THE TRACE CLASS CONJECTURE IN THE THEORY OF AUTOMORPHIC FORMS. II.

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Abstract. Let G be the group of real points of a reductive algebraic $\mathbb{Q}$-group satisfying the assumptions made in [H, Chapter I] and let Γ be an arithmetic subgroup of G. Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$ and denote by R_Γ^d the restriction of R_Γ to the discrete subspace. In this paper we prove that for every integrable rapidly decreasing function f on G, the operator $R_\Gamma^d(f)$ is of the trace class.

0. Introduction

Let G be the group of real points of a reductive algebraic $\mathbb{Q}$-group $\mathbf{G}$ satisfying the assumptions made in [H, Chapter I] and let Γ be an arithmetic subgroup of G. We fix a maximal compact subgroup K of G. Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$ and let $L^2_d(\Gamma\backslash G)$ be the smallest closed subspace of $L^2(\Gamma\backslash G)$ which contains all irreducible subrepresentations of R_Γ. Let $\widehat{G}$ be the unitary dual of G. For each class $\pi \in G$, let $(\pi, \mathcal{H}_\pi)$ denote a representative of π. Then $L^2_d(\Gamma\backslash G)$ is an invariant subspace of $L^2(\Gamma\backslash G)$ which decomposes discretely as

$$L^2_d(\Gamma\backslash G) = \bigoplus_{\pi\in\widehat{G}} m_\Gamma(\pi)\mathcal{H}_\pi. \tag{0.1}$$

It is well known that the multiplicities $m_\Gamma(\pi)$ are all finite (see, e.g. [BG]). Denote by R_Γ^d the restriction of R_Γ to the invariant subspace $L^2_d(\Gamma\backslash G)$. Let $\mathcal{C}^1(G)$ be Harish–Chandra's Schwartz space of integrable rapidly decreasing functions on G (see section 1 for its definition). For each $f \in \mathcal{C}^1(G)$, one can define

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$$R^d_\Gamma(f) = \int_G f(g) R^d_\Gamma(g)\, dg$$

which is a bounded operator on $L^2_d(\Gamma\backslash G)$. In [Mu], we proved that for every K–finite function $f \in \mathcal{C}^1(G)$, $R^d_\Gamma(f)$ is a trace class operator. This is important for the application of the Arthur trace formula [A]. The main purpose of this paper is to remove the K–finiteness assumption, i.e. to prove the following theorem:

Theorem 0.1. *For every $f \in \mathcal{C}^1(G)$, the operator $R^d_\Gamma(f)$ is of the trace class.*

This result settles the trace class conjecture in full generality. A different proof has recently been announced by Ji [J]. For simplicity, we have restricted attention to arithmetic subgroups. However, Theorem 0.1 holds, of course, also for other lattices including non-arithmetic lattices in $\mathbb{R}$-rank one semisimple Lie groups (see the remark at the end of section 8).

As in [Mu], the proof of Theorem 0.1 is reduced to the estimation of the eigenvalue counting function for Bochner Laplace operators on the locally symmetric space $\Gamma\backslash G/K$. In fact, by passing to a normal subgroup of finite index if necessary, we can always assume that Γ acts without fixed points on the symmetric space $X = G/K$ so that $\Gamma\backslash X$ is a Riemannian manifold. Let $\sigma\colon K \to \mathrm{GL}(V_\sigma)$ be an irreducible unitary representation of K. Then σ defines a homogeneous vector bundle $\widetilde{E}_\sigma$ over X and an associated locally homogeneous vector bundle $E_\sigma = \Gamma\backslash\widetilde{E}_\sigma$ over $\Gamma\backslash X$. Let ∇ be the push down of the canonical connection on the homogeneous vector bundle $\widetilde{E}_\sigma$ and let $\Delta_\sigma = \nabla^*\nabla$ be the Bochner Laplace operator. This is a second order elliptic differential operator, acting in $C^\infty(\Gamma\backslash X, E_\sigma)$. Regarded as operator in $L^2(\Gamma\backslash X, E_\sigma)$ with domain $C^\infty_c(\Gamma\backslash X, E_\sigma)$, Δ_σ is essentially self–adjoint and nonnegative, and we will also denote its unique self–adjoint extension by Δ_σ. The spectrum of Δ_σ is the union of a pure point spectrum $\mathrm{Spec}_{pp}(\Delta_\sigma)\colon 0 \le \lambda_1 \le \lambda_2 \le \cdots$ and an absolutely continuous spectrum $\mathrm{Spec}_{ac}(\Delta_\sigma)$. Then Theorem 0.1 is a consequence of

Theorem 0.2. *Let $N_\sigma(\lambda)$ be the number of linearly independent eigenfunctions of Δ_σ with eigenvalues less than λ. Let $\Omega_K \in Z(\mathfrak{k}_\mathbb{C})$ be the Casimir element of K. Put $\lambda_\sigma = \sigma(\Omega_K)$ and $m = \dim G$. Then there exists $C > 0$ such that for every $\sigma \in \widehat{K}$, one has*

$$N_\sigma(\lambda) \le C(1 + \lambda_\sigma^{4m} + \lambda^{4m}), \quad \lambda \ge 0.$$

This is an improved version of the estimate of the counting function obtained in [Mu] in the sense that the dependence of the right hand side on $\sigma \in \widehat{K}$ has been made explicit. Given $\pi \in \widehat{G}$, let $\lambda_\pi = \pi(\Omega)$ be the eigenvalue of the Casimir operator Ω of G in the Gårding space $\mathcal{H}_\pi^\infty$ of $\mathcal{H}_\pi$. As immediate consequence of Theorem 0.2 we get

Corollary 0.3. *Let $N \geq 3 \dim G$. Then*

$$\sum_{\pi \in \widehat{G}} \sum_{\sigma \in \widehat{K}} m_\Gamma(\pi) \dim \left(\mathcal{H}_\pi \otimes V_\sigma\right)^K \left(1 + \lambda_\pi^2 + \lambda_\sigma^2\right)^{-N} < \infty.$$

This is easily seen to imply Theorem 0.1. In order to prove Theorem 0.2, we proceed essentially as in [Mu], keeping track of the dependence of the various constants on $\sigma \in \widehat{K}$. We now shall explain our method in more detail. First, we recall that the discrete spectrum has a further decomposition

$$L^2_d\left(\Gamma\backslash X, E_\sigma\right) = L^2_{cus}\left(\Gamma\backslash X, E_\sigma\right) \oplus L^2_{res}\left(\Gamma\backslash X, E_\sigma\right)$$

into the direct sum of the space of cusp forms $L^2_{cus}(\Gamma\backslash X, E_\sigma)$ and the residual subspace $L^2_{res}(\Gamma\backslash X, E_\sigma)$. We start by establishing the estimate of Theorem 0.2 for cuspidal eigenfunctions, i.e. we estimate the number $N_{cus,\sigma}(\lambda)$ of linearly independent eigensections of Δ_σ which satisfy the cuspidal condition and have eigenvalues less than λ. Donnelly proved in [D1] that

$$\overline{\lim}_{\lambda\to\infty} \frac{N_{cus,\sigma}(\lambda)}{\lambda^{n/2}} \leq (4\pi)^{-n/2} \frac{\mathrm{V}ol(\Gamma\backslash G/K)}{\Gamma(n/2+1)} \dim(V_\sigma)$$

This implies that $N_{cus,\sigma}(\lambda) \leq C(1+\lambda^{n/2})$ for $\lambda \geq 0$ and some constant $C > 0$.

We need, however, an explicit bound of C in terms of σ. To treat this problem, we modify Donnelly's method [D1]. More precisely, we use the cut–off Laplacian $\Delta_{\sigma,T}$ introduced in [Mu]. Here $T \in \mathfrak{a}_0$ is the truncation parameter and the domain of the quadratic form which defines $\Delta_{\sigma,T}$ consists of those functions whose constant terms along all proper $\mathbb{Q}$ parabolic subgroups vanish above height T. Then $\Delta_{\sigma,T}$ has pure discrete spectrum. Since each cuspidal eigenfunctions of Δ_σ is in the domain of $\Delta_{\sigma,T}$, the cuspidal spectrum is a part of the spectrum of $\Delta_{\sigma,T}$. Now the eigenvalue counting function for $\Delta_{\sigma,T}$ can be estimated using the method of Donnelly [D1] combined with the principle of semigroup domination [DL]. In particular, this leads to an upper bound for $N_{cus,\sigma}(\lambda)$ of the form claimed by Theorem 0.2. To deal with the residual discrete spectrum, we proceed as in [Mu], using Langlands' description of $L^2_{res}\left(\Gamma\backslash X, E_\sigma\right)$ (cf. [L1]). It follows from Langlands' theory of Eisenstein series that $L^2_{res}(\Gamma\backslash X, E_\sigma)$ is spanned by iterated residues of cuspidal Eisenstein series [L1, Chapter 7]. In particular, this gives a decomposition

$$L^2_{res}(\Gamma\backslash X, E_\sigma) = \bigoplus_{\mathcal{C}} L^2_{res}(\Gamma\backslash X, E_\sigma; \mathcal{C})$$

where $\mathcal{C}$ runs over association classes of proper rational parabolic subgroups P of G. One of the important consequences of Langlands' theory is that the bound on the higher rank residual spectrum can be reduced to bounds on the number of real poles of the intertwining operator $C(s;\sigma)$ for the cuspidal rank one Eisenstein series. Using the factorization of $\det C(s;\sigma)$ derived in [Mu, Section 6], this problem can be reduced to the estimation of the rank one residual spectrum. At various stages of this reduction process we have to estimate the dimension of spaces of cusp forms associated to the reductive parts of the $\mathbb{Q}$ parabolic subgroups that are involved. This is where the bounds on the number of cuspidal eigenfunctions of Δ_σ enter again.
Finally, the rank one residual discrete spectrum is related to some part of the spectrum of $\Delta_{\sigma,T}$. Applying the estimates obtained for the counting function of $\Delta_{\sigma,T}$, the proof of Theorem 0.2 is completed.
Theorem 0.1 implies a corresponding adèlic version which we discuss in the final section 8.

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1. Preliminaries

1.1. Let $\mathbf{G}$ be a reductive linear algebraic group defined over $\mathbb{Q}$ which satisfies the assumptions made in [H, Ch.I,§1]. Let $G = \mathbf{G}(\mathbb{R})$ and let Γ be an arithmetic subgroup of G, i.e. Γ is a subgroup of $\mathbf{G}(\mathbb{Q})$ which is commensurable with $\rho(\mathbf{G}) \cap \mathrm{GL}_n(\mathbb{Z})$ for any injective $\mathbb{Q}$-morphism $\rho\colon \mathbf{G} \to \mathrm{GL}_n$. We note that the assumptions that are imposed on $\mathbf{G}$ imply that the intersection of Γ with the center Z of G is cocompact and Γ is commensurable with the product of its intersections with Z and the derived group $\mathcal{D}G^0$ of the identity component G^0 of G.
We fix a maximal compact subgroup K of G and set $X = G/K$. By θ we shall denote the Cartan involution of G with respect to K. According to [BS, Proposition 1.6] there is a unique choice for θ which is algebraic.
Throughout this paper we shall assume that Γ acts without fixed points on X. This can be always achieved by passing to a normal subgroup of finite index.

1.2. The connected component of a Lie group G will be denoted by G^0. The Lie algebra of a Lie group G, H,... is denoted by the corresponding lowercase German letter $\mathfrak{g}$, $\mathfrak{h}$,... By $Z(\mathfrak{g})$ we shall denote the center of the universal enveloping algebra $U(\mathfrak{g})$ of the complexified Lie algebra $\mathfrak{g} \otimes \mathbb{C}$. In particular, $Z(\mathfrak{g})$ contains the Casimir element Ω_G (or simply Ω) of G with respect to an admissible real-valued bilinear form F on $\mathfrak{g}$ (cf. [BG]).

1.3. Let $\mathbf{P}$ be parabolic subgroup of $\mathbf{G}$ which is defined over $\mathbb{Q}$. Let $P = \mathbf{P}(\mathbb{R})$. We shall call P a $\mathbb{Q}$-parabolic subgroup of G. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{P}$ and $\mathbf{L}$ the Levi $\mathbb{Q}$-subgroup of $\mathbf{P}$ which is stable under θ. Then $\mathbf{P} = \mathbf{L} \cdot \mathbf{N}$. Let $\mathbf{M} = \cap_\chi \ker(\chi^2)$ where χ runs over the set $X_{\mathbb{Q}}(\mathbf{L})$ of rational characters of $\mathbf{L}$. Then $\mathbf{M}$ is a $\mathbb{Q}$-subgroup of $\mathbf{L}$. We also note that $\mathbf{M}$ is a reductive linear algebraic $\mathbb{Q}$-group which satisfies the same assumptions as $\mathbf{G}$. Furthermore, let $\mathbf{S} = \mathbf{L} \cap R_d\mathbf{P}$ be the greatest torus of the split radical $R_d\mathbf{P}$ of $\mathbf{P}$ stable under θ. Put $A_P = \mathbf{S}(\mathbb{R})^0$, $M_P = \mathbf{M}(\mathbb{R})$ and $N_P = \mathbf{N}(\mathbb{R})$. Then we get a decomposition of P as

$$P = N_P A_P M_P \tag{1.1}$$

(or just NAM, if there is no danger of confusion) which is called Langlands decomposition of P and A_P is called the special split component of P. The rank of P is defined to be the dimension of A_P. The Weyl group of A_P is $W(A_P) = N_G(A_P)/Z_G(A_P)$. Furthermore, we set

$$\Gamma_P = \Gamma \cap P \quad \text{and} \quad \Gamma_M = N\Gamma \cap M.$$

Then Γ_M is an arithmetic subgroup of M_P. We also put

$$K_M = K \cap M \quad \text{and} \quad X_M = M/K_M.$$

Observe that $K \cap M = K \cap P$.
We have $G = PK$. Therefore, any $x \in G$ has a decomposition

$$x = namk$$

with $k \in K$, $m \in M$, $a \in A$, $n \in N$. The factor a is uniquely determined by x. Set

$$H_P(x) = \log a.$$

The roots of (P, A) will be denoted by Φ_P and Ψ_P will denote the set of simple roots of (P, A). For $\beta \in \Phi_P$, let

$$\mathfrak{n}_\beta = \big\{ Y \in \mathfrak{g} \mid [H, Y] = \beta(H)Y, \quad H \in \mathfrak{a} \big\}.$$

Then

$$\mathfrak{n} = \bigoplus_{\beta \in \Phi_P} \mathfrak{n}_\beta.$$

As usual, let

$$\rho_P = \frac{1}{2} \sum_{\beta \in \Phi_P} \dim(\mathfrak{n}_\beta)\beta.$$

For a given subset $F \subset \Psi_P$ we denote by P_F the $\mathbb{Q}$-parabolic subgroup of G which is associated to F. Note that $P \subset P_F$ and the Lie algebra $\mathfrak{a}_F$ of the split component of P_F consists of all $H \in \mathfrak{a}$ such that $\alpha(H) = 0$ for all $\alpha \in F$.

1.4. Next we recall the definition of Harish-Chandra's Schwartz space $\mathcal{C}^1(G)$ of integrable rapidly decreasing functions on G (cf. [BM, p.162], [Wa2, p.230]). The definition involves two spherical functions on G, σ and Ξ. The function $\sigma(g)$ can be defined as the geodesic distance between the cosets K and gK in the symmetric space G/K. Let P be a minimal parabolic subgroup of G. Then Ξ is the zonal spherical function defined by

$$\Xi(g) = \int_K e^{-\rho_P(H_P(gk))} dk.$$

For a given element $Y \in U(\mathfrak{g})$, let $L(Y)$ (resp. $R(Y)$) denote the right (resp. left) invariant differential operators on G associated to Y. Then $\mathcal{C}^1(G)$ is the space of all functions $f \in C^\infty(G)$ such that for all $Y_1, Y_2 \in U(\mathfrak{g})$ and $r \geq 0$

$$\sup_{g \in G} (1 + \sigma(g))^r \Xi(g)^{-2} |L(Y_1)R(Y_2)f(g)| < \infty.$$

We note that $\mathcal{C}^1(G)$ is contained in $L^1(G)$. This follows from [W2, Proposition 8.3.7.5].

1.5. Let P be a $\mathbb{Q}$-parabolic subgroup of G with unipotent radical N_P. Let f be a complex-valued, locally bounded, measurable function on $\Gamma\backslash G$. The constant term f^P of f along P is defined as

$$f^P(x) = \int_{\Gamma \cap N_P \backslash N_P} f(nx)\, dn$$

where the measure dn is normalized by the condition that the volume of $\Gamma \cap N_P \backslash N_P$ equals 1. The subspace of $L^2(\Gamma\backslash G)$ consisting of all f satisfying $f^P = 0$ for all $\mathbb{Q}$-parabolic subgroups $P \neq G$ is denoted by $L^2_{cus}(\Gamma\backslash G)$. This is the space of cusp forms in $L^2(\Gamma\backslash G)$. It is well known that $L^2_{cus}(\Gamma\backslash G)$ is contained in $L^2_d(\Gamma\backslash G)$ (cf. [L1, p. 41], [H, Theorem 2]).

