambda)^{-1}J_{Q|P}(\pi_v,\lambda). \tag{7.10}$$

the local normalized intertwining operator. The operators $R_{Q|P}(\pi_v,\lambda)$ satisfy properties $(R_1),...,(R_8)$ of [Ar4, Theorem 2.1]. Let $\pi = \otimes_v\pi_v \in \Pi_{\mathrm{dis}}(G(\mathbb{A}))$. There exists a finite set of places $S(\pi)$ of F, containing the Archimedean places, such that for all $v \notin S$, G/F_v and π_v are unramified. For $v \notin S(\pi)$, let $\mathbf{K}_v$ be hyperspecial and assume that $\phi_v \in \mathcal{H}_P(\pi_v)^{\mathbf{K}_v}$. Then by (R_8) we have

$$R_{Q|P}(\pi_v,\lambda)\phi_v = \phi_v, \quad s \notin S(\pi).$$

Hence the product (4.13) runs only over $v \in S(\pi)$ and therefore, it is well defined for all $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. So it suffices to consider the local intertwining operators $R_{Q|P}(\pi_v,\lambda)$. Let $P_0,...,P_k \in \mathcal{P}(M)$ and $\alpha_1,...,\alpha_k \in \Sigma_M$ such that $P = P_0$, $Q = P_k$ and $P_{i-1}\big|^{\alpha_i}P_i$ for $i = 1,...,k$. By [Ar4, (R_2)] we get

$$R_{Q|P}(\pi_v,\lambda) = R_{P_k|P_{k-1}}(\pi_v,\lambda) \circ R_{R_{k-1}|P_{k-2}}(\pi_v,\lambda) \circ \cdots \circ R_{P_1|P_0}(\pi_v,\lambda). \tag{7.11}$$

Hence we can assume that $Q, P \in \mathcal{P}(M)$ are adjacent along some root $\alpha \in \Sigma_M$. Then $R_{Q|P}(\pi_v, \lambda)$ depends only on $\lambda(\alpha^\vee)$. First consider the case $v < \infty$. Let q_v be the order of the residue field of F_v. By (R_6), $R_{Q|P}(\pi_v, \lambda)$ is a rational function of $q_v^{-\lambda(\alpha^\vee)}$. Furthermore, by [Sh1, Theorem 2.2.2] there exists a polynomial $Q_v(\pi_v, s)$ with $Q_v(\pi_v, 0) = 1$, such that

$$Q_v(\pi_v, q_v^{-\lambda(\alpha^\vee)}) J_{Q|P}(\pi_v, \lambda)$$

is a holomorphic and non-zero function of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. Moreover, the degree of the polynomial Q_v is independent of $\pi_v \in \Pi(M(F_v))$ By (7.10) it follows that

$$n_{Q|P}(\pi_v, \lambda) Q_v(\pi_v, q_v^{-\lambda(\alpha^\vee)}) R_{Q|P}(\pi_v, \lambda)$$

is holomorphic on $\mathfrak{a}^*_{M,\mathbb{C}}$. The normalizing factors $n_{Q|P}(\pi_v, \lambda)$ satisfy properties similar to the corresponding properties $(R_1), ..., (R_8)$ satisfied by the local intertwining operators [Ar4, Theorem 2.1]. In particular, there exists a rational function $n_\alpha(\pi_v, s)$ such that

$$n_{Q|P}(\pi_v, \lambda) = n_\alpha(\pi_v, q_v^{-\lambda(\alpha^\vee)}).$$

Now observe that for every $R > 0$ and $z \in \mathbb{C}$ the number of solutions of $q_v^{-s} = z$ in the disc $|s| \leq R$ is bounded by $1 + (2\pi)^{-1} \log(q_v) R$. Hence it suffices to estimate the number of zeros of $n_\alpha(\pi_v, s)$ and $Q_v(\pi_v, s)$, respectively. As mentioned above, the degree of $Q_v(\pi, s)$ is bounded independently of $\pi \in \Pi(M(F_v))$. The rational function $n_\alpha(\pi_v, s)$ has been described in [Mu2, (3.6)]. It follows from [Mu2, Lemma 3.1] and [Mu2, (3.6)] that there exists $C > 0$ such that for all $M \in \mathcal{L}(M_0)$ and all square integrable $\pi \in \Pi(M(F_v))$ the number of poles and zeros of $n_\alpha(\pi, s)$ is less than C. Now let π be tempered. It is known that π is an irreducible constituent of an induced representation $\mathrm{Ind}_R^M(\sigma)$, where M_R is an admissible Levi subgroup of M and $\sigma \in \Pi(M_R(F_v))$ is square integrable modulo A_R. Then by [Ar4, (2.2)] we are reduced to the square integrable case. In general, π is a Langlands quotient of an induced representation $\mathrm{Ind}_R^M(\sigma, \mu)$, where M_R is an admissible Levi subgroup of M, $\sigma \in \Pi_{\mathrm{temp}}(M_R(F_v))$, and μ is point in the chamber of $\mathfrak{a}^*_R/\mathfrak{a}^*_M$. Now we use [Ar4, (2.3)] to reduce to the tempered case, which proves that there exists $C > 0$ such that the number of poles and zeros of $n_\alpha(\pi, s)$ is less than C for all $\pi \in \Pi(M(F_v))$.

The case $v \in S_\infty$ has been already treated in section 6. See (6.11) and the text above (6.11).

Now we can summarize our results. Using (4.11), we obtain the following proposition.

Proposition 7.1. *Let $M \in \mathcal{L}(M_0)$. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup and $\sigma \in \Pi(\mathbf{K}_\infty)$. For every $R > 0$ there exists $C > 0$ such that for all $Q, P \in \mathcal{P}(M)$ and $\pi \in \Pi(M(\mathbb{A}))$ the number of singular hyperplanes of $R_{Q|P}(\pi, \lambda)\big|_{\mathcal{A}^2_\pi(P)^{K_f,\sigma}}$, which intersects the ball of radius R in $\mathfrak{a}^*_{M,\mathbb{C}}$, is bounded by C.*

Next we consider the global normalizing factors. By (4.11), $n_{Q|P}(\pi, \lambda)$ is the product of the normalizing factors $n_\alpha(\pi, \lambda(\alpha^\vee))$, $\alpha \in \Sigma_P \cap \Sigma_{\bar{Q}}$. Thus our problem is reduced to the estimation of the number of real poles, counted to multiplicity, of the meromorphic function $n_\alpha(\pi,s)$. Let $\Sigma_\alpha^{\mathbb{R}}(\pi)$ be the set of real poles of $n_\alpha(\pi,s)$. Let $\pi\in\Pi_{\rm dis}(M(\mathbb{A});K_f,\sigma)$. Then it follows from Corollary 5.2 that there exists $C>0$

$$\sum_{\rho\in\Sigma_\alpha^{\mathbb{R}}(\pi)} n(\rho)\le C(1+\log(1+\lambda_{\pi_\infty}^2)). \tag{7.12}$$

Now we can summarize our results as follows. For arbitrary $Q,P\in\mathcal{P}(M)$, the global normalizing factor $n_{Q|P}(\pi,\lambda)$ is the product of the functions $n_\alpha(\pi,\lambda(\alpha^\vee))$, $\alpha\in\Sigma_P\cap\Sigma_{\bar Q}$ (4.11). Note that $\#\Sigma_P\le\dim\mathfrak{n}_P$. Let $d_P:=\dim\mathfrak{n}_P$. Then it follows from (7.12) and Proposition (7.1) that there exists $C>0$ such for every $\pi\in\Pi_{\rm cus}(M(\mathbb{A});K_f,\sigma)$ and every $\phi\in\mathcal{A}^2_{{\rm cus},\pi}(P)$ the number of singular hyperplanes of the Eisenstein series $E(\phi,\lambda)$, counted with multiplicity, which are real and intersect the ball $\{\lambda\in\mathfrak{a}^*_{M,\mathbb{C}}\colon \|\lambda\|\le\|\rho_P\|\}$ is bounded by

$$C(1+\log(1+\lambda_{\pi_\infty}^2))^{d_P}, \tag{7.13}$$

Now recall from the beginning of this section that the residual spectrum is spanned by iterated residues of Eisenstein series $E(\phi,\lambda)$ with respect to complete flags of affine admissible subspaces of $\mathfrak{a}^*_{M,\mathbb{C}}$. For $\lambda\ge0$ let

$$\Pi_{\rm cus}(M(\mathbb{A});\lambda):=\{\pi\in\Pi_{\rm cus}(M(\mathbb{A}))\colon -\lambda_{\pi_\infty}\le\lambda\}.$$

We note that there exists $C\in\mathbb{R}$ such that $C\le-\lambda_{\pi_\infty}$ for all $\pi\in\Pi_{\rm dis}(M(\mathbb{A}))$. Using (7.1) and (7.5), it follows that there exist $C_0,C_1>0$ such that

$$\begin{aligned} N_{\rm res}^{K_f,\sigma}(\lambda)&\le C_1\sum_{P\supset P_0}(1+\log(1+\lambda^2))^{d_P}\\ &\quad\cdot\sum_{\tau\in\mathcal{F}_{M_P}(\sigma)}\ \sum_{\pi\in\Pi_{\rm cus}(M_P(\mathbb{A});\lambda+C_0)} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_{M_P,f}})\dim(\mathcal{H}_{\pi_\infty}(\tau)).\end{aligned} \tag{7.14}$$

