

ON THE SPECTRAL SIDE OF ARTHUR'S TRACE FORMULA – ABSOLUTE CONVERGENCE

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ABSTRACT. We derive a refinement of the spectral expansion of Arthur's trace formula. The expression is absolutely convergent with respect to the trace norm.

CONTENTS

1. Introduction	1
2. Combinatorial formulas	2
3. A refined spectral expansion	9
4. The Maass-Selberg relations	11
5. Proof of absolute convergence	13
References	19

1. INTRODUCTION

The trace formula is an important tool for the study of automorphic forms on arithmetic quotients. It was introduced by Selberg, who mostly considered the case of quotients of the upper half-plane ([Sel56]), and later developed by Arthur in his groundbreaking work on the subject, which deals with the adelic quotients $G(F)\backslash G(\mathbb{A})$ for a general reductive group G defined over a number field F . (See [Art05] for an excellent survey on the theory.) In essence, the trace formula is an equality between a sum of geometric distributions which are certain weighted orbital integrals, and a sum of spectral distributions which are suitably defined weighted traces of representations. For applications, it is important to have an explicit description of these distributions. In [Art82b] Arthur derived an expression for the spectral side of the non-invariant trace formula in terms of certain limits of intertwining operators. In this paper we explicate these terms further and write them as linear combinations of products of first-order derivatives of intertwining operators. The basis for this refinement is provided by the combinatorial identities for certain piecewise power series proved in the companion paper [FLc]. Applying these identities in the context of

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the trace formula we obtain Theorems 1 and 2 below, which are then used to explicate the spectral side in Corollary 1. We also explicate special cases of the Maass-Selberg relations in Theorem 4. We emphasize that our formula relies on Arthur's original expansion. We do not provide any shortcut for the derivation of the latter.

A key feature of our refined spectral expansion is its absolute convergence with respect to the trace norm. This relies on previous work by the third named author and generalizes earlier results in this direction ([Lan90, Mül89, Mül98, Ji98, Mül00, Mül02, MS04]). Remarkably, Arthur was able to finesse this difficulty in his work. This is partly because his emphasis is on comparing trace formulas on two different groups. However, for other applications of the trace formula the absolute convergence may be indispensable. An example is the work of the second and third named authors on Weyl's law with remainder for the groups $GL(n)$ [LM09]. (Note that in this case the absolute convergence had been already obtained in [MS04] by a different argument, which is special to $GL(n)$.) Another possible application of the refined spectral expansion is to the problem of limit multiplicities for $GL(n)$, which we plan to consider in a future paper.

A preliminary announcement of some of the results of this paper and [FLc] was made in [FLM09].¹

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2. COMBINATORIAL FORMULAS

2.1. Notation. Let G be a reductive group defined over a number field F . All algebraic subgroups of G considered in the following will be tacitly assumed to be defined over F . We will mostly use, with some minor modifications, the notation and conventions of [Art82a, Art82b]. In particular:

- \mathbb{A} is the ring of adèles of F , \mathbb{A}_f the ring of finite adèles and $F_\infty = \mathbb{R} \otimes_{\mathbb{Q}} F$.
- $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ is the universal enveloping algebra of the complexified Lie algebra of $G(F_\infty)$.
- \mathfrak{z} is the center of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.
- T_0 is a fixed maximal F -split torus.
- M_0 is the centralizer of T_0 , which is a minimal Levi subgroup defined over F .

¹Note the following typo on p. 15564 of [ibid.]: the *sum* over $s \in W(M)$ should be replaced by the *average* over $s \in W(M)$.

- A_0 is the identity component of $T_0(\mathbb{R})$, which is viewed as a subgroup of $T_0(\mathbb{A})$ via the diagonal embedding of \mathbb{R} into F_∞ .
- \mathcal{L} is the set of Levi subgroups containing M_0 , i.e. the (finite) set of centralizers of subtori of T_0 .
- $W_0 = N_{G(F)}(T_0)/M_0$ is the Weyl group of (G, T_0) , where $N_{G(F)}(H)$ is the normalizer of H in $G(F)$.
- For any $s \in W_0$ we choose a representative $w_s \in G(F)$.
- W_0 acts on \mathcal{L} by $sM = w_s M w_s^{-1}$.

For $M \in \mathcal{L}$ we use the following additional notation.

- T_M is the split part of the identity component of the center of M .
- $W(M) = N_{G(F)}(M)/M$, which can be identified with a subgroup of W_0 .
- $A_M = A_0 \cap T_M(\mathbb{R})$.
- \mathfrak{a}_M^* is the \mathbb{R} -vector space spanned by the lattice $X^*(M)$ of F -rational characters of M ; $\mathfrak{a}_{M, \mathbb{C}}^* = \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$.
- \mathfrak{a}_M is the dual space of \mathfrak{a}_M^* , which is spanned by the co-characters of T_M .
- $H_M : M(\mathbb{A}) \rightarrow \mathfrak{a}_M$ is the homomorphism given by $e^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_{\mathbb{A}} = \prod_v |\chi(m_v)|_v$ for any $\chi \in X^*(M)$.
- $M(\mathbb{A})^1 \subset M(\mathbb{A})$ is the kernel of H_M .
- $\mathcal{L}(M)$ is the set of Levi subgroups containing M .
- $\mathcal{P}(M)$ is the set of parabolic subgroups of G with Levi part M .
- $\mathcal{F}(M) = \mathcal{F}^G(M) = \coprod_{L \in \mathcal{L}(M)} \mathcal{P}(L)$ is the (finite) set of parabolic subgroups of G containing M .
- $W(M)$ acts on $\mathcal{P}(M)$ and $\mathcal{F}(M)$ by $sP = w_s P w_s^{-1}$.
- Σ_M is the set of reduced roots of T_M on the Lie algebra of G .
- For any $\alpha \in \Sigma_M$ we denote by $\alpha^\vee \in \mathfrak{a}_M$ the corresponding co-root.
- $L_{\text{disc}}^2(A_M M(F) \backslash M(\mathbb{A}))$ is the discrete part of $L^2(A_M M(F) \backslash M(\mathbb{A}))$, i.e. the closure of the sum of all irreducible subrepresentations of the regular representation of $M(\mathbb{A})$.
- $\Pi_{\text{disc}}(M(\mathbb{A}))$ denotes the countable set of equivalence classes of irreducible unitary representations of $M(\mathbb{A})$ which occur in the decomposition of $L_{\text{disc}}^2(A_M M(F) \backslash M(\mathbb{A}))$ into irreducibles.

For any $L \in \mathcal{L}(M)$ we identify \mathfrak{a}_L^* with a subspace of \mathfrak{a}_M^* . We denote by \mathfrak{a}_M^L the annihilator of \mathfrak{a}_L^* in \mathfrak{a}_M . For any integer $i \geq 0$ let

$$\mathcal{L}_i(M) = \{L \in \mathcal{L}(M) : \dim \mathfrak{a}_M^L = i\}$$

and

$$\mathcal{F}_i(M) = \bigcup_{L \in \mathcal{L}_i(M)} \mathcal{P}(L),$$

so that $\mathcal{F}(M) = \coprod_{i=0}^d \mathcal{F}_i(M)$ where d is the co-rank of M . We endow \mathfrak{a}_{M_0} with the structure of a Euclidean space by choosing a W_0 -invariant inner product. This choice fixes Haar measures on the spaces \mathfrak{a}_M^L and their duals $(\mathfrak{a}_M^L)^*$. We follow Arthur in the corresponding normalization of Haar measures on the groups $M(\mathbb{A})$ ([Art78, §1]).

For any $P \in \mathcal{P}(M)$ we use the following notation.

- $\mathfrak{a}_P = \mathfrak{a}_M$.
- N_P is the unipotent radical of P and M_P is the unique $L \in \mathcal{L}(M)$ (in fact the unique $L \in \mathcal{L}(M_0)$) such that $P \in \mathcal{P}(L)$.
- $\Sigma_P \subset \mathfrak{a}_P^*$ is the set of reduced roots of T_M on the Lie algebra of N_P .
- Δ_P is the subset of simple roots of P , which is a basis for $(\mathfrak{a}_P^G)^*$.
- $\mathfrak{a}_{P,+}^*$ is the closure of the Weyl chamber of P , i.e.

$$\mathfrak{a}_{P,+}^* = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Sigma_P\} = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta_P\}.$$

- δ_P is the modulus function of $P(\mathbb{A})$.
- v_{Δ_P} is the co-volume of the lattice spanned by Δ_P in $(\mathfrak{a}_P^G)^*$ and

$$\theta_P(\lambda) = v_{\Delta_P}^{-1} \prod_{\alpha \in \Delta_P} \langle \lambda, \alpha^\vee \rangle, \quad \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*.$$

- $P^\circ \in \mathcal{P}(M)$ is the parabolic subgroup opposite to P (with respect to M), i.e. $\Sigma_{P^\circ} = -\Sigma_P$ and $\Delta_{P^\circ} = -\Delta_P$.
- $\bar{\mathcal{A}}_2(P)$ is the Hilbert space completion of

$$\{\phi \in C^\infty(M(F)N_P(\mathbb{A})\backslash G(\mathbb{A})) : \delta_P^{-\frac{1}{2}}\phi(\cdot x) \in L_{\text{disc}}^2(A_M M(F)\backslash M(\mathbb{A})) \forall x \in G(\mathbb{A})\}$$

with respect to the inner product

$$(\phi_1, \phi_2) = \int_{A_M M(F)N_P(\mathbb{A})\backslash G(\mathbb{A})} \phi_1(g) \overline{\phi_2(g)} dg.$$

Let $\alpha \in \Sigma_M$. We say that two parabolic subgroups $P, Q \in \mathcal{P}(M)$ are *adjacent* along α , and write $P|^\alpha Q$, if $\Sigma_P \cap \Sigma_{Q^\circ} = \{\alpha\}$. Alternatively, P and Q are adjacent if the closure \overline{PQ} of PQ belongs to $\mathcal{F}_1(M)$. Any $R \in \mathcal{F}_1(M)$ is of the form \overline{PQ} for a unique unordered pair $\{P, Q\}$ of parabolic subgroups in $\mathcal{P}(M)$, namely P and Q are the maximal parabolic subgroups of R , and $P|^\alpha Q$ with $\alpha^\vee \in \Sigma_P^\vee \cap \mathfrak{a}_M^R$. Switching the order of P and Q changes α to $-\alpha$.

