THE ANALYTIC TORSION AND ITS ASYMPTOTIC BEHAVIOUR FOR SEQUENCES OF HYPERBOLIC MANIFOLDS OF FINITE VOLUME

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Abstract. In this paper we study the regularized analytic torsion of finite volume hyperbolic manifolds. We consider sequences of coverings X_i of a fixed hyperbolic orbifold X_0. Our main result is that for certain sequences of coverings and strongly acyclic flat bundles, the analytic torsion divided by the index of the covering, converges to the L^2-torsion. Our results apply to certain sequences of arithmetic groups, in particular to sequences of principal congruence subgroups of $\mathrm{SO}^0(d,1)(\mathbb{Z})$ and to sequences of principal congruence subgroups or Hecke subgroups of Bianchi groups.

1. Introduction

The aim of this paper is to extend the results of Bergeron and Venkatesh [BV] on the asymptotic equality of analytic and L^2-torsion for strongly acyclic representations from the compact to the finite volume case.

Therefore, we shall first recall the results of Bergeron and Venkatesh about the compact case. Let G be a semisimple Lie group of non-compact type. Let K be a maximal compact subgroup of G and let $\widetilde{X} = G/K$ be the associated Riemannian symmetric space endowed with a G-invariant metric. Let $\Gamma \subset G$ be a co-compact discrete subgroup. For simplicity we assume that Γ is torsion free. Let $X := \Gamma\backslash\widetilde{X}$. Then X is a compact locally symmetric manifold of non-positive curvature. Let τ be an irreducible finite dimensional complex representation of G. Let $E_\tau \to X$ be the flat vector bundle associated to the restriction of τ to Γ. By [MtM], E_τ can be equipped with a canonical Hermitian fibre metric, called admissible, which is unique up to scaling. Let $\Delta_P(\tau)$ be the Laplace operator on E_τ-valued p-forms with respect to the metric on X and in E_τ. Let $\zeta_p(s;\tau)$ be the zeta function of $\Delta_p(\tau)$ (see [Sh]). Then the analytic torsion $T_X(\tau) \in \mathbb{R}^+$ is defined by

$$T_X(\tau) := \exp\left(\frac{1}{2}\sum_{p=1}^{d}(-1)^p p \frac{d}{ds}\zeta_p(s;\tau)\big|_{s=0}\right). \tag{1.1}$$

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On the other hand there is the L^2-torsion $T_X^{(2)}(\tau)$ (see [Lo]). Since the heat kernels on $\widetilde{X}$ are G-invariant, one has

$$\log T_X^{(2)}(\tau) = \operatorname{vol}(X) t_{\widetilde{X}}^{(2)}(\tau), \tag{1.2}$$

where $t_{\widetilde{X}}^{(2)}(\tau)$ is a constant that depends only on $\widetilde{X}$ and τ. It is an interesting problem to see if the L^2-torsion can be approximated by the torsion of finite coverings $X_i \to X$. This problem has been studied by Bergeron and Venkatesh [BV] under a certain non-degeneracy condition on τ. Representations which satisfy this condition are called *strongly acyclic*. One of the main results of [BV] is as follows. Let $X_i \to X$, $i \in \mathbb{N}$, be a sequence of finite coverings of X. Let τ be strongly acyclic. Let $\operatorname{inj}(X_i)$ denote the injectivety radius of X_i and assume that $\operatorname{inj}(X_i) \to \infty$ as $i \to \infty$. Then by [BV, Theorem 4.5] one has

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{\operatorname{vol}(X_i)} = t_{\widetilde{X}}^{(2)}(\tau). \tag{1.3}$$

If $\operatorname{rk}_{\mathbb{C}}(G) - \operatorname{rk}_{\mathbb{C}}(K) = 1$, one can show that $t_{\widetilde{X}}^{(2)}(\tau) \neq 0$. Using the equality of analytic torsion and Reidemeister torsion [Mu2], Bergeron and Venkatesh [BV] used this result to study the growth of torsion in the cohomology of cocompact arithmetic groups. Furthermore, recently P. Scholze [Sch] has shown the existence of Galois representations associated with mod p cohomology of locally symmetric spaces for GL_n over a totally real or CM field. This makes it desirable to extend these results in various directions. Especially, one would like to extend (1.3) to the finite volume case. However, due to the presence of the continuous spectrum of the Laplace operators in the non-compact case, one encounters serious technical difficulties in attempting to generalize (1.3) to the finite volume case. In [Ra1] J. Raimbault has dealt with finite volume hyperbolic 3-manifolds. In [Ra2] he applied this to study the growth of torsion in the cohomology for certain sequences of congruence subgroups of Bianchi groups. His result generalized the exponential growth of torsion, obtained in [Pf2] for local systems induced from the even symmetric powers of the standard representation of $\mathrm{SL}_2(\mathbb{C})$, to all strongly acyclic local systems and furthermore they implied that the limit of the normalized torsion size exists. The main purpose of the present paper is to extend (1.3) to hyperbolic manifolds of finite volume and arbitrary dimension.

So from now on we let $G = \mathrm{Spin}(d,1)$, $K = \mathrm{Spin}(d)$ or $G = \mathrm{SO}^0(d,1)$ and $K = \mathrm{SO}(d)$ for $d > 1$. Then K is a maximal compact subgroup of G. Let $\widetilde{X} = G/K$. Choose an invariant Riemannian metric on $\widetilde{X}$. If the metric is suitably normalized, $\widetilde{X}$ is isometric to the d-dimensional hyperbolic space $\mathbb{H}^d$. Let $\Gamma \subset G$ be a torsion free lattice, i.e., Γ is a discrete, torsion free subgroup with $\operatorname{vol}(\Gamma\backslash G) < \infty$. Let $X = \Gamma\backslash\widetilde{X}$. Then X is an oriented d-dimensional hyperbolic manifold of finite volume. Let τ be an irreducible finite dimensional complex representation of G and let $E_\tau \to X$ be the flat vector bundle associated to τ as above, endowed with an admissible Hermitian fibre metric. The first problem is to define the analytic torsion if X is non-compact, which is the case we are interested in. Then the Laplace operator $\Delta_p(\tau)$ has a non-empty continuous spectrum and hence, the zeta function $\zeta_p(s;\tau)$ can not be defined in the usual way. It requires an additional regularization. We use the method introduced in [MP2]. One uses an appropriate height function to truncate X at sufficiently high level $Y > Y_0$ to get a compact submanifold $X(Y) \subset X$ with boundary $\partial X(Y)$. Let $K^{p,\tau}(t,x,y)$ be the kernel of the heat operator $\exp(-t\Delta_p(\tau))$. Then it follows that there exists $\alpha(t) \in \mathbb{R}$ such that $\int_{X(Y)} \operatorname{tr} K^{p,\tau}(x,x,t)\,dx - \alpha(t)\log Y$ has a limit as $Y \to \infty$. Then we put

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) := \lim_{Y\to\infty}\left(\int_{X(Y)} \operatorname{tr} K^{p,\tau}(t,x,x)\,dx - \alpha(t)\log Y\right). \tag{1.4}$$

As pointed out in [MP2, Remark 5.4], the regularized trace is not uniquely defined. It depends on the choice of truncation parameters on the manifold X. However, if a locally symmetric space $X_0 = \Gamma_0\backslash\widetilde{X}$ of finite volume is given and if truncation parameters on X_0 are fixed, then every locally symmetric manifold X which is a finite covering of X_0 is canonically equipped with truncation parameters: One simply pulls back the truncation on X_0 to a truncation on X via the covering map. This will be explained in detail in section 6 of the present paper.

We remark that we do not assume that the group Γ_0 is torsion-free. In fact, the typical example for Γ_0 in the arithmetic case will be $\Gamma_0 = \mathrm{SO}^0(d,1)(\mathbb{Z})$ or $\Gamma_0 = \mathrm{SL}_2(\mathcal{O}_D)$, where $\mathcal{O}_D$ is the ring of integers of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-D})$, $D \in \mathbb{N}$ being square-free. Then Γ will denote, for example, a principal congruence subgroup. However, we assume that Γ is not only a torsion-free lattice but also that Γ satisfies the following condition: For each Γ-cuspidal parabolic subgroup P' of G one has

$$\Gamma \cap P' = \Gamma \cap N_{P'}, \tag{1.5}$$

where $N_{P'}$ denotes the nilpotent radical of P'. This condition holds naturally, for example, for all principal congruence subgroups of sufficiently high level.

Let θ be the Cartan involution of G with respect to our choice of K. Let $\tau_\theta = \tau \circ \theta$. If $\tau \not\cong \tau_\theta$, it can be shown that $\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ is exponentially decreasing as $t \to \infty$ and admits an asymptotic expansion as $t \to 0$. Therefore, the regularized zeta function $\zeta_p(s;\tau)$ of $\Delta_p(\tau)$ can be defined as in the compact case by

$$\zeta_p(s;\tau) := \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) t^{s-1}\,dt. \tag{1.6}$$

The integral converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > d/2$ and admits a meromorphic extension to the whole complex plane. The zeta function is regular at $s = 0$. So in analogy with the compact case, the analytic torsion $T_X(\tau) \in \mathbb{R}^+$ can be defined by the same formula (1.1).

In even dimensions, $T_X(\tau)$ is rather trivial (see [MP2]). So we assume that $d = 2n+1$, $n \in \mathbb{N}$. To formulate our main result, we need to introduce some notation. We let Γ_0 be a fixed lattice in G and we let $X_0 := \Gamma_0\backslash\widetilde{X}$. We let Γ_i, $i \in \mathbb{N}$ be a sequence of finite index torsion-free subgroups of Γ_0. Then following Raimbault [Ra1], in definition 8.2 we define the condition on the sequence Γ_i to be cusp-uniform. This condition is, roughly spoken, a condition on the shape of the 2n-tori which form the cross-sections of the cusps of the manifolds $X_i := \Gamma_i\backslash\widetilde{X}$. For more details, we refer to section 8. We let $\ell(\Gamma_i)$ be the length of the shortest closed geodesic on X_i. We assume that truncation parameters on the orbifold X_0 are fixed and for each i and τ with $\tau \neq \tau_\theta$ we define the analytic torsion with respect to the induced truncation parameters on X_i as above. Then our main result can be stated as the following theorem.

Theorem 1.1. *Let Γ_0 be a lattice in G. Let Γ_i, $i \in \mathbb{N}$ be a sequence of finite-index subgroups of Γ_0 which is cusp-uniform. Assume that for $i \geq 1$ the group Γ_i is torsion free and satisfies* (1.5). *Let $\mathfrak{P}_{\Gamma_i} = \{P_{i,j},\ j = 1, \dots, \kappa(\Gamma_i)\}$ be a set of representatives of Γ_i-conjugacy classes of Γ_i-cuspidal parabolic subgroups of G and let $N_{P_{i,j}}$ denote the nilpotent radical of $P_{i,j}$. Assume that $\lim_{i\to\infty} \ell(\Gamma_i) = \infty$ and that*

$$\lim_{i\to\infty} \frac{1}{[\Gamma_0 : \Gamma_i]} \big(\kappa(\Gamma_i) + \sum_{j=1}^{\kappa(\Gamma_i)} \log[\Gamma_0 \cap N_{P_{i,j}} : \Gamma_i \cap N_{P_{i,j}}]\big) = 0. \tag{1.7}$$

Then for $X_i := \Gamma_i \backslash \widetilde{X}$ and every τ with $\tau \neq \tau_\theta$ one has

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{[\Gamma_0 : \Gamma_i]} = t^{(2)}_{\widetilde{X}}(\tau) \operatorname{vol}(X_0).$$

We remark that the condition (1.7) is independent of the choice of $\mathfrak{P}_{\Gamma_i}$. Furthermore, one immediately sees that it is satisfied, for example, if

$$\lim_{i\to\infty} \frac{\kappa(\Gamma_i) \log[\Gamma_0 : \Gamma_i]}{[\Gamma_0 : \Gamma_i]} = 0. \tag{1.8}$$

For hyperbolic 3-manifolds, Theorem 1.1 was proved by J. Raimbault [Ra1] under additional assumptions on the intertwining operators. We emphasize that we don't need this assumption.

For sequences of cusp uniform normal subgroups Γ_i of Γ_0 which exhaust Γ_0, the assumption (1.7) is easily verified and we have the following theorem for the case of normal subgroups.

Theorem 1.2. *Let Γ_0 be a lattice in G and let Γ_i, $i \in \mathbb{N}$, be a sequence of finite-index normal subgroups which is cusp uniform and such that each Γ_i, $i \geq 1$, is torsion-free and satisfies* (1.5). *If $\lim_{i\to\infty}[\Gamma_0 : \Gamma_i] = \infty$ and if each $\gamma_0 \in \Gamma_0 - \{1\}$ only belongs to finitely many Γ_i, then for each τ with $\tau \neq \tau_\theta$ one has*

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{[\Gamma : \Gamma_i]} = t^{(2)}_{\widetilde{X}}(\tau) \operatorname{vol}(X_0). \tag{1.9}$$

In particular, if under the same assumptions Γ_i is a tower of normal subgroups, i.e. $\Gamma_{i+1} \subset \Gamma_i$ for each i and $\cap_i \Gamma_i = \{1\}$, then (1.9) *holds.*

We shall now give applications of our main results to the case of arithmetic groups. Firstly let $\Gamma_0 := \mathrm{SO}^0(d,1)(\mathbb{Z})$. Then Γ_0 is a lattice in $\mathrm{SO}^0(d,1)$. For $q \in \mathbb{N}$ let $\Gamma(q)$ be the principal congruence subgroup of level q (see section 10). Using a result of Deitmar and Hoffmann [DH], it follows that the family of principal congruence subgroups is cusp uniform (see Lemma 10.1). Thus, Theorem 1.2 implies the following corollary.

Corollary 1.3. *For any finite-dimensional irreducible representation* τ *of* $\mathrm{SO}^0(d,1)$ *with* $\tau \neq \tau_\theta$ *the principal congruence subgroups* $\Gamma(q)$, $q \geq 3$, *of* $\Gamma_0 := \mathrm{SO}^0(d,1)(\mathbb{Z})$ *satisfy*

$$\lim_{q\to\infty} \frac{\log T_{X_q}(\tau)}{[\Gamma : \Gamma(q)]} = t^{(2)}_{\widetilde{X}}(\tau)\,\mathrm{vol}(X_0),$$

where $X_q := \Gamma(q)\backslash\mathbb{H}^d$ *and* $X_0 := \Gamma_0\backslash\mathbb{H}^d$.

Secondly, we give some specific applications in the 3-dimensional case. There is a natural isomorphism $\mathrm{Spin}(3,1) \cong \mathrm{SL}_2(\mathbb{C})$. If ρ is the standard-representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$, then the finite-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ are given as $\mathrm{Sym}^m\,\rho \otimes \mathrm{Sym}^n\,\overline{\rho}$, $m,n \in \mathbb{N}$. Here Sym^k denotes the k-th symmetric power and $\overline{\rho}$ denotes the complex-conjugate representation of ρ. One has $(\mathrm{Sym}^m\,\rho \otimes \mathrm{Sym}^n\,\overline{\rho})_\theta = \mathrm{Sym}^n\,\rho \otimes \mathrm{Sym}^m\,\overline{\rho}$. For $D \in \mathbb{N}$ square-free let $\mathcal{O}_D$ be the ring of integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-D})$ and let $\Gamma(D) := \mathrm{SL}_2(\mathcal{O}_D)$. Then $\Gamma(D)$ is a lattice in $\mathrm{SL}_2(\mathbb{C})$. If $\mathfrak{a}$ is a non-zero ideal in $\mathcal{O}_D$, let $\Gamma(\mathfrak{a})$ be the associated principal congruence subgroup of level $\mathfrak{a}$ (see section 11). Then Theorem 1.2 implies the following corollary.

Corollary 1.4. *If* $\mathfrak{a}_i$ *is a sequence of non-zero ideals in* $\mathcal{O}_D$ *such that each* $N(\mathfrak{a}_i)$ *is sufficiently large and such that* $\lim_{i\to\infty} N(\mathfrak{a}_i) = \infty$, *then for any representation* $\tau = \mathrm{Sym}^n\,\rho \otimes \mathrm{Sym}^m\,\bar{\rho}$ *with* $m \neq n$ *and for* $X_D := \Gamma(D)\backslash\mathbb{H}^3$ *and* $X_i := \Gamma(\mathfrak{a}_i)\backslash\mathbb{H}^3$ *one has*

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{[\Gamma(D) : \Gamma(\mathfrak{a}_i)]} = t^{(2)}_{\widetilde{X}}(\tau)\,\mathrm{vol}(X_D). \tag{1.10}$$

Finally, due to their arithmetic significance, in the 3-dimensional case we also want to treat Hecke subgroups of the Bianchi groups. These groups do not fall directly in the framework of our two main theorems, since their systole does not necessarily tend to infinity if their index in the Bianchi groups does. However, a slight modification of the proof of our main results will also give the corresponding statement for these groups. More precisely, for a non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_D$ let $\Gamma_0(\mathfrak{a})$ be the corresponding Hecke subgroup. Actually, since these groups are not torsion-free, we have to take a fixed torsion-free subgroup Γ' of $\Gamma(D)$ of finite index which satisfies assumption (1.5), for example a principal congruence subgroup of sufficiently high level, and consider the intersections $\Gamma_0'(\mathfrak{a}) := \Gamma_0(\mathfrak{a}) \cap \Gamma'$. Then we have the following theorem:

Theorem 1.5. *If* $\mathfrak{a}_i$ *is a sequence of non-zero ideals in* $\mathcal{O}_D$ *such that* $\lim_{i\to\infty} N(\mathfrak{a}_i) = \infty$, *then for any representation* $\tau = \mathrm{Sym}^n\,\rho \otimes \mathrm{Sym}^m\,\bar{\rho}$ *with* $m \neq n$ *and for* $X_D := \Gamma(D)\backslash\mathbb{H}^3$, $X_i' := \Gamma_0'(\mathfrak{a}_i)\backslash\mathbb{H}^3$ *one has*

$$\lim_{i\to\infty} \frac{\log T_{X_i'}(\tau)}{[\Gamma(D) : \Gamma_0'(\mathfrak{a}_i)]} = t^{(2)}_{\widetilde{X}}(\tau)\,\mathrm{vol}(X_D). \tag{1.11}$$

We shall now outline our method to prove our main results. Let $d = 2n+1$. We assume that the representation τ is not invariant under the Cartan involution. To indicate the dependence of the heat operator, the regularized trace and other quantities on the covering X_i, we use the subscript X_i. Let

$$K_{X_i}(t,\tau) := \frac{1}{2}\sum_{p=1}^{d}(-1)^p p \operatorname{Tr}_{\mathrm{reg};X_i}\left(e^{-t\Delta_{X_i,p}(\tau)}\right). \tag{1.12}$$

As observed above, $K_{X_i}(t,\tau)$ is exponentially decreasing as $t \to \infty$ and admits an asymptotic expansion as $t \to 0$. Thus the analytic torsion $T_{X_i}(\tau) \in \mathbb{R}^+$ can be defined by

$$\log T_{X_i}(\tau) = \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty K_{X_i}(t,\tau)t^{s-1}\,dt\right)\bigg|_{s=0}. \tag{1.13}$$

The integral converges for $\mathrm{Re}(s) > d/2$ and its value at $s = 0$ is defined by analytic continuation. For $T > 0$ write

$$\log T_{X_i}(\tau) = \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T K_{X_i}(t,\tau)t^{s-1}\,dt\right)\bigg|_{s=0} + \int_T^\infty K_{X_i}(t,\tau)t^{-1}\,dt. \tag{1.14}$$

Now we study the behaviour as $i \to \infty$ of the terms on the right hand side. We start with the second term. Our assumption about τ implies that the spectrum of the Laplacians $\Delta_{X_i,p}$, $i \in \mathbb{N}$, have a uniform positive lower bound. Using the definition (6.12) of the regularized trace, it follows that there exist constants $C_i, c > 0$ such that for $t \geq 10$ we have

$$|K_{X_i}(t,\tau)| \leq C_i e^{-ct}$$

The problem is to estimate C_i. In Proposition 7.2, we will show that there exists a constant C such that for each i and each $t \geq 10$ one has an estimation

$$|\operatorname{Tr}_{\mathrm{reg};X_i}\left(e^{-t\Delta_{X_i,p}(\tau)}\right)| \leq Ce^{-ct}\left(\operatorname{Tr}_{\mathrm{reg};X_i}\left(e^{-\Delta_{X_i,p}(\tau)}\right) + \mathrm{vol}(X_i)\right) \tag{1.15}$$

for each $p = 1,\dots,d$. This estimate is easy to prove in the compact case and one does not need the term $\mathrm{vol}(X_i)$ here. More precisely, if X_i is compact and if $\lambda_1(i) \leq \lambda_2(i) \leq \cdots$ are the eigenvalues of $\Delta_{X_i,p}(\tau)$, counted with multiplicity, then for $t \geq 2$ we have

$$\operatorname{Tr}\left(e^{-t\Delta_{X_i,p}(\tau)}\right) = \sum_{j=1}^\infty e^{-t\lambda_j(i)} \leq e^{-t\lambda_1(i)/2}\sum_{j=1}^\infty e^{-\lambda_j(i)} = e^{-t\lambda_1(i)/2}\operatorname{Tr}\left(e^{-\Delta_{X_i,p}(\tau)}\right),$$

and the assumption on τ implies that there is $c > 0$ such that $\lambda_1(i) \geq c$ for all $i \in \mathbb{N}$.

In the non-compact case, the proof of equation (1.15) is more difficult since one also has to deal with the contribution of the continuous spectrum to the regularized trace, which is given by the logarithmic derivative of certain intertwining operators. The key ingredient of our approach to treat the terms involving the intertwining operators is the factorization of the determinant of the intertwining operators, which we will study carefully under coverings in section 4. Our main result is Theorem 4.6.

To estimate $\operatorname{Tr}_{\mathrm{reg};X_i}\left(e^{-\Delta_{X_i,p}}\right)$ we use that the regularized trace of the heat operator, up to a minor term, is equal to the spectral side of the Selberg trace formula applied to the heat operator (see [MP2]). Then we apply the Selberg trace formula to express the regularized trace through the geometric side of the trace formula. More precisely, let $\widetilde{E}_\tau$

be the homogeneous vector bundle over $\widetilde{X} = G/K$ associated to $\tau|_K$ and let $\widetilde{\Delta}_p(\tau)$ be the Laplacian on $\widetilde{E}_\tau$-valued p-forms on $\widetilde{X}$. The heat operator $e^{-t\widetilde{\Delta}_p(\tau)}$ is a convolution operator with kernel $H_t^{\nu_p(\tau)}\colon G \to \operatorname{End}(\Lambda^p\mathfrak{p}^* \otimes V_\tau)$. Let $h_t^{\nu_p(\tau)}(g) = \operatorname{tr} H_t^{\nu_p(\tau)}(g)$, $g \in G$. Then by the trace formula we get

$$\operatorname{Tr}_{\mathrm{reg};X_i}\left(e^{-t\Delta_{X_i,p}(\tau)}\right) = I_{X_1}(h_t^{\tau,p}) + H_{X_1}(h_t^{\tau,p}) + T'_{X_1}(h_t^{\tau,p}) + S_{X_1}(h_t^{\tau,p}), \tag{1.16}$$

where I_{X_i}, H_{X_i}, T'_{X_i}, and S_{X_i} are distributions on G associated to the identity, the hyperbolic and the parabolic conjugacy classes of Γ_i, respectively. The distributions are described in section 8. For example, the identity contribution is given by

$$I_{X_i}(h_t^{\tau,p}) = \operatorname{vol}(X_i) h_t^{\tau,p}(1).$$

Now we put $t = 1$ and estimate each term on the right hand side of (1.16). In this way we can conclude that there exist $C, c > 0$ such that for $t \geq 10$ and all $i \in \mathbb{N}$ we have

$$|K_{X_i}(t,\tau)| \leq C(\operatorname{vol}(X_i) + \kappa(X_i) + \alpha(X_i))e^{-ct},$$

where $\alpha(X_i)$ is defined in terms of the lattices associated to the cross sections of the cusps of X_i (see (8.11)). Using the assumptions of Theorem 1.1, we finally get that there exist $C, c > 0$ such that

$$\frac{1}{\operatorname{vol}(X_i)}\left|\int_T^\infty K_{X_i}(t,\tau)t^{-1}\,dt\right| \leq Ce^{-cT} \tag{1.17}$$

for all $i \in \mathbb{N}$.

