ABSOLUTE CONVERGENCE OF THE SPECTRAL SIDE OF THE ARTHUR TRACE FORMULA FOR GL_n

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WITH APPENDIX BY
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0. INTRODUCTION

Let E be a number field and let G be a connected reductive algebraic group over E. Let $\mathbb{A}$ be the ring of adèles of E and let $G(\mathbb{A})$ be the group of points of G with values in $\mathbb{A}$. Let $G(\mathbb{A})^1$ be the intersection of the kernels of the maps $x \mapsto |\xi(x)|$, $x \in G(\mathbb{A})$, where ξ ranges over the group $X(G)_E$ of characters of G defined over E. Then the (noninvariant) trace formula of Arthur is an identity

$$\sum_{\mathfrak{o} \in \mathfrak{O}} J_{\mathfrak{o}}(f) = \sum_{\chi \in \mathfrak{X}} J_\chi(f), \quad f \in C_c^\infty(G(\mathbb{A})^1),$$

between distributions on $G(\mathbb{A})^1$. The left hand side is the *geometric side* and the right hand side the *spectral side* of the trace formula.

In this paper we are concerned with the spectral side of the trace formula. The distributions J_χ are initially defined in terms of truncated Eisenstein series. They are parametrized by the set of cuspidal data $\mathfrak{X}$ which consists of the Weyl group orbits of pairs (M_B, r_B), where M_B is the Levi component of a standard parabolic subgroup and r_B is an irreducible cuspidal automorphic representation of $M_B(\mathbb{A})^1$. In the fine χ-expansion of the spectral side the inner products of truncated Eisenstein series are replaced by terms containing generalized logarithmic derivatives of intertwining operators. This leads to an integral-series that is only known to be conditionally convergent. It is an open problem to prove that the fine χ-expansion is absolutely convergent and the main purpose of this paper is to settle this problem for the group GL_n.

To explain our results in more detail, we need to introduce some notation. We fix a Levi component M_0 of a minimal parabolic subgroup P_0

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of G. We assume that all parabolic subgroups considered in this paper contain M_0. Let P be a parabolic subgroup of G, defined over E, with unipotent radical N_P. Let M_P be the unique Levi component of P which contains M_0. We denote the split component of the center of M_P by A_P and its Lie algebra by $\mathfrak{a}_P$. For parabolic groups $P \subset Q$ there is a natural surjective map $\mathfrak{a}_P \to \mathfrak{a}_Q$ whose kernel we will denote by $\mathfrak{a}_P^Q$. Let $\mathcal{A}^2(P)$ be the space of automorphic forms on $N_P(\mathbb{A})M_P(E)\backslash G(\mathbb{A})$ which are square-integrable modulo $A_{P,\mathbb{Q}}(\mathbb{R})^0$, where $A_{P,\mathbb{Q}}$ is the split component of the center of the group obtained from M_P by restricting scalars from E to $\mathbb{Q}$. Let Q be another parabolic subgroup of G, defined over E, with Levi component M_Q, split component A_Q and corresponding Lie algebra $\mathfrak{a}_Q$. Let $W(\mathfrak{a}_P, \mathfrak{a}_Q)$ be the set of all linear isomorphisms from $\mathfrak{a}_P$ to $\mathfrak{a}_Q$ which are restrictions of elements of the Weyl group $W(A_0)$. The theory of Eisenstein series associates to each $s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$ an intertwining operator

$$M_{Q|P}(s, \lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}},$$

which for $\mathrm{Re}(\lambda)$ in a certain chamber, can be defined by an absolutely convergent integral and admits an analytic continuation to a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. Set

$$M_{Q|P}(\lambda) := M_{Q|P}(1, \lambda).$$

Let $\Pi(M_P(\mathbb{A})^1)$ be the set of equivalence classes of irreducible unitary representations of $M_P(\mathbb{A})^1$. Let $\chi \in \mathfrak{X}$ and $\pi \in \Pi(M_P(\mathbb{A})^1)$. Then (χ, π) singles out a certain subspace $\mathcal{A}^2_{\chi,\pi}(P)$ of $\mathcal{A}^2(P)$ [A3, p.1249]. Let $\overline{\mathcal{A}}^2_{\chi,\pi}(P)$ be the Hilbert space completion of $\mathcal{A}^2_{\chi,\pi}(P)$ with respect to the canonical inner product. For each $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ we have an induced representation $\rho_{\chi,\pi}(P, \lambda)$ of $G(\mathbb{A})$ in $\overline{\mathcal{A}}^2_{\chi,\pi}(P)$.

For each Levi subgroup L let $\mathcal{P}(L)$ be the set of all parabolic subgroups with Levi component L. If P is a parabolic subgroup, let Δ_P denote the set of simple roots of (P, A_P). Let L be a Levi subgroup which contains M_P. Set

$$\mathfrak{M}_L(P, \lambda) =$$
$$\lim_{\Lambda \to 0} \left(\sum_{Q_1 \in \mathcal{P}(L)} \mathrm{vol}(\mathfrak{a}^G_{Q_1}/\mathbb{Z}(\Delta^\vee_{Q_1})) M_{Q|P}(\lambda)^{-1} \frac{M_{Q|P}(\lambda + \Lambda)}{\prod_{\alpha \in \Delta_{Q_1}} \Lambda(\alpha^\vee)} \right),$$

