APPROXIMATION OF L^2-ANALYTIC TORSION FOR ARITHMETIC QUOTIENTS OF THE SYMMETRIC SPACE $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$

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Abstract. In this paper we define a regularized analytic torsion for arithmetic quotients of the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. We consider sequences of congruence subgroups of a fixed arithmetic subgroup.

1. Introduction

Let X be a compact oriented Riemannian manifold of dimension d. Let ρ be a finite dimensional representation of $\pi_1(X)$ and let $E_\rho \to X$ be the associated flat vector bundle. Pick a Hermitian fiber metric in E_ρ. Let $\Delta_p(\rho)$ be the Laplace operator on E_ρ-valued p-forms. Let $\zeta_p(s,\rho)$ be its zeta function [Sh]. Let $e^{-t\Delta_p(\rho)}$, $t>0$, be the heat operator and let $b_p(\rho)=\dim\ker\Delta_p(\rho)$. Then for $\mathrm{Re}(s)>d/2$ one has

$$\zeta_p(s,\rho)=\frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}\left(e^{-t\Delta_p(\rho)}\right)-b_p(\rho))t^{s-1}dt. \tag{1.1}$$

Then the analytic torsion $T_X(\rho)\in\mathbb{R}^+$, introduced by Ray and Singer [RS], is defined by

$$\log T_X(\rho):=\frac{1}{2}\sum_{p=1}^{d}(-1)^p p\frac{d}{ds}\zeta_p(s;\rho)\big|_{s=0}. \tag{1.2}$$

The corresponding L^2-invariant, the L^2-analytic torsion $T_X^{(2)}(\rho)$, was introduced by Lott [Lo] and Mathai [MV]. It is defined in terms of the von Neumann trace of the heat operators on the universal covering $\widetilde{X}$ of X.

The analytic torsion has been used by Bergeron and Venkatesh [BV] to study the growth of torsion in the cohomology of co-compact arithmetic groups. The approach of [BV] is based on the approximation of the L^2-torsion by the renormalized analytic torsion for sequences of coverings of a given compact locally symmetric space. Since many important arithmetic groups are not co-compact, it is desirable to extend these results to the non-compact case. The first problem is that the analytic torsion is not defined for non-compact manifolds. To cope with this problem we defined in [MzM] a regularized version of the analytic torsion for quotients of the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ by arithmetic groups.

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The goal of the present paper is to extend the result of Bergeron and Venkatesh [BV] on the approximation of the L^2-analytic torsion to this setting.

To begin with we recall the results of Bergeron and Venkatesh. Let G be a semisimple Lie group of non-compact type. Let K be a maximal compact subgroup of G and let $\widetilde{X} = G/K$ be the associated Riemannian symmetric space endowed with a G-invariant metric. Let $\Gamma \subset G$ be a co-compact discrete subgroup. For simplicity we assume that Γ is torsion free. Let $X := \Gamma\backslash\widetilde{X}$. Then X is a compact locally symmetric manifold of non-positive curvature. Let τ be an irreducible finite dimensional complex representation of G. Denote by $T_X(\tau)$ (resp. $T_X^{(2)}(\tau)$) the analytic torsion (resp. the L^2-torsion) taken with respect to the representation $\tau|_\Gamma$ of Γ. Since the heat kernels on $\widetilde{X}$ are G-invariant, one has

$$\log T_X^{(2)}(\tau) = \operatorname{vol}(X) t_{\widetilde{X}}^{(2)}(\tau), \tag{1.3}$$

where $t_{\widetilde{X}}^{(2)}(\tau)$ is a constant that depends only on $\widetilde{X}$ and τ. It is an interesting problem to see if the L^2-torsion can be approximated by the torsion of finite coverings $X_i \to X$. This problem has been studied by Bergeron and Venkatesh [BV] under a certain non-degeneracy condition on τ. Representations which satisfy this condition are called *strongly acyclic*. One of the main results of [BV] is as follows. Let $X_i \to X$, $i \in \mathbb{N}$, be a sequence of finite coverings of X. Let τ be strongly acyclic. Let $\operatorname{inj}(X_i)$ denote the injectivety radius of X_i and assume that $\operatorname{inj}(X_i) \to \infty$ as $i \to \infty$. Then by [BV, Theorem 4.5] one has

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{\operatorname{vol}(X_i)} = t_{\widetilde{X}}^{(2)}(\tau). \tag{1.4}$$

If $\operatorname{rank}_{\mathbb{C}}(G) - \operatorname{rank}_{\mathbb{C}}(K) = 1$, one can show that $t_{\widetilde{X}}^{(2)}(\tau) \neq 0$. Combined with the equality of analytic torsion and Reidemeister torsion [Mu2], Bergeron and Venkatesh [BV] used this result to study the growth of torsion in the cohomology of co-compact arithmetic groups. This makes it desirable to extend these results in various directions. Especially, one would like to extend (1.4) to the finite volume case. However, due to the presence of the continuous spectrum of the Laplace operators in the non-compact case, one encounters serious technical difficulties in attempting to generalize (1.4) to the finite volume case. In [Ra1] J. Raimbault has dealt with finite volume hyperbolic 3-manifolds. In [Ra2] he applied this to study the growth of torsion in the cohomology for certain sequences of congruence subgroups of Bianchi groups.

The main purpose of the present paper is to extend (1.4) to arithmetic quotients of

$$\widetilde{X} := \operatorname{SL}(n,\mathbb{R})/\operatorname{SO}(n).$$

In order to define the regularized analytic torsion in the non-compact case, we pass to the adelic framework. Let $G = \operatorname{SL}(n)$. Let $\mathbb{A}$ be the ring of adeles and $\mathbb{A}_f$ the ring of finite adeles. Let $K_\infty = \operatorname{SO}(n)$ be the usual maximal compact subgroup of $G(\mathbb{R}) = \operatorname{SL}(n,\mathbb{R})$. Given an open compact subgroup, $K_f \subset G(\mathbb{A}_f)$, let

$$X(K_f) := G(\mathbb{Q})\backslash(\widetilde{X} \times G(\mathbb{A}_f)/K_f) \tag{1.5}$$

be the associated adelic quotient. This is the adelic version of a locally symmetric space. Since $\mathrm{SL}(n)$ is simply connected, strong approximation holds for $\mathrm{SL}(n)$ and therefore, we have

$$X(K_f) = \Gamma\backslash\widetilde{X}, \tag{1.6}$$

where Γ is the projection of $(G(\mathbb{R}) \times K_f) \cap G(\mathbb{Q})$ onto $G(\mathbb{R})$. We will assume that K_f is neat so that $X(K_f)$ is a manifold. Let $\tau\colon G(\mathbb{R}) \to \mathrm{GL}(V_\tau)$ be a finite dimensional complex representation. The restriction of τ to $\Gamma \subset G(\mathbb{R})$ induces a flat vector bundle E_τ over $X(K_f)$. By [MM], E_τ, is isomorphic to the locally homogeneous vector bundle over $X(K_f)$, which is associated to $\tau|_{K_\infty}$. Moreover it can be equipped with a distinguished fiber metric, induced from an admissible inner product in V_τ. In this way we get a fiber metric in E_τ. Let $\Delta_p(\tau)$ be the twisted Laplace operator on p-forms with values in E_τ. If $X(K_f)$ is not compact, $\Delta_p(\tau)$ has continuous spectrum and therefore, the analytic torsion can not be defined by (1.2). In [MzM] we have introduced a regularized version of the analytic torsion. The starting point for the definition of the regularized analytic torsion in the non-compact case is formula (1.1). In [MzM] we introduced a regularized trace of the heat operator. It is defined as follows. Let $\widetilde{\Delta}_p(\tau)$ be the Laplace operator on $\widetilde{E}_\tau$-valued p-forms on $\widetilde{X}$. The heat operator $e^{-\widetilde{\Delta}_p(\tau)}$ is a convolution operator given by a kernel $H_t^{\tau,p}\colon G(\mathbb{R}) \to \mathrm{GL}(\Lambda^p\mathfrak{p}^\star \otimes V_\tau)$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$. Let $h_t^{\tau,p} \in C^\infty(G(\mathbb{R}))$ be defined by

$$h_t^{\tau,p}(g) = \operatorname{tr} H_t^{\tau,p}(g), \quad g \in G(\mathbb{R}).$$

Let $J_{\mathrm{geo}}(f)$, $f \in C_c^\infty(G(\mathbb{A}))$, be the geometric side of the (non-invariant) Arthur trace formula [Ar1]. By [FL1, Theorem 7.1], $J_{\mathrm{geo}}(f)$ is defined for all $f \in \mathcal{C}(G(\mathbb{A}), K_f)$, the adelic version of the Schwartz space (see section 2 for its definition). Let $\mathbf{1}_{K_f}$ be the characteristic function of K_f in $G(\mathbb{A}_f)$. Put

$$\chi_{K_f} := \frac{\mathbf{1}_{K_f}}{\mathrm{vol}(K_f)}. \tag{1.7}$$

Then $h_t^{\tau,p} \otimes \chi_{K_f}$ belongs to the Schwartz space $\mathcal{C}(G(\mathbb{A}), K_f)$, and in [MzM, (13.16)] we defined the regularized trace of the heat operator by

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) = J_{\mathrm{geo}}(h_t^{\tau,p} \otimes \chi_{K_f}). \tag{1.8}$$

If $X(K_f)$ is compact, this equality is just the content of the trace formula. For the motivation of this definition see [MzM].

In order to be able to use the Mellin transform to define a regularized zeta function similar to (1.1) one needs to know the asymptotic behavior of the regularized trace of the heat operator as $t \to \infty$ and $t \to 0$. Let θ be the Cartan involution of $G(\mathbb{R})$. Let $\tau_\theta := \tau \circ \theta$. Assume that $\tau \ncong \tau_\theta$. Then by [MzM, Theorem 1.2] there exists $c > 0$ such that $\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) = O(e^{-ct})$ as $t \to \infty$ for all $p = 0, \dots, d$. Furthermore, by [MzM, Theorem 1.1], $\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ admits an asymptotic expansion as $t \to 0$. This expansion contains logarithmic terms. Using these facts, the zeta function $\zeta_p(s,\tau)$ can be defined as in (1.1) with the trace of the heat operator replaced by the regularized trace. Due to the presence of log-terms in the asymptotic expansion for $t \to 0$, $\zeta_p(s,\tau)$ may have a pole at $s=0$. So the definition (1.2) of the analytic torsion has to be modified. Let $f(s)$ be a meromorphic function on $\mathbb{C}$. For $s_0 \in \mathbb{C}$ let $f(s) = \sum_{k \geq k_0} a_k (s-s_0)^k$ be the Laurent expansion of f at s_0. Put $\mathrm{FP}_{s=s_0} f(s) := a_0$. Now we define the analytic torsion $T_{X(K_f)}(\tau) \in \mathbb{C} \setminus \{0\}$ by

$$\log T_{X(K_f)}(\tau) = \frac{1}{2} \sum_{p=0}^{d} (-1)^p p \left(\mathrm{FP}_{s=0} \frac{\zeta_p(s;\tau)}{s} \right). \tag{1.9}$$

If the zeta functions are holomorphic at $s=0$, this is the same definition as before.

Now we can formulate our main result. Let $K(N) \subset \mathrm{SL}(n, \mathbb{A}_f)$ be the principal congruence subgroup of level $N \geq 3$. Put $X(N) := X(K(N))$. Note that $X(N) = \Gamma(N) \backslash \widetilde{X}$, where $\Gamma(N) \subset \mathrm{SL}(n,\mathbb{Z})$ is the principal congruence subgroup of level N. Then our main result is the following theorem

Theorem 1.1. *Let $\tau \in \mathrm{Rep}(G(\mathbb{R}))$. Assume that $\tau \not\cong \tau_\theta$. Then we have*

$$\lim_{N \to \infty} \frac{\log T_{X(N)}(\tau)}{\mathrm{vol}(X(N))} = t^{(2)}_{\widetilde{X}}(\tau).$$

We shall now briefly outline our method to prove Theorem 1.1. For technical reasons we work with $\mathrm{GL}(n)$ in place of $\mathrm{SL}(n)$. Let $K_f \subset \mathrm{GL}(n, \mathbb{A}_f)$ be an open compact subgroup. Then we define the corresponding adelic quotient $Y(K_f)$ as above by

$$Y(K_f) := \mathrm{GL}(n,\mathbb{Q}) \backslash (\widetilde{X} \times \mathrm{GL}(n, \mathbb{A}_f))/K_f.$$

We note that $Y(K_f)$ is the disjoint union of finitely many locally symmetric spaces $\Gamma_i \backslash \widetilde{X}$ for arithmetic subgroups $\Gamma_i \subset \mathrm{GL}(n,\mathbb{Q})$, $i = 1, \dots, l$. Now let $K(N) \subset \mathrm{GL}(n, \mathbb{A}_f)$ be the principal congruence subgroup of level N. Put $Y(N) := Y(K(N))$. Then $Y(N)$ is the disjoint union of $\varphi(N)$ copies of $X(N)$, where $\varphi(N)$ is Euler's function (see [Ar6, p. 13]). The disjoint union of $\varphi(N)$ copies of the flat E_τ over $X(N)$ is a flat bundle $\widehat{E}_\tau$ over $Y(N)$. Let $\Delta_{p,N}(\tau)$ be the Laplace operator on $\widehat{E}_\tau$-valued p-forms on $Y(N)$. We define the regularized trace of the heat operator $e^{-t\Delta_{p,N}(\tau)}$ as above by

$$\mathrm{Tr}\left(e^{-t\Delta_{p,N}(\tau)}\right) := J^{\mathrm{GL}(n)}_{\mathrm{geo}}(h^{\tau,p}_t \otimes \chi_{K(N)}),$$

where $J^{\mathrm{GL}(n)}_{\mathrm{geo}}$ is now the geometric side of the trace formula for $\mathrm{GL}(n,\mathbb{A})^1$ and $\chi_{K(N)}$ the normalized characteristic function of $K(N)$ in $\mathrm{GL}(n, \mathbb{A}_f)$. Using the regularized trace, we define the analytic torsion $T_{Y(N)}(\tau)$ in the same way as above. Comparing the trace formulas for $\mathrm{SL}(n)$ and $\mathrm{GL}(n)$, it follows that

$$\log T_{Y(N)}(\tau) = \varphi(N) \log T_{X(N)}(\tau).$$

Furthermore note that $\mathrm{vol}(Y(N)) = \varphi(N)\, \mathrm{vol}(X(N))$. Hence it suffices to show that

$$\lim_{N \to \infty} \frac{\log T_{Y(N)}(\tau)}{\mathrm{vol}(Y(N))} = t^{(2)}_{\widetilde{X}}(\tau). \tag{1.10}$$

To establish (1.10) we proceed as follows. Let

$$K_N(t,\tau) := \frac{1}{2}\sum_{p=1}^{d}(-1)^p p\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,N}(\tau)}\right). \tag{1.11}$$

As observed above, $K_N(t,\tau)$ is exponentially decreasing as $t\to\infty$ and admits an asymptotic expansion as $t\to 0$. Thus the analytic torsion can be defined by

$$\log T_{Y(N)}(\tau) = \mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^\infty \operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,N}(\tau)}\right)t^{s-1}dt\right). \tag{1.12}$$

Let $T>0$. We decompose the integral into the integrals over $[0,T]$ and $[T,\infty)$. The integral over $[T,\infty)$ is an entire function of s. Hence it follows that

$$\log T_{Y(N)}(\tau) = \mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T K_N(t,\tau)t^{s-1}dt\right)+\int_T^\infty K_N(t,\tau)t^{-1}dt. \tag{1.13}$$

To deal with the second integral, we show that there exist $C,c>0$ such that

$$\frac{1}{\operatorname{vol}(Y(N))}\left|\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,N}(\tau)}\right)\right| \le Ce^{-ct} \tag{1.14}$$

for all $t\ge 1$, $p=0,\dots,d$, and $N\in\mathbb{N}$. To prove (1.14) we use the definition (1.8) and the trace formula, which gives

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,N}(\tau)}\right) = J_{\mathrm{spec}}(h_t^{\tau,p}\otimes\chi_{K(N)}).$$

To estimate the right hand side we use the fine spectral expansion of [FLM1] and proceed as in [MzM]. However, the important new feature is that we need to control the dependence on N of all constants appearing in the estimations. The main ingredients of the spectral side of the trace formula are logarithmic derivatives of intertwining operators. Uniform estimations in N of the relevant integrals containing the logarithmic derivatives were obtained in [FLM2]. These are essential for our purpose. Using (1.14) it follows that $\operatorname{vol}(Y(N))^{-1}$ times the second integral in (1.13) is $O(e^{-cT})$, where the implied constants are independent of N.

To deal with the first term, we first show that, up to a term which is $O(e^{-cT})$, we can replace $h_t^{\tau,p}$ by a function with compact support $h_{t,T}^{\tau,p}$ with support depending on T and which coincides with $h_t^{\tau,p}$ in a neighborhood of $1\in G(\mathbb{R})^1$. The proof of this result uses again the fine expansion of the spectral side of the trace formula. Next we use the geometric side of the trace formula. Let J_{unip} be the unipotent contribution to the geometric side. Since $h_{t,T}^{\tau,p}$ has compact support, it follows that for sufficiently large N, the geometric side equals $J_{\mathrm{unip}}(h_{t,T}^{\tau,p}\otimes\chi_{K(N)})$. Next we apply the fine geometric expansion of [Ar4], which expresses $J_{\mathrm{unip}}(h_{t,T}^{\tau,p}\otimes\chi_{K(N)})$ as a finite sum of weighted orbital integrals $J_M(\mathcal{O},h_{t,T}^{\tau,p}\otimes\chi_{K(N)})$ (see (10.10)). Here $M\in\mathcal{L}$ and $\mathcal{O}$ runs over the set of unipotent elements in $M(\mathbb{Q})$ up to $M(\mathbb{Q}_S)$-conjugacy for $S=S(N)$ a suitable finite set of places. (If $G=\mathrm{GL}(n)$, the resulting equivalence classes are just the unipotent $M(\mathbb{Q})$-conjugacy classes in $M(\mathbb{Q})$.) The coefficients $a^M(S(N),\mathcal{O})$ appearing in the fine geometric expansion depend on a sufficiently large set $S(N)$ of places of $\mathbb{Q}$. Then by the decomposition formula (8.5) for weighted orbital integrals, the study of $J_M(\mathcal{O}, h_{t,T}^{\tau,p} \otimes \chi_{K(N)})$ can be reduced to the study of weighted orbital integrals at infinite place and at the finite places in $S(N)$. At the infinite place the weighted orbital integrals are of the form $J_M^L(\mathcal{O}_\infty, (h_{t,T}^{\tau,p})_Q)$, where $L \in \mathcal{L}(M)$, Q is a parabolic subgroup of G with Levi component L, and $(h_{t,T}^{\tau,p})_Q$ is defined by (8.4). These integrals have been studied in [MzM]. By [MzM, Propostion 12.3], $J_M^L(\mathcal{O}_\infty, (h_{t,T}^{\tau,p})_Q)$ has an asymptotic expansion as $t \to 0$. So we can form its partial Mellin transform (10.16), which is a meromorphic function of $s \in \mathbb{C}$. Then the constant term in the Laurent expansion is the contribution of $J_M^L(\mathcal{O}_\infty, (h_{t,T}^{\tau,p})_Q)$ to the first term on the right hand side of (1.13). It is just a constant depending on T, but not N. We are left with the finite orbital integrals $J_M^L(\mathcal{O}_{\mathrm{fin}}, (\chi_{K(N)})_Q)$. Again using the decomposition formula, the study of these integrals can be reduced to study of integrals of the form $J_M^{L_p}(\mathcal{O}_p, \mathbf{1}_{K(N)_p, Q_p})$ at primes $p|N$. Now the point is that in the case of $\mathrm{GL}(n)$ these integrals can be written as integrals over $N_p(\mathbb{Q}_p)$ with a certain weight factor, where N_p is the unipotent radical of some parabolic subgroup in L_p (see (8.8)). The analysis of the weight factors leads to an estimation of these integrals, depending on N. For $M \neq G$ or $M = G$ and $\mathcal{O} \neq 1$, they all decay in N like $O(N^{-(n-1)}(\log N)^a)$ for some fixed $a > 0$. The final step is to estimate the constants $a^M(S(N), \mathcal{O})$ appearing in the fine geometric expansion (10.10). For $\mathrm{GL}(n)$ such estimations were obtained in [Ma2]. The final result is that the contribution to first term of the right hand side of (1.13) of the weighted orbital integrals $J_M(\mathcal{O}, h_{t,T}^{\tau,p} \otimes \chi_{K(N)})$ with $M \neq Q$ times $\mathrm{vol}(Y(N))^{-1}$ decays like $N^{-(n-1)}(\log N)^a$ for some $a > 0$ independent of N. For the contribution of $(G, 1)$ we get $\mathrm{vol}(Y(N))(t_{\widetilde{X}}^{(2)}(\tau) + O(e^{-cT})$ This completes the proof of Theorem 1.1.

We expect that Theorem 1.1 holds more general for congruence subgroups of classical groups. The main obstacle to extend the theorem to other groups is the fine geometric expansion. At the moment, we only know how to estimate the coefficients $a^M(S(N), U)$ for $\mathrm{GL}(n)$. Nevertheless, we expect to be able to overcome this problem. Therefore, we will work in each section with the most general assumptions Thai are possible.