To deal with the first term on the right hand side of (1.14), put

$$k_t^\tau := \frac{1}{2}\sum_{p=1}^d (-1)^p p h_t^{\tau,p}. \tag{1.18}$$

Then by (1.12) and (1.16) we get

$$K_{X_i}(t,\tau) = I_{X_1}(k_t^\tau) + H_{X_1}(k_t^\tau) + T'_{X_1}(k_t^\tau) + S_{X_1}(k_t^\tau). \tag{1.19}$$

Now we take the partial Mellin transform of each term on the right hand side, take its derivative at $s = 0$, and study its behaviour as $i \to \infty$. For the contribution of the identity we get $\operatorname{vol}(X_i)(t_{\widetilde{X}}^{(2)}(\tau) + O(e^{-cT}))$. Using the assumptions of Theorem 1.1, it follows that the other terms, divided by $[\Gamma_0 : \Gamma_i]$, converge to 0. Thus we get

$$\lim_{i\to\infty}\frac{1}{[\Gamma_0:\Gamma_i]}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T K_{X_i}(t,\tau)t^{s-1}\,dt\right)\bigg|_{s=0} = \operatorname{vol}(X_0)(t_{\widetilde{X}}^{(2)}(\tau) + O(e^{-cT})). \tag{1.20}$$

Combining (1.20), (1.14) and (1.17), and using that $T > 0$ is arbitrary, Theorem 1.1 follows.

Theorem 1.2 is a simple consequence of Theorem 1.1. For the corollaries we only need to verify that the assumptions of the main theorems are satisfied.

The paper is organized as follows. In section 2 we fix some notation and collect some basic facts. In section 3 we recall some facts about Eisenstein series and intertwining operators. Section 4 deals with the factorization of the determinant of the C-matrix.

The main result is Theorem 4.6. In section 5 we consider Bochner-Laplace operators and establish some properties of their spectrum. In section 6 we introduce the regularized trace of the heat operator using the truncated heat kernel and express it in terms of spectral data of the corresponding Laplace operator. Section 7 deals with the estimation of the regularized trace of the heat operator for large time. The bound obtained in Proposition 7.2 involves the regularized trace of the heat operator at time $t = 1$. In section 8 we use the geometric side of the trace formula to study this term in detail. Of particular importance are the constants obtained from the contribution of the parabolic conjugacy classes which we need to estimate uniformly with respect to the covering. In section 9 we prove our main theorems. In the final sections 10 and 11 we apply our results to derive the corollaries.

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2. Preliminaries

We let $d = 2n+1$, $n \in \mathbb{N}$ and we let either $G = \mathrm{SO}^0(d,1)$, $K = \mathrm{SO}(d)$ or $G = \mathrm{Spin}(d,1)$, $K = \mathrm{Spin}(d)$. Then K is a maximal compact subgroup of G and if the quotient $\widetilde{X} := G/K$ is equipped with the G-invariant metric defined by (2.3), then $\widetilde{X}$ is isometric to the d-dimensional hyperbolic space. Let $G = NAK$ be the Iwasawa decomposition of G as in [MP2, section 2] and let M be the centralizer of A in K. Let $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{a}$, $\mathfrak{k}$, $\mathfrak{m}$ denote the Lie algebras of G, N, A K and M. Fix a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{m}$. Then

$$\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{b}$$

is a Cartan subalgebra of $\mathfrak{g}$. We can identify $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{so}(d+1,\mathbb{C})$. Let $e_1 \in \mathfrak{a}^*$ be the positive restricted root defining $\mathfrak{n}$. Then we fix $e_2, \dots, e_{n+1} \in i\mathfrak{b}^*$ such that the positive roots $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ are chosen as in [Kn2, page 684-685] for the root system D_{n+1}. We let $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ be the set of roots of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ which do not vanish on $\mathfrak{a}_{\mathbb{C}}$. The positive roots $\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ are chosen such that they are restrictions of elements from $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. For $j = 1, \dots, n+1$ let

$$\rho_j := n + 1 - j. \tag{2.1}$$

Then the half-sums of positive roots ρ_G and ρ_M, respectively, are given by

$$\rho_G := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \alpha = \sum_{j=1}^{n+1} \rho_j e_j; \quad \rho_M := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})} \alpha = \sum_{j=2}^{n+1} \rho_j e_j. \tag{2.2}$$

Put

$$\langle X, Y \rangle_\theta := -\frac{1}{2(d-1)} B(X, \theta(Y)), \quad X, Y \in \mathfrak{g}. \tag{2.3}$$

Let $\mathbb{Z}\left[\frac{1}{2}\right]^j$ be the set of all $(k_1, \dots, k_j) \in \mathbb{Q}^j$ such that either all k_i are integers or all k_i are half integers. Let $\mathrm{Rep}(G)$ denote the set of finite dimensional irreducible representations

τ of G. These are parametrized by their highest weights

$$\Lambda(\tau)=k_1(\tau)e_1+\cdots+k_{n+1}(\tau)e_{n+1};\;\; k_1(\tau)\geq k_2(\tau)\geq\cdots\geq k_n(\tau)\geq|k_{n+1}(\tau)|, \tag{2.4}$$

where $(k_1(\tau),\dots,k_{n+1}(\tau))$ belongs to $\mathbb{Z}\left[\frac{1}{2}\right]^{n+1}$ if $G=\mathrm{Spin}(d,1)$ and to $\mathbb{Z}^{n+1}$ if $G=\mathrm{SO}^0(d,1)$. Moreover, the finite dimensional irreducible representations $\nu\in\hat{K}$ of K are parametrized by their highest weights

$$\Lambda(\nu)=k_2(\nu)e_2+\cdots+k_{n+1}(\nu)e_{n+1};\;\; k_2(\nu)\geq k_3(\nu)\geq\cdots\geq k_n(\nu)\geq k_{n+1}(\nu)\geq 0, \tag{2.5}$$

where $(k_2(\nu),\dots,k_{n+1}(\nu))$ belongs to $\mathbb{Z}\left[\frac{1}{2}\right]^{n}$ if $G=\mathrm{Spin}(d,1)$ and to $\mathbb{Z}^n$ if $G=\mathrm{SO}^0(d,1)$. Finally, the finite dimensional irreducible representations $\sigma\in\hat{M}$ of M are parametrized by their highest weights

$$\Lambda(\sigma)=k_2(\sigma)e_2+\cdots+k_{n+1}(\sigma)e_{n+1};\;\; k_2(\sigma)\geq k_3(\sigma)\geq\cdots\geq k_n(\sigma)\geq|k_{n+1}(\sigma)|, \tag{2.6}$$

where $(k_2(\sigma),\dots,k_{n+1}(\sigma))$ belongs to $\mathbb{Z}\left[\frac{1}{2}\right]^{n}$, if $G=\mathrm{Spin}(d,1)$, and to $\mathbb{Z}^n$, if $G=\mathrm{SO}^0(d,1)$. For $\nu\in\hat{K}$ and $\sigma\in\hat{M}$ we denote by $[\nu:\sigma]$ the multiplicity of σ in the restriction of ν to M.

Let Ω, Ω_K and Ω_M be the Casimir elements of G, K and M, respectively, with respect to the inner product (2.3). Then by a standard computation one has

$$\Omega=H_1^2-2nH_1+\Omega_M \qquad \mathrm{mod}\ \mathfrak{n}U(\mathfrak{g}). \tag{2.7}$$

Let M' be the normalizer of A in K and let $W(A)=M'/M$ be the restricted Weyl-group. It has order two and it acts on the finite-dimensional representations of M as follows. Let $w_0\in W(A)$ be the non-trivial element and let $m_0\in M'$ be a representative of w_0. Given $\sigma\in\hat{M}$, the representation $w_0\sigma\in\hat{M}$ is defined by

$$w_0\sigma(m)=\sigma(m_0mm_0^{-1}),\quad m\in M.$$

Let $\Lambda(\sigma)=k_2(\sigma)e_2+\cdots+k_{n+1}(\sigma)e_{n+1}$ be the highest weight of σ as in (2.6). Then the highest weight $\Lambda(w_0\sigma)$ of $w_0\sigma$ is given by

$$\Lambda(w_0\sigma)=k_2(\sigma)e_2+\cdots+k_n(\sigma)e_n-k_{n+1}(\sigma)e_{n+1}. \tag{2.8}$$

Let $P:=NAM$. We equip $\mathfrak{a}$ with the norm induced from the restriction of the normalized Killing form on $\mathfrak{g}$. Let $H_1\in\mathfrak{a}$ be the unique vector which is of norm one and such that the positive restricted root, implicit in the choice of N, is positive on H_1. Let $\exp:\mathfrak{a}\to A$ be the exponential map. Every $a\in A$ can be written as $a=\exp\log a$, where $\log a\in\mathfrak{a}$ is unique. For $t\in\mathbb{R}$, we let $a(t):=\exp(tH_1)$. If $g\in G$, we define $n(g)\in N$, $H(g)\in\mathbb{R}$ and $\kappa(g)\in K$ by

$$g=n(g)a(H(g))\kappa(g).$$

Now let P' be any parabolic subgroup. Then there exists a $k_{P'}\in K$ such that $P'=N_{P'}A_{P'}M_{P'}$ with $N_{P'}=k_{P'}Nk_{P'}^{-1}$, $A_{P'}=k_{P'}Ak_{P'}^{-1}$, $M_{P'}=k_{P'}Mk_{P'}^{-1}$. We choose a set of $k_{P'}$'s, which will be fixed from now on. Let $k_P=1$. We let $a_{P'}(t):=k_{P'}a(t)k_{P'}^{-1}$. If $g\in G$, we define $n_{P'}(g)\in N_{P'}$, $H_{P'}(g)\in\mathbb{R}$ and $\kappa_{P'}(g)\in K$ by

$$g=n_{P'}(g)a_{P'}(H_{P'}(g))\kappa_{P'}(g) \tag{2.9}$$

and we define an identification $\iota_{P'}$ of $(0,\infty)$ with $A_{P'}$ by $\iota_{P'}(t) := a_{P'}(\log(t))$. For $Y > 0$, let $A^0_{P'}[Y] := \iota_{P'}(Y,\infty)$ and $A_{P'}[Y] := \iota_{P'}[Y,\infty)$. For $g \in G$ as in (2.9) we let $y_{P'}(g) := e^{H_{P'}(g)}$.

Let Γ be a discrete subgroup of G such that $\mathrm{vol}(\Gamma\backslash G) < \infty$. We do not assume at the moment that Γ is torsion-free. Let $X := \Gamma\backslash\widetilde{X}$. Let $\mathrm{pr}_X : G \to X$ be the projection. A parabolic subgroup P' of G is called a Γ-cuspidal parabolic subgroup if $\Gamma \cap N_{P'}$ is a lattice in $N_{P'}$. Let $\mathfrak{P}_\Gamma = \{P_1, \dots, P_{\kappa(\Gamma)}\}$ be a set of representatives of Γ-conjugacy-classes of Γ-cuspidal parabolic subgroups of G. Then for each $P' \in \mathfrak{P}_\Gamma$ one has

$$\Gamma \cap P' = \Gamma \cap (M_{P'}N_{P'}). \tag{2.10}$$

The number

$$\kappa(X) := \kappa(\Gamma) = \#\mathfrak{P}_\Gamma \tag{2.11}$$

is finite and equals the number of cusps of X. More precisely, for each $P_i \in \mathfrak{P}_\Gamma$ there exists a $Y_{P_i} > 0$ and there exists a compact connected subset $C = C(Y_{P_1}, \dots, Y_{P_{\kappa(\Gamma)}})$ of G such that in the sense of a disjoint union one has

$$G = \Gamma \cdot C \sqcup \bigsqcup_{i=1}^{\kappa(X)} \Gamma \cdot N_{P_i} A^0_{P_i}[Y_{P_i}]\, K \tag{2.12}$$

and such that

$$\gamma \cdot N_{P_i} A^0_{P_i}[Y_{P_i}]\, K \cap N_{P_i} A^0_{P_i}[Y_{P_i}]\, K \neq \emptyset \Leftrightarrow \gamma \in \Gamma \cap P_i. \tag{2.13}$$

For each $P_i \in \mathfrak{P}_\Gamma$ let

$$Y^0_{P_i}(\Gamma) = \inf\{Y_{P_i} \colon Y_{P_i} \in \mathbb{R}^+ \text{ satisfies (2.13)}\}. \tag{2.14}$$

Moreover, we define the height-function y_{Γ,P_i} on X by

$$y_{\Gamma,P_i}(x) := \sup\{y_{P_i}(g) \colon g \in G,\ \mathrm{pr}_X(g) = x\}. \tag{2.15}$$

By (2.12) and (2.13) the supremum is finite. For $Y_1, \dots, Y_{\kappa(X)} \in (0,\infty)$ we let

$$X(P_1, \dots, P_{\kappa(X)}; Y_1, \dots, Y_{\kappa(X)}) := \{x \in X \colon y_{\Gamma,P_i}(x) \le Y_i,\ i = 1, \dots, \kappa(X)\}. \tag{2.16}$$

If $Y \in (0,\infty)$, we write $X_{\mathfrak{P}_\Gamma}(Y)$ or $X(P_1, \dots, P_{\kappa(X)}; Y)$ for $X(P_1, \dots, P_{\kappa(X)}; Y, \dots, Y)$, i.e.

$$X_{\mathfrak{P}_\Gamma}(Y) := X(P_1, \dots, P_{\kappa(X)}; Y) := \{x \in X \colon y_{\Gamma,P_i}(x) \le Y,\ i = 1, \dots, \kappa(X)\}. \tag{2.17}$$

For later purposes we now recall the interpretation of the semisimple elements in terms of closed geodesics. For further details we refer, for example, to [Pf1, section 3]. We let Γ_{s} denote the semisimple elements of Γ which are not G-conjugate to an element of K. By $\mathrm{C}(\Gamma)_{\mathrm{s}}$ we denote the set of Γ-conjugacy classes of elements of Γ_{s}. Then for each $\gamma \in \Gamma_{\mathrm{s}}$ there exists a unique geodesic $\widetilde{c}_\gamma$ in $\widetilde{X}$ which is stabilized by γ. If one lets

$$\ell(\gamma) = \inf_{x \in \widetilde{X}} d(x, \gamma x), \tag{2.18}$$

then $\ell(\gamma) > 0$ and the infimum is attained exactly by the points in $\widetilde{X}$ lying on $\widetilde{c}_\gamma$. Let $\mathcal{C}(X)$ denote the set of closed geodesics of X. For $\gamma \in \Gamma_{\mathrm{s}}$ let c_γ be the projection of the segment of $\widetilde{c}_\gamma$ from x_0 to γx_0, x_0 a point on $\widetilde{c}_\gamma$, to X. Then one can show that c_γ depends only on the Γ-conjugacy class of γ and that the assignment $\gamma \mapsto c_\gamma$ induces a bijection between $\mathrm{C}(\Gamma)_{\mathrm{s}}$ and $\mathcal{C}(X)$. For $c \in \mathcal{C}(X)$ let $\ell(c)$ denote its length. Then there exists a constant C_X such that for each R one can estimate

$$\#\{c \in \mathcal{C}(X) \colon \ell(c) \leq R\} \leq C_X e^{2nR}. \tag{2.19}$$

In particular, if one sets

$$\ell(\Gamma) := \ell(X) := \inf\{\ell(c) \colon c \in \mathcal{C}(X)\}, \tag{2.20}$$

then $\ell(\Gamma) > 0$.

Measures are normalized as follows. We normalize the Haar-measure on K such that K has volume 1. We fix an isometric identification of $\mathbb{R}^{2n}$ with $\mathfrak{n}$ with respect to the inner product $\langle\cdot,\cdot\rangle_\theta$. We give $\mathfrak{n}$ the measure, induced from the Lebesgue measure under this identification. Moreover, we identify $\mathfrak{n}$ and N by the exponential map and we will denote by dn the Haar measure on N, induced from the measure on $\mathfrak{n}$ under this identification. We normalize the Haar measure on G by setting

$$\int_G f(g)dg = \int_N \int_{\mathbb{R}} \int_K e^{-2nt} f(na(t)k)dkdtdn. \tag{2.21}$$

If P' is a parabolic subgroup of G, the measures on $N_{P'}$ and $A_{P'}$ will be the measures induced from N and A via the conjugation with $k_{P'}$. Let f be integrable over $\Gamma\backslash G$. Then identifying f with a measurable function on G it follows from (2.21), (2.12) and (2.13) that for every $Y \geq Y_0$ one has

$$\int_{\Gamma\backslash G} f(x)dx = \int_{C(Y)} f(g)dg + \sum_{i=1}^{\kappa(\Gamma)} \int_{\Gamma\cap N_{P_i}\backslash N_{P_i}} \int_{\log Y}^{\infty} \int_K e^{-2nt} f(n_{P_i} a_{P_i}(t)k) dn_{P_i} dt dk \tag{2.22}$$

For $\sigma \in \hat{M}$ and $\lambda \in \mathbb{C}$ let $\pi_{\sigma,\lambda}$ be the principal series representation of G parametrized as in [MP2, section 2.7]. In particular, the representations $\pi_{\sigma,\lambda}$ are unitary iff $\lambda \in \mathbb{R}$. We denote by $\Theta_{\sigma,\lambda}$ the global character of $\pi_{\sigma,\lambda}$. For $\sigma \in \hat{M}$ with highest weight $\Lambda(\sigma)$ as in (2.6) let $\sigma(\Omega_M)$ be the Casimir eigenvalue of σ and let

$$c(\sigma) := \sigma(\Omega_M) - n^2 = \sum_{j=2}^{n+1} (k_j(\sigma) + \rho_j)^2 - \sum_{j=1}^{n+1} \rho_j^2, \tag{2.23}$$

where the second equality follows from a standard computation.

3. Eisenstein series

In this section we recall the definition and some basic properties of the Eisenstein series. Let Γ be a discrete subgroup of G such that $\mathrm{vol}(\Gamma\backslash G)$ is finite. Furthermore, for convenience we assume in this section that Γ is torsion-free and that for each Γ-cuspidal parabolic subgroup P' of G one has

$$\Gamma \cap P' = \Gamma \cap N_{P'}. \tag{3.1}$$

Let $\mathfrak{P}_\Gamma$ be a fixed set of representatives of Γ-conjugacy classes of Γ-cuspidal parabolic subgroups of G. Let $P_i \in \mathfrak{P}_\Gamma$. For $\sigma \in \hat{M}$ we define a representation σ_{P_i} of M_{P_i} by

$$\sigma_{P_i}(m_{P_i}) := \sigma(k_{P_i}^{-1} m_{P_i} k_{P_i}), \quad m_{P_i} \in M_{P_i}. \tag{3.2}$$

Now let $\nu \in \hat{K}$ and $\sigma_P \in \hat{M}$ such that $[\nu : \sigma] \neq 0$. Then we let $\mathcal{E}_{P_i}(\sigma, \nu)$ be the set of all continuous functions Φ on G which are left-invariant under $N_{P_i} A_{P_i}$ such that for all $x \in G$ the function $m \mapsto \Phi_{P_i}(mx)$ belongs to $L^2(M_{P_i}, \sigma_{P_i})$, the σ_{P_i}-isotypical component of the right regular representation of M_{P_i}, and such that for all $x \in G$ the function $k \mapsto \Phi_{P_i}(xk)$ belongs to the ν-isotypical component of the right regular representation of K. The space $\mathcal{E}_{P_i}(\sigma, \nu)$ is finite dimensional and in fact one has

$$\dim(\mathcal{E}_{P_i}(\sigma, \nu)) = \dim(\sigma) \dim(\nu). \tag{3.3}$$

We define an inner product on $\mathcal{E}_{P_i}(\sigma, \nu)$ as follows. Any element of $\mathcal{E}_{P_i}(\sigma, \nu)$ can be identified canonically with a function on K. For $\Phi, \Psi \in \mathcal{E}_{P_i}(\sigma, \nu)$ put

$$\langle \Phi, \Psi \rangle := \operatorname{vol}(\Gamma \cap N_{P_i} \backslash N_{P_i}) \int_K \Phi(k) \bar{\Psi}(k) dk. \tag{3.4}$$

Define the Hilbert space $\mathcal{E}_{P_i}(\sigma)$ by

$$\mathcal{E}_{P_i}(\sigma) := \bigoplus_{\substack{\nu \in \hat{K} \\ [\nu:\sigma] \neq 0}} \mathcal{E}_{P_i}(\sigma, \nu).$$

For $\Phi_{P_i} \in \mathcal{E}_{P_i}(\sigma, \nu)$ and $\lambda \in \mathbb{C}$ let

$$\Phi_{P_i,\lambda}(g) := e^{(\lambda+n)(H_{P_i}(x))} \Phi_{P_i}(g). \tag{3.5}$$

Let $x \in \Gamma\backslash G$, $x = \Gamma g$. Then the Eisenstein series $E(\Phi_{P_i} : \lambda : x)$ is defined by

$$E(\Phi_{P_i} : \lambda : x) := \sum_{\gamma \in (\Gamma \cap N_{P_i})\backslash \Gamma} \Phi_{P_i,\lambda}(\gamma g). \tag{3.6}$$

On $\Gamma\backslash G \times \{\lambda \in \mathbb{C} \colon \operatorname{Re}(\lambda) > n\}$ the series (3.6) is absolutely and locally uniformly convergent. As a function of λ, it has a meromorphic continuation to $\mathbb{C}$ with only finitely many poles in the strip $0 < \operatorname{Re}(\lambda) \leq n$ which are located on $(0, n]$ and it has no poles on the line $\operatorname{Re}(\lambda) = 0$. By (2.7), for $\sigma \in \hat{M}$ with $[\nu : \sigma] \neq 0$ and $\Phi_{P_i} \in \mathcal{E}(\sigma, \nu)$ one has

$$\Omega \Phi_{P_i,\lambda} = (\lambda^2 + c(\sigma)) \Phi_{P_i,\lambda}, \tag{3.7}$$

where $c(\sigma)$ is as in (2.23). Since Ω is G-invariant it follows that

$$\Omega E(\Phi_{P_i} : \lambda : x) = (\lambda^2 + c(\sigma)) E(\Phi_{P_i} : \lambda : x). \tag{3.8}$$

Let

$$\boldsymbol{\mathcal{E}}(\sigma, \nu) := \bigoplus_{P_i \in \mathfrak{P}_\Gamma} \mathcal{E}_{P_i}(\sigma, \nu); \quad \boldsymbol{\mathcal{E}}(\sigma) := \bigoplus_{P_i \in \mathfrak{P}_\Gamma} \mathcal{E}_{P_i}(\sigma).$$

By (3.3) one has

$$\dim \boldsymbol{\mathcal{E}}(\sigma, \nu) = \kappa(\Gamma) \dim(\sigma) \dim(\nu). \tag{3.9}$$

Let $P_i, P_j \in \mathfrak{P}_\Gamma$ and let $\sigma \in \hat{M}$. For $\Phi_{P_i} \in \mathcal{E}_{P_i}(\sigma,\nu)$, $i = 1,2$, and $g \in G$ let

$$E_{P_j}(\Phi_{P_i} : g : \lambda) := \frac{1}{\operatorname{vol}\left(\Gamma \cap N_{P_j} \backslash N_{P_j}\right)} \int_{\Gamma \cap N_{P_j} \backslash N_{P_j}} E(\Phi_{P_i} : ng : \lambda) dn$$

be the constant term of $E(\Phi_{P_i} : - : \lambda)$ along P_j. Then there exists a meromorphic function

$$C_{P_i|P_j}(\sigma : \nu : \lambda) : \mathcal{E}_{P_i}(\sigma,\nu) \longrightarrow \mathcal{E}_{P_j}(w_0\sigma,\nu),$$

such that for $P_i, P_j \in \mathfrak{P}_\Gamma$ one has

$$E_{P_j}(\Phi_{P_i} : g : \lambda) = \delta_{i,j}\Phi_{P_i,\lambda}(g) + (C_{P_i|P_j}(\sigma : \nu : \lambda)\Phi_{P_i})_{-\lambda}(g). \tag{3.10}$$