2.2. Intertwining operators. Fix a maximal compact subgroup $K = K_\infty K_f$ of $G(\mathbb{A}) = G(F_\infty)G(\mathbb{A}_f)$ which is admissible with respect to M_0 . For any $P \in \mathcal{P}(M)$ let

- $H_P : G(\mathbb{A}) \rightarrow \mathfrak{a}_P$ be the extension of H_M to a left $N_P(\mathbb{A})$ - and right K -invariant map,
- $\mathcal{A}^2(P)$ the dense subspace of $\bar{\mathcal{A}}^2(P)$ consisting of its K - and \mathfrak{z} -finite vectors, i.e. the space of automorphic forms ϕ on $N_P(\mathbb{A})M(F)\backslash G(\mathbb{A})$ such that $\delta_P^{-\frac{1}{2}}\phi(\cdot k)$ is a square-integrable automorphic form on $A_M M(F)\backslash M(\mathbb{A})$ for all $k \in K$,
- $\rho(P, \lambda)$, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, the induced representation of $G(\mathbb{A})$ on $\bar{\mathcal{A}}^2(P)$ given by

$$(\rho(P, \lambda, y)\phi)(x) = \phi(xy)e^{\langle \lambda, H_P(xy) - H_P(x) \rangle}.$$

It is isomorphic to $\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} (L_{\text{disc}}^2(A_M M(F)\backslash M(\mathbb{A})) \otimes e^{\langle \lambda, H_M(\cdot) \rangle})$.

For $P, Q \in \mathcal{P}(M)$ let

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \rightarrow \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*,$$

be the standard *intertwining operator* [Art82b, §1], which is the meromorphic continuation in λ of the integral

$$[M_{Q|P}(\lambda)\phi](x) = \int_{N_Q(\mathbb{A}) \cap N_P(\mathbb{A}) \backslash N_Q(\mathbb{A})} \phi(nx) e^{\langle \lambda, H_P(nx) - H_Q(x) \rangle} dn, \quad \phi \in \mathcal{A}^2(P), x \in G(\mathbb{A}).$$

These operators satisfy the following properties.

- (1) $M_{P|P}(\lambda) \equiv \text{Id}$ for all $P \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$.
- (2) For any $P, Q, R \in \mathcal{P}(M)$ we have $M_{R|P}(\lambda) = M_{R|Q}(\lambda) \circ M_{Q|P}(\lambda)$ for all $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. In particular, $M_{Q|P}(\lambda)^{-1} = M_{P|Q}(\lambda)$.
- (3) $M_{Q|P}(\lambda)^* = M_{P|Q}(-\bar{\lambda})$ for any $P, Q \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. In particular, $M_{Q|P}(\lambda)$ is unitary for $\lambda \in \mathfrak{ia}_M^*$.
- (4) If $P|^\alpha Q$ then $M_{Q|P}(\lambda)$ depends only on $\langle \lambda, \alpha^\vee \rangle$.

Arthur's expression for the spectral side of the trace formula involves certain limits of these intertwining operators. Let $P \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{ia}_M^*$. For $Q \in \mathcal{P}(M)$ and $\Lambda \in \mathfrak{ia}_M^*$ define

$$\mathcal{M}_Q(P, \lambda, \Lambda) = M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda) = M_{P|Q}(\lambda) M_{Q|P}(\lambda + \Lambda).$$

Then $(\mathcal{M}_Q(P, \lambda, \cdot))_{Q \in \mathcal{P}(M)}$ is a (G, M) -family with values in the space of operators on $\mathcal{A}^2(P)$ [Art82b, p. 1310]. Therefore the limit

$$\mathcal{M}_M(P, \lambda) = \lim_{\Lambda \rightarrow 0} \sum_{Q \in \mathcal{P}(M)} \frac{\mathcal{M}_Q(P, \lambda, \Lambda)}{\theta_Q(\Lambda)}$$

exists. More generally, for any $L \in \mathcal{L}(M)$ and $Q \in \mathcal{P}(L)$ the restriction $\mathcal{M}_Q(P, \lambda, \cdot)$ of $\mathcal{M}_{Q_1}(P, \lambda, \cdot)$ to \mathfrak{ia}_L^* does not depend on $Q_1 \in \mathcal{P}(M)$ provided that $Q_1 \subset Q$, and the limit

$$\mathcal{M}_L(P, \lambda) = \lim_{\substack{\Lambda \in \mathfrak{ia}_L^* \\ \Lambda \rightarrow 0}} \sum_{Q \in \mathcal{P}(L)} \frac{\mathcal{M}_Q(P, \lambda, \Lambda)}{\theta_Q(\Lambda)}$$

exists.

2.3. The main formulas. Our main result is an explicit evaluation of the limit $\mathcal{M}_L(P, \lambda)$ in terms of first order derivatives of the intertwining operators $M_{P_1|P_2}$. To describe our two formulas we need some more notation.

A *flag* \mathfrak{f} is an ascending chain $Q_0 \subset \cdots \subset Q_m = G$ of parabolic subgroups of G such that Q_{i-1} is maximal in Q_i for $i = 1, \dots, m$ (or equivalently $\dim \mathfrak{a}_{Q_i} = \dim \mathfrak{a}_{Q_{i-1}} - 1$). The length m of the chain is therefore the co-rank of Q_0 . We denote by $\mathfrak{G}(L)$ the set of flags with $Q_0 \in \mathcal{P}(L)$. For any flag $\mathfrak{f} \in \mathfrak{G}(L)$ choose for $i = 0, \dots, m-1$ an auxiliary vector μ_i in the relative interior of $\mathfrak{a}_{Q_i,+}^*$ such that the lattice spanned by μ_0, \dots, μ_{m-1} has co-volume one in the vector space $(\mathfrak{a}_{Q_0}^G)^*$.

Let $\mathfrak{s} : \mathcal{F}(M) \rightarrow \mathcal{P}(M)$ be a map such that $\mathfrak{s}(Q) \subset Q$ for all Q ; in particular $\mathfrak{s}(P) = P$ for $P \in \mathcal{P}(M)$. We call \mathfrak{s} a *selector*. For any smooth function f on \mathfrak{a}_M^* and $\mu \in \mathfrak{a}_M^*$ denote by $D_\mu f$ the directional derivative of f along $\mu \in \mathfrak{a}_M^*$. Then the expression

$$\partial_{\mathfrak{f}}^{\mathfrak{s}}(P, \lambda) = \frac{1}{m!} M_{\mathfrak{s}(Q_0)|P}(\lambda)^{-1} D_{\mu_0} M_{\mathfrak{s}(Q_0)|\mathfrak{s}(Q_1)}(\lambda) \cdots \\ D_{\mu_i} M_{\mathfrak{s}(Q_i)|\mathfrak{s}(Q_{i+1})}(\lambda) \cdots D_{\mu_{m-1}} M_{\mathfrak{s}(Q_{m-1})|\mathfrak{s}(Q_m)}(\lambda) M_{\mathfrak{s}(G)|P}(\lambda)$$

does not depend on the choice of the auxiliary vectors μ_i (cf. [FLc, §4, 7]), although it depends in general on the choice of \mathfrak{s} .