2. Reduction to the semisimple case

Let G and Γ be as in section 1.1. In this section we indicate how the proof of Theorem 0.1 can be reduced to the case of a connected semisimple Lie group with finite center. Let $\Gamma' \subset \Gamma$ be a subgroup of finite index. Then $L^2_d(\Gamma\backslash G)$ is a subspace of $L^2_d(\Gamma'\backslash G)$ and it suffices to prove Theorem 0.1 for Γ'. The converse is also true if we include finite-dimensional unitary representations of Γ. Suppose that $\Gamma' \subset \Gamma$ is a normal subgroup and set $H = \Gamma'\backslash\Gamma$. Let $\widehat{H}$ be the set of irreducible unitary representations of H. For any $(\rho, V_\rho) \in \widehat{H}$, let $L_\rho \to \Gamma\backslash X$ be the flat unitary vector bundle defined by $\Gamma \to H \to \mathrm{GL}(V_\rho)$. Then for any $\sigma \in \widehat{K}$, one has

$$L^2(\Gamma\backslash X, E_\sigma \otimes L_\rho) = \left(L^2(\Gamma'\backslash X, E_\sigma) \otimes V_\rho\right)^H.$$

Let ∇^ρ be the product of the canonical connection ∇ on E_σ and the flat connection on L_ρ. Let

$$\Delta_{\sigma,\rho} = (\nabla^\rho)^* \nabla^\rho$$

be the associated Bochner Laplace operator. Then $\Delta_{\sigma,\rho}$ has similar properties as Δ_σ. Let $N_{\Gamma,\sigma}(\lambda, \rho)$ be the eigenvalue counting function of $\Delta_{\sigma,\rho}$ and let $N_{\Gamma',\sigma}(\lambda)$ be the eigenvalue counting function of Δ_σ acting in $L^2(\Gamma'\backslash X, E_\sigma)$.

Proposition 2.1. *We have*

$$N_{\Gamma',\sigma}(\lambda) = \sum_{\rho \in \widehat{H}} \dim(V_\rho) N_{\Gamma,\sigma}(\lambda, \rho), \quad \lambda \geq 0.$$

Proof. The proof is an immediate consequence of the following analogue of the Peter-Weyl theorem:

$$L^2(\Gamma'\backslash X, E_\sigma) = \bigoplus_{\rho \in \widehat{H}} \left(V_\rho^* \otimes L^2(\Gamma\backslash X, E_\sigma \otimes L_\rho)\right).$$

□

All our arguments work equally well if we include a flat unitary vector bundle as above. Hence, in order to prove Theorem 0.1, we can always replace Γ by any discrete subgroup of G which is commensurable with Γ. First, this means that we can confine ourself to the case of a connected group $\mathbf{G}$. Furthermore, the lattice Γ is commensurable with the product of its intersections with the center Z of G and with the derived group $\mathcal{D}G^0$ of the identity component G^0 of G. Since the intersection of Γ with Z is cocompact in Z, it is sufficient to consider the intersection of Γ with $\mathcal{D}G^0$. Thus we can assume that $\mathbf{G}$ is a connected semisimple linear algebraic group and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic subgroup.

3. Heat kernels on the universal covering

In this section, we study the heat kernels of Bochner Laplace operators on G/K. Let (σ, V_σ) be an irreducible unitary representation of K on a complex vector space V_σ. Let $\widetilde{E}_\sigma = (G \times V_\sigma)/K$ be the associated homogeneous vector bundle over $X = G/K$. We equip $\widetilde{E}_\sigma$ with the G–invariant Hermitian fibre metric which is induced by the inner product in V_σ. Let $C^\infty(\widetilde{E}_\sigma)$, $C_c^\infty(\widetilde{E}_\sigma)$ and $L^2(\widetilde{E}_\sigma)$ denote the space of smooth sections, the space of compactly supported smooth sections and the Hilbert space of square integrable sections of $\widetilde{E}_\sigma$, respectively. Then we have

$$C^\infty(\widetilde{E}_\sigma) = (C^\infty(G) \otimes V_\sigma)^K, \quad L^2(\widetilde{E}_\sigma) = (L^2(G) \otimes V_\sigma)^K \tag{3.1}$$

and similarly for $C_c^\infty(\widetilde{E}_\sigma)$. Let $\widetilde{\nabla}$ be the canonical connection on $\widetilde{E}_\sigma$. Recall that $\widetilde{\nabla}$ is the unique G–invariant connection on $\widetilde{E}_\sigma$ such that if s is a C^∞ cross–section, $Y \in \mathfrak{p}$ and $\pi\colon G \to G/K$ is the canonical projection, then

$$\widetilde{\nabla}_{\pi_*(Y)}(s) = \frac{d}{dt} s(\exp(tY)K)$$

where π_* is the differential of π at the identity element $e \in G$. Let $\widetilde{\Delta}_\sigma = \widetilde{\nabla}^*\widetilde{\nabla}$ be the Bochner Laplace operator of $\widetilde{\nabla}$. Let R be the right regular representation of G on $C^\infty(G)$. Then with respect to the identification (3.1), we have

$$\widetilde{\nabla}^*\widetilde{\nabla} = -R(\Omega) \otimes \mathrm{Id} + \mathrm{Id} \otimes (\Omega_K) \tag{3.2}$$

where Ω and Ω_K are the Casimir elements of G and K, respectively [Mi]. Since σ is irreducible, there exists $\lambda_\sigma \in \mathbb{R}$ such that

$$\sigma(\Omega_K) = \lambda_\sigma \mathrm{Id}. \tag{3.3}$$

In fact $\lambda_\sigma \geq 0$. Thus by (3.2), we get

$$\widetilde{\Delta}_\sigma = -R(\Omega) \otimes \mathrm{Id} + \lambda_\sigma \mathrm{Id}. \tag{3.4}$$

The operator $\widetilde{\Delta}_\sigma\colon C_c^\infty(\widetilde{E}_\sigma) \to L^2(\widetilde{E}_\sigma)$ is essentially self–adjoint and we will also denote its unique self–adjoint extension by $\widetilde{\Delta}_\sigma$. Then $\widetilde{\Delta}_\sigma \geq 0$. Let $\exp(-t\widetilde{\Delta}_\sigma)$ be the heat semigroup. This is a smoothing operator in $L^2(\widetilde{E}_\sigma)$ which commutes with the representation of G on $L^2(\widetilde{E}_\sigma)$. Therefore, it is of the form

$$(e^{-t\widetilde{\Delta}_\sigma}\varphi)(g) = \int_G h_t(g^{-1}g_1)\varphi(g_1)dg_1, \quad \varphi \in (L^2(G)\otimes V_\sigma)^K,\ g \in G, \tag{3.5}$$

where $h_t \colon G \to \operatorname{End}(V_\sigma)$ is in L^2 and satisfies the covariance property

$$h_t(g) = \sigma(k)h_t(k^{-1}gk')\sigma(k')^{-1}, \quad \text{for } g \in G,\ k, k' \in K. \tag{3.6}$$

In order to get estimates for h_t, we relate it to the heat kernel of G with respect to the left invariant metric on G as in [BM]. Let $X_1, \cdots, X_p$ be an orthonormal basis for $\mathfrak{p}$ with respect to $B|\mathfrak{p}\times\mathfrak{p}$ and let $Y_1, \cdots, Y_k$ be an orthonormal basis for $\mathfrak{k}$ with respect to $-B|\mathfrak{k}\times\mathfrak{k}$. Then we have

$$\Omega = \sum_{i=1}^p X_i^2 - \sum_{i=1}^k Y_i^2 \quad \text{and} \quad \Omega_K = -\sum_{i=1}^k Y_i^2.$$

Let

$$P = -\Omega + 2\Omega_K = -\sum_{i=1}^p X_i^2 - \sum_{i=1}^k Y_i^2. \tag{3.7}$$

Then $R(P)$ is the Laplace operator Δ_G on G with respect to the left invariant metric. The heat semigroup $e^{-t\Delta_G}$ is represented by a smooth kernel p_t, i.e.

$$\left(e^{-t\Delta_G}f\right)(g) = \int_G p_t(g^{-1}g')f(g')dg', \quad f \in L^2(G),\ g \in G, \tag{3.8}$$

where $p_t \in C^\infty(G)\cap L^2(G)$. In fact, p_t belongs to $L^1(G)$ (see, e.g. [N]) so that (3.8) can be written as

$$e^{-t\Delta_G} = R(p_t).$$

Let

$$Q = \int_K R(k)\otimes\sigma(k)\ dk$$

be the orthogonal projection of $L^2(G)\otimes V_\sigma$ onto its K–invariant subspace $(L^2(G)\otimes V_\sigma)^K$. By (3.4) and (3.7) one has

$$\begin{aligned}
\widetilde{\Delta}_\sigma &= -Q(R(\Omega)\otimes \mathrm{Id})Q + \lambda_\sigma \mathrm{Id}_{L^2(\widetilde{E}_\sigma)} \\
&= Q(R(P)\otimes \mathrm{Id})Q - 2Q(R(\Omega_K)\otimes \mathrm{Id})Q + \lambda_\sigma \mathrm{Id}_{L^2(\widetilde{E}_\sigma)} \\
&= Q(\Delta_G\otimes \mathrm{Id})Q - \lambda_\sigma \mathrm{Id}_{L^2(\widetilde{E}_\sigma)}.
\end{aligned}$$

Hence, we get

$$e^{-t\widetilde{\Delta}_\sigma} = Q(e^{-t\Delta_G} \otimes \mathrm{Id})Q \cdot e^{t\lambda_\sigma}$$

which implies that

$$h_t(g) = e^{t\lambda_\sigma} \int_K \int_K p_t(k^{-1}gk')\sigma(kk'^{-1})\, dk\, dk'. \tag{3.9}$$

Since $p_t \in L^1(G)$, it follows that

$$h_t \in \left(L^1(G) \otimes \mathrm{End}(V_\sigma)\right)^{K\times K},$$

i.e. $h_t \in L^1(G) \otimes \mathrm{End}(V_\sigma)$ and it satisfies the covariance property (3.6). Thus, by (3.9), h_t can be estimated in terms of p_t. In order to estimate p_t, we use the following result of Borel [B1] (see also Borel und Harish–Chandra [BH]): every real semisimple Lie group has a discrete torsion free subgroup Γ with $\Gamma\backslash G$ compact.

On Riemannian manifolds admitting a properly discontinuous group of isometries with compact quotient, Donnelly [D2] has constructed a fundamental solution of the heat equation which is uniquely determined by the properties it satisfies. Thus, let $H(t,g,g')$ be the fundamental solution of the heat equation $\partial/\partial t + \Delta_G$ constructed by Donnelly [D2]. In addition to the usual properties of a fundamental solution of the heat equation, $H(t,g,g')$ satisfies

$$\left\| \frac{\partial^i}{\partial t^i} \nabla_g^j \nabla_{g'}^k H(t,g,g') \right\| \le C t^{-n/2-i-j-k} \exp\left(-\frac{d^2(g,g')}{4t}\right) \tag{3.10}$$

for $0 < t \le T$ and $0 \le i,j,k \le 1$. Here $d(g,g')$ is the geodesic distance of $g, g' \in G$ with respect to the left invariant metric on G. It follows from (3.10) that $H(t,g,g')$ represents $e^{-t\Delta_G}$ and therefore, it coincides with p_t, i.e.

$$p_t(g^{-1}g') = H(t,g,g').$$

This implies that p_t satisfies

$$\left\| \frac{\partial^i}{\partial t^i} \nabla^j p_t(g) \right\| \le C t^{-n/2-i-j} \exp\left(-\frac{d^2(g,1)}{4t}\right) \tag{3.11}$$

for $0 < t \le T$, $0 \le i,j \le 1$ and $g \in G$. By (3.9) and (3.11), we obtain

Theorem 3.1. *For $\sigma \in \widehat{K}$, let $h_t^\sigma(g)$ be the kernel of $\exp(-t\widetilde{\Delta}_\sigma)$. Let ρ be the geodesic distance on $X = G/K$. There exists $C > 0$ such that for every $\sigma \in \widehat{K}$ one has*

$$\left\| \frac{\partial^i}{\partial t^i} \nabla^j h_t^\sigma(g) \right\| \le C(1+\lambda_\sigma^i) e^{t\lambda_\sigma} t^{-n/2-i-j} \exp\left(-\frac{\rho^2(gK,K)}{4t}\right) \tag{3.12}$$

for $0 < t \leq T$, $0 \leq i, j \leq 1$ *and* $g \in G$.

Proof. First we note that one has

$$d(g, 1) \geq \rho(gK, K) \tag{3.13}$$

for any $g \in G$. Let $Y \in \mathfrak{p}$, $\| Y \| = 1$. Since σ is unitary, it follows from (3.9), (3.11) and (3.13) that for $0 \leq i, j \leq 1$:

$$\begin{aligned}
\left\| \frac{\partial^i}{\partial t^i} \nabla_Y^j h_t^\sigma(g) \right\| &\leq C \int_{K \times K} \left\| \frac{\partial^i}{\partial t^i} \nabla_Y^j p_t(k^{-1} g k') \right\| dk\, dk' \\
&= C \int_{K \times K} \left\| \frac{\partial^i}{\partial t^i} \left(\nabla^j_{Ad(k')Y} p_t \right) \left(k^{-1} g k' \right) \right\| dk\, dk' \\
&\leq C_1 (1 + \lambda_\sigma^i) e^{t\lambda_\sigma} t^{-n/2 - i - j} \exp\left(- \frac{\rho^2(gK, K)}{4t} \right).
\end{aligned}$$

□

One also can use the principal of semigroup domination as in [DL] to estimate h_t^σ. Let $H_\sigma(t, x, y)$ be the heat kernel for $\widetilde{\Delta}_\sigma$. Then

$$H_\sigma(t, gK, g'K) = h_t^\sigma(g^{-1} g'). \tag{3.14}$$

Let $|H_\sigma|$ be the pointwise norm of the homomorphism

$$H_\sigma(t, x, y) \in \operatorname{Hom}\left((\widetilde{E}_\sigma)_y, (\widetilde{E}_\sigma)_x \right).$$

Let $\epsilon > 0$ and set $|H_\sigma|_\epsilon = \left(|H_\sigma|^2 + \epsilon \right)^{1/2}$. Then $|H_\sigma|_\epsilon(t, x, y)$ is a smooth function of $x, y \in X$ and $t \in \mathbb{R}^+$. Since $(\partial/\partial t) H_\sigma = -\widetilde{\Delta}_\sigma H_\sigma$, the Bochner formula implies

$$\begin{aligned}
|H_\sigma|_\epsilon \frac{\partial}{\partial t} |H_\sigma|_\epsilon = \frac{1}{2} \frac{\partial}{\partial t} (|H_\sigma|_\epsilon)^2 &= \langle \frac{\partial}{\partial t} H_\sigma, H_\sigma \rangle \\
&= \langle -\widetilde{\Delta}_\sigma H_\sigma, H_\sigma \rangle = -|\nabla H_\sigma|^2 - \frac{1}{2} \Delta_0 |H_\sigma|^2 \\
&= -|\nabla H_\sigma|^2 - \frac{1}{2} \Delta_0 |H_\sigma|_\epsilon^2
\end{aligned}$$

where Δ_0 is the Laplacian on functions of X. Employing Bochner's formula again, we obtain

$$|H_\sigma|_\epsilon \frac{\partial}{\partial t} |H_\sigma|_\epsilon = |\nabla |H_\sigma|_\epsilon|^2 - |\nabla H_\sigma|^2 - |H_\sigma|_\epsilon \Delta_0 |H_\sigma|_\epsilon.$$

Since $|\nabla |H_\sigma|_\epsilon| \leq |\nabla H_\sigma|$, it follows that for each $\epsilon > 0$, one has

$$\left(\frac{\partial}{\partial t} + \Delta_0 \right) |H_\sigma|_\epsilon \leq 0.$$

Now let $\epsilon \to 0$. Then, considered as distribution, we get

$$\left(\frac{\partial}{\partial t} + \Delta_0\right) |H_\sigma| \le 0.$$

Using (3.12), one can extend the proof of Theorem 4.3 in [DL] to our setting which implies

Proposition 3.2. *For every $T > 0$ there exists $C > 0$ such that for every $\sigma \in \widehat{K}$, one has*

$$|H_\sigma(t,x,y)| \le C t^{-n/2} \exp\left(-\frac{\rho^2(x,y)}{4t}\right) \tag{3.15}$$

for $0 < t \le T$, $x, y \in X$.

Compared with (3.12), the improvement is that the factor $e^{t\lambda_\sigma}$ does not occur in (3.14). We claim that the same is true for derivatives of H_σ. This would improve our estimates.