For a given P let $M=M_P$. As in (3.2) there exist finitely many lattices $\Gamma_{M,i}\subset M(F_\infty)$, $i=1,...,k$, such that

$$A_MM(F)\backslash M(\mathbb{A})/K_{M,f}\cong\bigsqcup_{i=1}^k(\Gamma_{M,i}\backslash M(F_\infty)^1). \tag{7.15}$$

Thus we get an isomorphism of $M(F_\infty)$-modules

$$L^2_{\rm cus}(A_MM(F)\backslash M(\mathbb{A}))^{K_{M,f}}\cong\bigoplus_{i=1}^k L^2_{\rm cus}(\Gamma_{M,i}\backslash M(F_\infty)^1). \tag{7.16}$$

And hence for $\tau\in\Pi(K_{M,\infty})$ we get

$$L^2_{\rm cus}(A_MM(F)\backslash M(\mathbb{A}))^{K_{M,f},\tau}\cong\bigoplus_{i=1}^k L^2_{\rm cus}(\Gamma_{M,i}\backslash(M(F_\infty)^1\otimes V_\tau)^{K_{M,\infty}}. \tag{7.17}$$

Let $\widetilde{X}_M=A_M\backslash M(F_\infty)/K_{M,\infty}$. Let $E_{\tau,i}\to\Gamma_{M,i}\backslash\widetilde{X}_M$ be the locally homogeneous vector bundle associated to τ. Let $N_{\rm cus}^{\Gamma_{M,i}}(\lambda;\tau)$ be the eigenvalues counting function for the Casimir operator acting in $L^2_{\text{cus}}(\Gamma_{M,i}\backslash\widetilde{X}_M, E_{\tau,i})$. It follows from (7.17) that

$$\sum_{\pi\in\Pi_{\text{cus}}(M(\mathbb{A});\lambda)} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_{M,f}})\dim(\mathcal{H}_{\pi_\infty}(\tau)) = \sum_{i=1}^{k} N_{\text{cus}}^{\Gamma_{M,i}}(\lambda;\tau). \tag{7.18}$$

Let $m_M := \dim\widetilde{X}_M$. Then by [Do, Theorem 9.1] we get

$$N_{\text{cus}}^{\Gamma_{M,i}}(\lambda;\tau) \ll 1+\lambda^{m_M/2}. \tag{7.19}$$

Thus by (7.18) it follows that there exists $C>0$ such that

$$\sum_{\tau\in\mathcal{F}_M(\sigma)}\sum_{\pi\in\Pi_{\text{cus}}(M(\mathbb{A});\lambda+C_0)} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_{M,f}})\dim(\mathcal{H}_{\pi_\infty}(\tau)) \le C(1+\lambda^{m_M/2}). \tag{7.20}$$

Let $n := \dim\widetilde{X}$ and $m_0(G) := \max\{m_{M_P}\colon P\neq G\}$. Note that $d_P\le r$ for all P. Then by (7.14) and (7.20) we obtain

$$N_{\text{res}}^{K_f,\sigma}(\lambda) \le C(1+\log(1+\lambda^2))^n(1+\lambda^{m_0(G)/2}). \tag{7.21}$$

Now observe that $\widetilde{X}\cong\widetilde{X}_M\times A_M\times N_P(F_\infty)$ [Bo3, Sect. 4.2, (3)]. Hence $m_M\le n-2$, and therefore $m_0(G)\le n-2$. Thus we get

$$N_{\text{res}}^{K_f,\sigma}(\lambda) \le C(1+\lambda^{(n-1)/2}),\quad \lambda\ge 0, \tag{7.22}$$

where $C>0$ depends on K_f and σ. This completes the proof of the second statement of Theorem 1.4.

Next we wish to extend this result to any Levi subgroup $L\in\mathcal{L}(M)$. Recall that for any pair of elements $Q\in\mathcal{P}(L)$ and $R\in\mathcal{P}^L(M)$ there exists a unique $P\in\mathcal{P}(M)$ such that $P\subset Q$ and $P\cap L=R$. Then P is denoted by $Q(R)$. Let $R,R'\in\mathcal{P}^L(M)$, $\pi\in\Pi_{\text{dis}}(M(\mathbb{A}))$, and $Q\in\mathcal{P}(L)$. Then for any $k\in\mathbf{K}$ and $\phi\in\mathcal{A}^2_\pi(Q(R))$, the function ϕ_k on $M(\mathbb{A})$, which is defined by $\phi_k(m):=\phi(mk)$, $m\in M(\mathbb{A})$, belongs to $\mathcal{A}^2_\pi(R)$, and one has

$$(M_{Q(R')|Q(R)}(\pi,\lambda)\phi)_k = M_{R'|R}(\pi,\lambda)\phi_k) \tag{7.23}$$

[Ar9, (1.3)]. Furthermore, the normalizing factors satisfy

$$n_{Q(R')|Q(R)}(\pi,\lambda) = n_{R'|R}(\pi,\lambda) \tag{7.24}$$

[Ar4, Sect. 2]. Thus the considerations above continue to hold for each standard Levi subgroup L of G. For $M\in\mathcal{L}(M_0)$ and $\sigma\in\Pi(\mathbf{K}_\infty)$ let $\sigma_M := \sigma|_{\mathbf{K}_\infty\cap M(F_\infty)}$. Denote by $N_{M,\text{res}}^{K_{M,f},\sigma_M}(\lambda)$ the counting function of the residual spectrum for M with respect to $(K_{M,f},\sigma_M)$. Let $m_0(M) := \max\{m_{M_R}\colon R\in\mathcal{P}^M,\ R\neq M\}$. Then, summarizing our results, we get

$$N_{M,\text{res}}^{K_{M,f},\sigma_M}(\lambda) \le C(1+\log(1+\lambda^2))^{r_M}\lambda^{m_0(M)/2} \tag{7.25}$$

for $\lambda\ge 0$. As above it follows that $m_0(M)\le\dim\widetilde{X}_M-2$, and we obtain the following proposition.

Proposition 7.2. *Assume that G satisfies condition (L). Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and $\sigma \in \Pi(\mathbf{K}_\infty)$. Let $M \in \mathcal{L}$ and $m_M := \dim \widetilde{X}_M$. Then there exists $C > 0$ such that*

$$N_{M,\text{res}}^{K_{M,f},\sigma_M}(\lambda) \leq C(1 + \lambda^{(m_M-1)/2}) \tag{7.26}$$

for $\lambda \geq 0$.

8. The spectral side of the trace formula

In this section we apply the spectral side of the (non-invariant) trace formula of Arthur [Ar1], [Ar2], to the heat kernel. The goal is to prove that the leading term of the asymptotic expansion as $t \to 0$ is given by the trace of the heat operator, restricted to the point spectrum.

To begin with we briefly recall the structure of the spectral side. Let $L \supset M$ be Levi subgroups in $\mathcal{L}$, $P \in \mathcal{P}(M)$, and let $m = \dim \mathfrak{a}_L^G$ be the co-rank of L in G. Denote by $\mathfrak{B}_{P,L}$ the set of m-tuples $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee)$ of elements of $\Sigma_P^\vee$ whose projections to $\mathfrak{a}_L$ form a basis for $\mathfrak{a}_L^G$. For any $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee) \in \mathfrak{B}_{P,L}$ let $\text{vol}(\underline{\beta})$ be the co-volume in $\mathfrak{a}_L^G$ of the lattice spanned by $\underline{\beta}$ and let

$$\begin{aligned}\Xi_L(\underline{\beta}) &= \{(Q_1, \dots, Q_m) \in \mathcal{F}_1(M)^m : \quad \beta_i^\vee \in \mathfrak{a}_M^{Q_i},\ i = 1, \dots, m\} \\ &= \{\langle P_1, P_1' \rangle, \dots, \langle P_m, P_m' \rangle) : \quad P_i|^{\beta_i} P_i',\ i = 1, \dots, m\}.\end{aligned}$$

Given $Q, P \in \mathcal{P}(M)$, let $M_{Q|P}(\lambda) \colon \mathcal{A}^2(P) \to \mathcal{A}^2(Q)$, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, be the intertwining operator defined by (4.4).