Now we let

$$C_{P_i|P_j}(\sigma_{P_i},\lambda) := \bigoplus_{\substack{\nu \in \hat{K} \\ [\nu:\sigma] \neq 0}} C_{P_i|P_j}(\sigma,\nu,\lambda),$$

where σ_{P_i} is defined by (3.2). Furthermore, let

$$\mathbf{C}(\sigma,\lambda) : \boldsymbol{\mathcal{E}}(\sigma) \to \boldsymbol{\mathcal{E}}(w_0\sigma); \quad \mathbf{C}(\sigma,\nu,\lambda) : \boldsymbol{\mathcal{E}}(\sigma,\nu) \to \boldsymbol{\mathcal{E}}(w_0\sigma,\nu)$$

be the maps built from the maps $C_{P_i|P_j}(\sigma,\lambda)$, resp. $C_{P_i|P_j}(\sigma,\nu,\lambda)$. Then one has

$$\mathbf{C}(w_0\sigma,\lambda)\mathbf{C}(\sigma,-\lambda) = \operatorname{Id}; \quad \mathbf{C}(\sigma,\lambda)^* = \mathbf{C}(w_0\sigma,\bar{\lambda}). \tag{3.11}$$

Let $\sigma \in \hat{M}$ and $\nu \in \hat{K}$. If $\sigma = w_0\sigma$, let $\overline{\mathcal{E}}_{P_i}(\sigma,\nu) := \mathcal{E}_{P_i}(\sigma,\nu)$, $\overline{\boldsymbol{\mathcal{E}}}(\sigma,\nu) := \boldsymbol{\mathcal{E}}(\sigma,\nu)$, $\overline{\mathbf{C}}(\sigma : \nu : s) := \mathbf{C}(\sigma : \nu : s)$. If $\sigma \neq w_0\sigma$, let $\overline{\mathcal{E}}_{P_i}(\sigma,\nu) := \mathcal{E}_{P_i}(\sigma,\nu) \oplus \mathcal{E}_{P_i}(w_0\sigma,\nu)$ $\overline{\boldsymbol{\mathcal{E}}}(\sigma,\nu) := \boldsymbol{\mathcal{E}}(\sigma,\nu) \oplus \boldsymbol{\mathcal{E}}(\boldsymbol{w_0\sigma},\nu)$ and

$$\overline{\mathbf{C}}(\sigma,\nu,s) : \ \overline{\boldsymbol{\mathcal{E}}}(\sigma,\nu) \to \overline{\boldsymbol{\mathcal{E}}}(\sigma,\nu); \quad \overline{\mathbf{C}}(\sigma,\nu,s) := \begin{pmatrix} 0 & \mathbf{C}(w_0\sigma,\nu,s) \\ \mathbf{C}(\sigma,\nu,s) & 0 \end{pmatrix}. \tag{3.12}$$

Let R_σ (resp. $R_{w_0\sigma}$) denote the right regular representation of K on $\boldsymbol{\mathcal{E}}(\sigma)$ (resp. $\boldsymbol{\mathcal{E}}(w_0\sigma)$). Then $\mathbf{C}(\sigma,s)$ is an intertwining operator between R_σ and $R_{w_0\sigma}$. Thus if ν is a finite-dimensional representation of K on V_ν, we can define $\widetilde{\boldsymbol{C}}(\sigma,\nu,s)$ as the restriction of $(\boldsymbol{C}(\sigma,s) \otimes \operatorname{Id})$ to a map from $(\boldsymbol{\mathcal{E}}(\sigma) \otimes V_\nu)^K$ to $(\boldsymbol{\mathcal{E}}(w_0\sigma) \otimes V_\nu)^K$. For later purpose we need the following Lemma.

Lemma 3.1. *In the sense of meromorphic functions one has*

$$\operatorname{Tr}\left(\widetilde{\boldsymbol{C}}(\sigma,\nu,s)^{-1}\frac{d}{ds}\widetilde{\boldsymbol{C}}(\sigma,\nu,s)\right) = \frac{1}{\dim(\nu)}\operatorname{Tr}\left(\boldsymbol{C}(\sigma,\nu,s)^{-1}\frac{d}{ds}\boldsymbol{C}(\sigma,\nu,s)\right)$$

for each $\sigma \in \hat{M}$, $\nu \in \hat{K}$ *with* $[\nu : \sigma] \neq 0$.

Proof. Let P_1 be the projection form $\boldsymbol{\mathcal{E}}(\sigma)$ to $\boldsymbol{\mathcal{E}}(\sigma,\nu)$ and let P_2 be the projection from $(\boldsymbol{\mathcal{E}}(\sigma) \otimes V_\nu)$ to $(\boldsymbol{\mathcal{E}}(\sigma) \otimes V_\nu)^K$. Then using that $\check{\nu} \cong \nu$ we have

$$P_1 = \dim(\nu)\int_K \chi_\nu(k)R_\sigma(k); \quad P_2 = \int_K R_\sigma(k) \otimes \nu(k)dk,$$

where χ_ν is the character of ν. Thus one has

$$\begin{aligned}
&\operatorname{Tr}\left(\widetilde{\boldsymbol{C}}(\sigma,\nu,s)^{-1}\frac{d}{ds}\widetilde{\boldsymbol{C}}(\sigma,\nu,s)\right)=\operatorname{Tr}\left(\boldsymbol{C}(\sigma,s)^{-1}\frac{d}{ds}\boldsymbol{C}(\sigma,s)\otimes\operatorname{Id}\circ P_2\right)\\
=&\operatorname{Tr}\left(\int_K\boldsymbol{C}(\sigma,s)^{-1}\frac{d}{ds}\boldsymbol{C}(\sigma,s)\circ R_\sigma(k)\otimes\nu(k)dk\right)\\
=&\operatorname{Tr}\left(\int_K\boldsymbol{C}(\sigma,s)^{-1}\frac{d}{ds}\boldsymbol{C}(\sigma,s)\circ\chi_\nu(k)R_\sigma(k)dk\right)\\
=&\frac{1}{\dim(\nu)}\operatorname{Tr}\left(\boldsymbol{C}(\sigma,s)^{-1}\frac{d}{ds}\boldsymbol{C}(\sigma,s)\circ P_1\right)=\frac{1}{\dim(\nu)}\operatorname{Tr}\left(\boldsymbol{C}(\sigma,\nu,s)^{-1}\frac{d}{ds}\boldsymbol{C}(\sigma,\nu,s)\right),
\end{aligned}$$

which concludes the proof of the proposition. □

4. Factorization of the C-matrix

We let Γ be a discrete subgroup of G satisfying (3.1) and we keep the notations of the previous section. By the results of Müller, in particular [Mu1, equation (6.8)], the determinant of the matrix $\overline{\mathbf{C}}(\sigma,\nu,\lambda)$ factorizes into a product of an exponential factor and an infinite Weierstrass product involving its zeroes and poles. For the case of a hyperbolic surface, this factorization was first established by Selberg (see[Se, page 656]).

While the poles and zeroes of the C-matrices are easily seen to be independent of the choice of $\mathfrak{P}_\Gamma$, the exponential factor depends on $\mathfrak{P}_\Gamma$ or, equivalently, on the choice of truncation parameters. This fact will become particularly crucial if one lets the manifold X vary. In [Mu1], the manifold X and the set $\mathfrak{P}_\Gamma$ were fixed. Therefore, for the purposes of the present article we have to go through the arguments of the paper [Mu1] which led to equation (6.8) in this paper and to keep track of the precise choices of truncation parameters.

Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$. If ν is a finite dimensional representation of K, let $L^2(\Gamma\backslash G)_\nu$ denote the ν-isotypical component of the restriction of R_Γ to K. Let $C_c^\infty(\Gamma\backslash G)_\nu:=C_c^\infty(\Gamma\backslash G)\cap L^2(\Gamma\backslash G)_\nu$. Then it is easy to see that $C_c^\infty(\Gamma\backslash G)_\nu$ is dense in $L^2(\Gamma\backslash G)_\nu$.

Now let Δ_ν be the differential operator in $C^\infty(\Gamma\backslash G)_\nu$, which is induced by $-R_\Gamma(\Omega)$. If we regard it as an operator in $L^2(\Gamma\backslash G)_\nu$ with domain $C_c^\infty(\Gamma\backslash G)_\nu$, it is symmetric, essentially selfadjoint and satisfies $\Delta_\nu\geq-\nu(\Omega_K)$, where $\nu(\Omega_K)\in\mathbb{R}^+$ is the Casimir eigenvalue of ν. This follows easily from the considerations in the next section 5. The closure of Δ_ν will be denoted by $\overline{\Delta_\nu}$. One has

$$\sigma(\overline{\Delta_\nu})\subset(-\nu(\Omega_K),\infty). \tag{4.1}$$

We fix a smooth function ϕ on $\mathbb{R}$ with values in $[0,1]$ such that $\phi(t)=0$ for $t\leq0$ and $\phi(t)=1$ for $t\geq1$. If $P_i\in\mathfrak{P}_\Gamma$, then for $Y\in(0,\infty)$ we let

$$\psi_{P_i,Y}(n_{P_i}a_{P_i}(t)k):=\phi(t-\log Y),\quad n_{P_i}\in N_{P_i},\ t\in\mathbb{R}.$$

Now let $Y_{P_i} \in (0, \infty)$, $i = 1, \dots, \kappa(\Gamma)$, such that $Y_{P_i} \geq Y^0_{P_i}(\Gamma)$, where $Y^0_{P_i}(\Gamma)$ is defined by (2.14). For $\Phi_{P_i} \in \mathcal{E}(\sigma_{P_i}, \nu)$ we define a function $\theta(\Phi_{P_i} : Y_{P_i} : \lambda : x)$ on $\Gamma\backslash G$ by

$$\theta(\Phi_{P_i} : Y_{P_i} : \lambda : x) := \sum_{\gamma \in \Gamma \cap N_{P_i} \backslash \Gamma} \psi_{P_i, Y_{P_i}}(\gamma g) \Phi_{P_i, \lambda}(\gamma g); \quad x = \Gamma g. \tag{4.2}$$

By (2.13) at most one summand in this sum can be non-zero . We let

$$H(\Phi_{P_i} : Y_{P_i} : \lambda : x) := (\Delta_\nu + c(\sigma_{P_i}) + \lambda^2)\theta(\Phi_{P_i} : Y_{P_i} : \lambda : x).$$

Then by (3.7) one has $H(\Phi_{P_i} : Y_{P_i} : \lambda : x) \in C^\infty_c(\Gamma\backslash G)_\nu$. Moreover, the Eisenstein series can be characterized by the following Proposition, which for $\dim X = 2$ is due to Colin de Verdière [CV].

Proposition 4.1. *For* $P_i \in \mathfrak{P}_\Gamma$, $Y_{P_i} \geq Y^0_{P_i}(\Gamma)$ *and* $\lambda \in \mathbb{C}$ *with* $\lambda^2 + c(\sigma) \notin (-\infty, \nu(\Omega_K))$ *and* $\mathrm{Re}(\lambda) > 0$ *one has*

$$E(\Phi_{P_i} : \lambda : x) = \theta(\Phi_{P_i} : Y_{P_i} : \lambda : x) - (\overline{\Delta}_\nu + \lambda^2 + c(\sigma))^{-1}(H(\Phi_{P_i} : Y_{P_i} : \lambda : x)).$$

Proof. This was proved in general in [Mu1, Proposition 4.7]. For the convenience of the reader we recall the proof. Denote the right hand side by $\widetilde{E}(\Phi_{P_i} : \lambda : x)$. By definition it satisfies $(\Delta_\nu + \lambda^2 + c(\sigma))\widetilde{E}(\Phi_{P_i} : \lambda : x) = 0$. By (3.8), $E(\Phi_{P_i} : \lambda : x)$ satisfies the same differential equation. By [Mu1, Lemma 4.5], $E(\Phi_{P_i} : \lambda) - \theta(\Phi_{P_i} : Y_{P_i} : \lambda)$ is square integrable for $\mathrm{Re}(\lambda) > n$. Hence, $u := E((\Phi_{P_i} : \lambda) - \widetilde{E}(\Phi_{P_i} : \lambda)$ is square integrable for $\mathrm{Re}(\lambda) > n$ and satisfies $(\Delta_\nu + \lambda^2 + c(\sigma))u = 0$. Since Δ_ν is essentially self-adjoint, it follows that $E((\Phi_{P_i} : \lambda) = \widetilde{E}(\Phi_{P_i} : \lambda)$ for $\mathrm{Re}(\lambda) > n$. The proposition follows by the uniqueness the analytic continuation. ☐

Lemma 4.2. *There exists a constant* C_1 *which is independent of* Γ *and* $\mathfrak{P}_\Gamma$ *such that for all* $\lambda \in \mathbb{C}$ *with* $\lambda^2 + c(\sigma) \notin (-\infty, \nu(\Omega_K))$ *and* $\mathrm{Re}(\lambda) > 0$*, all* $Y_{P_i} \geq Y^0_{P_i}(\Gamma)$*, and all* $\Phi_{P_i} \in \mathcal{E}_{P_i}(\sigma, \nu)$*,* $P_i \in \mathfrak{P}_\Gamma$*, one has*

$$\|H(\Phi_{P_i} : Y_{P_i} : \lambda : x)\|_{L^2(\Gamma\backslash G)} \leq C_1 e^{\mathrm{Re}(\lambda)(\log Y_{P_i} + 2)} \|\Phi_{P_i}\|_{\mathcal{E}_{P_i}(\sigma,\nu)}.$$

Proof. There exists a unique $\Phi_P \in \mathcal{E}_P(\sigma, \nu)$ such that $\Phi_{P_i,\lambda}(g) = \Phi_{P,\lambda}(\kappa^{-1}_{P_i} g \kappa_{P_i})$. Since Δ_ν commutes with the right-action of G on $\Gamma\backslash G$, it follows from (2.22) that

$$\begin{aligned}
&\int_{\Gamma\backslash G} |H(\Phi_{P_i} : Y_{P_i} : \lambda : x)|^2 dx \\
&= \mathrm{vol}(\Gamma \cap N_{P_i} \backslash N_{P_i}) \int_{\log Y_{P_i}}^{\log Y_{P_i}+1} \int_K e^{-2nt} |(\Delta_\nu + c(\sigma) + \lambda^2)\psi_{P_i, Y_{P_i}}(a_{P_i}(t))\Phi_{P_i,\lambda}(a_{P_i}(t)k)|^2 \, dk dt \\
&= \mathrm{vol}(\Gamma \cap N_{P_i} \backslash N_{P_i}) \int_{\log Y_{P_i}}^{\log Y_{P_i}+1} \int_K e^{-2nt} |(\Delta_\nu + c(\sigma) + \lambda^2)\psi_{P, Y_{P_i}}(a(t))\Phi_{P,\lambda}(a(t)k)|^2 \, dk dt.
\end{aligned}$$

Now using (2.7) and (3.8) one obtains

$$\begin{aligned}
&(\Delta_\nu + c(\sigma) + \lambda^2)(\psi_{P, Y_{P_i}}(a(t))\Phi_{P,\lambda}(a(t)k)) \\
&= - e^{(\lambda+n)t}\Phi_P(k)(\phi''(t - \log Y_{P_i}) + 2\lambda\phi'(t - \log Y_{P_i})).
\end{aligned}$$

This proves the proposition. □

Corollary 4.3. *There exists a constant* C_1 *which is independent of* Γ *and* $\mathfrak{P}_\Gamma$ *such that for all* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda^2) + c(\sigma) \geq \nu(\Omega_K) + 1$ *and* $\mathrm{Re}(\lambda) > 0$*, all* $Y_{P_i} \geq Y^0_{P_i}(\Gamma)$ *and all* $\Phi_{P_i} \in \mathcal{E}_{P_i}(\sigma, \nu)$*,* $P_i \in \mathfrak{P}_\Gamma$*, one has*

$$\|(\overline{\Delta}_\nu + \lambda^2 + c(\sigma))^{-1} H(\Phi_{P_i} : Y_{P_i} : \lambda : x)\|_{L^2(\Gamma\backslash G)} \leq C_1 e^{\mathrm{Re}(\lambda)(\log Y_{P_i} + 2)} \|\Phi_{P_i}\|_{\mathcal{E}_{P_i}(\sigma,\nu)}$$

Proof. By [Ka, V,§3.8] one can estimate the operator norm of the resolvent by

$$\|(\overline{\Delta}_\nu + \lambda^2 + c(\sigma))^{-1}\| \leq \frac{1}{\mathrm{dist}(-\lambda^2 - c(\sigma), \mathrm{spec}(\overline{\Delta}_\nu))},$$

where the estimate holds without any constant. Applying the previous Lemma and (4.1), the corollary follows. □

In the following proposition we estimate the coefficients of the C-matrix.

Proposition 4.4. *There exists a constant* C_2*, which is independent of* Γ *and* $\mathfrak{P}_\Gamma$ *such that for all* $P_i, P_j \in \mathfrak{P}_\Gamma$*, all* $Y_{P_i}, Y_{P_j} \in (0, \infty)$ *with* $Y_{P_i} \geq Y^0_{P_i}(\Gamma)$*,* $Y_{P_j} \geq Y^0_{P_j}(\Gamma)$*, all* $\Phi_{P_i} \in \overline{\mathcal{E}}_{P_i}(\sigma, \nu)$*,* $\Phi_{P_j} \in \overline{\mathcal{E}}_{P_j}(\sigma, \nu)$ *and all* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda^2) + c(\sigma) \geq \nu(\Omega_K) + 1$ *and* $\mathrm{Re}(\lambda) > 0$*, one has*

$$|\left\langle C_{P_i|P_j}(\sigma, \nu, \lambda)(\Phi_{P_i}), \Phi_{P_j} \right\rangle_{\overline{\mathcal{E}}_{P_j}(\sigma,\nu)}| \leq C_2 e^{\mathrm{Re}(\lambda)(\log Y_{P_i} + \log Y_{P_j} + 4)} \|\Phi_{P_i}\|_{\overline{\mathcal{E}}_{P_i}(\sigma,\nu)} \cdot \|\Phi_{P_j}\|_{\overline{\mathcal{E}}_{P_j}(\sigma,\nu)}.$$

Proof. By the definition (3.10) of the constant term it follows that for each $t \in \mathbb{R}$ and each $k \in K$ one has

$$\begin{aligned} C_{P_i|P_j}(\sigma, \nu, \lambda)(\Phi_{P_i})(k) =& e^{(\lambda - n)t} (C_{P_i|P_j}(\sigma, \nu, \lambda)(\Phi_{P_i}))_{-\lambda}(a_{P_j}(t)k) \\ =& e^{(\lambda-n)t} \big(E_{P_j}(\Phi_{P_i} : a_{P_j}(t)k : \lambda) - \delta_{i,j} \Phi_{P_i,\lambda}(a_{P_j}(t)k) \big). \end{aligned}$$

Moreover, by (2.12) and (2.13), for $t \geq \log Y_{P_j} + 1$ one has

$$\theta(\Phi_{P_i} : Y_{P_i} : \lambda : a_{P_j}(t)k) = \delta_{i,j} \Phi_{P_i,\lambda}(a_{P_j}(t)k).$$

Thus by Proposition 4.1 for $t \geq \log Y_{P_j} + 1$ one has

$$\begin{aligned} & E_{P_j}(\Phi_{P_i} : a_{P_j}(t)k : \lambda) - \delta_{i,j} \Phi_{P_i,\lambda}(a_{P_j}(t)k) \\ =& - \frac{1}{\mathrm{vol}\left(\Gamma \cap N_{P_j} \backslash N_{P_j}\right)} \int_{\Gamma \cap N_{P_j} \backslash N_{P_j}} (\overline{\Delta}_\nu + c(\sigma) + \lambda^2)^{-1} (H(\Phi_{P_i} : Y_{P_i} : \lambda : n_{P_j} a_{P_j}(t)k)) \, dn_{P_j}. \end{aligned}$$

Combining these equations, it follows that for each $t \geq \log Y_{P_j} + 1$ one has

$$\begin{aligned} & \left\langle C_{P_i|P_j}(\sigma, \nu, \lambda)(\Phi_{P_i}), \Phi_{P_j} \right\rangle_{\overline{\mathcal{E}}_{P_j}(\sigma,\nu)} \\ &= \mathrm{vol}(\Gamma \cap N_{P_j} \backslash N_{P_j}) \int_K \overline{\Phi_{P_j}}(k) C_{P_i|P_j}(\sigma, \nu, \lambda)(\Phi_{P_i})(k) dk = -e^{(\lambda-n)t} \\ (4.3) \quad & \times \int_K \overline{\Phi_{P_j}}(k) \int_{\Gamma \cap N_{P_j} \backslash N_{P_j}} (\overline{\Delta}_\nu + c(\sigma) + \lambda^2)^{-1} (H(\Phi_{P_i} : Y_{P_i} : \lambda : n_{P_j} a_{P_j}(t)k)) \, dn_{P_j} dk. \end{aligned}$$

Now we define a function $\widetilde{f}_{P_j,\lambda}$ on G by

$$\widetilde{f}_{P_j,\lambda}(n_{P_j}a_{P_j}(t)k) = e^{(n+\lambda)t}\chi_{[\log Y_{P_j},\, \log Y_{P_j}+1]}(t)\Phi_{P_j}(k),$$

where $\chi_{[\log Y_{P_j},\, \log Y_{P_j}+1]}(t)$ denotes the characteristic function of the interval $[\log Y_{P_j}, \log Y_{P_j} + 1]$. Then we define a function $f_{P_j,\lambda}$ on $\Gamma\backslash G$ by

$$f_{P_j,\lambda}(x) = \sum_{\gamma\in\Gamma\cap P_j\backslash\Gamma} \widetilde{f}_{P_j,\lambda}(\gamma g), \quad x = \Gamma g.$$

By (2.13), at most one summand in this sum can be nonzero. Integrating equation (4.3) over t in the interval $[\log Y_{P_j},\ \log Y_{P_j} + 1]$ and using (2.22), we obtain

$$\begin{aligned} &\left| \left\langle C_{P_i|P_j}(\sigma,\nu,\lambda)(\Phi_{P_i}), \Phi_{P_j}\right\rangle_{\overline{\mathcal{E}}_{P_j}(\sigma,\nu)} \right| \\ =&\left| \left\langle (\overline{\Delta}_\nu + c(\sigma) + \lambda^2)^{-1}(H(\Phi_{P_i} : Y_{P_i} : \lambda)), f_{P_j,\lambda}\right\rangle_{L^2(\Gamma\backslash G)} \right|. \end{aligned}$$

Now observe that

$$\|f_{P_j,\lambda}\|_{L^2(\Gamma\backslash G)} \leq e^{\operatorname{Re}(\lambda)(\log Y_{P_j}+1)}\|\Phi_{P_j}\|_{\overline{\mathcal{E}}_{P_j}(\sigma,\nu)}.$$

Applying Corollary 4.3, the Proposition follows. □

Summarizing our results, we obtain the following refinement of [Mu1, Lemma 6.1].

Corollary 4.5. *Let* $\bar{d}(\sigma,\nu) := \dim\overline{\mathcal{E}}_P(\sigma,\nu)$. *For each* $P_i \in \mathfrak{P}_\Gamma$ *let* $Y_{P_i} \geq Y^0_{P_i}(\Gamma)$ *be given. Put*

$$q_1 := \prod_{i=1}^{\kappa(\Gamma)} e^{2(\log Y_{P_i}+2)\bar{d}(\sigma,\nu)}.$$

There exists a constant $C > 0$ *which is independent of* Γ, $\mathfrak{P}_\Gamma$, *and* Y_{P_i}, $i = 1, ..., \kappa(\Gamma)$, *such that for all* $\lambda \in \mathbb{C}$ *satisfying* $\operatorname{Re}(\lambda^2) + c(\sigma) \geq \nu(\Omega_K) + 1$ *and* $\operatorname{Re}(\lambda) > 0$, *one has*

$$|\det(\overline{\mathbf{C}}(\sigma,\nu,\lambda))| \leq Cq_1^{\operatorname{Re}(\lambda)}.$$

Proof. If one chooses for each $i = 1,\ldots,\kappa(\Gamma)$ an orthonormal base of $\mathcal{E}_{P_i}(\sigma,\nu)$ resp. $\mathcal{E}_{P_i}(w_0\sigma,\nu)$ and applies the preceding proposition, the corollary follows immediately from the Leibniz formula for the determinant. □

Applying the previous Corollary we can restate the factorization of the C-matrix, [Mu1, equation 6.8] with an expression for the exponential factor in terms of the truncation parameters that will be sufficient for our later considerations.