Theorem 1. *For $M \in \mathcal{L}$, $P \in \mathcal{P}(M)$, $L \in \mathcal{L}(M)$ and any selector $\mathfrak{s} : \mathcal{F}(M) \rightarrow \mathcal{P}(M)$ we have*

$$\mathcal{M}_L(P, \lambda) = \sum_{\mathfrak{f} \in \mathfrak{O}(L)} \partial_{\mathfrak{f}}^{\mathfrak{s}}(P, \lambda).$$

A consequence of this formula is that $\mathcal{M}_L(P, \lambda)$ can be expressed in terms of first order derivatives of the operators $M_{P_1|P_2}$ for pairs of *adjacent* parabolic subgroups P_1 and P_2 . Indeed, for any $Q, Q' \in \mathcal{P}(M)$ there exists a sequence

$$Q = P_0 |^{\alpha_1} P_1 \cdots P_{k-1} |^{\alpha_k} P_k = Q'$$

of adjacent parabolic subgroups starting with Q and ending with Q' . By the product rule, this implies

$$(1) \quad D_\mu M_{Q|Q'}(\lambda) = \sum_{j=1}^k M_{Q|P_{j-1}}(\lambda) D_\mu M_{P_{j-1}|P_j}(\lambda) M_{P_j|Q'}(\lambda).$$

We now give a more elegant expression for $\mathcal{M}_L(P, \lambda)$ which has the same feature. Let again $m = \dim \mathfrak{a}_L^G$ be the co-rank of L in G . Denote by $\mathfrak{B}_{P,L}$ the set of m -tuples $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee)$ of elements of Σ_P^\vee whose projections to \mathfrak{a}_L form a basis for \mathfrak{a}_L^G , and let $\text{vol}(\underline{\beta})$ be the co-volume in \mathfrak{a}_L^G of the lattice spanned by this basis. For any $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee) \in \mathfrak{B}_{P,L}$ let

$$\Xi_L(\underline{\beta}) = \{(Q_1, \dots, Q_m) \in \mathcal{F}_1(M)^d : \beta_i^\vee \in \mathfrak{a}_M^{Q_i}, i = 1, \dots, m\} \\ = \{(\overline{P_1 P'_1}, \dots, \overline{P_m P'_m}) : P_i |^{\beta_i} P'_i, i = 1, \dots, m\}.$$

For a pair $P_1 |^\alpha P_2$ of adjacent parabolic subgroups in $\mathcal{P}(M)$ write

$$\delta_{P_1|P_2}(\lambda) = D_\varpi M_{P_1|P_2}(\lambda) : \mathcal{A}^2(P_2) \rightarrow \mathcal{A}^2(P_1),$$

where $\varpi \in \mathfrak{a}_M^*$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$. Equivalently, if $M_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)$ for a meromorphic function Φ of one complex variable, we have $\delta_{P_1|P_2}(\lambda) = \Phi'(\langle \lambda, \alpha^\vee \rangle)$.

For any m -tuple $\mathcal{X} = (Q_1, \dots, Q_m) \in \Xi_L(\underline{\beta})$ with $Q_i = \overline{P_i P'_i}$, $P_i |^{\beta_i} P'_i$, denote by $\Delta_{\mathcal{X}}(P, \lambda)$ the expression

$$\frac{\text{vol}(\underline{\beta})}{m!} M_{P_1|P}(\lambda)^{-1} \delta_{P_1|P'_1}(\lambda) M_{P'_1|P_2}(\lambda) \cdots \delta_{P_{m-1}|P'_{m-1}}(\lambda) M_{P'_{m-1}|P_m}(\lambda) \delta_{P_m|P'_m}(\lambda) M_{P'_m|P}(\lambda).$$

We need one further combinatorial ingredient. Let $\underline{\mu} = (\mu_1, \dots, \mu_m) \in (\mathfrak{a}_M^*)^m$. Then for any $\underline{\beta} = (\beta_1, \dots, \beta_m) \in \mathfrak{B}_{P,L}$ there exists an m -tuple $(Q_1, \dots, Q_m) \in \Xi_L(\underline{\beta})$ and $\mu \in \mathfrak{a}_L^*$ such that $\mu - \mu_i \in \mathfrak{a}_{Q_i,+}^*$ for all $i = 1, \dots, m$. The vector $\mu \in \mathfrak{a}_L^*$ is in fact uniquely determined by the linear equations $\langle \mu, \beta_i^\vee \rangle = \langle \mu_i, \beta_i^\vee \rangle$, $i = 1, \dots, m$. Moreover, for $\underline{\mu}$ in general position (i.e. away from a finite set of hyperplanes) the m -tuple (Q_1, \dots, Q_m) is unique. More precisely (cf. [FLc, Remark 8.3]), a non-trivial linear dependency modulo \mathfrak{a}_M^L

$$v := c_1 \alpha_1^\vee + \dots + c_k \alpha_k^\vee \in \mathfrak{a}_M^L, \quad c_1, \dots, c_k \in \mathbb{Z}, \quad \alpha_1, \dots, \alpha_k \in \Sigma_M, \quad k \geq 1,$$

is called minimal if any proper subsequence of $(\alpha_1^\vee, \dots, \alpha_k^\vee)$ is linearly independent modulo \mathfrak{a}_M^L .² The conditions on $\underline{\mu}$ are that for any minimal dependency as above we have

- (1) $\sum_{i=1}^k c_i \langle \mu_{\pi(i)}, \alpha_i^\vee \rangle \neq 0$ for any non-constant function $\pi : \{1, \dots, k\} \rightarrow \{1, \dots, m\}$, and
- (2) if $v \neq 0$ then $\langle \mu_j, v \rangle \neq 0$ for $j = 1, \dots, m$.

For such an m -tuple $\underline{\mu}$ we obtain a map $\mathcal{X}_{L,\underline{\mu}} : \mathfrak{B}_{P,L} \rightarrow \mathcal{F}_1(M)^m$ with $\mathcal{X}_{L,\underline{\mu}}(\underline{\beta}) \in \Xi_L(\underline{\beta})$ for all $\underline{\beta} \in \mathfrak{B}_{P,L}$. The second formula for $\mathcal{M}_L(P, \lambda)$ is

Theorem 2. *Let $M \in \mathcal{L}$, $P \in \mathcal{P}(M)$, $L \in \mathcal{L}(M)$ and $\underline{\mu} \in (\mathfrak{a}_M^*)^m$ be in general position. Then we have*

$$(2) \quad \mathcal{M}_L(P, \lambda) = \sum_{\underline{\beta} \in \mathfrak{B}_{P,L}} \Delta_{\mathcal{X}_{L,\underline{\mu}}(\underline{\beta})}(P, \lambda).$$

Remark 1. The Theorem is in fact easy to prove for $m = 1$, where it reduces to the calculation of a first order derivative by the product rule (1). In this case, the image of \mathfrak{a}_L^* in $\mathfrak{a}_M^*/\mathfrak{a}_G^*$ is a line and the set $\mathcal{P}(L)$ consists of two elements R and R° . For $Q, Q' \in \mathcal{P}(M)$ with $Q \subset R$, $Q' \subset R^\circ$ we can write $\mathcal{M}_L(P, \lambda) = M_{P|Q}(\lambda) D_{|\alpha_R^\vee| \varpi_R} M_{Q|Q'}(\lambda) M_{Q'|P}(\lambda)$, where α_R is the unique element of Δ_R and $\varpi_R \in \mathfrak{a}_L^*$ is such that $\langle \varpi_R, \alpha_R^\vee \rangle = 1$. If now

$$Q = P_0 |^{\alpha_1} P_1 \dots P_{k-1} |^{\alpha_k} P_k = Q'$$

is a sequence of adjacent parabolic subgroups then we obtain from (1) that

$$\mathcal{M}_L(P, \lambda) = \sum_{j=1}^k M_{P|P_{j-1}}(\lambda) D_{|\alpha_R^\vee| \varpi_R} M_{P_{j-1}|P_j}(\lambda) M_{P_j|P}(\lambda).$$

Here, only the terms with $\alpha_j \notin \Sigma_M^L$ contribute to the sum, and if P_0, \dots, P_k is a *minimal* sequence of adjacent parabolic subgroups (a gallery) then each root $\alpha \in \Sigma_Q \setminus \Sigma_{L \cap Q}^L$ appears precisely once. In this case we can rewrite the result as

$$\mathcal{M}_L(P, \lambda) = \sum_{\beta \in \Sigma_P \setminus \Sigma_{L \cap P}^L} \Delta_{Q_1(\beta)}(P, \lambda),$$

where $Q_1(\beta) = \overline{P_{j-1} P_j}$ for the unique $1 \leq j \leq k$ with $\alpha_j = \pm \beta$. In Theorem 2 the minimal sequence of parabolic subgroups is obtained by taking the chambers $\mathfrak{a}_{P_j,+}^*$ intersected by $-\mu_1 + \mathfrak{a}_L^*$.

²Of course, the integrality assumption on c_1, \dots, c_k entails no loss of generality.

On the other hand, the case $m = 2$ is already much less evident. In this case, we can rewrite (2) in a more geometric way. Note that the space $\mathfrak{a}_L^*/\mathfrak{a}_G^*$ is two-dimensional and for each root $\beta \in \Sigma_L$ the line $\langle \lambda, \beta^\vee \rangle = 0$ is the union of two rays $\mathfrak{a}_{Q,+}^*$, $Q \in \mathcal{F}_1(L)$. Assume for simplicity that $L = M$. Suppose that $\mu_1 - \mu_2 \in \mathfrak{a}_{P_0,+}^*$. We can write \mathfrak{a}_M^* as the union of closed convex cones $\mathfrak{a}_{P_0,+}^* \cup \mathfrak{a}_{P_0^\circ,+}^* \cup \mathcal{C} \cup \mathcal{C}'$ where no two cones intersect in their interior. We can then reorder the right-hand side of (2) as the sum over all unordered pairs $\{\beta_1, \beta_2\}$ of roots in Σ_P , where for each such pair we sum $\Delta_{\mathcal{X}}(P, \lambda)$ over the two pairs $\mathcal{X} = (Q_1, Q_2) \in \Xi_M(\beta_1, \beta_2)$ for which the associated rays $\mathfrak{a}_{Q_i,+}^*$ are both contained in either \mathcal{C} or \mathcal{C}' .

Remark 2. There is also an important special case considered already in [Art82b, §7]. Namely, suppose that the operators $M_{Q|P}(\lambda)$ act as scalars and moreover that there exist meromorphic functions $\phi_\alpha : \mathbb{C} \rightarrow \mathbb{C}$, one for each $\alpha \in \Sigma_P$, such that $M_{P'|P}(\lambda) = \phi_\alpha(\langle \lambda, \alpha^\vee \rangle)$ for all pairs $P|^\alpha P'$ adjacent along α . In this case the expression $\Delta_{\mathcal{X}}(\lambda)$ does not depend on $\mathcal{X} \in \Xi_L(\beta)$ and Theorem 2 reduces to [Art82b, Corollary 7.3].