The paper is organized as follows. In section 2 we fix notations and recall some basic facts. In section 3 we state some facts concerning heat kernels on symmetric spaces. In section 4 we recall the definition of the regularized trace of the heat operator on $Y(K_f)$ and we introduce the analytic torsion. In section 5 we review the refined expansion of the spectral side of the Arthur trace formula. The spectral side of the trace formula is used in section 6 to study the large time behavior of the regularized trace of the heat operator. The main point is to derive estimations which are uniform in K_f. In section 7 we study the behavior of the regularized trace as $t \to 0$. We use again the spectral side of the trace formula to show that, up to an exponentially decreasing term, we can replace the heat kernel by a compactly supported function. In section 8 we use the geometric side, applied to the modified test function. It turns out that for principal congruence subgroups $K(N)$ of sufficient high level $N \in \mathbb{N}$, only the unipotent contribution to the geometric side occurs. Then we use Arthur's fine geometric expansion, which expresses the unipotent contribution in terms of weighted orbital integrals. In section 9 we derive estimations for

p-adic weighted orbital integrals. In the section 10 we prove our main result GL(n). Based on this result, we prove Theorem 1.1 in the final section (11).

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2. Preliminaries

Let G be a reductive algebraic group defined over $\mathbb{Q}$. We fix a minimal parabolic subgroup P_0 of G defined over $\mathbb{Q}$ and a Levi decomposition $P_0 = M_0 \cdot N_0$, both defined over $\mathbb{Q}$. If $G = \mathrm{GL}(n)$, we choose P_0 to be the supgroup of upper triangular matrices of G, N_0 its unipotent radical, and M_0 the group of diagonal matrices in G.

Let $\mathcal{F}$ be the set of parabolic subgroups of G which contain M_0 and are defined over $\mathbb{Q}$. Let $\mathcal{L}$ be the set of subgroups of G which contain M_0 and are Levi components of groups in $\mathcal{F}$. For any $P \in \mathcal{F}$ we write

$$P = M_P N_P,$$

where N_P is the unipotent radical of P and M_P belongs to $\mathcal{L}$. Let $M \in \mathcal{L}$. Denote by A_M the $\mathbb{Q}$-split component of the center of M. Put $A_P = A_{M_P}$. Let $L \in \mathcal{L}$ and assume that L contains M. Then L is a reductive group defined over $\mathbb{Q}$ and M is a Levi subgroup of L. We shall denote the set of Levi subgroups of L which contain M by $\mathcal{L}^L(M)$. We also write $\mathcal{F}^L(M)$ for the set of parabolic subgroups of L, defined over $\mathbb{Q}$, which contain M, and $\mathcal{P}^L(M)$ for the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. Each of these three sets is finite. If $L = G$, we shall usually denote these sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$.

Let $X(M)_{\mathbb{Q}}$ be the group of characters of M which are defined over $\mathbb{Q}$. Put

$$\mathfrak{a}_M := \mathrm{Hom}(X(M)_{\mathbb{Q}}, \mathbb{R}). \tag{2.15}$$

This is a real vector space whose dimension equals that of A_M. Its dual space is

$$\mathfrak{a}_M^* = X(M)_{\mathbb{Q}} \otimes \mathbb{R}.$$

We shall write,

$$\mathfrak{a}_P = \mathfrak{a}_{M_P},\ A_0 = A_{M_0} \quad \text{and} \quad \mathfrak{a}_0 = \mathfrak{a}_{M_0}. \tag{2.16}$$

For $M \in \mathcal{L}$ let $A_M(\mathbb{R})^0$ be the connected component of the identity of the group $A_M(\mathbb{R})$. Let $W_0 = N_{\mathbf{G}(\mathbb{Q})}(A_0)/M_0$ be the Weyl group of (G, A_0), where $N_{G(\mathbb{Q})}(H)$ is the normalizer of H in $G(\mathbb{Q})$. For any $s \in W_0$ we choose a representative $w_s \in G(\mathbb{Q})$. Note that W_0 acts on $\mathcal{L}$ by $sM = w_s M w_s^{-1}$. For $M \in \mathcal{L}$ let $W(M) = N_{\mathbf{G}(\mathbb{Q})}(M)/M$, which can be identified with a subgroup of W_0.

For any $L \in \mathcal{L}(M)$ we identify $\mathfrak{a}_L^*$ with a subspace of $\mathfrak{a}_M^*$. We denote by $\mathfrak{a}_M^L$ the annihilator of $\mathfrak{a}_L^*$ in $\mathfrak{a}_M$. We set

$$\mathcal{L}_1(M) = \{L \in \mathcal{L}(M) : \dim \mathfrak{a}_M^L = 1\}$$

and

$$\mathcal{F}_1(M) = \bigcup_{L\in\mathcal{L}_1(M)} \mathcal{P}(L). \tag{2.17}$$

We shall denote the simple roots of (P, A_P) by Δ_P. They are elements of $X(A_P)_{\mathbb{Q}}$ and are canonically embedded in $\mathfrak{a}_P^*$. Let $\Sigma_P \subset \mathfrak{a}_P^*$ be the set of reduced roots of A_P on the Lie algebra of G. For any $\alpha \in \Sigma_M$ we denote by $\alpha^\vee \in \mathfrak{a}_M$ the corresponding co-root. Let P_1 and P_2 be parabolic subgroups with $P_1 \subset P_2$. Then $\mathfrak{a}_{P_2}^*$ is embedded into $\mathfrak{a}_{P_1}^*$, while $\mathfrak{a}_{P_2}$ is a natural quotient vector space of $\mathfrak{a}_{P_1}$. The group $M_{P_2} \cap P_1$ is a parabolic subgroup of M_{P_2}. Let $\Delta_{P_1}^{P_2}$ denote the set of simple roots of $(M_{P_2} \cap P_1, A_{P_1})$. It is a subset of Δ_{P_1}. For a parabolic subgroup P with $P_0 \subset P$ we write $\Delta_0^P := \Delta_{P_0}^P$.

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_{\text{fin}}$ the ring of finite adeles of $\mathbb{Q}$. We fix a maximal compact subgroup $\mathbf{K} = \prod_\nu K_\nu = K_\infty \mathbf{K}_{\text{fin}}$ of $G(\mathbb{A}) = G(\mathbb{R})G(\mathbb{A}_{\text{fin}})$. We assume that the maximal compact subgroup $\mathbf{K} \subset G(\mathbb{A})$ is admissible with respect to M_0 [Ar5, §1].

Let $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ be the homomorphism given by

$$e^{\langle \chi, H_M(m)\rangle} = |\chi(m)|_{\mathbb{A}} = \prod_v |\chi(m_v)|_v \tag{2.18}$$

for any $\chi \in X(M)$. Let

$$M(\mathbb{A})^1 := \{m \in M(\mathbb{A})\colon H_M(m) = 0\}.$$

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G(\mathbb{R})$ and K_∞, respectively. Let θ be the Cartan involution of $G(\mathbb{R})$ with respect to K_∞. It induces a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We fix an invariant bi-linear form B on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. This choice defines a Casimir operator Ω on $G(\mathbb{R})$, and we denote the Casimir eigenvalue of any $\pi \in \Pi(G(\mathbb{R}))$ by λ_π. Similarly, we obtain a Casimir operator Ω_{K_∞} on K_∞ and write λ_τ for the Casimir eigenvalue of a representation $\tau \in \Pi(K_\infty)$ (cf. [BG, §2.3]). The form B induces a Euclidean scalar product $(X, Y) = -B(X, \theta(Y))$ on $\mathfrak{g}$ and all its subspaces. For $\tau \in \Pi(K_\infty)$ we define $\|\tau\|$ as in [CD, §2.2]. Note that the restriction of the scalar product $(\cdot,\cdot)$ on $\mathfrak{g}$ to $\mathfrak{a}_0$ gives $\mathfrak{a}_0$ the structure of a Euclidean space. In particular, this fixes Haar measures on the spaces $\mathfrak{a}_M^L$ and their duals $(\mathfrak{a}_M^L)^*$. We follows Arthur in the corresponding normalization of Haar measures on the groups $M(A)$ ([Ar1, §1]).

Finally we introduce the space of Schwartz functions $\mathcal{C}(G(\mathbb{A})^1)$ from [FL1]. For any compact open subgroup K_f of $G(\mathbb{A}_f)$ the space $G(\mathbb{A})^1/K_f$ is the countable disjoint union of copies of

$$G(\mathbb{R})^1 = G(\mathbb{R}) \cap G(\mathbb{A})^1 \tag{2.19}$$

and therefore, it is a differentiable manifold. Any element $X \in \mathcal{U}(\mathfrak{g}_\infty^1)$ of the universal enveloping algebra of the Lie algebra $\mathfrak{g}_\infty^1$ of $G(\mathbb{R})^1$ defines a left invariant differential operator $f \mapsto f * X$ on $G(\mathbb{A})^1/K_f$. Let $\mathcal{C}(G(\mathbb{A})^1; K_f)$ be the space of smooth right K_f-invariant functions on $G(\mathbb{A})^1$ which belong, together with all their derivatives, to $L^1(G(\mathbb{A})^1)$. The space $\mathcal{C}(G(\mathbb{A})^1; K_f)$ becomes a Fréchet space under the seminorms

$$\|f * X\|_{L^1(G(\mathbb{A})^1)}, \quad X \in \mathcal{U}(\mathfrak{g}^1_\infty).$$

Denote by $\mathcal{C}(G(\mathbb{A})^1)$ the union of the spaces $\mathcal{C}(G(\mathbb{A})^1; K_f)$ as K_f varies over the compact open subgroups of $G(\mathbb{A}_f)$ and endow $\mathcal{C}(G(\mathbb{A})^1)$ with the inductive limit topology.

3. Heat kernels

Since the heat kernel of the twisted Laplace operators plays a key role in the paper, we summarize some basic facts about Bochner-Laplace operators on global Riemannian symmetric spaces and their heat kernels. In this section we assume that G is a connected semisimple group and $G(\mathbb{R})$ is of noncompact type. Then $G(\mathbb{R})$ is a semisimple real Lie group of noncompact type. Let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup and

$$\widetilde{X} = G(\mathbb{R})/K_\infty$$

the associated Riemannian symmetric space. Let $\Gamma \subset G(\mathbb{R})$ be a torsion free lattice and let $X = \Gamma\backslash\widetilde{X}$. Let ν be a finite-dimensional unitary representation of K_∞ on $(V_\nu, \langle\cdot,\cdot\rangle_\nu)$. Let

$$\widetilde{E}_\nu := G(\mathbb{R}) \times_\nu V_\nu$$

be the associated homogeneous vector bundle over $\tilde{X}$. Then $\langle\cdot,\cdot\rangle_\nu$ induces a $G(\mathbb{R})$-invariant metric $\tilde{h}_\nu$ on $\tilde{E}_\nu$. Let $\widetilde{\nabla}^\nu$ be the connection on $\tilde{E}_\nu$ induced by the canonical connection on the principal K_∞-fiber bundle $G(\mathbb{R}) \to G(\mathbb{R})/K_\infty$. Then $\widetilde{\nabla}^\nu$ is $G(\mathbb{R})$-invariant. Let

$$E_\nu := \Gamma\backslash\widetilde{E}_\nu$$

be the associated locally homogeneous vector bundle over X. Since $\tilde{h}_\nu$ and $\widetilde{\nabla}^\nu$ are $G(\mathbb{R})$-invariant, they push down to a metric h_ν and a connection ∇^ν on E_ν. Let $C^\infty(\widetilde{X}, \widetilde{E}_\nu)$ resp. $C^\infty(X, E_\nu)$ denote the space of smooth sections of $\widetilde{E}_\nu$, resp. E_ν. Let

$$C^\infty(G(\mathbb{R}),\nu) := \{f : G(\mathbb{R}) \to V_\nu \colon f \in C^\infty,\ f(gk) = \nu(k^{-1})f(g), \quad \forall g \in G(\mathbb{R}),\ \forall k \in K_\infty\}, \tag{3.1}$$

Let $L^2(G(\mathbb{R}),\nu)$ be the corresponding L^2-space. There is a canonical isomorphism

$$\widetilde{A}\colon C^\infty(\widetilde{X}, \widetilde{E}_\nu) \cong C^\infty(G(\mathbb{R}),\nu), \tag{3.2}$$

(see [Mia, p. 4]). $\widetilde{A}$ extends to an isometry of the corresponding L^2-spaces. Let

$$C^\infty(\Gamma\backslash G(\mathbb{R}),\nu) := \{f \in C^\infty(G(\mathbb{R}),\nu)\colon f(\gamma g) = f(g)\ \forall g \in G(\mathbb{R}), \forall \gamma \in \Gamma\} \tag{3.3}$$

and let $L^2(\Gamma\backslash G(\mathbb{R}),\nu)$ be the corresponding L^2-space. The isomorphism (3.2) descends to isomorphisms

$$A\colon C^\infty(X, E_\nu) \cong C^\infty(\Gamma\backslash G(\mathbb{R}),\nu), \quad L^2(X, E_\nu) \cong L^2(\Gamma\backslash G(\mathbb{R}),\nu). \tag{3.4}$$

Let $\widetilde{\Delta}_\nu = \widetilde{\nabla^\nu}^* \widetilde{\nabla}^\nu$ be the Bochner-Laplace operator of $\widetilde{E}_\nu$. This is a $G(\mathbb{R})$-invariant second order elliptic differential operator whose principal symbol is given by

$$\sigma_{\widetilde{\Delta}_\nu}(x,\xi) = \|\xi\|_x^2 \cdot \mathrm{Id}_{E_{\nu,x}}, \quad x \in \widetilde{X},\ \xi \in T_x^*(\widetilde{X}).$$

Since $\widetilde{X}$ is complete, $\widetilde{\Delta}_\nu$ with domain the smooth compactly supported sections is essentially self-adjoint [LM, p. 155]. Its self-adjoint extension will be denoted by $\widetilde{\Delta}_\nu$ too. Let $\Omega \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ and $\Omega_{K_\infty} \in \mathcal{Z}(\mathfrak{k})$ be the Casimir operators of $\mathfrak{g}$ and $\mathfrak{k}$, respectively, where the latter is defined with respect to the restriction of the normalized Killing form of $\mathfrak{g}$ to $\mathfrak{k}$. Then with respect to the isomorphism (3.2) we have

$$\widetilde{\Delta}_\nu = -R(\Omega) + \nu(\Omega_{K_\infty}), \tag{3.5}$$

where R denotes the right regular representation of $G(\mathbb{R})$ in $C^\infty(G(\mathbb{R}),\nu)$ (see [Mia, Proposition 1.1]).

Let $e^{-t\widetilde{\Delta}_\nu}$, $t>0$, be the heat semigroup generated by $\widetilde{\Delta}_\nu$. It commutes with the action of $G(\mathbb{R})$. With respect to the isomorphism (3.2) we may regard $e^{-t\widetilde{\Delta}_\nu}$ as bounded operator in $L^2(G(\mathbb{R}),\nu)$, which commutes with the action of $G(\mathbb{R})$. Hence it is a convolution operator, i.e., there exists a smooth map

$$H_t^\nu\colon G(\mathbb{R}) \to \mathrm{End}(V_\nu) \tag{3.6}$$

such that

$$(e^{-t\widetilde{\Delta}_\nu}\phi)(g) = \int_{G(\mathbb{R})} H_t^\nu(g^{-1}g')(\phi(g'))\,dg', \quad \phi \in L^2(G(\mathbb{R}),\nu).$$

The kernel H_t^ν satisfies

$$H_t^\nu(k^{-1}gk') = \nu(k)^{-1} \circ H_t^\nu(g) \circ \nu(k'),\ \forall k,k' \in K, \forall g \in G. \tag{3.7}$$

Moreover, proceeding as in the proof of [BM, Proposition 2.4] it follows that H_t^ν belongs to $(\mathscr{C}^q(G(\mathbb{R})) \otimes \mathrm{End}(V_\nu))^{K_\infty \times K_\infty}$ for all $q>0$, where $\mathscr{C}^q(G(\mathbb{R}))$ is Harish-Chandra's Schwartz space of L^q-integrable rapidly decreasing functions on $G(\mathbb{R})$.

Let π be a unitary representation of $G(\mathbb{R})$ on a Hilbert space $\mathcal{H}_\pi$. Define a bounded operator on $\mathcal{H}_\pi \otimes V_\nu$ by

$$\tilde{\pi}(H_t^\nu(g)) := \int_{G(\mathbb{R})} \pi(g) \otimes H_t^\nu(g)\,dg. \tag{3.8}$$

Then relative to the splitting

$$\mathcal{H}_\pi \otimes V_\nu = (\mathcal{H}_\pi \otimes V_\nu)^{K_\infty} \oplus \left((\mathcal{H}_\pi \otimes V_\nu)^{K_\infty}\right)^\perp,$$

$\tilde{\pi}(H_t^\nu)$ has the form

$$\begin{pmatrix} \pi(H_t^\nu) & 0 \\ 0 & 0 \end{pmatrix},$$

where $\pi(H_t^\nu)$ acts on $(\mathcal{H}_\pi \otimes V_\nu)^{K_\infty}$. Assume that π is irreducible. Let $\pi(\Omega)$ be the Casimir eigenvalue of π. Then as in [BM, Corollary 2.2] it follows from (3.5) that

$$\pi(H_t^\nu) = e^{t(\pi(\Omega)-\nu(\Omega_{K_\infty}))}\,\mathrm{Id}, \tag{3.9}$$

where Id is the identity on $(\mathcal{H}_\pi \otimes V_\nu)^{K_\infty}$. Put

$$h_t^\nu(g) := \operatorname{tr} H_t^\nu(g), \quad g \in G(\mathbb{R}). \tag{3.10}$$

Then $h_t^\nu \in \mathscr{C}^q(G(\mathbb{R}))$ for all $q > 0$. Let π be a unitary representation of $G(\mathbb{R})$. Put

$$\pi(h_t^\nu) = \int_{G(\mathbb{R})} h_t^\nu(g)\pi(g)\,dg.$$

Assume that $\pi(H_t^\nu)$ is a trace class operator. Then it follows as in [BM, Lemma 3.3] that $\pi(h_t^\nu)$ is a trace class operator and

$$\operatorname{Tr}\pi(h_t^\nu) = \operatorname{Tr}\pi(H_t^\nu). \tag{3.11}$$

Now assume that π is a unitary admissible representation. Let $A : \mathcal{H}_\pi \to \mathcal{H}_\pi$ be a bounded operator which is an intertwining operator for $\pi|_K$. Then $A \circ \pi(h_t^\nu)$ is again a finite rank operator. Define an operator $\tilde{A}$ on $\mathcal{H}_\pi \otimes V_\nu$ by $\tilde{A} := A \otimes \mathrm{Id}$. Then by the same argument as in [BM, Lemma 5.1] one has

$$\operatorname{Tr}\left(\tilde{A} \circ \tilde{\pi}(H_t^\nu)\right) = \operatorname{Tr}\left(A \circ \pi(h_t^\nu)\right). \tag{3.12}$$

Together with (3.9) we obtain

$$\operatorname{Tr}\left(A \circ \pi(h_t^\nu)\right) = e^{t(\pi(\Omega)-\nu(\Omega_{K_\infty}))}\operatorname{Tr}\left(\tilde{A}|_{(\mathcal{H}_\pi\otimes V_\nu)^K}\right). \tag{3.13}$$

Next we consider the twisted Laplace operator. Let τ be an irreducible finite dimensional representation of $G(\mathbb{R})$ on V_τ. Let F_τ be the flat vector bundle over X associated to the restriction of τ to Γ. Let $\widetilde{E}_\tau$ be the homogeneous vector bundle over $\widetilde{X}$ associated to $\tau|_{K_\infty}$ and let $E_\tau := \Gamma\backslash\widetilde{E}_\tau$. There is a canonical isomorphism

$$E_\tau \cong F_\tau \tag{3.14}$$

[MM, Proposition 3.1]. By [MM, Lemma 3.1], there exists an inner product $\langle\cdot,\cdot\rangle$ on V_τ such that

1. $\langle\tau(Y)u, v\rangle = -\langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{k}$, $u, v \in V_\tau$
2. $\langle\tau(Y)u, v\rangle = \langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{p}$, $u, v \in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\tau|_{K_\infty}$ is unitary with respect to this inner product, it induces a metric on E_τ, and by (3.14) on F_τ, which we also call admissible. Let $\Lambda^p(F_\tau) = \Lambda^p T^*(X) \otimes F_\tau$. By (3.14) $\Lambda^p(F_\tau)$ is isomorphic to the locally homogeneous vector bundle associated to the representation

$$\nu_p(\tau) := \Lambda^p \operatorname{Ad}^* \otimes \tau :\ K_\infty \to \mathrm{GL}(\Lambda^p\mathfrak{p}^* \otimes V_\tau). \tag{3.15}$$

The space of smooth section of $\Lambda^p(F_\tau)$ is the space $\Lambda^p(X,F_\tau)$ of F_τ-valued p-forms. By (3.3) there is a canonical isomorphism

$$\Lambda^p(X,F_\tau)\cong C^\infty(\Gamma\backslash G,\nu_p(\tau)). \tag{3.16}$$