Theorem 4.6. *Let* σ_j, $j = 1,\ldots,l$ *denote the poles of* $\det(\overline{\mathbf{C}}(\sigma,\nu,\lambda))$ *in the interval* $(0,n]$ *and let* η *run through the poles of* $\det(\overline{\mathbf{C}}(\sigma,\nu,\lambda))$ *in the half-plane* $\operatorname{Re}(\lambda) \leq 0$, *both counted with multiplicity. Then one has*

$$\det(\overline{\mathbf{C}}(\sigma,\nu,\lambda)) = \det(\overline{\mathbf{C}}(\sigma,\nu,0))q^\lambda \prod_{j=1}^{l}\frac{\lambda+\sigma_j}{\lambda-\sigma_j}\prod_\eta\frac{\lambda+\bar{\eta}}{\lambda-\eta}.$$

Moreover, if for each $P_i \in \mathfrak{P}_\Gamma$ a $Y_{P_i} \in (0,\infty)$ with $Y_{P_i} \geq Y^0_{P_i}(\Gamma)$ is given, then q can be written as

$$q = e^a \prod_{i=1}^{\kappa(\Gamma)} e^{2(\log Y_{P_i}+2)\tilde{d}(\sigma,\nu)}, \tag{4.4}$$

where $a \in \mathbb{R}$, $a \leq 0$.

Proof. Using the previous Corollary instead of [Mu1, Lemma 6.1], one can proceed exactly as in [Mu1, section 6] to obtain the Theorem. □

5. Twisted Laplace operators

Let ν be a finite dimensional unitary representation of K on $(V_\nu, \langle\cdot,\cdot\rangle_\nu)$. Let

$$\tilde{E}_\nu := G \times_\nu V_\nu$$

be the associated homogeneous vector bundle over $\tilde{X}$. Then $\langle\cdot,\cdot\rangle_\nu$ induces a G-invariant metric $\tilde{B}_\nu$ on $\tilde{E}_\nu$. Let

$$E_\nu := \Gamma\backslash(G \times_\nu V_\nu)$$

be the associated locally homogeneous bundle over X. Since $\tilde{B}_\nu$ is G-invariant, it can be pushed down to a fiber metric B_ν on E_ν. Let

$$C^\infty(G,\nu) := \{f : G \to V_\nu \colon f \in C^\infty,\ f(gk) = \nu(k^{-1})f(g),\ \forall g \in G,\ \forall k \in K\}. \tag{5.1}$$

Let

$$C^\infty(\Gamma\backslash G,\nu) := \{f \in C^\infty(G,\nu) \colon f(\gamma g) = f(g),\ \forall g \in G,\ \forall \gamma \in \Gamma\}. \tag{5.2}$$

Let $C^\infty(X, E_\nu)$ denote the space of smooth sections of E_ν. Then there is a canonical isomorphism

$$A : C^\infty(X, E_\nu) \cong C^\infty(\Gamma\backslash G, \nu)$$

(see [Mi1, p. 4]). There is also a corresponding isometry for the space $L^2(X, E_\nu)$ of L^2-sections of E_ν.

Let τ be an irreducible finite dimensional representation of G on V_τ. Let E_τ be the flat vector bundle associated to the restriction of τ to Γ. Let $\widetilde{E}_\tau \to \widetilde{X}$ be the homogeneous vector bundle associated to $\tau|_K$. Then by [MtM] there is canonical isomorphism

$$E_\tau \cong \Gamma\backslash\widetilde{E}_\tau.$$

By [MtM], there exists an inner product $\langle\cdot,\cdot\rangle$ on V_τ such that

(1) $\langle\tau(Y)u, v\rangle = -\langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{k}$, $u, v \in V_\tau$

(2) $\langle\tau(Y)u, v\rangle = \langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{p}$, $u, v \in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\tau|_K$ is unitary with respect to this inner product, it induces a fiber metric on $\widetilde{E}_\tau$, and hence on E_τ. This fiber metric will also be called admissible. Let $\Lambda^p(X, E_\tau)$ be the space of E_τ-valued p-forms. This is the space of smooth sections of the vector bundle $\Lambda^p(E_\tau) := \Lambda^p T^*X \otimes E_\tau$. Let

$$d_p(\tau)\colon \Lambda^p(X, E_\tau) \to \Lambda^{p+1}(X, E_\tau) \tag{5.3}$$

be the exterior derivative and let

$$\Delta_p(\tau) = d_p(\tau)^* d_p(\tau) + d_{p-1}(\tau) d_{p-1}(\tau)^* \tag{5.4}$$

be the Laplace operator on E_τ-valued p-forms. This operator can be expressed in the locally homogeneous setting as follows. Let $\nu_p(\tau)$ be the representation of K defined by

$$\nu_p(\tau) := \Lambda^p \operatorname{Ad}^* \otimes \tau :\ K \to \operatorname{GL}(\Lambda^p \mathfrak{p}^* \otimes V_\tau). \tag{5.5}$$

There is a canonical isomorphism

$$\Lambda^p(E_\tau) \cong \Gamma \backslash (G \times_{\nu_p(\tau)} (\Lambda^p \mathfrak{p}^* \otimes V_\tau)), \tag{5.6}$$

which induces an isomorphism

$$\Lambda^p(X, E_\tau) \cong C^\infty(\Gamma \backslash G, \nu_p(\tau)). \tag{5.7}$$

There is a corresponding isometry of the L^2-spaces. Let $\tau(\Omega)$ be the Casimir eigenvalue of τ. With respect to the isomorphism (5.7) on has

$$\Delta_p(\tau) = -R_\Gamma(\Omega) + \tau(\Omega) \operatorname{Id} \tag{5.8}$$

(see [MtM, (6.9)]). Next we want to show that the discrete spectrum of the operators $\Delta_p(\tau)$ is greater or equal than $1/4$ for each p and each $\tau \in \operatorname{Rep}(G)$ satisfying $\tau \neq \tau_\theta$. This was already stated in [MP2, Lemma 7.3]. However, as it was kindly brought to our attention by Martin Olbrich, the parametrization of the complementary series used in the proof of that Lemma was incorrect. Therefore we shall now correct the part of the argument leading to the proof of [MP2, Lemma 7.3] which involved the complementary series. We let $\hat{G}_{\text{un}}$ denote the unitary dual of G.

Lemma 5.1. *Let $\tau \in \operatorname{Rep}(G)$ such that $\tau \neq \tau_\theta$. Let $\pi \in \hat{G}_{\text{un}}$ belong to the complementary series. Let $p \in \{0, \dots, d\}$. Then if $[\pi : \nu_p(\tau)] \neq 0$ one has $-\pi(\Omega) + \tau(\Omega) \geq 1$.*

Proof. Let τ be a finite-dimensional irreducible representation of G of highest weight $\Lambda(\tau) = \tau_1 e_1 + \dots + \tau_{n+1} e_{n+1}$ as in (2.4) and assume that $\tau \neq \tau_\theta$. Let $p \in \{0, \dots, d\}$ and let $\sigma \in \hat{M}$ such that $[\nu_p(\tau) : \sigma] \neq 0$. Assume that $\sigma = w_0 \sigma$. Let $\Lambda(\sigma) = k_2(\sigma) e_2 + \dots + k_{n+1}(\sigma) e_{n+1}$ be the highest weight of σ as in (2.6). It was shown in the proof of [MP2, Lemma 7.1] that $\tau_{j-1} + 1 \geq |k_j(\sigma)|$ for every $j \in \{2, \dots, n+1\}$. Let $c(\sigma)$ be as in (2.23) and let $l \in \{1, \dots, n\}$ be minimal with the property that $k_{l+1}(\sigma) = 0$. Using

$\rho_{j-1} = \rho_j + 1$ and [MP1, equation 2.20], it follows that one can estimate

$$\begin{aligned} c(\sigma) = \sum_{j=2}^{l}(k_j(\sigma)+\rho_j)^2 - \sum_{j=1}^{n+1}\rho_j^2 \leq \sum_{j=2}^{l}(\tau_{j-1}+\rho_{j-1})^2 - \sum_{j=1}^{n+1}\rho_j^2 \\ =\tau(\Omega) - \sum_{j=l}^{n+1}(\tau_j+\rho_j)^2. \end{aligned} \tag{5.9}$$

We parametrize the principal series representations as above. Then if π belongs to the complementary series, by [KS, Proposition 49, Proposition 53] and our parametrization there exists a $\sigma \in \hat{M}$, $\sigma = w_0\sigma$ and a $\lambda \in (0, n-l+1)$, where l is minimal with the property that $k_{l+1}(\sigma) = 0$, such that $\pi_{\sigma,i\lambda}$ is unitarizable with unitarization π. We write $\pi = \pi^c_{\sigma,i\lambda}$. If $[\pi^c_{\sigma,i\lambda} : \nu_p(\tau)] \neq 0$, then by Frobenius reciprocity [Kn1, page 208] one has $[\nu_p(\tau) : \sigma] \neq 0$. Thus, since $\sigma = w_0\sigma$, it follows easily from the branching laws for restrictions of representations from G to K and from K to M, [GW][Theorem 8.1.3, Theorem 8.1.4] that all $k_j(\tau)$ defined as in (2.4) are integral. By [MP1, Corollary 2.4] one has

$$-\pi^c_{\sigma,i\lambda}(\Omega) + \tau(\Omega) = -\lambda^2 - c(\sigma) + \tau(\Omega) \tag{5.10}$$

and if we apply equation (5.9) and the condition $|\tau_{n+1}| \geq 1$, it follows that

$$-\pi^c_{\sigma,i\lambda}(\Omega) + \tau(\Omega) \geq \sum_{j=l}^{n+1}(\tau_j+\rho_j)^2 - (n-l+1)^2 = \sum_{j=l}^{n+1}(\tau_j+\rho_j)^2 - \rho_l^2 \geq \tau_{n+1}^2 \geq 1$$

and the Lemma is proved. ☐

Corollary 5.2. *Let $\tau \in \mathrm{Rep}(G)$, $\tau \neq \tau_\theta$. For $p \in \{0, \ldots, d\}$ let λ_0 be an eigenvalue of $\Delta_p(\tau)$. Then one has $\lambda_0 \geq \frac{1}{4}$.*

Proof. Using the preceding Lemma, one can proceed exactly as in the proof of [MP2, Lemma 7.3] to establish the corollary. ☐

6. The regularized trace under coverings

Let $X = \Gamma\backslash\mathbb{H}^d$ be a finite-volume hyperbolic manifold. For τ a finite-dimensional irreducible representation of G let $e^{-t\Delta_p(\tau)}$ be the heat operator associated to the Laplace operator (5.4) acting on the locally homogeneous vector-bundle E_τ over X. To begin with recall the definition of the regularized trace of the heat operators $e^{-t\Delta_p(\tau)}$ introduced in [MP2]. Let

$$K_X^{\tau,p}(t;x,y) \in C^\infty(X_1 \times X_1, E_\tau \boxtimes E_\tau^*)$$

be the kernel of $e^{-t\Delta_p(\tau)}$. If a set $\mathfrak{P}_\Gamma$ of representatives of Γ-cuspidal parabolic subgroups of X is fixed, then according to (2.17), one obtains compact smooth manifolds $X_{\mathfrak{P}_\Gamma}(Y)$

with boundary which exhaust X. Using the Maass-Selberg relations, one can show that there is an asymptotic expansion

$$\int_{X_{\mathfrak{P}_\Gamma}(Y)} \operatorname{Tr} K_X^{\tau,p}(t;x,x)dx = \alpha_{-1}(t)\log Y + \alpha_0(t) + o(1), \tag{6.1}$$

as $Y \to \infty$. Now recall that on a compact manifold the trace of the heat operator is given by the integral of the pointwise trace of the heat kernel. Based on this observation one defines the regularized trace $\operatorname{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)})$ as the constant term of the asymptotic expansion (6.1). However, this definition depends on the choice of the set $\mathfrak{P}_\Gamma$ of representatives of Γ-cuspidal parabolic subgroups of G or equivalently on the choice of a truncation parameter on the manifold X, see [MP2, Remark 5.4]. Therefore, this definition is not suitable if one wants to study the regularized trace for families of hyperbolic manifolds.

To overcome this problem, we remark that if $\pi\colon X_1 \to X_0$ is a finite covering of X_0 and if truncation parameters on the manifold X_0 are given, then there is a canonical way to truncate the manifold X_1, putting $X_1(Y) := \pi^{-1}(X_0(Y))$. Thus one only has to fix truncation parameters for the manifold X_0 or equivalently a set $\mathfrak{P}_{\Gamma_0}$ of representatives of Γ_0-cuspidal parabolic subgroups of G. To make this approach rigorous, we first need to discuss some facts about height functions.

Let Γ_0 be a discrete subgroup of G of finite covolume. We emphasize that we do not assume that Γ_0 is torsion-free. Let $\mathfrak{P}_{\Gamma_0} := \{P_{0,1}, \dots, P_{0,\kappa(X_0)}\}$ be a fixed set of Γ_0-cuspidal parabolic subgroups of G. Each $P_{0,l}$, $l = 1, \dots, \kappa(X_0)$, has a Langlands decomposition $P_{0,l} = N_{0,l}A_{0,l}M_{0,l}$. If P' is any Γ_0-cuspidal parabolic subgroup of G, there exists $\gamma' \in \Gamma_0$ and a unique $l' \in \{1, \dots, \kappa(\Gamma_0)\}$ such that $\gamma' P' \gamma'^{-1} = P_{0,l'}$. Write

$$\gamma' = n_{0,l'}\iota_{P_{0,l'}}(t_{P'})k_{0,l'}, \tag{6.2}$$

$n_{0,l'} \in N_{P_{0,l'}}$, $t_{P'} \in (0,\infty)$, $\iota_{P_{0,l'}}(t_{P'}) \in A_{P_{0,l(P')}}$ as above, and $k_{0,l'} \in K$. Since $P_{0,l'}$ equals its normalizer in G, the projection of the element γ' to $(\Gamma_0 \cap P_{0,l'})\backslash\Gamma_0$ is unique. Moreover, since $P_{0,l'}$ is Γ_0-cuspidal, one has $\Gamma_0 \cap P_{0,l'} = \Gamma_0 \cap N_{P_{0,l'}}M_{P_{0,l'}}$. Thus $t_{P'}$ depends only on $\mathfrak{P}_{\Gamma_0}$ and P'.

Now we let $\Gamma_1 \subset \Gamma_0$ be a subgroup of finite index. Then a parabolic subgroup P' of G is Γ_0-cuspidal iff it is Γ_1-cuspidal. We assume for simplicity that Γ_1 satisfies (3.1). Let $X_0 = \Gamma_0\backslash\widetilde{X}$, $X_1 = \Gamma_1\backslash\widetilde{X}$. Let $\pi : X_1 \to X_0$ be the covering map and let $\operatorname{pr}_{X_0} : G \to X_0$ and $\operatorname{pr}_{X_1} : G \to X_1$ be the corresponding projections. Let $\mathfrak{P}_{\Gamma_1} = \{P_1, \dots, P_{\kappa(X_1)}\}$ be a set of representatives of Γ_1-cuspidal parabolic subgroups. Then for each $j \in \{1, \dots, \kappa(X_1)\}$ let $l(j) \in \{1, \cdots, \kappa(\Gamma_0)\}$, $\gamma_j \in \Gamma_0$, and $t_j := t_{P_j}$ be as in (6.2) with respect to P_j. Fix $Y(\Gamma_0) \in (0,\infty)$ such that for each $P_{0,l} \in \mathfrak{P}_{\Gamma_0}$, $l = 1, \dots, \kappa(\Gamma_0)$, one has

$$Y(\Gamma_0) \geq Y^0_{\Gamma_0}(P_{0,l}), \tag{6.3}$$

where $Y^0_{\Gamma_0}(P_{0,l})$ is defined by (2.14). Then the following Lemma holds.

Lemma 6.1. *For each $P_j \in \mathfrak{P}_{\Gamma_1}$ let $Y^0_{P_j}(\Gamma_1)$ be defined by* (2.14). *Then one has*

$$Y^0_{P_j}(\Gamma_1) \leq t_j^{-1} Y(\Gamma_0). \tag{6.4}$$

Let $X_0(Y) := X_0(P_{0,1},\dots,P_{0,\kappa(X_0)},Y)$. Then for Y sufficiently large one has

$$\pi^{-1}(X_0(Y)) = X_1(P_1,\dots,P_{\kappa(X_1)};t_1^{-1}Y,\dots,t_{\kappa(X_1)}^{-1}Y).$$

Proof. Since $P_{0,l(j)} = k_{0,l(j)}P_jk_{0,l(j)}^{-1}$, and since the adjoint action by $k_{0,l(j)}$ is an isometry form the Lie-algebra of A_{P_j} to the Lie-algebra of $A_{P_{0,l(j)}}$, it follows that for every $j = 1,\dots,\kappa(X_1)$ and every $g \in G$ one has

$$y_{P_{0,l(j)}}(\gamma_j g k_{0,l(j)}^{-1}) = t_j y_{P_j}(g). \tag{6.5}$$

This implies (6.4). Indeed, if $g \in G$ and $\gamma \in \Gamma_1$ satisfy $y_{P_j}(g) > t_j^{-1}Y(\Gamma_0)$ and $y_{P_j}(\gamma g) > t_j^{-1}Y(\Gamma_0)$, then by (6.5) and the choice of $Y(\Gamma_0)$ one has $\gamma \in \gamma_j^{-1}(\Gamma_0 \cap P_{l(j)})\gamma_j = \Gamma_0 \cap P_j$.

To prove the second part of the lemma, let $x \in X_1 - X_1(P_1,\dots,P_{\kappa(X_1)};t_1^{-1}Y,\dots,t_{\kappa(X_1)}^{-1}Y)$. By (2.16) there exists $j \in \{1,\cdots,\kappa(\Gamma_1)\}$ such that $y_{\Gamma_1,P_j}(x) > t_j^{-1}Y$. Then by (2.15) there exists $g \in G$ satisfying $\mathrm{pr}_{X_1}(g) = x$ and $y_{P_j}(g) > t_j^{-1}Y$. Now observe that $\mathrm{pr}_{X_0}(\gamma_j g k_{0,l(j)}^{-1}) = x$. Using (6.5) and (2.15), it follows that $y_{\Gamma_0,P_{0,l(j)}}(\pi(x)) > Y$, i.e., $x \in \pi^{-1}(X_0 - X_0(Y))$. Thus we have shown that

$$X_1 - X_1(P_1,\dots,P_{\kappa(X_1)};t_1^{-1}Y,\dots,t_{\kappa(X_1)}^{-1}Y) \subseteq \pi^{-1}(X_0 - X_0(Y)). \tag{6.6}$$

It remains to prove the opposite inclusion. Fix $l \in \{1,\dots,\kappa(X_0)\}$. Since $P_{0,l}$ equals its normalizer in G, it follows that

$$\#\{P_j \in \mathfrak{P}_{\Gamma_1} \colon \gamma_j P_j \gamma_j^{-1} = P_{0,l}\} = \#\left(\Gamma_1\backslash\Gamma_0/\Gamma_0 \cap P_{0,l}\right) \tag{6.7}$$

and the γ_j with $\gamma_j P_j \gamma_j^{-1} = P_{0,l}$ form a set of representatives of equivalence classes in the double coset (6.7). For each γ_j with $\gamma_j P_j\gamma_j^{-1} = P_{0,l}$ let $\mu_{i,j} \in \Gamma_1\backslash\Gamma_0$, $i = 1,\dots,r(j)$, be such that the orbit of $\Gamma_1\gamma_j$ under the action of $\Gamma_0 \cap P_{0,l}$ is given by the $\Gamma_1\mu_{i,j}$, $i = 1,\dots,r(j)$. Then

$$[\Gamma_0 : \Gamma_1] = \sum_{\substack{j\in\{1,\dots,\kappa(\Gamma_1)\}\\ \gamma_j P_j \gamma_j^{-1} = P_{0,l}}} r(j). \tag{6.8}$$

Write $\mu_{i,j} = \gamma_j p_{i,j}$ with $p_{i,j} \in \Gamma_0 \cap P_{0,l}$. Choose $Y_{P_j} \in (0,\infty)$, $j = 1,\dots,\kappa(X_1)$, such that (2.12) and (2.13) hold for Γ_1. Let $Y > \max\{t_j^{-1}Y_{P_j} \colon j = 1,\dots,\kappa(X_1)\}$. Let $x_0 \in X_0 - X_0(Y)$. Then there exists a $P_{0,l} \in \mathfrak{P}_{\Gamma_0}$ such that $y_{\Gamma_0,P_{0,l}}(x_0) > Y$. Thus there exists $g_0 \in G$ such that $x_0 = \mathrm{pr}_{X_0}(g_0)$ and $y_{P_{0,l}}(g_0) > Y$. By (2.10) one has $y_{P_{0,l}}(p_{i,j}g_0) > Y$. We claim that

$$\pi^{-1}(x_0) = \left\{\mathrm{pr}_{X_1}(\gamma_j^{-1}p_{i,j}g_0) \colon \gamma_j P_j \gamma_j^{-1} = P_{0,l}, : i = 1,\dots,r(j)\right\}. \tag{6.9}$$

Obviously, each $\mathrm{pr}_{X_1}(\gamma_j^{-1}p_{i,j}g_0)$ is contained in $\pi^{-1}(x_0)$. On the other hand, assume that $\mathrm{pr}_{X_1}(\gamma_j^{-1}p_{i,j}g_0) = \mathrm{pr}_{X_1}(\gamma_{j'}^{-1}p_{i',j'}g_0) =: x_1$, where $\gamma_j P_j \gamma_j^{-1} = P_{0,l} = \gamma_{j'}P_{j'}\gamma_{j'}^{-1}$. By (2.10) and (6.5) one obtains

$$y_{\Gamma_1,P_j}(x_1) > t_j^{-1}Y > Y_{P_j}, \quad y_{\Gamma_1,P_j'}(x_1) > t_{j'}^{-1}Y > Y_{P_{j'}}. \tag{6.10}$$

Applying (2.12), (2.13) one obtains $j = j'$ and hence $i = i'$. Thus, since $\#\{\pi^{-1}(x_0)\} = [\Gamma_0 : \Gamma_1]$, (6.9) follows from (6.8). Applying (6.9) and (6.10) one obtains

$$\pi^{-1}(X_0 - X_0(Y)) \subseteq X_1 - X_1(P_1, \dots, P_{\kappa(X_1)}; t_1^{-1}Y, \dots, t_{\kappa(X_1)}^{-1}Y).$$

and together with (6.6) the lemma follows. □

Let $\Delta_{X_1,p}(\tau)$ be the Laplace operator on E_τ-valued p-forms on X_1. Using the preceding Lemma, we can give an invariant definition of the regularized trace of $e^{-t\Delta_{X_1,p}(\tau)}$ provided the set $\mathfrak{P}_{\Gamma_0}$ is fixed. We fix a set $\mathfrak{P}_{\Gamma_1}$ of representatives of Γ_1-cuspidal parabolic subgroups of G. Then by Lemma 6.1 we have

$$\int_{\pi^{-1}(X_0(Y))} \operatorname{Tr} K_{X_1}^{\tau,p}(t;x,x)dx = \int_{X_1(P_1,\dots,P_{\kappa(X_1)};t_1^{-1}Y,\dots,t_{\kappa(X_1)}^{-1}Y)} \operatorname{Tr} K_{X_1}^{\tau,p}(t;x,x)dx. \tag{6.11}$$

Now arguing exactly as in [MP2, section 5] and applying Lemma 6.1, we obtain

$$\begin{aligned}
\int_{\pi^{-1}X_0(Y)} \operatorname{Tr} K_{X_1}^{\tau,p}(t;x,x)dx\,dx = &\sum_{\substack{\sigma\in\hat{M}\\ [\nu_p(\tau):\sigma]\neq 0}} \sum_{P_j\in\mathfrak{P}_{\Gamma_1}} \frac{e^{-t(\tau(\Omega)-c(\sigma))}\dim(\sigma)\log(t_j^{-1}Y)}{\sqrt{4\pi t}} + \sum_j e^{-t\lambda_j} \\
&+ \sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))} \frac{\operatorname{Tr}(\widetilde{\boldsymbol{C}}(\sigma,\nu,0))}{4} \\
&- \frac{1}{4\pi} \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} \int_{\mathbb{R}} e^{-t\left(\lambda^2+\tau(\Omega)-c(\sigma)\right)} \operatorname{Tr}\left(\widetilde{\boldsymbol{C}}(\sigma,\nu,-i\lambda)\frac{d}{dz}\widetilde{\boldsymbol{C}}(\sigma,\nu,i\lambda)\right) d\lambda + o(1),
\end{aligned}$$

as $Y \to \infty$. Here the λ_j in the first row are the eigenvalues of $\Delta_{X_1,p}(\tau)$, counted with multiplicity. It follows that the integral on the left-hand side of (6.11) admits an asymptotic expansion in Y as Y goes to infinity. Note that, since the factor factor $\tau(\Omega)$ comes from equation (5.8), the last equation coincides with [MP2, equation 5.7] up to the occurrence of the t_j's in the first sum. This occurrence is caused by the different choices of truncation parameters. The appearance of the t_j's is exactly the reason why the above integral is independent of the choice of $\mathfrak{P}_{\Gamma_1}$ and depends only on the choice of $\mathfrak{P}_{\Gamma_0}$.