2.4. Deduction of the main formulas. Before going further, we first verify that Theorems 1 and 2 are direct consequences of the combinatorial results of [FLc]. We refer to [ibid.] for terminology and facts about polyhedral fans which will be used below.

For each pair (G, M) of a reductive group G over F and a Levi subgroup $M \in \mathcal{L}$ of co-rank d we consider the hyperplane arrangement $\mathfrak{H} = \mathfrak{H}(G, M)$ in \mathfrak{a}_M^* given by the root hyperplanes

$$H_\alpha = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle = 0\}, \quad \alpha \in \Sigma_M.$$

Recall that this gives rise to a polyhedral fan $\Sigma(G, M)$ whose chambers are the closures of the connected components of $\mathfrak{a}_M^* \setminus \bigcup_{\alpha \in \Sigma_M} H_\alpha$. In fact, $P \mapsto \mathfrak{a}_{P,+}^*$ defines an order reversing bijection between $\mathcal{F}^G(M)$ and $\Sigma(G, M)$. Under this bijection the set $\mathcal{F}_i(M)$ corresponds to the set $\Sigma_i(G, M)$ of cones of codimension i . In particular, the chambers of $\Sigma(G, M)$ correspond to $\mathcal{P}(M)$ and two chambers $\mathfrak{a}_{P,+}^*$ and $\mathfrak{a}_{Q,+}^*$ are adjacent if and only if P and Q are adjacent; for $P|^\alpha Q$ the hyperplane H_α is spanned by the wall $\mathfrak{a}_{P,+}^* \cap \mathfrak{a}_{Q,+}^* = \mathfrak{a}_{PQ,+}^*$. The fan $\Sigma(G, M)$ is simplicial: for each $P \in \mathcal{P}(M)$ there are exactly d adjacent parabolic subgroups, indexed by Δ_P . The core of $\Sigma(G, M)$ is \mathfrak{a}_G^* .

Dually, we can also think of $\Sigma(G, M)$ as the normal fan of the root zonotope $\mathcal{Z}(G, M)$ which is by definition the Minkowski sum of the intervals $[0, \alpha^\vee]$, $\alpha \in \Sigma_M$. The faces of $\mathcal{Z}(G, M)$ correspond to the cones of $\Sigma(G, M)$, namely to $\mathcal{F}(M)$. For example, when $G = \mathrm{GL}(n)$ and M is a maximal torus, the root zonotope is the well-known permutahedron (cf. [Zie95, p. 17-18, 200], [Pos09]).

Changing G and M is reflected by standard operations on the fan $\Sigma(G, M)$ or dually on $\mathcal{Z}(G, M)$. Namely, for any $Q \in \mathcal{F}(M)$ the restricted fan $\Sigma(G, M)^{\supset \mathfrak{a}_{Q,+}^*}$ with respect to $\mathfrak{a}_{Q,+}^*$ ([FLc, §2]) is $\Sigma(L, M)$ where L is the Levi subgroup of Q . Dually, $\mathcal{Z}(L, M)$ is up to translation the face corresponding to Q in $\mathcal{Z}(G, M)$ (viewed as a zonotope in its own right). On the other hand, if $U = \mathfrak{a}_M^L$ for $L \in \mathcal{L}(M)$ then the induced fan $\Sigma(G, M)^\sharp$ on $U^\perp = \mathfrak{a}_L^*$ ([FLc, §7]) is $\Sigma(G, L)$. Once again, $\mathcal{Z}(G, L)$ is the projection of $\mathcal{Z}(G, M)$ along U .

In the language of [FLc], Arthur's notion of a (G, M) -family becomes a $\Sigma(G, M)$ -piecewise smooth function, i.e. a $\Sigma(G, M)$ -piecewise element of the ring of smooth functions on \mathfrak{a}_M^* (considered as a flat module over $\text{Sym}(\mathfrak{a}_M)$). The push-forward $\delta_{\Sigma(G, M)}$ of a (G, M) -family $(c_P(\lambda))_{P \in \mathcal{P}(M)}$ (viewed as a $\Sigma(G, M)$ -piecewise smooth function) is precisely the function denoted by $c_M(\lambda)$ on [Art82b, p. 1297]. Moreover, the operations described subsequently in [loc. cit.] correspond to the restriction of a $\Sigma(G, M)$ -piecewise smooth function c to $\Sigma(G, M)^{\triangleright \mathfrak{a}_{Q,+}^*} = \Sigma(M_Q, M)$ and the operation $c^\#$ with respect to $U = \mathfrak{a}_M^L$, respectively.

Recall the notion of a *compatible family* with respect to a fan [FLc, Definition 4.1]. In the case of the fan $\Sigma(G, M)$, a compatible family is a collection $\mathcal{A} = (\mathcal{A}_P)_{P \in \mathcal{P}(M)}$ of power series $\mathcal{A}_P \in \mathcal{E}[[\mathfrak{a}_M^*]]$, where \mathcal{E} is a finite-dimensional \mathbb{C} -algebra, subject to the conditions $\mathcal{A}_P(0) = 1_{\mathcal{E}}$ for all $P \in \mathcal{P}(M)$ and $\mathcal{A}_{P_1} \mathcal{A}_{P_2}^{-1} \in \mathcal{E}[[\alpha^\vee]]$ for all pairs $P_1 |^\alpha P_2$ of adjacent parabolic subgroups. Fix $\lambda \in \mathfrak{a}_{M, \mathbb{C}}^*$ in general position. For each $Q \in \mathcal{P}(M)$ let \mathcal{A}_Q be the Taylor expansion at $\Lambda = 0$ of the restriction of the operator $\mathcal{M}_Q(P, \lambda, \Lambda)$ to a fixed (finite-dimensional) (\mathfrak{g}, K) -isotypic subspace V of $\mathcal{A}^2(P)$. By the functional equations we obtain a $\Sigma(G, M)$ -compatible family with values in the algebra $\mathcal{E} = \text{End}(V)$ and

$$\mathcal{A}_{P_1} \mathcal{A}_{P_2}^{-1} = M_{P_1|P}(\lambda)^{-1} M_{P_1|P_2}(\lambda + \Lambda) M_{P_2|P}(\lambda), \quad P_1, P_2 \in \mathcal{P}(M).$$

Using the dictionary above, Theorems 1 and 2 are then obtained by applying [FLc, Theorem 7.4] and [FLc, Theorem 8.2] respectively to \mathcal{A} , $\Sigma(G, M)$ and $U = \mathfrak{a}_M^L$.

3. A REFINED SPECTRAL EXPANSION

In this section, we apply Theorems 1 and 2 to derive a refinement of the spectral side of Arthur's trace formula. Fix an open subgroup K_0 of K_f . The space $G(\mathbb{A})/K_0$ is a discrete union of countably many copies of $G(F_\infty)$ and in particular a differentiable manifold. Let $C^\infty(G(\mathbb{A}); K_0)$ be the space of smooth functions on $G(\mathbb{A})/K_0$, viewed as right- K_0 -invariant functions on $G(\mathbb{A})$. We consider the topological vector space $\mathcal{C}(G(\mathbb{A}), K_0)$ of all functions $h \in C^\infty(G(\mathbb{A}), K_0)$ such that $|h * X|_{L^1(G(\mathbb{A}))} < \infty$ for all $X \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ with the topology induced by the seminorms $|h * X|_{L^1(G(\mathbb{A}))}$. For any $h \in \mathcal{C}(G(\mathbb{A}), K_0)$ the image of the operator $\rho(P, \lambda, h)$ lies in the smooth and K_0 -invariant part of $\bar{\mathcal{A}}^2(P)$.

The main technical statement of this paper is the following Theorem.

Theorem 3. *Fix $K_0 \subset K_f$ and let $M \in \mathcal{L}$, $P \in \mathcal{P}(M)$ and $L \in \mathcal{L}(M)$. Then for any $\beta \in \mathfrak{B}_{P, L}$ and $\mathcal{X} \in \Xi_L(\beta)$ the seminorm*

$$\int_{\mathfrak{ia}_L^*} \|\Delta_{\mathcal{X}}(P, \lambda) \rho(P, \lambda, h)\|_1 \, d\lambda$$

on $\mathcal{C}(G(\mathbb{A}), K_0)$ is continuous, where $\|\cdot\|_1$ denotes the trace norm on $\bar{\mathcal{A}}^2(P)$. Similarly, for any selector $\mathfrak{s} : \mathcal{F}(M) \rightarrow \mathcal{P}(M)$ and any flag $\mathfrak{f} \in \mathfrak{G}(L)$ the seminorm

$$\int_{\mathfrak{ia}_L^*} \|\partial_{\mathfrak{f}}^{\mathfrak{s}}(P, \lambda) \rho(P, \lambda, h)\|_1 \, d\lambda$$

is continuous.

Implicit here is that for almost all $\lambda \in \mathfrak{ia}_L^*$ the operator $\Delta_{\mathcal{X}}(P, \lambda)\rho(P, \lambda, h)$ extends to a trace class operator on $\bar{\mathcal{A}}^2(P)$.

Remark 3. The case $P = G$ essentially amounts to the trace-class conjecture of Selberg which asserts that $\rho(G, h)$ is of trace class. It was settled in [Mül89] for K -finite test functions and independently in [Mül98] and [Ji98] in the general case.