Let $\Delta_p(\tau)$ be the Laplace operator in $\Lambda^p(X,F_\tau)$. Let R_Γ be the right regular representation of $G(\mathbb{R})$ in $C^\infty(\Gamma\backslash G,\nu_p(\tau))$ and Ω the Casimir element of $G(\mathbb{R})$. By [MM] it follows that with respect to the isomorphism (3.16) we have

$$\Delta_p(\tau)=-R_\Gamma(\Omega)+\tau(\Omega). \tag{3.17}$$

Let $\widetilde{\Delta}_p(\tau)$ be the lift of $\Delta_p(\tau)$ to the universal covering $\widetilde{X}$. It acts in the space $\Lambda^p(\widetilde{X},\widetilde{F}_\tau)$ of p-forms on $\widetilde{X}$ with values in the pull back $\widetilde{F}_\tau$ of F_τ. Then by (3.2) we have

$$\Lambda^p(\widetilde{X},\widetilde{F}_\tau)\cong C^\infty(G(\mathbb{R}),\nu_p(\tau)).$$

and with respect to this isomorphism we also have

$$\widetilde{\Delta}_p(\tau)=-R(\Omega)+\tau(\Omega),$$

where R is the regular representation of $G(\mathbb{R})$ in $C^\infty(G(\mathbb{R}),\nu_p(\tau))$. Using (3.5) we obtain

$$\widetilde{\Delta}_p(\tau)=\widetilde{\Delta}_{\nu_p(\tau)}+\tau(\Omega)-\nu_p(\tau)(\Omega_{K_\infty}). \tag{3.18}$$

We note that $\widetilde{\Delta}_p(\tau)$ is a formally self-adjoint, non-negative, elliptic second order differential operator. Regarded as operator in the Hilbert space $L^2\Lambda^p(X,F_\tau)$ of square integrable F_τ-valued p-forms on X with domain the space of compactly supported smooth p-forms, it has a unique self-adjoint extension which we also denote by $\widetilde{\Delta}_p(\tau)$. This is a non-negative self-adjoint operator in $\Lambda^p(\widetilde{X},\widetilde{F}_\tau)$. Let $e^{-t\widetilde{\Delta}_p(\tau)}$, $t>0$, be heat semigroup generated by $\widetilde{\Delta}_p(\tau)$. It is well known that $e^{-t\widetilde{\Delta}_p(\tau)}$ is an integral operator with a smooth kernel. Since $\widetilde{\Delta}_p(\tau)$ commutes with the action of $G(\mathbb{R})$, $e^{-t\widetilde{\Delta}_p(\tau)}$ is a convolution operator with kernel

$$H_t^{\tau,p}\colon G(\mathbb{R})\to\operatorname{End}(\Lambda^p\mathfrak{p}^\star\otimes V_\tau), \tag{3.19}$$

which belongs to $C^\infty\cap L^2$, and satisfies the covariance property

$$H_t^{\tau,p}(k^{-1}gk')=\nu_p(\tau)(k)^{-1}H_t^{\tau,p}(g)\nu_p(\tau)(k') \tag{3.20}$$

with respect to the representation (3.15). Moreover, for all $q>0$ we have

$$H_t^{\tau,p}\in(\mathcal{C}^q(G(\mathbb{R}))\otimes\operatorname{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau))^{K_\infty\times K_\infty}, \tag{3.21}$$

where $\mathcal{C}^q(G(\mathbb{R}))$ denotes Harish-Chandra's L^q-Schwartz space (see [MP2, Sect. 4]). Let $h_t^{\tau,p}\in C^\infty(G(\mathbb{R}))$ be defined by

$$h_t^{\tau,p}(g)=\operatorname{tr}H_t^{\tau,p}(g),\quad g\in G(\mathbb{R}). \tag{3.22}$$

Then $h_t^{\tau,p}\in\mathcal{C}^q(G(\mathbb{R}))$ for all $q>0$.

4. Analytic torsion

We briefly recall the definition of the analytic torsion. For details we refer to [MzM]. Let G be a reductive algebraic group over $\mathbb{Q}$. Let $K_\infty \subset G\mathbb{R})^1$ be a maximal compact subgroup and $\widetilde{X} = G(\mathbb{R})^1/K_\infty$. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup. Let

$$X(K_f) := G(\mathbb{Q})\backslash(\widetilde{X} \times G(\mathbb{A}_f))/K_f \tag{4.1}$$

be the adelic quotient. It is the disjoint union of finitely many components $\Gamma_i\backslash\widetilde{X}$, where $\Gamma_i \subset G(\mathbb{Q})$, $i = 1, \dots, m$, are arithmetic subgroups. Let $\tau \in \operatorname{Rep}(G(\mathbb{R})^1)$. Denote by $E_{\tau;i}$ the locally flat vector bundle over $\Gamma_i\backslash\widetilde{X}$, associated to $\tau|_{\Gamma_i}$. Let E_τ be the disjoint union of the $E_{\tau;i}$. Then E_τ is a flat vector bundle over $X(K_f)$. Let $\Delta_p(\tau)$ the Laplace operator on E_τ-valued p-forms over $X(K_f)$. Let $h_t^{\tau,p}$ be the function defined by (3.22) and let χ_{K_f} be the normalized characteristic function of K_f in $G(\mathbb{A}_f)$ defined by (1.7). Put

$$\phi_t^{\tau,p} := h_t^{\tau,p} \otimes \chi_{K_f}. \tag{4.2}$$

Then $\phi_t^{\tau,p}$ belongs to the adelic Schwartz space $\mathcal{C}(G(\mathbb{A})^1; K_f)$ (see section 2). Let $J_{\text{geo}}(f)$, $f \in C_1^\infty(G(\mathbb{A})^1)$ be the geometric side of the Arthur trace formula [Ar1]. The distribution J_{geo} extends to $\mathcal{C}(G(\mathbb{A})^1; K_f)$ (see [FL1]). In [MzM, (13.17)] we defined the regularized trace of the heat operator $e^{-t\Delta_p(\tau)}$ by

$$\operatorname{Tr}_{\text{reg}}\left(e^{-t\Delta_p(\tau)}\right) := J_{\text{geo}}(\phi_t^{\tau,p}). \tag{4.3}$$

For the motivation for this definition we refer to [MzM]. We only note that if $X(K_f)$ is compact, then $e^{-t\Delta_p(\tau)}$ is a trace class operator and the regularized trace is the usual trace, which is equal to the spectral side of the trace formula. So in this case, (4.3) is just the trace formula. To define the zeta function $\zeta_p(s,\tau)$ through the Mellin transform of the regularized trace of the heat operator, we need to determine the asymptotic behavior of $\operatorname{Tr}_{\text{reg}}(e^{-t\Delta_p(\tau)})$ as $t \to \infty$ and $t \to 0$. This requires additional assumptions.

From now on we assume that $G = \operatorname{GL}(n)$ or $\operatorname{SL}(n)$. Let θ be the Cartan involution of $G(\mathbb{R})^1$. Let $\tau_\theta = \tau \circ \theta$. Assume that $\tau \neq \tau_\theta$. Then by [MzM, Proposition 13.4] and the trace formula we have

$$\operatorname{Tr}_{\text{reg}}\left(e^{-t\Delta_p(\tau)}\right) = O(e^{-ct}) \tag{4.4}$$

as $t \to \infty$. The existence of an asymptotic expansion as $t \to 0$ follows from [MzM, Theorem 1.1]. Assume that K_f is contained in $K(N)$ for some $N \geq 3$. Then there is an asymptotic expansion

$$\operatorname{Tr}_{\text{reg}}\left(e^{-t\Delta_p(\tau)}\right) \sim t^{-d/2}\sum_{j=0}^\infty a_j t^j + t^{-(d-1)/2}\sum_{j=0}^\infty\sum_{i=0}^{r_j} b_{ij} t^{j/2}(\log t)^i \tag{4.5}$$

as $t \to 0$. Thus, under the assumptions above, the integral

$$\zeta_p(s,\tau) := \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}_{\text{reg}}\left(e^{-t\Delta_p(\tau)}\right) t^{s-1}dt \tag{4.6}$$

converges absolutely and uniformly on compact subsets of the half-plane $\operatorname{Re}(s) > d/2$, and admits a meromorphic extension to the entire complex plane. Due to the logarithmic terms in the expansion (4.5), the zeta function $\zeta_p(s,\tau)$ may have a pole at $s=0$. The analytic torsion is then defined by (1.9).

In the case of $G = \mathrm{GL}(3)$ we are able to determine the coefficients of the log-terms. This shows that the zeta functions definitely have a pole at $s=0$. However, the combination $\sum_{p=1}^{5}(-1)^p p\zeta_p(s;\tau)$ turns out to be holomorphic at $s=0$ (see [MzM, sect. 14] and we can define the logarithm of the analytic torsion by

$$\log T_{X(K_f)}(\tau) = \frac{d}{ds}\left(\frac{1}{2}\sum_{p=1}^{5}(-1)^p p\zeta_p(s;\tau)\right)\Big|_{s=0}.$$

5. Review of the spectral side of the trace formula

In this section G is an arbitrary reductive algebraic group over $\mathbb{Q}$. Arthur's (non-invariant) trace formula is the equality

$$J_{\mathrm{geo}}(f) = J_{\mathrm{spec}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1), \tag{5.1}$$

of two distributions on $G(\mathbb{A})^1$, namely the equality of the geometric side $J_{\mathrm{geo}}(f)$ and the spectral side $J_{\mathrm{spec}}(f)$ of the trace formula. In this section we recall the definition of the spectral side, and in particular the refinement of the spectral expansion obtained in [FLM1], which we need for our purpose. Combining [FLM1] and [FL1], it follows that (5.1) extends continuously to $f \in \mathcal{C}(G(\mathbb{A})^1)$.

The main ingredient of the spectral side are logarithmic derivatives of intertwining operators. We briefly recall the structure of the intertwining operators.

Let $P \in \mathcal{P}(M)$. Let U_P be the unipotent radical of P. Recall that we denote by $\Sigma_P \subset \mathfrak{a}_P^*$ the set of reduced roots of A_M of the Lie algebra $\mathfrak{u}_P$ of U_P. Let Δ_P be the subset of simple roots of P, which is a basis for $(\mathfrak{a}_P^G)^*$. Write $\mathfrak{a}_{P,+}^*$ for the closure of the Weyl chamber of P, i.e.

$$\mathfrak{a}_{P,+}^* = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee\rangle \ge 0 \text{ for all } \alpha \in \Sigma_P\} = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee\rangle \ge 0 \text{ for all } \alpha \in \Delta_P\}.$$

Denote by δ_P the modulus function of $P(\mathbb{A})$. Let $\bar{\mathcal{A}}_2(P)$ be the Hilbert space completion of

$$\{\phi \in C^\infty(M(\mathbb{Q})U_P(\mathbb{A})\backslash G(\mathbb{A})) : \delta_P^{-\frac{1}{2}}\phi(\cdot x) \in L^2_{\mathrm{disc}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A})),\ \forall x \in G(\mathbb{A})\}$$

with respect to the inner product

$$(\phi_1,\phi_2) = \int_{A_M(\mathbb{R})^0 M(\mathbb{Q})\mathbf{U}_P(\mathbb{A})\backslash \mathbf{G}(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}\, dg.$$

Let $\alpha \in \Sigma_M$. We say that two parabolic subgroups $P, Q \in \mathcal{P}(M)$ are *adjacent* along α, and write $P|^\alpha Q$, if $\Sigma_P \cap -\Sigma_Q = \{\alpha\}$. Alternatively, P and Q are adjacent if the group $\langle P, Q\rangle$ generated by P and Q belongs to $\mathcal{F}_1(M)$ (see (2.17) for its definition). Any $R \in \mathcal{F}_1(\mathcal{M})$ is of the form $\langle P, Q\rangle$, where P, Q are the elements of $\mathcal{P}(M)$ contained in R. We have $P|^{\alpha}Q$ with $\alpha^\vee \in \Sigma_P^\vee \cap \mathfrak{a}_M^R$. Interchanging P and Q changes α to $-\alpha$.

For any $P \in \mathcal{P}(M)$ let $H_P\colon G(\mathbb{A}) \to \mathfrak{a}_P$ be the extension of H_M to a left $U_P(\mathbb{A})$-and right $\mathbf{K}$-invariant map. Denote by $\mathcal{A}^2(P)$ the dense subspace of $\bar{\mathcal{A}}^2(P)$ consisting of its $\mathbf{K}$- and $\mathfrak{z}$-finite vectors, where $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathfrak{g}\otimes\mathbb{C}$. That is, $\mathcal{A}^2(P)$ is the space of automorphic forms ϕ on $U_P(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A})$ such that $\delta_P^{-\frac{1}{2}}\phi(\cdot k)$ is a square-integrable automorphic form on $A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A})$ for all $k \in \mathbf{K}$. Let $\rho(P,\lambda)$, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, be the induced representation of $G(\mathbb{A})$ on $\bar{\mathcal{A}}^2(P)$ given by

$$(\rho(P,\lambda,y)\phi)(x) = \phi(xy)e^{\langle\lambda, H_P(xy)-H_P(x)\rangle}.$$

It is isomorphic to the induced representation

$$\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\left(L^2_{\mathrm{disc}}(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A}))\otimes e^{\langle\lambda,H_M(\cdot)\rangle}\right).$$

For $P, Q \in \mathcal{P}(M)$ let

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

be the standard *intertwining operator* [Ar9, §1], which is the meromorphic continuation in λ of the integral

$$[M_{Q|P}(\lambda)\phi](x) = \int_{U_Q(\mathbb{A})\cap U_P(\mathbb{A})\backslash U_Q(\mathbb{A})} \phi(nx)e^{\left\langle\lambda, H_P(nx)-H_Q(x)\right\rangle}\, dn, \quad \phi\in\mathcal{A}^2(P),\ x \in G(\mathbb{A}).$$

Given $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$, let $\mathcal{A}^2_\pi(P)$ be the space of all $\phi\in\mathcal{A}^2(P)$ for which the function $M(\mathbb{A}) \ni x \mapsto \delta_P^{-\frac{1}{2}}\phi(xg)$, $g\in G(\mathbb{A})$, belongs to the π-isotypic subspace of the space $L^2(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A}))$. For any $P \in \mathcal{P}(\mathcal{M})$ we have a canonical isomorphism of $G(\mathbb{A}_f)\times(\mathfrak{g}_{\mathbb{C}},K_\infty)$-modules

$$j_P : \operatorname{Hom}(\pi, L^2(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A})))\otimes \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi) \to \mathcal{A}^2_\pi(P).$$

If we fix a unitary structure on π and endow $\operatorname{Hom}(\pi, L^2(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A})))$ with the inner product $(A,B) = B^*A$ (which is a scalar operator on the space of π), the isomorphism j_P becomes an isometry.

Suppose that $P|^\alpha Q$. The operator $M_{Q|P}(\pi,s) := M_{Q|P}(s\varpi)|_{\mathcal{A}^2_\pi(P)}$, where $\varpi\in\mathfrak{a}^*_M$ is such that $\langle\varpi,\alpha^\vee\rangle = 1$, admits a normalization by a global factor $n_\alpha(\pi,s)$ which is a meromorphic function in s. We may write

$$M_{Q|P}(\pi,s)\circ j_P = n_\alpha(\pi,s)\cdot j_Q \circ (\operatorname{Id}\otimes R_{Q|P}(\pi,s)) \tag{5.2}$$

where $R_{Q|P}(\pi,s) = \otimes_v R_{Q|P}(\pi_v,s)$ is the product of the locally defined normalized intertwining operators and $\pi = \otimes_v\pi_v$ [Ar9, §6], (cf. [Mu2, (2.17)]). In many cases, the normalizing factors can be expressed in terms automorphic L-functions [Sha1], [Sha2]. For example, let $G = \mathrm{GL}(n)$. Then the global normalizing factors n_α can be expressed in terms of Rankin-Selberg L-functions. The known properties of these functions are collected and analyzed in [Mu1, §§4,5]. Write $M \simeq \prod_{i=1}^r \mathrm{GL}(n_i)$, where the root α is trivial on $\prod_{i\geq 3}\mathrm{GL}(n_i)$, and let $\pi\simeq\otimes\pi_i$ with representations $\pi_i\in\Pi_{\mathrm{disc}}(\mathrm{GL}(n_i,\mathbb{A}))$. Let $L(s,\pi_1\times\tilde{\pi}_2)$

be the completed Rankin-Selberg L-function associated to π_1 and π_2. It satisfies the functional equation

$$L(s,\pi_1\times\tilde{\pi}_2)=\epsilon(\frac{1}{2},\pi_1\times\tilde{\pi}_2)N(\pi_1\times\tilde{\pi}_2)^{\frac{1}{2}-s}L(1-s,\tilde{\pi}_1\times\pi_2) \tag{5.3}$$

where $|\epsilon(\frac{1}{2},\pi_1\times\tilde{\pi}_2)|=1$ and $N(\pi_1\times\tilde{\pi}_2)\in\mathbb{N}$ is the conductor. Then we have

$$n_\alpha(\pi,s)=\frac{L(s,\pi_1\times\tilde{\pi}_2)}{\epsilon(\frac{1}{2},\pi_1\times\tilde{\pi}_2)N(\pi_1\times\tilde{\pi}_2)^{\frac{1}{2}-s}L(s+1,\pi_1\times\tilde{\pi}_2)}. \tag{5.4}$$

We now turn to the spectral side. Let $L\supset M$ be Levi subgroups in $\mathcal{L}$, $P\in\mathcal{P}(M)$, and let $m=\dim\mathfrak{a}_L^G$ be the co-rank of L in G. Denote by $\mathfrak{B}_{P,L}$ the set of m-tuples $\underline{\beta}=(\beta_1^\vee,\dots,\beta_m^\vee)$ of elements of $\Sigma_P^\vee$ whose projections to $\mathfrak{a}_L$ form a basis for $\mathfrak{a}_L^G$. For any $\underline{\beta}=(\beta_1^\vee,\dots,\beta_m^\vee)\in\mathfrak{B}_{P,L}$ let $\operatorname{vol}(\underline{\beta})$ be the co-volume in $\mathfrak{a}_L^G$ of the lattice spanned by $\underline{\beta}$ and let

$$\begin{aligned}\Xi_L(\underline{\beta})&=\{(Q_1,\dots,Q_m)\in\mathcal{F}_1(M)^m:\quad \beta_i^\vee\in\mathfrak{a}_M^{Q_i},\ i=1,\dots,m\}\\&=\{\langle P_1,P_1'\rangle,\dots,\langle P_m,P_m'\rangle):\quad P_i|^{\beta_i}P_i',\ i=1,\dots,m\}.\end{aligned}$$

For any smooth function f on $\mathfrak{a}_M^*$ and $\mu\in\mathfrak{a}_M^*$ denote by $D_\mu f$ the directional derivative of f along $\mu\in\mathfrak{a}_M^*$. For a pair $P_1|^\alpha P_2$ of adjacent parabolic subgroups in $\mathcal{P}(M)$ write

$$\delta_{P_1|P_2}(\lambda)=M_{P_2|P_1}(\lambda)D_\varpi M_{P_1|P_2}(\lambda):\mathcal{A}^2(P_2)\to\mathcal{A}^2(P_2), \tag{5.5}$$

where $\varpi\in\mathfrak{a}_M^*$ is such that $\langle\varpi,\alpha^\vee\rangle=1$. [1] Equivalently, writing $M_{P_1|P_2}(\lambda)=\Phi(\langle\lambda,\alpha^\vee\rangle)$ for a meromorphic function Φ of a single complex variable, we have

$$\delta_{P_1|P_2}(\lambda)=\Phi(\langle\lambda,\alpha^\vee\rangle)^{-1}\Phi'(\langle\lambda,\alpha^\vee\rangle).$$

For any m-tuple $\mathcal{X}=(Q_1,\dots,Q_m)\in\Xi_L(\underline{\beta})$ with $Q_i=\langle P_i,P_i'\rangle$, $P_i|^{\beta_i}P_i'$, denote by $\Delta_{\mathcal{X}}(P,\lambda)$ the expression

$$\frac{\operatorname{vol}(\underline{\beta})}{m!}M_{P_1'|P}(\lambda)^{-1}\delta_{P_1|P_1'}(\lambda)M_{P_1'|P_2'}(\lambda)\cdots\delta_{P_{m-1}|P_{m-1}'}(\lambda)M_{P_{m-1}'|P_m'}(\lambda)\delta_{P_m|P_m'}(\lambda)M_{P_m'|P}(\lambda). \tag{5.6}$$

In [FLM1, pp. 179-180] we defined a (purely combinatorial) map $\mathcal{X}_L:\mathfrak{B}_{P,L}\to\mathcal{F}_1(M)^m$ with the property that $\mathcal{X}_L(\underline{\beta})\in\Xi_L(\underline{\beta})$ for all $\underline{\beta}\in\mathfrak{B}_{P,L}$.[2]

For any $s\in W(M)$ let L_s be the smallest Levi subgroup in $\mathcal{L}(M)$ containing w_s. We recall that $\mathfrak{a}_{L_s}=\{H\in\mathfrak{a}_M\mid sH=H\}$. Set

$$\iota_s=|\det(s-1)_{\mathfrak{a}_M^{L_s}}|^{-1}.$$

For $P\in\mathcal{F}(M_0)$ and $s\in W(M_P)$ let $M(P,s):\mathcal{A}^2(P)\to\mathcal{A}^2(P)$ be as in [Ar3, p. 1309]. $M(P,s)$ is a unitary operator which commutes with the operators $\rho(P,\lambda,h)$ for $\lambda\in i\mathfrak{a}_{L_s}^*$. Finally, we can state the refined spectral expansion.