where λ and Λ are constrained to lie in $i\mathfrak{a}^*_L$, and for each $Q_1 \in \mathcal{P}(L)$, Q is a group in $\mathcal{P}(M_P)$ which is contained in Q_1. Then $\mathfrak{M}_L(P, \lambda)$ is an unbounded operator which acts on the Hilbert space $\overline{\mathcal{A}}^2_{\chi,\pi}(P)$. In the special case that $L = M$ and $\dim \mathfrak{a}^G_L = 1$, the operator $\mathfrak{M}_L(P, \lambda)$

has a simple description. Let P be a parabolic subgroup with Levi component M. Let α be the unique simple root of (P, A_P) and let $\tilde{\omega}$ be the element in $(\mathfrak{a}_M^G)^*$ such that $\tilde{\omega}(\alpha^\vee) = 1$. Let $\overline{P}$ be the opposite parabolic group of P. Then

$$\mathfrak{M}_L(P, z\tilde{\omega}) = -\operatorname{vol}(\mathfrak{a}_M^G/\mathbb{Z}\alpha^\vee) M_{\overline{P}|P}(z\tilde{\omega})^{-1} \cdot \frac{d}{dz} M_{\overline{P}|P}(z\tilde{\omega}).$$

Let $f \in C_c^\infty(G(\mathbb{A})^1)$. Then Arthur [A4, Theorem 8.2] proved that $J_\chi(f)$ equals the sum over Levi subgroups M containing M_0, over L containing M, over $\pi \in \Pi(M(\mathbb{A})^1)$, and over $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$, a certain subset of the Weyl group, of the product of

$$|W_0^M||W_0|^{-1}|\det(s-1)_{\mathfrak{a}_M^L}|^{-1}|\mathcal{P}(M)|^{-1}$$

a factor to which we need not pay too much attention , and of

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \sum_{P\in\mathcal{P}(M)} \operatorname{tr}(\mathfrak{M}_L(P,\lambda) M_{P|P}(s,0)\rho_{\chi,\pi}(P,\lambda,f))\, d\lambda. \tag{0.1}$$

So far, it is only known that $\sum_{\chi\in\mathfrak{X}} |J_\chi(f)| < \infty$ and the goal is to show that the integral–sum obtained by summing (0.1) over $\chi \in \mathfrak{X}$ and $\pi \in \Pi(M(\mathbb{A})^1)$ is absolutely convergent with respect to the trace norm. For a given Levi subgroup M let $\mathcal{L}(M)$ be the set of all Levi subgroups L with $M \subset L$. Put $M(P,s) = M_{P|P}(s,0)$. Denote by $\| T \|_1$ the trace norm of a trace class operator T. Let $\mathcal{C}^1(G(\mathbb{A})^1)$ be the space of integrable rapidly decreasing functions on $G(\mathbb{A})^1$ (see [Mu4, §1.3] for its definition). Then our main result is the following theorem.

Theorem 0.1. *Let $G = \mathrm{GL}_n$. Then the sum over all $M \in \mathcal{L}(M_0)$, $L \in \mathcal{L}(M)$, $\chi \in \mathfrak{X}$, $\pi \in \Pi(M(\mathbb{A})^1)$, and $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ of the product of*

$$|W_0^M||W_0|^{-1}|\det(s-1)_{\mathfrak{a}_M^L}|^{-1}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} |\mathcal{P}(M)|^{-1} \sum_{P\in\mathcal{P}(M)} \| \mathfrak{M}_L(P,\lambda) M(P,s)\rho_{\chi,\pi}(P,\lambda,f) \|_1 \, d\lambda$$

is convergent for all $f \in \mathcal{C}^1(G(\mathbb{A})^1)$.

By Theorem 0.1, the spectral side for GL_n can now be rewritten in the following way. Denote by $\Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$ the set of all $\pi \in \Pi(M(\mathbb{A})^1)$ which are equivalent to an irreducible subrepresentation of the regular representation of $M(\mathbb{A})^1$ in $L^2(M(E)\backslash M(\mathbb{A})^1)$. As in Section 7 of [A3], we shall identify any representation of $M(\mathbb{A})^1$ with a representation of $M(\mathbb{A})$ which is trivial on $A_{M,\mathbb{Q}}(\mathbb{R})^0$, where $A_{M,\mathbb{Q}}$ is the split component of the center of the group $\mathrm{Res}_{E/\mathbb{Q}}\,\mathrm{GL}_n$ obtained from G

by restricting scalars from E to $\mathbb{Q}$. For any parabolic group P, let $\mathcal{A}^2_\pi(P) = \oplus_\chi \mathcal{A}^2_{\chi,\pi}(P)$ and for $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, let $\rho_\pi(P,\lambda)$ be the induced representation of $G(\mathbb{A})$ in $\overline{\mathcal{A}}^2_\pi(P)$, the Hilbert space completion of $\mathcal{A}^2_\pi(P)$. Given $M \in \mathcal{L}$, $L \in \mathcal{L}(M)$, $P \in \mathcal{P}(M)$, $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ and a function $f \in \mathcal{C}^1(G(\mathbb{A})^1)$, let

$$J^L_{M,P}(f,s) = \sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)} \int_{i\mathfrak{a}^*_L/i\mathfrak{a}^*_G} \operatorname{tr}(\mathfrak{M}_L(P,\lambda) M_{P|P}(s,0)\rho_\pi(P,\lambda,f))\, d\lambda.$$

By Theorem 0.1 this integral-series is absolutely convergent with respect to the trace norm. Furthermore for $M \in \mathcal{L}$ and $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ set

$$a_{M,s} = |\mathcal{P}(M)|^{-1}|W_0^M||W_0|^{-1}|\det(s-1)_{\mathfrak{a}^L_M}|^{-1}.$$

Then for all functions f in $\mathcal{C}^1(G(\mathbb{A})^1)$, the spectral side of the Arthur trace formula equals

$$\sum_{M\in\mathcal{L}} \sum_{L\in\mathcal{L}(M)} \sum_{P\in\mathcal{P}(M)} \sum_{s\in W^L(\mathfrak{a}_M)_{\mathrm{reg}}} a_{M,s} J^L_{M,P}(f,s).$$

Note that all sums in this expression are finite.