4. The cuspidal spectrum

Let (σ, V_σ) be an irreducible unitary representation of K on V_σ and let

$$E_\sigma = \Gamma\backslash\widetilde{E}_\sigma \to \Gamma\backslash X$$

be the push down of the homogeneous vector bundle $\widetilde{E}_\sigma$ associated to (σ, V_σ). Then the space of C^∞-sections of E_σ is given by

$$C^\infty(\Gamma\backslash X, E_\sigma) = (C^\infty(\Gamma\backslash G) \otimes V_\sigma)^K,$$

and similarly, the space of L^2-sections of E_σ equals

$$L^2(\Gamma\backslash X, E_\sigma) = (L^2(\Gamma\backslash G) \otimes V_\sigma)^K. \tag{4.1}$$

Since $\widetilde{\Delta}_\sigma$ is G-invariant, it induces an elliptic operator Δ_σ acting on $C^\infty(\Gamma\backslash X, E_\sigma)$. If ∇ is the push down of the canonical connection $\widetilde{\nabla}$ on $\widetilde{E}_\sigma$, then

$$\Delta_\sigma = \nabla^*\nabla.$$

This operator is essentially self–adjoint on $L^2(\Gamma\backslash X, E_\sigma)$ and our goal is to study the point spectrum of Δ_σ. By (4.1), the subspace of $L^2(\Gamma\backslash X, E_\sigma)$ which is spanned by the L^2-eigensections of Δ_σ equals

$$L^2_d(\Gamma\backslash X, E_\sigma) = (L^2_d(\Gamma\backslash G) \otimes V_\sigma)^K.$$

Let

$$L^2_{cus}(\Gamma\backslash X, E_\sigma) = (L^2_{cus}(\Gamma\backslash G) \otimes V_\sigma)^K.$$

Then $L^2_{cus}(\Gamma\backslash X, E_\sigma)$ is a subspace of $L^2_d(\Gamma\backslash X, E_\sigma)$ and the corresponding part of the spectrum of Δ_σ is called the *cuspidal spectrum*. The main purpose of this section is to study the cuspidal spectrum. To this end, we use the so called cut–off Laplacian $\Delta_{\sigma,T}$ introduced in [Mu] where T is the truncation parameter.

First we recall the definition of $\Delta_{\sigma,T}$. Let $P_1, \dots, P_l$ be a set of representatives of the Γ–conjugacy classes of rank one $\mathbb{Q}$–parabolic subgroups of G. Let A_j be the special split component of P_j and let $\mathfrak{a}_j$ be the Lie algebra of A_j. Put

$$\mathfrak{a}_0 = \bigoplus_j \mathfrak{a}_j.$$

For each $\mathbb{Q}$–parabolic subgroup P of G with special split component A_P, there exists a well defined linear map

$$I_P \colon \mathfrak{a}_0 \to \mathfrak{a}_P$$

[OW], [Mu]. If $P = P_j$, $1 \le j \le l$, then I_{P_j} is the projection of $\mathfrak{a}_0$ onto the summand $\mathfrak{a}_j$. Let ρ_j be the half-sum of the positive roots of (P_j, A_j) and let $T_\rho \in \mathfrak{a}_0$ be the vector whose i–th component equals H_{ρ_i}– the image of ρ_i under the canonical identification of $\mathfrak{a}_i^*$ and $\mathfrak{a}_i$. Let $t \gg 0$ and put $T = tT_\rho$. Let $H^1(\Gamma\backslash X, E_\sigma)$ be the Sobolev space, i.e. the completion of $C_c^\infty(\Gamma\backslash X, E_\sigma)$ with respect to the norm

$$\| \varphi \|_{H^1}^2 = \int_{\Gamma\backslash X} \left(|\varphi(x)|^2 + |\nabla\varphi(x)|^2 \right) dx.$$

For T as above, let $H_T^1(\Gamma\backslash X, E_\sigma)$ be the subspace of $H^1(\Gamma\backslash X, E_\sigma)$ defined by

$$\begin{aligned} H_T^1(\Gamma\backslash X, E_\sigma) = \{\varphi \in H^1(\Gamma\backslash X, E_\sigma) \mid \varphi^{P_i}(a_i m_i) = 0 \\ \text{for } \log(a_i) > T_i,\ i = 1, \dots, m\}. \end{aligned}$$

Let $\mathcal{H}_T$ be the closure of $H_T^1(\Gamma\backslash X, E_\sigma)$ in $L^2(\Gamma\backslash X, E_\sigma)$. Consider the quadratic form

$$q(\varphi) = \| \nabla\varphi \|^2, \quad \varphi \in H_T^1(\Gamma\backslash X, E_\sigma).$$

Since $H_T^1(\Gamma\backslash X, E_\sigma)$ is a closed subspace of $H^1(\Gamma\backslash X, E_\sigma)$, the quadratic form q is represented by a nonnegative self–adjoint operator $\Delta_{\sigma,T}$. This is our cut–off Laplacian. If $G = SL(2, \mathbb{R})$ and $\sigma = \mathbf{1}$, this operator was first introduced by Lax and Phillips [LP]. In [Mu, Theorem 3.23] we proved that the spectrum of $\Delta_{\sigma,T}$ is discrete consisting of a sequence of eigenvalues $0 \le \lambda_1 < \lambda_2 < \cdots$ of finite multiplicity. Furthermore, if φ is a cuspidal eigenfunction of Δ_σ, i.e. $\varphi \in L_{cus}^2(\Gamma\backslash X, E_\sigma)$ and $\Delta_\sigma\varphi = \lambda\varphi$, then it follows immediately from the description of $\Delta_{\sigma,T}$ [Mu, p. 490] that φ is contained in the domain of $\Delta_{\sigma,T}$ and $\Delta_{\sigma,T}\varphi = \lambda\varphi$. Thus the cuspidal spectrum $\mathrm{Spec}_{cus}(\Delta)$ of Δ is contained in the spectrum of $\Delta_{\sigma,T}$. Let $\mathcal{E}(\lambda_i)$ be the eigenspace for the eigenvalue λ_i of $\Delta_{\sigma,T}$ and let

$$N_{T,\sigma}(\lambda) = \sum_{\lambda_i \leq \lambda} \dim \mathcal{E}(\lambda_i). \tag{4.2}$$

Then, in order to estimate the counting function $N_{cus,\sigma}(\lambda)$ of the cuspidal spectrum, it is sufficient to estimate $N_{T,\sigma}(\lambda)$. By [Mu, Theorem 3.23], we have

$$N_{T,\sigma}(\lambda) \leq C(1+\lambda^{n/2}), \quad \lambda \geq 0,$$

where $n = \dim X$. The main purpose of this section is to determine explicitly the dependence of C on $\sigma \in \widehat{K}$. To this end, we introduce a covering of $\Gamma\backslash X$ by special neighborhoods constructed in [BS] and use Neumann bracking as in [Mu]. We briefly recall the definition of these neighborhoods. For details we refer to [BS], [Z]. Let P be a $\mathbb{Q}$ parabolic subgroup of G with special split component A and Langlands decomposition $P = NAM$. Let $K_M = K \cap M$ and set $X_M = M/K_M$. There is a canonical isomorphism

$$\mu_P \colon N \times X_M \times A \simeq X \tag{4.3}$$

and a canonical action of P on $N \times X_M \times A$ which is defined as follows: Let $p = nam$ where $n \in N,\ a \in A$ and $m \in M$. Then

$$p \cdot (n_1, z, a_1) = (nman_1a^{-1}m^{-1}, mz, aa_1).$$

With this definition, μ_P commutes with the action of P. Set

$$e(P) = N \times X_M,\ \Gamma_P = \Gamma \cap P \quad \text{and} \quad e'(P) = \Gamma_P\backslash e(P).$$

There is a canonical fibration

$$\pi_P \colon e'(P) \to \Gamma_M\backslash X_M$$

with fibre $\Gamma \cap N\backslash N$. Given $\tau \in \mathfrak{a}$, put

$$A_\tau = \{a \in A \mid \alpha(\log a) > \alpha(\tau),\ \alpha \in \Psi_P\}.$$

Then for an open subset $\widetilde{Y}$ of $e(P)$ and $\tau \in \mathfrak{a}$, put

$$\widetilde{W}(\widetilde{Y}, \tau) = \mu_P(\widetilde{Y} \times A_\tau).$$

Now assume that $Y \subset e'(P)$ is an open subset and $\widetilde{Y}$ its inverse image under the canonical projection $e(P) \to e'(P)$. Then $\widetilde{W}(\widetilde{Y}, \tau)$ is Γ_P–invariant and we put

$$W(Y, \tau) = \Gamma_P\backslash\widetilde{W}(\widetilde{Y}, \tau).$$

Let $\pi\colon X \to \Gamma\backslash X$ be the canonical projection. Then, for $\tau \in \mathfrak{a}$ sufficiently large, we have $W(Y,\tau) = \pi(\widetilde{W}(\widetilde{Y},\tau))$ and μ_P induces an isomorphism

$$\mu'_P\colon Y \times A_\tau \simeq W(Y,\tau).$$

(see [Mu, Lemma 3.16]). An open set in $\Gamma\backslash X$ of the form $W(Y_P,\tau_P)$ with $Y_P \subset e'(P)$ a relatively compact open subset and $\tau_P \in \mathfrak{a}_P$ is called a special neighborhood. Note that for $P = G$, $W(Y_P,\tau_P) = Y_P$ is a relatively compact open subset of $\Gamma\backslash X$. Let $\omega_M \subset \Gamma_M\backslash X_M$ be a relatively compact open subset and put $Y_P = \pi_P^{-1}(\omega_M)$. Then $\widetilde{W}(\widetilde{Y}_P,\tau_P)$ is N_P–invariant and therefore, the cuspidal condition makes sense on $W(Y_P,\tau_P)$ (see [Mu, p. 493]). Hence, for any such $W = W(Y_P,\tau_P)$ we can define the space $L^2_{cusp}(W,E_\sigma)$ of cusp forms in $L^2(W,E_\sigma)$. To define this space, let $P_F = N_F A_F M_F$, $F \subset \Psi_P$, be a $\mathbb{Q}$ parabolic subgroup that belongs to P, i.e. $N_F \subset N$, $A_F \subset A$ and $M_F \supset M$. Given $\varphi \in L^2(W,E_\sigma)$, the function

$$\varphi^{P_F}(x) = \int_{\Gamma\cap N_F\backslash N_F} \varphi(nx)\, dn$$

is well defined and belongs to $L^2(W,E_\sigma)$. Then

$$L^2_{cus}(W,E_\sigma) = \left\{\varphi \in L^2(W,E_\sigma) \mid \varphi^{P_F} = 0 \text{ for all } F \subset \Psi_P\right\}.$$

Furthermore, let $H^1(W,E_\sigma)$ be the Sobolev space of W and let

$$H^1_{cus}(W,E_\sigma) = L^2_{cus}(W,E_\sigma) \cap H^1(W,E_\sigma).$$

In [Mu, Proposition 3.18], we constructed a finite covering of $\Gamma\backslash X$ by special neighborhoods which is adapted to our purpose. Namely, let $Q_i = N_i A_i M_i$ $(1 \le i \le q)$ be a set of representatives of the Γ–conjugacy classes of $\mathbb{Q}$ parabolic subgroups of G. Then there exist $\tau_i \in \mathfrak{a}_i$ and relatively compact open subsets $\omega_i \subset \Gamma_{M_i}\backslash X_{M_i}$, $1 \le i \le q$, such that the following conditions hold:

1) Let $Y_i = \pi_{Q_i}^{-1}(\omega_i)$. Then the canonical projection $\Gamma_P\backslash X \to \Gamma\backslash X$ induces an inclusion $W(Y_i,\tau_i) \to \Gamma\backslash X$ and

$$\{W(Y_i,\tau_i) \mid i = 1,\dots,q\}$$

is a covering of $\Gamma\backslash X$.

2) Set $W_i = W(Y_i,\tau_i)$, $i = 1,\dots,q$. The map

$$\varphi \longmapsto (\varphi|W_1,\dots,\varphi|W_q)$$

defines an embedding

$$H^1_T(\Gamma\backslash X, E_\sigma) \subset \bigoplus_{i=1}^{q} H^1_{cus}(W_i, E_\sigma). \tag{4.4}$$

In general, W_i is a manifold with corners. In order to be able to apply Donnelly's method, we have to smooth out corners. This can be done in a way that preserves the invariance under N_i, i.e. the resulting manifold W'_i satisfies $N_i W'_i = W'_i$ which gurantees that the cuspidal condition still makes sense for W'_i.
Now let W be any of the smooth manifolds W'_i, $i = 1, \dots, q$. Let $\Delta_{W,\sigma}$ be the self–adjoint operator in $L^2(W, E_\sigma)$ which is defined by the quadratic form $\varphi \longmapsto \| \nabla\varphi \|^2$ on the Sobolev space $H^1(W, E_\sigma)$. Note that this is the self–adjoint extension of the Bochner Laplace operator $\nabla^*\nabla$ which satisfies Neumann boundary conditions where a section s of E_σ over W is said to satisfy Neumann boundary conditions if $\nabla_\nu s(x) = 0$ for $x \in \partial W$. Here ν is the unit normal vector field along ∂W.
Then $L^2_{cus}(W, E_\sigma)$ which is defined as above, is an invariant subspace for $\Delta_{W,\sigma}$ and it follows as in [D1] that $\Delta_{W,\sigma}$ has pure discrete spectrum in $L^2_{cus}(W, E_\sigma)$. Let $N_{W,\sigma}(\lambda)$ be the number of linearly independent cuspidal eigenfunctions of $\Delta_{W,\sigma}$ with eigenvalues less than λ. Using (4.4), it follows by Neumann bracketing that there exists $C > 0$ such that for all $\sigma \in \widehat{K}$, one has

$$N_{T,\sigma}(\lambda) \le C \sum_{i=1}^{q} N_{W'_i,\sigma}(\lambda), \quad \lambda \ge 0. \tag{4.5}$$

It remains to estimate $N_{W,\sigma}(\lambda)$. We have to distinguish two cases. First suppose that W is a relatively compact domain in $\Gamma\backslash X$ with smooth boundary. This corresponds to a special neighborhood which is defined with respect to $Q_i = G$. Then $N_i = \{0\}$ so that no cuspidal condition has to be imposed. By Corallary 4.5 of [DL] one has

$$\mathrm{Tr}(e^{-t\Delta_{W,\sigma}}) \le \dim(V_\sigma)(1 + C_1 \mathrm{Vol}(W) t^{-n/2})$$

for $0 < t \le T$ and $C_1 > 0$ independent of σ. Let $0 \le \mu_1 \le \mu_2 \le \cdots$ be the eigenvalues of $\Delta_{W,\sigma}$. Then we can estimate the counting function as follows

$$\begin{aligned} N_{W,\sigma}(\lambda) &= \sum_{\mu_i \le \lambda} 1 \le e \sum_{\mu_i \le \lambda} e^{-\mu_i/\lambda} \le e\,\mathrm{Tr}(e^{-\Delta_{W,\sigma}/\lambda}) \\ &\le e \dim(V_\sigma)(1 + C_2 \lambda^{n/2}) \end{aligned}$$

Let $\lambda_\sigma = \sigma(\Omega_K)$ be the eigenvalue of the Casimir operator Ω_K in V_σ. By Weyl's dimension formula there exists $C_3 > 0$ such that

$$\dim(V_\sigma) \leq C_3(1 + \lambda_\sigma^l) \tag{4.6}$$

for all $\sigma \in \widehat{K}$ where $l \leq \frac{1}{2}\dim K$. Thus we proved

Proposition 4.1. *Let $W \subset \Gamma\backslash X$ be a compact submanifold having a smooth boundary. Let (σ, V_σ) be an irreducible unitary representation of K with Casimir eigenvalue λ_σ. Let $\Delta_{W,\sigma}$ be the Bochner Laplace operator $\nabla^*\nabla$ on $C^\infty(W, E_\sigma|W)$ with Neumann boundary conditions and let $N_{W,\sigma}(\lambda)$ be the eigenvalue counting function for $\Delta_{W,\sigma}$. There exists $C > 0$ such that for every $\sigma \in \widehat{K}$ one has*

$$N_{W,\sigma}(\lambda) \leq C(1 + \lambda_\sigma^p + \lambda^n),\ \lambda \geq 0, \tag{4.7}$$

where $n = \dim X$ and $p = \dim K$.

Next we have to consider one of the noncompact special neighborhoods ${W'}_i$. To estimate the counting function in this case, we proceed as in [D1] using Theorem 3.1 and keeping track of the constants. We provide some details. Let P be a fixed proper $\mathbb{Q}$ parabolic subgroup of G with special split component A_P and Langlands decomposition $P = N_P A_P M_P$. Let $W = W(Y_P, \tau_P)$ be a special neighborhood with respect to P. Thus $\tau_P \in \mathfrak{a}_P$ and $Y_P = \pi_P^{-1}(\omega_P)$ where $\omega_P \subset \Gamma_{M_P}\backslash X_{M_P}$ is a relatively compact open subset. Then $W \subset \Gamma_P\backslash X$.