We assume from now on that the set $\mathfrak{P}_{\Gamma_0}$ is fixed. By the above considerations we are let to the following definition of the regularized trace of the heat operator for finite coverings of X_0.

Definition 6.2. Let $X_1 = \Gamma_1\backslash\widetilde{X}$ be a finite covering of X_0 and assume that Γ_1 is torsion free and satisfies (3.1). Let $\Delta_{X_1,p}(\tau)$ be the Laplace operator on E_τ-valued p-forms on X_1.

For any choice of a set $\mathfrak{P}_{\Gamma_1}$ of representatives of Γ_1-cuspidal parabolic subgroups we put

$$\begin{aligned}\operatorname{Tr}_{\mathrm{reg};X_1}(e^{-t\Delta_{X_1,p}(\tau)}) := &-\sum_{\substack{\sigma\in\hat{M}\\ [\nu_p(\tau):\sigma]\neq 0}}\sum_{P_j\in\mathfrak{P}_{\Gamma_1}}\frac{e^{-t(\tau(\Omega)-c(\sigma))}\dim(\sigma)\log(t_j)}{\sqrt{4\pi t}}\\ &+\sum_j e^{-t\lambda_j}+\sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))}\frac{\operatorname{Tr}(\widetilde{\boldsymbol{C}}(\sigma,\nu,0))}{4}\\ &-\frac{1}{4\pi}\sum_{\substack{\sigma\in\hat{M}\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))}\int_{\mathbb{R}} e^{-t\lambda^2}\operatorname{Tr}\left(\widetilde{\boldsymbol{C}}(\sigma,\nu_p(\tau),-i\lambda)\frac{d}{dz}\widetilde{\boldsymbol{C}}(\sigma,\nu,i\lambda)\right)d\lambda,\end{aligned}\tag{6.12}$$

where the notation is as above.

If one expresses $\operatorname{Tr}_{\mathrm{reg};X_1}(e^{-t\Delta_{X_1,p}(\tau)})$ using the geometric side of the trace formula, then it becomes again transparent that the summands $\log t_j$ compensate the ambiguity caused by the choice of $\mathfrak{P}_{\Gamma_1}$ so that $\operatorname{Tr}_{\mathrm{reg};X_1}(e^{-t\Delta_{X_1,p}(\tau)})$ depends only on the choice of $\mathfrak{P}_{\Gamma_0}$. For further details we refer the reader to section 8, in particular to equations (8.9) and (8.12).

7. Exponential decay of the regularized trace for large time

In this section we estimate the regularized trace for large time and with respect to coverings. Let Γ_0 be a lattice in G and put $X_0 = \Gamma_0\backslash\widetilde{X}$. Let $X_1 = \Gamma_1\backslash\widetilde{X}$ be a finite covering of X_0 such that Γ_1 is torsion-free and satisfies (3.1). We assume that a set $\mathfrak{P}_{\Gamma_0}$ of representatives of Γ_0-cuspidal parabolic subgroups is fixed. We define the regularized trace according to Definition 6.2. To begin with we establish the following lemma.

Lemma 7.1. *For every $\sigma\in(0,\infty)$ one has*

$$\int_{\mathbb{R}}\frac{\sigma}{\sigma^2+\lambda^2}e^{-t\lambda^2}d\lambda=\sqrt{4\pi t}\,e^{t\sigma^2}\int_\sigma^\infty e^{-tu^2}du.$$

Proof. Put

$$f(\sigma):=\int_{\mathbb{R}}\frac{\sigma}{\sigma^2+\lambda^2}e^{-t\lambda^2}d\lambda=\int_{\mathbb{R}}e^{-t\sigma^2\lambda^2}\frac{1}{1+\lambda^2}d\lambda.$$

Then

$$\begin{aligned}f'(\sigma)=&-2t\sigma\int_{\mathbb{R}}e^{-t\sigma^2\lambda^2}\frac{\lambda^2}{1+\lambda^2}d\lambda=-2t\sigma\left(\int_{\mathbb{R}}e^{-t\sigma^2\lambda^2}d\lambda-\int_{\mathbb{R}}e^{-t\sigma^2\lambda^2}\frac{1}{1+\lambda^2}d\lambda\right)\\ =&-\sqrt{4\pi t}+2t\sigma f(\sigma).\end{aligned}$$

The general solution of this differential equation on $(0,\infty)$ is given by

$$y(\sigma)=e^{t\sigma^2}\left(\sqrt{4\pi t}\int_\sigma^\infty e^{-tu^2}du+C\right)$$

and since f satisfies $\lim_{\sigma\to\infty}f(\sigma)=0$, the Lemma follows. ∎

The following proposition is our main result concerning the large time estimation of the regularized trace of the heat kernel.

Proposition 7.2. *Let τ be such that $\tau_\theta \ncong \tau$. There exist constants $C, c > 0$ such that for all finite covers X_1 of X_0 one has*

$$\left|\mathrm{Tr}_{\mathrm{reg};X_1}\left(e^{-t\Delta_{X_1,p}(\tau)}\right)\right| \leq Ce^{-ct}(\mathrm{Tr}_{\mathrm{reg};X_1}(e^{-\Delta_{X_1,p}(\tau)}) + \mathrm{vol}(X_1))$$

for $t \geq 10$.

Proof. Let $\overline{\boldsymbol{C}}(\sigma, \nu_p(\tau), s)$ be as in (3.12). For each $\sigma \in \hat{M}$ one has $c(\sigma) = c(w_0\sigma)$. Thus by Lemma 3.1 the last line of 6.12 can be rewritten as

$$-\frac{1}{4\pi\dim(\nu_p(\tau))}\sum_{\substack{\sigma\in\hat{M}/W(A)\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))}\int_{\mathbb{R}} e^{-t\lambda^2}\,\mathrm{Tr}\left(\overline{\boldsymbol{C}}(\sigma,\nu_p(\tau),-i\lambda)\frac{d}{dz}\overline{\boldsymbol{C}}(\sigma,\nu,i\lambda)\right)\,d\lambda.$$

We have

$$\mathrm{Tr}\left(\overline{\boldsymbol{C}}(\sigma,\nu_p(\tau),-i\lambda)\frac{d}{dz}\overline{\boldsymbol{C}}(\sigma,\nu_p(\tau),i\lambda)\right) = \frac{d}{dz}\log\det\overline{\boldsymbol{C}}(\sigma,\nu_p(\tau),i\lambda).$$

Let $\sigma_1, \ldots, \sigma_l \in (0, n]$, $n = (d-1)/2$, be the poles of $\det\overline{\boldsymbol{C}}(\sigma, \nu_p(\tau), s)$ in the half-plane $\mathrm{Re}(s) \geq 0$. Poles occur only if $\sigma = w_0\sigma$. Let η run over the poles of $\det\overline{\boldsymbol{C}}(\sigma, \nu_p(\tau), s)$ in the half-plane $\mathrm{Re}(s) < 0$, both counted with multiplicity. For $\sigma \in \hat{M}$, put

$$\overline{\sigma} = \begin{cases}\sigma, & \sigma = w_0\sigma;\\ \sigma\oplus w_0\sigma, & \sigma\neq w_0\sigma.\end{cases} \tag{7.1}$$

Let $Y(\Gamma_0)$ be as in (6.3). By Lemma 6.1 we have $t_j^{-1}Y(\Gamma_0) \geq Y^0_{P_j}(\Gamma_1)$ for $j = 1, \cdots, \kappa(\Gamma_1)$. Using Theorem 4.6 and (3.3) we get

$$\begin{aligned}
&\frac{1}{\dim(\nu_p(\tau))}\,\mathrm{Tr}\left(\overline{\boldsymbol{C}}(\sigma,\nu_p(\tau),-i\lambda)\frac{d}{dz}\overline{\boldsymbol{C}}(\sigma,\nu_p(\tau),i\lambda)\right)\\
=&2\dim(\overline{\sigma})\left(-\sum_{j=1}^{\kappa(\Gamma_1)}\log t_j + (Y(\Gamma_0)+2)\kappa(\Gamma_1)\right)\\
&+a(\sigma,\nu) + \frac{1}{\dim(\nu_p(\tau))}\left(\sum_{j=1}^{l}\frac{2\sigma_j}{\lambda^2+\sigma_j^2} + \sum_\eta\frac{2\,\mathrm{Re}(\eta)}{(\lambda-\mathrm{Im}(\eta))^2+\mathrm{Re}(\eta)^2}\right),
\end{aligned}$$

where $a(\sigma,\nu) \in \mathbb{R}$, $a(\sigma,\nu) \leq 0$. Let $\sigma_{pp}(\Delta_{X_1,p}(\tau))$ denote the pure point spectrum of $\Delta_{X_1,p}(\tau)$. Then $\sigma_{pp}(\Delta_{X_1,p}(\tau))$ is the union of the cuspidal spectrum $\sigma_{cusp}(\Delta_{X_1,p}(\tau))$ and the residual spectrum $\sigma_{res}(\Delta_{X_1,p}(\tau))$. For a given eigenvalue $\lambda \in \sigma_{pp}(\Delta_{X_1,p}(\tau))$, let $m(\lambda)$ denote its multiplicity. Put

$$I_1(t,\nu_p(\tau)) := \sum_{\lambda\in\sigma_{cusp}(\Delta_{X_1,p}(\tau))} m(\lambda)e^{-t\lambda},$$

$$I_2(t,\nu_p(\tau)) := \sum_{\lambda\in\sigma_{res}(\Delta_{X_1,p}(\tau))} m(\lambda)e^{-t\lambda} - \frac{1}{2\pi\dim\nu_p(\tau)} \sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu_p(\tau):\sigma]\neq 0}} \sum_{j=1}^{l} e^{-t(\tau(\Omega-c(\sigma)))} \int_{\mathbb{R}} e^{-t\lambda^2}\frac{\sigma_j}{\lambda^2+\sigma_j^2}d\lambda,$$

$$I_3(t,\nu_p(\tau)) := -\sum_{\substack{\sigma\in\hat{M}/W(A)\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))}\Bigg(\frac{a(\sigma,\nu)}{4\sqrt{\pi t}} + \frac{1}{\sqrt{4\pi t}}\kappa(\Gamma_1)\dim(\bar{\sigma})(Y(\Gamma_0)+2) + \frac{1}{2\pi\dim(\nu_p(\tau))}\int_{\mathbb{R}} e^{-t\lambda^2}\sum_{\eta}\frac{\mathrm{Re}(\eta)}{\mathrm{Re}(\eta)^2+(\lambda-\mathrm{Im}(\eta))^2}\,d\lambda\Bigg),$$

and

$$I_4(t,\nu_p(\tau)) := \sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))}\frac{\mathrm{Tr}(\widetilde{\boldsymbol{C}}(\sigma,\nu,0))}{4}.$$

Then it follows from (6.12) that we have

$$\mathrm{Tr}_{\mathrm{reg};X_1}(e^{-t\Delta_{X_1,p}(\tau)}) = I_1(t,\nu_p(\tau)) + I_2(t,\nu_p(\tau)) + I_3(t,\nu_p(\tau)) + I_4(t,\nu_p(\tau)). \tag{7.2}$$

To estimate $I_1(t,\nu_p(\tau))$ we apply Corollary 5.2. It follows that for $t\geq 2$ we have

$$|I_1(t,\nu_p(\tau))| \leq e^{-\frac{t}{8}} I_1(1,\nu_p(\tau)). \tag{7.3}$$

To deal with $I_2(t,\nu_p(\tau))$ observe that to each $\lambda_j\in\sigma_{res}(\Delta_p(\tau))$ there correspond a $\sigma\in\hat{M}$ satisfying $\sigma=w_0\sigma$ and $[\nu_p(\tau):\sigma]\neq 0$, and a pole σ_j of $\det C(\sigma,\nu_p(\tau),s)$ in $(0,n]$ such that

$$\lambda_j = -\sigma_j^2 + \tau(\Omega) - c(\sigma). \tag{7.4}$$

Moreover, the multiplicity of σ_j divided by $\dim(\nu_p(\tau))$ equals the multiplicity of the eigenvalue λ_j. Let μ_j be the sequence of the σ_j's, where the multiplicity of each μ_j is the multiplicity of σ_j divided by $\dim(\nu_p(\tau))$. Put

$$h_{\mu_j}(t) := 1 - \frac{\sqrt{t}}{\sqrt{\pi}}\int_{\mu_j}^{\infty} e^{-tu^2}du = 1 - \frac{1}{\sqrt{\pi}}\int_{\sqrt{t}\mu_j}^{\infty} e^{-u^2}du.$$

Using Lemma 7.1, we get

$$I_2(t,\nu_p(\tau)) = \sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))}\sum_j\left(e^{t\mu_j^2}-\frac{1}{2\pi}\int_{\mathbb{R}}e^{-t\lambda^2}\frac{\mu_j}{\lambda^2+\mu_j^2}d\lambda\right)$$
$$= \sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu_p(\tau):\sigma]\neq 0}}\sum_j e^{-t(\tau(\Omega)-c(\sigma)-\mu_j^2)}h_{\mu_j}(t).$$

Now observe that $1\geq h_{\mu_j}(t)\geq\frac{1}{2}$. Moreover, by Corollary 5.2 it follows that for every μ_j we have

$$-\mu_j^2+\tau(\Omega)-c(\sigma)\geq\frac{1}{4}. \tag{7.5}$$

Thus for each $t\geq 10$ we get

$$|I_2(t,\nu_p(\tau))|\leq e^{-\frac{t}{8}}\sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu_p(\tau):\sigma]\neq 0}}\sum_j e^{-\frac{t(\tau(\Omega)-c(\sigma)-\mu_j^2)}{2}}h_{\mu_j}(t)$$
$$\leq e^{-\frac{t}{8}}\sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu_p(\tau):\sigma]\neq 0}}\sum_j e^{-(\tau(\Omega)-c(\sigma)-\mu_j^2)}e^{-1}h_{\mu_j}(t)$$
$$\leq e^{-\frac{t}{8}}\sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu_p(\tau):\sigma]\neq 0}}\sum_j e^{-(\tau(\Omega)-c(\sigma)-\mu_j^2)}h_{\mu_j}(1)=e^{-\frac{t}{8}}I_2(1,\nu_p(\tau)). \tag{7.6}$$

Next we deal with $I_3(t,\nu_p(\tau))$. By [MP2, Lemma 7.1] we have

$$\tau(\Omega)-c(\sigma)\geq\frac{1}{4} \tag{7.7}$$

for all $\sigma\in\hat{M}$ with $[\nu_p(\tau):\sigma]\neq 0$. Then since $a(\sigma,\nu)\leq 0$, $\mathrm{Re}(\eta)<0$, for each $t\geq 2$ we can estimate

$$|I_3(t,\nu_p(\tau))|\leq e^{-\frac{t}{8}}\sum_{\substack{\sigma\in\hat{M}/W(A)\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-(\tau(\Omega)-c(\sigma))}\dim(\bar{\sigma})\kappa(\Gamma_1)(Y(\Gamma_0)+2)\frac{1}{\sqrt{4\pi}}$$
$$-e^{-\frac{t}{8}}\sum_{\substack{\sigma\in\hat{M}/W(A)\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-(\tau(\Omega)-c(\sigma))}\left(\frac{a(\sigma,\nu)}{4\sqrt{\pi}}\right.$$
$$\left.+\frac{1}{2\pi\dim(\nu_p(\tau))}\int_{\mathbb{R}}e^{-\lambda^2}\sum_\eta\frac{\mathrm{Re}(\eta)}{\mathrm{Re}(\eta)^2+(\lambda-\mathrm{Im}(\eta))^2}d\lambda\right)$$
$$=2e^{-\frac{t}{8}}\sum_{\substack{\sigma\in\hat{M}/W(A)\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-(\tau(\Omega)-c(\sigma))}\dim(\bar{\sigma})(Y(\Gamma_0)+2)\kappa(\Gamma_1)\frac{1}{\sqrt{4\pi}}+e^{-\frac{t}{8}}I_3(1,\nu_p(\tau)).$$

By [Ke, Proposition 3.3] there exists $C(d) > 0$ such that

$$\kappa(X) \leq C(d)\,\mathrm{vol}(X) \tag{7.8}$$

for all complete hyperbolic manifolds of finite volume and dimension d. Thus for each $t \geq 2$ we obtain

$$|I_3(t, \nu_p(\tau))| \leq e^{-\frac{t}{8}}(I_3(1, \nu_p(\tau)) + C_2\,\mathrm{vol}(X_1)), \tag{7.9}$$

where C_2 depends only on Γ_0 and $\mathfrak{P}_{\Gamma_0}$. To estimate $I_4(t, \nu_p(\tau))$ we recall that $\widetilde{\boldsymbol{C}}(\sigma, \nu, 0)^2 = \mathrm{Id}$. Hence there exist natural numbers $c_1(\Gamma, \sigma, \nu)$, $c_2(\Gamma, \sigma, \nu)$ such that

$$c_1(\Gamma_1, \sigma, \nu) + c_2(\Gamma_1, \sigma, \nu) = \dim\left(\boldsymbol{\mathcal{E}}(\sigma, \nu) \otimes V_\nu\right)^K = \kappa(X_1)\dim(\sigma),$$

and

$$\mathrm{Tr}(\widetilde{\boldsymbol{C}}(\sigma, \nu, 0)) = c_1(\Gamma_1, \sigma, \nu) - c_2(\Gamma_1, \sigma, \nu).$$

Using (7.7) and (7.8) we obtain for $t \geq 2$:

$$\begin{aligned} &|I_4(t, \nu_p(\tau))| \\ &\leq e^{-\frac{t}{8}}(I_4(1, \nu_p(\tau)) + 2c_2(\Gamma_1, \sigma, \nu)) \leq e^{-\frac{t}{8}}(I_4(1, \nu_p(\tau)) + 2C(d)\dim(\sigma)\,\mathrm{vol}(X_1)). \end{aligned} \tag{7.10}$$

Combining (7.2), (7.3), (7.6), (7.9) and (7.10), the proof of the proposition is complete. □

8. Geometric side of the trace formula

To study the behaviour of the analytic torsion under coverings we will apply the trace formula to the regularized trace of the heat operator. In this section we recall the structure of the geometric side of the trace formula and study the parabolic contribution.

Let the assumptions be the same as at the beginning of the previous section. Let $\tau \in \mathrm{Rep}(G)$ and assume that $\tau \neq \tau_\theta$. Let $\widetilde{E}_\tau$ be the homogeneous vector bundle over $\widetilde{X} = G/K$, associated to $\tau|_K$, equipped with an admissible Hermitian metric (see section 5). Let $\widetilde{\Delta}_p(\tau)$ be the Laplace operator on $\widetilde{E}_\tau$-valued p-forms. The on $C^\infty(G, \nu_p(\tau))$ one has

$$\widetilde{\Delta}_p(\tau) = -\Omega + \tau(\Omega), \tag{8.1}$$

see [MtM, (6.9)] Let

$$H_t^{\tau,p}\colon G \to \mathrm{End}(\Lambda^p\mathfrak{p}^* \otimes V_\tau) \tag{8.2}$$

be the kernel of the heat operator $e^{-t\widetilde{\Delta}_p(\tau)}$. Let

$$h_t^{\tau,p} = \mathrm{tr}\, H_t^{\tau,p}. \tag{8.3}$$

We apply the trace formulas in [MP2, section 6] to express the regularized trace as a sum of distributions evaluated at $h_t^{\tau,p}$. The terms appearing on the geometric side of the trace formula are associated to the different types of Γ-conjugacy classes. We briefly recall their definition. For further details, we refer the reader to [MP2, section 6] and the references therein. In order to indicate the dependence of the distributions on the manifold X_1, we shall use X_1 as a subscript. The contribution of the identity to the trace formula is given by

$$I_{X_1}(h_t^{\tau,p}) := \mathrm{vol}(X_1) h_t^{\tau,p}(1). \tag{8.4}$$

The hyperbolic contribution is given by

$$H_{X_1}(h_t^{\tau,p}) := \int_{\Gamma_1\backslash G} \sum_{\gamma\in\Gamma_{1,s}-\{1\}} h_t^{\tau,p}(x^{-1}\gamma x)\, dx, \tag{8.5}$$

where $\Gamma_{1,\mathrm{s}}$ are the semisimple elements of Γ_1. By [Wa, Lemma 8.1] the integral converges absolutely. Moreover, arguing as in the cocompact case [Wal], if G_γ resp. $(\Gamma_1)_\gamma$ denote the centralizers of γ in G resp. Γ_1, one has

$$H_{X_1}(h_t^{\tau,p}) = \sum_{[\gamma]\in \mathrm{C}(\Gamma_1)_s-[1]} \mathrm{vol}((\Gamma_1)_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} h_t^{\tau,p}(x^{-1}\gamma x)dx,$$

where $\mathrm{C}(\Gamma_1)_{\mathrm{s}}$ are the Γ_1-conjugacy classes of semisimple elements of Γ_1. Now the latter sum can also be written as a sum over the set $\mathrm{C}(\Gamma_0)_s$ of non elliptic semisimple conjugacy classes of the group Γ_0 as follows. For each $\gamma\in\Gamma_0$ let $c_{\Gamma_1}(\gamma)$ be the number of fixed points of γ on Γ_0/Γ_1. This number clearly depends only on the Γ_0-conjugacy class of γ. Then if Γ_γ is the centralizer of γ in Γ_0, one has

$$H_{X_1}(h_t^{\tau,p}) = \sum_{[\gamma]\in \mathrm{C}(\Gamma_0)_{\mathrm{s}}-[1]} c_{\Gamma_1}(\gamma)\, \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} h_t^{\tau,p}(x^{-1}\gamma x)dx, \tag{8.6}$$

see [Co, page 152-153]. This expression will be used when we treat the Hecke subgroups of the Bianchi groups.