We shall now explain the main steps of the proof of Theorem 0.1. The proof relies on Theorem 0.1 of [Mu4]. In this theorem the absolute convergence of the spectral side of the trace formula has been reduced to a problem about local components of automorphic representations. So the main issue of the present paper is to verify that for GL_n, the assumptions of Theorem 0.1 of [Mu4] are satisfied.

Let $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ be a standard Levi subgroup and let $P, Q \in \mathcal{P}(M)$. We shall identify $\mathfrak{a}^*_M$ with $\mathbb{R}^r$. Given a place v of E and an irreducible unitary representation $\pi_v = \otimes_{i=1}^r \pi_{v,i}$ of $M(E_v)$, let $J_{Q|P}(\pi_v, \mathbf{s})$, $\mathbf{s} \in \mathbb{C}^r$, be the local intertwining operator between the induced representations $I^G_P(\pi_v[\mathbf{s}])$ and $I^G_Q(\pi_v[\mathbf{s}])$, where $\mathbf{s} = (s_1, ..., s_r)$ and $\pi_v[\mathbf{s}] = \otimes_i(\pi_{v,i}|\det|^{s_i})$. It follows from results of Shahidi [Sh5] that there exist normalizing factors $r_{Q|P}(\pi_v,\mathbf{s})$, which are defined in terms of Rankin-Selberg L-functions, such that the normalized intertwining operators

$$R_{Q|P}(\pi_v,\mathbf{s}) = r_{Q|P}(\pi_v,\mathbf{s})^{-1} J_{Q|P}(\pi_v,\mathbf{s})$$

satisfy the properties of Theorem 2.1 of [A7]. If $v < \infty$ and K_v is an open compact subgroup of $G(E_v)$, denote by $R_{Q|P}(\pi_v,\mathbf{s})_{K_v}$ the restriction of $R_{Q|P}(\pi_v,\mathbf{s})$ to the subspace $\mathcal{H}_P(\pi_v)^{K_v}$ of K_v-invariant vectors in the Hilbert space $\mathcal{H}_P(\pi_v)$ of the induced representation. If $v|\infty$, let

$K_v \subset G(E_v)$ be the standard maximal compact subgroup. For every $\sigma_v \in \Pi(K_v)$ we denote by $\| \sigma_v \|$ the norm of the highest weight of σ_v. Given $\pi_v \in \Pi(G(E_v))$ and $\sigma_v \in \Pi(K_v)$, let $R_{Q|P}(\pi_v, \mathbf{s})_{\sigma_v}$ be the restriction of $R_{Q|P}(\pi_v, \mathbf{s})$ to the σ_v-isotypical subspace of $\mathcal{H}_P(\pi_v)$. Finally for any place v, let $\Pi_{\text{disc}}(M(E_v))$ be the subspace of all π_v in $\Pi(M(E_v))$ such that there exists an automorphic representation π in the discrete spectrum of $M(\mathbb{A})$ whose local component at v is equivalent to π_v. Then the main result that we need to prove Theorem 0.1 is the following proposition.

Proposition 0.2. *Let v be a place of E. For all $M \in \mathcal{L}$ and $P, Q \in \mathcal{P}(M)$ the following holds.*

1) If $v < \infty$, then for every open compact subgroup K_v of $\mathrm{GL}_n(E_v)$ and every multi-index $\alpha \in \mathbb{N}^r$ there exists $C > 0$ such that

$$\| D_{\mathbf{u}}^{\alpha} R_{Q|P}(\pi_v, i\mathbf{u})_{K_v} \| \leq C \tag{0.2}$$

for all $\pi_v \in \Pi_{\text{disc}}(M(E_v))$ and $\mathbf{u} \in \mathbb{R}^r$.

2) If $v|\infty$, then for every multi-index $\alpha \in \mathbb{N}^r$ there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$\| D_{\mathbf{u}}^{\alpha} R_{Q|P}(\pi_v, i\mathbf{u})_{\sigma_v} \| \leq C(1 + \| \sigma_v \|)^N \tag{0.3}$$

for all $\mathbf{u} \in \mathbb{R}^r$, $\sigma_v \in \Pi(K_v)$ and $\pi_v \in \Pi_{\text{disc}}(M(E_v))$.

The normalization used in [Mu4] differs slightly from the normalization by L-functions. However, it is easy to compare the two normalizations and it follows that Proposition 0.2 holds also with respect to the normalization used in [Mu4]. Together with Theorem 0.1 of [Mu4], this implies Theorem 0.1. Actually in [Mu4] we considered only reductive algebraic groups G defined over $\mathbb{Q}$. However, passing to the group $G' = \mathrm{Res}_{E/\mathbb{Q}} G$ which is obtained from G by restriction of scalars, it follows immediately that the results of [Mu4] can also be applied to reductive algebraic groups defined over a number field.