Recall that $L_{\text{disc}}^2(A_M M(F) \backslash M(\mathbb{A}))$ splits as the completed direct sum of its π -isotypic components for $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$. We have a corresponding decomposition of $\bar{\mathcal{A}}^2(P)$ as a direct sum of Hilbert spaces $\hat{\oplus}_{\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))} \bar{\mathcal{A}}_{\pi}^2(P)$. Similarly, we have the algebraic direct sum decomposition $\mathcal{A}^2(P) = \oplus_{\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))} \mathcal{A}_{\pi}^2(P)$, where $\mathcal{A}_{\pi}^2(P)$ is the K -finite part of $\bar{\mathcal{A}}_{\pi}^2(P)$. We further decompose

$$\mathcal{A}_{\pi}^2(P) = \oplus_{\tau \in \widehat{K_{\infty}}} \mathcal{A}_{\pi}^2(P)^{\tau}$$

according to isotypic subspaces for the action of K_{∞} . Let $\mathcal{A}_{\pi}^2(P)^{K_0}$ be the subspace of K_0 -invariant functions in $\mathcal{A}_{\pi}^2(P)$, and similarly for $\mathcal{A}_{\pi}^2(P)^{K_0, \tau}$ for any $\tau \in \widehat{K_{\infty}}$. The latter space is always finite-dimensional. For any $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$ let λ_{π} denote the Casimir eigenvalue of π_{∞} . Let also λ_{τ} be the Casimir eigenvalue of $\tau \in \widehat{K_{\infty}}$.

Theorem 3 will be proved by reducing it to the following statement in the co-rank one situation.

Proposition 1. *Let $P, Q \in \mathcal{P}(M)$ with $P|^{\alpha}Q$ and let $\varpi \in \mathfrak{a}_M^*$ be such that $\langle \varpi, \alpha^{\vee} \rangle = 1$. Then there exist C, N and N_1 such that*

$$(3) \quad \int_{\mathbb{R}} \|\delta_{P|Q}(s\varpi)|_{\mathcal{A}_{\pi}^2(Q)^{K_0, \tau}}\| (1 + |s|)^{-N} ds \leq C(1 + \lambda_{\tau}^2 + \lambda_{\pi}^2)^{N_1}$$

for any $\tau \in \widehat{K_{\infty}}$ and $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$.

The following refinement of Arthur's spectral expansion is a consequence of Theorem 3. Let $\mathcal{C}(G(\mathbb{A}))$ be the inductive limit of $\mathcal{C}(G(\mathbb{A}), K_0)$ over the open subgroups K_0 of K_f . For any $s \in W(M)$ let L_s be the smallest Levi subgroup in $\mathcal{L}(M)$ containing w_s . It is also characterized by the condition $\mathfrak{a}_{L_s} = \{H \in \mathfrak{a}_M \mid sH = H\}$ ([Art82b, p. 1299], cf. [OT92, Theorem 6.27]). We set

$$\iota_s = \left| \det(s - 1)_{\mathfrak{a}_{L_s}^{L_s}} \right|^{-1}.$$

For $P \in \mathcal{F}(M_0)$ and $s \in W(M_P)$ let $M(P, s) : \mathcal{A}^2(P) \rightarrow \mathcal{A}^2(P)$ be as in [Art82b, p. 1309]. This is a unitary operator which commutes with the operators $\rho(P, \lambda, h)$ for $\lambda \in \mathfrak{ia}_{L_s}^*$.

Corollary 1. *For any $h \in C_c^{\infty}(G(\mathbb{A}))$ the spectral side of Arthur's trace formula is given by*

$$\sum_{[P]} \frac{1}{|W(M_P)|} \sum_{s \in W(M_P)} \iota_s \int_{\mathfrak{ia}_{L_s}^*} \text{tr}(\mathcal{M}_L(P, \lambda) M(P, s) \rho(P, \lambda, h)) d\lambda$$

where the sum is over representatives in $\mathcal{F}(M_0)$ of associate classes of parabolic subgroups. We can also write it as

$$\sum_{[P]} \frac{1}{|W(M_P)|} \sum_{s \in W(M_P)} \iota_s \sum_{\underline{\beta} \in \mathfrak{B}_{P, L_s}} \int_{\mathfrak{ia}_{L_s}^*} \mathrm{tr}(\Delta_{\mathcal{X}_{L_s, \underline{\beta}}}(P, \lambda) M(P, s) \rho(P, \lambda, h)) d\lambda$$

or as

$$\sum_{[P]} \frac{1}{|W(M_P)|} \sum_{s \in W(M_P)} \iota_s \sum_{\mathfrak{f} \in \mathfrak{G}(L_s)} \int_{\mathfrak{ia}_{L_s}^*} \mathrm{tr}(\partial_{\mathfrak{f}}^{\mathfrak{s}}(P, \lambda) M(P, s) \rho(P, \lambda, h)) d\lambda$$

with $\underline{\mu}$ and \mathfrak{s} as in Theorem 1 and 2 above. In all these expressions the sums are finite and the integrals are absolutely convergent with respect to the trace norm and define distributions on $\mathcal{C}(G(\mathbb{A}))$.

We note that in the case $G = \mathrm{GL}(n)$ the absolute convergence statement of Corollary 1 (however without the more explicit formulas) was established by a different method in [MS04].

Remark 4. It is natural to ask whether there is an analogous result for the geometric side, namely whether the sum of weighted orbital integrals extends continuously to $\mathcal{C}(G(\mathbb{A}))$. This would yield a trace formula identity for a large class of test functions. Such functions play a role in Langlands' idea of "beyond endoscopy." At any rate, at least for the semisimple part of the geometric side the answer is positive [FLb] (cf. also [FL11], where the full geometric side is discussed for $G = \mathrm{GL}(2)$).

4. THE MAASS-SELBERG RELATIONS

As a byproduct of Theorems 1 and 2 we also get an explication of the Maass-Selberg relations in the so-called singular case. Let χ be a cuspidal datum of G (i.e. a $G(F)$ -conjugacy class of pairs (L, σ) consisting of a Levi subgroup L of G defined over F and an irreducible cuspidal representation σ of $L(\mathbb{A}^1)$). Furthermore, let $M \in \mathcal{L}$ be of co-rank d in G , $P \in \mathcal{P}(M)$ and π a representation of $M(\mathbb{A}^1)$. We also need to fix a minimal parabolic subgroup $P_0 \in \mathcal{P}(M_0)$. Let $T_0 \in \mathfrak{a}_{M_0}$ be the vector defined in [Art81, Lemma 1.1]. Recall the definition of the operator $\omega_{\chi, \pi}^T(P, \lambda)$ on $\mathcal{A}_{\chi, \pi}^2(P)$ for $T \in \mathfrak{a}_{M_0}$ and $\lambda \in \mathfrak{ia}_M^*$ ([Art82a, p. 1278], [Art82b, §2]). It is given as the sum over all $s \in W(M, \pi) := \{s \in W(M) : s\pi = \pi\}$ of

$$\lim_{\Lambda \rightarrow s\lambda - \lambda} \sum_{Q \in \mathcal{P}(M)} e^{\langle \Lambda, Y_Q(T) \rangle} \frac{M_{Q|P}(\lambda)^{-1} M_{Q|P}(s, s^{-1}(\lambda + \Lambda))}{\theta_Q(\Lambda)}.$$

Here, $M_{Q|P}(s, \lambda)$ are the variants of the intertwining operators defined in [Art82b, p. 1292], and $Y_Q(T)$ is the projection of $t^{-1}(T - T_0) + T_0$ to \mathfrak{a}_M , where $t \in W_0$ is such that $tQ \supset P_0$.

For $\varphi \in \mathcal{A}_{\chi, \pi}^2(P)$ let $E(g, \varphi, \lambda)$ be the Eisenstein series associated to φ and λ , which is given by the meromorphic continuation of

$$\sum_{\gamma \in P(F) \backslash G(F)} \varphi(\gamma g) e^{\langle \lambda, H_P(\gamma g) \rangle}.$$

By Arthur's asymptotic inner product formula for truncated Eisenstein series ([Art82c, Corollary 9.2], [Art82a, p. 1279]) we have

$$(\Lambda^T E(\cdot, \varphi_1, \lambda), E(\cdot, \varphi_2, \lambda))_{G(F)\backslash G(\mathbb{A})^1} = (\omega_{\chi, \pi}^T(P, \lambda)\varphi_1, \varphi_2)_{\mathcal{A}_{\chi, \pi}^2} + e(\varphi_1, \varphi_2, \lambda, T)$$

for all $\varphi_1, \varphi_2 \in \mathcal{A}_{\chi, \pi}^2(P)$, where the error term $e(\varphi_1, \varphi_2, \lambda, T)$ is exponentially small in $\min_{\alpha \in \Delta_0} \langle \alpha, T \rangle$ for T in the positive Weyl chamber with respect to P_0 . In the case where π is cuspidal (and χ is the cuspidal datum associated to M and π) we have $e(\varphi_1, \varphi_2, \lambda, T) = 0$ for T sufficiently regular ([Lan76], [Art80, §4]). (An alternative approach to these results is contained in [JLR99] for cuspidal π and in [Lap11] for the general case.)