[1] Note that this definition differs slightly from the definition of $\delta_{P_1|P_2}$ in [FLM1].

[2] The map $\mathcal{X}_L$ depends in fact on the additional choice of a vector $\underline{\mu}\in(\mathfrak{a}_M^*)^m$ which does not lie in an explicit finite set of hyperplanes. For our purposes, the precise definition of $\mathcal{X}_L$ is immaterial.

Theorem 5.1 ([FLM1]). *For any $h \in C_c^\infty(G(\mathbb{A})^1)$ the spectral side of Arthur's trace formula is given by*

$$J_{\mathrm{spec}}(h) = \sum_{[M]} J_{\mathrm{spec},M}(h), \tag{5.7}$$

M ranging over the conjugacy classes of Levi subgroups of G (represented by members of $\mathcal{L}$), where

$$J_{\mathrm{spec},M}(h) = \frac{1}{|W(M)|} \sum_{s\in W(M)} \iota_s \sum_{\underline{\beta}\in\mathfrak{B}_{P,L_s}} \int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} \mathrm{tr}(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\lambda) M(P,s)\rho(P,\lambda,h))\, d\lambda \tag{5.8}$$

with $P \in \mathcal{P}(M)$ arbitrary. The operators are of trace class and the integrals are absolutely convergent with respect to the trace norm and define distributions on $\mathcal{C}(G(\mathbb{A})^1)$.

Note that the term corresponding to $M = G$ is $J_{\mathrm{spec},G}(h) = \mathrm{tr}\, R_{\mathrm{disc}}(h)$. Next assume that M is the Levi subgroup of a maximal parabolic subgroup P. Furthermore, let $L = M$. Let $\bar{P}$ be the opposite parabolic subgroup to P. Then up to a constant, the contribution to the spectral side is given by

$$\sum_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} \int_{i\mathfrak{a}^*} \mathrm{tr}(M_{\bar{P}|P}(\pi,\lambda)^{-1} \frac{d}{dz} M_{\bar{P}|P}(\pi,\lambda) M(P,s)\rho(P,\pi,\lambda,h))\, d\lambda.$$

6. Large time behavior of the regularized trace

The purpose of this section is to improve (4.4) so that the estimations are uniform with respect to K_f. To this end we use the trace formula (5.1). By Theorem 5.1, J_{spec} is a distribution on $\mathcal{C}(G(\mathbb{A}); K_f)$ and by [FL1, Theorem 7.1], J_{geo} is continuous on $\mathcal{C}(G(\mathbb{A}); K_f)$. This implies that (5.1) holds for $\phi_t^{\tau,p}$. Using the definition (4.3) of the regularized trace and the trace formula we get

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) = J_{\mathrm{spec}}(\phi_t^{\tau,p}). \tag{6.1}$$

Now we apply Theorem 5.1 to study the asymptotic behavior as $t \to \infty$ of the right hand side. Let $M \in \mathcal{L}$ and $P \in \mathcal{P}(M)$. Recall that $L^2_{\mathrm{dis}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))$ splits as the completed direct sum of its π-isotypic components for $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$. We have a corresponding decomposition of $\bar{\mathcal{A}}^2(P)$ as a direct sum of Hilbert spaces $\hat{\oplus}_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))} \bar{\mathcal{A}}^2_\pi(P)$. Similarly, we have the algebraic direct sum decomposition

$$\mathcal{A}^2(P) = \bigoplus_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))} \mathcal{A}^2_\pi(P),$$

where $\mathcal{A}^2_\pi(P)$ is the $\mathbf{K}$-finite part of $\bar{\mathcal{A}}^2_\pi(P)$. For $\sigma \in \widehat{K_\infty}$ let $\mathcal{A}^2_\pi(P)^\sigma$ be the σ-isotypic subspace. Then $\mathcal{A}^2_\pi(P)$ decomposes as

$$\mathcal{A}^2_\pi(P) = \bigoplus_{\sigma\in\widehat{K_\infty}} \mathcal{A}^2_\pi(P)^\sigma.$$

Let $\mathcal{A}^2_\pi(P)^{K_f}$ be the subspace of K_f-invariant functions in $\mathcal{A}^2_\pi(P)$, and for any $\sigma \in \widehat{K_\infty}$ let $\mathcal{A}^2_\pi(P)^{K_f,\sigma}$ be the σ-isotypic subspace of $\mathcal{A}^2_\pi(P)^{K_f}$. Recall that $\mathcal{A}^2_\pi(P)^{K_f,\sigma}$ is finite dimensional. Let $M_{Q|P}(\pi,\lambda)$ denote the restriction of $M_{Q|P}(\lambda)$ to $\mathcal{A}^2_\pi(P)$. Recall that the operator $\Delta_\chi(P,\lambda)$, which appears in the formula (5.8), is defined by (5.6). Its definition involves the intertwining operators $M_{Q|P}(\lambda)$. If we replace $M_{Q|P}(\lambda)$ by its restriction $M_{Q|P}(\pi,\lambda)$ to $\mathcal{A}^2_\pi(P)$, we obtain the restriction $\Delta_\chi(P,\pi,\lambda)$ of $\Delta_\chi(P,\lambda)$ to $\mathcal{A}^2_\pi(P)$. Similarly, let $\rho_\pi(P,\lambda)$ be the induced representation in $\bar{\mathcal{A}}^2_\pi(P)$. Fix $\beta \in \mathfrak{B}_{P,L_s}$ and $s \in W(M)$. Then for the integral on the right of (5.8) with $h = \phi_t^{\tau,p}$ we get

$$\sum_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))} \int_{i(\mathfrak{a}^G_{L_s})^*} \operatorname{Tr}\left(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\lambda)M(P,\pi,s)\rho_\pi(P,\lambda,\phi_t^{\tau,p})\right)\, d\lambda. \tag{6.2}$$

Let $P,Q \in \mathcal{P}(M)$ and $\nu \in \Pi(K_\infty)$. Denote by $\widetilde{M}_{Q|P}(\pi,\nu,\lambda)$ the restriction of

$$M_{Q|P}(\pi,\lambda)\otimes \mathrm{Id}\colon \mathcal{A}^2_\pi(P)\otimes V_\nu \to \mathcal{A}^2_\pi(P)\otimes V_\nu$$

to $(\mathcal{A}^2_\pi(P)^{K_f}\otimes V_\nu)^{K_\infty}$. Denote by $\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)$ and $\widetilde{M}(P,\pi,\nu,s)$ the corresponding restrictions. Let $m(\pi)$ denote the multiplicity with which π occurs in the regular representation of $M(\mathbb{A})$ in $L^2_{\mathrm{dis}}(M(\mathbb{Q})\backslash M(\mathbb{A}))$. Then

$$\rho_\pi(P,\lambda) \cong \oplus_{i=1}^{m(\pi)} \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi,\lambda). \tag{6.3}$$

Let $\pi = \pi_\infty \otimes \pi_f$, where π_∞ and π_f are irreducible unitary representations of $M(\mathbb{R})$ and $M(\mathbb{A}_f)$, respectively. Then

$$\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi,\lambda) = \operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty,\lambda)\otimes \operatorname{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f,\lambda).$$

Let $\mathcal{H}(\pi_\infty)$ and $\mathcal{H}(\pi_f)$ denote the Hilbert space of π_∞ and π_f, respectively. Let $\omega(\pi_\infty,\lambda)$ be the Casimir eigenvalue of the induced representation $\operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty,\lambda)$ and let Π_{K_f} be the orthogonal projection of $\mathcal{H}(\pi_f)$ onto the subspace $\mathcal{H}(\pi_f)^{K_f}$ of K_f-invariant vectors. Then by (3.17) it follows that

$$\operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty,\lambda,h_t^{\tau,p}) = e^{t(\tau(\Omega)-\omega(\pi_\infty,\lambda))}\,\mathrm{Id},$$

where Id is the identity on $(\mathcal{H}(\pi_\infty)\otimes\Lambda^p\mathfrak{p}^\star\otimes V_\tau)^{K_\infty}$. Furthermore,

$$\operatorname{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f,\lambda,\chi_{K_f}) = \Pi_{K_f}.$$

Let $\Pi_{K_f,\nu_p(\tau)}$ denote the orthogonal projection onto $\bar{\mathbb{A}}^2_\pi(P)^{K_f,\nu_p(\tau)}$. Then it follows that

$$\rho_\pi(P,\lambda,\phi_t^{\tau,p}) = e^{t(\tau(\Omega)-\omega(\pi_\infty,\lambda))}\Pi_{K_f,\nu_p(\tau)}. \tag{6.4}$$

Fix positive restricted roots of $\mathfrak{a}_P$ and let $\rho_{\mathfrak{a}_P}$ denote the corresponding half-sum of these roots. For $\xi\in\Pi(M(\mathbb{R}))$ and $\lambda\in\mathfrak{a}_P^*$ let

$$\pi_{\xi,\lambda} := \operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\xi\otimes e^{i\lambda})$$

be the unitary induced representation. Let $\xi(\Omega_M)$ be the Casimir eigenvalue of ξ. Define a constant $c(\xi)$ by

$$c(\xi) := -\langle \rho_{\mathfrak{a}_P}, \rho_{\mathfrak{a}_P} \rangle + \xi(\Omega_M). \tag{6.5}$$

Then for $\lambda \in \mathfrak{a}_P^*$ one has

$$\pi_{\xi,\lambda}(\Omega) = -\|\lambda\|^2 + c(\xi) \tag{6.6}$$

(see [Kn, Theorem 8.22]). Let

$$\mathcal{T} := \{\nu \in \Pi(K_\infty) \colon [\nu_p(\tau) \colon \nu] \neq 0\}. \tag{6.7}$$

Using (6.4) and (3.13), it follows that (6.2) is equal to

$$\begin{split} \sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))} \sum_{\nu \in \mathcal{T}} e^{-t(\tau(\Omega) - c(\pi_\infty))} \\ \int_{i(\mathfrak{a}_{L_s}^G)^*} e^{-t\|\lambda\|^2} \operatorname{Tr}\left(\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P, \pi, \nu, \lambda) \widetilde{M}(P, \pi, \nu, s)\right) \, d\lambda. \end{split} \tag{6.8}$$

Using that $M(P, \pi, s)$ is unitary, it follows that (6.8) can be estimated by

$$\begin{split} \sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))} \sum_{\nu \in \mathcal{T}} \dim\left(\mathcal{A}_\pi^2(P)^{K_f, \nu}\right) \\ \cdot e^{-t(\tau(\Omega) - c(\pi_\infty))} \int_{i(\mathfrak{a}_{L_s}^G)^*} e^{-t\|\lambda\|^2} \|\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P, \pi, \nu, \lambda)\| \, d\lambda. \end{split} \tag{6.9}$$

First we estimate the integral in (6.9). Let $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee)$ and $\mathcal{X}_{L_s}(\underline{\beta}) = (Q_1, \dots, Q_m) \in \Xi_{L_s}(\underline{\beta})$ with with $Q_i = \langle P_i, P_i' \rangle$, $P_i|^{\beta_i} P_i'$, $i = 1, \dots, m$. Using the definition (5.6) of $\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P, \pi, \nu, \lambda)$, it follows that we can bound the integral by a constant multiple of

$$\dim(\nu) \int_{i(\mathfrak{a}_{L_s}^G)^*} e^{-t\|\lambda\|^2} \prod_{i=1}^m \left\| \delta_{P_i|P_i'}(\lambda) \Big|_{\mathcal{A}_\pi^2(P_i')^{K_f, \nu}} \right\| \, d\lambda. \tag{6.10}$$

We introduce new coordinates $s_i := \langle \lambda, \beta_i^\vee \rangle$, $i = 1, \dots, m$, on $(\mathfrak{a}_{L_s, \mathbb{C}}^G)^*$. Using (5.2), we can write

$$\delta_{P_i|P_i'}(\lambda) = \frac{n_{\beta_i}'(\pi, s_i)}{n_{\beta_i}(\pi, s_i)} + j_{P_i'} \circ (\operatorname{Id} \otimes R_{P_i|P_i'}(\pi, s_i)^{-1} R_{P_i|P_i'}'(\pi, s_i)) \circ j_{P_i'}^{-1}. \tag{6.11}$$

In [FLM2, Definition 5.2, Definition 5.9] two conditions, called (TWN) and (BD), for an arbitrary reductive group have been formulated, which imply appropriate estimations for the terms on the right. Furthermore, in [FLM2, Prop. 5.5, Prop. 5.15] it was shown that the conditions (TWN) and (BD) both hold for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. Assume that the conditions (TWN) and (BD) hold for G. Then as in [FLM2, (22)] this implies that for any

$\epsilon > 0$ and sufficiently large k and m one has

(6.12)
$$\int_{i(\mathfrak{a}^G_{L_s})^*} (1+\|\lambda\|)^{-k} \prod_{i=1}^m \left\| \delta_{P_i|P_i'}(\lambda)\Big|_{\mathcal{A}^2_\pi(P_i')^{K_f,\nu}} \right\| \, d\lambda \ll_{\varepsilon,\mathcal{T}} \Lambda_M(\pi_\infty; G_M)^m \operatorname{level}(K_f; G_M^+)^\varepsilon.$$

for all $\nu \in \mathcal{T}$. To estimate $\Lambda_M(\pi_\infty; G_M)$ we first recall the definition of Vogan's definition of a norm on $\|\cdot\|$ on $\Pi(K_\infty)$. Let χ_μ be the highest weight of an arbitrary irreducible constituent of $\mu|_{K^0_\infty}$ with respect to a maximal torus of K^0_∞ and the choice of a system of positive roots. Let ρ be the half sum of all positive roots with multiplicities. For $\mu \in \Pi(K_\infty)$ the norm $\|\mu\|$ is defined by $\|\mu\| = \|\chi_\mu + 2\rho\|^2$. A minimal K_∞-type of a representation of $G(\mathbb{R})$ is then a K_∞-type minimizing $\|\cdot\|$. For $\pi \in \Pi(M(\mathbb{A}))$ denote by λ_{π_∞} the Casimir eigenvalue of the restriction of π_∞ to $M(\mathbb{R})^1$. Let

$$\Lambda_\pi = \min_\tau \sqrt{\lambda^2_{\pi_\infty} + \lambda^2_\tau}, \tag{6.13}$$

where τ runs over the lowest $\mathbf{K}_\infty$-types of the induced representation $\operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty)$. Then by [FLM2, (10)] we have

$$1 \le \Lambda_M(\pi; G_M) \le 1 + \Lambda^2_\pi. \tag{6.14}$$

Now observe that $\dim \mathcal{A}^2_\pi(P)^{K_f,\nu} = 0$, unless $[\operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty)|_{K_\infty} : \nu] > 0$. Thus for a minimal K_∞-type τ of $\operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty)$ one has $\lambda^2_\tau \le \lambda^2_\nu$. Since $\mathcal{T}$ is finite, there exists $C > 0$ such that

$$\Lambda_\pi \le C(1 + |\lambda_{\pi_\infty}|) \tag{6.15}$$

for all $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$ with $\dim \mathcal{A}^2_\pi(P)^{K_f,\nu} \neq 0$. Thus it follows that for $t \ge 1$, (6.9) can be estimated by a constant times

$$\sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))} \sum_{\nu \in \mathcal{T}} \dim\left(\mathcal{A}^2_\pi(P)^{K_f,\nu}\right) e^{-t(\tau(\Omega) - c(\pi_\infty))} (1 + |\lambda_{\pi_\infty}|)^m \operatorname{level}(K_f; G_M^+)^\varepsilon. \tag{6.16}$$

To continue with the estimation, we need the following lemma.

Lemma 6.1. *Let $P = MAN$ be a parabolic subgroup of G and let $K^M_\infty = M(\mathbb{R}) \cap K_\infty$. Let $(\tau, V_\tau) \in \operatorname{Rep}(G(\mathbb{R}))$. Assume that $\tau \not\cong \tau_\theta$. There exists $\delta > 0$ such that for all $(\xi, W_\xi) \in \Pi(M(\mathbb{R})^1)$ satisfying $\dim(W_\xi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^{K^M_\infty} \neq 0$ one has*

$$\tau(\Omega) - c(\xi) \ge \delta.$$

Proof. First consider the case $P = G$. In the proof of Lemma 4.1 in [BV] it is shown that there exists $\delta > 0$ such that

$$\tau(\Omega) - \pi(\Omega) \ge \delta \tag{6.17}$$

for each irreducible unitary representation π of $G(\mathbb{R})$ for which

$$\operatorname{Hom}_{K_\infty}(\Lambda^p \mathfrak{p} \otimes V_\tau, \pi) \neq 0.$$

In fact, the proof goes through for every unitary representation π of $G(\mathbb{R})$ such that $\pi(\Omega)$ is a scalar (see [BV, §II.6].

Now let $P = MAN$ be a proper parabolic subgroup of G. Let $\xi \in \Pi(M(\mathbb{R})^1)$ with $\dim(W_\xi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^{K_\infty^M} \neq 0$ and $\lambda \in \mathfrak{a}^\star$. Consider the induced representation $\pi_{\xi,\lambda}$. By Frobenius reciprocity and the assumption on ξ we have

$$\dim\left(W_\xi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau\right)^{K_\infty^M} = \dim\left(\mathcal{H}_{\xi,\lambda} \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau\right)^{K_\infty} \neq 0.$$

Recall that $\pi_{\xi,\lambda}(\Omega)$ is a scalar given by (6.6). Thus by (6.17) it follows that

$$\tau(\Omega) - \pi_{\xi,\lambda}(\Omega) \geq \delta.$$

Using (6.6) we obtain

$$\tau(\Omega) - c(\xi) \geq \delta - \|\lambda\|^2$$

for every $\lambda \in \mathfrak{a}^\star$. Hence $\tau(\Omega) - c(\xi) \geq \delta$, which proves the lemma. $\square$

Given $\lambda > 0$, let

$$\Pi_{\text{dis}}(M(\mathbb{A}); \lambda) := \left\{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))\colon |\lambda_{\pi_\infty}| \leq \lambda\right\}.$$

Let $d = \dim M(\mathbb{R})^1/K_\infty^M$. As in [Mu1, Proposition 3.5] it follows that for every $\nu \in \Pi(K_\infty)$ there exists $C > 0$ such that

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))_\lambda} \dim \mathcal{A}^2_\pi(P)^{K_f,\nu} \leq C(1 + \lambda^{d/2}) \tag{6.18}$$

for all $\lambda \geq 0$.

Put

$$\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}} = \bigoplus_{\nu \in \mathcal{T}} \mathcal{A}^2_\pi(P)^{K_f,\nu},$$

where $\mathcal{T}$ is defined by (6.7).