The main analytic ingredients in the proof of Proposition 0.2 are a non-trivial uniform bound toward the Ramanujan hypothesis on the Langlands parameters of local components of cuspidal automorphic representations [LRS] and the determination of the residual spectrum [MW]. Furthermore Corollary A.3 is important for the proof of (0.3).

Let us explain this in more detail. First note that any local component π_v of a cuspidal automorphic representation π of $\mathrm{GL}_m(\mathbb{A})$ is generic [Sk]. This implies that π_v is equivalent to a fully induced representation [JS3], i.e.,

$$\pi_v \cong I_{P(E_v)}^{G(E_v)}(\tau_1[t_1], \ldots, \tau_r[t_r]),$$

where P is a standard parabolic subgroup of type $(n_1, ..., n_r)$, τ_i are tempered representations of $\mathrm{GL}_{n_i}(E_v)$ and the t_i's are real numbers satisfying

$$t_1 > t_2 > \cdots > t_r.$$

Here $\tau_i[t_i]$ is the representation $g \mapsto \tau_i(g)|\det(g)|^{t_i}$. For a unitary generic representation π_v the parameters t_i satisfy $|t_i| < 1/2$. In [LRS], Luo, Rudnick and Sarnak proved that for an unramified π_v which is the local component of a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$, one has

$$\max_i |t_i| < \frac{1}{2} - \frac{1}{m^2+1}. \tag{0.4}$$

First we extend this result of Luo, Rudnick and Sarnak to all local components of cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$. Then we use the description of the residual spectrum of $\mathrm{GL}_m(\mathbb{A})$, given by Mœglin and Waldspurger [MW], to prove similar bounds for the local components of all automorphic representations in the residual spectrum of $\mathrm{GL}_m(\mathbb{A})$ (cf. Proposition 3.5 for the precise statement). As a consequence, it follows that for every local component π_v of an automorphic representation π in the discrete spectrum of $M(\mathbb{A})^1$ the normalized intertwining operator $R_{Q|P}(\pi_v, \mathrm{s})$ is holomorphic in the domain $\mathrm{Re}(s_i - s_j) > 2/(n^2+1)$, $1 \leq i < j \leq r$. This is the key result which is needed to prove Proposition 0.2. Combined with Corollary A.3 it immediately implies (0.3). For a finite place v we use that by Theorem 2.1 of [A7] any matrix coefficient of $R_{Q|P}(\pi_v, \mathrm{s})$ is a rational function of $q_v^{s_i - s_j}$, $i < j$. Together with the above result, this implies (0.2).

In an earlier version of this paper, the first two authors were only able to establish (0.3) for a fixed K_v-type, so that Theorem 0.1 could only be proved for K-finite functions $f \in \mathcal{C}^1(G(\mathbb{A})^1)$. With the help of the appendix which was kindly provided by E. Lapid, the K-finiteness assumption could be lifted.

To extend the results of this paper to other reductive groups G one would need, in particular, the existence of non-trivial uniform bounds on the local components of cuspidal automorphic representations of $G(\mathbb{A})$. For a discussion of this problem we refer to [Sa]. Also note that [CL] is a step in this direction.

The paper is organized as follows. In section 2 we compare the two different normalizations of intertwining operators and we prove some estimate for conductors. In section 3 we estimate the (continuous) Langlands parameters of local components of cuspidal automorphic representations of GL_m which generalizes results of Luo, Rudnick and

Sarnak [LRS] to the case of ramified representations. Then we use the description of the residual spectrum of GL_m by Mœglin and Waldspurger [MW] to obtain estimations for the Langlands parameters of all local components of automorphic representations in the discrete spectrum of GL_m. We use these results in section 4 to prove Proposition 0.2 and Theorem 0.1. In the appendix, the normalized intertwining operators for real Lie groups are studied. The main results is Corollary A.3 which proves estimations for derivatives of matrix coefficients of intertwining operators along the imaginary axis, under the assumption that the intertwining operators are holomorphic in a fixed strip containing the imaginary axis.

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1. Preliminaries

1.1. Let E be a number field and let $\mathbb{A}$ denote the ring of adèles of E. Fix a positive integer n and let G be the group GL_n considered as algebraic group over E. By a parabolic subgroup of G we will always mean a parabolic subgroup which is defined over E. Let P_0 be the subgroup of upper triangular matrices of G. The Levi subgroup M_0 of P_0 is the group of diagonal matrices in G. A parabolic subgroup P of G is called standard, if $P \supset P_0$. By a Levi subgroup we will mean a subgroup of G which contains M_0 and is the Levi component of a parabolic subgroup of G. If $M \subset L$ are Levi subgroups, we denote the set of Levi subgroups of L which contain M by $\mathcal{L}^L(M)$. Furthermore, let $\mathcal{F}^L(M)$ denote the set of parabolic subgroups of L defined over E which contain M, and let $\mathcal{P}^L(M)$ be the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. If $L = G$, we shall denote these sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$. Write $\mathcal{L} = \mathcal{L}(M_0)$. Suppose that $P \in \mathcal{F}^L(M)$. Then

$$P = N_P M_P,$$

where N_P is the unipotent radical of P and M_P is the unique Levi component of P which contains M.