Next we describe the distributions associated to the parabolic conjugacy classes. Firstly let

$$T'_{X_1}(h_t^{\tau,p}) := \kappa(X_1) \int_K \int_N h_t^{\tau,p}(knk^{-1}) \log\|\log n\|\, dkdn. \tag{8.7}$$

We note that T is a non-invariant distribution which depends on X_1 only via the number of cusps of X_1. Now let P' be any Γ_0-cuspidal parabolic subgroup of G, or equivalently a Γ_1-cuspidal parabolic subgroup of G. Let $\mathfrak{n}_{P'}$ denote the Lie algebra of $N_{P'}$. Then $\exp : \mathfrak{n}_{P'} \to N_{P'}$ is an isomorphism and we denote its inverse by log. We equip $\mathfrak{n}_{P'}$ with the inner product obtained by restriction of the inner product in (2.3). By $\|\cdot\|$ we denote the corresponding norm. Let

$$\Lambda_{P'}(\Gamma_1) := \log(\Gamma_1\cap N_{P'});\quad \Lambda^0_{P'}(\Gamma_1) := \mathrm{vol}(\Lambda_{P'}(\Gamma_1))^{-\frac{1}{2n}}\Lambda_{P'}(\Gamma_1).$$

Then $\Lambda_{P'}(\Gamma_1)$ and $\Lambda^0_{P'}(\Gamma_1)$ are lattices in $\mathfrak{n}_{P'}$ and $\Lambda^0_{P'}(\Gamma_1)$ is unimodular. Then for $\mathrm{Re}(s)>0$ the Epstein-type zeta function $\zeta_{P';\Gamma_1}$, defined by

$$\zeta_{P';\Gamma_1}(s) := \sum_{\eta\in\Lambda_{P'}(\Gamma_1)-\{0\}} \|\eta\|^{-2n(1+s)}, \tag{8.8}$$

converges and $\zeta_{P';\Gamma_1}$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at 0. Let $C(\Lambda_{P'}(\Gamma_1))$ be the constant term of $\zeta_{P';\Gamma_1}$ at $s=0$. Now as before let $\mathfrak{P}_{\Gamma_1}$ be a set of representatives of Γ_1-cuspidal parabolic subgroups and for each $P_j \in \mathfrak{P}_{\Gamma_1}$ let t_j be as in the previous sections. Then put

$$S_{X_1}(h_t^{\tau,p}) := \sum_{P_j\in\mathfrak{P}_{\Gamma_1}} \Bigg(C(\Lambda_{P_j}(\Gamma_1)) \frac{\operatorname{vol}(\Lambda_{P_j}(\Gamma_1))}{\operatorname{vol}(S^{2n-1})} \sum_{\sigma\in\hat{M}} \frac{\dim(\sigma)}{2\pi} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(h_t^{\tau,p})\, d\lambda \\ - \sum_{\substack{\sigma\in\hat{M}\\ [\nu_p(\tau):\sigma]\neq 0}} \frac{e^{-t(\tau(\Omega)-c(\sigma))}\dim(\sigma)\log(t_j)}{\sqrt{4\pi t}} \Bigg).$$

Comparing the Definition 6.2 and [MP2, Definition 5.1], it follows from [MP2, Theorem 6.1] that

$$\operatorname{Tr}_{\mathrm{reg};X_1}(e^{-t\Delta_{X_1,p}(\tau)}) = I_{X_1}(h_t^{\tau,p}) + H_{X_1}(h_t^{\tau,p}) + T'_{X_1}(h_t^{\tau,p}) + S_{X_1}(h_t^{\tau,p}). \tag{8.9}$$

We now study the distribution $S_{X_1}(h_t^{\tau,p})$ in more detail. By [MP2, Proposition 4.1] we have

$$\Theta_{\sigma,\lambda}(h_t^{\tau,p}) = e^{-t(\lambda^2+\tau(\Omega)-c(\sigma))}$$

for $[\nu_p(\tau):\sigma]\neq 0$, and $\Theta_{\sigma,\lambda}(h_t^{\nu}) = 0$ otherwise. Thus we can rewrite

$$S_{X_1}(h_t^{\tau,p}) := \sum_{\substack{\sigma\in\hat{M}\\ [\nu_p(\tau):\sigma]\neq 0}} \frac{e^{-t(\tau(\Omega)-c(\sigma))}\dim(\sigma)}{\sqrt{4\pi t}} \left(\sum_{P_j\in\mathfrak{P}_{\Gamma_1}} C(\Lambda_{P_j}(\Gamma_1)) \frac{\operatorname{vol}(\Lambda_{P_j}(\Gamma_1))}{\operatorname{vol}(S^{2n-1})} - \log(t_j) \right).$$

Let Λ be a lattice in $\mathbb{R}^{2n}$. The associated Epstein zeta function

$$\zeta_\Lambda(s) := \sum_{\lambda\in\Lambda-\{0\}} \|\lambda\|^{-2n(1+s)} \tag{8.10}$$

converges for $\operatorname{Re}(s)>0$ and admits a meromorphic extension to $\mathbb{C}$. Let $C(\Lambda)$ denote the constant term of the Laurent expansion of $\zeta_\Lambda(s)$ at $s=0$. The following lemma describes the behaviour of $C(\Lambda)$ under scaling.

Lemma 8.1. *Let Λ be a lattice in $\mathbb{R}^{2n}$. Let $\mu\in(0,\infty)$ and put $\Lambda' := \mu\Lambda$. Then one has*

$$C(\Lambda') = \mu^{-2n}\left(C(\Lambda) - \frac{\operatorname{vol}(S^{2n-1})\log\mu}{\operatorname{vol}(\Lambda)} \right).$$

Proof. Let $R(\Lambda)$ be the residue of ζ_Λ at 0. Then one has

$$C(\Lambda') = \mu^{-2n}(C(\Lambda) - R(\Lambda)2n\log\mu).$$

Moreover, by [Ter, Chapter 1.4, Theorem 1] one has

$$R(\Lambda) = \frac{\operatorname{vol}(S^{2n-1})}{2n\operatorname{vol}(\Lambda)}$$

and the lemma follows. □

Now we let P' be any Γ_0-cuspidal parabolic subgroup of G. Following section 6, we let $l' \in \{1, \dots, \kappa(X_0)\}$ such that there exists $\gamma' \in \Gamma_0$ with $\gamma' P' \gamma'^{-1} = P_{0,l'}$. As in (6.2) we write $\gamma' = n_{0,l'} a_{0,l'}(\log t_{P'}) k_{0,l'}$. If Γ_1 is a finite index subgroup of Γ_0, we define a lattice $\tilde{\Lambda}_{P'}(\Gamma_1)$ in $\mathfrak{n}_{P_{0,l(P')}}$ as

$$\tilde{\Lambda}_{P'}(\Gamma_1) := \log(\gamma'(\Gamma_1 \cap N_{P'})\gamma'^{-1}).$$

If Γ_1 is normal in Γ_0, one has $\tilde{\Lambda}_{P'}(\Gamma_1) = \Lambda_{P_{0,l'}}(\Gamma_1)$. Since γ' is unique in $\Gamma_0/(\Gamma_0 \cap P')$ and $\Gamma_0 \cap P' = \Gamma_0 \cap (M_{P'} N_{P'})$, the isometry class of $\tilde{\Lambda}_{P'}(\Gamma_1)$ is independent of the choice of γ' having the required property. Let $\hat{\Lambda}_{P'}(\Gamma_1)$ be the unimodular lattice corresponding to $\tilde{\Lambda}_{P'}(\Gamma_1)$, i.e.

$$\hat{\Lambda}_{P'}(\Gamma_1) := (\operatorname{vol}(\tilde{\Lambda}_{P'}(\Gamma_1)))^{-\frac{1}{2n}} \cdot \tilde{\Lambda}_{P'}(\Gamma_1).$$

With respect to the norms induced by the Killing form, the lattice $\Lambda_{P'}(\Gamma_1)$ in $\mathfrak{n}_{P'}$ is isometric to the lattice $t_{P'}^{-1}\tilde{\Lambda}_{P'}(\Gamma_1)$ in $\mathfrak{n}_{P_{0,l(P')}}$. Thus the preceding Lemma implies that

$$\frac{C(\Lambda_{P_j}(\Gamma_1)) \operatorname{vol}(\Lambda_{P_j}(\Gamma_1))}{\operatorname{vol}(S^{2n-1})} = \frac{C(\tilde{\Lambda}_{P_j}(\Gamma_1)) \operatorname{vol}(\tilde{\Lambda}_{P_j}(\Gamma_1))}{\operatorname{vol}(S^{2n-1})} + \log t_j.$$

Now define

$$\alpha(X_1) := \alpha(\Gamma_1) := \sum_{j=1}^{\kappa(X_1)} \frac{C(\tilde{\Lambda}_{P_j}(\Gamma_1)) \operatorname{vol}(\tilde{\Lambda}_{P_j}(\Gamma_1))}{\operatorname{vol}(S^{2n-1})}. \tag{8.11}$$

Then, putting everything together, we can write

$$S_{X_1}(h_t^{\tau,p}) = \alpha(X_1) \sum_{\substack{\sigma \in \hat{M} \\ [\nu_p(\tau):\sigma] \neq 0}} \frac{e^{-t(\tau(\Omega) - c(\sigma))} \dim(\sigma)}{\sqrt{4\pi t}}. \tag{8.12}$$

Finally, for each $l = 1, \dots, \kappa(\Gamma_0)$, we let $\mathcal{P}(\mathfrak{n}_{P_{0,l}})$ be the set of isometry classes of unimodular lattices in $\mathfrak{n}_{P_{0,l}}$ equipped with the standard topology, i.e., with the topology induced by identification of $\mathcal{P}(\mathfrak{n}_{P_{0,l}})$ with $\operatorname{SO}(2n) \backslash \operatorname{SL}_{2n}(\mathbb{R}) / \operatorname{SL}_{2n}(\mathbb{Z})$. Now in order to control the constant $\alpha(\Gamma_i)$ for sequences of finite coverings, we make the following definition.

Definition 8.2. Let Γ_i be a sequence of finite index subgroups of Γ_0. Let $\mathfrak{P}_{\Gamma_0}$ be a fixed set of representatives Γ_0-cuspidal parabolic subgroups of Γ_0. Then the sequence Γ_i is called cusp uniform if for each $l = 1, \dots, \kappa(\Gamma_0)$ there exists a compact set $\mathcal{K}_l$ in $\mathcal{P}(\mathfrak{n}_{P_{0,l}})$ such that for each Γ_0-cuspidal parabolic P' the lattices $\hat{\Lambda}_{P'}(\Gamma_i)$, $i \in \mathbb{N}$, belong to $\mathcal{K}_l$.

We can reformulate the condition of cusp-uniformity in a simpler way as follows. We let $\mathcal{P}(\mathfrak{n})$ be the space of isometry classes of unimodular lattices in $\mathfrak{n}$, equipped with the topology as above. For each parabolic subgroup P' of G there exists a $g_{P'} \in G$ with $g_{P'} P' g_{P'}^{-1} = P$. Let Γ be a discrete subgroup of G of finite covolume. If P' is Γ-cuspidal, we let

$$\Lambda_{P|P'}(\Gamma) := \operatorname{vol}\left(\log(g_{P'}(\Gamma \cap N_{P'})g_{P'}^{-1})\right)^{\frac{1}{2n}} \log(g_{P'}(\Gamma \cap N_{P'})g_{P'}^{-1}). \tag{8.13}$$

This a unimodular lattice in $\mathfrak{n}$ and since the image of $g_{P'}$ in $P\backslash G$ is unique, the isometry class of $\Lambda_{P'}(\Gamma)$ is independent of the choice of $g_{P'}$ with $g_{P'}P'g_{P'}^{-1} = P$.

Lemma 8.3. *The following conditions are equivalent:*

1. *The sequence Γ_i is cusp-uniform.*
2. *For each Γ_0-cuspidal parabolic subgroup P' of G there exists a compact set $\mathcal{K}_{P'}$ in $\mathcal{P}(\mathfrak{n}_{P'})$ such that $\Lambda^0_{P'}(\Gamma_i) \in \mathcal{K}_{P'}$ for every i.*
3. *There exists a compact set $\mathcal{K}_P$ in $\mathcal{P}(\mathfrak{n}_P)$ such that for each Γ_0-cuspidal parabolic subgroup P' of G one has $\Lambda_{P|P'}(\Gamma_i) \in \mathcal{K}_P$ for each $i \in \mathbb{N}$.*

Proof. By the preceding arguments all lattices are isometric. □

Lemma 8.4. *Let $\mathcal{K}$ be a compact set of unimodular lattices in $\mathbb{R}^{2n}$. Then the constant term of the Laurent expansion of the Epstein zeta functions $\zeta_\Lambda(s)$ at $s = 0$ is bounded on $\mathcal{K}$.*

Proof. By [Ter, Chapt.I, §1.4, Theorem 1] the analytic continuation of $\zeta_\Lambda(s)$ is given by

$$\pi^{-s}\Gamma(s)\zeta_\Lambda(s) = \frac{2}{ns} - \frac{2}{n(1+s)} + \left(\int_1^\infty (t^{\frac{n}{2}(1+s)-1} + t^{-\frac{n}{2}s-1}) \sum_{\lambda\in\Lambda-\{0\}} e^{-t\pi\|\lambda\|^2}\,dt\right). \tag{8.14}$$

Now for a lattice Λ in $\mathbb{R}^{2n}$, let $\lambda_1(\Lambda)$ denote the smallest norm of a non-zero vector in Λ. Let $\mathbb{B}(R)$ denote the ball in $\mathbb{R}^{2n}$ around the origin of radius R. Then it follows from [BHW, Theorem 2.1] that for each $R > 0$ we have

$$\#\{\mathbb{B}(R) \cap \Lambda\} \le \left(\frac{2R}{\lambda_1(\Lambda)} + 1\right)^{2n}.$$

If $\mathcal{K}$ is a compact set of unimodular lattices in $\mathbb{R}^{2n}$, then by Mahler's criterion there exists a constant μ such that $\lambda_1(\Lambda) \ge \mu$ for each $\Lambda \in \mathcal{K}$. Thus for each $\Lambda \in \mathcal{K}$ and for each $t \in [1,\infty)$ we have

$$\sum_{\lambda\in\Lambda-\{0\}} e^{-t\pi\|\lambda\|^2} \le e^{-\frac{t\pi\mu^2}{2}} \sum_{\lambda\in\Lambda-\{0\}} e^{-\frac{\pi\|\lambda\|^2}{2}} \le e^{-\frac{t\pi\mu^2}{2}} \sum_{k=1}^\infty e^{-\frac{\pi(\mu k)^2}{2}} \#\{\mathbb{B}(\mu(k+1)) \cap \Lambda\}$$
$$\le e^{-\frac{t\pi\mu^2}{2}} \sum_{k=1}^\infty e^{-\frac{\pi(\mu k)^2}{2}} (2k+3)^{2n} =: C_1 e^{\frac{-t\pi\mu^2}{2}},$$

where C_1 is a constant which is independent of Λ. Applying (8.14), the Lemma follows. □

Now we can control the behaviour of the constants, appearing in the definitions of the terms $T'_{X_i}(h_t^{\tau,p})$ and $\mathcal{S}_{X_i}(h_t^{\tau,p})$, under sequences of coverings $X_i = \Gamma_i\backslash\widetilde{X}$ of X_0. As always we assume that a set $\mathfrak{P}_{\Gamma_0}$ of representatives of Γ_0-cuspidal parabolic subgroups of G is fixed. For each i we let $\mathfrak{P}_{\Gamma_i} = \{P_{i,j},\ j = 1, \dots, \kappa(\Gamma_i)\}$ be a set of representatives of Γ_i-conjugacy classes of Γ_i-cuspidal parabolic subgroups. We can estimate $\alpha(\Gamma_i)$ as follows.

Proposition 8.5. *Let Γ_i be cusp-uniform sequence of finite index subgroups of Γ_0. Then there exists a constant $c_1(\Gamma_0)$ such that*

$$|\alpha(\Gamma_i)| \leq c_1(\Gamma_0)\kappa(\Gamma_i) + c_1(\Gamma_0)\sum_{j=1}^{\kappa(\Gamma_i)} \log[\Gamma_0 \cap N_{P_{i,j}} : \Gamma_i \cap N_{P_{i,j}}].$$

In particular, there exists a constant $c_2(\Gamma_0)$ such that we have

$$|\alpha(\Gamma_i)| \leq c_2(\Gamma_0)\kappa(\Gamma_i)\log[\Gamma_0 : \Gamma_i].$$

Proof. By Lemma 8.1, for each $P_{i,j} \in \mathfrak{P}_{\Gamma_i}$ one has

$$C(\tilde{\Lambda}_{P_{i,j}}(\Gamma_i))\operatorname{vol}(\tilde{\Lambda}_{P_{i,j}}(\Gamma_i)) = C(\hat{\Lambda}_{P_{i,j}}(\Gamma_i)) - \frac{\operatorname{vol}(S^{2n-1})\log\operatorname{vol}(\tilde{\Lambda}_{P_{i,j}}(\Gamma_i)}{2n}.$$

By assumption the lattices $\hat{\Lambda}_{P_{i,j}}(\Gamma_i)$, $i \in \mathbb{N}$, lie in a compact subset of $\mathcal{P}(\mathfrak{n}_{P_{0,l(j)}})$. Thus by Lemma 8.4 there exists a constant $c_1'(\Gamma_0)$ such that for each i one has $|C(\hat{\Lambda}_{P_{i,j}}(\Gamma_i))| \leq c_1'(\Gamma_0)$. Since $\tilde{\Lambda}_{P_{i,j}}(\Gamma_0) = \Lambda_{P_{0,l(j)}}(\Gamma_0)$, the lattice $\tilde{\Lambda}_{P_{i,j}}(\Gamma_i)$ is a sublattice of $\Lambda_{P_{0,l(j)}}(\Gamma_0)$ of index $[\Gamma_0 \cap N_{P_{i,j}} : \Gamma_i \cap N_{P_{i,j}}]$. Therefore one has

$$\operatorname{vol}(\tilde{\Lambda}_{P_{i,j}}(\Gamma_i)) = \operatorname{vol}(\Lambda_{P_{0,l(j)}}(\Gamma_0))[\Gamma_0 \cap N_{P_{i,j}} : \Gamma_i \cap N_{P_{i,j}}] \leq c_1''(\Gamma_0)[\Gamma_0 \cap N_{P_{i,j}} : \Gamma_i \cap N_{P_{i,j}}],$$

where $c_1''(\Gamma_0)$ is a constant which is independent of i. This proves the first estimate. The second estimate follows immediately from the first one. ☐

In the next proposition we estimate the number of cusps and the behaviour of the constant $\alpha(\Gamma_i)$ under sequences of normal coverings.

Proposition 8.6. *Let Γ_i be a sequence of normal subgroups of Γ_0 of finite index $[\Gamma_0 : \Gamma_i]$ such that $[\Gamma_0 : \Gamma_i] \to \infty$ as $i \to \infty$ and such that each $\gamma_0 \in \Gamma_0$, $\gamma_0 \neq 1$, belongs only to finitely many Γ_i. Assume that each Γ_i satisfies assumption* (3.1). *Then one has*

$$\lim_{i\to\infty} \frac{\kappa(\Gamma_i)}{[\Gamma_0 : \Gamma_i]} = 0.$$

If in addition the sequence Γ_i is cusp-uniform, then one has

$$\lim_{i\to\infty} \frac{|\alpha(\Gamma_i)|}{[\Gamma_0 : \Gamma_i]} = 0.$$

Proof. Using that each Γ_i, $i \geq 1$, satisfies (3.1) and Γ_i is normal in Γ_0, one has

$$\#\{\Gamma_i\backslash\Gamma_0/\Gamma_0 \cap P_{0,l}\} = \frac{[\Gamma_0 : \Gamma_i]}{[\Gamma_0 \cap P_{0,l} : \Gamma_i \cap P_{0,l}]} \leq \frac{[\Gamma_0 : \Gamma_i]}{[\Gamma_0 \cap N_{P_{0,l}} : \Gamma_i \cap N_{P_{0,l}}]}$$

for each $l = 1, \dots, \kappa(\Gamma_0)$. Thus using (6.7), one can estimate

$$\frac{\kappa(\Gamma_i)}{[\Gamma_0 : \Gamma_i]} = \frac{\sum_{P_{0,l}\in\mathfrak{P}_{\Gamma_0}} \#\{\Gamma_i\backslash\Gamma_0/\Gamma_0 \cap P_{0,l}\}}{[\Gamma_0 : \Gamma_i]} \leq \sum_{P_0\in\mathfrak{P}_{\Gamma_0}} \frac{1}{[\Gamma_0 \cap N_{P_{0,l}} : \Gamma_i \cap N_{P_{0,l}}]}.$$

Moreover, for each $l = 1, \dots, \kappa(\Gamma_0)$ and each $j = 1, \dots, \kappa(\Gamma_i)$ one has

$$\Gamma_0 \cap N_{P_{i,j}} = \gamma_j(\Gamma_0 \cap N_{P_{0,l(j)}})\gamma_j^{-1}, \quad \Gamma_i \cap N_{P_{i,j}} = \gamma_j(\Gamma_i \cap N_{P_{0,l(j)}})\gamma_j^{-1},$$

where the second equality is due to the assumption that Γ_i is normal in Γ_0. Thus applying (6.7), one can estimate

$$\begin{aligned}
&\frac{1}{[\Gamma_0 : \Gamma_i]} \sum_{j=1}^{\kappa(\Gamma_i)} \log[\Gamma_0 \cap N_{P_{i,j}} : \Gamma_i \cap N_{P_{i,j}}] \\
&= \frac{\sum_{P_{0,l} \in \mathfrak{P}_{\Gamma_0}} \#\{\Gamma_i \backslash \Gamma_0 / \Gamma_0 \cap P_{0,l}\} \log\left[\Gamma_0 \cap N_{P_{0,l}} : \Gamma_i \cap N_{P_{0,l}}\right]}{[\Gamma_0 : \Gamma_i]} \\
&\leq \sum_{P_0 \in \mathfrak{P}_{\Gamma_0}} \frac{\log\left[\Gamma_0 \cap N_{P_{0,l}} : \Gamma_i \cap N_{P_{0,l}}\right]}{\left[\Gamma_0 \cap N_{P_{0,l}} : \Gamma_i \cap N_{P_{0,l}}\right]}.
\end{aligned}$$

The condition that each $\gamma_0 \in \Gamma_0 - \{1\}$, $\gamma_0 \neq 1$, belongs only to finitely many Γ_i implies that $[\Gamma_0 \cap N_{P_{0,l}} : \Gamma_i \cap N_{P_{0,l}}]$ goes to ∞ as $i \to \infty$. Thus the first statement and together with the previous proposition also the second one are proved. ☐

9. Proof of the main results

We keep the assumptions of the previous sections. So Γ_0 is a lattice in G and Γ_1 is a torsion-free subgroup of finite index of Γ_0, which satisfies (3.1). We let $X_0 := \Gamma_0 \backslash \widetilde{X}$ and $X_i := \Gamma_i \backslash \widetilde{X}$. We assume that a set $\mathfrak{P}_{\Gamma_0}$ of representatives of Γ_0-conjugacy classes of Γ_0-cuspidal parabolic subgroups of G is fixed. Then for each $\tau \in \mathrm{Rep}(G)$, $\tau \neq \tau_\theta$, let $\mathrm{Tr}_{\mathrm{reg};X_1}(e^{-t\Delta_{X_1,p}(\tau)})$ be the the regularized trace of $e^{-t\Delta_{X_1,p}(\tau)}$, as defined by 6.2. It follows from Proposition (7.2) that there exist constants $C, c > 0$ such that

$$\left| \mathrm{Tr}_{\mathrm{reg};X_1}\left(e^{-t\Delta_{X_1,p}(\tau)}\right) \right| \leq Ce^{-ct}, \tag{9.1}$$

for $t \geq 1$. Applying [Proposition 6.9][MP2], it follows immediately from the definition of $\mathrm{Tr}_{\mathrm{reg};X_1}(e^{-t\Delta_{X_1,p}(\tau)})$ that there is an asymptotic expansion

$$\mathrm{Tr}_{\mathrm{reg};X_1}(e^{-t\Delta_{X_1,p}(\tau)}) \sim \sum_{j=0}^{\infty} a_j t^{j-\frac{d}{2}} + \sum_{j=0}^{\infty} b_j t^{j-\frac{1}{2}} \log t + \sum_{j=0}^{\infty} c_j t^j$$

as $t \to +0$. Put

$$K_{X_1}(t,\tau) := \frac{1}{2} \sum_{p=1}^{d} (-1)^p p \, \mathrm{Tr}_{\mathrm{reg};X_1}\left(e^{-t\Delta_{X_1,p}(\tau)}\right).$$

Then it follows that we can define the analytic torsion $T_{X_1}(\tau)$ by

$$\log T_{X_1}(\tau) = \frac{d}{ds}\left(\frac{1}{\Gamma(s)} \int_0^\infty K_{X_1}(t,\tau) t^{s-1}\, dt\right)\bigg|_{s=0},$$

where the integral converges in the half-plane $\mathrm{Re}(s) > d/2$ and is defined near $s = 0$ by analytic continuation. Let $T > 0$. Then it follows from (9.1) that $\int_T^\infty K_{X_1}(t,\tau)t^{s-1}\,dt$ is an entire function of s. Therefore we have

$$\log T_{X_1}(\tau) = \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T K_{X_1}(t,\tau)t^{s-1}\,dt\right)\Bigg|_{s=0} + \int_T^\infty K_{X_1}(t,\tau)t^{-1}\,dt. \tag{9.2}$$

To proceed further, we first need to estimate the integrand of the hyperbolic term (8.5). Recall that for a lattice Γ in G we denote by $\ell(\Gamma)$ the length of the shortest closed geodesic of $\Gamma\backslash\widetilde{X}$.