For any smooth function f on $\mathfrak{a}_M^*$ and $\mu \in \mathfrak{a}_M^*$ denote by $D_\mu f$ the directional derivative of f along $\mu \in \mathfrak{a}_M^*$. For a pair $P_1|^\alpha P_2$ of adjacent parabolic subgroups in $\mathcal{P}(M)$ write

$$\delta_{P_1|P_2}(\lambda) = M_{P_2|P_1}(\lambda) D_\varpi M_{P_1|P_2}(\lambda) : \mathcal{A}^2(P_1) \to \mathcal{A}^2(P_2), \tag{8.1}$$

where $\varpi \in \mathfrak{a}_M^*$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$. [1] Equivalently, writing $M_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)$ for a meromorphic function Φ of a single complex variable, we have

$$\delta_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)^{-1} \Phi'(\langle \lambda, \alpha^\vee \rangle).$$

Recall that for $P, Q \in \mathcal{P}(M)$, $\langle P, Q \rangle$ denotes the group generated by P and Q. For any m-tuple $\mathcal{X} = (Q_1, \dots, Q_m) \in \Xi_L(\underline{\beta})$ with $Q_i = \langle P_i, P_i' \rangle$, $P_i|^{\beta_i} P_i'$, denote by $\Delta_{\mathcal{X}}(P, \lambda)$ the expression

(8.2)

$$\frac{\text{vol}(\underline{\beta})}{m!} M_{P_1'|P}(\lambda)^{-1} \delta_{P_1|P_1'}(\lambda) M_{P_1'|P_2'}(\lambda) \cdots \delta_{P_{m-1}|P_{m-1}'}(\lambda) M_{P_{m-1}'|P_m'}(\lambda) \delta_{P_m|P_m'}(\lambda) M_{P_m'|P}(\lambda).$$

[1] Note that this definition differs slightly from the definition of $\delta_{P_1|P_2}$ in [FLM1].

Recall the (purely combinatorial) map $\mathcal{X}_L : \mathfrak{B}_{P,L} \to \mathcal{F}_1(M)^m$ with the property that $\mathcal{X}_L(\underline{\beta}) \in \Xi_L(\underline{\beta})$ for all $\underline{\beta} \in \mathfrak{B}_{P,L}$ as defined in [FLM1, pp. 179–180].[2]

For any $s \in W(M)$ let L_s be the smallest Levi subgroup in $\mathcal{L}(M)$ containing w_s. We recall that $\mathfrak{a}_{L_s} = \{H \in \mathfrak{a}_M \mid sH = H\}$. Set

$$\iota_s = |\det(s-1)_{\mathfrak{a}_M^{L_s}}|^{-1}.$$

For $P \in \mathcal{F}(M_0)$ and $s \in W(M_P)$ let $M(P,s) : \mathcal{A}^2(P) \to \mathcal{A}^2(P)$ be as in [Ar3, p. 1309]. $M(P,s)$ is a unitary operator which commutes with the operators $\rho(P,\lambda,h)$ for $\lambda \in \mathrm{i}\mathfrak{a}_{L_s}^*$. Finally, we can state the refined spectral expansion.

Theorem 8.1 ([FLM1])**.** *For any $h \in C_c^\infty(G(\mathbb{A})^1)$ the spectral side of Arthur's trace formula is given by*

$$J_{spec}(h) = \sum_{[M]} J_{spec,M}(h), \tag{8.3}$$

M ranging over the conjugacy classes of Levi subgroups of G (represented by members of $\mathcal{L}$), where

$$J_{spec,M}(h) = \frac{1}{|W(M)|} \sum_{s \in W(M)} \iota_s \sum_{\underline{\beta} \in \mathfrak{B}_{P,L_s}} \int_{\mathrm{i}(\mathfrak{a}_{L_s}^G)^*} \mathrm{tr}(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\lambda) M(P,s) \rho(P,\lambda,h))\, d\lambda \tag{8.4}$$

with $P \in \mathcal{P}(M)$ arbitrary. The operators are of trace class and the integrals are absolutely convergent with respect to the trace norm and define distributions on $\mathcal{C}(G(\mathbb{A})^1)$.

Now we apply the trace formula to the heat kernel. We recall its definition. For details see [MM1, § 3]. Recall that the underlying symmetric space is

$$\widetilde{X} = G(F_\infty)^1 / \mathbf{K}_\infty,$$

where $G(F_\infty)^1 = G(\mathbb{A})^1 \cap G(F_\infty)$. Note that $G(F_\infty)^1$ is semisimple and

$$G(F_\infty) = G(F_\infty)^1 \cdot A_G. \tag{8.5}$$

Given $\nu \in \Pi(\mathbf{K}_\infty)$, let $\widetilde{E}_\nu \to \widetilde{X}$ be the associated homogeneous vector bundle. Let $\widetilde{\Delta}_\nu = (\widetilde{\nabla}^\nu)^* \widetilde{\nabla}^\nu$ be the Bochner-Laplace operator acting in the space $C^\infty(\widetilde{X}, \widetilde{E}_\nu)$ of smooth sections of $\widetilde{E}_\nu$. This is a $G(F_\infty)^1$-invariant second order elliptic differential operator. Since $\widetilde{X}$ is complete, $\widetilde{\Delta}_\nu$, regarded as operator in $L^2(\widetilde{X}, \widetilde{E}_\nu)$ with domain the smooth compactly supported sections, is essentially self-adjoint [LaM, p. 155]. Its self-adjoint extension will also be denoted by $\widetilde{\Delta}_\nu$. Let $\Omega \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ and $\Omega_{\mathbf{K}_\infty} \in \mathcal{Z}(\mathfrak{k})$ be the Casimir operators of $\mathfrak{g}$ and $\mathfrak{k}$, respectively, where the latter is defined with respect to the restriction of the normalized Killing form of $\mathfrak{g}$ to $\mathfrak{k}$. Then with respect to the isomorphism (3.13) we have

$$\widetilde{\Delta}_\nu = -R(\Omega) + \nu(\Omega_{\mathbf{K}_\infty}), \tag{8.6}$$

[2] The map $\mathcal{X}_L$ depends in fact on the additional choice of a vector $\underline{\mu} \in (\mathfrak{a}_M^*)^m$ which does not lie in an explicit finite set of hyperplanes. For our purposes, the precise definition of $\mathcal{X}_L$ is immaterial.

where R denotes the right regular representation of $G(F_\infty)$ in $C^\infty(G(F_\infty),\nu)$ (see [Mia, Proposition 1.1]).

Let $e^{-t\widetilde{\Delta}_\nu}$, $t>0$, be the heat semigroup generated by $\widetilde{\Delta}_\nu$. It commutes with the action of $G(F_\infty)^1$. With respect to the isomorphism (3.13) we may regard $e^{-t\widetilde{\Delta}_\nu}$ as a bounded operator in $L^2(G(F_\infty)^1,\nu)$, which commutes with the action of $G(F_\infty)^1$. Hence it is a convolution operator, i.e., there exists a smooth map

$$H_t^\nu\colon G(F_\infty)^1\to \operatorname{End}(V_\nu) \tag{8.7}$$

such that

$$(e^{-t\widetilde{\Delta}_\nu}\phi)(g)=\int_{G(F_\infty)^1} H_t^\nu(g^{-1}g')(\phi(g'))\,dg',\quad \phi\in L^2(G(F_\infty)^1,\nu).$$

The kernel H_t^ν satisfies

$$H_t^\nu(k^{-1}gk')=\nu(k)^{-1}\circ H_t^\nu(g)\circ\nu(k'),\ \forall k,k'\in\mathbf{K}_\infty,\forall g\in G(F_\infty)^1. \tag{8.8}$$

Moreover, proceeding as in the proof of [BM, Proposition 2.4] it follows that H_t^ν belongs to $(\mathscr{C}^q(G(F_\infty)^1)\otimes\operatorname{End}(V_\nu))^{K_\infty\times K_\infty}$ for all $q>0$, where $\mathscr{C}^q(G(F_\infty)^1)$ is Harish-Chandra's Schwartz space of L^q-integrable rapidly decreasing functions on $G(F_\infty)^1$. Put

$$h_t^\nu(g):=\operatorname{tr} H_t^\nu(g),\quad g\in G(F_\infty)^1. \tag{8.9}$$

Then $h_t^\nu\in\mathscr{C}^q(G(F_\infty)^1)$ for all $q>0$. We extend h_t^ν to a function on $G(F_\infty)$ by

$$h_t^\nu(ag)=h_t^\nu(g),\quad g\in G(F_\infty)^1,\ a\in A_G.$$

Let $\mathbf{1}_{K_f}\colon G(\mathbb{A}_f)\to\mathbb{C}$ be the characteristic function of K_f. Put

$$\chi_{K_f}:=\frac{\mathbf{1}_{K_f}}{\operatorname{vol}(K_f)} \tag{8.10}$$

and

$$\phi_t^\nu(g):=h_t^\nu(g_\infty)\chi_{K_f}(g_f) \tag{8.11}$$

for $g=g_\infty\cdot g_f\in G(\mathbb{A})=G(F_\infty)\cdot G(\mathbb{A}_f)$. Now observe that all derivatives of ϕ_t^ν belong to $L^1(G(\mathbb{A})^1)$. Thus ϕ_t^ν belongs to $\mathcal{C}(G(\mathbb{A});K_f)$ (see section 2 for its definition). By Theorem 8.1, J_{spec} is a distribution on $\mathcal{C}(G(\mathbb{A});K_f)$. Thus we can insert ϕ_t^ν into the trace formula and by Theorem 8.1 we get

$$J_{\text{spec}}(\phi_t^\nu)=\sum_{[M]} J_{\text{spec},M}(\phi_t^\nu), \tag{8.12}$$

where the sum ranges over the conjugacy classes of Levi subgroups of G and $J_{\text{spec},M}(\phi_t^{\tau,p})$ is given by (8.4). To analyze these terms, we proceed as in [MM1, Section 13]. Recall that the operator $\Delta_{\mathcal{X}}(P,\lambda)$, which appears in the formula (8.4), is defined by (8.2). Its definition involves the intertwining operators $M_{Q|P}(\lambda)$. If we replace $M_{Q|P}(\lambda)$ by its restriction