Suppose that $M \subset M_1 \subset L$ are Levi subgroups of G. If $Q \in \mathcal{P}^L(M_1)$ and $R \in \mathcal{P}^{M_1}(M)$, there is a unique group $Q(R) \in \mathcal{P}^L(M)$ which is contained in Q and whose intersection with M_1 is R.

Let $M \in \mathcal{L}$ and denote by A the split component of the center of M. Then A is defined over E. Let $X(M)_E$ be the group of characters of M defined over E and set

$$\mathfrak{a}_M = \mathrm{Hom}(X(M)_E, \mathbb{R}).$$

Then $\mathfrak{a}_M$ is a real vector space whose dimension equals that of A. Its dual space is

$$\mathfrak{a}_M^* = X(M)_E \otimes \mathbb{R}.$$

For any $M \in \mathcal{L}$ there exists a partition $(n_1, ..., n_r)$ of n such that

$$M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}.$$

Then $\mathfrak{a}_M^*$ can be canonically identified with $(\mathbb{R}^r)^*$ and the Weyl group $W(\mathfrak{a}_M)$ coincides with the group S_r of permutations of the set $\{1, ..., r\}$.

1.2. Let H be a reductive algebraic group defined over $\mathbb{Q}$, let F be a local field of characteristic 0 and let K be an open compact subgroup of $H(F)$. We shall denote by $\Pi(H(\mathbb{A}))$ (resp. $\Pi(H(F))$, $\Pi(K)$, etc.) the set of equivalence classes of irreducible unitary representations of $H(\mathbb{A})$ (resp. $H(F)$, K, etc.).

1.3. Let F be a local field of characteristic zero. If π is an admissible representation of $\mathrm{GL}_m(F)$, we shall denote by $\widetilde{\pi}$ the contragredient representation to π. Let π_i, $i = 1, ..., r$, be irreducible admissible representations of the group $\mathrm{GL}_{n_i}(F)$. Then $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ is an irreducible admissible representation of

$$M(F) = \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F).$$

For $\mathbf{s} \in \mathbb{C}^r$ let $\pi_i[s_i]$ be the representation of $\mathrm{GL}_{n_i}(F)$ which is defined by

$$\pi_i[s_i](g) = |\det(g)|^{s_i} \pi_i(g), \quad g \in \mathrm{GL}_{n_i}(F).$$

Let

$$I_P^G(\pi, \mathbf{s}) = \mathrm{Ind}_{P(F)}^{G(F)}(\pi_1[s_1] \otimes \cdots \otimes \pi_r[s_r])$$

be the induced representation and denote by $\mathcal{H}_P(\pi)$ the Hilbert space of the representation $I_P^G(\pi, \mathbf{s})$. Sometimes we will denote $I_P^G(\pi, \mathbf{s})$ by $I_P^G(\pi_1[s_1], ..., \pi_r[s_r])$.

2. Normalizing factors for local intertwining operators

Let F be a local field of characteristic 0. If F is non-Archimedean, let $\mathcal{O}$ be the ring of integers of F and let $\mathfrak{P}$ be the unique maximal ideal of $\mathcal{O}$. Let q be the number of elements of the residue field $\mathcal{O}/\mathfrak{P}$. Let $K = \mathrm{GL}_n(\mathcal{O})$. If F is Archimedean, let K be the standard maximal compact subgroup of $\mathrm{GL}_n(F)$, i.e., $K = \mathrm{O}(n)$, if $F = \mathbb{R}$, and $K = \mathrm{U}(n)$, if $F = \mathbb{C}$.

Let $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ be a standard Levi subgroup. We identify $\mathfrak{a}_M$ with $\mathbb{R}^r$. Let $P_1, P_2 \in \mathcal{P}(M)$. Given $\pi \in \Pi(M(F))$, let

$$J_{P_2|P_1}(\pi, \mathbf{s}), \quad \mathbf{s} \in \mathbb{C}^r,$$

be the intertwining operator which intertwines the induced representations $I_{P_1}^G(\pi, \mathbf{s})$ and $I_{P_2}^G(\pi, \mathbf{s})$. The intertwining operator $J_{P_2|P_1}(\pi, \mathbf{s})$ is defined by an integral over $N_{P_1}(F) \cap N_{\overline{P}_2}(F)$ which converges for $\mathrm{Re}(\mathbf{s})$ in a certain chamber of $\mathfrak{a}_M^*$. It follows from [A7] and [CLL, §15] that the intertwining operators can be normalized in a suitable way. This means that there exist scalar valued meromorphic functions $r_{P_2|P_1}(\pi, \mathbf{s})$ of $\mathbf{s} \in \mathbb{C}^r$ such that the normalized intertwining operators

$$R_{P_2|P_1}(\pi, \mathbf{s}) = r_{P_2|P_1}(\pi, \mathbf{s})^{-1} J_{P_2|P_1}(\pi, \mathbf{s})$$

satisfy the properties of Theorem 2.1 of [A7]. The method used in [A7] works for every reductive group G. For GL_n, however, it follows from results of Shahidi [Sh1], [Sh5] that local intertwining operators can be normalized by L-functions.