Lemma 6.2. *For every $R \geq 0$ we have*

$$\sum_{\substack{\pi \in \Pi_{\text{dis}}(M(\mathbb{A})) \\ -\lambda_{\pi_\infty} \leq R}} \dim(\mathcal{A}^2_\pi(P)^{K_f,\sigma}) < \infty.$$

Proof. By passing to a subgroup of finite index, we may assume that $K_f = \prod_{p<\infty} K_p$. Let $K_{M,f} = K_f \cap M(\mathbb{A}_f)$ and $K_{M.\infty} = K_\infty \cap M(\mathbb{R})$. For $\pi \in \Pi(M(\mathbb{A}))$ and $\tau \in \Pi(K_{M,\infty})$ let $\mathcal{H}_{\pi_\infty}(\tau)$ denote the τ-isotypical subspace of the representation space $\mathcal{H}_{\pi_\infty}$. Let

$$m_\pi = \dim \operatorname{Hom}(\pi, L^2_{\text{dis}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A})), \tag{6.19}$$

i.e., m_π is the multiplicity with which π occurs in regular representation of $M(\mathbb{A})$ in $L^2(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))$. Arguing as in the proof of Proposition 3.5 in [Mu1], it suffices to show that for every $\tau \in \Pi(K_{M,\infty})$

$$\sum_{\substack{\pi \in \Pi_{\text{dis}}(M(\mathbb{A})) \\ -\lambda_{\pi_\infty} \leq R}} m_\pi \dim(\mathcal{H}_{\pi_f}^{K_{M,f}}) \cdot \dim(\mathcal{H}_{\pi_\infty}(\tau)) < \infty.$$

Let $\Gamma_M \subset M(\mathbb{R})$ be an arithmetic subgroup. Let $\Omega_{M(\mathbb{R})^1}$ be the Casmir element of $M(\mathbb{R})^1$ and let A_τ be the differential operator in $C^\infty(\Gamma_M\backslash M(\mathbb{R})^1;\tau)$ which is induced by $-\Omega_{M(\mathbb{R})^1}$. Let $\bar{A}_\tau$ be the self-adjoint extension of A_τ in L^2. Proceeding as in the proof of Lemma 3.2 of [Mu1], it follows that it suffices to show that for every $R\geq 0$, the number of eigenvalues λ_i of $\bar{A}_\tau$ (counted with multiplicities), satisfying $\lambda_i\leq R$, is finite. Let Δ_τ be the Bochner-Laplace operator and let λ_τ be the Casimir eigenvalue of τ. Then by ? we have

$$A_\tau = \Delta_\tau - \lambda_\tau \operatorname{Id}. \tag{6.20}$$

Using that $\Delta_\tau\geq 0$ and the counting function of the eigenvalues of Δ_τ has a polynomial bound (see [Mu3]), the lemma follows. □

Let $\delta>0$ be as in Lemma 6.1. Put $c=\delta/2$. It follows from Lemma 6.1 that for $t\geq 1$, (6.16) can be estimated by

$$e^{-ct}\sum_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))}\sum_{\nu\in\mathcal{T}}\dim\left(\mathcal{A}^2_\pi(P)^{K_f,\nu}\right)e^{-t(\tau(\Omega)-c(\pi_\infty))/2}(1+|\lambda_{\pi_\infty}|)^m\operatorname{level}(K_f;G_M^+)^\varepsilon, \tag{6.21}$$

where $m\in\mathbb{N}$ is sufficiently large. Now observe that $\tau(\Omega)\geq 0$. Thus by (6.5) we get

$$\tau(\Omega)-c(\pi_\infty)\geq -\lambda_{\pi_\infty}. \tag{6.22}$$

By Lemma 6.2, there are only finitely many $\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))$ with $\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}}\neq 0$ and $-\lambda_{\pi_\infty}\leq 0$. Decompose the sum over π in (6.21) in two summands $\Sigma_1(t)$ and $\Sigma_2(t)$, where in $\Sigma_1(t)$ the summation runs over all π with $-\lambda_{\pi_\infty}\leq 0$. Using (6.22) it follows that for $-\lambda_{\pi_\infty}>0$ we have

$$\tau(\Omega)-c(\pi_\infty)\geq|\lambda_{\pi_\infty}|$$

Thus for every $l\in\mathbb{N}$, K_f and $t\geq 1$ we have

$$\Sigma_2(t)\ll_l e^{-ct}\sum_{\substack{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))\\ -\lambda_{\pi_\infty}>0}}\sum_{\nu\in\mathcal{T}}\dim\left(\mathcal{A}^2_\pi(P)^{K_f,\nu}\right)(1+|\lambda_{\pi_\infty}|)^{-l}\operatorname{level}(K_f,G_M^+)^\varepsilon. \tag{6.23}$$

To estimate $\Sigma_1(t)$ we note that it follows from the proof of Lemma 6.2 that there exists $C_1\in\mathbb{R}$, which depends on τ, but is independent of K_f, such that $C_1\leq -\lambda_{\pi_\infty}$ for all $\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))$ with $\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}}\neq 0$. Thus for every $l\in\mathbb{N}$, K_f, and $t\geq 1$ we get

$$\Sigma_1(t)\ll_l e^{-ct}\sum_{\substack{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))\\ -\lambda_{\pi_\infty}\leq 0}}\sum_{\nu\in\mathcal{T}}\dim\left(\mathcal{A}^2_\pi(P)^{K_f,\nu}\right)(1+|\lambda_{\pi_\infty}|)^{-l}\operatorname{level}(K_f,G_M^+)^\varepsilon. \tag{6.24}$$

Putting everything together we obtain the following lemma.

Lemma 6.3. *Suppose that G satisfies properties (TWN)* [FLM2, Definition 5.2] *and (BD)* [FLM2, Definition 5.9]. *Let $\tau\in\operatorname{Rep}(G(\mathbb{R}))$. Assume that $\tau\not\cong\tau_\theta$. Let M be a proper Levi subgroup of G. There exists $c>0$, independent of K_f, and for every $l\in\mathbb{N}$ and $\varepsilon>0$ there exists $C>0$, which is independent of K_f, such that*

$$|J_{\mathrm{spec},M}(\phi_t^{\tau,p})|\leq Ce^{-ct}\sum_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))}\sum_{\nu\in\mathcal{T}}\dim\left(\mathcal{A}^2_\pi(P)^{K_f,\nu}\right)(1+|\lambda_{\pi_\infty}|)^{-l}\operatorname{level}(K_f,G_M^+)^\varepsilon.$$

for $t \geq 1$ *and* $p = 0, \dots, d$.

We now specialize to the case of principal congruence subgroups. Fix a faithful $\mathbb{Q}$-rational representation $\rho\colon G \to \mathrm{GL}(V)$ and a lattice Λ in the representation space V such that the stabilizer of $\widehat{\Lambda} = \widehat{Z} \otimes \Lambda \subset \mathbb{A}_f \otimes V$ in $G(\mathbb{A}_f)$ is the group K_f. Since the maximal compact subgroups of $\mathrm{GL}(\mathbb{A}_f \otimes V)$ are precisely the stabilizers of lattices, it is easy to see that such a lattice exists. For $N \in \mathbb{N}$ let

$$K(N) = \{g \in G(\mathbb{A}_f)\colon \rho(g)v \equiv v \mod N\widehat{\Lambda},\ v \in V\}$$

be the principal congruence subgroup of level N, which is a factorizable normal open subgroup of $\mathbf{K}_f$. Let

$$Y(N) := G(\mathbb{Q})\backslash(\widetilde{X} \times G(\mathbb{A}_f)/K(N) \tag{6.25}$$

be the adelic quotient associated to $K(N)$. Fix $P = M \cdot U \in \mathcal{P}(M)$. By (6.3) have

$$\begin{aligned} \dim \mathcal{A}^2_\pi(P)^{K(N),\nu} &= m_\pi \dim \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi)^{(K(N),\nu} \\ &= m_\pi \dim \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R}}(\pi_\infty)^\nu \dim \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f)^{K(N)}. \end{aligned} \tag{6.26}$$

Note that $\dim \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R}}(\pi_\infty)^\nu$ is bounded by $(\dim \nu)^2$. Let $\mathbf{K}_f \subset G(\mathbb{A}_f)$ be the standard maximal compact subgroup. Let Λ be a set of coset representatives for the double cosets $(P(\mathbb{A}_f) \cap \mathbf{K}_f)\backslash \mathbf{K}_f/K(N)$. Since $K(N)$ is a normal subgroup of $\mathbf{K}_f$ of finite index, it follows from [Re, Lemme, III.2] that the map $\varphi \mapsto (\varphi(g))_{g\in\Lambda}$ defines an isomorphism

$$\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f)^{K(N)} \cong \oplus_{g\in\Lambda}(\pi_f)^{P(\mathbb{A}_f)\cap K(N)}.$$

Thus we get

$$\dim \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f)^{K(N)} \leq [\mathbf{K}_f\colon (\mathbf{K}_f \cap P(\mathbb{A}_f))K(N)] \dim(\pi_f^{K_M(N)}).$$

Using the factorization $\mathbf{K}_f \cap P(\mathbb{A}_f) = (K_f \cap M(\mathbb{A}_f))(\mathbf{K}_f \cap U(\mathbb{A}_f))$, we can write

$$\begin{aligned} [\mathbf{K}_f\colon (\mathbf{K}_f \cap P(\mathbb{A}_f))K(N)] = \mathrm{vol}(K_M(N)) \,\mathrm{vol}(K(N))^{-1}[\mathbf{K}_f \cap U(\mathbb{A}_f)\colon K(N) \cap U(\mathbb{A}_f)]^{-1} \\ [K(N) \cap P(\mathbb{A}_f)\colon (K(N) \cap M(\mathbb{A}_f))(K(N) \cap U(\mathbb{A}_f)]. \end{aligned}$$

The index $[K(N) \cap P(\mathbb{A}_f)\colon (K(N) \cap M(\mathbb{A}_f))(K(N) \cap U(\mathbb{A}_f)]$ is bounded independently of N. Furthermore, identifying U with its Lie algebra $\mathfrak{u}$ via the exponential map, which is an isomorphism of affine varieties, it follows that there exist $C_1, C_2 > 0$ such that

$$C_1 N^{-\dim U} \leq [\mathbf{K}_f \cap U(\mathbb{A}_f)\colon K(N) \cap U(\mathbb{A}_f)]^{-1} \leq C_2 N^{-\dim U}$$

for all $N \in \mathbb{N}$. Therefore there exist $C > 0$, independent of N, such that

$$\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f)^{K(N)} \leq C N^{-\dim U} \,\mathrm{vol}(K(N))^{-1} \,\mathrm{vol}(K_M(N)) \dim \pi_f^{K_M(N)}.$$

Let

$$\phi^{\tau,p}_{t,N} = h^{\tau,p}_t \otimes \chi_{K(N)}. \tag{6.27}$$

Then $\phi_{t,N}^{\tau,p} \in \mathcal{C}(G(\mathbb{A})^1, K(N))$. Combined with Lemma 6.3 and (6.26) it follows that there exists $C > 0$ such that

(6.28)
$$\begin{aligned}\frac{1}{\operatorname{vol}(Y(N))}|J_{\text{spec},M}(\phi_{t,N}^{\tau,p})| \leq & Ce^{-ct} N^{-\dim U+\varepsilon} \\ & \cdot \operatorname{vol}(K_M(N)) \sum_{\pi\in\Pi_{\text{dis}}(M(\mathbb{A}))^{\mathcal{T}}} m_\pi (1+|\lambda_{\pi_\infty}|)^{-l} \dim \pi_f^{K_M(N)}.\end{aligned}$$

for all $t \geq 1$ and $N \in \mathbb{N}$. For an open compact subgroup $K_{M,f} \subset M(\mathbb{A}_f)$ let $\mu_{K_f}^M$ be the measure on $\Pi(M(\mathbb{R})^1)$ defined by

$$\begin{aligned}\mu_{K_f}^M = & \frac{\operatorname{vol}(K_{M,f})}{\operatorname{vol}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)} \\ & \cdot \sum_{\pi\in\Pi(M(\mathbb{A})^1)} \dim \operatorname{Hom}_{M(\mathbb{A})^1)}(\pi, L^2(M(\mathbb{Q})\backslash M(\mathbb{A})^1)) \dim \pi_f^{K_{M,f}} \delta_{\pi_\infty}.\end{aligned}$$

It follows from [FLM2, Lemma 7.7], together with [FLM2, Proposition 5.5] and [FLM2, Theorem 5.15] that the collection of measures $\{\mu_{K_M(N)}^M\}_{N\in\mathbb{N}}$ is polynomially bounded in the sense of [FLM2, Definition 6.2]. For $l \in \mathbb{N}$ let $g_{l,\mathcal{T}}$ be the non-negative function on $\Pi(G(\mathbb{R}))$ defined by

$$g_{l,\mathcal{T}}(\pi) := \begin{cases} (1+|\lambda_\pi|)^{-l}, & \text{if } \pi \in \Pi(G(\mathbb{R}))_{\mathcal{T}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then it follows from [FLM2, Proposition 6.1, (4)] that there exists $l \in \mathbb{N}$, which depends only on $\mathcal{T}$, such that

$$\mu_{K_f}^M(g_{l,\mathcal{T}}) = \frac{\operatorname{vol}(K_M(N))}{\operatorname{vol}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)} \sum_{\pi\in\Pi_{\text{dis}}(M(\mathbb{A}))^{\mathcal{T}}} (1+|\lambda_{\pi_\infty}|)^{-l} m_\pi \dim \pi_f^{K_M(N)} \tag{6.29}$$

is bounded independently of $N \in \mathbb{N}$. Together with (6.28) we obtain the following lemma.

Lemma 6.4. *Suppose that G satisfies properties (TWN)* [FLM2, Definition 5.2] *and (BD)* [FLM2, Definition 5.9]. *Let $M \in \mathcal{L}$, $M \neq G$. Let $P = M \cdot U \in \mathcal{P}(M)$ and let $\tau \in \operatorname{Rep}(G(\mathbb{R}))$ such that $\tau \not\cong \tau_\theta$. There exist $C, c, \delta > 0$ such that*

$$\frac{1}{\operatorname{vol}(Y(N))}|J_{\text{spec},M}(\phi_{t,N}^{\tau,p})| \leq Ce^{-ct} N^{-\delta} \tag{6.30}$$

for all $t \geq 1$, $p = 0, \dots, d$, and $N \in \mathbb{N}$.

Now we consider the case $M = G$. Then by definition of $\phi_{t,N}^{\tau,p}$ we have

$$J_{\text{spec},G}(\phi_{t,N}^{\tau,p}) = \sum_{\pi\in\Pi_{\text{dis}}(G(\mathbb{A})^1)} m_\pi \operatorname{Tr} \pi(\phi_{t,N}^{\tau,p}) = \sum_{\pi\in\Pi_{\text{dis}}(G(\mathbb{A})^1)} m_\pi \dim(\pi_f^{K(N)}) \operatorname{Tr} \pi_\infty(h_t^{\tau,p}). \tag{6.31}$$

Now observe that by [MP2, (4.18), (4.19)] we have

$$\operatorname{Tr}\pi_\infty(h_t^{\tau,p}) = e^{t(\pi_\infty(\Omega)-\tau(\Omega))}\dim(\mathcal{H}_{\pi_\infty}\otimes\Lambda^p\mathfrak{p}^\star\otimes V_\tau)^{K_\infty}.$$

Furthermore, for $\nu\in\Pi(K_\infty)$ we have

$$[\pi_\infty|_{K_\infty}:\nu]\le(\dim\nu)$$

(see [Kn, Theorem 8.1]). Thus there exists $C>0$ such that

$$\frac{1}{\operatorname{vol}(Y(N))}|J_{\mathrm{spec},G}(\phi_{t,N}^{\tau,p})|\le C\operatorname{vol}(K(N))\sum_{\pi\in\Pi_{\mathrm{dis}}(G(\mathbb{A})^1)^{\mathcal{T}}} m_\pi\dim(\pi_f^{K(N)})e^{t(\pi_\infty(\Omega)-\tau(\Omega))}$$

for all $t>0$ and $N\in\mathbb{N}$. As above, put $\lambda_{\pi_\infty}:=\pi_\infty(\Omega)$. If we argue as in the proof of Lemma 6.3, it follows that there exists $c>0$ and for all $l\in\mathbb{N}$ there exist $C_l>0$ such that

$$\begin{aligned}\frac{1}{\operatorname{vol}(Y(N))}|J_{\mathrm{spec},G}(\phi_{t,N}^{\tau,p})|\le& C_le^{-ct}\operatorname{vol}(K(N))\\&\cdot\sum_{\pi\in\Pi_{\mathrm{dis}}(G(\mathbb{A})^1)^{\mathcal{T}}} m_\pi(1+|\lambda_{\pi_\infty}|)^{-l}\dim(\pi_f^{K(N)})\end{aligned}\tag{6.32}$$

for all $t\ge1$ and $N\in\mathbb{N}$. Using that (6.29) for $M=G$, we get

Lemma 6.5. *Let $\tau\in\operatorname{Rep}(G(\mathbb{R}))$ such that $\tau\not\cong\tau_\theta$. There exist $C,c>0$ such that*

$$\frac{1}{\operatorname{vol}(Y(N))}|J_{\mathrm{spec},G}(\phi_{t,N}^{\tau,p})|\le Ce^{-ct}\tag{6.33}$$

for all $t\ge1$, $p=0,\dots,d$, and $N\in\mathbb{N}$.

Combining Lemmas 6.4, Lemma 6.5 and (?) it follows that there exist $C,c>0$ such that

$$\frac{1}{\operatorname{vol}(Y(N))}|J_{\mathrm{spec}}(\phi_{t,N}^{\tau,p})|\le Ce^{-ct}\tag{6.34}$$

for all $t\ge1$, $p=0,\dots,d$, and $N\in\mathbb{N}$.

Let $\Delta_{p,N}(\tau)$ be the Laplace operator on E_τ-valued p-forms. By (6.1) we have

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,N}(\tau)}\right)=J_{\mathrm{spec}}(\phi_{t,N}^{\tau,p})$$

and by (6.34) we obtain

Proposition 6.6. *Suppose that G satisfies properties (TWN)* [FLM2, Definition 5.2] *and (BD)* [FLM2, Definition 5.9]*. There exist $C,c>0$ such that*

$$\frac{1}{\operatorname{vol}(Y(N))}|\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,N}(\tau)}\right)|\le Ce^{-ct}$$

for all $t\ge1$, $p=0,\dots,d$, and $N\in\mathbb{N}$.

Recall that by [FLM2, Prop. 5.5, Prop 5.15] the properties (TWN) and (BD) are satisfied for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. Hence we get the following corollary.

Corollary 6.7. *Let $G = \mathrm{GL}(n)$ or $\mathrm{SL}(n)$. There exist $C, c > 0$ such that*

$$\frac{1}{\mathrm{vol}(Y(N))} | \operatorname{Tr}_{\mathrm{reg}} \left(e^{-t\Delta_{p,N}(\tau)} \right) | \leq C e^{-ct}$$

for all $t \geq 1$, $p = 0, \dots, d$, and $N \in \mathbb{N}$.

7. Modification of the heat kernel

In order to study the short time behavior of the regularized trace of the heat operator with the help of the trace formula, we need to show that we can replace $h_t^{\tau,p}$ by an appropriate compactly supported test function without changing the asymptotic behavior as $t \to 0$. We introduced such a modification of $h_t^{\tau,p}$ already in [MzM]. The main purpose of this section is to establish estimations which are uniform in the lattice.

In this section we assume that $G = \mathrm{GL}(n)$. Let $G(\mathbb{R})^1$ be defined by (2.19). Let $d(x, y)$ denote the geodesic distance of $x, y \in \widetilde{X}$. On $G(\mathbb{R})^1$ we introduce the function r by

$$r(g) := d(gK_\infty, K_\infty), \quad g \in G(\mathbb{R})^1.$$

For $R > 0$ let

$$B_R := \{g \in G(\mathbb{R})^1 \colon r(g) < R\}. \tag{7.1}$$

We need the following auxiliary lemma.

Lemma 7.1. *There exist $C, c > 0$ such that*

$$\int_{G(\mathbb{R})^1} e^{-r^2(g)/t} dg \leq C e^{ct}$$

for $t > 0$.

Proof. Note that $r(g)$ is bi-K_∞-invariant. Thus using the Cartan decomposition $G(\mathbb{R})^1 = K_\infty A^+ K_\infty$, we get

$$\int_{G(\mathbb{R})^1} e^{-r^2(g)/t} dg = \int_{A^+} e^{-r^2(a)/t} \delta(a) da,$$

where

$$\delta(\exp H) = \prod_{\alpha \in \Delta +} (\sinh \alpha(H))^{m_\alpha}, \quad H \in \mathfrak{a}^+$$

(see [He, Chapt. I, Theorem 5.8]). Let $a = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in A^+$. Then $\lambda_j > 1$, $j = 1, \dots, n$, and

$$r(a)^2 = \sum_{j=1}^n (\log \lambda_j)^2.$$

Furthermore, note that

$$\int_0^\infty e^{-(\log \lambda)^2/t} e^{c\lambda} d\lambda = \frac{\sqrt{\pi t}}{2} \exp(c^2 t)(1 - \mathrm{erf}(c\sqrt{t})),$$

where $\mathrm{erf}(x)$ is the error function (see [GR, 3.322,2]. This proves the claim. $\square$

Let $f \in C^\infty(\mathbb{R})$ such that $f(u) = 1$, if $|u| \leq 1/2$, and $f(u) = 0$, if $|u| \geq 1$. Let $\varphi_R \in C_c^\infty(G(\mathbb{R})^1)$ be defined by

$$\varphi_R(g) := f\left(\frac{r(g)}{R}\right). \tag{7.2}$$

Then we have $\operatorname{supp}\varphi_R \subset B_R$. Extend φ_R to $G(\mathbb{R})$ by

$$\varphi_R(g_\infty z) = \varphi_R(g_\infty), \quad g_\infty \in G(\mathbb{R})^1,\ z \in A_G(\mathbb{R})^0.$$

Define $\widetilde{h}_{t,R}^{\tau,p} \in C^\infty(G(\mathbb{R}))$ by

$$\widetilde{h}_{t,R}^{\tau,p}(g_\infty) := \varphi_R(g_\infty) h_t^{\tau,p}(g_\infty), \quad g_\infty \in G(\mathbb{R}). \tag{7.3}$$

Then the restriction of $\widetilde{h}_{t,R}^{\tau,p} \otimes \chi_{K(N)}$ to $G(\mathbb{A})^1$ belongs to $C_c^\infty(G(\mathbb{A})^1)$. Let $K(N) \subset \mathrm{GL}(n,\mathbb{A}_f)$ be the principal congruence subgroup of level N and let $Y(N)$ be the adelic quotient defined by (6.25).

Proposition 7.2. *There exist constants $C_1, C_2, C_3 > 0$ such that*

$$\frac{1}{\mathrm{vol}(Y(N))}\left|J_{\mathrm{spec}}(h_t^{\tau,p} \otimes \chi_{K(N)}) - J_{\mathrm{spec}}(\widetilde{h}_{t,R}^{\tau,p} \otimes \chi_{K(N)})\right| \leq C_1 e^{-C_2 R^2/t + C_3 t}$$

for all $N \in \mathbb{N}$, $p = 0, \dots, d$, $t > 0$ and $R \geq 1$.