$M_{Q|P}(\pi,\lambda)$ to $\mathcal{A}^2_\pi(P)$, we obtain the restriction $\Delta_{\mathcal{X}}(P,\pi,\lambda)$ of $\Delta_{\mathcal{X}}(P,\lambda)$ to $\mathcal{A}^2_\pi(P)$. Similarly, let $\rho_\pi(P,\lambda)$ be the induced representation in $\bar{\mathcal{A}}^2_\pi(P)$. Fix $s\in W(M)$ and $\beta\in\mathfrak{B}_{P,L_s}$. Then for the integral on the right of (8.4) with $h=\phi_t^\nu$ we get

$$\sum_{\pi\in\Pi_{\rm dis}(M(\mathbb{A}))}\int_{i(\mathfrak{a}^G_{L_s})^*}\operatorname{Tr}\left(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\lambda)M(P,\pi,s)\rho_\pi(P,\lambda,\phi_t^\nu)\right)\,d\lambda. \tag{8.13}$$

In order to deal with the integrand, we need the following result. Let π be a unitary admissible representation of $G(F_\infty)$. Let $A\colon\mathcal{H}_\pi\to\mathcal{H}_\pi$ be a bounded operator which is an intertwining operator for $\pi|_{\mathbf{K}_\infty}$. Then $A\circ\pi(h_t^\nu)$ is a finite rank operator. Define an operator $\tilde{A}$ on $\mathcal{H}_\pi\otimes V_\nu$ by $\tilde{A}:=A\otimes\operatorname{Id}$. Then by [MM1, (9.13)] we have

$$\operatorname{Tr}(A\circ\pi(h_t^\nu))=e^{t(\pi(\Omega)-\nu(\Omega_{\mathbf{K}_\infty}))}\operatorname{Tr}\left(\tilde{A}|_{(\mathcal{H}_\pi\otimes V_\nu)^{\mathbf{K}_\infty}}\right). \tag{8.14}$$

We will apply this to the induced representation $\rho_\pi(P,\lambda)$. Let $m(\pi)$ denote the multiplicity with which π occurs in the regular representation of $M(\mathbb{A})$ in $L^2_{\rm dis}(A_MM(F)\backslash M(\mathbb{A}))$. Then

$$\rho_\pi(P,\lambda)\cong\oplus_{i=1}^{m(\pi)}\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi,\lambda). \tag{8.15}$$

Fix positive restricted roots of $\mathfrak{a}_P$ and let $\rho_{\mathfrak{a}_P}$ denote the corresponding half-sum of these roots. For $\xi\in\Pi(M(F_\infty))$ and $\lambda\in\mathfrak{a}_P^*$ let

$$\pi_{\xi,\lambda}:=\operatorname{Ind}_{P(F_\infty)}^{G(F_\infty)}(\xi\otimes e^{i\lambda})$$

be the unitary induced representation. Let $\xi(\Omega_M)$ be the Casimir eigenvalue of ξ. Define a constant $c(\xi)$ by

$$c(\xi):=-\langle\rho_{\mathfrak{a}_P},\rho_{\mathfrak{a}_P}\rangle+\xi(\Omega_M). \tag{8.16}$$

Then for $\lambda\in\mathfrak{a}_P^*$ one has

$$\pi_{\xi,\lambda}(\Omega)=-\|\lambda\|^2+c(\xi) \tag{8.17}$$

(see [Kn, Theorem 8.22]). Denote by $\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)$ resp. $\widetilde{M}(P,\pi,\nu,s)$ the extensions of $\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)$ resp. $M(P,\pi,\nu,s)$ to operators on $\mathcal{A}^2_\pi(P)\otimes V_\nu$ as above. Using the definition of ϕ_t^ν, (8.15) and (8.14), it follows that (8.13) is equal to

(8.18)
$$\sum_{\pi\in\Pi_{\rm dis}(M(\mathbb{A}))}e^{tc(\pi_\infty)}\cdot\int_{i(\mathfrak{a}^G_{L_s})^*}e^{-t\|\lambda\|^2}\operatorname{Tr}\left(\left(\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)\widetilde{M}(P,\pi,\nu,s)\right)|_{(\mathcal{A}^2_\pi(P)^{K_f}\otimes V_\nu)^{\mathbf{K}_\infty}}\right)\,d\lambda.$$

Since $M(P,\pi,s)$ is unitary, (8.18) can be estimated by

(8.19)
$$\sum_{\pi\in\Pi_{\rm dis}(M(\mathbb{A}))}\dim\left(\mathcal{A}^2_\pi(P)^{K_f,\nu}\right)\cdot e^{tc(\pi_\infty)}\int_{i(\mathfrak{a}^G_{L_s})^*}e^{-t\|\lambda\|^2}\|\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)\|_{1,\mathcal{A}^2_\pi(P)^{K_f,\nu}}\,d\lambda.$$

For $\pi \in \Pi(M(\mathbb{A}))$ denote by λ_{π_∞} the Casimir eigenvalue of the restriction of π_∞ to $M(F_\infty)^1$. Given $\lambda > 0$, let

$$\Pi_{\text{dis}}(M(\mathbb{A});\lambda) := \{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))\colon |\lambda_{\pi_\infty}| \le \lambda\}.$$

For the estimation of (8.19) we need the following auxiliary result.

Lemma 8.2. *Let $d_M = \dim M(F_\infty)^1/\mathbf{K}_\infty^M$. For every open compact subgroup $K_f \subset G(\mathbb{A}_f)$ and every $\nu \in \Pi(\mathbf{K}_\infty)$ there exists $C > 0$ such that*

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A});\lambda)} \dim \mathcal{A}^2_\pi(P)^{K_f,\nu} \le C(1+\lambda^{d_M/2}) \tag{8.20}$$

for all $\lambda \ge 0$.

Proof. By (7.5) it suffices to fix $\tau \in \Pi(K_{M,\infty})$ and to estimate the sum

$$\sum_{\Pi_{\text{dis}}(M(\mathbb{A}),\lambda)} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_{M,f}})\dim(\mathcal{H}_{\pi_\infty}(\tau)).$$

To estimate the sum over $\Pi_{\text{cus}}(M(\mathbb{A}),\lambda)$ we use [Mu3, Lemma 3.2], which holds for general reductive groups. Thus we get

$$\sum_{\pi \in \Pi_{\text{cus}}(M(\mathbb{A});\lambda)} \dim \mathcal{A}^2_\pi(P)^{K_f,\nu} \le C(1+\lambda^{d_M/2}). \tag{8.21}$$

Next observe that for any $\tau \in \Pi(K_{M,\infty})$, by definition of the counting function we have

$$\sum_{\pi \in \Pi_{\text{res}}(M(\mathbb{A});\lambda)} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_{M,f}})\dim(\mathcal{H}_{\pi_\infty}(\tau)) = N_{M,\text{res}}^{K_{M,f},\tau}(\lambda).$$

By Proposition 7.2 the right hand side is bounded by a constant $C > 0$ times $1+\lambda^{(d_M-1)/2}$. By (7.5) we get

$$\sum_{\pi \in \Pi_{\text{res}}(M(\mathbb{A});\lambda)} \mathcal{A}^2_\pi(P)^{K_f,\nu} \le C(1+\lambda^{(d_M-1)/2}).$$

Together with (8.21) the lemma follows. ∎

Next we estimate the integral in (8.19). We use the notation introduced above. Let $\underline{\beta} = (\beta_1^\vee,\dots,\beta_m^\vee)$ and $\mathcal{X}_{L_s}(\underline{\beta}) = (Q_1,\dots,Q_m) \in \Xi_{L_s}(\underline{\beta})$ with $Q_i = \langle P_i, P_i'\rangle$, $P_i|^{\beta_i}P_i'$, $i = 1,\dots,m$. Using the definition (8.2) of $\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)$, it follows that we can bound the integral by a constant multiple of

$$\dim(\nu)\int_{i(\mathfrak{a}_{L_s}^G)^*} e^{-t\|\lambda\|^2}\prod_{j=1}^m \left\| \delta_{P_j|P_j'}(\lambda)\Big|_{\mathcal{A}^2_\pi(P_j')^{K_f,\nu}}\right\| \, d\lambda, \tag{8.22}$$

where $\delta_{P_j|P_j'}(\lambda)$ is defined by (8.1). We introduce new coordinates $s_j := \langle \lambda, \beta_j^\vee\rangle$, $j = 1,\dots,m$, on $(\mathfrak{a}_{L_s,\mathbb{C}}^G)^*$. Using (4.9) and (8.1), we can write

$$\delta_{P_i|P_i'}(\lambda) = \frac{n_{\beta_i}'(\pi,s_i)}{n_{\beta_i}(\pi,s_i)} + j_{P_i'}\circ(\operatorname{Id}\otimes R_{P_i|P_i'}(\pi,s_i)^{-1}R'_{P_i|P_i'}(\pi,s_i))\circ j_{P_i'}^{-1}. \tag{8.23}$$