The normalizing factors defined by the Rankin-Selberg L-functions can be described as follows. Fix a nontrivial continuous character ψ of the additive group F^+ of F and equip F with the Haar measure which is selfdual with respect to ψ. First assume that P is a standard maximal parabolic subgroup with Levi component $M = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$. Let $\pi_i \in \Pi(\mathrm{GL}_{n_i}(F))$, $i = 1, 2$. If F is non-Archimedean, let $L(s, \pi_1 \times \pi_2)$ and $\epsilon(s, \pi_1 \times \pi_2, \psi)$ be the Rankin-Selberg L-function and the ϵ-factor, respectively, as defined in [JPS]. If F is Archimedean, let the L-function and the ϵ-factor be defined by using the Langlands parametrization (cf. [A7], [Sh2]). Then the normalizing factor can be regarded as a function $r_{\overline{P}|P}(\pi_1 \otimes \pi_2, s)$ of one complex variable which is given by

$$r_{\overline{P}|P}(\pi_1 \otimes \pi_2, s) = \frac{L(s, \pi_1 \times \widetilde{\pi}_2)}{L(1+s, \pi_1 \times \widetilde{\pi}_2)\epsilon(s, \pi_1 \times \widetilde{\pi}_2, \psi)}. \tag{2.1}$$

For arbitrary rank, the normalizing factors are products of normalizing factors associated to rank one groups in M. Let e_i, $i = 1, \dots, r$, denote the standard basis of $(\mathbb{R}^r)^*$. Then there exist $\sigma_1, \sigma_2 \in S_r$ such that the set of roots of (P_1, A_M) and (P_2, A_M), respectively, are given by

$$\Sigma_{P_k} = \{e_i - e_j \mid 1 \leq i, j \leq r,\ \sigma_k(i) < \sigma_k(j)\}, \quad k = 1, 2. \tag{2.2}$$

Put

$$\mathrm{I}(\sigma_1, \sigma_2) = \{(i,j) \mid 1 \leq i, j \leq r,\ \sigma_1(i) < \sigma_1(j),\ \sigma_2(i) > \sigma_2(j)\}.$$

Then

$$\Sigma_{P_1} \cap \Sigma_{\overline{P}_2} = \{e_i - e_j \mid (i,j) \in \mathrm{I}(\sigma_1, \sigma_2)\}.$$

Let $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ where $\pi_i \in \Pi(\mathrm{GL}_{n_i}(F))$, $i = 1, ..., r$. For $\mathbf{s} = (s_1, ..., s_r) \in \mathbb{C}^r$ set

$$r_{P_2|P_1}(\pi, \mathbf{s}) := \prod_{(i,j) \in \mathrm{I}(\sigma_1, \sigma_2)} \frac{L(s_i - s_j, \pi_i \times \widetilde{\pi}_j)}{L(1 + s_i - s_j, \pi_i \times \widetilde{\pi}_j)\epsilon(s_i - s_j, \pi_i \times \widetilde{\pi}_j, \psi)}. \tag{2.3}$$

Since the Rankin-Selberg L-factors are meromorphic functions, it follows that $r_{P_2|P_1}(\pi, \mathbf{s})$ are meromorphic functions of $\mathbf{s} \in \mathbb{C}^r$ and as explained in [A7, §4] and [AC, p.87], they satisfy all properties that are requested for normalizing factors.

In order to be able to apply the results of [Mu4] we have to compare the normalizing factors $r_{Q|P}(\pi, \mathbf{s})$ with those used in [Mu4] which we denote by $\widetilde{r}_{Q|P}(\pi, \mathbf{s})$. If F is Archimedean, the normalizing factors $\widetilde{r}_{Q|P}(\pi, \mathbf{s})$ are defined as the Artin L-factors and therefore, coincide with the $r_{Q|P}(\pi, \mathbf{s})$. Assume that F is non-Archimedean. By the construction of the normalizing factors it suffices to consider the case where P is maximal, $Q = \overline{P}$ and π is square integrable. Let P be a standard maximal parabolic subgroup of GL_m with Levi component

$$M = \mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2}.$$

Then the normalizing factor may be regarded as a function $\widetilde{r}_{\overline{P}|P}(\pi, s)$ of one complex variable s. We recall the construction of $\widetilde{r}_{\overline{P}|P}(\pi, s)$ for square integrable representations π [CLL]. It follows from [Si1], [Si2] that for every $\pi \in \Pi_2(M(F))$ there exists a rational function $U_P(\pi, z)$ such that the Plancherel measure $\mu(\pi, s)$ is given by

$$\mu(\pi, s) = U_P(\pi, q^{-s}).$$

The rational function $U_P(\pi, z)$ is of the form

$$U_P(\pi, z) = a \prod_{i=1}^{r} \frac{(1 - \alpha_i z)(1 - \overline{\alpha}_i^{-1} z)}{(1 - \beta_i z)(1 - \overline{\beta}_i^{-1} z)},$$

where $|\alpha_i| \le 1$, $|\beta_i| \le 1$, $i = 1, ..., r$, and $a \in \mathbb{C}$ is a constant such that

$$a \prod_{i=1}^{r} \frac{\alpha_i}{\beta_i} > 0.$$

Let $b \in \mathbb{C}$ be such that

$$|b|^2 \prod_{i=1}^{r} \frac{\overline{\alpha_i}}{\overline{\beta_i}} = a$$

and set

$$V_P(\pi, z) = b \prod_{i=1}^{r} \frac{(1-\alpha_i z)}{(1-\beta_i z)}. \tag{2.4}$$

Then the normalizing factor $\widetilde{r}_{\overline{P}|P}(\pi, s)$ is defined by

$$\widetilde{r}_{\overline{P}|P}(\pi, s) = V_P(\pi, q^{-s})^{-1}.$$

By definition we have

$$\mu_P(\pi, s) = \left(\overline{\widetilde{r}_{\overline{P}|P}(\pi, -\overline{s})}\widetilde{r}_{\overline{P}|P}(\pi, s)\right)^{-1},$$

which is one of the main conditions that the normalizing factors have to satisfy.