Proof. Let $\psi_R := 1 - \varphi_R$. Then

$$J_{\mathrm{spec}}(h_t^{\tau,p} \otimes \chi_{K(N)}) - J_{\mathrm{spec}}(\widetilde{h}_{t,R}^{\tau,p} \otimes \chi_{K(N)}) = J_{\mathrm{spec}}(\psi_R h_t^{\tau,p} \otimes \chi_{K(N)}).$$

Now we use the refined spectral expansion (5.8). Let $M \in \mathcal{L}$ and let $J_{\mathrm{spec},M}$ be the distribution on the right hand side of (5.8), which corresponds to M. Let

$$\Delta_G = -\Omega + 2\Omega_{K_\infty},$$

where Ω (resp. Ω_{K_∞}) denotes the Casimir operator of $G(\mathbb{R})^1$(resp. K_∞). Observe that $\psi_R h_t^{\tau,p} \otimes \chi_{K(N)}$ belongs to $\mathcal{C}(G(\mathbb{A})^1)$ and the proof of Lemma 7.2 and Corollary 7.4 in [FLM1] extends to $h \in \mathcal{C}(G(\mathbb{A})^1)$. Thus there exists $k \geq 1$ such that for any $\varepsilon > 0$ we have

$$\begin{aligned} \frac{1}{\mathrm{vol}(Y(N))} J_{\mathrm{spec},M}(\psi_R h_t^{\tau,p} \otimes \chi_{K(N)}) &= \frac{1}{\mathrm{vol}(G(\mathbb{Q})\backslash G(A)^1)} J_{\mathrm{spec},M}(\psi_R h_t^{\tau,p} \otimes \mathbf{1}_{K(N)}) \\ &\ll_{\mathcal{T},\varepsilon} \|(\mathrm{Id} + \Delta_G)^k(\psi_R h_t^{\tau,p})\|_{L^1(G(\mathbb{R})^1)} N^{(\dim M - \dim G)/2 + \varepsilon} \end{aligned} \tag{7.4}$$

for all $N \in \mathbb{N}$, $p = 0, \dots, d$, $t > 0$, and $R > 0$.

Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})^1$ and let $Y_1, \dots, Y_r$ be an orthonormal basis of $\mathfrak{g}$. Then $\Delta_G = -\sum_i Y_i^2$. Denote by ∇ the canonical connection on $G(\mathbb{R})^1$. Then it follows that there exists $C > 0$ such that

$$|(\mathrm{Id} + \Delta_G)^k h(g)| \leq C \sum_{l=0}^{2k} \|\nabla^l h(g)\|, \quad g \in G(\mathbb{R})^1,$$

for all $h \in C^\infty(G(\mathbb{R})^1)$. Let $m = \dim G(\mathbb{R})^1$. By [Mu1, Proposition 2.1] it follows that for every $T > 0$ and $j \in \mathbb{N}$ there exist $C_2, C_3 > 0$ such that

$$\|\nabla^j h_t^{\tau,p}(g)\| \leq C_2 t^{-(m+j)/2} e^{-C_3 r^2(g)/t}, \quad g \in G(\mathbb{R})^1,$$

for all $0 < t \leq T$. Using the semigroup property and arguing as in the proof of Corollary 1.6 in [Do2], it follows that there exist $A_1, A_2, A_3 > 0$ such that

$$\|\nabla^j h_t^{\tau,p}(g)\| \leq A_1 t^{-(m+j)/2} e^{-A_2 r^2(g)/t + A_3 t}, \quad g \in G(\mathbb{R})^1, \tag{7.5}$$

for all $t > 0$. Now observe that for every $j \in \mathbb{N}$ there exists $C_j > 0$ such that

$$\|\nabla^j \psi_R\| \leq C_j$$

for all $R \geq 1$. Since ψ_R vanishes on B_R, it follows from (7.5) that there exist $C_4, C_5, C_6 > 0$ such that

$$\sum_{l=0}^{2k} \|\nabla^l(\psi_R h_t^{\tau,p})(g)\| \leq C_4 e^{-C_5 R^2/t + A_3 t} e^{-C_6 r^2(g)/t}$$

for all $g \in G(\mathbb{R})^1$, $t > 0$, and $R \geq 1$. Using Lemma 7.1, it follows that there exist $C_1, C_2, C_3 > 0$ such that

$$\|(\mathrm{Id} + \Delta_G)^k(\psi_R h_t^{\tau,p})\|_{L^1(G(\mathbb{R})^1)} \leq C_1 e^{-C_2 R^2/t + C_3 t} \tag{7.6}$$

for all $t > 0$ and $R \geq 1$. Combined with (7.4) it follows that for every $\varepsilon > 0$ we have

$$\frac{1}{\mathrm{vol}(Y(N))} J_{\mathrm{spec},M}(\psi_R h_t^{\tau,p} \otimes \chi_{K(N)}) \ll_\varepsilon e^{-cR^2/t} N^{(\dim M - \dim G)/2 + \varepsilon}$$

for all $N \in \mathbb{N}$, $p = 0, \dots, d$, and $t > 0$. and $R \geq 1$. Especially, there exist $C_1, C_2, C_3 > 0$ such that

$$\frac{1}{\mathrm{vol}(Y(N))} |J_{\mathrm{spec},M}(\psi_R h_t^{\tau,p} \otimes \chi_{K(N)})| \leq C_1 e^{-C_2 R^2/t + C_3 t} \tag{7.7}$$

for all $N \in \mathbb{N}$, $p = 0, \dots, d$, and $t > 0$, and $R \geq 1$.

It remains to consider the case $M = G$. Then we have

$$\begin{aligned} J_{\mathrm{spec},G}(\psi_R h_t^{\tau,p} \otimes \chi_{K(N)}) &= \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A})^1)} m_\pi \operatorname{Tr} \pi(\psi_R h_t^{\tau,p} \otimes \chi_{K(N)}) \\ &= \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A})^1)} m_\pi \dim(\pi_f^{K(N)}) \operatorname{Tr} \pi_\infty(\psi_R h_t^{\tau,p}). \end{aligned}$$

For $\nu \in \Pi(K_\infty)$ denote by $\mathcal{H}_{\pi_\infty}(\nu)$ the ν-isotypic subspace. Let

$$\mathcal{H}_{\pi_\infty}^{\mathcal{T}} = \sum_{\nu \in \mathcal{T}} \mathcal{H}_{\pi_\infty}(\nu).$$

Then for every $k \in \mathbb{N}$ we have

$$|\operatorname{Tr} \pi_\infty(\psi_R h_t^{\tau,p})| \leq \|(\mathrm{Id} + \pi_\infty(\Delta_G))^{-k}\|_{1,\mathcal{H}_{\pi_\infty}^{\mathcal{T}}} \|(\mathrm{Id} + \Delta_G)^k(\psi_R h_t^{\tau,p})\|_{L^1(G(\mathbb{A})^1)}.$$

Now observe that $\pi_\infty(\Delta_G)$ acts on $\mathcal{H}_{\pi_\infty}(\nu)$ by the scalar $-\lambda_{\pi_\infty}+2\lambda_\nu$, where λ_{π_∞} and λ_ν are the Casimir eigenvalues of π_∞ and π_ν, respectively. Furthermore, by [Mu2, Lemma 6.1] we have

$$-\lambda_{\pi_\infty}+\lambda_\nu\geq 0 \tag{7.8}$$

for $\mathcal{H}_{\pi_f}^{K(N)}\neq 0$ and $\mathcal{H}_{\pi_\infty}(\nu)\neq 0$. Moreover $\lambda_\nu\geq 0$. Thus $1-\lambda_{\pi_\infty}+2\lambda_\nu>0$ and we get

$$\|(\operatorname{Id}+\pi_\infty(\Delta_G))^{-k}\|_{1,\mathcal{H}^{\mathcal{T}}_{\pi_\infty}}\leq\sum_{\nu\in\mathcal{T}}\dim(\nu)(1-\lambda_{\pi_\infty}+2\lambda_\nu)^{-k}.$$

Using (7.8) we get

$$(1-\lambda_{\pi_\infty}+2\lambda_\nu)^2\geq\frac{1}{4}(1+\lambda_{\pi_\infty}^2+\lambda_\nu^2)\geq\frac{1}{4}(1+|\lambda_{\pi_\infty}|)^2.$$

Thus we get

$$\|(\operatorname{Id}+\pi_\infty(\Delta_G))^{-k}\|_{1,\mathcal{H}^{\mathcal{T}}_{\pi_\infty}}\leq\frac{1}{4}\dim(\mathcal{H}^{\mathcal{T}}_{\pi_\infty})(1+|\lambda_{\pi_\infty}|)^{-k}.$$

Together with (7.6) it follows that for every $k\in\mathbb{N}$ there exists $C_k>0$ such that

$$|\operatorname{Tr}\pi_\infty(\psi_R h_t^{\tau,p})|\leq C_k e^{-C_2R^2/t+C_3t}(1+|\lambda_{\pi_\infty}|)^{-k}$$

for all $t>0$ and $R\geq 1$. This gives

$$\begin{aligned}\frac{1}{\operatorname{vol}(Y(N))}&|J_{\mathrm{spec},G}(\psi_R h_t^{\tau,p}\otimes\chi_{K(N)})|\\&\leq C_k e^{-C_2R^2/t+C_3t}\operatorname{vol}(K(N))\sum_{\pi\in\Pi\mathrm{dis}(G(\mathbb{A})^1)}m_\pi\dim(\pi_f^{K(N)})(1+|\lambda_{\pi_\infty}|)^{-k}\end{aligned}$$

for all $t>0$ and $R\geq 1$. As above it follows from [FLM2, Proposition 6.1, (4)] that there exists $k\in\mathbb{N}$, which depends only on $\mathcal{T}$, such that $\operatorname{vol}(K(N))$ times the sum is bounded independently of $N\in\mathbb{N}$. Hence there exist $C_1,C_2,C_3>0$ such that

$$\frac{1}{\operatorname{vol}(Y(N))}|J_{\mathrm{spec},G}(\psi_R h_t^{\tau,p}\otimes\chi_{K(N)})|\leq C_1e^{-C_2R^2/t+C_3t}$$

for all $t>0$, $p=0,\dots,d$, $N\in\mathbb{N}$, and $R\geq 1$. This completes the proof of the proposition. □

Proposition 7.2 allows us to replace $h_t^{\tau,p}$ by a compactly supported function.

8. The geometric side of the trace formula

In this section we assume that $G=\mathrm{GL}(n)$. To study the behavior of the regularized trace for small time, we use the geometric side J_{geo} of the Arthur trace formula. Consider the equivalence relation on $G(\mathbb{Q})$ defined by $\gamma\sim\gamma'$ whenever the semisimple parts of γ and γ' are $G(\mathbb{Q})$-conjugate, and denote by $\mathcal{O}_G$ the set of all resulting equivalence classes. They are indexed by the conjugacy classes of semisimple elements of $G(\mathbb{Q})$. Then the coarse geometric expansion of J_{geo} is

$$J_{\text{geo}}(f) = \sum_{\mathfrak{o}\in\mathcal{O}_G} J_{\mathfrak{o}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1), \tag{8.1}$$

where the distributions are the value at $T = 0$ of a polynomial $J_{\mathfrak{o}}^T(f)$ defined in [Ar1]. When $\mathfrak{o}$ consists of the unipotent elements of $G(\mathbb{Q})$, we write $J_{\text{unip}}(f)$ for $J_{\mathfrak{o}}(f)$.

Fix $R \geq 1$ and put $\varphi := \varphi_R$. Put

$$\widetilde{h}_t^{\tau,p} := \varphi h_t^{\tau,p}. \tag{8.2}$$

Lemma 8.1. *There exists $N_0 \in \mathbb{N}$ such that*

$$J_{\text{geo}}(\widetilde{h}_t^{\tau,p} \otimes \chi_{K(N)}) = J_{\text{unip}}(\widetilde{h}_t^{\tau,p} \otimes \chi_{K(N)})$$

for all $N \geq N_0$.

Proof. By definition, the support of $\widetilde{h}_t^{\tau,p}$ is contained in B_R. Then the support of $\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N)}$ is contained in $B_R K(N) \subset B_R\mathbf{K}$, and therefore there are only finitely many classes $\mathfrak{o} \in \mathcal{O}_G$ that contribute to the geometric side of the trace formula (8.1) for the the functions $\widetilde{h}_t^{\tau,p} \otimes \chi_{K(N)}$. Moreover, the only class $\mathfrak{o} \in \mathcal{O}_G$ for which the union of the $G(\mathbb{A})$-conjugacy classes of elements of $\mathfrak{o}$ meets $G(\mathbb{R})K(N)$ for infinitely many $N \in \mathbb{N}$ is the unipotent class. For assume that $\mathfrak{o}$ has this property. Let $\gamma \in \mathfrak{o}$ and let $q \in \mathbb{Q}[X]$ be the characteristic polynomial of the linear map $\gamma - \mathrm{Id} \in \mathrm{End}(\mathbb{C}^n)$. The assumption on $\mathfrak{o}$ implies that every coefficient of q, except the leading coefficient 1, is either arbitrarily close to 0 at some prime p or has absolute value < 1 at infinitely many places. Therefore, necessarily, $q = X^n$, and γ is unipotent. Therefore, the geometric side reduces to $J_{\text{unip}}(\widetilde{h}_t^{\tau,p} \otimes \chi_{K(N)})$ for all but finitely many $N \in \mathbb{N}$. □

To analyze $J_{unip}(f)$ we use Arthur's fine geometric expansion [Ar4, Corollaries 8.3] to express $J_{\text{unip}}(f)$ in terms of weighted orbital integrals. To state the result we recall some facts about weighted orbital integrals. Let S be a finite set of places of $\mathbb{Q}$ containing ∞. Set

$$\mathbb{Q}_S = \prod_{v\in S} \mathbb{Q}_v, \quad \text{and} \quad G(\mathbb{Q}_S) = \prod_{v\in S} G(\mathbb{Q}_v).$$

Let $M \in \mathcal{L}$ and $\gamma \in M(\mathbb{Q}_S)$. The general weighted orbital integrals $J_M(\gamma, f)$ defined in [Ar5] are distributions on $G(\mathbb{Q}_S)$. If γ is such that $M_\gamma = G_\gamma$, then, as the name suggests, $J_M(\gamma, f)$ is given by an integral of the form

$$J_M(\gamma, f) = \left|D(\gamma)\right|^{1/2} \int_{G_\gamma(\mathbb{Q}_S)\backslash G(\mathbb{Q}_S)} f(x^{-1}\gamma x) v_M(x)\, dx,$$

where $D(\gamma)$ is the discriminant of γ [Ar5, p. 231] and $v_M(x)$ is the weight function associated to the (G, M)-family $\{v_P(\lambda, x)\colon P \in \mathcal{P}(M)\}$ defined in [Ar5, p.230]. For general γ

the definition is more complicated. In this case, $J_M(\gamma, f)$ is obtained as a limit of a linear combination of integrals as above. For more details we refer to [Ar8]. Let

$$G(\mathbb{Q}_S)^1 = G(\mathbb{Q}_S) \cap G(\mathbb{A})^1$$

and write $C_c^\infty(G(\mathbb{Q}_S)^1)$ for the space of functions on $G(\mathbb{Q}_S)^1$ obtained by restriction of functions in $C_c^\infty(G(\mathbb{Q}_S))$. If γ belongs to the intersection of $M(\mathbb{Q}_S)$ with $G(\mathbb{Q}_S)^1$, one can obviously define the corresponding weighted orbital integral as linear form on $C_c^\infty(G(\mathbb{Q}_S)^1)$.

Since for $\mathrm{GL}(n)$ all conjugacy classes are stable (in the sense that for any finite set S, two unipotent elements in $G(\mathbb{Q})$ are conjugate in $G(\mathbb{Q}_S)$ if and only if they are conjugate in $G(\mathbb{Q})$), the expression of $J_{\mathrm{unip}}(f)$ in terms of weighted orbital integrals simplifies. For $M \in \mathcal{L}$ let $(\mathcal{U}_M(\mathbb{Q}))$ be the (finite) set of unipotent conjugacy classes of $M(\mathbb{Q})$. Let $F \in C_c^\infty(G(\mathbb{Q}_S)^1)$ and denote by $\mathbf{1}_{K^S}$ the characteristic function of the standard maximal compact subgroup of $G(\mathbb{A}^S)$. Then by [Ar4, Corollary 8.3] there exist constants $a(S, \mathcal{O})$ which depend on the normalization of measures such that

$$J_{\mathrm{unip}}(F \otimes \mathbf{1}_{K^S}) = \mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1)F(1) + \sum_{(M,\mathcal{O}) \neq (G,\{1\})} a^M(S, \mathcal{O}) J_M(\mathcal{O}, F), \tag{8.3}$$

where M runs over $\mathcal{L}$ and $\mathcal{O}$ over $(\mathcal{U}_M(\mathbb{Q}))$. To deal with the S-adic integral, we note that $J_M(\mathcal{O}, f)$ can be decomposed into a sum of products of integrals at ∞ and at the finite places $S_f = S \setminus \{\infty\}$. Suppose that $F = F_\infty \otimes F_f = F_\infty \otimes \bigotimes_{p \in S_f} F_p$ with $F_v \in C^\infty(G(\mathbb{Q}_v))$. Let $L \in \mathcal{L}(M)$ and $Q = LV \in \mathcal{P}(L)$. Define

$$F_{\infty,Q}(m) = \delta_Q(m)^{1/2} \int_{K_\infty} \int_{V(\mathbb{R})} F_\infty(k^{-1} m v k)\, dk\, dv, \quad m \in M(\mathbb{R}), \tag{8.4}$$

and define $F_{f,Q}$ in a similar way. Then for every pair of Levi subgroups $L_1, L_2 \in \mathcal{L}(M)$ there exist constants $d_M^G(L_1, L_2) \in \mathbb{C}$ such that

$$J_M(\mathcal{O}, F) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) J_M^{L_1}(\mathcal{O}_\infty, F_{\infty,Q_1}) J_M^{L_2}(\mathcal{O}_f, F_{f,Q_2}) \tag{8.5}$$

(see [Ar3],[Ar10, (18.7)]) where $Q_i \in \mathcal{P}(L_i)$, and $\mathcal{O}_f = (\mathcal{O}_v)_{v \in S_f}$, where for each $v \in S$, $\mathcal{O}_v \subseteq M(\mathbb{Q}_v)$ denotes the $M(\mathbb{Q}_v)$-conjugacy class of $\mathcal{O}$. The coefficients $d_M^G(L_1, L_2)$ are independent of S and they vanish unless the natural map $\mathfrak{a}_M^{L_1} \oplus \mathfrak{a}_M^{L_2} \longrightarrow \mathfrak{a}_M^G$ is an isomorphism. In case the coefficient does not vanish, it depends on the chosen measures on $\mathfrak{a}_M^{L_1}$, $\mathfrak{a}_M^{L_2}$ and $\mathfrak{a}_M^G$.

We shall apply (8.3) and (8.5) with test functions F satisfying $F_f = \mathbf{1}_{K(N)}$. In this case we can choose the set of places $S = S(N)$ quite explicitly and also have a good upper bound for the global coefficients $a^M(S(N), \mathcal{O})$ that occur in (8.3). Namely we have

Lemma 8.2. *(1) Let $S(N) = \{\infty\} \cup \{p : \; p|N\}$. Then (8.3) with $S = S(N)$ holds for $F = F_\infty \otimes \mathbf{1}_{K(N)}$.*

(2) There exist constants $a, b > 0$ such that for all N, all M and all unipotent orbits $\mathcal{O}$ in M we have

$$|a^M(S(N), \mathcal{O})| \leq a(1 + \log N)^b$$

with $S(N)$ as in the first part.

Proof. The first statement is contained in [Ar4, Corollary 8.3]. The second statement follows from [Ma1], see also [Ma2, §6]. □

In the following we write

$$N = \prod_p p^{e_p}$$

for the prime factorization of N. Then $\mathbf{1}_{K(N)} = \bigotimes_p \mathbf{1}_{K(p^{e_p})}$ with $K(p^{e_p})$ the principal congruence subgroup of level p^{e_p} in $\mathbf{K}_p = \mathrm{GL}_n(\mathbb{Z}_p)$.

If $L_2 = G$ (we can always reduce to the case of $\mathrm{GL}(n)$), we split the finite orbital integral $J_M^G(\mathcal{O}_f, \mathbf{1}_{K(N)})$ further, until we arrive at

$$J_M^G(\mathcal{O}_f, \mathbf{1}_{K(N)}) = \sum_{\underline{L} \in \mathcal{L}(M)^{|S(N)_f|}} d_M^G(\underline{L}) \prod_{p \in S(N)_f} J_M^{L_p}(\mathcal{O}_p, \mathbf{1}_{\mathbf{1}_{K(p^{e_p})}, Q_p}), \tag{8.6}$$

where $\underline{L}$ runs over all tuples $(L_p)_{p \in S(N)_f}$ of Levi subgroups $L_p \in \mathcal{L}(M)$, and $d_M^G(\underline{L})$ are certain constants satisfying $d_M^G(\underline{L}) = 0$ unless the natural map

$$\bigoplus_{p \in S(N)_f} \mathfrak{a}_0^{L_p} \to \mathfrak{a}_0^G \tag{8.7}$$

is an isomorphism. Moreover, the parabolic subgroups $Q_p \in \mathcal{P}(L_p)$ are unique and chosen as explained in [Ar10, §17-18].