As before, by the functional equations for the operators $M_{Q|P}(s, \lambda)$ [Art82b, (1.2)] the Taylor expansions \mathcal{A}_Q of $M_{Q|P}(\lambda)^{-1}M_{Q|P}(s, s^{-1}(\lambda + \Lambda))M_{P|P}(s^{-1}, \lambda)$ at $\Lambda = 0$ form a compatible family (when restricted to a finite-dimensional K_0 -fixed and τ -isotypic subspace as before) and we have

$$\mathcal{A}_{P_1}\mathcal{A}_{P_2}^{-1} = M_{P_1|P}(\lambda)^{-1}M_{P_1|P_2}(\lambda + \Lambda)M_{P_2|P}(\lambda).$$

For adjacent parabolic subgroups $Q|^\beta Q'$ in $\mathcal{P}(M)$ let $t \in W_0$ be such that $tQ \supset P_0$. Then $t\beta \in \mathfrak{a}_{tM}^*$ lifts to a unique simple root $\alpha_{\overline{QQ'}}$ in Δ_0 and we have

$$Y_Q(T) - Y_{Q'}(T) = \langle \alpha_{\overline{QQ'}}, T - T_0 \rangle \beta^\vee.$$

Therefore, the collection $Y(T) = (Y_Q(T))_{Q \in \mathcal{P}(M)}$ defines a piecewise linear function on the fan $\Sigma(G, M)$ and the Taylor series c_Q of $e^{\langle \Lambda, Y_Q(T) \rangle}$ at $\Lambda = 0$ form a scalar-valued compatible family. If $\langle \alpha, T - T_0 \rangle > 0$ for all $\alpha \in \Delta_0$, as we may assume, the $Y_Q(T)$ form even a positive A_M -orthogonal set, i.e. they are precisely the vertices of their convex hull, which is a polytope in \mathfrak{a}_M . (Cf. [FLc, §4] and also [Art81, §7] for more details.)

We conclude that (the restrictions to suitable finite-dimensional subspaces of) the operators $c_Q \mathcal{A}_Q$ form a compatible family. Using [FLc], we can evaluate the limit in the definition of $\omega_{\chi, \pi}^T(P, \lambda)$ at the point $\Lambda = 0$, or equivalently in the case where $s\lambda = \lambda$ for all $s \in W(M, \pi)$. The resulting expressions are explicit, although combinatorially complicated polynomials in T .

To describe them, let $\mathfrak{s} : \mathcal{F}(M) \rightarrow \mathcal{P}(M)$ be a selector and $\mathfrak{f} \in \mathfrak{G}(M)$. Set

$$\begin{aligned} \partial_{\mathfrak{f}}^{\mathfrak{s}; T}(P, \lambda) &= \frac{1}{d!} M_{\mathfrak{s}(Q_0)|P}(\lambda)^{-1} \prod_{i=1}^d [D_{\mu_{i-1}} M_{\mathfrak{s}(Q_{i-1})|\mathfrak{s}(Q_i)}(\lambda) \\ &\quad + \langle \mu_i, Y_{\mathfrak{s}(Q_{i-1})}(T) - Y_{\mathfrak{s}(Q_i)}(T) \rangle M_{\mathfrak{s}(Q_{i-1})|\mathfrak{s}(Q_i)}(\lambda)] M_{\mathfrak{s}(G)|P}(\lambda) \end{aligned}$$

with $\mu_i \in \mathfrak{a}_{Q_i,+}^*$ as in §2.3. We can rewrite this expression as the sum over all $0 \leq k \leq d$ and all indices $1 \leq i_1 < \dots < i_k \leq d$ of

$$\frac{1}{d!} \prod_{i \neq i_1, \dots, i_k} \langle \mu_i, Y_{\mathfrak{s}(Q_{i-1})}(T) - Y_{\mathfrak{s}(Q_i)}(T) \rangle M_{\mathfrak{s}(Q_{i_1-1})|P}(\lambda)^{-1} \\ \prod_{j=1}^{k-1} \left(D_{\mu_{i_{j-1}}} M_{\mathfrak{s}(Q_{i_{j-1}})|\mathfrak{s}(Q_{i_j})}(\lambda) M_{\mathfrak{s}(Q_{i_j})|\mathfrak{s}(Q_{i_{j+1}-1})}(\lambda) \right) D_{\mu_{i_k-1}} M_{\mathfrak{s}(Q_{i_k-1})|\mathfrak{s}(Q_{i_k})}(\lambda) M_{\mathfrak{s}(Q_{i_k})|P}(\lambda).$$

For $T = T_0$ only $k = d$ contributes and we get $\partial_{\mathfrak{f}}^{s;T_0}(P, \lambda) = \partial_{\mathfrak{f}}^s(P, \lambda)$. Also, define

$$\Delta_{\mathcal{X}}^T(P, \lambda) = \frac{\text{vol}(\underline{\beta})}{d!} M_{P_1|P}(\lambda)^{-1} \prod_{i=1}^d \left[\left(\delta_{P_i|P'_i}(\lambda) + \left\langle \alpha_{\overline{P_i P'_i}}, T - T_0 \right\rangle M_{P_i|P'_i}(\lambda) \right) M_{P'_i|P_{i+1}}(\lambda) \right],$$

where in the last factor P_{d+1} is replaced by P . We can rewrite this analogously as the sum over all $0 \leq k \leq d$ and all indices $1 \leq i_1 < \dots < i_k \leq d$ of

$$\frac{\text{vol}(\underline{\beta})}{d!} \prod_{i \neq i_1, \dots, i_k} \left\langle \alpha_{\overline{P_i P'_i}}, T - T_0 \right\rangle M_{P_{i_1}|P}(\lambda)^{-1} \prod_{j=1}^k \left(\delta_{P_{i_j}|P'_{i_j}}(\lambda) M_{P'_{i_j}|P_{i_{j+1}}}(\lambda) \right)$$

with the convention that $P_{i_{k+1}}$ should be replaced by P in the last factor. This gives a combinatorial expression for the polynomial $\Delta_{\mathcal{X}}^T(P, \lambda)$ as a sum of monomials in the root coordinates $\langle \alpha, T - T_0 \rangle$, $\alpha \in \Delta_0$. Again we have $\Delta_{\mathcal{X}}^{T_0}(P, \lambda) = \Delta_{\mathcal{X}}(P, \lambda)$. We can now apply the combinatorial formulas of [FLc, Corollary 4.3 and Theorem 8.1] to the operators $c_Q \mathcal{A}_Q$, $Q \in \mathcal{P}(M)$, to get the following result.

Theorem 4. *Assume that $\lambda \in \mathfrak{ia}_M^*$ is singular in the sense that $s\lambda = \lambda$ for all $s \in W(M, \pi)$. Then*

$$\begin{aligned} \omega_{\chi, \pi}^T(P, \lambda) &= \sum_{s \in W(M, \pi)} \sum_{\mathfrak{f} \in \mathfrak{G}(M)} \partial_{\mathfrak{f}}^{s;T}(P, \lambda) M(P, s) \\ &= \sum_{s \in W(M, \pi)} \sum_{\underline{\beta} \in \mathfrak{B}_{P, M}} \Delta_{\mathcal{X}_{M, \underline{\beta}}}^T(P, \lambda) M(P, s). \end{aligned}$$

The leading term of this polynomial is simply the volume of the convex hull of the $Y_Q(T)$ times the operator $\sum_{s \in W(M, \pi)} M(P, s)$, and the Theorem provides a combinatorial formula for this volume (cf. [FLc, Pos09]). On the other hand, the value at $T = T_0$ is given by the formulas of Theorems 1 and 2 for $L = M$.

5. PROOF OF ABSOLUTE CONVERGENCE

In this section we give the proofs of our analytic results Theorem 3 and Corollary 1.

5.1. Reduction of Theorem 3 to Proposition 1. Fix $M \in \mathcal{L}$, $P \in \mathcal{P}(M)$, $L \in \mathcal{L}(M)$ as above and let m be the co-rank of L in G . Using (1) the operators $\partial_{\mathfrak{f}}^s(P, \lambda)$ can be expressed as linear combinations of the operators $\Delta_{\mathcal{X}}(P, \lambda)$. Therefore it is enough to show the first part of Theorem 3.

Fix $\underline{\beta} \in \mathfrak{B}_{P,L}$ and $\mathcal{X} \in \Xi_L(\underline{\beta})$. Let

$$\Delta = \text{Id} - \Omega + 2\Omega_{K_\infty}$$

where Ω (resp. Ω_{K_∞}) is the Casimir operator of $G(F_\infty)$ (resp. K_∞). The operator $\Delta_{\mathcal{X}}(P, \lambda)\rho(P, \lambda, \Delta^{2k})^{-1}$, $k \in \mathbb{N}$ is defined on $\mathcal{A}^2(P)$. We will show the convergence of

$$(4) \quad \int_{\mathfrak{ia}_L^*} \|\Delta_{\mathcal{X}}(P, \lambda)\rho(P, \lambda, \Delta^{2k})^{-1}\|_1 d\lambda$$

for sufficiently large k . In particular for almost all λ , $\Delta_{\mathcal{X}}(P, \lambda)\rho(P, \lambda, \Delta^{2k})^{-1}$ extends to a trace-class, and a fortiori bounded, operator on $\bar{\mathcal{A}}^2(P)$. Since

$$\begin{aligned} \|\Delta_{\mathcal{X}}(P, \lambda)\rho(P, \lambda, h)\|_1 &\leq \|\Delta_{\mathcal{X}}(P, \lambda)\rho(P, \lambda, \Delta^{2k})^{-1}\|_1 \|\rho(P, \lambda, \Delta^{2k} * h)\| \\ &\leq \|\Delta_{\mathcal{X}}(P, \lambda)\rho(P, \lambda, \Delta^{2k})^{-1}\|_1 \|h * \Delta^{2k}\|_{L^1(G(\mathbb{A}))} \end{aligned}$$

this will imply Theorem 3.