Let π_1 and π_2 be tempered representations of $\mathrm{GL}_{m_1}(F)$ and $\mathrm{GL}_{m_2}(F)$, respectively. By Corollary 6.1.2 of [Sh1] the Plancherel measure is given by

$$\mu(\pi_1 \otimes \pi_2, s) = q^{f(\widetilde{\pi}_1 \times \pi_2)} \frac{L(1+s, \pi_1 \times \widetilde{\pi}_2)}{L(s, \pi_1 \times \widetilde{\pi}_2)} \frac{L(1-s, \widetilde{\pi}_1 \times \pi_2)}{L(-s, \widetilde{\pi}_1 \times \pi_2)},$$

where $f(\widetilde{\pi}_1 \times \pi_2) \in \mathbb{Z}$ is the conductor of $\widetilde{\pi}_1 \times \pi_2$. Using the description of the Rankin-Selberg L-functions for tempered representations [JPS] (see also section 3), it follows that

$$L(s, \pi_1 \times \widetilde{\pi}_2) = \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} \left(1 - a_{ij} q^{-s}\right)^{-1}, \tag{2.5}$$

with complex numbers a_{ij} satisfying $|a_{ij}| < 1$. Furthermore by [JPS], the ϵ-factor $\epsilon(s, \pi_1 \times \pi_2, \psi)$ has the following form

$$\epsilon(s, \pi_1 \times \pi_2, \psi) = c(\pi_1 \times \pi_2, \psi) q^{-f(\pi_1 \times \pi_2, \psi)s}, \tag{2.6}$$

with $c(\pi_1 \times \pi_2, \psi) \in \mathbb{C} - \{0\}$ and $f(\pi_1 \times \pi_2, \psi) \in \mathbb{Z}$. Let

$$c(\psi) = \max\{r \mid \mathfrak{P}^{-r} \subset \ker \psi\}$$

Then

$$f(\pi_1 \times \pi_2, \psi) = n_1 n_2 c(\psi) + f(\pi_1 \times \pi_2), \tag{2.7}$$

with $f(\pi_1 \times \pi_2) \in \mathbb{Z}$ independent of ψ. For simplicity assume that $c(\psi) = 0$. By Lemma 6.1 of [Sh1] we have

$$|c(\pi_1 \times \pi_2)| = q^{f(\widetilde{\pi}_1 \times \pi_2)/2}. \tag{2.8}$$

Thus the ϵ-factor can be written as

$$\epsilon(s, \pi_1 \times \pi_2, \psi) = W(\pi_1 \times \pi_2) q^{(1/2-s)f(\pi_1 \times \pi_2)},$$

where the root number $W(\pi_1 \times \pi_2)$ satisfies $|W(\pi_1 \times \pi_2)| = 1$. Finally, observe that $f(\pi_1 \times \widetilde{\pi}_2) = f(\widetilde{\pi}_1 \times \pi_2)$. Using (2.5), (2.6) and (2.8) it follows that the constant b in (2.4) can be chosen to be $\epsilon(0, \pi_1 \times \widetilde{\pi}_2, \psi)$ and

$$\widetilde{r}_{\overline{P}|P}(\pi_1 \otimes \pi_2, s) = \frac{L(s, \pi_1 \times \widetilde{\pi}_2)}{L(1+s, \pi_1 \times \widetilde{\pi}_2)\epsilon(0, \pi_1 \times \widetilde{\pi}_2, \psi)}. \tag{2.9}$$

Comparing (2.1) and (2.9), it follows that

$$r_{\overline{P}|P}(\pi_1 \otimes \pi_2, s) = \frac{\epsilon(0, \pi_1 \times \widetilde{\pi}_2, \psi)}{\epsilon(s, \pi_1 \times \widetilde{\pi}_2, \psi)} \widetilde{r}_{\overline{P}|P}(\pi_1 \otimes \pi_2, s). \tag{2.10}$$

This can be extended to parabolic groups of arbitrary rank in the usual way. Let M be a standard Levi subgroup of GL_n of type $(n_1, ..., n_r)$ and let $P_1, P_2 \in \mathcal{P}(M)$. Using the product formula for $\widetilde{r}_{P_2|P_1}(\pi, \mathbf{s})$ [A7, p.29] and the corresponding product formula (2.3) for $r_{P_2|P_1}(\pi, \mathbf{s})$, we extend (2.10) to all tempered representations π of $M(F)$. Finally, if π is any irreducible unitary representation of $M(F)$, it can be written as a Langlands quotient $\pi = J_R^M(\tau, \mu)$, where R is a parabolic subgroup of M, τ is a tempered representation of $M_R(F)$ and μ is a point in the chamber of $\mathfrak{a}_R^*/\mathfrak{a}_M^*$ attached to R. Then

$$r_{P_2|P_1}(\pi, \mathbf{s}) = r_{P_2(R)|P_1(R)}(\tau, \mathbf{s} + \mu)$$

and a similar formula holds for $\widetilde{r}_{P_2|P_1}(\pi, \mathbf{s})$ [A7, (2.3)]. Let $\sigma_1, \sigma_2 \in S_r$ be attached to P_1, P_2 such that Σ_{P_i} is given by (2.2). Then we get