It follows from [Ar5] (see also [LM]) that each local integral can be written as (using that $K(p^{e_p})$ is normal in K_p)

$$J_M^{L_p}(\mathcal{O}_p, \mathbf{1}_{K(p^{e_p}), Q_p}) = \int_{N_p(\mathbb{Q}_p)} \mathbf{1}_{K(p^{e_p}), Q_p}(n) w_{M, \mathcal{O}_p}^{L_p}(n) \, dn, \tag{8.8}$$

where $P_p = M_p N_p \subset L_v$ is a standard parabolic subgroup with $M_p \subset M$ such that $\mathcal{O}_p$ is induced from the trivial orbit in M_p to M, i.e., P_p is a Richardson parabolic for $\mathcal{O}_p$ in M. The function $w_{M,\mathcal{O}_p}^{L_p}$ is a certain weight function on $N_p(\mathbb{Q}_p)$ of the form

$$w_{M,\mathcal{O}_p}^{L_p} = Q(\log \|q_1(X)\|_p, \ldots, \log \|q_r(X)\|_p), \tag{8.9}$$

where $n = \mathrm{Id} + X$ with X a nilpotent upper triangular matrix, $q_1, \ldots, q_r$ are polynomials in X with image in some affine space, and Q is a polynomial. Note that $Q, q_1, \ldots, q_r$ only depend on $\mathcal{O}$, M, and L_p (as a Levi subgroup of G defined over $\mathbb{Q}$), but not on the place p.

9. Bounds for p-adic orbital integrals

In this section we still assume that $G = \mathrm{GL}(n)$. We deal with the orbital integrals of the form $J_M^L(\mathcal{O}, \mathbf{1}_{K(N),Q})$, $Q \in \mathcal{P}(L)$, which arise in (8.5) for our type of test functions.

We first make the following observation: Let $Q = LV$ be a semistandard parabolic subgroup. Since $K(N) \cap V(\mathbb{A}_f) = V(N\hat{\mathbb{Z}})$, we have

$$\int_{V(\mathbb{A}_f)} \mathbf{1}_{K(N)}(v)\, dv = N^{-\dim V}. \tag{9.1}$$

By the definition (8.4) and the fact that $K(N)$ is a normal subgroup in $\mathbf{K}_f$ we have

$$\mathbf{1}_{K(N),Q}(m) = \delta_Q(m)^{1/2} \int_{V(\mathbb{A}_f)} \mathbf{1}_{K(N)}(mv)\, dv$$

for any $m \in L(\mathbb{A}_f)$. Hence $\mathbf{1}_{K(N),Q}(m) = 0$ unless $m \in K^L(N) = K(N) \cap L(\mathbb{A}_f)$. Now if $m \in K^L(N)$, we have $mv \in K(N)$ if and only if $v \in K(N)$. Hence

$$\mathbf{1}_{K(N),Q}(m) = N^{-\dim V} \mathbf{1}_{K^L(N)}(m). \tag{9.2}$$

It therefore suffices to bound $J_M^L(\mathcal{O}, \mathbf{1}_{K^L(N)})$. Moreover, since L is isomorphic to a direct product of finitely many smaller $\mathrm{GL}(m)$'s, it suffices to consider the case $Q_2 = G = \mathrm{GL}(n)$. Moreover, the formulas similar to (9.1) and (9.2) hold for the local integrals at p for the functions $\mathbf{1}_{K(p^{e_p})}$ with the necessary adjustments.

We now use (8.6) to find an upper bound for the orbital integrals.

Lemma 9.1. *If $d_M^G(\underline{L}) \neq 0$, then at most $\dim \mathfrak{a}_M^G$-many elements of $\underline{L}$ are not equal to M. Moreover, if $L_p = M$, then*

$$J_M^{L_p}(\mathcal{O}_p, \mathbf{1}_{K(p^{e_p}),Q_p}) = J_M^M(\mathcal{O}_p, \mathbf{1}_{K(p^{e_p}),Q_p}) = p^{-\frac{e_p}{2}\dim \mathrm{Ind}_M^G \mathcal{O}} \tag{9.3}$$

Proof. The first assertion is clear from the fact that the map in (8.7) is an isomorphism if $d_M^G(\underline{L}) \neq 0$. For the second assertion let $Q_p = MV$ be the Iwasawa decomposition of Q_p and let $P^M = L^M U^M$ be a Richardson parabolic in M for $\mathcal{O}_p$ with $T_0 \subseteq L^M$, that is, $\mathcal{O}$ is induced from the trivial orbit in L^M to M. Then $L^M U^M V =: L^M U^G$ is a Richardson parabolic for the induced orbit $\mathrm{Ind}_M^G \mathcal{O}_p$. Since $K(p^{e_p})$ is a normal subgroup in K_p, we can compute the invariant orbital integral $J_M^M(\mathcal{O}, \mathbf{1}_{K(p^{e_p}),Q_p})$ as (see [LM] for the first equality)

$$\begin{aligned} J_M^M(\mathcal{O}, \mathbf{1}_{K(p^{e_p}),Q_p}) = \int_{U^M(\mathbb{Q}_p)} \mathbf{1}_{K(p^{e_p}),Q_p}(u)\, du &= \int_{U^M(\mathbb{Q}_p)} \int_{V(\mathbb{Q}_p)} \mathbf{1}_{K(p^{e_p})}(uv)\, dv\, du \\ &= \int_{U^G(\mathbb{Q}_p)} \mathbf{1}_{K(p^{e_p})}(u)\, du. \end{aligned}$$

Since $\dim U^G = \dim \mathrm{Ind}_M^G \mathcal{O}/2$, the equation (9.3) follows from (9.1). $\square$

It follows from (8.8) that each of the local weighted orbital integrals can be written as (again using that $K(p^{e_p})$ is normal in $\mathbf{K}_p$)

$$J_M^{L_p}(\mathcal{O}, \mathbf{1}_{K(p^{e_p}),Q_p}) = \int_{N_p(\mathbb{Q}_p)} \mathbf{1}_{K(p^{e_p}),Q_p}(n) w_{M,\mathcal{O}}^{L_p}(n)\, dn$$

where $P_p = M_pN_p \subseteq L_p$ is a semistandard parabolic subgroup with $M_p \subseteq M$ such that P_p is a Richardson parabolic for $\mathcal{O}$ in M. The function $w_{L_p,\mathcal{O}}$ is a certain weight function on $N_p(\mathbb{Q}_p)$ of the form

$$w_{M,\mathcal{O}}^{L_p}(n) = Q(\log \|q_1(X)\|_p, \ldots, \log \|q_r(X)\|_p),$$

where $n = \mathrm{Id} + X$ with $X \in \mathrm{Mat}_{n\times n}(\mathbb{Q}_p)$ a nilpotent matrix, $q_1, \ldots, q_r$ polynomials in X with image in some finite dimensional affine space, and Q is a polynomial. The polynomials $Q, q_1, \ldots, q_r$ only depend on $\mathcal{O}$, M, and L_p (as a Levi subgroup of G defined over $\mathbb{Q}$), but not on the prime p. Hence there are overall only finitely many possibilities for those polynomials independent of the level N. Now if $n \in K(p^{e_p}) \cap N_p(\mathbb{Q}_p)$ we can write $n = \mathrm{Id} + p^{e_p}Y$ with $Y \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$ a nilpotent matrix. Hence setting $n' = \mathrm{Id} + Y$ we get

$$\begin{aligned}\left|J_M^{L_p}(\mathcal{O}, \mathbf{1}_{K(p^{e_p}),Q_p})\right| &\le p^{-e_p \dim V_p} \int_{N_p(\mathbb{Q}_p)} \mathbf{1}_{K^{L_p}(p^{e_p})}(n) |w_{M,\mathcal{O}}^{L_p}(n)|\, dn \\ &\le p^{-e_p \dim V_p} p^{-e_p \dim N_p} \int_{N_p(\mathbb{Q}_p)} \mathbf{1}_{K_p^{L_p}}(n') Q'(\log p^{e_p}, |\log \|q_1(Y)\|_p|, \ldots, |\log \|q_r(Y)\|_p|)\, dn'\end{aligned}$$

with Q' a suitable polynomial only depending on $Q, q_1, \ldots, q_r$ and n but not on N.

Lemma 9.2. *There exist absolute constants $r, p > 0$ (independent of p, N) such that*

$$\int_{N_p(\mathbb{Q}_p)} \mathbf{1}_{K_p^{L_p}}(n') Q'(\log p^{e_p}, |\log \|q_1(Y)\|_p|, \ldots, |\log \|q_p(Y)\|_p|)\, dn' \le C(1 + \log p^{e_p})^r.$$

Proof. There exists another polynomial $\tilde{Q}$ and some integer $j > 0$ such that

$$\begin{aligned}&Q'(\log p^{e_p}, |\log \|q_1(Y)\|_p|, \ldots, |\log \|q_p(Y)\|_p|) \\ &\qquad\qquad \le (1 + \log p^{e_p})^j \tilde{Q}(|\log \|q_1(Y)\|_p|, \ldots, |\log \|q_p(Y)\|_p|)\end{aligned}$$

for all n'. We can assume that $\tilde{Q}$ is independent of p and does only depend on Q'. But now by [Ma2, §10] there exists a constant $C > 0$ such that

$$\int_{N_p(\mathbb{Q}_p)} \mathbf{1}_{K_p^{L_p}}(n') \tilde{Q}(\log p^{e_p}, |\log \|q_1(Y)\|_p|, \ldots, |\log \|q_p(Y)\|_p|)\, dn' \le C$$

and C can be chosen to depend only on $\tilde{Q}$ and n but not on p. ∎

Together with the discussion previous to the lemma this immediately implies:

Corollary 9.3. *With the notation as before, we have*

$$\left|J_M^{L_p}(\mathcal{O}, \mathbf{1}_{K(p^{e_p}),Q_p})\right| \leq C p^{-\frac{e_p}{2}\operatorname{Ind}_M^G \mathcal{O}}(1+\log p^{e_p})^r$$

with r and C chosen to depend only on n but not on p or N.

Proof. It remains to note that $\dim V_p + \dim N_p$ equals half the dimension of the induced class $\operatorname{Ind}_M^G \mathcal{O}$. $\square$

The estimate in the corollary can also be written as

$$\left|J_M^{L_p}(\mathcal{O}, \mathbf{1}_{K(p^{e_p}),Q_p})\right| \leq C|N|_p^{\dim \operatorname{Ind}_M^G \mathcal{O}}(1-\log|N|_p)^r.$$

Combining this with the second assertion of Lemma 9.1 we get

$$\left|J_M^{L_p}(\mathcal{O}, \mathbf{1}_{K(p^{e_p}),Q_p})\right| \begin{cases} \leq C|N|_p^{\dim \operatorname{Ind}_M^G \mathcal{O}}(1-\log|N|_p)^r & \text{if } L_p \neq M, \\ = |N|_p^{\dim \operatorname{Ind}_M^G \mathcal{O}} & \text{if } L_p = M. \end{cases}$$

Further using the first assertion of Lemma 9.1 we have for any tuple $\underline{L} = \{L_p\}_{p\in S(N)_f}$ with $d_M^G(\underline{L}) \neq 0$ that

$$\begin{aligned} \left|\prod_{p\in S(N)_f} J_M^{L_p}(\mathcal{O}, \mathbf{1}_{K^{L_p}(p^{e_p}),Q_p})\right| &\leq N^{-\dim \operatorname{Ind}_M^G \mathcal{O}/2} C^{\dim \mathfrak{a}_M^G} \prod_{p\in S(N)_f : L_p\neq M} (1-\log|N|_p)^r \\ &\leq cN^{-\dim \operatorname{Ind}_M^G \mathcal{O}/2}(\log N)^{r(n-1)} \end{aligned}$$

for some absolute constant $c>0$ independent of N. The first assertion of Lemma 9.1 also implies that the number of tuples $\underline{L}$ with $d_M^G(\underline{L}) \neq 0$ is bounded by $|S(N)_f|^{\dim \mathfrak{a}_M^G}$. Since the number of elements in $S(N)_f$ is equal to the number $\omega(N)$ of prime factors of N, and $\omega(N) \leq \log_2 N \leq 2\log N$, we get that for any $N \geq 2$ we have

$$\left|J_M^{L_2}(\mathcal{O}, \mathbf{1}_{K(N),Q_2})\right| \leq c' N^{-\dim \operatorname{Ind}_M^G \mathcal{O}/2}(\log N)^{(r+1)(n-1)} \tag{9.4}$$

for some absolute constant $c' > 0$.

10. Proof of the main result for GL(n)

Let $G = \mathrm{GL}(n)$. Let $K(N) \subset \mathrm{GL}(n, \mathbb{A}_f)$ be the principal congruence subgroup and

$$Y(N) := X(K(N))$$

the associated adelic quotient (4.1). Let $\tau \in \operatorname{Rep}(G(\mathbb{R})^1)$ satisfying $\tau \ncong \tau_\theta$. Let E_τ be the associated flat vector bundle over $Y(N)$ as defined in section (4). Let $\Delta_{p,Y(N)}(\tau)$ be the Laplace operator on E_τ-valued p-forms on $Y(N)$. For $t > 0$ let $e^{-t\Delta_{p,Y(N)}(\tau)}$ be the heat operator. The regularized trace $\operatorname{Tr}_{\mathrm{reg}}(e^{-t\Delta_{p,Y(N)}(\tau)})$ of the heat operator $e^{-t\Delta_{p,Y(N)}(\tau)}$ is defined by (4.3). By (4.4) and (4.5) the zeta function $\zeta_{p,N}(s;\tau)$ is defined by

$$\zeta_{p,N}(s;\tau) := \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) t^{s-1} dt. \tag{10.1}$$

The integral converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > d/2$, and admits a meromorphic extension to the entire complex plane. Then the analytic torsion $T_{Y(N)}(\tau) \in \mathbb{R}^+$ is defined by

$$\log T_{Y(N)}(\tau) = \frac{1}{2}\sum_{p=0}^{d}(-1)^p p \left(\mathrm{FP}_{s=0}\frac{\zeta_{p,Y(N)}(s;\tau)}{s}\right) \tag{10.2}$$

(see [MzM, (13.38)]). Let $T > 0$. We write

$$\begin{aligned}\int_0^\infty \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) t^{s-1}dt = &\int_0^T \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) t^{s-1}dt\\ &+\int_T^\infty \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) t^{s-1}dt.\end{aligned} \tag{10.3}$$

We first deal with the second integral on the right hand side. Note that the integral is an entire function of s. Therefore, we have

$$\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_T^\infty \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) t^{s-1}dt\right) = \int_T^\infty \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) t^{-1}dt.$$

Using Proposition 6.6 it follows that there exist $C, c > 0$ such that

$$\frac{1}{\mathrm{vol}(Y(N))}\left|\int_T^\infty \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) t^{-1}dt\right| \le Ce^{-cT} \tag{10.4}$$

for all $T \ge 1$, $p = 0, \dots, d$, and $N \in \mathbb{N}$.

Now we turn to the first integral on the right hand side of (10.3). Recall that

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) = J_{\mathrm{spec}}(h_t^{\tau,p} \otimes \chi_{K(N)}).$$

For $R > 0$ let $\varphi_R \in C_c^\infty(G(\mathbb{R})^1)$ be the function defined by (7.2). By Proposition 7.2 we have

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) = J_{\mathrm{spec}}(\varphi_R h_t^{\tau,p} \otimes \chi_{K(N)}) + r_R(t), \tag{10.5}$$

where $r_R(t)$ is a function of $t \in [0, T]$ which satisfies

$$\frac{1}{\mathrm{vol}(Y(N))}|r_R(t)| \le C_1 e^{-C_2R^2/t + C_3t} \tag{10.6}$$

for $0 \le t \le T$. This implies that $\int_0^T r_R(t)t^{s-1}dt$ is holomorphic in $s \in \mathbb{C}$ and

$$FP_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T r_R(t)t^{s-1}dt\right) = \int_0^T r_R(t)t^{-1}dt.$$

Moreover

$$\begin{aligned}\frac{1}{\mathrm{vol}(Y(N))}\left|\int_0^T r_R(t)t^{-1}dt\right| &\le C_1\int_0^T e^{-C_2R^2/t+C_3t}t^{-1}dt\\ &\le C_1 e^{-C_4R^2/T+C_3T}\int_0^{T/R^2} e^{-C_4/t}t^{-1}dt.\end{aligned} \tag{10.7}$$

Now put $R = T^2$ and let

$$h_{t,T}^{\tau,p} := \varphi_{T^2} h_t^{\tau,p}. \tag{10.8}$$

Then it follows from (10.5) and (10.7) that there exist $C, c > 0$ such that

$$\begin{aligned}\frac{1}{\operatorname{vol}(Y(N))}\bigg|\mathrm{FP}_{s=0}&\left(\frac{1}{s\Gamma(s)}\int_0^T \operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right)t^{s-1}dt\right)\\ &-\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_{\mathrm{spec}}(h_{t,T}^{\tau,p}\otimes\chi_{K(N)})t^{s-1}dt\right)\bigg| \le Ce^{-cT}\end{aligned} \tag{10.9}$$

for $T \geq 1$, $p = 0, \dots, d$, and $N \in \mathbb{N}$. Using the trace formula, we are reduced to deal with

$$\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_{\mathrm{geo}}(h_{t,T}^{\tau,p}\otimes\chi_{K(N)})t^{s-1}dt\right).$$

Let $\varphi \in C_c^\infty(G(\mathbb{R})^1)$ be such that $\varphi(g) = 1$ in a neighborhood of $1 \in G(\mathbb{R})^1$. Put

$$\widetilde{h}_t^{\tau,p} = \varphi h_t^{\tau,p}.$$

We consider test functions with $\widetilde{h}_t^{\tau,p}$ at the infinite place. By Lemma 8.1 there exists $N_0 \in \mathbb{N}$ such that

$$J_{\mathrm{geo}}(\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N)}) = J_{\mathrm{unip}}(\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N)})$$

for $N \geq N_0$. Let $S(N)$ be as in Lemma 8.2. By the fine geometric expansion (8.3) and the definition of $h_{t,T}^{\tau,p}$ we have

$$\begin{aligned}J_{\mathrm{unip}}(\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N)}) =& \operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K(N))\widetilde{h}_t^{\tau,p}(1)\\ &+\sum_{(M,\mathcal{O})\neq(G,\{1\})} a^M(S(N),\mathcal{O})J_M(\mathcal{O},\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N)}).\end{aligned} \tag{10.10}$$

Concerning the volume factor, we used that $\chi_{K(N)} = \mathbf{1}_{K(N)}/\operatorname{vol}(K(N))$. To begin with we consider the first term on the right hand side. Note that $\widetilde{h}_t^{\tau,p}(1) = h_t^{\tau,p}(1)$. Furthermore, by [MP2, (5.11)] there is an asymptotic expansion

$$h_t^{\tau,p}(1) \sim \sum_{j=0}^\infty a_j t^{-d/2+j} \tag{10.11}$$

as $t \to 0$. Furthermore, by [MP2, (5.16)] there exists $c > 0$ such that

$$h_t^{\tau,p}(1) = O(e^{-ct}) \tag{10.12}$$

as $t \to \infty$. From (10.11) and (10.12) follows that the integral

$$\int_0^\infty h_t^{\tau,p}(1)t^{s-1}dt \tag{10.13}$$

converges in the half-plane $\operatorname{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$. The same is true for the integral over $[0, T]$ and we get

$$\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T h_t^{\tau,p}(1)t^{s-1}\right) = \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty h_t^{\tau,p}(1)t^{s-1}\right)\bigg|_{s=0} + O(e^{-cT}) \tag{10.14}$$

Recall the definition of the $L^{(2)}$-analytic torsion [Lo], [MV]. For $t>0$ let

$$K^{(2)}(t,\tau):=\sum_{p=1}^{d}(-1)^p p h_t^{\tau,p}(1).$$

Put

$$t_{\widetilde{X}}^{(2)}(\tau):=\frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty K^{(2)}(t,\tau)t^{s-1}dt\right)\bigg|_{s=0}.$$

Then by [MP2, (5.20)], the $L^{(2)}$-analytic torsion $T_{Y(N)}^{(2)}(\tau)\in\mathbb{R}^+$ is given by

$$\log T_{Y(N)}^{(2)}(\tau)=\operatorname{vol}(Y(N))\cdot t_{\widetilde{X}}^{(2)}(\tau).$$

To summarize, we get

$$\frac{1}{2}\sum_{p=1}^{d}(-1)^p p\,\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T h_t^{\tau,p}(1)t^{s-1}dt\right)=t_{\widetilde{X}}^{(2)}(\tau)+O(e^{-cT}) \tag{10.15}$$

for $T\geq 1$.