First we construct an interior parametrix for the Neumann heat kernel on W' satisfying the cuspidal conditions. The interior parametrix is obtained by restricting the corresponding heat kernel on $\Gamma_P\backslash X$ to W. Let $H_\sigma(t,x,y)$ be the heat kernel for $\widetilde{\Delta}_\sigma$ on X. Then the fundamental solution $F_\sigma(t,x,y)$ for Δ_σ on $\Gamma_P\backslash X$ is given by

$$F_\sigma(t,x,y) = \sum_{\gamma\in\Gamma_P} H_\sigma(t,x,\gamma y). \tag{4.8}$$

By Proposition 3.2, we have

$$\sum_{\gamma\in\Gamma_P} |H_\sigma(t,x,\gamma y)| \leq Ct^{-n/2} \sum_{\gamma\in\Gamma_P} e^{-\rho^2(x,\gamma y)/4t}$$

By [Wa1, Theorem 9] this series is uniformly convergent on compact subsets. Using Theorem 3.1 and (3.14), the same holds for all derivatives up to order 1. Thus, $F_\sigma(t,x,y)$ is a smooth kernel of $(t,x,y) \in \mathbb{R}^+ \times X \times X$ which represents $\exp(-t\Delta_\sigma)$ on $\Gamma_P\backslash X$. Now we have to modify $F_\sigma(t,x,y)$ by projecting onto the cuspidal subspace. Let Ψ be the set of simple roots of (P,A). Given any subset $\theta \subset \Psi$, let

$P_\theta = N_\theta A_\theta M_\theta$ be the $\mathbb{Q}$ parabolic subgroup belonging to P which is defined by θ. A smooth section of E_σ over $\Gamma_P\backslash X$ may be identified with a C^∞–function $\varphi\colon \Gamma_P\backslash G \to V_\sigma$ satisfying

$$\varphi(gk) = \sigma(k)^{-1}\varphi(g), \quad g \in G,\ k \in K.$$

Set

$$(T(P,\theta)\varphi)(g) = \int_{\Gamma\cap N_\theta\backslash N_\theta} \varphi(ng)\, dn.$$

Then $T(P,\theta)\varphi$ is left invariant under $\Gamma_P \cdot N_\theta$. In particular, $T(P,\theta)$ is a projection on $C^\infty(\Gamma_P\backslash X, E_\sigma)$. For any subset $\theta \subset \Psi$ set

$$\mathcal{L}_\theta = T(P, \Psi - \theta).$$

and let

$$\begin{aligned}\overline{F}_\sigma(t,x,y) &= \left(\prod_{\alpha\in\psi}(1-\mathcal{L}_\alpha(y))\right) F_\sigma(t,x,y)\\ &= \sum_{\theta\subset\Psi}(-1)^{|\theta|}\mathcal{L}_\theta(y)F_\sigma(t,x,y).\end{aligned} \tag{4.9}$$

Here the projectors $\mathcal{L}_\theta(y)$ act on the third argument y of $F_\sigma(t,x,y)$. By Lemma 5.1 in [D1] together with the isometry invariance of $H_\sigma(t,x,y)$ it follows that

$$\mathcal{L}_\theta(x)\overline{F}_\sigma(t,x,y) = 0, \quad \theta \subset \Psi.$$

Thus $\overline{F}_\sigma(t,x,y)$ is the fundamental solution for Δ_σ on $\Gamma_P\backslash X$ satisfying the cuspidal conditions. The heat kernel $\overline{F}_\sigma(t,x,y)$ can be estimated as in [D1]. Let $\log\colon A_P \to \mathfrak{a}_P$ be the inverse of the exponential map. Let $\mathrm{pr}_{A_P}\colon N_P \times X_{M_P} \times A_P \to A_P$ be the projection and let $r_P\colon X \to \mathfrak{a}_P$ be defined by

$$r_P(x) = \log\left(\mathrm{pr}_{A_P}(\mu_P^{-1}(x))\right) \tag{4.10}$$

where μ_P is the isomorphism (4.3). Furthermore, for $x, y \in \Gamma_P\backslash X$ we set

$$\rho_N(x,y) = \inf_{n_1,n_2\in N} \rho(xn_1, yn_2).$$

Thus $\rho_N(x,y)$ is the distance of the N-orbits through x and y, respectively. In other words, if we denote by $\pi : \Gamma_P\backslash X \to \Gamma_P N\backslash X$ the canonical projection, then $\rho_N(x,y)$ is the geodesic distance of $\pi(x)$, $\pi(y)$. Then, with a slight modification, the analogous statement to Lemma 5.5 in [D1] is

Lemma 4.2. *For any fixed simple root α, let*

$$F_{\sigma,\alpha}(t,x,y) = (1-\mathcal{L}_\alpha(y))\, F_\sigma(t,x,y).$$

Let $T>0$ be fixed. There exists $C>0$ such that for every $\sigma\in\widehat{K}$ and $0<t\le T$ one has

$$\begin{aligned}|F_{\sigma,\alpha}(t,x,y)| \le\, & C(1+\lambda_\sigma)e^{t\lambda_\sigma}t^{-n/2}\min\left(e^{-\alpha(r(x))},\ e^{-\alpha(r(y))}\right)\\ &\times \max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))},\ \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right).\\ &\times \max_{\substack{\nu\in\Psi\\ n_\nu=1}}\max\left(\nu(r(x)),\nu(r(y))\right)\exp\left(-\frac{\rho_N^2(x,y)}{32t}\right),\end{aligned}$$

uniformly for x,y in any Siegel domain $\mathfrak{S}_c(P)$. Here $\lambda_\sigma=\sigma(\Omega_K)$ and $r=r_P$. Moreover, n_β is the dimension of the root space $\mathfrak{n}_\beta$ and n_ν is the multiplicity of the simple root ν.

Proof. We identify $x,y\in\Gamma_P\backslash X$ with points $x,y\in X$ which realize the geodesic distance from x to y. Let $\psi=\Psi-\{\alpha\}$. Then by definition

$$F_{\sigma,\alpha}(t,x,y)=\sum_{\gamma\in\Gamma_P}\left\{H_\sigma(t,x,\gamma y)-\int_{\Gamma\cap N_\psi\backslash N_\psi} H_\sigma(t,nx,\gamma y)\,dn\right\}.$$

where we have identified $\Gamma\cap N_\psi\backslash N_\psi$ with a fundamental domain of $\Gamma\cap N_\psi$ in N_ψ. Using (3.15), we can estimate the term coming from $\gamma=1$. This implies

$$\left|F_{\sigma,\alpha}(t,x,y)-\sum_{\substack{\gamma\in\Gamma_P\\ \gamma\ne 1}}\left\{H_\sigma(t,x,\gamma y)-\int_{\Gamma\cap N_\psi\backslash N_\psi} H_\sigma(t,nx,\gamma y)\,dn\right\}\right| \le Ct^{-n/2}e^{-\rho^2(x,y)/4t}.$$

Now use the mean value theorem combined with (3.12) and (3.14) to estimate the terms with $\gamma\ne 1$. Then we get

$$\begin{aligned}\left|F_{\sigma,\alpha}(t,x,y)\right| \le\, & C_2(1+\lambda_\sigma)e^{t\lambda_\sigma}t^{-n/2}\sum_{\substack{\gamma\in\Gamma_P\\ \gamma\ne 1}}\left(1+\frac{\rho_N(x,\gamma y)}{t}\right)e^{-\rho_N^2(x,\gamma y)/8t}\\ &\times \min(\operatorname{diam}(x),\operatorname{diam}(y)) + C_1 t^{-n/2}e^{-\rho_N^2(x,y)/4t}.\end{aligned}$$

Here $\operatorname{diam}(x)$ is the diameter of $\Gamma\cap N_\psi\backslash N_\psi$ at x. As in [D1], this implies the lemma.

□

Since the projections $\mathcal{L}_\beta(y)$ are L^∞–bounded, it follows from (4.9) that

$$\begin{aligned} |\overline{F}_\sigma(t,x,y)| \leq C(1+\lambda_\sigma)e^{t\lambda_\sigma}t^{-n/2}\min_{\alpha\in\Psi}\min\left(e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}\right) \\ \times \max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right) e^{-\rho_N^2(x,y)/32t}. \end{aligned} \tag{4.11}$$

uniformly for $x,y\in\mathfrak{S}_c(P)$, $0<t\leq T$, and $\sigma\in\widehat{K}$.
Now let $W=W(Y_\rho,\tau_\rho)$ be a special neighborhood as above and let W' be obtained from W by smoothing out corners such that $NW'=W'$. We use $\overline{F}_\sigma(t,x,y)$ as an interior parametrix and apply the single layer potential construction to modify $\overline{F}_\sigma(t,x,y)$ as in [D1, section 6], [F, Chapt. 5], yielding a fundamental solution $Q_\sigma(t,x,y)$ for the heat equation $\frac{\partial}{\partial t}+\Delta_\sigma$ on W' which satisfies Neumann boundary conditions and cuspidal interior conditions. The kernel $Q_\sigma(t,x,y)$ can be estimated similarly to (4.11):

$$\begin{aligned} \left|Q_\sigma(t,x,y)\right| \leq C_1(1+\lambda_\sigma)e^{t\lambda_\sigma}t^{-n/2}\min_{\alpha\in\Psi}\min\left(e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}\right) \\ \times \max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right)\exp\left(-\frac{C_2\rho_N^2(x,y)}{t}\right) \\ \times\left(1+C_3\exp\left(-C_4(\rho^2(x,\partial W')+\rho^2(y,\partial W'))/t\right)\right) \end{aligned} \tag{4.12}$$

for $0<t\leq T$. However, since ρ_N and not ρ occurs in the estimate (4.12), it is not obvious that $Q_\sigma(t,x,y)$ satisfies Neumann boundary conditions. In order to verify that it does satisfy Neumann boundary conditions, we proceed as follows. First we take the full heat kernel $F_\sigma(t,x,y)$ and employ the single layer potential construction to get the fundamental solution $F_{\sigma,N}(t,x,y)$ for the heat equation $\frac{\partial}{\partial t}+\Delta_\sigma$ on W' which satisfies Neumann boundary conditions. Let $\overline{F}_{\sigma,N}(t,x,y)$ be defined by (4.9) with F_σ replaced by $F_{\sigma,N}$. Since the boundary of W' fibres through $N\cap\Gamma\backslash N$, it follows from the iterative construction of the boundary correction term that

$$Q_\sigma(t,x,y)=\overline{F}_{\sigma,N}(t,x,y)$$

which shows that $Q_\sigma(t,x,y)$ indeed satisfies Neumann boundary conditions. This implies

Theorem 4.3. *Let W' be as above. Given an irreducible unitary representation (σ,V_σ) of K, let $\Delta_{W',\sigma}$ be the Bochner Laplace operator with Neumann boundary conditions and cuspidal interior conditions on W'.*

Then $\exp(-t\Delta_{W',\sigma})$ *is a trace class operator for* $t > 0$ *and there exists* $C > 0$ *such that for every* (σ, V_σ) *we have*

$$\operatorname{Tr}(e^{-t\Delta_{W',\sigma}}) \le C(1+\lambda_\sigma)e^{t\lambda_\sigma}\dim(V_\sigma)(1+t^{-n/2})$$

for $0 < t \le T$.

Proof. By construction, $Q_\sigma(t,x,y)$ represents the heat semigroup $\exp(-t\Delta_{W',\sigma})$. Hence, using (4.12) we get

$$\begin{aligned}
\operatorname{Tr}(e^{-t\Delta_{W',\sigma}}) &= \int_{W'} \operatorname{Tr}\, Q_\sigma(t,x,x)dx \le \dim(V_\sigma)\int_{W'} \left|Q_\sigma(t,x,x)\right| dx \\
&\le C_5(1+\lambda_\sigma)e^{t\lambda_\sigma}\dim(V_\sigma)t^{-n/2}\int_{r(W')} \min_{\alpha\in\Psi}(e^{-\alpha(r)/2})\, dr \\
&\le C(1+\lambda_\sigma)\dim(V_\sigma)e^{t\lambda_\sigma}t^{-n/2}
\end{aligned}$$

for $0 < t \le T$.

□

Corollary 4.4. *Let* $N_{W',\sigma}(\lambda)$ *be the counting function for* $\Delta_{W',\sigma}$. *There exists* $C > 0$ *such that*

$$N_{W',\sigma}(\lambda) \le C(1+\lambda_\sigma^{m/2+1}+\lambda^{m/2+1}), \quad \sigma \in \widehat{K},$$

where $\lambda_\sigma = \sigma(\Omega_K)$ *and* $m = \dim G$.

Proof. By Theorem 4.3, we have

$$\operatorname{Tr}\left(e^{-t(\Delta_{W',\sigma}+\lambda_\sigma)}\right) \le C(1+\lambda_\sigma)\dim(V_\sigma)(1+t^{-n/2}) \tag{4.13}$$

for $0 < t \le 1$. Let $0 \le \mu_1 \le \mu_2 \le \cdots$ be the eigenvalues of $\Delta_{W',\sigma}$ where each eigenvalue is repeated according to its multiplicity. Using (4.13) together with (4.6), we get

$$\begin{aligned}
N_{W',\sigma}(\lambda) &= \sum_{\mu_i\le\lambda} 1 = \sum_{\mu_i+\lambda_\sigma\le\lambda+\lambda_\sigma} 1 \\
&\le e\sum_{\mu_i+\lambda_\sigma\le\lambda+\lambda_\sigma} e^{-(\mu_i+\lambda_\sigma)/(\lambda+\lambda_\sigma)} \\
&\le e\operatorname{Tr}\left(e^{-(\Delta_{W',\sigma}+\lambda_\sigma)/(\lambda+\lambda_\sigma)}\right) \\
&\le C_1(1+\lambda_\sigma)\dim(V_\sigma)(1+(\lambda+\lambda_\sigma)^{n/2}) \\
&\le C_2(1+\lambda_\sigma^{m/2+1}+\lambda^{m/2+1})
\end{aligned}$$

for $\lambda \ge 0$ and $m = \dim G$.

□

Combining (4.5), Proposition 4.1 and Corollary 4.4, we obtain

Theorem 4.5. *Let $N_{T,\sigma}(\lambda)$ be the eigenvalue counting function for the cut–off Laplacian $\Delta_{\sigma,T}$. There exists $C > 0$ such that for every $\sigma \in \widehat{K}$ one has*

$$N_{T,\sigma}(\lambda) \leq C(1 + \lambda_\sigma^{m/2+1} + \lambda^{m/2+1}), \quad \lambda \geq 0,$$

where $\lambda_\sigma = \sigma(\Omega_K)$ and $m = \dim G$.

Since the cuspidal spectrum $\mathrm{Spec}_{cus}(\Delta_\sigma)$ of Δ_σ is contained in the spectrum of $\Delta_{\sigma,T}$ this implies

Corollary 4.6. *Given any irreducible unitary representation (σ, V_σ) of K, let $N_{cus,\sigma}(\lambda)$ be the number of eigenvalues, counted with multiplicity, less than λ of the Bochner Laplace operator $\Delta_\sigma = \nabla^*\nabla$, acting in the space of cusp forms $L^2_{cus}(\Gamma\backslash G, \sigma)$. There exists $C > 0$ such that for every $\sigma \in \widehat{K}$:*

$$N_{cus,\sigma}(\lambda) \leq C(1 + \lambda_\sigma^{m/2+1} + \lambda^{m/2+1}), \quad \lambda \geq 0,$$

where $m = \dim G$.

5. The residual spectrum

Let $L^2_{res}(\Gamma\backslash G)$ be the orthogonal complement of the space cusp forms $L^2_{cus}(\Gamma\backslash G)$ in $L^2_d(\Gamma\backslash G)$. This is the residual subspace. Thus the discrete spectrum has a further decomposition which is given by

$$L^2_d(\Gamma\backslash G) = L^2_{cus}(\Gamma\backslash G) \oplus L^2_{res}(\Gamma\backslash G). \tag{5.1}$$

Let (σ, V_σ) be any irreducible unitary representation of K. Then it follows that $L^2_d(\Gamma\backslash X, E_\sigma)$ has a corresponding decomposition. Let

$$L^2_{res}(\Gamma\backslash G, \sigma) = (L^2_{res}(\Gamma\backslash G) \otimes V_\sigma)^K. \tag{5.2}$$

This space can be identified with a subspace of $L^2_d(\Gamma\backslash X, E_\sigma)$ which we denote by $L^2_{res}(\Gamma\backslash X, E_\sigma)$. Thus (5.1) induces the decomposition

$$L^2_d(\Gamma\backslash X, E_\sigma) = L^2_{cus}(\Gamma\backslash X, E_\sigma) \oplus L^2_{res}(\Gamma\backslash X, E_\sigma). \tag{5.3}$$

To complete the proof of Theorem 0.2, we have to estimate the number of eigenvalues of Δ_σ in the residual subspace $L^2_{res}(\Gamma\backslash X, E_\sigma)$. To this end we use Langlands' description of the residual spectrum [L1] and proceed as in [Mu]. We may simply repeat the arguments given in [Mu] keeping track of the constants at each step. For the convenience of the reader we include some details.