Now assume that G satisfies property (L). By [FL1, Prop. 3.8], G satisfies property (TWN+) (tempered winding numbers, strong version). This means that for any proper Levi subgroup M of G defined over $\mathbb{Q}$, and any root $\alpha \in \Sigma_M$, and $T \in \mathbb{R}$ the following estimate holds

$$\int_T^{T+1} |n'_\alpha(\pi, it)| dt \ll \log(|T| + \Lambda(\pi_\infty; p^{\mathrm{sc}}) + \mathrm{level}(\pi; p^{\mathrm{sc}})) \tag{8.24}$$

for all $\Pi_{\mathrm{dis}}(M(\mathbb{A}))$. In the proof of Corollary 5.2 it was proved that there exists $C > 0$ such that for all $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}); K_f, \sigma)$ one has

$$\mathrm{level}(\pi; p^{\mathrm{sc}}) \leq C, \quad \mathit{and} \quad \Lambda(\pi_\infty; p^{\mathrm{sc}}) \leq C(1 + \lambda_{\pi_\infty}^2). \tag{8.25}$$

Hence we get

$$\int_T^{T+1} |n'_\alpha(\pi, it)| dt \ll \log(|T| + 1 + \lambda_{\pi_\infty}^2) \tag{8.26}$$

For all $T \in \mathbb{R}$ and $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}); K_f, \sigma)$.

Lemma 8.3. *There exists $C > 0$ such that*

$$\int_{\mathbb{R}} \left| \frac{n'_\alpha(\pi, i\lambda)}{n_\alpha(\pi, i\lambda)} \right| e^{-t\lambda^2} d\lambda \leq C(1 + \log(1 + \lambda_{\pi_\infty}^2)) \frac{1 + |\log t|}{\sqrt{t}} \tag{8.27}$$

for all $0 < t \leq 1$ and $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}); K_f, \sigma)$.

Proof. By (5.12) it follows that $|n_\alpha(\pi, i\lambda)| = 1$ for $\lambda \in \mathbb{R}$. Furthermore, by (8.26) we have

$$\int_0^\lambda |n'_\alpha(\pi, iu)| du \leq C|\lambda| \log(|\lambda| + 1 + \lambda_{\pi_\infty}^2) \tag{8.28}$$

for $\lambda \in \mathbb{R}$ and $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}); K_f, \sigma)$. Hence, using integration by parts, the integral on the left hand side of the claimed inequality equals

$$2t \int_{\mathbb{R}} \int_0^\lambda |n'_\alpha(\pi, iu)| du \lambda e^{-t\lambda^2} d\lambda.$$

By (8.28) we get

$$\begin{aligned} \int_{\mathbb{R}} |n'_\alpha(\pi, i\lambda)| e^{-t\lambda^2} d\lambda &\leq Ct \int_{\mathbb{R}} \log(|\lambda| + 1 + \lambda_{\pi_\infty}^2) \lambda^2 e^{-t\lambda^2} d\lambda \\ &\leq C_1(1 + \log(1 + \lambda_{\pi_\infty}^2)) \frac{1 + |\log t|}{\sqrt{t}} \end{aligned}$$

for all $0 < t \leq 1$ and $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}); K_f, \sigma)$. □

Let $l_s = \dim(A_{L_s}/A_G)$. Combining (8.23), Lemma 8.3 and Proposition 6.2 it follows that there exists $C > 0$

(8.29)

$$\int_{i(\mathfrak{a}_{L_s}^G)^*} e^{-t\|\lambda\|^2} \prod_{i=1}^m \left\| \delta_{P_i|P'_i}(\lambda) \Big|_{\mathcal{A}^2_\pi(P'_i)^{K_f,\nu}} \right\| d\lambda \leq C(1 + |\log t|)^{l_s} t^{-l_s/2} \left(1 + \log(1 + \lambda_{\pi_\infty}^2)\right)^{l_s}$$

for all $0 < t \leq 1$ and $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$ with $\mathcal{A}^2_\pi(P)^{K_f,\nu}$. Now we can estimate (8.19). Note that $c(\pi_\infty) = \lambda_{\pi_\infty} - \|\rho_{\mathfrak{a}_P}\|^2$. Using (8.29), it follows that (8.13) can be estimated by a constant multiple of

$$(1 + |\log t|)^{l_s} t^{-l_s/2} \sum_{\pi\in\Pi_{\text{dis}}(M(\mathbb{A}))} \dim \mathcal{A}^2_\pi(P)^{K_f,\nu} \left(1 + \log(1 + \lambda^2_{\pi_\infty})\right)^{l_s} e^{t\lambda_{\pi_\infty}}. \tag{8.30}$$

Let $X_M = M(F_\infty)^1/K_{M,\infty}$ and $m = \dim X_M$. Using Lemma 8.2 it follows that for every $\varepsilon > 0$ there exists $C > 0$ such that the series is bounded by $Ct^{-m/2-\varepsilon}$ for $0 < t \leq 1$. Together with (8.30) this yields the following proposition.

Proposition 8.4. *Let $M \in \mathcal{L}$. Let $m = \dim X_M$ and $l = \max_{s\in W(M)} \dim(A_{L_s}/A_G)$. For every $\varepsilon > 0$ there exists $C > 0$ such that*

$$|J_{spec,M}(\phi^\nu_t)| \leq Ct^{-(m+l)/2-\varepsilon}$$

for all $0 < t \leq 1$.

Now we distinguish two cases. First assume that $M = G$. Then $A_M = A_G$. Let R^1_{dis} be the restriction of the regular representation R^1 of $G(\mathbb{A})^1$ in $L^2(G(F)\backslash G(\mathbb{A})^1)$ to the discrete subspace. Then $J_{\text{spec},G}(\phi^\nu_t) = Tr(R^1_{\text{dis}}(\phi^{\nu,1}_t)$. Let R_{dis} be the regular representation of $G(\mathbb{A})$ in $L^2(A_G G(F)\backslash G(\mathbb{A}))$. Then the operator $R_{\text{dis}}(\phi^\nu_t)$ is isomorphic to $R^1(\phi^{\nu,1}_t)$. Thus

$$J_{\text{spec},G}(\phi^{\nu,1}_t) = \operatorname{Tr}(R_{\text{dis}}(\phi^\nu_t)). \tag{8.31}$$

Given $\pi \in \Pi_{\text{dis}}(G(\mathbb{A}))$, let $m(\pi)$ denote the multiplicity with which π occurs in the regular representation of $G(\mathbb{A})$ in $L^2(A_G G(F)\backslash G(\mathbb{A}))$. Then, using Corollary 2.2 in [BM], we get

$$J_{\text{spec},G}(\phi^{\nu,1}_t) = \sum_{\pi\in\Pi_{\text{dis}}(G(\mathbb{A}),\xi_0)} m(\pi) \dim(\mathcal{H}^{K_f}_{\pi_f}) \dim(\mathcal{H}_{\pi_\infty} \otimes V_\nu)^{K_\infty} e^{t\lambda_{\pi_\infty}}. \tag{8.32}$$

Now assume that M is a proper Levi subgroup. Let $P = M \ltimes N$. Let $\widetilde{X} = G(F_\infty)^1/\mathbf{K}_\infty$. Then

$$\widetilde{X} \cong X_M \times A_M/A_G \times N(F_\infty).$$

Since $l = \max_{s\in W(M)} \dim(A_{L_s}/A_G) \leq \dim(A_M/A_G)$, it follows that $m + l \leq \dim \widetilde{X} - 1$. Thus using this together with Proposition 8.4, we get

Theorem 8.5. *Suppose that G satisfies property (L). Let $n = \dim \widetilde{X}$. For every open compact subgroup K_f of $G(\mathbb{A}_f)$ and every $\nu \in \Pi(\mathbf{K}_\infty)$ the spectral side of the trace formula, evaluated at $\phi^{\nu,1}_t$, satisfies*

$$\begin{aligned} J_{spec}(\phi^{\nu,1}_t) = &\sum_{\pi\in\Pi_{\text{dis}}(G(\mathbb{A}))} m(\pi) \dim(\mathcal{H}^{K_f}_{\pi_f}) \dim(\mathcal{H}_{\pi_\infty} \otimes V_\nu)^{K_\infty} e^{t\lambda_{\pi_\infty}} \\ &+ O(t^{-(n-1)/2}) \end{aligned} \tag{8.33}$$

as $t \to 0^+$.