Lemma 9.1. *Let $h_t^{\tau,p} \in C^\infty(G)$ be defined by (8.3). For each $T \in (0,\infty)$ there exists a constant $C > 0$, depending on T and X_0 only, such that for all hyperbolic manifolds $X_1 = \Gamma_1\backslash\widetilde{X}$, which are finite coverings of X_0, and all $g \in G$ one has*

$$\left|\sum_{\gamma\in\Gamma_{1,s}-\{1\}} h_t^{\tau,p}(g^{-1}\gamma g)\right| \le Ce^{-\frac{\ell(\Gamma_0)^2}{32t}}e^{-\frac{\ell(\Gamma_1)^2}{8t}}$$

for all $t \in (0,T]$.

Proof. Let $\nu_p(\tau)$ be the representation of K defined by (5.5). Let $\widetilde{E}_{\nu_p(\tau)}$ be the associated homogeneous vector bundle over $\widetilde{X}$ equipped with the canonical metric connection [MP2, section 4]. Let $\widetilde{\Delta}_{\nu_p(\tau)}$ be the Bochner-Laplace operator acting on $C^\infty(\widetilde{X},\widetilde{E}_{\nu_p(\tau)})$. Then on $C^\infty(G,\nu_p(\tau))$, the action of this operator is given by

$$\widetilde{\Delta}_{\nu_p(\tau)} = -R(\Omega) + \nu_p(\tau)(\Omega_K),$$

where Ω_K is the Casimir eigenvalue of $\mathfrak{k}$ with respect to the restriction of the normalized Killing form $\mathfrak{g}$ to $\mathfrak{k}$, see [Mi1, Proposition 1.1]. Thus by (8.1) there exists an endomorphism $E_p(\tau)$ of $\Lambda^p\mathfrak{p}^*\otimes V_\tau$ such that

$$\widetilde{\Delta}_p(\tau) = \widetilde{\Delta}_{\nu_p(\tau)} + E_p(\tau).$$

Moreover $E_p(\tau)$ commutes with $\widetilde{\Delta}_{\nu_p(\tau)}$. Let

$$H_t^{\nu_p(\tau)} \colon G \to \mathrm{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau)$$

be the kernel of the heat operator $e^{-t\widetilde{\Delta}_{\nu_p(\tau)}}$. Then it follows that

$$H_t^{\tau,p} = e^{-tE_p(\tau)}\circ H_t^{\nu_p(\tau)}. \tag{9.3}$$

Let $H_t^0(g)$ be the heat kernel for the Laplacian on functions on $\widetilde{X}$. Using (9.3) and [MP1, Proposition 3.1] it follows that there exist constants $C > 0$ and $c \in \mathbb{R}$ such that

$$\|H_t^{\tau,p}(g)\| \le Ce^{ct}H_t^0(g), \quad g\in G,\ t>0.$$

Hence we get

$$|h_t^{\tau,p}(g)| \le C\dim(\tau)e^{ct}H_t^0(g), \quad g\in G,\ t>0.$$

By [Do1] there exists $C_1 > 0$ which depends only on T such that for each $t \in (0, T]$ one has

$$H_t^0(g) \leq C_1' t^{-d/2} \exp\left(-\frac{d^2(gK, K1)}{4t}\right)$$

for $0 < t \leq T$. The constant C_1' depends only on T. Thus we get

$$\begin{aligned}\sum_{\gamma \in \Gamma_{1,s} - \{1\}} |h_t^{\tau,p}(g^{-1}\gamma g)| &\leq C_2 t^{-d/2} e^{cT} \sum_{\gamma \in \Gamma_{1,s} - \{1\}} e^{-d^2(\gamma gK, gK)/(4t)} \\ &\leq C_3 e^{-\ell(\Gamma_i)^2/(8t)} e^{-\ell(\Gamma_0)^2/(32t)} \sum_{\gamma \in \Gamma_{0,s} - \{1\}} e^{-d^2(\gamma gK, gK)/(16T)},\end{aligned} \tag{9.4}$$

where C_2, C_3 are constants which depend only on T. It remains to show that the last sum converges and can be estimated independently of g . For $r \in (0, \infty)$ and $x \in \widetilde{X}$ we let $B_r(x)$ be the metric ball of radius r around x. There exists a constant $C > 0$ such that

$$\operatorname{vol}(B_r(x)) \leq C e^{2nr} \tag{9.5}$$

for all $r \in (0, \infty)$. It easily follows from (2.12) and (2.13) that there exists an $\epsilon > 0$ such that for all $x \in \widetilde{X}$ and all $\gamma \in \Gamma_{0,\mathrm{s}}$, $\gamma \neq 1$ one has $B_\epsilon(x) \cap \gamma B_\epsilon(x) = \emptyset$. Thus for each $x \in \widetilde{X}$ the union

$$\bigsqcup_{\gamma \in \Gamma_{0,\mathrm{s}} \colon d(x, \gamma x) \leq R} \gamma B_\epsilon(x)$$

is disjoint and contained in $B_{\epsilon+R}(x)$. Using (9.5) it follows that there exists a constant $C_{X_0} > 0$, depending on X_0, such that for all $R \in (0, \infty)$ and all $x \in \widetilde{X}$ one has

$$\#\{\gamma \in \Gamma_{0,\mathrm{s}} \colon d(x, \gamma x) \leq R\} \leq C_{X_0} e^{2nR}.$$

Applying (9.4) the Lemma follows. □

Applying the preceding lemma we obtain the following estimate for the regularized trace which is uniform with respect to coverings.

Proposition 9.2. *There exists a constant $C > 0$ such that for each hyperbolic manifold $X_1 = \Gamma_1 \backslash \widetilde{X}$, which is a finite covering of X_0, and for which Γ_1 satisfies* (3.1)*, one has*

$$|\operatorname{Tr}_{\mathrm{reg};X_1}\left(e^{-\Delta_{X_1,p}(\tau)}\right)| \leq C(\operatorname{vol}(X_1) + \kappa(X_1) + \alpha(X_1)),$$

where $\kappa(X_1)$ is the number of cusps of X_1 and $\alpha(X_1)$ is as in (8.11)*.*

Proof. We put $t = 1$ in (8.9) and estimate the terms on the right hand side. The identity contribution (8.4) can be estimated by $C_1 \operatorname{vol}(X_1)$. By (8.7), the third term can be estimated by $C_2\kappa(X_1)$. Using (8.12), it follows that the forth term is bounded by $C_3\alpha(X_1)$. Finally, (8.5) and Lemma 9.1 imply that the hyperbolic term is bounded by $C_4 \operatorname{vol}(X_1)$. The constants $C_i > 0$, $i = 1, \cdots, 4$, are all independent of X_1. This finishes the proof. □

Now we can deal with the second integral in (9.2). Using Proposition 7.2, Proposition 9.2, assumption (1.7) and Proposition 8.5, it follows that there exists a $C, c > 0$ such that for all finite coverings $\pi\colon X_1 \to X_0$ as above we have

$$\frac{1}{\operatorname{vol}(X_1)}\left|\int_T^\infty K_{X_1}(t,\tau)t^{-1}\,dt\right| \leq Ce^{-cT} \tag{9.6}$$

for all $T \geq 10$.

It remains to treat the first term on the right hand side of (9.2). For this purpose we use the geometric side of the trace formula as it is given in (8.9). Therefore, put

$$k_t^\tau := \frac{1}{2}\sum_{p=1}^{d}(-1)^p p h_t^{\tau,p}. \tag{9.7}$$

It follows from [MP2, section 9] that the Mellin transform $\int_0^\infty k_t^\tau(1)t^{s-1}\,dt$ converges absolutely and uniformly on compact subsets of $\operatorname{Re}(s) > d/2$, and admits a meromorphic extension to $\mathbb{C}$, which is holomorphic at $s = 0$. Let

$$t_{\widetilde{X}}^{(2)}(\tau) := \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty k_t^\tau(1)t^{s-1}\,dt\right)\bigg|_{s=0}. \tag{9.8}$$

Then in analogy to the compact case (1.2), the L^2-torsion $T_{X_1}^{(2)}(\tau) \in \mathbb{R}^+$ is given by

$$\log T_{X_1}^{(2)}(\tau) = \operatorname{vol}(X_1)t_{\widetilde{X}}^{(2)}(\tau).$$

For details we refer to [MP2, section 9]. Furthermore, it follows from [MP2, equation 9.4] that there exist $C, c > 0$ such that

$$\left|\int_T^\infty k_t^\tau(1)t^{-1}\,dt\right| \leq Ce^{-cT}$$

for $T > 0$. Hence we get

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T I_{X_1}(k_t^\tau)t^{s-1}\,dt\right)\bigg|_{s=0} = \operatorname{vol}(X_1)\cdot\left(t_{\widetilde{X}}^{(2)}(\tau) + O\left(e^{-cT}\right)\right). \tag{9.9}$$

Now let Γ_i, $i \in \mathbb{N}$, be a sequence of torsion-free subgroups of finite index of Γ_0, which satisfy the assumptions of Theorem 1.1. Firstly, by (9.9) we have

$$\lim_{i\to\infty}\frac{1}{[\Gamma_0 : \Gamma_i]}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T I_{X_i}(k_t^\tau)t^{s-1}\,dt\right)\bigg|_{s=0} = \operatorname{vol}(X_0)\cdot\left(t_{\widetilde{X}}^{(2)}(\tau) + O\left(e^{-cT}\right)\right). \tag{9.10}$$

Let $(\Gamma_i)_s$ be the set of semi-simple elements in Γ_i. By (8.5) the hyperbolic contribution is given by

$$H_{X_i}(k_t^\tau) = \int_{\Gamma_i\backslash G}\sum_{\gamma\in(\Gamma_i)_s-\{1\}} k_t^\tau(g^{-1}\gamma g)\,d\dot{g}.$$

It follows from Lemma 9.1 that

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T H_{X_i}(k_t^\tau)t^{s-1}\,dt\right)\bigg|_{s=0} = \int_0^T H_{X_i}(k_t^\tau)t^{-1}\,dt$$

and that there exists a constant C_2, depending on T, such that

$$\left|\int_0^T H_{X_i}(k_t^\tau)t^{-1}\,dt\right| \le C_2\operatorname{vol}(X_i)e^{-\frac{\ell(\Gamma_i)^2}{8T}}.$$

Hence if $\ell(\Gamma_i)\to\infty$ as $i\to\infty$, one has

$$\lim_{i\to\infty}\frac{1}{[\Gamma_0\colon\Gamma_i]}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T H_{X_i}(k_t^\tau)t^{s-1}\,dt\right)\Big|_{s=0}=0. \tag{9.11}$$

Next we study the term associated to $T'_{X_i}(k_t^\tau)$, defined in (8.7). We let $J_{X_i}(k_t^\tau)$ and $\mathcal{I}_{X_i}(k_t^\tau)$ be defined according to [MP2, (6.13), (6.15)], where the subindex X_i indicates that these distributions depend on the manifold X_i. Then by definition we have

$$T'_{X_i}(k_t^\tau)=\kappa(X_i)\mathcal{I}_{X_i}(k_t^\tau)+J_{X_i}(k_t^\tau).$$

Using the results of [MP2, section 6], it follows that there is an asymptotic expansion

$$T'_{X_i}(k_t^\tau)\sim\sum_{k=0}^\infty a_k t^{k-(d-2)/2}+\sum_{k=0}^\infty b_k t^{k-1/2}\log t+c_0$$

as $t\to 0$. Thus for $\operatorname{Re}(s)>(d-2)/2$, the integral

$$\int_0^T T'_{X_i}(k_t^\tau)t^{s-1}\,dt$$

converges and has a meromorphic extension to $\mathbb{C}$, which at $s=0$ has at most a simple pole. Applying the definition of T'_{X_i} it follows that there exists a function $\phi(T,\tau)$ such that

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T T'_{X_i}(k_t^\tau)t^{s-1}\,dt\right)\Big|_{s=0}=\phi(T,\tau)\cdot\kappa(X_i).$$

Thus if $\lim_{i\to\infty}\kappa(X_i)/[\Gamma_0:\Gamma_i]=0$, we obtain

$$\lim_{i\to\infty}\frac{1}{[\Gamma_0\colon\Gamma_i]}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T T'_{X_i}(k_t^\tau)t^{s-1}\,dt\right)\Big|_{s=0}=0. \tag{9.12}$$

Finally, by (8.12) the integral

$$\int_0^T t^{s-1}\mathcal{S}_{X_i}(k_t^\tau)dt$$

converges absolutely for $s\in\mathbb{C}$ with $\operatorname{Re}(s)>\frac{1}{2}$ and has a meromorphic extension to $\mathbb{C}$ with an at most a simple pole at $s=0$. Moreover, it follows from (8.12) that there exists a function $\psi(T,\tau)$ such that

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T \mathcal{S}_{X_i}(k_t^\tau)t^{s-1}\,dt\right)\Big|_{s=0}=\psi(T,\tau)\cdot\alpha(\Gamma_i),$$

where $\alpha(\Gamma_i)$ is as in (8.11). By assumption (1.7) and Proposition 8.5 it follows that

$$\lim_{i\to\infty}\frac{1}{[\Gamma_0:\Gamma_i]}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T \mathcal{S}_{X_i}(k_t^\tau)t^{s-1}\,dt\right)\Big|_{s=0}=0.$$

Combined with (9.10), (9.11), and (9.12) we get

$$\lim_{i\to\infty}\frac{1}{[\Gamma_0:\Gamma_i]}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T K_{X_1}(t,\tau)t^{s-1}\,dt\right)\Bigg|_{s=0}=\operatorname{vol}(X_0)\cdot(t^{(2)}_{\widetilde{X}}(\tau)+O(e^{-cT})). \tag{9.13}$$

Finally, combining (9.13), (9.2) and (9.6), and using that $T>0$ is arbitrary, Theorem 1.1 follows. □

Now assume that Γ_i is normal in Γ_0 and each $\gamma\in\Gamma_0$ belongs only to finitely many Γ_i. Note that $\ell(\gamma)$ depends only on the Γ_0-conjugacy class. Since by (2.19), for each $R>0$ there are only finitely many conjugacy classes $[\gamma]\in\mathrm{C}(\Gamma_{0,\mathrm{s}})$ with $\ell(\gamma)\le R$, one has $\lim_{i\to\infty}\ell(\Gamma_i)=\infty$. Thus, if one applies Proposition 8.6 and the preceding arguments, Theorem 1.2 follows.

10. Principal congruence subgroups of $\mathrm{SO}^0(d,1)$

In this section we apply Theorem 1.2 to the case of principal congruence subgroups of $\mathrm{SO}^0(d,1)$ and prove Corollary 1.3. Therefore, throughout this section we let $G:=\mathrm{SO}^0(d,1)$, d odd, $d=2n+1$. Let $K=\mathrm{SO}(d)$, regarded as a subgroup of G. Then K is a maximal compact subgroup of G.

We realize the standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ as follows. Denote by $E_{i,j}$ the matrix in $\mathfrak{g}$ whose entry at the i-th row and j-th column is equal to 1 and all of whose other entries are equal to 0 and let $H_1:=E_{1,2}+E_{2,1}$. Let $\mathfrak{a}:=\mathbb{R}H_1$ and let

$$\mathfrak{n}=\left\{X(v):=\begin{pmatrix}0&0&v^t\\0&0&v^t\\v&-v&0\end{pmatrix},\quad v\in\mathbb{R}^{d-1}\right\}. \tag{10.1}$$

Then for the standard ordering of the restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$, $\mathfrak{n}$ is the direct sum of the positive restricted root spaces. We let

$$\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k}$$

be the associated Iwasawa decomposition. Let N be the connected Lie group with Lie algebra $\mathfrak{n}$ and let $A:=\exp(\mathfrak{a})$. Let M be the centralizer of A in K. Then

$$P=MAN$$

is a parabolic subgroup of G.

For $v\in\mathbb{R}^{d-1}$ one has

$$\exp(X(v))=1+X(v)+\frac{X^2(v)}{2}=\begin{pmatrix}1+\|v\|^2/2&-\|v\|^2/2&v^t\\\|v\|^2/2&1-\|v\|^2/2&v^t\\v&-v&I_{d-1}\end{pmatrix}, \tag{10.2}$$

where I_{d-1} denotes the unit-matrix and where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{d-1}$. We have $N=\exp(\mathfrak{n})$.

The group G is an algebraic group defined over $\mathbb{Q}$ and we let $\Gamma_0 := G(\mathbb{Z})$ be its integral points. By [BoHa], Γ_0 is a lattice in G. It follows from (10.2) that

$$\log(\Gamma_0 \cap N) = \left\{ \begin{pmatrix} 0 & 0 & v^t \\ 0 & 0 & v^t \\ v & -v & 0 \end{pmatrix}, \quad v \in \mathbb{Z}^{d-1}, \quad \|v\|^2 \in 2\mathbb{Z} \right\}. \tag{10.3}$$

In particular, P is a Γ_0-cuspidal parabolic subgroup of G.

Now for $q \in \mathbb{N}$ we let $\Gamma(q)$ be the principal congruence subgroup of level q, i.e.

$$\Gamma(q) = \{A \in \Gamma_0 \colon A \equiv I \mod(q)\}.$$

Then $\Gamma(q)$ coincides with the kernel of the canonical map $\Gamma_0 \to G(\mathbb{Z}/q\mathbb{Z})$. In particular, $\Gamma(q)$ is a normal subgroup of Γ_0. If $q \geq 3$, then the group $\Gamma(q)$ is neat in the sense of Borel, see [Bo, 17.4]. In particular, $\Gamma(q)$ is torsion free and satisfies (3.1).

In the following Lemma we verify the cusp-uniformity of the groups $\Gamma(q)$. The Lemma is just a special case of Lemma 4 of the paper [DH] of Deitmar and Hoffmann who treated the more general case of families of strictly bounded depth in algebraic $\mathbb{Q}$-groups of arbitrary real rank. However, for the convenience of the reader we shall now recall the proof of Deitmar and Hoffmann in our situation.

Lemma 10.1. *Let P' be a Γ_0-cuspidal parabolic subgroup defined over $\mathbb{Q}$ with nilpotent radical $N_{P'}$. Let $\mathfrak{n}_{P'}$ be the Lie-algebra of $N_{P'}$. Then there exists a lattice $\Lambda^+_{\mathfrak{n}_{P'}}$ in $\mathfrak{n}_{P'}$ such that*

$$q\Lambda^+_{\mathfrak{n}_{P'}} \subseteq \log(\Gamma(q) \cap N_{P'}) \subseteq \frac{q}{4}\Lambda^+_{\mathfrak{n}_{P'}}$$

for each $q \in \mathbb{N}$. In particular, the sequence $\Gamma(q)$, $q \in \mathbb{N}$, is cusp-uniform.

Proof. Let $\mathrm{Mat}_{(d+1)\times(d+1)}(\mathbb{Z})$ be the integral $(d+1)\times(d+1)$-matrices. Then by (10.1) $\mathfrak{n} \cap \mathrm{Mat}_{(d+1)\times(d+1)}(\mathbb{Z})$ is a lattice in $\mathfrak{n}$. We choose $g \in G(\mathbb{Q})$ such that $P' = gPg^{-1}$. Then $\mathfrak{n}_{P'} = g\mathfrak{n}g^{-1}$ and thus

$$\Lambda^+_{\mathfrak{n}_{P'}} := 2(\mathfrak{n}_{P'} \cap \mathrm{Mat}_{(d+1)\times(d+1)}(\mathbb{Z}))$$

is a lattice in $\mathfrak{n}_{P'}$. By (10.2), one has $\exp(Y) = 1 + Y + \frac{Y^2}{2}$ for each $Y \in \mathfrak{n}_{P'}$ and thus the first inclusion is clear. Moreover, by (10.1), if $k \geq 3$ one has $Y^k = 0$ for each $Y \in \mathfrak{n}_{P'}$ and thus for each $n_{P'} \in N_{P'}$ one has

$$\log n_{P'} = (n_{P'} - 1) - \frac{1}{2}(n_{P'} - 1)^2$$

and this gives the second inclusion. The second statement follows from Mahler's criterion and Lemma 8.3. ∎

It is obvious that every $\gamma_0 \in \Gamma_0$ belongs only to finitely many $\Gamma(q)$. If we use equation (10.3), we easily see that $[\Gamma_0 \cap N : \Gamma(q) \cap N]$ goes to infinity if q does and so $[\Gamma_0 : \Gamma(q)]$ goes to infinity if $q \to \infty$. Thus applying Lemma 10.1, Corollary 1.3 follows from Theorem 1.2.

11. Principal congruence subgroups and Hecke subgroups of Bianchi groups

We finally turn to the proofs of Corollary 1.4 and Theorem 1.5. We let $F := \mathbb{Q}(\sqrt{-D})$, $D \in \mathbb{N}$ square-free, be an imaginary quadratic number field. Let $\mathcal{O}_D$ be the ring of integers of F, i.e. $\mathcal{O}_D = \mathbb{Z} + \sqrt{-D}\mathbb{Z}$ if $D \equiv 1, 2$ modulo 4, $\mathcal{O}_D = \mathbb{Z} + \frac{1+\sqrt{-D}}{2}\mathbb{Z}$ if $D \equiv 3$ modulo 4. We let $\Gamma(D) := \mathrm{SL}_2(\mathcal{O}_D)$ be the associated Bianchi-group. Then $X_D := \Gamma(D)\backslash\mathbb{H}^3$ is of finite volume. More precisely, one has

$$\mathrm{vol}(X_D) = \frac{|\delta_F|^{\frac{3}{2}}\zeta_F(2)}{4\pi^2},$$

where ζ_F is the Dedekind zeta function of F and δ_F is is the discriminant of F, see [Hu], [Sa, Proposition 2.1]. Let $\mathfrak{a}$ be any nonzero ideal in $\mathcal{O}_D$ and let $N(\mathfrak{a})$ denote its norm. Then the associated principal congruence subgroup $\Gamma(\mathfrak{a})$ is defined as

$$\Gamma(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(D) \colon a - 1 \in \mathfrak{a}; d - 1 \in \mathfrak{a}; b, c \in \mathfrak{a} \right\}.$$

Moreover, the associated Hecke subgroup $\Gamma_0(\mathfrak{a})$ is defined as

$$\Gamma_0(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(D) \colon c \in \mathfrak{a} \right\}.$$

Let P be the parabolic subgroup given by the upper triangular matrices in $\mathrm{SL}_2(\mathbb{C})$. Then the Langlands decomposition $P = MAN$ is given by

$$M = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \theta \in [0, 2\pi) \right\}$$

and

$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbb{R}, \lambda > 0 \right\}; \quad N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{C} \right\}.$$

We recall that by [Ba, Corollary 5.2] the canonical map from $\mathrm{SL}_2(\mathcal{O}_D)$ to $\mathrm{SL}_2(\mathcal{O}_D/\mathfrak{a})$ is surjective. Thus the sequence

$$1 \to \Gamma(\mathfrak{a}) \to \Gamma(D) \to \mathrm{SL}_2(\mathcal{O}_D/\mathfrak{a}) \to 1$$

is exact and taking the prime-decomposing of $\mathfrak{a}$ it follows as in [Sh, Chapter 1.6] for the $\mathrm{SL}_2(\mathbb{R})$-case that

$$[\Gamma(D) : \Gamma(\mathfrak{a})] = N(\mathfrak{a})^3 \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 - \frac{1}{N(\mathfrak{p})^2}\right). \tag{11.1}$$

It also follows that the sequence

$$1 \to \Gamma(\mathfrak{a}) \to \Gamma_0(\mathfrak{a}) \to P(\mathcal{O}_D/\mathfrak{a}) \to 1$$

is exact. Moreover the order of $P(\mathcal{O}_D/\mathfrak{a})$ is $N(\mathfrak{a})\phi(\mathfrak{a})$, where

$$\phi(\mathfrak{a}) := \#\{(\mathcal{O}_D/\mathfrak{a})^*\} = N(\mathfrak{a})\prod_{\mathfrak{p}|\mathfrak{a}}\left(1 - N(\mathfrak{p})^{-1}\right). \tag{11.2}$$

Thus one obtains

$$[\Gamma(D) : \Gamma_0(\mathfrak{a})] = N(\mathfrak{a})\prod_{\mathfrak{p}|\mathfrak{a}}(1 + N(\mathfrak{p})^{-1}). \tag{11.3}$$

Here the products in (11.1), (11.2) and (11.3) are taken over all prime ideals $\mathfrak{p}$ in $\mathcal{O}_D$ dividing $\mathfrak{a}$.