To begin with we recall some facts from Langlands' theory of Eisenstein series [L1], [H]. Let P be any $\mathbb{Q}$–parabolic subgroup of G with special split component A_P and Langlands decomposition $P = N_P A_P M_P$. Let

$K_M = K \cap M$ and let $\sigma_M\colon K_M \to GL(V_\sigma)$ be the restriction of σ to K_M. Given a character $\chi\colon Z(\mathfrak{m}) \to \mathbb{C}$, set

$$\begin{aligned} &L^2_{cus}(\Gamma_M\backslash M, \sigma, \chi) = \\ &\left\{\varphi \in L^2_{cus}(\Gamma_M\backslash M, \sigma_M) \mid D\varphi = \chi(D)\varphi \text{ for all } D \in Z(\mathfrak{m})\right\}. \end{aligned} \tag{5.4}$$

This is a finite dimensional Hilbert space of automorphic forms. Let $\mathfrak{a}^*$ be the dual of the Lie algebra $\mathfrak{a}$ and let

$$(\mathfrak{a}^*)^+ = \{\lambda \in \mathfrak{a}^* \mid \langle\lambda, \alpha\rangle > 0 \text{ for all } \alpha \in \Psi_P\}.$$

Let $\Phi \in L^2_{cus}(\Gamma_M\backslash M, \sigma, \chi)$ be given. Then we extend Φ to a function $\Phi\colon \Gamma_P \cdot N_P\backslash G \to V_\sigma$ by

$$\Phi(namk) = \sigma(k)^{-1}\Phi(m). \tag{5.5}$$

Let $\Lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and suppose that $\mathrm{Re}(\Lambda) \in \rho_P + (\mathfrak{a}^*)^+$. Then the Eisenstein series attached to Φ is defined as

$$E(P|A, \Phi, \Lambda, g) = \sum_{\gamma\in\Gamma_P\backslash\Gamma} e^{(\Lambda+\rho_P)(H_P(\gamma g))}\Phi(\gamma g).$$

This series is absolutely and uniformly convergent on compact subsets of $(\rho_P + (\mathfrak{a}^*)^+ + \sqrt{-1}\mathfrak{a}^*) \times G$. As a function of Λ, it can be continued to a meromorphic function of $\Lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and all singularities of $E(P|A, \Phi, \Lambda)$ lie along hyperplanes.

Let P_i, $i = 1, 2$, be two $\mathbb{Q}$ parabolic subgroups of G with special split components A_i and Langlands decomposition $P_i = N_iA_iM_i$. Then (P_1, A_1) and (P_2, A_2) are said to be associate if there exists $g \in G(\mathbb{Q})$ such that $Ad(g)\mathfrak{a}_1 = \mathfrak{a}_2$. The set of all such isomorphisms is denoted by $W(\mathfrak{a}_1, \mathfrak{a}_2)$. Set $W(\mathfrak{a}) = W(\mathfrak{a}, \mathfrak{a})$.

Let $\chi \in \widehat{Z}(\mathfrak{m}_1)$ and $\Phi \in L^2_{cus}(\Gamma_{M_1}\backslash M_1, \sigma, \chi)$. We denote the constant term of $E(P_1|A_1, \Phi, \Lambda)$ along P_2 by $E^{P_2}(P_1|A_1, \Phi, \Lambda)$. If (P_1, A_1) and (P_2, A_2) are not associate and $\mathrm{rank} P_2 \geq \mathrm{rank} P_1$, then

$$E^{P_2}(P_1|A_1, \Phi, \Lambda) = 0.$$

If (P_1, A_1) and (P_2, A_2) are associated, then

$$E^{P_2}(P_1|A_1, \Phi, \Lambda, x) = \sum_{w\in W(\mathfrak{a}_1,\mathfrak{a}_2)} e^{(w\Lambda+\rho_2)(H_2(x))}(c_{P_2|P_1}(w, \Lambda)\Phi)(x) \tag{5.6}$$

where $\rho_2 = \rho_{P_2}$, $H_2 = H_{P_2}$ and

$$c_{P_2|P_1}(w, \Lambda)\colon L^2_{cus}(\Gamma_{M_1}\backslash M_1, \sigma, \chi) \to L^2_{cus}(\Gamma_{M_2}\backslash M_2, \sigma, {}^w\chi)$$

is a linear operator which is a meromorphic function of $\Lambda \in \mathfrak{a}^*_{\mathbb{C}}$. This operator is called an *intertwining operator*.

It is convenient to combine the intertwining operators into matrices of such operators. If $\mathcal{O} \in \widehat{Z}(\mathfrak{m})/W(A)$, put

$$L^2_{cus}(\Gamma_M\backslash M, \sigma, \mathcal{O}) = \bigoplus_{\chi\in\mathcal{O}} L^2_{cus}(\Gamma_M\backslash M, \sigma, \chi).$$

Furthermore, the relation of association is an equivalence relation, hence breaks up the set of all $\mathbb{Q}$–parabolic subgroups of G into equivalence classes. Fix one such class $\mathcal{C}$. Choose a complete set of representatives $P_1, \dots, P_r$ of the $G(\mathbb{Q})$ conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_r$ in $\mathcal{C}$. For each $1 \le i \le r$, let $P_{il}, l = 1, \dots, r_i$, be a complete set of representatives of the Γ–conjugacy classes in $\mathcal{C}_i$. Then

$$\{P_{il} \mid 1 \le i \le r,\ 1 \le l \le r_i\}$$

is a set of representatives for $\mathcal{C}/\Gamma$. Let $y_{il} \in K$ be such that

$$P_{il} = y_{il} P_i y_{il}^{-1}.$$

Let A_i be a split component of P_i, $1 \le i \le r$, and set $A_{il} = y_{il} A_i y_{il}^{-1}$. Then A_{il} is a split component of P_{il}. Let $P_{il} = N_{il} A_{il} M_{il}$ be the Langlands decomposition of P_{il}. Let

$$\mathcal{O} = \{\mathcal{O}_{il} \mid 1 \le i \le r,\ 1 \le l \le r_i\} \tag{5.7}$$

be a set of associate orbits with $\mathcal{O}_{il} \in \widehat{Z}(\mathfrak{m}_{il})/W(A_{il})$. For each $1 \le i \le r$, let $\mathcal{O}_i = \{\mathcal{O}_{il} \mid 1 \le l \le r_i\}$ and set

$$\mathcal{E}_{cus}(\sigma, \mathcal{O}_i) = \bigoplus_l L^2_{cus}\left(\Gamma_{M_{il}}\backslash M_{il}, \sigma, \mathcal{O}_{il}\right). \tag{5.8}$$

Given $w \in W(\mathfrak{a}_i, \mathfrak{a}_j)$ and $\Lambda \in \mathfrak{a}^*_{i,\mathbb{C}}$, define $C_{ji}(w, \Lambda_i)$ to be the matrix

$$\left(c_{P_{jk}|P_{il}}(y_{jk} w y_{il}^{-1}, {}^{y_{il}}\Lambda)\right)$$

where $1 \le l \le r_i,\ 1 \le k \le r_j$. This matrix defines a linear operator

$$C_{ji}(w, \Lambda) \colon \mathcal{E}_{cus}(\sigma, \mathcal{O}_i) \to \mathcal{E}_{cus}(\sigma, \mathcal{O}_j). \tag{5.9}$$

For $\Phi_i = \{\Phi_{il} \mid 1 \le l \le r_i\} \in \mathcal{E}_{cus}(\sigma, \mathcal{O}_i)$ and $\Lambda \in \mathfrak{a}^*_{i,\mathbb{C}}$, define

$$E(P_i, \Phi_i, \Lambda) = \sum_{l=1}^{r_i} E(P_{il}|A_{il}, \Phi_{il},\ {}^{y_{il}}\Lambda). \tag{5.10}$$

Then all Eisenstein series $E(P_i, \Phi_i, \Lambda),\ 1 \le i \le r,\ \Phi_i \in \mathcal{E}_{cus}(\sigma, \mathcal{O}_i)$ and intertwining operators $C_{ji}(w, \Lambda),\ 1 \le i, j \le r,\ w \in W(\mathfrak{a}_i, \mathfrak{a}_j)$ are meromorphic functions of $\Lambda \in \mathfrak{a}^*_{\mathbb{C}}$ all of whose singularities lie along hyperplanes. Moreover, the following functional equations hold

$$E(P_i,\Phi_i,\Lambda)=E(P_j,C_{ji}(w,\Lambda)\Phi_i,w\Lambda)$$
$$C_{ji}(w_1w_2,\Lambda)=C_{jk}(w_1,w_2\Lambda)C_{ki}(w_2,\Lambda)$$

for $1\le i,j,k\le r$, $w\in W(\mathfrak{a}_i,\mathfrak{a}_j)$, $w_1\in W(\mathfrak{a}_j,\mathfrak{a}_k)$ and $w_2\in W(\mathfrak{a}_k,\mathfrak{a}_i)$. For the proof of these statements see [L1] or [H, Theorems 8,9]. Using the inner product formula for truncated Eisenstein series [L2], we obtain

Proposition 5.1. *The singular hyperplanes of any $E(P_i,\Phi,\Lambda)$, $1\le i\le r$, $\Phi\in\mathcal{E}(\sigma,\mathcal{O}_i)$, are contianed in the union of the singular hyperplanes of $C_{ji}(w,\Lambda)\colon\mathcal{E}_{cus}(\sigma,\mathcal{O}_i)\to\mathcal{E}_{cus}(\sigma,\mathcal{O}_j)$, $w\in W(\mathfrak{a}_i,\mathfrak{a}_j)$, $1\le i,j\le r$.*

Next we recall the iterated residue construction. Let $P=NAM$ be a $\mathbb{Q}$–parabolic subgroup of G. Given $\alpha\in\Phi_P$ and $c\in\mathbb{R}$, set

$$H(\alpha,c)=\{\Lambda\in\mathfrak{a}^*_{\mathbb{C}}\mid\langle\Lambda,\alpha\rangle=c\}\,.$$

An affine subspace $\mathcal{H}\subset\mathfrak{a}^*_{\mathbb{C}}$ is called admissible if $\mathcal{H}$ is the intersection of such hyperplans. Suppose that $\mathcal{H}_1\supset\mathcal{H}_2$ are two admissible affine subspaces of $\mathfrak{a}^*_{\mathbb{C}}$ such that $\mathcal{H}_2$ is of codimension one in $\mathcal{H}_1$. Let $F(\Lambda)$ be a meromorphic function on $\mathcal{H}_1$ whose singularities lie along hyperplans which are admissible subspaces of $\mathfrak{a}^*_{\mathbb{C}}$. Choose a real unit vector Λ_0 in $\mathcal{H}_1$ which is normal to $\mathcal{H}_2$. Then we can define a meromorphic function $\mathrm{Res}_{\mathcal{H}_2}F$ on $\mathcal{H}_2$ by

$$\mathrm{Res}_{\mathcal{H}_2}F(\Lambda)=\frac{\delta}{2\pi i}\int_0^1F(\Lambda+\delta e^{2\pi i\theta}\Lambda_0)d\theta$$

where $\delta>0$ is sufficiently small so that $F(\Lambda+z\Lambda_0)$ has no singularities for $0<|z|<2\delta$. The singularities of $\mathrm{Res}_{\mathcal{H}_2}F$ lie on the intersections of $\mathcal{H}_2$ with the singular hyperplanes of F that are different from $\mathcal{H}_2$. Now consider a complete flag

$$\mathfrak{a}^*_{\mathbb{C}}=\mathcal{H}_p\supset\mathcal{H}_{p-1}\supset\cdots\supset\mathcal{H}_1\supset\mathcal{H}_0=\{\Lambda\},\quad p=\dim\mathfrak{a}^*,$$

of affine admissible subspaces of $\mathfrak{a}^*_{\mathbb{C}}$ and let $\Lambda_i\in\mathcal{H}_i$ be a real unit vector which is normal to $\mathcal{H}_{i-1}, i=1,\dots,p$. We call $\mathcal{F}=\{\mathcal{H}_i,\Lambda_i\}$ an admissible flag. Let F be a meromorphic function on $\mathfrak{a}^*_{\mathbb{C}}$ whose singularities lie along admissible hyperplanes of $\mathfrak{a}^*_{\mathbb{C}}$. Then we define F_i inductively by

$$F_p=F,\ F_i=\mathrm{Res}_{\mathcal{H}_i}F_{i+1},\ i=0,\dots,p-1.$$

Set

$$\mathrm{Res}_{\mathcal{F}}F=F_0.$$

This is the iterated residue of F.

Let $\chi \in \widehat{Z}(\mathfrak{m})$ and $\Phi \in L^2_{cus}(\Gamma_M\backslash M, \sigma, \chi)$. It follows from [H, Lemma 118] that all singular hyperplanes of $E(P|A, \Phi, \Lambda)$ are admissible. Let $H(\alpha_i, c_i)$, $i = 0, \dots, p-1$, $p = \dim \mathfrak{a}$, be a set of singular hyperplanes of $E(P|A, \Phi, \Lambda)$ such that $\cap_{i=0}^{p-1} H(\alpha_i, c_i) = \{\Lambda_0\}$. Set $\mathcal{H}_i = \cap_{j\geq i} H(\alpha_i, c_i)$, $i = 0, \dots, p-1$, and $\mathcal{H}_p = \mathfrak{a}^*_{\mathbb{C}}$. Choose real unit vectors $\Lambda_i \in \mathcal{H}_i$ normal to $\mathcal{H}_{i-1}$. Then $\mathcal{F} = \{\mathcal{H}_i, \Lambda_i\}$ is an admissible flag. Let $\varphi \in C_c^\infty(\mathfrak{a})$ and let $\widehat{\varphi}(\Lambda)$ be its Fourier transform. This is a holomorphic function on $\mathfrak{a}^*_{\mathbb{C}}$. Put

$$\Psi = \operatorname{Res}_{\mathcal{F}}[E(P|A, \Phi, \Lambda)\widehat{\varphi}(\Lambda)]. \tag{5.11}$$

It is clear that Ψ depends only on the derivatives of $\widehat{\varphi}$ at Λ_0. Let ${}^+\mathfrak{a}^*$ be the positive cone in $\mathfrak{a}^*$ spanned by the simple roots of (P, A). If $\Lambda_0 \in {}^+\mathfrak{a}^*$ then Ψ is square integrable and satisfies [H, Lemma 118]

$$R(\Omega)\Psi = \left(|\Lambda_0|^2 - |\rho_P|^2 + \chi(\Omega_M)\right)\Psi.$$

By (3.4) it follows that Ψ is also an eigenfunction of Δ_σ:

$$\Delta_\sigma\Psi = \left(|\rho_P|^2 - |\Lambda_0|^2 - \chi(\Omega_M) + \lambda_\sigma\right)\Psi \tag{5.12}$$

and by construction, Ψ is not a cusp form. This implies that Ψ is contained in $L^2_{res}(\Gamma\backslash X, E_\sigma)$.