9. Geometric side of the trace formula

As before, G is a reductive group over a number field F. In this section we consider the geometric side of the Arthur trace formula $J_{\rm geom}$ evaluated at ϕ_t^ν and determine the asymptotic behavior of $J_{\rm geom}(\phi_t^\nu)$ as $t\to 0$. The geometric side $J_{\rm geom}$ of the trace formula was introduced in [Ar1]. See also [Ar5]. For $f\in C_c^\infty(G(\mathbb{A})^1)$, Arthur has defined $J_{\rm geom}(f)$ as the value at a point $T_0\in\mathfrak{a}_0$, specified in [Ar3, Lemma 1.1], of a polynomial $J^T(f)$ on $\mathfrak{a}_0$. By [FL3, Theorem 7.1], $J_{\rm geom}(f)$ is absolutely convergent for all $f\in\mathcal{C}(G(\mathbb{A});K_f)$. Let $\phi_t^\nu\in\mathcal{C}(G(\mathbb{A});K_f)$ be the function which is defined by (8.11). Then $J_{\rm geom}(\phi_t^\nu)$ is well defined. In [MM2, (1.5)], the regularized trace of the heat operator $e^{-t\Delta_\nu}$ was defined as

$$\operatorname{Tr}_{\rm reg}\left(e^{-t\Delta_\nu}\right):=J_{\rm geom}(\phi_t^\nu).$$

Then in [MM2, Theorem 1.1] an asymptotic expansion of $\operatorname{Tr}_{\rm reg}\left(e^{-t\Delta_\nu}\right)$ as $t\to 0$ has been established. For our purpose we need to know the precise form of the term of order $t^{-n/2}$, where $n=\dim X$. To this end we briefly recall the derivation of the asymptotic expansion. The first step is to replace ϕ_t^ν by an appropriate compactly supported function $\tilde\phi_t^\nu$ with support concentrated near the identity element. Such a function is constructed as follows.

Let $d(\cdot,\cdot):\tilde X\times\tilde X\longrightarrow[0,\infty)$ be the geodesic distance on $\tilde X$, and put $r(g_\infty)=d(g_\infty x_0,x_0)$ where $x_0=\mathbf{K}_\infty\in\tilde X$ is the base point. Let $0<a<b$ be sufficiently small real numbers and let $\beta:\mathbb{R}\longrightarrow[0,\infty)$ be a smooth function supported in $[-b,b]$ such that $\beta(y)=1$ for $0\le|y|\le a$, and $0\le\beta(y)\le 1$ for $|y|>a$. Define

$$\psi_t^\nu(g_\infty)=\beta(r(g_\infty))h_t^\nu(g_\infty). \tag{9.1}$$

and

$$\tilde\phi_t^\nu(g)=\psi_t^\nu(g_\infty)\chi_{K_f}(g_f) \tag{9.2}$$

for $g=g_\infty\cdot g_f\in G(\mathbb{A})=G(F_\infty)\cdot G(\mathbb{A}_f)$. Then $\tilde\phi_t^\nu\in C_c^\infty(G(\mathbb{A})^1)$ and $\psi_t^\nu\in C_c^\infty(G(F_\infty)^1)$. By [MM1, Proposition 12.1] there is some $c>0$ such that for every $0<t\le 1$ we have

$$\left|J_{\rm geom}(\phi_t^\nu)-J_{\rm geom}(\tilde\phi_t^\nu)\right|\ll e^{-c/t}. \tag{9.3}$$

We note that in [MM1, Sect. 12] we made the assumption that $G=\mathrm{GL}(n)$ or $G=\mathrm{SL}(n)$. However, the proof of the proposition holds without any restriction on G. The next result reduces the considerations to the unipotent contribution to the geometric side. Before we state it, we recall the coarse geometric expansion of Arthur's trace formula [Ar5, Sect. 10]: Two elements $\gamma_1,\gamma_2\in G(F)$ are called coarsely equivalent if their semisimple parts (in the Jordan decomposition) are conjugate in $G(F)$. Then for any $f\in C_c^\infty(G(\mathbb{A})^1)$ we have

$$J_{\rm geom}(f)=\sum_{\mathfrak{o}}J_{\mathfrak{o}}(f),$$

where $\mathfrak{o}$ runs over the coarse equivalence classes in $G(F)$, and the distribution $J_{\mathfrak{o}}$ is supported in the set of all $g\in G(\mathbb{A})^1$ whose semisimple part is conjugate in $G(\mathbb{A})$ to some semisimple element in $\mathfrak{o}$. If $\mathfrak{o}\neq\mathfrak{o}'$, the supports of $J_{\mathfrak{o}}$ and $J_{\mathfrak{o}'}$ are disjoint. Note that the set of unipotent elements in $G(F)$ constitute a single equivalence class $\mathfrak{o}_{\rm unip}$ and we write

$J_{\text{unip}} = J_{\mathfrak{o}_{\text{unip}}}$. Assume that K_f is neat. If the support of β is sufficiently small then by [MM2, Prop. 3.1] we have

$$J_{\text{geom}}(\tilde{\phi}_t^\nu) = J_{\text{unip}}(\tilde{\phi}_t^\nu). \tag{9.4}$$

By (9.3) and (9.4) the problem is reduced to the study of the asymptotic behavior of $J_{\text{unip}}(\tilde{\phi}_t^\nu)$ as $t \searrow 0$. For this purpose we use Arthur's fine geometric expansion of J_{unip}. In order to state it we need to introduce some notation.

Let S be a finite set of places of F, which includes the archimedean places, such that $K_v = \mathbf{K}_v$ for $v \notin S$. Let $G(F_S)^1 = G(F_S) \cap G(\mathbb{A})^1$. Let $M \in \mathcal{L}$. Following Arthur, we introduce an equivalence relation on the set of unipotent elements in $M(F)$ that depends on the set S: Two unipotent elements $u, v \in M(F)$ are (M, S)-equivalent if and only if u and v are conjugate in $M(F_S)$. We denote the equivalence class of u by $[u]_S \subseteq M(F)$ and let $\mathcal{U}_S^M$ denote the set of all such equivalence classes.

Note that two equivalent unipotent elements define the same unipotent conjugacy class in $M(F_S)$, so we can view $\mathcal{U}_S^M$ also as the set of unipotent conjugacy classes in $M(F_S)$ which have at least one F-rational representative, and we denote the corresponding conjugacy class by $[u]_S$ as well.

Remark 9.1. *(i) If $T \subseteq S$, then we get a well-defined map $\mathcal{U}_S^M \ni [u]_S \mapsto [u]_T \in \mathcal{U}_T^M$.*

(ii) If $G = \mathrm{GL}(n)$, the equivalence relation is independent of S and is the same as conjugation in $M(F)$.

For $[u]_S \in \mathcal{U}_S^M$ and $f_S \in C_c^\infty(G(F_S)^1)$, Arthur associates a weighted orbital integral $J_M^G([u]_S, f_S)$ [Ar6] which is a distribution supported on the $G(F_S)$-conjugacy class induced from $[u]_S \subseteq M(F_S)$. Let $\mathbf{1}_{\mathbf{K}^S} \in C_c^\infty(G(\mathbb{A}^S))$ be the characteristic function of $\mathbf{K}^S$, if $f_S \in C_c^\infty(G(F_S)^1)$. Put $f = f_S \mathbf{1}_{\mathbf{K}^S} \in C_c^\infty(G(\mathbb{A})^1)$. By [Ar7, Corollary 8.3] there exist unique constants $a^M([u]_S, S) \in \mathbb{C}$ and conjugacy classes $[u]_S \in \mathcal{U}_S^M$, such that for all $f_S \in C_c^\infty(G(F_S)^1)$ we have

$$J_{\text{unip}}(f) = \sum_{M \in \mathcal{L}} \sum_{[u]_S \in \mathcal{U}_S^M} a^M([u]_S, S) J_M^G([u]_S, f_S). \tag{9.5}$$

In fact, Corollary 8.3 in [Ar7] is stated only for reductive groups over $\mathbb{Q}$. However, at the end of the article, Arthur explains that all results of the article hold equally well for reductive groups over a number field F.

In general, there is not much known about the coefficients $a^M([u]_S, S)$. However, for our purpose we only need to know $a^G(1, S)$, which by [Ar7, Corollary 8.5] is given by

$$a^G(1, S) = \mathrm{vol}(G(F) \backslash G(\mathbb{A})^1). \tag{9.6}$$

Write $S = S_\infty \sqcup S_0$. Then $K_f = K_{S_0} \mathbf{K}^S$. Recall that by (9.2)

$$\tilde{\phi}_t^\nu = \frac{1}{\mathrm{vol}(K_f)} \psi_t^\nu \cdot \mathbf{1}_{K_{S_0}} \cdot \mathbf{1}_{\mathbf{K}^S}.$$

Then by (9.5) we get

$$J_{\text{unip}}(\tilde{\phi}_t^\nu) = \sum_{M\in\mathcal{L}} \sum_{[u]_S\in\mathcal{U}_S^M} \frac{a^M([u]_S,S)}{\operatorname{vol}(K_f)} J_M^G([u]_S, \psi_t^\nu\cdot \mathbf{1}_{K_{S_0}}). \tag{9.7}$$

Using (9.6), the term that corresponds to $(G,1)$ equals

$$\frac{\operatorname{vol}(G(F)\backslash G(\mathbb{A})^1)}{\operatorname{vol}(K_f)} h_t^\nu(1). \tag{9.8}$$

To deal with the weighted orbital integrals in general, we use Arthur's splitting formula [Ar5, (18.7)], which we recall next. Let S be any finite set of places of F which not necessarily contains the archimedean places. Let $L\in\mathcal{L}(M)$ and $Q\in\mathcal{P}(L)$. Given $f_S\in G(F_S)$ let

$$f_{S,Q}(m) = \delta_Q(m)^{1/2}\int_{\mathbf{K}_S}\int_{N_Q(F_S)} f_S(k^{-1}mnk)\,dn\,dk, \quad m\in L.$$