Next we consider the weighted orbital integrals on the right hand side of (10.10). Note that by definition of $\chi_{K(N)}$ we have

$$J_M(\mathcal{O},\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N)})=\frac{1}{\operatorname{vol}(K(N))}J_M(\mathcal{O},\widetilde{h}_t^{\tau,p}\otimes\mathbf{1}_{K(N)}).$$

To deal with the integral on the right hand side, we use the decomposition formula (8.5). For $L\in\mathcal{L}(M)$, $Q\in\mathcal{P}(L)$, and $\mathcal{O}\in(\mathcal{U}_M(\mathbb{Q}))$ consider the integral $J_M^L(\mathcal{O},(\widetilde{h}_t^{\tau,p})_Q)$. Unfolding the definition $(\tilde{h}_t^{\tau,p})_Q$, the local weighted orbital integral $J_M^L((\mathcal{O},\tilde{h}_t^{\tau,p})_Q)$ can be written as a non-invariant integral over the unipotent radical of a suitable semistandard parabolic subgroup in G. More precisely, there is a semistandard parabolic subgroup $R=M_RU_R\subseteq M$ which is a Richardson parabolic for $\mathcal{O}$ in M. If $Q=LV$ is the Levi decomposition of Q, we get

$$J_M^L(\mathcal{O},(\tilde{h}_t^{\tau,p})_Q)=\int_{V(\mathbb{R})}\int_{U_R(\mathbb{R})}\tilde{h}_t^{\tau,p}(uv)w(u)\,du\,dv$$

where w is a certain weight function depending on the class $\mathcal{O}$, and the groups M and L. This weight function on $U_R(\mathbb{R})$ satisfies a certain "log-homogeneity" property as explained in [MzM, §6-7]. Note that $M_RU_RV=:M_RV'$ is a Richardson parabolic for the induced class $\operatorname{Ind}_M^G\mathcal{O}$ in G. Extending w trivially to all of $V'(\mathbb{R})$ (and writing w for the extension again), we get

$$J_M^L(\mathcal{O},(\tilde{h}_t^{\tau,p})_Q)=\int_{V'(\mathbb{R})}\tilde{h}_t^{\tau,p}(v)w(v)\,dv$$

and this extended w is again log-homogeneous. It follows from [MzM, §12] that this integral admits an asymptotic expansion as $t\to 0$. This implies that the integral

$$\int_0^T J_M^L(\mathcal{O},(\tilde{h}_t^{\tau,p})_Q)t^{s-1}dt \tag{10.16}$$

converges absolutely and uniformly on compact subsets of $\mathrm{Re}(s) > d/2$ and admits a meromorphic extension to $s \in \mathbb{C}$. Put

$$A_M^L(\mathcal{O}_\infty, T) := \mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_M^L(\mathcal{O}, (\widetilde{h}_t^{\tau,p})_Q) t^{s-1} dt\right). \tag{10.17}$$

By (8.5) it follows that the Mellin transform of $J_M(\mathcal{O}, \widetilde{h}_t^{\tau,p} \otimes \mathbf{1}_{K(N)})$ as a function of t is a meromorphic function on $\mathbb{C}$, and we get

$$\begin{aligned} &\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_M(\mathcal{O}, \widetilde{h}_t^{\tau,p} \otimes \mathbf{1}_{K(N)}) t^{s-1} dt\right) \\ &\qquad = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) A_M^{L_1}(\mathcal{O}_\infty, T) J_M^{L_2}(\mathcal{O}_f, \mathbf{1}_{K(N), Q_2}). \end{aligned} \tag{10.18}$$

Denote by $J_{\mathrm{unip}-\{1\}}(\widetilde{h}_t^{\tau,p} \otimes \mathbf{1}_{K(N)})$ the sum on the right hand side of (10.10) with the term $\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K(N))\widetilde{h}_t^{\tau,p}(1)$ removed. Combining (10.18), Lemma 8.2, and (9.4), we obtain

Proposition 10.1. *For every $T \geq 1$ there exist constants $C(T), a > 0$, a independent of T, such that for all $N \geq 2$ we have*

$$\left|\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_{unip-\{1\}}(\tilde{h}_t^{\tau,p} \otimes \mathbf{1}_{K(N)}) t^{s-1}\, dt\right)\right| \leq C(T) N^{-(n-1)} (\log N)^a.$$

Now we can turn to the proof of the main Theorem. Let

$$K_N(t,\tau) := \frac{1}{2}\sum_{p=1}^d (-1)^p p \,\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right).$$

Let $T > 0$. By (10.1), (10.2) and (10.3) we have

$$\log T_{Y(N)}(\tau) = \mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T K_N(t,\tau) t^{s-1} dt\right) + \int_T^\infty K_N(t,\tau) t^{-1} dt. \tag{10.19}$$

By (10.4) there exist $C, c > 0$ such that

$$\frac{1}{\mathrm{vol}(Y(N))}\left|\int_T^\infty K_N(t,\tau) t^{-1} dt\right| \leq C e^{-cT} \tag{10.20}$$

for all $T \geq 1$ and $N \in \mathbb{N}$. Let $h_{t,T}^{\tau,p} \in C_c^\infty(G(\mathbb{R})^1)$ be defined by (10.8). Put

$$K_N(t,\tau;T) := \frac{1}{2}\sum_{p=1}^d (-1)^p p J_{\mathrm{geo}}(h_{t,T}^{\tau,p} \otimes \chi_{K(N)}).$$

By (10.9) and the trace formula it follows that there exist $C, c > 0$ such that

$$\begin{aligned} \frac{1}{\operatorname{vol}(Y(N))}\Bigg|\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T K_N(t,\tau)t^{s-1}dt\right) \\ - \mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T K_N(t,\tau;T)t^{s-1}dt\right)\Bigg| \leq Ce^{-cT} \end{aligned} \tag{10.21}$$

for all $T \geq 1$ and $N \in \mathbb{N}$. Let

$$K_{\mathrm{unip}-\{1\},N}(t,\tau;T) := \frac{1}{2}\sum_{p=1}^{d}(-1)^p p J_{\mathrm{unip}-\{1\}}(h_{t,T}^{\tau,p} \otimes \chi_{K(N)}). \tag{10.22}$$

By Lemma 8.1 and (10.10) it follows that for every $T \geq 1$ there exists $N_0(T) \in \mathbb{N}$ such that

$$K_N(t,\tau;T) = \frac{\operatorname{vol}(Y(N))}{2}\sum_{p=1}^{d}(-1)^p p h_{t,T}^{\tau,p}(1) + K_{\mathrm{unip}-\{1\},N}(t,\tau;T)$$

for $N \geq N_0(T)$. Using (10.15) and (10.21) it follows that for every $T \geq 1$ there exists $N_0(T) \in \mathbb{N}$ such that

$$\begin{aligned} &\frac{1}{\operatorname{vol}(Y(N))}\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T K_N(t,\tau)t^{s-1}dt\right) \\ &= t_{\widetilde{X}}^{(2)}(\tau) + \frac{1}{\operatorname{vol}(Y(N))}\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T K_{\mathrm{unip}-\{1\},N}(t,\tau)t^{s-1}dt\right) \\ &\quad + O(e^{-cT}). \end{aligned} \tag{10.23}$$

for $N \geq N_0(T)$. Applying Proposition 10.1 we get that for every $T \geq 1$ there exist constants $C_1(T), C_2, a, c > 0$ and $N_0(T) \in \mathbb{N}$ such that

$$\begin{aligned} \left|\frac{1}{\operatorname{vol}(Y(N))}\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T K_N(t,\tau)t^{s-1}dt\right) - t_{\widetilde{X}}^{(2)}(\tau)\right| \\ \leq C_1(T)N^{-(n-1)}(\log N)^a + C_2e^{-cT} \end{aligned} \tag{10.24}$$

for $N \geq N_0(T)$. Combined with (10.19) and (10.20) it follows that

$$\lim_{N\to\infty}\frac{\log T_{Y(N)}(\tau)}{\operatorname{vol}(Y(N))} = t_{\widetilde{X}}^{(2)}(\tau). \tag{10.25}$$

11. Proof of the main result for SL(n)

The following section is due to Werner Hoffmann. In order to deduce Theorem 1.1 from (10.25), we need to compare the trace formulas for GL(n) and SL(n). This is the purpose of the current section. Let $K_f \subset \mathrm{GL}(n, \mathbb{A}_f)$ be an open compact subgroup. Consider the action of $\mathrm{SL}(n, \mathbb{A})K_f$ on $\mathrm{GL}(n, \mathbb{Q})\backslash \mathrm{GL}(n, \mathbb{A})^1$. Then we get

$$\mathrm{GL}(n, \mathbb{Q})\backslash \mathrm{GL}(n, \mathbb{A})^1 \cong \bigcup_g (g^{-1} \mathrm{GL}(n, \mathbb{Q})g \cap \mathrm{SL}(n, \mathbb{A})K_f)\backslash \mathrm{SL}(n, \mathbb{A})K_f, \tag{11.1}$$

where g runs over a set of representatives of

$$\mathrm{GL}(n, \mathbb{Q})\backslash \mathrm{GL}(n, \mathbb{A})^1 / \mathrm{SL}(n, \mathbb{A})K_f \cong \mathrm{GL}(n, \mathbb{Q}) \mathrm{SL}(n, \mathbb{A})\backslash \mathrm{GL}(n, \mathbb{A})^1/K_f.$$

Let

$$U = \{\det k \colon k \in K\} \cong K_f/(K_f \cap \mathrm{SL}(n, \mathbb{A})).$$

The determinant induces a bijection

$$\det\colon\ \mathrm{GL}(n, \mathbb{Q}) \mathrm{SL}(n, \mathbb{A})\backslash \mathrm{GL}(n, \mathbb{A})^1/K \cong \mathbb{Q}^\times\backslash\mathbb{A}^1/U. \tag{11.2}$$

Comparing the archimedean components we get

$$g^{-1} \mathrm{GL}(n, \mathbb{Q})g \cap \mathrm{SL}(n, \mathbb{A})K = g^{-1} \mathrm{SL}(n, \mathbb{Q})g.$$

Factorizing both sides of (11.1) by K_f, we get

$$\mathrm{GL}(n, \mathbb{Q})\backslash \mathrm{GL}(n, \mathbb{A})^1/K_f \cong \bigcup_g g^{-1} \mathrm{SL}(n, \mathbb{Q})g\backslash \mathrm{SL}(n, \mathbb{A})/(\mathrm{SL}(n, \mathbb{A}) \cap K_f). \tag{11.3}$$

Now recall that for $\mathrm{SL}(n)$ strong approximation holds, i.e., for every open compact subgroup K_f' of $\mathrm{SL}(n, \mathbb{A}_f)$ we have

$$\mathrm{SL}(n, \mathbb{A}) = \mathrm{SL}(n, \mathbb{Q})K_f' \mathrm{SL}(n, \mathbb{R}).$$

Put $K_f' = \mathrm{SL}(n, \mathbb{A}) \cap gK_f g^{-1}$ and conjugate both sides with g^{-1}. It follows that

$$\mathrm{SL}(n, \mathbb{A}) = g^{-1} \mathrm{SL}(n, \mathbb{Q})g(\mathrm{SL}(n, \mathbb{A}) \cap K_f) \mathrm{SL}(n, \mathbb{R}).$$

Thus $\mathrm{SL}(n, \mathbb{R})$ acts transitively from the right on each component of the decomposition (11.3). The stabilizer of the double coset of e in the g-component is

$$\Gamma_{g,K_f} = \{g_\infty^{-1}\gamma_\infty g_\infty \mid \gamma \in \mathrm{SL}(n, \mathbb{Q}),\ g_f^{-1}\gamma_f g_f \in K_f\}, \tag{11.4}$$

where $\mathrm{SL}(n, \mathbb{Q})$ is embedded diagonally in $\mathrm{SL}(n, \mathbb{A})$. Thus we get an isomorphism of right $\mathrm{SL}(n, \mathbb{R})$-spaces

$$\mathrm{GL}(n, \mathbb{Q})\backslash \mathrm{GL}(n, \mathbb{A})^1/K_f \cong \bigcup_g (\Gamma_{g,K_f}\backslash \mathrm{SL}(n, \mathbb{R})), \tag{11.5}$$

and the union is disjoint. This isomorphism induces an isomorphism of $\mathrm{SL}(n, \mathbb{R})$-modules

$$L^2(\mathrm{GL}(n, \mathbb{Q})\backslash \mathrm{GL}(n, \mathbb{A})^1/K_f) \cong \bigoplus_{g\in I} L^2(\Gamma_{g,K_f}\backslash \mathrm{SL}(n, \mathbb{R})), \tag{11.6}$$

where I is a set of representatives of $\mathrm{GL}(n, \mathbb{Q}) \mathrm{SL}(n, \mathbb{A})\backslash \mathrm{GL}(n, \mathbb{A})^1/K_f$ and a given ϕ corresponds to $(\phi_g)_{g\in I}$ with ϕ_g defined by

$$\phi_g(x) = \phi(gx), \quad x \in \mathrm{SL}(n, \mathbb{R}).$$

Now note that the right regular representation R of $\mathrm{GL}(n,\mathbb{A})^1$ in $L^2(\mathrm{GL}(n,\mathbb{Q})\backslash \mathrm{GL}(n,\mathbb{A})^1)$ induces a representation of the convolution algebra $L^2(K_f\backslash \mathrm{GL}(n,\mathbb{A})^1/K_f)$ in the Hilbert space $L^2(\mathrm{GL}(n,\mathbb{Q})\backslash \mathrm{GL}(n,\mathbb{A})^1/K_f)$. For $h\in L^2(K_f\backslash \mathrm{GL}(n,\mathbb{A})^1/K_f)$ let

$$K_h(x,y):=\sum_{\gamma\in \mathrm{GL}(n,\mathbb{Q})} h(x^{-1}\gamma y).$$

Then we have

$$(R(h)\phi)(x)=\int_{\mathrm{GL}(n,\mathbb{Q})\backslash \mathrm{GL}(n,\mathbb{A})^1/K_f} K_h(x,y)\phi(y)dy.$$

With respect to the isomorphism (11.5) the kernel K_h is given by the components

$$\Gamma_{g_1,K_f}\backslash \mathrm{SL}(n,\mathbb{R})\times\Gamma_{g_2,K_f}\backslash \mathrm{SL}(n,\mathbb{R})\ni(x,y)\mapsto K_h(g_1x,g_2y).$$

If h acts on the right hand side of (11.6) by these integral kernels, (11.6) becomes an isomorphism of $L^1(K_f\backslash \mathrm{GL}(n,\mathbb{A})^1/K_f)$-modules. Especially assume that $h=h_\infty\otimes\chi_{K_f}$. Then it follows from (11.4) that

$$K_h(gx,gy)=\sum_{\gamma\in\Gamma_{g,K_f}} h_\infty(x^{-1}\gamma y).$$

Now we turn to the trace formula. We briefly recall the definition of the distribution $J^T(f)$, $f\in C_c^\infty(G(\mathbb{A})^1)$. For details see [Ar1]. Let $P=M_PN_P$ be a standard parabolic subgroup of G and let Q be a parabolic subgroup containing P. Let τ_Q^P and $\widehat{\tau}_P^P$ denote the characteristic functions of the set

$$\{X\in\mathfrak{a}_0\colon \langle\alpha,X\rangle>0 \text{ for all } \alpha\in\Delta_P^Q\}$$

and

$$\{X\in\mathfrak{a}_0\colon \langle\varpi,X\rangle>0 \text{ for all } \varpi\in\hat{\Delta}_P^Q\},$$

respectively. If $Q=G$, we will suppress the superscript. Moreover we put $\tau_0:=\tau_0^G$ and $\hat{\tau}_0:=\widehat{\tau}_0^G$. Let

$$K_P(x,y)=\int_{N(\mathbb{Q})\backslash N(\mathbb{A})}\sum_{\gamma\in P(\mathbb{Q})} h(x^{-1}\gamma ny)dn=\int_{N(\mathbb{A})P(\mathbb{Q})} f(x^{-1}py)dp. \tag{11.7}$$

For $T\in\mathfrak{a}_0^+$ Arthur's distribution is defined by

$$J^T(h)=\int_{\mathrm{GL}(n,\mathbb{Q})\backslash \mathrm{GL}(n,\mathbb{A})^1/K_f}\sum_P(-1)^{n-\dim A_P}K_P(x,x)\widehat{\tau}_P(H_P(x)-T_P)dx,$$

where P runs over all $\mathbb{Q}$-rational parabolic subgroups of $\mathrm{GL}(n)$ and the truncation parameter T_P is chosen in such a way that

$$\mathrm{Ad}(\delta)(H_P(x)-T_P)=H_{\delta P\delta^{-1}}(x)-T_{\delta P\delta^1}$$

for all $\delta\in\mathrm{GL}(n,\mathbb{Q})$. Note that this definition differs from the usual definition, but it is easy to check that it agrees with the usual definition. Furthermore, for $d(T)>d_0$, the sum over P is finite. Using the decomposition (11.5), it follows that

$$J^T(h)=\sum_g \int_{\Gamma_{g,K_f}\backslash \operatorname{SL}(n,\mathbb{R})} \sum_P (-1)^{n-\dim A_P} K_P(gx,gx)\widehat{\tau}_P(H_P(gx)-T_P)dx.$$

Now assume that $h=h_\infty\otimes\chi_{K_f}$. Then the integrand $f(g^{-1}x^{-1}pxg)$ in $K_P(gx,gx)$ is nonzero, only if $p\in N(\mathbb{A})P(\mathbb{Q})\cap G(\mathbb{R})gK_fg^{-1}$. We may decompose the integral (11.7) into a sum over $P(\mathbb{Q})\cap gK_fg^{-1}$ and an integral over

$$P(\mathbb{Q})\cap gK_fg^{-1}\backslash N(\mathbb{A})P(\mathbb{Q})\cap gK_f\cong N(\mathbb{Q})\cap gK_fg^{-1}\backslash N(\mathbb{A})\cap gK_fg^{-1}.$$

Put

$$P_g=g_\infty^{-1}(P(\mathbb{R})\cap \operatorname{SL}(n,\mathbb{R}))g_\infty,\quad N_g=g_\infty^{-1}N(\mathbb{R})g_\infty.$$

Then we get

$$J^T(h)=\sum_g \int_{\Gamma_{g,K_f}\backslash \operatorname{SL}(n,\mathbb{R})} \sum_P (-1)^{n-\dim A_P} K_{P,g,K_f}(x,x)\widehat{\tau}_{P_g}(H_{P_g}(x)-T_{P_g})dx,$$

where

$$K_{P,g,K_f}(x,y)=\int_{\Gamma_{g,K_f}\cap N_g\backslash N_g}\sum_{\gamma\in\Gamma_{g,K_f}\cap P_g} h_\infty(x^{-1}\gamma n y)dn$$

and

$$\operatorname{Ad}(g_\infty)(H_{P_g}(x)-T_{P_g})=H_P(x)-T_P.$$

Now let $K_n(N)\subset \operatorname{GL}(n,\mathbb{A}_f)$ be the principal congruence subgroup of level N. In this case we have $U=K_1(N)$ and the components of $\operatorname{GL}(n,\mathbb{Q})\backslash\operatorname{GL}(n,\mathbb{A})^1/K_n(N)$ are parametrized by

$$\mathbb{Q}^\times\backslash\mathbb{A}^1/K_1(N)\cong(\mathbb{Z}/N\mathbb{Z})^\times. \tag{11.8}$$

The isomorphism (11.8) is explicitly given as follows. Let $a\in\mathbb{N}$ with $(a,N)=1$. Then its residue class $a+N\mathbb{Z}$ corresponds to the idele class $\tilde a$ with components $\tilde a_\infty=a$ and for each prime p, $\tilde a_p\in\mathbb{Q}_p$ equals p^{r_p}, where $a=\prod_p p^{r_p}$. Let $\tilde g=\operatorname{diag}(\tilde a,1,\dots,1)\in\operatorname{GL}(n,\mathbb{A})^1$ be the representative that corresponds to $\tilde a$ and let $g=(a,1,\dots,1)\in\operatorname{GL}(n,\mathbb{Q})$. Then for $\gamma\in\operatorname{SL}(n,\mathbb{Q})$ we have

$$\tilde g_p^{-1}\gamma_p\tilde g_p\in K_n(N)_p\Longleftrightarrow((g^{-1}\gamma g)^{-1})_p\in K_n(N)_p.$$

Thus

$$\Gamma_{\tilde g,K_n(N)}=\Gamma(N)$$

for all $\tilde g$, where $\Gamma(N)\subset\operatorname{SL}(n,\mathbb{Z})$ is the principal congruence subgroup of level N. Thus we have an isomorphism of right $\operatorname{SL}(n,\mathbb{R})$-spaces

$$\operatorname{GL}(n,\mathbb{Q})\backslash\operatorname{GL}(n,\mathbb{A})^1/K_n(N)\cong\bigsqcup_{(\mathbb{Z}/N\mathbb{Z})^\times}\Gamma(N)\backslash\operatorname{SL}(n,\mathbb{R}). \tag{11.9}$$

Let $\varphi(N)=\#(\mathbb{Z}/N\mathbb{Z})^\times)$. Then for $h=h_\infty\otimes\chi_{K_f}$ it follows that

$$J^T(h)=\varphi(N)\int_{\Gamma(N)\backslash\operatorname{SL}(n,\mathbb{R})}\sum_P(-1)^{n-\dim A_P}K_{P,N}(x,x)\widehat{\tau}_P(H_P(x)-T_P)dx,$$

where

$$K_{P,N}(x,y)=\int_{\Gamma(N)\cap N(\mathbb{R})\backslash N(\mathbb{R})}\sum_{\gamma\in\Gamma(N)\cap P(\mathbb{R})}h_\infty(x^{-1}\gamma ny)dn.$$

Let

$$X(N)=\Gamma(N)\backslash\operatorname{SL}(n,\mathbb{R})/\operatorname{SO}(n)$$

and let $\Delta_{p,X(N)}(\tau)$ be the Laplace operator on E_τ-valued p-forms on $X(N)$. Then it follows from the definition of the regularized trace (4.3) that

$$\operatorname{Tr}_{\text{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right)=\varphi(N)\operatorname{Tr}_{reg}\left(e^{-t\Delta_{p,X(N)}(\tau)}\right)$$

for all $N\geq 3$. Using the definition (1.9) of the analytic torsion, we obtain

$$\log T_{Y(N)}(\tau)=\varphi(N)\log T_{X(N)}(\tau). \tag{11.10}$$

Furthermore, by (11.9) we have

$$\operatorname{vol}(Y(N))=\varphi(N)\operatorname{vol}(X(N)). \tag{11.11}$$

Combining (10.25), (11.10), and (11.11), we obtain Theorem 1.1.

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