The main statement of Langlands' theory asserts that this construction gives all of the residual spectrum. More precisely, we have [L1, Theorem 7.1]

Theorem 5.2. *The residual subspace $L^2_{res}(\Gamma\backslash X, E_\sigma)$ is spanned by iterated residues of the form (5.11) at singular points $\Lambda_0 \in \{\Lambda \in {}^+\mathfrak{a}^*_P \mid |\Lambda| \leq |\rho_P|\}$ where $P = NAM$ runs over a set of representatives of the Γ–conjugacy classes of proper $\mathbb{Q}$–parabolic subgroups of G, $\Phi \in L^2_{cus}(\Gamma_M\backslash M, \sigma, \chi)$, $\chi \in \widehat{Z}(\mathfrak{m})$ and $\varphi \in C_0^\infty(\mathfrak{a})$.*

Thus each association class $\mathcal{C}$ of proper $\mathbb{Q}$–parabolic subgroups defines a subspace $L^2_{res}(\Gamma\backslash X, E_\sigma; \mathcal{C})$ of $L^2_{res}(\Gamma\backslash X, E_\sigma)$ consisting of all iterated residues associated to parabolic groups $P \in \mathcal{C}$. In this way, we get a decomposition

$$L^2_{res}(\Gamma\backslash X, E_\sigma) = \bigoplus_{\mathcal{C}} L^2_{res}(\Gamma\backslash X, E_\sigma; \mathcal{C})$$

where $\mathcal{C}$ runs over assocItation classes of proper $\mathbb{Q}$–parabolic subgroups of G. Let $N_{\mathcal{C},\sigma}(\lambda)$ be the eigenvalue counting function of Δ_σ acting in the space $L^2_{res}(\Gamma\backslash X, E_\sigma; \mathcal{C})$. It follows from Theorem 5.2 that in order to estimate $N_{\mathcal{C},\sigma}(\lambda)$, it is sufficient to count the number of iterated residues associated to $\mathcal{C}$ with eigenvalues less than λ. Let $\mathcal{O} = \{\mathcal{O}_{il}\}$ be a set of associated orbits as in (5.7). Then $-\chi(\Omega_M)$ has the same value $\mu(\mathcal{O})$ for all $\chi \in \mathcal{O}_{il},\ 1 \le i \le r,\ 1 \le l \le r_i$. Since the singular points Λ_0 that contribute to the residual spectrum satisfy $|\Lambda_0| \le |\rho_P|$, it follows from 5.12 that the eigenvalue of an iterated residue 5.11 with $\Phi \in L^2_{cus}(\Gamma_{M_{il}}\backslash M_{il}, \mathcal{O}_{il})$ is bounded from below by $\mu(\mathcal{O}) + \lambda_\sigma$. Thus it remains to count all intersections of singular hyperplanes of $C_{ji}(w,\Lambda)\colon \mathcal{E}_{cus}(\sigma,\mathcal{O}_i) \to \mathcal{E}_{cus}(\sigma,\mathcal{O}_j)$, $w \in W(\mathfrak{a}_i,\mathfrak{a}_j)$, such that the intersection consists of a single point which lies in $\{\Lambda \in {}^+\mathfrak{a}_i^* \mid |\Lambda| \le |\rho_{P_i}|\}$, together with the orders of the singularities of $C_{ji}(w,\Lambda)$ along the corresponding singular hyperplanes. More precisely, let $\mathfrak{h} = \{H_1, ..., H_p\}$, $p = \dim \mathfrak{a}_i$, be a set of singular hyperplanes of $C_{ji}(w,\Lambda)$ such that

$$\bigcap_k H_k = \{\Lambda_0\} \quad \text{and} \quad \Lambda_0 \in \{\Lambda \in {}^+\mathfrak{a}_i^* \mid |\Lambda| \le |\rho_{P_i}|\}. \tag{5.13}$$

Let $m(H_k)$ denote the order of the pole of $C_{ji}(w,\Lambda)$ at H_k and put

$$m(\mathfrak{h}) = \sum_{k=1}^{p} m(H_k).$$

Let $\mathcal{S}_{ji}(w)$ be the set of all systems $\mathfrak{h}$ of singular hyperplanes of $C_{ji}(w,\Lambda)$ such that (5.13) holds. Put

$$m_{ji}(w,\sigma,\mathcal{O}) = \sum_{\mathfrak{h}\in\mathcal{S}_{ji}(w)} m(\mathfrak{h})$$

and

$$m(\sigma,\mathcal{O}) = \sum_{i,j}\ \sum_{w\in W(\mathfrak{a}_i,\mathfrak{a}_j)} m_{ji}(w,\sigma,\mathcal{O}).$$

Then by Proposition 5.1 we get

$$N_{\mathcal{C},\sigma}(\lambda) \le \sum_{i,l}\ \sum_{\substack{\mathcal{O}\\ \mu(\mathcal{O})+\lambda_\sigma \le \lambda}} m(\sigma,\mathcal{O}) \dim(L^2_{cus}(\Gamma_{M_{il}}\backslash M_{il},\ \sigma, \mathcal{O}_{il})). \tag{5.14}$$

First, we estimate the dimension of the spaces of cusp forms that enter (5.14). Given an irreducible unitary representation (τ, V_τ) of K_M, let $[\sigma\colon\tau] = \dim(V_\sigma \otimes V_\tau)^{K_M}$. Then

$$\sigma_M = \bigoplus_{\tau\in\widehat{K}_M} [\sigma\colon\tau]\tau.$$

Let $\Omega_{K_M} \in Z(\mathfrak{k}_M)$ be the Casimir element of K_M. For each $\tau \in \widehat{K}_M$, one has

$$\tau(\Omega_{K_M}) = \lambda_\tau \mathrm{Id},$$

where $\lambda_\tau \geq 0$. Let P_τ be the orthogonal projection of V_σ onto V_τ. Then we have

$$\sigma(\Omega_{K_M}) = \sum_{\tau \in \widehat{K}_M} [\sigma \colon \tau] \lambda_\tau P_\tau.$$

Next note that $\Omega_K = \Omega_{K_M} + \Omega'$ where $\Omega' = -\sum_{i=1}^q Y_i^2$ and $Y_1, \dots, Y_q$ is an orthonormal basis of $\mathfrak{k}_M^\perp$. Thus $\sigma(\Omega')$ is a symmmetric nonnegative operator in V_σ. Since $\sigma(\Omega_K) = \lambda_\sigma \mathrm{Id}$, this implies

$$\lambda_\sigma \geq \lambda_\tau \text{ for all } \tau \in \widehat{K} \text{ with } [\sigma \colon \tau] > 0. \tag{5.15}$$

Note that $\mu(\mathcal{O}) + \lambda_\tau$ is an eigenvalue of the Bochner Laplace operator in $L^2(\Gamma_M \backslash X_M, E_\tau)$. Hence by Corollary 4.6 and (5.15), we get

$$\begin{aligned} \sum_{i,l} \sum_{\substack{\mathcal{O} \\ \mu(\mathcal{O}) + \lambda_\sigma \leq \lambda}} & \dim(L^2_{cus}(\Gamma_{M_{il}} \backslash M_{il}, \sigma, \mathcal{O}_{il})) \\ & \leq C \dim(V_\sigma)(1 + \lambda_\sigma^{m_1} + \lambda^{m_1}) \\ & \leq C(1 + \lambda_\sigma^q + \lambda^q) \end{aligned} \tag{5.16}$$

where $m_1 = \frac{1}{2} \dim M_{il} + 1$, $q = \frac{1}{2} \dim M_{il} + \frac{1}{2} \dim K + 1$ and $C > 0$ is independent of σ.

It remains to bound $m(\sigma, \mathcal{O})$, i.e. the number of intersections of real singular hyperplanes of the intertwining operators $C_{ji}(w, \Lambda)$ that are reduced to a point in the bounded domain $\{\Lambda \in {}^+\mathfrak{a}^* \mid |\Lambda| \leq |\rho_P|\}$. The first step is the reduction to rank one parabolic groups. We proceed as in [Mu, section 8], using that the intertwining operators for higher rank parabolic subgroups can be expressed as products of intertwining operators of rank one parabolic subgroups of the Levi components of the higher rank groups (see [H, chapter V]).

Let $\mathcal{C}$ be an association class of $\mathbb{Q}$–parabolic subgroups of G. Let $P = NAM$ be any element in $\mathcal{C}$. Denote by $\mathcal{L}$ the set of Weyl chambers in $\mathfrak{a}$. There is a one-to-one correspondence between $\mathcal{C}/G(\mathbb{Q})$ and $\mathcal{L}/W(A)$ (cf. [H, chapter V, section 4]). To each $C \in \mathcal{L}$, one can associate a unique $P_C \in \mathcal{L}$ with $P_C = N_C AM$. Let $\mathcal{L} = \{C_1, \dots, C_q\}$ and set $P_i = P_{C_i}$, $i = 1, \dots, q$. This set contains a set of representatives for $\mathcal{C}/G(\mathbb{Q})$. Let $\mathcal{C}_i$ be the $G(\mathbb{Q})$–conjugacy class containing P_i, $1 \leq i \leq q$. Note that all P_i have the same split component A. We choose a set of representatives P_{il}, $1 \leq l \leq q_i$, for the Γ–conjugacy classes in $\mathcal{C}_i$.

Let $y_{il} \in G(\mathbb{Q})$ such that $P_{il} = y_{il} P_i y_{il}^{-1}$ and $A_{il} = y_{il} A y_{il}^{-1}$. Then A_{il} is a split component of P_{il}. Let $P_{il} = N_{il} A_{il} M_{il}$ be the corresponding Langlands decomposition. Let $\mathcal{O} = \{\mathcal{O}_{il} \mid 1 \le i \le q,\ 1 \le l \le q_i\}$ be a set of associate orbits where $\mathcal{O}_{il} \in \widehat{Z}(\mathfrak{m}_{il})/W(A_{il})$. Then for each $w \in W(\mathfrak{a})$, we can define intertwining operators

$$C_{ji}(w, \Lambda)\colon \mathcal{E}_{cus}(\sigma, \mathcal{O}_i) \to \mathcal{E}_{cus}(\sigma, \mathcal{O}_j)$$

as above. As explained in Chapter V, section 4 of [H], the functional equation implies that there exists k, $1 \le k \le q$, such that

$$C_{ji}(w, \Lambda) = C_{jk}(1, w\Lambda) C_{ki}(w, \Lambda)$$

and $C_{ki}(w, \Lambda)$ is entire. Hence, we only have to consider $C_{ji}(1, \Lambda)$. Furthermore, by Lemma 117 in [H] there exists a sequence $i = i_1, \dots, i_p = j$, $1 \le i_l \le q$, $p \le q$, such that the chambers C_{i_l} and $C_{i_{l+1}}$ are adjacent for all $l = 1, \dots, p-1$ and

$$C_{ji}(1, \Lambda) = C_{i_p i_{p-1}}(1, \Lambda) \cdots C_{i_3 i_2}(1, \Lambda) C_{i_2 i_1}(1, \Lambda). \tag{5.17}$$

Therefore, it suffices to bound the number of singular hyperplanes of $C_{ji}(1, \Lambda)$ for adjacent chambers C_i and C_j. In the proof of Lemma 116 in [H] it is shown that for adjacent chambers, the entries of $C_{ji}(w, \Lambda)$ can be expressed in terms of intertwining operators for rank one parabolic groups. Assume that $i = 1$, and $j = 2$. Since C_1 and C_2 are adjacent, there exists a unique $\mathbb{Q}$–parabolic subgroup $P' = N'A'M'$ of G such that (P', A') dominates (P_1, A) and (P_2, A), i.e. $P' \supset P_i$ and $A_i \supset A'$, $i = 1, 2$. Moreover, $\overline{(\mathfrak{a})^+} = \overline{C_1} \cap \overline{C_2}$. Therefore, we have that $\operatorname{rank}(P') = \operatorname{rank}(P_1) - 1$. By Lemma 2 in [H], P_i, $i = 1, 2$, determines a $\mathbb{Q}$–parabolic subgroup ${}^*P_i = {}^*N_i {}^*A_i {}^*M_i$ of M' such that

$${}^*M_i = M_i,\ A_i = {}^*A_i A',\ N_i = {}^*N_i N',\ i = 1, 2. \tag{5.18}$$

In particular, $\dim {}^*A_i = 1$, i.e. ${}^*P_i$, $i = 1, 2$, are rank one $\mathbb{Q}$–parabolic subgroups of M'. Let ${}^*\mathcal{C}_i$ be the $M'(\mathbb{Q})$ conjugacy class of $\mathbb{Q}$–parabolic subgroups of M' containing ${}^*P_i$. Associated to ${}^*C_1$ and ${}^*C_2$, there are intertwining operators ${}^*C_{21}({}^*w, {}^*\Lambda)$ where ${}^*w \in W({}^*\mathfrak{a}_1)$ and ${}^*\Lambda \in (\mathfrak{a}_1)^*_{\mathbb{C}}$. These operators act in spaces $\mathcal{E}^*_{cus}(\sigma, \mathcal{O}_i)$ which are similar to (5.8). Now each entry of $C_{21}(1, \Lambda)$ is equal to the product of a holomorphic function and some entry of ${}^*C_{21}(w, {}^*\Lambda)$, $w \in W({}^*\mathfrak{a}_1)$, where ${}^*\Lambda$ is the restriction of Λ to ${}^*\mathfrak{a}_1 \subset \mathfrak{a}_1$. For details see [H, pp. 124–125], [Mu, pp. 524–525]. This implies

Lemma 5.3. *There exists $c > 0$ which is independent of σ and $\mathcal{O}$, such that for any adjacent chambers C_i and C_j, the number of real singular hyerplanes of $C_{ji}(1, \Lambda)$ which intersect $\{\Lambda \in {}^+\mathfrak{a}_i^* \mid |\Lambda| \le |\rho_P|\}$*

is bounded by the number of real poles of ${}^*C_{ji}(w, {}^*\Lambda)$, $w \in W({}^*\mathfrak{a}_i)$, *in the interval* $[-c, |\rho|]$.

Thus, using (5.17), we have reduced our problem to the rank one case. Namely, let $m_1(\sigma, \mathcal{O})$ be the maximum of the number of poles, counted with their order, of the intertwining operators ${}^*C_{ji}(w, {}^*\Lambda)$, $w \in W({}^*\mathfrak{a}_i)$, in the interval $[-c, |\rho|]$ where $c \gg 0$. Let $p = \dim \mathfrak{a}_i$. Then

$$m(\sigma, \mathcal{O}) \leq m_1(\sigma, \mathcal{O})^p. \tag{5.19}$$

6. The rank one residual spectrum

In this section we study the poles of rank one cuspidal Eisenstein series in the half–plane $\mathrm{Re}(s) > 0$. The corresponding residues span a subspace of $L^2_{res}(\Gamma\backslash X, E_\sigma)$ which is the direct sum of all $L^2_{res}(\Gamma\backslash X, E_\sigma; \mathcal{C})$ where $\mathcal{C}$ runs over association classes of rank one $\mathbb{Q}$–parabolic subgroups.

Let $\mathcal{C}$ be an association class of rank one (i.e. maximal) $\mathbb{Q}$–parabolic subgroups of G. Let $P = NAM$ be an element of $\mathcal{C}$ and let $P^- = N^-AM$ be the opposite $\mathbb{Q}$–parabolic subgroup. Then any element of $\mathcal{C}$ is $G(\mathbb{Q})$–conjugate either to P or to P^-. Moreover, P and P^- are $G(\mathbb{Q})$–conjugate if and only if $-1 \in W(A)$. Let $P_1, \cdots, P_r$ be a set of representatives of the $G(\mathbb{Q})$–conjugacy classes in $\mathcal{C}$. Then $r = 1$ or 2. Let $\mathcal{C}_i$ be the $G(\mathbb{Q})$ conjugacy class containing P_i. Let P_{il}, $l = 1, \ldots, r_i$, be a set of representatives of the Γ–conjugacy classes in $\mathcal{C}_i$. Then $\{P_{il} \mid 1 \leq i \leq r,\ 1 \leq l \leq r_i\}$ is a set of representatives of the Γ–conjugacy classes in $\mathcal{C}$. Let

$$\{\mathcal{O}_{il} \mid 1 \leq i \leq r,\ 1 \leq l \leq r_i\}$$

be a set of associated orbits with $\mathcal{O}_{il} \in \widehat{Z}(\mathfrak{m}_{il})/W(A_{il})$. For $\mathcal{O}_i = \{\mathcal{O}_{il} \mid 1 \leq l \leq r_i\}$, let $\mathcal{E}_{cus}(\sigma, \mathcal{O}_i)$ be defined by (5.8) and let $C_{ji}(w, \Lambda)$ be the corresponding intertwining operators (5.9). Set

$$\mathcal{E}_{cus}(\sigma, \mathcal{O}) = \bigoplus_{i=1}^{r} \mathcal{E}_{cus}(\sigma, \mathcal{O}_i). \tag{6.1}$$

Let α_i be the simple root of (P_i, A_i), $i = 1, \ldots, r$, and put $\lambda_i = \alpha_i/|\alpha_i|$. Then $\rho_i = |\rho|\lambda_i$ (note that $|\rho_1| = |\rho_2|$ which we denote by $|\rho|$).