Suppose that $S=S_1\cup S_2$ with S_1, S_2 non-empty and disjoint, and that f_S is the restriction of a product $f_{S_1}f_{S_2}$ to $G(F_S)^1$ with $f_{S_j}\in C^\infty(G(F_{S_j}))$, $j=1,2$. Then the splitting formula states that

$$J_M^G([u]_S, f_S) = \sum_{L_1,\,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) J_M^{L_1}([u]_{S_1}, f_{S_1,Q_1}) J_M^{L_2}([u]_{S_2}, f_{S_2,Q_2}), \tag{9.9}$$

where the notation is as follows: The $d_M^G(L_1,L_2)\in\mathbb{R}$ are certain constants which depend only on M, L_1, L_2, G but not on S. In fact, $d_M^G(L_1,L_2)$ is non-zero only if the natural map $\mathfrak{a}_M^{L_1}\oplus\mathfrak{a}_M^{L_2}\longrightarrow\mathfrak{a}_M^G$ is an isomorphism. The Q_j are arbitrary elements in $\mathcal{P}(L_j)$ and $[u]_{S_j}\in\mathcal{U}_{S_j}^M$ is the image of $[u]_S$ under the canonical map $\mathcal{U}_S^M\to\mathcal{U}_{S_j}^M$. Finally, $J_M^{L_j}([u]_{S_j},\cdot)$ denotes the S_j-adic distribution which is supported on the $L_j(F_{S_j})$-conjugacy class which is induced from $[u]_{S_j}\subseteq M(F_{S_j})$ and is defined as in [Ar6].

We apply the splitting formula to the weighted orbital integral on the right of (9.7) with $S_1=S_\infty$ and $S_2=S_0$. We obtain

$$J_M^G([u]_S, \psi_t^\nu\cdot\mathbf{1}_{K_{S_0}}) = \sum_{L_1,\,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) J_M^{L_1}([u]_\infty, \psi_{t,Q_1}^\nu) J_M^{L_2}([u]_{S_0}, \mathbf{1}_{K_{S_0},Q_2}). \tag{9.10}$$

This is a finite sum with $d_M^G(L_1,L_2)$ and $J_M^{L_2}([u]_{S_0},\mathbf{1}_{K_{S_0},Q_2})$ independent of t. The asymptotic expansion in t of weighted orbital integrals of the form $J_M^{L_1}([u]_\infty,\psi_{t,Q_1}^\nu)$ has been determined in [MM2, Prop. 7.2]. This Corollary has been proved for groups over $\mathbb{Q}$. However, the proof can be easily extended to reductive groups over F, either by repeating the arguments or using restriction of scalars,

We recall the proposition. Let $M\in\mathcal{L}$, $P_1=M_1N_1\in\mathcal{F}(M)$ and $\mathcal{O}\subset M(F_\infty)$ a unipotent conjugacy class in $M(F_\infty)$. Let $d_{\mathcal{O}}=\dim\mathcal{O}^{G(F_\infty)^1}$ be the dimension of the unipotent orbit in $G(F_\infty)^1$ induced from $M(F_\infty)$, and let $r_M^{M_1} = \dim \mathfrak{a}_M^{M_1}$. Then there exist constants $b_{ij} = b_ij(M,\mathcal{O}) \in \mathbb{C}$, $j \geq 0$, $0 \leq i \leq r_M^{M_1}$, such that for $0 < t \leq 1$

$$J_M^{M_1}(\mathcal{O}, (\psi_t^\nu)_{P_1}) \sim t^{-n/2+d_{\mathcal{O}}^G/4} \sum_{j=0}^{\infty} \sum_{i=0}^{r_M^{M_1}} b_{ij} t^{j/2} (\log t)^i. \tag{9.11}$$

If K_f is neat, then $d_{\mathcal{O}}^G > 0$ for $\mathcal{O} \neq 1$. Combining (9.3)–(9.11) it follows that for every $\nu \in \Pi(K_\infty)$ there exist $\varepsilon > 0$ such that

$$J_{\text{geom}}(\phi_t^\nu) = \operatorname{vol}(X(K_f)) h_t^\nu(1) + O(t^{-n/2+\varepsilon}) \tag{9.12}$$

for all $0 < t \leq 1$. By [Mu3, Lemma 2.3] we have

$$h_t^\nu(1) = \frac{\dim(\nu)}{(4\pi)^{n/2}} t^{-n/2} + O(t^{-(n-1)/2}) \tag{9.13}$$

as $t \searrow 0$. Together with (9.12) we obtain the following proposition.

Proposition 9.2. *Let G be a reductive group over a number field F. Let K_f be an open compact subgroup of $G(\mathbb{A})$. Assume that K_f is neat. Then for every $\nu \in \Pi(K_\infty)$ there exists $\varepsilon > 0$ such that we have*

$$J_{\text{geom}}(\phi_t^\nu) = \frac{\dim(\nu) \operatorname{vol}(X(K_f))}{(4\pi)^{n/2}} t^{-n/2} + O(t^{-n/2+\varepsilon})$$

for all $0 < t \leq 1$.

10. Proof of the main theorem

First we establish the adelic version of the Weyl law, which is Theorem 1.4. Let G_0 be a reductive algebraic group over a number field F and let $G = \operatorname{Res}_{F/\mathbb{Q}}(G_0)$ be the reductive group over $\mathbb{Q}$ which is obtained from G_0 by restriction of scalars. We shall use the (non-invariant) Arthur trace formula for reductive groups over F to deduce the Weyl law for G_0. Then we use the properties of the restriction of scalars to show that this is equivalent to the Weyl law for G.

To begin with we recall that the coarse Arthur trace formula over F is the identity

$$J_{\text{spec}}(f) = J_{\text{geom}}(f), \quad f \in \mathcal{C}(G(\mathbb{A}_F)^1).$$

Applied to ϕ_t^ν we get the equality

$$J_{\text{spec}}(\phi_t^\nu) = J_{\text{geom}}(\phi_t^\nu), \quad t > 0.$$

Assume that G_0 satisfies property (L). Let $K_{0,f}$ be an open compact subgroup of $G_0(\mathbb{A}_{F,f})$. We assume that $K_{0,f}$ is neat. Combining Theorem 8.5 and Proposition 9.2, we obtain

(10.1)
$$\sum_{\pi\in\Pi_{\mathrm{dis}}(G_0(\mathbb{A}_F))} m(\pi)\dim(\mathcal{H}_{\pi_f}^{K_{0,f}})\dim\left(\mathcal{H}_{\pi_\infty}\otimes V_\nu\right)^{K_\infty}\ e^{t\lambda_{\pi_\infty}} = \frac{\dim(\nu)\ \mathrm{vol}(X(K_{0,f}))}{(4\pi)^{n/2}} t^{-n/2} + O(t^{-(n-1)/2})$$

as $t\searrow 0$, where $X(K_{0,f})$ is defined by (3.10). Let $N_{\mathrm{dis}}^{K_{0,f},\nu}(\lambda)$ be the adelic counting function defined by (1.16). Applying Karamata's theorem [Fe, p. 446], we obtain

$$N_{\mathrm{dis}}^{K_{0,f},\nu}(\lambda) = \frac{\dim(\nu)\,\mathrm{vol}(X(K_{0,f}))}{(4\pi)^{n/2}\Gamma(n/2+1)}\lambda^{n/2} + o(\lambda^{n/2}) \tag{10.2}$$

as $\lambda\to\infty$. By Proposition 7.2 we get

$$N_{\mathrm{cus}}^{K_{0,f},\nu}(\lambda) = \frac{\dim(\nu)\,\mathrm{vol}(X(K_{0,f}))}{(4\pi)^{n/2}\Gamma(n/2+1)}\lambda^{n/2} + o(\lambda^{n/2}). \tag{10.3}$$

This is the first part of the Weyl law for G_0. The second part is the estimation of the counting function of the residual spectrum which follows from Proposition 7.2 for $M = G$.