Let $\mathbb{P}^1(F)$ be the one-dimensional projective space over F. As usual, we write ∞ for the element $[1, 0] \in \mathbb{P}^1(F)$. Then $\mathrm{SL}_2(F)$ acts naturally on $\mathbb{P}^1(F)$ and by [EGM, Chapter 7.2, Proposition 2.2] one has

$$\kappa(\Gamma(D)) = \#\left(\Gamma(D)\backslash\mathbb{P}^1(F)\right).$$

Using [EGM, Chapter 7.2, Theorem 2.4], it follows that $\kappa(\Gamma(D)) = d_F$, where d_F is the class number of F. The group P is the stabilizer of ∞ in $\mathrm{SL}_2(\mathbb{C})$. For each $\eta \in \mathbb{P}^1(F)$ we fix a $B_\eta \in \mathrm{SL}_2(F)$ with $B_\eta\eta = \infty$. We let $B_\infty = \mathrm{Id}$. Then $P_\eta := B_\eta^{-1}PB_\eta$ is the stabilizer of η in $\mathrm{SL}_2(\mathbb{C})$ and the $\Gamma(D)$-cuspidal parabolic subgroups of G are given as P_η. We let $N_\eta := B_\eta^{-1}NB_\eta$. If $\eta \in \mathbb{P}^1(F)$, we let $\Gamma(D)_\eta$, $\Gamma(\mathfrak{a})_\eta$, $\Gamma_0(\mathfrak{a})_\eta$ be the stabilizers of η in $\Gamma(D)$ resp. $\Gamma(\mathfrak{a})$ resp. $\Gamma_0(\mathfrak{a})$.

The following Proposition is an immediate consequence of the finiteness of the class number.

Proposition 11.1. *The set of all principal congruence subgroups $\Gamma(\mathfrak{a})$ and all Hecke subgroups $\Gamma_0(\mathfrak{a})$, $\mathfrak{a}$ a non-zero ideal in $\mathcal{O}_D$, is cusp-uniform.*

Proof. Let $\mathcal{J}_F$ be the ideal group of F, i.e. the group of all finitely generated non-zero $\mathcal{O}_D$-modules in F. We regard F^* as a subgroup of $\mathcal{J}_F$ by identifying F^* with the group of fractional principal ideals. Let $\mathcal{I}_F := \mathcal{J}_F/F^*$ be the ideal class group. Then $\#\mathcal{I}_F = d_F < \infty$, see [Ne, chapter I.6]. Now for $\eta \in \mathbb{P}^1(F)$, B_η as above, write $B_\eta = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}_2(F)$ and let $\mathfrak{u}$ be the $\mathcal{O}_D$-module generated by γ and δ and let $\mathfrak{b} := \mathfrak{u}^{-2} \cap \gamma^{-2}\mathfrak{a}$. It is easy to see that $\mathfrak{b} \neq 0$. Then proceeding as in [EGM, Chapter 8.2, Lemma 2.2], one obtains

$$B_\eta\Gamma(\mathfrak{a})_\eta B_\eta^{-1} \cap N = \left\{\begin{pmatrix}1 & \omega'\\ 0 & 1\end{pmatrix};\ \omega' \in \mathfrak{a}\mathfrak{u}^{-2}\right\};\ B_\eta\Gamma_0(\mathfrak{a})_\eta B_\eta^{-1} \cap N = \left\{\begin{pmatrix}1 & \omega''\\ 0 & 1\end{pmatrix};\ \omega'' \in \mathfrak{b}\right\}.$$

Let P' be a $\Gamma(D)$-cuspidal parabolic subgroup of G and let $\Lambda_{P|P'}(\Gamma(\mathfrak{a}))$ and $\Lambda_{P|P'}(\Gamma_0(\mathfrak{a}))$ denote the set of lattices defined as in (8.13). Since $\mathfrak{a}\mathfrak{u}^{-2}$ and $\mathfrak{b}$ belong to $\mathcal{J}_F$, and $\mathcal{I}_F$ is finite, it follows that $\Lambda_{P|P'}(\Gamma(\mathfrak{a}))$, and $\Lambda_{P|P'}(\Gamma_0(\mathfrak{a}))$ are finite sets. Applying the third criterion of Lemma 8.3, the proposition follows. ☐

The groups $\Gamma(\mathfrak{a})$ are torsion-free and satisfy (3.1) for $N(\mathfrak{a})$ sufficiently large. This was shown for example in the proof of Lemma 4.1 in [Pf2]. Since $[\Gamma(D) : \Gamma(\mathfrak{a})]$ tends to ∞ if $N(\mathfrak{a})$ tends to ∞ and since each $\gamma_0 \in \Gamma(D)$, $\gamma_0 \neq 1$, is contained in only finitely many

$\Gamma(\mathfrak{a})$, Corollary 1.4 follows from Proposition 11.1 and Theorem 1.2.

We finally turn to Theorem 1.5. The Hecke groups $\Gamma_0(\mathfrak{a})$ are never torsion-free and never satisfy (1.5). However, we may take a finite index subgroup Γ' of Γ_D, for example a fixed principal congruence subgroup of sufficiently high level, which is torsion free and satisfies assumption (1.5). Then for each non zero ideal $\mathfrak{a}$ of $\mathcal{O}_D$ we let

$$\Gamma_0'(\mathfrak{a}) := \Gamma_0(\mathfrak{a}) \cap \Gamma'.$$

This group satisfies now the required assumptions and if $n_0 := [\Gamma(D) : \Gamma']$, then

$$[\Gamma_0(\mathfrak{a}) : \Gamma_0'(\mathfrak{a})] \leq n_0 \tag{11.4}$$

for each non-zero ideal $\mathfrak{a}$. Thus since the set of all $\Gamma_0(\mathfrak{a})$ is cusp uniform by the preceding lemma, also the set of all $\Gamma_0'(\mathfrak{a})$, $\mathfrak{a}$ a non-zero ideal in $\mathcal{O}_D$, is cusp uniform. Now, as in [AC, page 15], for an ideal $\mathfrak{b}$ of $\mathcal{O}_D$ we let

$$\phi_u(\mathfrak{b}) := \#((\mathcal{O}_D/\mathfrak{b})^*/\mathcal{O}_D^*).$$

Then by [AC, Theorem 7] one has

$$\kappa(\Gamma_0(\mathfrak{a})) = d_F \sum_{\mathfrak{b}|\mathfrak{a}} \phi_u(\mathfrak{b} + \mathfrak{b}^{-1}\mathfrak{a}). \tag{11.5}$$

Now as in [FGT, Lemma 5.7], on the set of ideals in $\mathcal{O}_D$, we introduce the multiplicative function κ given by

$$\kappa(\mathfrak{p}^k) := \begin{cases} N(\mathfrak{p})^{\frac{k}{2}} + N(\mathfrak{p})^{\frac{k}{2}-1} & k \equiv 0(2), \\ 2N(\mathfrak{p})^{\frac{k-1}{2}} & k \equiv 1(2), \end{cases}$$

where $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_D$. Using (11.5), it easily follows that

$$\kappa(\Gamma_0(\mathfrak{a})) \leq d_F \kappa(\mathfrak{a}),$$

where one has equality if one replaces ϕ_u by ϕ in (11.5). Now observe that

$$\kappa(\mathfrak{a}) \leq 2N(\mathfrak{a})^{1/2} \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 + N(\mathfrak{p})^{-1}\right).$$

Using (11.3), we obtain

$$\frac{\kappa(\Gamma_0(\mathfrak{a}))}{[\Gamma(D) : \Gamma_0(\mathfrak{a})]} \leq \frac{2d_F}{\sqrt{N(\mathfrak{a})}}.$$

Now by (11.3) we have the trivial bound $[\Gamma(D) : \Gamma_0(\mathfrak{a})] \leq N(\mathfrak{a})^2$. It follows that

$$\lim_{N(\mathfrak{a})\to\infty} \frac{\kappa(\Gamma_0(\mathfrak{a})) \log[\Gamma(D) : \Gamma_0(\mathfrak{a})]}{[\Gamma(D) : \Gamma_0(\mathfrak{a})]} = 0.$$

Thus every sequence $\Gamma_0(\mathfrak{a})$ satisfies assumption (1.8) for $N(\mathfrak{a}) \to \infty$. As above, if $P_{0,1}, \dots, P_{0,d_F}$ are fixed representatives of $\Gamma(D)$-cuspidal parabolic subgroups of $\mathrm{SL}_2(\mathbb{C})$, then

$$\kappa(\Gamma_0'(\mathfrak{a})) = \sum_{j=1}^{d_F} \#\{\Gamma_0(\mathfrak{a})' \backslash \Gamma(D)/\Gamma(D) \cap P_{0,j}\}$$

and there is a similar formula for $\kappa(\Gamma_0(\mathfrak{a}))$. Thus one has $\kappa(\Gamma_0'(\mathfrak{a})) \leq n_0 \kappa(\Gamma_0(\mathfrak{a}))$ and putting everything together, it follows that the sequence $\Gamma_0'(\mathfrak{a})$ satisfies condition (1.8).

It remains to prove that the contribution of the semisimple conjugacy classes to the analytic torsion goes to zero for towers of Hecke subgroups. In order to prove this, we consider the formula (8.6). According to section 8, for $\gamma \in \Gamma(D)$ we let $c_{\Gamma_0(\mathfrak{a})}(\gamma)$ be the number of fixed points of γ on $\Gamma(D)/\Gamma_0(\mathfrak{a})$. To begin with, as in [FGT] we let

$$\tilde{\Gamma}(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a - d \in \mathfrak{a} \colon b, c \in \mathfrak{a} \right\}.$$

Now we define a multiplicative function $c(\cdot, \cdot)$ on the ideals of $\mathcal{O}_D$ by putting

$$c(\mathfrak{p}^k, \mathfrak{p}^r) := \begin{cases} N(\mathfrak{p})^{(k+r)/2}, & k - r \text{ odd}, \ k - r > 0 \\ 2N(\mathfrak{p})^{(k+r-1)/2}, & k - r \text{ even}, \ k - r > 0 \\ N(\mathfrak{p})^k + N(\mathfrak{p})^{k-1}, & k \leq r, \end{cases}$$

if $\mathfrak{p}$ is a prime ideal and $k, r \in \mathbb{N}^0$. Then the following proposition and its proof were kindly provided by Tobias Finis.

Proposition 11.2. *Let $\gamma \in \Gamma(D)$ and let $\mathfrak{b}$ be the largest divisor of $\mathfrak{a}$ such that $\gamma \in \tilde{\Gamma}(\mathfrak{b})$. Then one has*

$$c_{\Gamma_0(\mathfrak{a})}(\gamma) \leq c(\mathfrak{a}, \mathfrak{b}).$$

In particular, if $\nu(\mathfrak{a})$ denotes the number of prime divisors of $\mathfrak{a}$, one can estimate

$$c_{\Gamma_0(\mathfrak{a})}(\gamma) \leq 2^{\nu(\mathfrak{a})} \sqrt{N(\mathfrak{a}) N(\mathfrak{b})}.$$

Proof. We can identify the quotient $\Gamma(D)/\Gamma_0(\mathfrak{a})$ with the projective line $\mathbb{P}^1(\mathcal{O}_D/\mathfrak{a})$ and for a given $\gamma \in \Gamma(D)$ we have to estimate the number of its fixed points $N(\gamma, \mathfrak{a})$ on $\mathbb{P}^1(\mathcal{O}_D/\mathfrak{a})$. By the strong approximation theorem we have

$$N(\gamma, \mathfrak{a}) = \prod_{\mathfrak{p}} N(\gamma, \mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}), \quad \mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}.$$

So it suffices to study $N(\gamma, \mathfrak{p}^k)$ for a prime ideal $\mathfrak{p}$ of $\mathcal{O}_D$. First assume that γ is scalar modulo $\mathfrak{p}^k$. Then every point of $\mathbb{P}^1(\mathcal{O}_D/\mathfrak{p}^k)$ is a fixed point of γ. The number of elements of the projective line $\mathbb{P}^1(\mathcal{O}_D/\mathfrak{p}^k)$ equals $N(\mathfrak{p})^k + N(\mathfrak{p})^{k-1}$. Thus in this case the lemma is proved. Next assume that γ is not scalar modulo $\mathfrak{p}^k$. Let $r < k$ be the maximal integer such that γ is scalar modulo $\mathfrak{p}^r$. We work over the completion $\mathcal{O}_\mathfrak{p}$ of $\mathcal{O}$ at $\mathfrak{p}$. Let π be the corresponding prime element. Then we have $\mathcal{O}_\mathfrak{p}/\pi^l \cong \mathcal{O}/\mathfrak{p}^l$ for every l. Over $\mathcal{O}_\mathfrak{p}$ we have the decomposition

$$\gamma = a + \pi^r \eta,$$

where a is a scalar matrix and η is not scalar modulo π. A vector $v \in \mathcal{O}_{\mathfrak{p}}^2$ which is not divisible by π is an eigenvector of γ modulo π^k if and only if it is an eigenvector of η modulo π^{k-r}. If we consider the canonical map $\mathbb{P}^1(\mathcal{O}/\mathfrak{p}^k) \to \mathbb{P}^1(\mathcal{O}/\mathfrak{p}^{k-r})$, then the preimage of each element in $\mathbb{P}^1(\mathcal{O}/\mathfrak{p}^{k-r})$ has $N(\mathfrak{p})^r$ elements. Thus if n denotes the number of eigenvalues of η in $\mathbb{P}^1(\mathcal{O}/\mathfrak{p}^{k-r})$, we have $N(\gamma, \mathfrak{p}^k) = N(\mathfrak{p})^r n$. It remains to estimate n.

To this end, we may assume that η has an eigenvalue. Otherwise there is nothing to prove. Then adding a scalar matrix and performing a base change over $\mathcal{O}_{\mathfrak{p}}$, which does not change the number n, we may assume that η has the eigenvalue 0 with eigenvector $(1, 0)^t$. Since we assumed that η is not scalar modulo π, after a base change we may assume that η is of the form

$$\eta = \begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix},$$

where $d \in \mathcal{O}_{\mathfrak{p}}$. Now a set of representatives of eigenvectors in $\mathbb{P}^1(\mathcal{O}/\mathfrak{p}^{k-r})$ of this matrix is given by all classes of vectors represented by $(1, y)$, where y is chosen modulo $\mathfrak{p}^{k-r}$ and satisfies $y^2 - dy \equiv 0$ modulo $\mathfrak{p}^{k-r}$. Thus n is the number of solutions of the quadratic congruence for $y \in \mathcal{O}/\mathfrak{p}^{k-r}$. Let $\nu_{\mathfrak{p}}$ be the valuation corresponding to $\mathfrak{p}$. Then this congruence is equivalent to $\nu_{\mathfrak{p}}(y) + \nu_{\mathfrak{p}}(y - d) \geq k - r$. This implies that at least one summand is $\geq (k - r)/2$. We distinguish two cases. First, we assume that $\nu_{\mathfrak{p}}(d) < (k - r)/2$. Then exactly one summand is $\geq (k - r)/2$ and the other has the valuation $\nu_{\mathfrak{p}}(d)$. Thus in this case n is 2 times the number of all representatives whose valuation is $\geq k - r - \nu_{\mathfrak{p}}(d)$, i.e. $n = 2N(\mathfrak{p})^{\nu_p(d)}$. Secondly, we assume that $\nu_{\mathfrak{p}}(d) \geq (k - r)/2$. Then the congruence is equivalent to $\nu_{\mathfrak{p}}(y) \geq (k - r)/2$. Thus in this case one has $n = N(\mathfrak{p})^{\lfloor \frac{k-r}{2} \rfloor}$. In all cases we obtain $n \leq N(\mathfrak{p})^{(k-r)/2}$ if $k - r$ is even and $n \leq 2N(\mathfrak{p})^{(k-r-1)/2}$ if $k - r$ is odd. Putting everything together, the first estimate follows. This estimate immediately implies the second one. □

Remark 11.3. Proposition 11.2 also follows from more general estimates which are the content of a paper of Tobias Finis and Erez Lapid that is in preparation. Related results are also obtained in [A++].

The following Lemma is due to Finis, Grunewald and Tirao.

Lemma 11.4. *For every $\delta > 0$ there is a constant $C > 0$ such that for all non zero ideals $\mathfrak{b}$ of $\mathcal{O}_D$ and all $R > 0$ the number of elements in $[\gamma] \in \mathrm{C}(\Gamma(D))_s$ which satisfy $\ell(\gamma) \leq R$ and which belong to $\tilde{\Gamma}(\mathfrak{b})$ is bounded by $N(\mathfrak{b})^{-2} e^{(2+\delta)R}$.*

Proof. This follows directly from [FGT, Lemma 5.10]. □

Now we take a sequence $\mathfrak{a}_i$ of ideals such that $N(\mathfrak{a}_i)$ tends to infinity with i and we let $\Gamma_i := \Gamma_0'(\mathfrak{a}_i)$, $X_i := \Gamma_i \backslash \mathbb{H}^3$. We need to estimate the hyperbolic contribution $H_{X_i}(h_t^\tau)$. We use formula (8.6), and apply the Fourier inversion formulas of Harish-Chandra to the invariant orbital integrals using that the Fourier transform of h_t^τ can be computed explicitly. This was carried out in [MP2]. If we combine [MP2, (10.4)] for the special case of dimension

3 with equation (8.6), we obtain:

$$H_{X_i}(h_t^\tau)=\sum_{k=0}^{1}(-1)^{k+1}e^{-t\lambda_{\tau,k}^2}\sum_{[\gamma]\in\mathrm{C}(\Gamma(D))_s-[1]}c_{\Gamma_i}(\gamma)\frac{\ell(\gamma)}{n_\Gamma(\gamma)}L_{\mathrm{sym}}(\gamma;\sigma_{\tau,k})\frac{e^{-\ell(\gamma)^2/4t}}{(4\pi t)^{\frac{1}{2}}}. \tag{11.6}$$

Here the $\lambda_{\tau,k}\in(0,\infty)$ are as in [MP2, (8.4)] and the $\sigma_{\tau,k}\in\hat{M}$ are determined by their highest weight $\Lambda_{\sigma_{\tau,k}}$ given as in [MP2, (8.5)]. Moreover, $n_\Gamma(\gamma)$ is the period of the closed geodesic corresponding to γ and $L_{\mathrm{sym}}(\gamma;\sigma_{\tau,k})$ is as in [MP2, (6.2), (10.3)]. By [MP2, (10.11)] and the definition of $L_{\mathrm{sym}}(\gamma;\sigma_{\tau,k})$, there exists a constant C_0 such that for all $\gamma\in\Gamma(D)_s-\{1\}$ one has

$$\frac{\ell(\gamma)}{n_\Gamma(\gamma)}|L_{\mathrm{sym}}(\gamma;\sigma_{\tau,k})|\leq C_0.$$

Thus together with equation (11.4), Proposition 11.2 and Lemma 11.4, it follows that there exist constants C_1,C_2 such that for each i we can estimate

$$\begin{aligned}H_{X_i}(h_t^\tau)&\leq C_1 2^{\nu(\mathfrak{a})}\sum_{\mathfrak{b}|\mathfrak{a}}\sqrt{N(\mathfrak{b})N(\mathfrak{a})}\sum_{\substack{[\gamma]\in\mathrm{C}(\Gamma(D))_s-[1]\\ \gamma\in\tilde{\Gamma}(\mathfrak{b})}}\frac{e^{-\frac{\ell(\gamma)^2}{4t}}}{(4\pi t)^{\frac{1}{2}}}\\
&\leq C_1 2^{\nu(\mathfrak{a})}\sum_{\mathfrak{b}|\mathfrak{a}}\sqrt{N(\mathfrak{ab})}\sum_{k=1}^{\infty}\Bigg(\frac{e^{-\frac{(k\ell(\Gamma(D)))^2}{4t}}}{(4\pi t)^{\frac{1}{2}}}\\
&\times\#\{[\gamma]\in\mathrm{C}(\Gamma(D))_s\colon\gamma\in\tilde{\Gamma}(\mathfrak{b})\colon k\ell(\Gamma(D))\leq\ell(\gamma)\leq(k+1)\ell(\Gamma(D))\}\Bigg)\\
&\leq C_2 2^{\nu(\mathfrak{a})}\sqrt{N(\mathfrak{a})}\sum_{\mathfrak{b}|\mathfrak{a}}N(\mathfrak{b})^{-\frac{3}{2}}\sum_{k=1}^{\infty}\frac{ke^{-\frac{(k\ell(\Gamma(D)))^2}{4t}}}{(4\pi t)^{\frac{1}{2}}}e^{(2+\delta)k\ell(\Gamma(D))}.\end{aligned}$$

Let $\mathfrak{a}=\mathfrak{p}_1^{k_1}\cdot\dots\cdot\mathfrak{p}_{\nu(\mathfrak{a})}^{k_{\nu(\mathfrak{a})}}$ be the prime ideal decomposition of $\mathfrak{a}$. Then we have

$$2^{\nu(\mathfrak{a})}\sum_{\mathfrak{b}|\mathfrak{a}}N(\mathfrak{b})^{-\frac{3}{2}}\leq 2^{\nu(\mathfrak{a})}\prod_{j=1}^{\nu(\mathfrak{a})}\frac{1}{1-N(\mathfrak{p}_j)^{-\frac{3}{2}}}\leq 4^{\nu(\mathfrak{a})}.$$

Now note that there are only finitely many prime ideals with a given norm. This implies that for every $\epsilon>0$ there exists $C(\epsilon)>0$ such that for all $\mathfrak{a}$ we have $2^{\nu(\mathfrak{a})}\leq C(\epsilon)N(\mathfrak{a})^\epsilon$. Hence the right hand side is $O(N(\mathfrak{a})^\epsilon)$ as $N(\mathfrak{a})\to\infty$ for any $\epsilon>0$, where the implied constant depends on ϵ. Thus there exist constants $c,C_3,C_4>0$ such that we have

$$H_{X_i}(h_t^\tau)\leq C_3 2^{\nu(\mathfrak{a})}\sqrt{N(\mathfrak{a})}\sum_{\mathfrak{b}|\mathfrak{a}}N(\mathfrak{b})^{-\frac{3}{2}}e^{-\frac{c}{t}}\leq C_4N(\mathfrak{a})^{\frac{3}{4}}e^{-\frac{c}{t}}. \tag{11.7}$$

Applying (11.3), it follows that for ever $T\in(0,\infty)$ one has

$$\lim_{i\to\infty}\frac{1}{[\Gamma(D):\Gamma_i]}\int_0^T t^{-1}H_{X_i}(h_t^\tau)dt=0. \tag{11.8}$$

Thus the analog of equation (9.11) is also verified for the present sequence Γ_i of subgroups derived from Hecke subgroups. Since it was shown above that this sequence is cusp uniform and satisfies condition 1.8, the proof of Theorem 1.1 given in section 9 can be carried over to the present case. Thus also Theorem 1.5 is proved.

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