If $r = 1$, then $W(A_1) = \{\pm 1\}$. Define

$$C(s) = C_{11}(-1, s\lambda_1),\ s \in \mathbb{C}. \tag{6.2}$$

If $r = 2$, then $W(A_i) = \{1\}$, $i = 1, 2$, and $W(\mathfrak{a}_1, \mathfrak{a}_2) = \{w\}$. Define

$$C(s) = \begin{pmatrix} 0 & C_{12}(w^{-1}, s\lambda_2) \\ C_{21}(w, s\lambda_1) & 0 \end{pmatrix}. \tag{6.3}$$

In either case

$$C(s)\colon \mathcal{E}_{cus}(\sigma, \mathcal{O}) \to \mathcal{E}_{cus}(\sigma, \mathcal{O}) \tag{6.4}$$

is a linear operator that is a meromorphic function of $s \in \mathbb{C}$ and satisfies

$$C(s)C(-s) = \mathrm{Id},\ C(s)^* = C(\overline{s}),\ s \in \mathbb{C}. \tag{6.5}$$

The poles of $C(s)$ in the half–plane $\mathrm{Re}(s) \geq 0$ are all simple and contained in the interval $(0,\ |\rho|]$ (see [H, chapter IV]). For $\Phi \in \mathcal{E}_{cus}(\sigma, \mathcal{O})$ and $\Phi = \{\Phi_i \mid 1 \leq i \leq r\}$ set

$$E(\Phi, s) = \sum_{i=1}^{r} E(P_i, \Phi_i, s\lambda_i)$$

where the Eisenstein series on the right hand side are defined by (5.10). Then each $E(\Phi, s)$ is a meromorphic function of $s \in \mathbb{C}$ and satisfies the following functional equation:

$$E(C(s)\Phi, -s) = E(\Phi, s),\ s \in \mathbb{C}.$$

The poles of $E(\Phi, s)$ are contained in the poles of $C(s)$, and all poles of $E(\Phi, s)$ in $\mathrm{Re}(s) > 0$ are simple and contained in $(0, |\rho|]$ (cf. [H], Ch. IV, Theorem 7.) For any pole $s_0 \in (0, |\rho|]$ of $E(\Phi, s)$, the residue

$$\Psi = \mathrm{Res}_{s=s_0} E(\Phi, s)$$

is square integrable and satisfies

$$-R(\Omega)\Psi = (-s_0^2 + |\rho|^2 + \mu)\Psi$$

where $\mu = -\chi(\Omega_{M_{il}})$, $\chi \in \mathcal{O}_{il}$, which is independent of i, l. By (3.4), Ψ is an eigenfunction of Δ_σ with eigenvalue $-s_0^2 + |\rho|^2 + \mu + \lambda_\sigma$. Then Ψ belongs to $L^2_{res}(\Gamma\backslash X, E_\sigma; \mathcal{C})$ and this space is spanned by all such residues where $\Phi \in \mathcal{E}_{cus}(\sigma, \mathcal{O})$, $s_0 \in (0, |\rho|]$ is a pole of $C(s)$ and $\mathcal{O}$ runs over all orbits. Thus in order to estimate $N_{\mathcal{C},\sigma}(\lambda)$ for $\mathrm{rank}(\mathcal{C}) = 1$, it is sufficient to bound the number of poles of $C(s)$ in $(0, |\rho|]$. Using (4.11) in [Mu] and Corollary 3.15 in [Mu], we get

Proposition 6.1. *Let $C(s; \sigma, \mathcal{O})\colon \mathcal{E}_{cus}(\sigma, \mathcal{O}) \to \mathcal{E}_{cus}(\sigma, \mathcal{O})$ be the intertwining operator with respect to $\sigma \in \widehat{K}$ and a set of associated orbits $\mathcal{O}$. Let $\mu(\mathcal{O})$ be the common eigenvalue of the Casimir elements $\Omega_{M_{il}}$ with respect to $\mathcal{O}_{il}$. Then for all σ and $\mathcal{O}$, the number $n(\sigma, \mathcal{O})$ of poles, counted with their order, of $C(s; \sigma, \mathcal{O})$ in $(0, |\rho|]$ can be estimated by*

$$n(\sigma, \mathcal{O}) \leq \dim(\mathcal{E}_{cus}(\sigma, \mathcal{O})) N_{T,\sigma}(|\rho|^2 + \mu(\mathcal{O}) + \lambda_\sigma)$$

where $N_{T,\sigma}(\lambda)$ *is the eigenvalue counting function for the cut–off Laplacian* $\Delta_{\sigma,T}$.

Remark. The cut–off Laplacian used in [Mu] is equal to $\Delta_{\sigma,T} - \lambda_\sigma \mathrm{Id}$. This gives rise to the additional λ_σ in the counting function on the right hand side.

Now we apply Theorem 4.5 to estimate the counting function $N_{T,\sigma}(\lambda)$ and (5.16) to bound $\dim \mathcal{E}_{cus}(\sigma,\mathcal{O})$.

This leads to

Corollary 6.2. *Let* $m = \dim G$. *There exists* $C > 0$ *such that*

$$m(\sigma,\mathcal{O}) \leq C(1 + \lambda_\sigma^{2m} + (\mu(\mathcal{O}) + \lambda_\sigma)^{2m})$$

for all $\sigma \in \widehat{K}$ *and orbits* $\mathcal{O}$.

Note that by (5.15), $\mu(\mathcal{O}) + \lambda_\sigma \geq 0$. Corollary 6.2 is the improved version of Corollary 3.25 in [Mu].

Corollary 6.3. *Let* $C(s;\sigma,\mathcal{O})$ *be as above. There exists* $C > 0$ *such that the number of poles, counted with multiplicity, of* $\det C(s;\sigma,\mathcal{O})$ *in the half–plane* $\mathrm{Re}(s) > 0$ *is bounded by*

$$C(1 + \lambda_\sigma^{3m} + (\mu(\mathcal{O}) + \lambda_\sigma)^{3m})$$

for $m = \dim G$.

Proof. Let s_0, $\mathrm{Re}(s_0) > 0$, be a pole of $\det C(s;\sigma,\mathcal{O})$. Then s_0 is also a pole of $C(s;\sigma,\mathcal{O})$. Since all poles of $C(s;\sigma,\mathcal{O})$ in $\mathrm{Re}(s) > 0$ are simple, the order of $\det C(s;\sigma,\mathcal{O})$ at s_0 does not exceed $d = \dim \mathcal{E}_{cus}(\sigma,\mathcal{O})$. Using (5.16), we get the desired result. □

7. The poles of the rank one intertwining operators

Let the notation be the same as above. In this section we investigate the poles of the rank one intertwining operator $C(s) = C(s;\sigma,\mathcal{O})$ in the half–plane $\mathrm{Re}(s) \leq 0$. Let $\mu = \mu(\mathcal{O})$ be the common eigenvalue

$$-\chi(\Omega_{M_{ij}}),\ \chi \in \mathcal{O}_{ij},\ 1 \leq i \leq r,\ 1 \leq j \leq r_i.$$

We first study the poles of $\det C(s)$. The analogous result to [Mu, Theorem 7.1] is

Theorem 7.1. *There exists* $C > 0$ *such that for all* $\sigma \in \widehat{K}$ *and all associated orbits* $\mathcal{O}$ *one has*

$$\left| \int_{-\Lambda}^{\Lambda} \frac{d}{ds} \log \det C(i\lambda; \sigma, \mathcal{O})\, d\lambda \right| \leq C(\Lambda^{2m} + \mu^{2m} + \lambda_\sigma^{2m})$$

for $\Lambda \geq 0$ *and* $m = \dim G$.

Proof. We fix a truncation parameter $T = tT_\rho$ with $t \gg 0$ as in section 3. Recall that $T_\rho \in \mathfrak{a}_0$ is the vector whose i–th component equals H_{ρ_i}. Set

$$C_t(s) = e^{-2st|\rho|} C(s). \tag{7.1}$$

By the functional equation (6.5) for $C(s)$, it follows that $C_t(i\lambda)$ is unitary for $\lambda \in \mathbb{R}$ and hence, can be diagonalized. For $\Lambda \geq 0$, let $n(\Lambda)$ be the number of points $w \in [0, \Lambda]$ such that $C_t(iw)$ has at least one eigenvalue equal to -1. Using (3.7) in [Mu], we obtain

$$\begin{aligned} \left| \int_{-\Lambda}^{\Lambda} \frac{d}{ds} \log \det C(i\lambda; \sigma, \mathcal{O})\, d\lambda \right| & \\ \leq \left| \int_{-\Lambda}^{\Lambda} \frac{d}{ds} \log \det C_t(i\lambda; \sigma, \mathcal{O})\, d\lambda \right| & \\ + 4t|\rho|\Lambda \dim \mathcal{E}_{cus}(\sigma, \mathcal{O}) & \\ \leq 4 \dim \mathcal{E}_{cus}(\sigma, \mathcal{O})\ (2\pi n(\Lambda) + t|\rho|\Lambda), & \end{aligned} \tag{7.2}$$

for $\Lambda \geq 0$. Let $w \in [0, \Lambda]$ and assume that $C_t(iw)$ has at least one eigenvalue equal to -1. Thus there exists $\Phi \in \mathcal{E}_{cus}(\sigma, \mathcal{O})$, $\Phi \neq 0$, such that $C(iw)\Phi = -e^{2iwt|\rho|}\Phi$. Let Λ^T be the truncation operator [Mu, (3.10)]. Put $\varphi = \Lambda^T E(\Phi, w)$. By (3.4) and [Mu, Lemma 3.14] we get

$$\Delta_{\sigma,T}\varphi = \left(w^2 + |\rho|^2 + \mu + \lambda_\sigma\right) \varphi.$$

Now observe that different values of w give linearly independent eigenfunctions. Hence

$$n(\Lambda) \leq N_{T,\sigma}(\Lambda^2 + |\rho|^2 + \mu + \lambda_\sigma). \tag{7.3}$$

Using Corollary 4.6 and (5.16), the claimed estimate follows from (7.2). □

In [Mu, section 6] we established some key results concerning the analytic properties of the rank one intertwining operators. In particular we have

Proposition 7.2. *Let* $v_1, \dots, v_l \in [0, |\rho|]$ *be the poles of* $\det C(s)$ *in the half–plane* $\mathrm{Re}(s) \geq 0$, *and let* η_k, $k \in \mathbb{N}$, *be the poles of* $\det C(s)$ *in the half–plane* $\mathrm{Re}(s) \leq 0$, *both repeated according to their multiplicity. We fix a truncation parameter* $T = tT_\rho$, $t \gg 0$. *Then for* $\lambda \in \mathbb{R}$, *one has*

$$\begin{aligned} \frac{d}{ds} \log \det C(i\lambda) = \log q + \sum_{j=1}^{l} \frac{2v_j}{\lambda^2 + v_j^2} \\ + \sum_{k=1}^{\infty} \frac{2\,\mathrm{Re}(\eta_k)}{(\mathrm{Re}(\eta_k))^2 + (\lambda - \mathrm{Im}(\eta_k))^2}. \end{aligned} \tag{7.4}$$

where $\log q \leq 2(t+1) \dim \mathcal{E}_{cus}(\sigma, \mathcal{O}) + 1$.

We now can proceed as in [Mu, p. 520] to estimate the real poles of $\det C(s)$ in $\mathrm{Re}(s) < 0$. Note that for any $v > 0$, one has

$$\int_{-1}^{1} \frac{v}{v^2 + \lambda^2}\, d\lambda < \int_{-\infty}^{\infty} \frac{1}{1 + \lambda^2}\, d\lambda = \pi.$$

Hence, by (7.4) and Theorem 7.1, we get

$$\begin{aligned} \left| 2\log q + \int_{-1}^{1} \sum_{k=1}^{\infty} \frac{2\,\mathrm{Re}(\eta_k)}{(\mathrm{Re}(\eta_k))^2 + (\lambda - \mathrm{Im}(\eta_k))^2}\, d\lambda \right| \\ \leq 2\pi l + C(1 + \mu^{2n} + \lambda_\sigma^m). \end{aligned}$$

Here l is the number of poles, counted with multiplicity, of $\det C(s)$ in $\mathrm{Re}(s) \geq 0$. Thus l can be estimated by Corollary 6.3 which leads to

$$\begin{aligned} \left| 2\log q + \int_{-1}^{1} \sum_{k=1}^{\infty} \frac{2\,\mathrm{Re}(\eta_k)}{(\mathrm{Re}(\eta_k))^2 + (\lambda - \mathrm{Im}(\eta_k))^2}\, d\lambda \right| \\ \leq C_1(1 + \mu^{3m} + \lambda_\sigma^{3m}). \end{aligned} \tag{7.5}$$

We distinguish two cases:

1) $q \geq 1$. Then we have $0 \leq \log q \leq 2(t+1) \dim(\mathcal{E}_{cus}(\sigma, \mathcal{O})) + 1$.
Since $\mathrm{Re}(\eta_k) < 0$ for all k, (7.5) together with (5.16) leads to

$$\int_{-1}^{1} \sum_{k=1}^{\infty} \frac{|\mathrm{Re}(\eta_k)|}{(\mathrm{Re}(\eta_k))^2 + (\mathrm{Im}(\eta_k) - \lambda)^2}\, d\lambda \leq C_2(1 + \mu^{3m} + \lambda_\sigma^{3m}). \tag{7.6}$$

2) $q < 1$. Then $\log q < 0$. Since

$$\sum_{k=1}^{\infty} \frac{\mathrm{Re}(\eta_k)}{(\mathrm{Re}(\eta_k))^2 + (\mathrm{Im}(\eta_k) - \lambda)^2} < 0,$$

(7.6) holds in this case too.

Let $C > 0$ and denote by $N(c, \sigma, \mathcal{O})$ the number of poles, counted with multiplicity, of $\det C(s; \sigma, \mathcal{O})$ in $[-c, 0]$. Then

$$N(c,\sigma,\mathcal{O}) \int_{-1/c}^{1/c} \frac{1}{1+\lambda^2}\, d\lambda \leq \sum_{\substack{\eta_k \in \mathbb{R} \\ -c \leq \eta_k \leq 0}} \int_{-1}^{1} \frac{|\eta_k|}{|\eta_k|^2 + \lambda^2}\, d\lambda$$
$$\leq \int_{-1}^{1} \sum_{k=1}^{\infty} \frac{|\operatorname{Re}(\eta_k)|}{\left(\operatorname{Re}(\eta_k)\right)^2 + (\lambda - \operatorname{Im}(\eta_k))^2}\, d\lambda.$$

and by (7.6), this implies

$$N(c, \sigma, \mathcal{O}) \leq C_2(1 + \mu^{3m} + \lambda_\sigma^{3m})$$

for a constant $C_2 > 0$ that is independent of $\sigma, \mathcal{O}$. Combined with Corollary 6.3, we get

Theorem 7.3. *Let $c > 0$. There exists $C > 0$ such that for all $\sigma \in \widehat{K}$ and orbits $\mathcal{O}$, the number of poles, counted with multiplicity, of $\det C(s; \sigma, \mathcal{O})$ in $[-c, |\rho|]$ is bounded by $C(1 + \mu^{3m} + \lambda_\sigma^{3m})$ where $\mu = \mu(\mathcal{O})$ is the eigenvalue of the Casimir operator and $m = \dim G$.*

Choose an orthonormal basis $\Phi_1, \ldots, \Phi_d$ in $\mathcal{E}(\sigma, \mathcal{O})$ and set

$$C_{ij}(s; \sigma, \mathcal{O}) = \langle C(s; \sigma, \mathcal{O})\Phi_i, \Phi_j \rangle, \quad i, j = 1, \ldots, d.$$

Let s_0 be a pole of $C(s; \sigma, \mathcal{O})$ and let $\nu_{ij}(s_0; \sigma, \mathcal{O})$ be the order of $C_{ij}(s; \sigma, \mathcal{O})$ in s_0. Set

$$\nu(s_0; \sigma, \mathcal{O}) = \max_{i,j} \nu_{ij}(s_0; \sigma, \mathcal{O}).$$

If s_0 is not a pole of $C(s; \sigma, \mathcal{O})$, set $\nu(s_0; \sigma, \mathcal{O}) = 0$.

Corollary 7.4. *Let $c > 0$. There exists $C > 0$ such that for all $\sigma \in \widehat{K}$ and $\mathcal{O}$, one has*

$$\sum_{-c \leq s_0 \leq |\rho|} \nu(s_0; \sigma, \mathcal{O}) \leq C(1 + \mu^{3m} + \lambda_\sigma^{3m}), \quad \mu = \mu(\mathcal{O}).$$

Proof. We proceed exactly as in the proof of Corollary 7.6 in [Mu], using Corollary 4.6 in place of Theorem 9.1 in [D1]. □

Using the description of the residual subspace $L^2_{res}(\Gamma\backslash X, E_\sigma)$ together with (5.14), (5.16) and (5.19), we obtain an estimate for $N_{res,\sigma}(\lambda)$ of the form claimed by Theorem 0.2. Together with Corollary 4.6, this completes the proof of Theorem 0.2.