It remains to show the convergence of (4). The operator $\rho(P, \lambda, \Delta)$ acts on $\mathcal{A}_\pi^2(P)^{K_{0,\tau}}$ by the scalar $\mu(\pi, \lambda, \tau) = 1 + \|\lambda\|^2 - \lambda_\pi + 2\lambda_\sigma$. By [Mül02, (6.9)], this scalar satisfies

$$(5) \quad |\mu(\pi, \lambda, \tau)|^2 \geq \frac{1}{4}(1 + \|\lambda\|^2 + \lambda_\pi^2 + \lambda_\tau^2).$$

Suppose that $\mathcal{X} = (\overline{P_1 P'_1}, \dots, \overline{P_m P'_m})$ with $P_i |^{\alpha_i} P'_i$, $i = 1, \dots, m$. Using the inequality

$$\|A\|_1 \leq \dim V \|A\|$$

for any linear operator A on a finite-dimensional Hilbert space V , and the unitarity of $M_{Q|P}(\lambda)$, we reduce the problem to the convergence of

$$\sum_{\tau \in \widehat{K_\infty}} \sum_{\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))} \dim(\mathcal{A}_\pi^2(P)^{K_{0,\tau}}) \int_{\mathfrak{ia}_L^*} |\mu(\pi, \lambda, \tau)|^{-2k} \prod_{i=1}^m \|\delta_{P_i|P'_i}(\lambda)|_{\mathcal{A}_\pi^2(P'_i)^{K_{0,\tau}}}\| d\lambda$$

for sufficiently large k . By integrating first over \mathfrak{ia}_G^* we may replace the integral over \mathfrak{ia}_L^* by an integral over \mathfrak{ia}_L^G . Recall that $\delta_{P_i|P'_i}(\lambda)$ depends only on $\langle \lambda, \alpha_i^\vee \rangle$. Let $\varpi_1, \dots, \varpi_m$ be the basis of $(\mathfrak{a}_L^G)^*$ dual to $\alpha_1^\vee, \dots, \alpha_m^\vee$; thus, the coordinates of λ with respect to $\varpi_1, \dots, \varpi_m$ are $\langle \lambda, \alpha_i^\vee \rangle$. Using these coordinates Proposition 1 reduces the statement to the convergence of

$$\sum_{\tau \in \widehat{K_\infty}} \sum_{\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))} \dim(\mathcal{A}_\pi^2(P)^{K_{0,\tau}}) (1 + \lambda_\pi^2 + \lambda_\tau^2)^{-k}$$

for k sufficiently large. This in turn follows from [Mül98, Corollary 0.3].

5.2. Proof of Proposition 1. The operator $M_{Q|P}(\lambda)$ depends only on $s = \langle \lambda, \alpha^\vee \rangle$. For convenience we denote its restriction to $\mathcal{A}_\pi^2(P)$ by $M_{Q|P}(\pi, s)$. Since $M_{Q|P}(\lambda)$ is unitary for $\lambda \in \mathfrak{ia}_M^*$ we may replace the left-hand side of (3) by

$$\int_{\mathfrak{i}\mathbb{R}} \|M_{Q|P}(\pi, s)^{-1} M'_{Q|P}(\pi, s)|_{\mathcal{A}_\pi^2(P)^{K_0, \tau}}\| (1 + |s|)^{-N} ds.$$

We have a canonical isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_C, K_\infty)$ -modules

$$j_P : \text{Hom}(\pi, L^2(A_M M(F) \backslash M(\mathbb{A}))) \otimes \text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi) \rightarrow \mathcal{A}_\pi^2(P).$$

The operator $M_{Q|P}(\pi, s)$ admits a normalization by a global factor $n_\alpha(\pi, s)$ which is a meromorphic function in s . We write

$$M_{Q|P}(\pi, s) \circ j_P = n_\alpha(\pi, s) \cdot j_Q \circ (\text{Id} \otimes N_{Q|P}(\pi, s))$$

where $N_{Q|P}(\pi, s) = \otimes_v N_{Q|P}(\pi_v, s)$ is the product of the locally defined normalized intertwining operators and $\pi = \otimes_v \pi_v$ ([Art82b, §6], cf. [Mül02, (2.17)]). Consequently we have

$$M_{Q|P}(\pi, s)^{-1} M'_{Q|P}(\pi, s) = \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \text{Id} + j_P \circ (\text{Id} \otimes N_{Q|P}(\pi, s)^{-1} N'_{Q|P}(\pi, s)) \circ j_P^{-1}.$$

By [Mül02, Theorem 5.3] there exist $C > 0$, $N, N_1 \in \mathbb{N}$ such that

$$\int_{\mathfrak{i}\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-N} ds \leq C(1 + \Lambda_\pi^2)^{N_1}$$

for all $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$ with $\mathcal{A}_\pi^2(P)^{K_0} \neq 0$. Here, as in [ibid.],

$$\Lambda_\pi = \min_{\tau \in W_P(\pi_\infty)} \sqrt{\lambda_\pi^2 + \lambda_\tau^2}$$

where $W_P(\pi_\infty)$ denotes the set of minimal K_∞ -types of the induced representation $\text{Ind}_P^G(\pi_\infty)$.

To deal with the term involving the normalized intertwining operator, we may assume, by passing to a finite index subgroup if necessary, that $K_0 = \prod_{v < \infty} K_v$ where K_v is an open compact subgroup of $G(F_v)$ and K_v is hyperspecial for almost all v . Let $N_{Q|P}(\pi_v, s)_{K_v}$ denote the restriction of $N_{Q|P}(\pi_v, s)$ to the subspace of K_v -invariant vectors, and for $\tau = \otimes_{v|\infty} \tau_v \in \widehat{K_\infty}$ let $N_{Q|P}(\pi_v, s)_{\tau_v}$ be the restriction of $N_{Q|P}(\pi_v, s)$ to the τ_v -isotypic subspace. We recall from [Art89] that there exists a finite set S of places of F , which contains the archimedean ones and depends only on K_0 , such that

$$N_{Q|P}(\pi_v, s)_{K_v} = \text{Id}, \quad v \notin S.$$

Thus $N_{Q|P}(\pi, s)^{-1} N'_{Q|P}(\pi, s) = \sum_{v \in S} N_{Q|P}(\pi_v, s)^{-1} N'_{Q|P}(\pi_v, s)$ on $\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi)^{K_0}$. Let $\Pi(M(F_v))$ be the set of equivalence classes of irreducible representations of the local group $M(F_v)$. Using the unitarity of $N_{Q|P}(\pi_v, s)$ for $s \in \mathfrak{i}\mathbb{R}$ it remains to show the existence of $C > 0$ and $N_1 \in \mathbb{N}$ such that for all $v \in S$ and $\pi \in \Pi(M(F_v))$

$$(6) \quad \int_{\mathfrak{i}\mathbb{R}} \|N'_{Q|P}(\pi_v, s)_{K_v}\| (1 + |s|)^{-N} ds \leq C$$

if v is non-archimedean, and

$$(7) \quad \int_{\mathbb{i}\mathbb{R}} \|N'_{Q|P}(\pi_v, s)_{\tau_v}\| (1 + |s|)^{-N} ds \leq C(1 + \|\tau_v\|)^{N_1}$$

for all $\tau_v \in \widehat{K}_v$ if v is archimedean.

Since $\|(a_{ij})\| \leq (\sum |a_{ij}|^2)^{\frac{1}{2}} \leq \sum |a_{ij}|$, the left-hand sides of (6) and (7) are bounded by

$$\sum_{i,j} \int_{\mathbb{i}\mathbb{R}} |(N'_{Q|P}(\pi_v, s)e_i, e_j)| (1 + |s|)^{-N} ds,$$

where e_i is an orthonormal basis for $\text{Ind}(\pi_v)^{K_v}$ (in the p -adic case) or $\text{Ind}(\pi_v)^\tau$ (in the archimedean case). Note that $\dim \text{Ind}(\pi_v)^{K_v}$ is bounded independently of π_v in the p -adic case and $\dim \text{Ind}(\pi_v)^\tau \leq (\deg \tau_v)^2$ for $v|\infty$. Let $\|\tau_v\|$ be the norm of the highest weight of τ_v . By Weyl's dimension formula, $\deg \tau_v$ is bounded polynomially in $\|\tau_v\|$.

We now appeal to the following Lemma.

Lemma 1. *Let C be either the imaginary axis or the unit circle. Let $f(z)$ be a scalar valued rational function of degree $\leq m$ such that $|f(z)| \leq 1$ for all $z \in C$. Then*

$$(8) \quad \oint_C |f'(z)| |dz| \leq 8m$$

We are grateful to Benjamin Weiss for communicating to us the following simple proof.