Next we show that Theorem 1.4 is compatible with the restriction of scalars. To begin with we recall some facts about the Weil restriction of scalars [We], [Bo2]. By [We, Theorem 1.3.2] we have

$$G(\mathbb{Q}_v) = \prod_{w\mid v} G_0(F_w). \tag{10.4}$$

for all places v of $\mathbb{Q}$. In particular, we get

$$G(\mathbb{A}_{\mathbb{Q}}) = G_0(\mathbb{A}_F),\quad G(\mathbb{R}) = G_0(F_\infty) = \prod_{w\in S_\infty} G_0(F_w),\quad G(\mathbb{Q}) = G_0(F). \tag{10.5}$$

Therefore we obtain a bijection of the automorphic representations of G_0 with those of G. Also the regular representation of $G(\mathbb{A}_{\mathbb{Q}})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}}))$ is equivalent to the regular representation of $G_0(\mathbb{A}_F)$ on $L^2(G_0(F)\backslash G_0(\mathbb{A}_F))$. Furthermore, by [Bo2, 5.2], the map $P_0\mapsto \mathrm{Res}_{F/\mathbb{Q}}(P_0)$ induces a bijection between parabolic subgroups of G_0, defined over F, and parabolic subgroups of G, defined over $\mathbb{Q}$, and (10.4) and (10.5) continue to hold for F-parabolic subgroups of G_0. Let $P_0 = M_{P_0}N_{P_0}$ be a F-parabolic subgroup of G_0 and $P = \mathrm{Res}_{F/\mathbb{Q}}(P_0)$. Let $f\in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_Q)^1)$ and $\tilde f\in L^2(G_0(F)\backslash G_0(\mathbb{A}_F)^1)$ correspond to each other. Then

$$\int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A}_{\mathbb{Q}})} f(nx)dn = \int_{N_{P_0}(F)\backslash N_{P_0}(\mathbb{A}_F)} \tilde f(n_0x)dn_0. \tag{10.6}$$

Hence we get

$$L^2_{\mathrm{cus}}(G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})^1)\cong L^2_{\mathrm{cus}}(G_0(F)\backslash G_0(\mathbb{A}_F)^1).$$

The same holds for the residual spectrum. It follows that the counting functions for G and G_0 coincide. Thus (10.2) and (10.3) hold for the counting function of $G = \operatorname{Res}_{F/\mathbb{Q}}(G_0)$. This proves Theorem 1.4.

Next we deduce Theorem 1.2 from Theorem 1.4. To this end we express the counting function in a different way. Let $\sigma \in \Pi(\mathbf{K}_\infty)$. Let $L^2(A_G\Gamma\backslash G(\mathbb{R})),\sigma)$ be defined as in (1.3). Given $\pi \in \Pi(G(\mathbb{R}))$ let $m_\Gamma(\pi)$ be the multiplicity with which π occurs in the regular representation R_Γ in $L^2(A_G\Gamma\backslash G(\mathbb{R}))$. Let $L^2_{\rm dis}(A_G\Gamma\backslash G(\mathbb{R}))$ the span of all irreducible subrepresentations. Then

$$(L^2_{\rm dis}(A_G\Gamma\backslash G(\mathbb{R}))\otimes V_\sigma)^{\mathbf{K}_\infty} = \bigoplus_{\pi\in\Pi(G(\mathbb{R}))} m_\Gamma(\pi)(\mathcal{H}_\pi\otimes V_\sigma)^{\mathbf{K}_\infty}. \tag{10.7}$$

For $\tau \in \Pi(G(\mathbb{R}))$ let λ_τ be the Casimir eigenvalue of τ, i.e., the eigenvalue of $R_\Gamma(\Omega_{G(\mathbb{R})})$ on $\mathcal{H}_\tau$. Then it follows that $(\mathcal{H}_\tau\otimes V_\sigma)^{K_\infty}$ is an eigenspace of $\Delta_\sigma = -R_\Gamma(\Omega_{G(\mathbb{R})})$ with eigenvalue $-\lambda_\tau$. It follows that

$$N_{\Gamma,\rm dis}(\lambda;\sigma) := \sum_{\substack{\pi\in\Pi_(G(\mathbb{R}))\\ -\lambda_\pi\le\lambda}} m_\Gamma(\pi)\dim(\mathcal{H}_\pi\otimes V_\sigma)^{\mathbf{K}_\infty}. \tag{10.8}$$

There are similar formulas for $N_{\Gamma,\rm cus}(\lambda,\sigma)$ and $N_{\Gamma,\rm res}(\lambda,\sigma)$

Now we establish the relation between the adelic and real counting functions. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup. Let $\Gamma_i \subset G(\mathbb{Q})$, $i = 1,...,l$, be determined by (3.2). The relation between the classical and adelic counting functions is described by the following lemma.

Lemma 10.1. *Let $\sigma \in \Pi(\mathbf{K}_\infty)$. Then*

$$N^{K_f,\sigma}_{\rm dis}(\lambda) = \sum_{i=1}^{l} N_{\Gamma_i,\rm dis}(\lambda,\sigma)$$

for $\lambda \ge 0$. The same equality holds for the counting functions of the cuspidal and residual spectrum.

Proof. Given $\tau \in \Pi(G(\mathbb{R}))$, let $m(\tau)$ be the multiplicity with which τ occurs in the regular representation By (3.7) with respect to $F = \mathbb{Q}$ we have

$$\begin{aligned}\sum_{\substack{\tau\in\Pi_{\rm dis}(G(\mathbb{R}))\\ -\lambda_\tau\le\lambda}} m(\tau)(\mathcal{H}_\tau\otimes V_\nu)^{\mathbf{K}_\infty} &= \sum_{\substack{\tau\in\Pi_{\rm dis}(G(\mathbb{R}))\\ -\lambda_\tau\le\lambda}}\ \sum_{\substack{\pi\in\Pi_{\rm dis}(G(\mathbb{A}))\\ \pi_\infty=\tau}} m(\pi)\dim(\mathcal{H}^{K_f}_{\pi_f})\dim(\mathcal{H}_{\pi_\infty}\otimes V_\nu)^{\mathbf{K}_\infty}\\ &= \sum_{\substack{\pi\in\Pi_{\rm dis}(G(\mathbb{A}))\\ -\lambda_{\pi_\infty}\le\lambda}} m(\pi)\dim(\mathcal{H}^{K_f}_{\pi_f})\dim(\mathcal{H}_{\pi_\infty}\otimes V_\nu)^{\mathbf{K}_\infty}\\ &= N^{K_f,\sigma}_{\rm dis}(\lambda).\end{aligned}$$

Combined with (3.8) it follows that

$$N_{\text{dis}}^{K_f,\sigma}(\lambda) = \sum_{\substack{\tau\in\Pi_{\text{dis}}(G(\mathbb{R}))\\ -\lambda_\tau\le\lambda}} m(\tau)(\mathcal{H}_\tau\otimes V_\nu)^{K_\infty} = \sum_{i=1}^{l}\sum_{\substack{\tau\in\Pi(G(\mathbb{R}))\\ -\lambda_\tau\le\lambda}} m_{\Gamma_i}(\tau)(\mathcal{H}_\tau\otimes V_\nu)^{K_\infty}$$

$$= \sum_{i=1}^{l} N_{\Gamma_i,\text{dis}}(\lambda,\nu). \tag{10.9}$$

□

Let K_f and Γ_i, $i=1,...,l$, be as above. By Lemma 10.1 we have

$$N_{\text{cus}}^{K_f,\nu}(\lambda) = \sum_{i=1}^{l} N_{\Gamma_i,\text{cus}}(\lambda;\nu). \tag{10.10}$$

Furthermore, by (3.2) we have

$$\operatorname{vol}(X(K_f)) = \operatorname{vol}(A_G G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f) = \sum_{i=1}^{l}\operatorname{vol}(\Gamma_i\backslash\widetilde{X}), \tag{10.11}$$

where $\widetilde{X} = A_G\backslash G(\mathbb{R})/K_\infty$. Thus by (10.2) we obtain

$$\lim_{\lambda\to\infty}\sum_{i=1}^{l}\frac{N_{\Gamma_i,\text{cus}}(\lambda;\nu)}{\lambda^{n/2}} = \sum_{i=1}^{l}\frac{\dim(\nu)\operatorname{vol}(\Gamma_i\backslash\widetilde{X})}{(4\pi)^{n/2}\Gamma(n/2+1)}. \tag{10.12}$$

Now we argue as in [LV, Sect. 6.3] (1.10). By (1.10) we have

$$\limsup_{\lambda\to\infty}\frac{N_{\Gamma_i,\text{cus}}(\lambda;\nu)}{\lambda^{n/2}} \le \frac{\dim(\nu)\operatorname{vol}(\Gamma_i\backslash\widetilde{X})}{(4\pi)^{n/2}\Gamma(n/2+1)}$$

for $i=1,...,l$. Combined with (10.12) it follows that

$$N_{\Gamma_i,\text{cus}}(\lambda;\nu) = \frac{\dim(\nu)\operatorname{vol}(\Gamma_i\backslash\widetilde{X})}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)}\lambda^{n/2} + o(\lambda^{n/2}) \tag{10.13}$$

for $i=1,...,l$.

Now let $\Gamma\subset G(\mathbb{Q})$ be a congruence subgroup. By the definition of a congruence subgroup (see sect. 3) there exists a compact open subgroup $K_f\subset G(\mathbb{Q})$ such that $\Gamma = K_f\cap G(\mathbb{Q})$. Let Γ_i, $i=1,...,l$, be defined by (3.1). Then $\Gamma=\Gamma_1$ and the first part of Theorem 1.2 follows from (10.13).

To establish the second part of Theorem 1.2, we observe that by Lemma 10.1 we have

$$N_{\text{res}}^{K_f,\nu}(\lambda) = \sum_{i=1}^{l} N_{\Gamma_i,\text{res}}(\lambda,\nu) \tag{10.14}$$

for $\lambda \geq 0$. Since each summand on the right hand side is ≥ 0, and $\Gamma = \Gamma_1$, (7.22) yields

$$N_{\Gamma,\mathrm{res}}(\lambda, \nu) \leq C(1 + \lambda^{(n-1)/2}), \quad \lambda \geq 0. \tag{10.15}$$

This completes the proof of Theorem 1.2.

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