8. The trace class conjecture

To begin with, we prove Corollary 0.3. First note that for any $\pi \in \widehat{G}$ and any $\sigma \in \widehat{K}$, one has

$$-\lambda_\pi + \lambda_\sigma \geq 0, \quad \text{if} \quad \dim(\mathcal{H}_\pi \otimes V_\sigma)^K \neq 0. \tag{8.1}$$

Indeed, by (3.4), $-\lambda_\pi + \lambda_\sigma$ is an eigenvalue of $\Delta_\sigma \geq 0$. Let $\sigma \in \widehat{K}$ be fixed. By (8.1) and Theorem 0.2 we get

$$\begin{aligned} &\sum_{\substack{\pi \in \widehat{G} \\ \lambda_\pi^2 + \lambda_\sigma^2 \leq \lambda^2}} m_\Gamma(\pi) \dim(\mathcal{H}_\pi \otimes V_\sigma)^K \\ &\leq \sum_{\substack{\pi \in \widehat{G} \\ -\lambda_\pi + \lambda_\sigma \leq \sqrt{2}\lambda}} m_\Gamma(\pi) \dim(\mathcal{H}_\pi \otimes V_\sigma)^K = N_\sigma(\sqrt{2}\lambda) \\ &\leq C(1 + \lambda_\sigma^{4m} + \lambda^{4m}). \end{aligned} \tag{8.2}$$

Since $\lambda_\sigma \leq \lambda$, (8.2) leads to

$$\sum_{\sigma \in \widehat{K}} \sum_{\substack{\pi \in \widehat{G} \\ \lambda_\pi^2 + \lambda_\sigma^2 \leq \lambda^2}} m_\Gamma(\pi) \dim(\mathcal{H}_\pi \otimes V_\sigma)^K \leq C(1 + \lambda^{4m}) \sum_{\substack{\sigma \in \widehat{K} \\ \lambda_\sigma \leq \lambda}} 1. \tag{8.3}$$

Let $\mathfrak{h} \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$ and let Φ_c be the set of roots of $(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let $\Phi_c^+ \subset \Phi_c$ be a set positive roots. By $L \subset \mathfrak{h}_\mathbb{C}^*$ we denote the additive subgroup of integral linear functions, i.e. $\lambda \in L$ iff $\lambda(H_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi_c$. Let $L^+ \subset L$ be the dominant integral linear functions with respect to Φ_c^+. Recall that the irreducible finite–dimensional representations of $\mathfrak{k}_\mathbb{C}$ are parametrized by L^+. If (σ_μ, E_μ) is an irreducible representation of $\mathfrak{k}_\mathbb{C}$ with highest weight $\mu \in L^+$, then the eigenvalue of the Casimir operator Ω_K on E_μ is given by

$$\lambda_{\sigma_\mu} = \| \mu + \rho_c \|^2 - \| \rho_c \|^2$$

where $\rho_c = \frac{1}{2} \sum_{\alpha \in \Phi_c^+} \alpha$. Furthermore, L is a lattice in the real vector space $F \subset \mathfrak{h}_\mathbb{C}^*$ spanned by the roots. Let $l = \dim F$. Then we get

$$\sum_{\substack{\sigma\in\widehat{K}\\ \lambda_\sigma\leq\lambda}} 1 \leq C \sum_{\substack{\mu\in L\\ \|\mu\|^2\leq\lambda}} 1 \leq C \left(\sum_{\mu\in L-\{0\}} \|\, \mu \,\|^{-2l}\right) \lambda^l = C_1\lambda^l.$$

Combined with (8.3), we obtain

$$\sum_{\sigma\in\widehat{K}} \sum_{\substack{\pi\in\widehat{G}\\ \lambda_\pi^2+\lambda_\sigma^2\leq\lambda^2}} m_\Gamma(\pi)\dim(\mathcal{H}_\pi\otimes V_\sigma)^K \leq C_2(1+\lambda^{4m+l}). \tag{8.4}$$

Let $N \geq m$. Using (8.4), we get

$$\begin{aligned}
&\sum_{\pi\in\widehat{G}}\sum_{\sigma\in\widehat{K}} m_\Gamma(\pi)\dim(\mathcal{H}_\pi\otimes V_\sigma)^K(1+\lambda_\pi^2+\lambda_\sigma^2)^{-N}\\
&=\sum_{k=0}^{\infty} \sum_{\substack{\pi,\sigma\\ k^2\leq\lambda_\pi^2+\lambda_\sigma^2<(k+1)^2}} m_\Gamma(\pi)\dim(\mathcal{H}_\pi\otimes V_\sigma)^K(1+\lambda_\pi^2+\lambda_\sigma^2)^{-N}\\
&\leq C\sum_{k=0}^{\infty}(1+(k+1)^{(4m+l)})(1+k^2)^{-N} < \infty.
\end{aligned}$$

We are now ready to prove Theorem 0.1. Given $\pi\in\widehat{G}$ and $\sigma\in\widehat{K}$, let $\mathcal{H}_\pi(\sigma)$ be the σ–isotypical subspace of $\mathcal{H}_\pi$, i.e., $\mathcal{H}_\pi(\sigma)$ is the sum of all irreducible subspaces of the representation $(\pi|K,\mathcal{H}_\pi)$ of K that are equivalent to σ. Note that

$$\dim\mathcal{H}_\pi(\sigma) = \dim(\mathcal{H}_\pi\otimes V_\sigma)^K$$

For all (π,σ) such that $\mathcal{H}_\pi(\sigma)\neq\{0\}$, we choose an orthonormal basis $e_{\pi,\sigma,i}$, $i\in I(\pi,\sigma)$, for $\mathcal{H}_\pi(\sigma)$. Each $e_{\pi,\sigma,i}$ belongs to the Gårding space $\mathcal{H}_\pi^\infty$ of $\mathcal{H}_\pi$ and the Casimir operators Ω and Ω_K of G and K, respectively, act on $e_{\pi,\sigma,i}$ by

$$R(\Omega)e_{\pi,\sigma,i} = \lambda_\pi e_{\pi,\sigma,i},\ R(\Omega_K)e_{\pi,\sigma,i} = \lambda_\sigma e_{\pi,\sigma,i}.$$

Let $\mathcal{C}^1(G)$ be Harish-Chandra's Schwartz space of integrable rapidly decreasing functions on G (cf. section 1), and let $f\in\mathcal{C}^1(G)$. Then the Hilbert–Schmidt norm of $R_\Gamma^d(f)$ is given by

$$\begin{aligned}\| R_\Gamma^d(f) \|_{HS}^2 &= \sum_{\sigma\in\widehat{K}}\sum_{\pi\in\widehat{G}} m_\Gamma(\pi) \sum_{i\in I(\pi,\sigma)} (R_\Gamma(f)e_{\pi,\sigma,i},\ R_\Gamma(f)e_{\pi,\sigma,i})\\ &= \sum_{\sigma\in\widehat{K}}\sum_{\pi\in\widehat{G}} m_\Gamma(\pi)(1+\lambda_\pi^2+\lambda_\sigma^2)^{-N}\\ \times \sum_{i\in I(\pi,\sigma)} &\int_G \left((\Omega^2+\Omega_K^2+\mathrm{Id})^N f\right)(g)(R_\Gamma(g)e_{\pi,\sigma,i}, R_\Gamma(f)e_{\pi,\sigma,i})\,dg.\end{aligned}$$

By Corollary (0.3) we obtain

$$\| R_\Gamma^d(f) \|_{HS}^2 \leq C \| (\Omega^2+\Omega_K^2+\mathrm{Id})^N f \|_{L^1(G)} \| f \|_{L^1(G)} .$$

for some constant $C>0$. Thus $R_\Gamma^d(f)$ is a Hilbert–Schmidt operator. The proof shows that the same conclusion holds for any $f\in C_c^p(G)$ with $p\geq N$. Fix an integer $p\geq 3\dim G$ and let $f\in\mathcal{C}^1(G)$. According to Lemma 4.5 in [W1] there exist $h\in\mathcal{C}^1(G)$, $\mu_1\in C_c^p(G)$ and $\mu_2\in C_c^\infty(G)$ such that

$$f=h*\mu_1+f*\mu_2.$$

Thus

$$R_\Gamma^d(f)=R_\Gamma^d(h)\circ R_\Gamma^d(\mu_1)+R_\Gamma^d(f)\circ R_\Gamma^d(\mu_2).$$

All operators on the right hand side are Hilbert–Schmidt operators and hence, $R_\Gamma^d(f)$ is a trace class operator. This completes the proof of Theorem (0.1).

Remark. For simplicity, we have restricted attention to arithmetic subgroups Γ. But in fact, this is no restriction at all. First, let G be a connected semisimple Lie group with finite center and suppose that the $\mathbb{R}$-rank of G is one. Let Γ be a discrete subgroup of G such that the volume of $\Gamma\backslash G$ with respect to any invariant measure is finite. Then for any $f\in\mathcal{C}^1(G)$, the restriction of $R_\Gamma(f)$ to the subspace of cusp forms $L^2_{cus}(\Gamma\backslash G)$ is of the trace class [W1, Theorem 4.3]. Moreover, it is not difficult to estimate the residual spectrum in the rank one case by the methods we used to deal with the rank one spectrum in general. As above, this implies that $R_\Gamma^d(f)$ is of the trace class. On the other hand, if $\mathbf{G}$ is a connected semisimple $\mathbb{R}$-group with $\mathrm{rank}_{\mathbb{R}}\mathbf{G}(\mathbb{R})\geq 2$ and Γ is an irreducible lattice in $\mathbf{G}(\mathbb{R})$, then by Margulis [Ma, Chapter IX], Γ is an arithmetic subgroup in the following sense: There exists a cannected non-commutative almost $\mathbb{Q}$-simple $\mathbb{Q}$-group $\mathbf{H}$ and an $\mathbb{R}$-epimorphism

$\tau : \mathbf{H} \to \mathbf{G}$ such that the Lie group $(\ker \tau)(\mathbb{R})$ is compact and the subgroups $\tau(\mathbf{H}(\mathbb{Z}))$ and Γ are commensurable. It is not difficult to extend Theorem 0.1 to such lattices.

9. The adèlic version

In this section we denote by G a connected reductive linear algebraic group defined over $\mathbb{Q}$. We fix an embedding of G into GL_n. For a given place v of $\mathbb{Q}$, let $G(\mathbb{Q}_v)$ be the group of $\mathbb{Q}_v$–rational points of G. Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$ and let $G(\mathbb{A})$ be the corresponding adèle–valued group. If f stands for the set of finite places of $\mathbb{Q}$ and $\mathbb{A}_f$ is the corresponding ring of finite adèles, then

$$G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f).$$

For any prime p, let

$$G(\mathbb{Z}_p) = \mathrm{GL}_n(\mathbb{Z}_p) \cap G(\mathbb{Q}_p).$$

This is an open compact subgroup of $G(\mathbb{Q}_p)$. Then any open compact subgroup K'_f of $G(\mathbb{A}_f)$ contains a subgroup of finite index K_f of the form

$$K_f = \prod_p K_p$$

where K_p is an open compact subgroup of $G(\mathbb{Q}_p)$ and for almost all primes p, one has $K_p = G(\mathbb{Z}_p)$. For any such K_f, the double coset space

$$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})K_f$$

is known to be finite [B2, Section 5]. If $x_1 = 1, x_2, ..., x_l$ is a set of representatives in $G(\mathbb{A}_f)$ of the cosets, the groups

$$\Gamma_i = (G(\mathbb{R}) \times x_i K_f x_i^{-1}) \cap G(\mathbb{Q}), \quad 1 \le i \le l,$$

are arithmetic subgroups of $G(\mathbb{R})$ and

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_f \cong \bigsqcup_{i=1}^{l} (\Gamma_i \backslash G(\mathbb{R})). \tag{9.1}$$

Let Z be the $\mathbb{Q}$–split component of the center of G and let $Z(\mathbb{R})^0$ be the connected component of 1 in $Z(\mathbb{R})$. Let dg be the Tamagawa measure on $G(\mathbb{A})$ (cf. [A1, p.329]). Then $Z(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A})$ has finite volume with respect to dg. Furthermore, it follows from (9.1) that

$$L^2(Z(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \cong \bigoplus_{i=1}^{l} L^2(Z(\mathbb{R})^0 \Gamma_i\backslash G(\mathbb{R})) \qquad (9.2)$$

as $G(\mathbb{R})$–modules. Let $G^1 = Z\backslash G$ and $G_1 = Z(\mathbb{R})^0\backslash G(\mathbb{R})$. Then $G^1(\mathbb{R})$ is the quotient of G_1 by a finite group and Γ_i, $1 \le i \le l$, is a discrete subgroup of G_1 of finite covolume whose intersection with the center of G_1 is cocompact. Thus (G_1, Γ_i), $1 \le i \le l$, satisfy the assumptions made in section 1.1 and therefore, our results apply to this setting.
Let R be the right regular representation of $G(\mathbb{A})$ on the Hilbert space $L^2(Z(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))$ and let $L^2_d(Z(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))$ be the smallest closed invariant subspace that contains all irreducible subrepresentations of R. Let $\widehat{G(\mathbb{A})}$ be the unitary dual of $G(\mathbb{A})$. Then

$$L^2_d\left(Z(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A})\right) = \bigoplus_{\pi \in \widehat{G(\mathbb{A})}} m(\pi)\mathcal{H}_\pi.$$

Any $\pi \in \widehat{G(\mathbb{A})}$ that occurs in R is a restricted tensor product $\pi = \otimes_v \pi_v$ of irreducible representations π_v of the local group $G(\mathbb{Q}_v)$. Thus

$$\pi = \pi_\infty \otimes \pi_f$$

where π_∞ and π_f are irreducible representations of $G(\mathbb{R})$ and $G(\mathbb{A}_f)$, respectively. Let $\mathcal{H}_{\pi_f}$ be the representation space of π_f and let $\mathcal{H}_{\pi_f}^{K_f}$ denote the subspace of K_f-invariant vectors in $\mathcal{H}_{\pi_f}$. This is a finite-dimensional subspace and

$$\mathcal{H}_\pi^{K_f} = \mathcal{H}_{\pi_\infty} \otimes \mathcal{H}_{\pi_f}^{K_f}.$$

Hence it follows that as $G(\mathbb{R})$-modules, we have

$$L^2_d(Z(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \cong \bigoplus_{i=1}^{l} L^2_d(Z(\mathbb{R})^0 \Gamma_i\backslash G(\mathbb{R})) \qquad (9.3)$$

Let

$$h = \prod_p h_p \qquad (9.4)$$

be a function on $G(\mathbb{A}_f)$ which satisfies the following properties:

1): For any prime p, the function h_p is locally constant with compact support,

2): For almost all primes p, the function h_p is the characteristic function of $G(\mathbb{Z}_p)$.

The space of all linear combinations of such functions will be denoted by $\mathcal{H}(G(\mathbb{A}_f))$. This is the Hecke algebra of locally constant functions of compact support on $G(\mathbb{A}_f)$. Let R_f denote the restriction of the regular representation to $G(\mathbb{A}_f)$. Then for any $h \in \mathcal{H}(G(\mathbb{A}_f))$ we can define the Hecke operator

$$R_f(h) = \int_{G(\mathbb{A}_f)} h(g) R_f(g)\, dg.$$

Since R_f is unitary, $R_f(h)$ is a bounded operator on the Hilbert space $L^2(Z(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))$. Moreover, there exists an open compact subgroup K_f of $G(\mathbb{A}_f)$, which depends on h, such that h is right K_f-invariant. Hence $R_f(h)$ maps the space $L^2(Z(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))$ into the subspace of K_f-invariant functions.

Let $h_\infty \in \mathcal{C}^1(G(\mathbb{R}))$ and $h_f \in \mathcal{H}(G(\mathbb{A}_f))$. Set

$$h = h_\infty \times h_f \tag{9.5}$$

and let

$$R(h) = \int_{G(\mathbb{A})} h(g) R(g)\, dg.$$

Then $R(h) = R_\infty(h_\infty) \cdot R_f(h_f)$ and $R_\infty(h_\infty)$ commutes with the orthogonal projection P_f of $L^2(Z(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))$ onto the subspace of K_f-invariant functions. Thus

$$R(h) = P_f R_\infty(h_\infty) P_f\, R_f(h_f).$$

Let R^d denote the restriction of R to the discrete subspace (9.3). By (9.3) and Theorem 0.1 it follows that $P_f R^d_\infty(h_\infty) P_f$ is a trace class operator. Since $R_f(h_f)$ is bounded, $R^d(h)$ is also a trace class operator. Let $\mathcal{C}^1(G(\mathbb{A}))$ be the space of all linear combinations of functions of the form (9.5). Then we have proved

Theorem 9.1. *For any $h \in \mathcal{C}^1(G(\mathbb{A}))$, the operator $R^d(h)$ is a trace class operator.*

In [Mu] we proved Theorem 9.1 under the additional assumption that h is K-finite for a maximal compact subgroup of $G(\mathbb{A})$. This assumption has now been eliminated.

One can also include characters of $Z(\mathbb{R})^0$. Let ξ be a quasi-character of $Z(\mathbb{R})^0$. Denote by $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \xi)$ the space of functions ϕ on $G(\mathbb{Q})\backslash G(\mathbb{A})$ such that

$$\phi(zg) = \xi(z)\phi(g), \quad z \in Z(\mathbb{R})^0,\ g \in G(\mathbb{A})$$

and which are square integrable modulo $Z(\mathbb{R})^0$. Here the latter condition is well defined, since $G(\mathbb{A})$ is the direct product of $Z(\mathbb{R})^0$ with a normal subgroup $G(\mathbb{A})^1$ which contains $G(\mathbb{Q})$. Then $G(\mathbb{A})$ acts by right translations on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\xi)$. We denote this representation by R_ξ. Let R_ξ^d be the restriction of R_ξ to the corresponding discrete subspace. Then as above, it follows that for any $h\in\mathcal{C}^1(G(\mathbb{A}))$, the operator $R_\xi^d(h)$ is a trace class operator.

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