Proof. Assume first that f takes real values on C . Then the left-hand side of (8) is the total variation of f on C , i.e. $\sum_{j=1}^k |f(z_j) - f(z_{j-1})|$ where $z_j, j = 1, \dots, k$ are the extrema of f on C and we set $z_0 = z_k$. Since $k \leq 2m$, we get

$$\oint_C |f'(z)| |dz| \leq 4m$$

in this case. The general case follows immediately. \square

Remark 5. Let C be as in Lemma 1. Borwein and Erdélyi proved the following stronger inequality ([BE96]). Let $a_1, \dots, a_m \in C$ and define

$$\phi_{\geq}(z) = \prod_{j:|a_j| \geq 1} \frac{1 - \bar{a}_j z}{z - a_j}$$

if C is the unit circle and

$$\phi_{\geq}(z) = \prod_{j:\text{Re } a_j \geq 0} \frac{z - \bar{a}_j}{z + a_j}$$

if C is the imaginary axis. Then for any f such that $|f(z)| \leq 1$ on C and $\prod_{j=1}^m (z - a_j)f(z)$ is a polynomial of degree $\leq m$ we have

$$|f'(z)| \leq \max(|\phi'_{>}(z)|, |\phi'_{<}(z)|)$$

on C . Estimating the maximum by the sum and integrating over C we obtain Lemma 1 with 8 replaced by 2π which is best possible.

Going back to the proof of Proposition 1 we recall that the operators $N_{Q|P}(\pi_v, s)_{K_v}$ are unitary on the imaginary axis, and therefore their matrix coefficients are bounded by 1. Using Lemma 1 and the discussion preceding it remains to show the following

Lemma 2. *Let $P|^\alpha Q \in \mathcal{P}(M)$ and $\pi_v \in \Pi(M(F_v))$.*

(1) *Suppose that v is p -adic and $(\text{Ind}_P^G \pi_v)^{K_v} \neq 0$. Then any matrix coefficient*

$$(N_{Q|P}(\pi_v, s)\varphi_1, \varphi_2), \quad \varphi_1, \varphi_2 \in (\text{Ind } \pi_v)^{K_v},$$

is of the form $f(q^s)$ for some rational function f with $\deg f$ bounded in terms of K_v only.

(2) *Suppose that v is archimedean and let $\tau \in \widehat{K_v}$. Then any matrix coefficient*

$$f(s) = (N_{Q|P}(\pi_v, s)\varphi_1, \varphi_2), \quad \varphi_1, \varphi_2 \in (\text{Ind } \pi_v)^\tau,$$

is a rational function with $\deg f \leq c(1 + \|\tau\|)$ where c depends only on G .

Proof. We argue as in [MS04]. The rationality of f in both cases follows from [Art89, Theorem 2.1]. Suppose first that v is p -adic. In the following, the notation will be relative to F_v . (In particular, M_0 is a minimal Levi defined over F_v and so on.) We will also consider the normalized intertwining operators $N_{P_2|P_1}(\pi, \lambda)$ for general $P_1, P_2 \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}_M^*$. This differs a little from our previous notation since the operator $N_{Q|P}(\pi, s)$ is now written as $N_{Q|P}(\pi, s\varpi)$ where $\varpi \in \mathfrak{a}_M^*$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$. Note that P and Q are not necessarily adjacent anymore since the split rank may grow under base field extension. Write π_v as a Langlands quotient $J_{P_1}^M(\sigma_v, \mu)$ where $P_1 \in \mathcal{F}^M(M_0)$, σ_v is a tempered representation of M_{P_1} and μ is in the relative interior of $\mathfrak{a}_{P_1,+}^* \subset (\mathfrak{a}_0^M)^*$. Therefore, π_v is a quotient of $\text{Ind}_{P_2}^M(\delta_v, \mu)$ where $P_2 \in \mathcal{F}^M(M_0)$, $P_2 \subset Q$, δ_v is a square-integrable representation of M_{P_2} and $\mu \in \mathfrak{a}_{P_2,+}^*$. Let $P' = P_2 N_P \in \mathcal{F}(M_0)$ and $Q' = P_2 N_Q$. Then, as explained in [Art89, p. 30], we have a commutative diagram

$$\begin{array}{ccc} \text{Ind}_{P'}(\delta, \mu + s\varpi) & \xrightarrow{N_{Q'|P'}(\delta, \mu + s\varpi)} & \text{Ind}_{Q'}(\delta, \mu + s\varpi) \\ \downarrow & & \downarrow \\ \text{Ind}_P(\pi, s\varpi) & \xrightarrow{N_{Q|P}(\pi, s\varpi)} & \text{Ind}_Q(\pi, s\varpi) \end{array}$$

Therefore, any matrix coefficient of $N_{Q|P}(\pi_v, s)$ is also a matrix coefficient of $N_{Q'|P'}(\delta_v, \mu + s)$. Hence, we are reduced to the case where π is square-integrable. However, up to a twist by an unramified character there are only finitely many square-integrable representations such that $(\text{Ind } \pi)^{K_v} \neq 0$. The p -adic case follows.

In the archimedean case $\deg f$ is the number of poles of f since $|f(s)| \leq 1$ on the imaginary axis. By [MS04, Proposition A.2] this number is bounded by $c(1 + \|\tau\|)$ where c depends on G only. \square

This completes the proof of Proposition 1, and therefore also of Theorem 3.

Remark 6. For applications of the trace formula, such as the problem of limit multiplicities, it will be of interest to make the bounds of Lemma 2 effective in K_v in the p -adic case.

5.3. Finally we show Corollary 1. Consider the spectral side of Arthur's trace formula whose fine expansion was obtained in [Art82b]. For a test function $f \in C_c^\infty(G(\mathbb{A}))$ it is given by an absolutely convergent sum

$$(9) \quad \sum_{\chi \in \mathfrak{X}} J_\chi(f)$$

where χ ranges over the set \mathfrak{X} of all cuspidal data of G . To describe the distributions J_χ we recall that the decomposition

$$L^2(M(F) \backslash M(\mathbb{A})^1) = \bigoplus_{\chi \in \mathfrak{X}} \hat{L}^2(M(F) \backslash M(\mathbb{A}))_\chi$$

according to cuspidal data gives rise to a decomposition $\mathcal{A}^2(P) = \bigoplus \mathcal{A}_\chi^2(P)$.

Arthur's expansion for J_χ is

$$(10) \quad J_\chi(h) = \sum_{[P], s \in W(M_P)} \frac{\iota_s}{|W(M_P)|} \int_{\mathfrak{ia}_{L_s}^*} \operatorname{tr} \left(\mathcal{M}_{L_s}(P, \lambda) M(P, s) \rho(P, \lambda, h) \Big|_{\mathcal{A}_\chi^2(P)} \right) d\lambda$$

for any bi- K -finite $h \in C_c^\infty(G(\mathbb{A}))$, where P ranges over parabolic subgroups up to association and the integral is absolutely convergent with respect to the trace norm. Implicit here is that the operator $\mathcal{M}_{L_s}(P, \lambda) \rho(P, \lambda, h)$ extends to a trace class operator on $\mathcal{A}_\chi^2(P)$.

This expression is a slight reformulation of [Art82b, Theorems 8.1 and 8.2].³ To explain this, suppose that $t \in W_0$ and $P \in \mathcal{P}(M)$. The map $t : \mathcal{A}^2(P) \rightarrow \mathcal{A}^2(tP)$ given by $t\phi(x) = \phi(w_t^{-1}x)$ is an isometry which intertwines $\rho(P, \lambda)$ with $\rho(tP, t\lambda)$ and satisfies $t\mathcal{A}_\chi^2(P) = \mathcal{A}_\chi^2(tP)$ for all $\chi \in \mathfrak{X}$. We also have $tM_{Q|P}(\lambda) = M_{tQ|tP}(t\lambda)t$ for any $Q \in \mathcal{P}(M)$. For any $s \in W(M)$ we have $tst^{-1} \in W(tM)$, $L_{tst^{-1}} = tL_s$ and $tM(P, s) = M(tP, tst^{-1})t$ (cf. [Art82b, (1.4), (1.5)]). Hence,

$$(11) \quad \mathcal{M}_{L_{tst^{-1}}}(tP, t\lambda) M(tP, tst^{-1}) \rho(tP, t\lambda, h) t = t \mathcal{M}_{L_s}(P, \lambda) M(P, s) \rho(P, \lambda, h), \quad \lambda \in \mathfrak{ia}_L^*.$$

Also, for all $Q \in \mathcal{P}(L)$ and $P' \in \mathcal{P}(M)$ we have

$$M_{P|P'}(\lambda) \mathcal{M}_Q(P', \lambda, \Lambda) = \mathcal{M}_Q(P, \lambda, \Lambda) M_{P|P'}(\lambda + \Lambda)$$

and therefore

$$(12) \quad M_{P|P'}(\lambda) \mathcal{M}_{L_s}(P', \lambda) M(P', s) \rho(P', \lambda, h) = \mathcal{M}_{L_s}(P, \lambda) M(P, s) \rho(P, \lambda, h) M_{P|P'}(\lambda).$$

The equivalence between (10) and [Art82b, Theorems 8.1 and 8.2] now follows from (11), (12) and the fact that the orbit of M under W_0 is of size $\frac{|W_0|}{|W_0^M|} |W(M)|^{-1}$.

Corollary 1 now follows from (9), (10) and Theorem 3. The passage from bi- K -finite functions to compactly supported functions is explained in [Art82b, p. 1326] using [Art82a, Proposition 2.3].

Remark 7. It is tempting to contemplate whether one can use our results to simplify the argument of [Art82a, Art82b] for the derivation of the spectral side of the trace formula. However, this will probably require a more flexible formula for the Maass-Selberg relations

³Cf. [Art05, p. 137] for the reason for the restriction to bi- K -finite functions.

which is valid not only for singular parameters. Moreover, some control over the error term in Arthur's asymptotic inner formula for truncated Eisenstein series is also necessary. Fortunately, for the upshot of the spectral expansion this is not essential.

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