Lemma 2.1. *For all irreducible unitary representation $\pi = \otimes_{i=1}^r \pi_i$ of $M(F)$ we have*

$$r_{P_2|P_1}(\pi, \mathbf{s}) = \prod_{(i,j) \in I(\sigma_1, \sigma_2)} \frac{\epsilon(0, \pi_i \times \widetilde{\pi}_j, \psi)}{\epsilon(s_i - s_j, \pi_i \times \widetilde{\pi}_j, \psi)} \widetilde{r}_{P_2|P_1}(\pi, \mathbf{s}), \quad \mathbf{s} \in \mathbb{C}^r.$$

Since we will be concerned with logarithmic derivatives of normalizing factors, we need estimates for $f(\pi_1 \times \pi_2)$. Let $f(\pi_i)$ be the conductor of π_i, $i = 1, 2$. Then by Theorem 1 of [BH] and Corollary (6.5) of [BHK] we have

$$0 \le f(\pi_1 \times \pi_2) \le n_1 f(\pi_1) + n_2 f(\pi_2) - \inf\{f(\pi_1), f(\pi_2)\} \tag{2.11}$$

for all admissible smooth representations π_i of $\mathrm{GL}_{n_i}(F)$, $i=1,2$. Furthermore, by Corollary (6.5) of [BHK], we have $f(\pi_1\times\pi_2)=0$ if and only if there exists a quasicharacter χ of $F^\times$ such that both $\pi_1\otimes\chi\circ\det$ and $\pi_2\otimes\chi^{-1}\circ\det$ are unramified principal series representations.

By (2.11) it suffices to estimate the conductors $f(\pi_i)$, $i=1,2$.

Given an open compact subgroup $K\subset\mathrm{GL}_m(F)$, let

$$\Pi(\mathrm{GL}_m(F);K)=\big\{\pi\in\Pi(\mathrm{GL}_m(F))\bigm|\pi^K\neq\{0\}\big\}.$$

Lemma 2.2. *For every open compact subgroup K of $\mathrm{GL}_m(F)$ there exists $C>0$ such that $f(\pi)\le C$ for all $\pi\in\Pi(\mathrm{GL}_m(F);K)$.*

Proof. In the first step we reduce the proof to the case of square-integrable representations. Let $\pi\in\Pi(\mathrm{GL}_m(F))$. Then there exist a parabolic subgroup P of GL_m of type $(m_1,...,m_r)$, tempered representations τ_j of $\mathrm{GL}_{m_j}(F)$ and real numbers t_i with $t_1>\cdots>t_r$ such that π is isomorphic to the Langlands quotient $J_P^{\mathrm{GL}_m}(\tau_1[t_1],...,\tau_r[t_r])$. By Theorem 3.4 of [J] it follows that

$$f(\pi)=f\left(I_P^{\mathrm{GL}_m}(\tau_1[t_1],...,\tau_r[t_r])\right)=\sum_j f(\tau_j).$$

Furthermore a tempered representation τ of $\mathrm{GL}_d(F)$ is full induced: $\tau=I_Q^{\mathrm{GL}_d}(\sigma_1,...,\sigma_l)$, where Q is a parabolic subgroup of GL_d of type $(d_1,...,d_l)$ and σ_i is a square-integrable representation of $\mathrm{GL}_{d_i}(F)$, $i=1,...,l$. Then by (3.2.3) of [J] we get

$$f(\tau)=\sum_j f(\sigma_j).$$

Next we relate the K-invariant subspaces. We may assume that $K\subset\mathrm{GL}_m(\mathcal{O}_F)$ is a congruence subgroup. Suppose that π is a subquotient of an induced representation $I_P^{\mathrm{GL}_m}(\sigma)$, where P is a parabolic subgroup of GL_m of type $(m_1,...,m_h)$, $\sigma=\otimes_i\rho_i$ and ρ_i is an admissible representation of $\mathrm{GL}_{m_i}(F)$. If $\pi^K\neq\{0\}$, then $I_P^{\mathrm{GL}_m}(\sigma)^K\neq\{0\}$. Furthermore we have

$$\begin{aligned}I_P^{\mathrm{GL}_m}(\sigma)^K&=\left(I_{\mathrm{GL}_m(\mathcal{O}_F)\cap P}^{\mathrm{GL}_m(\mathcal{O}_F)}(\sigma)\right)^K\\&\hookrightarrow\bigoplus_{\mathrm{GL}_m(\mathcal{O}_F)/K}I_{K\cap P}^K(\sigma)^K\\&\cong\bigoplus_{\mathrm{GL}_m(\mathcal{O}_F)/K}\sigma^{K\cap P}.\end{aligned}$$

Now observe that

$$K \cap P = \prod_{i=1}^{h} K_i,$$

where $K_i \subset \mathrm{GL}_{m_i}(\mathcal{O}_F)$ are congruence subgroups. Then

$$\sigma^{K\cap P} \cong \otimes_{i=1}^{h} \rho_i^{K_i}.$$

Thus if $\pi^K \neq \{0\}$, then $\rho_i^{K_i} \neq \{0\}$ for all i, $1 \leq i \leq h$. Combined with the above relations of the conductors, we reduce to the case of square-integrable representations.

Let $\mathbf{1}$ denote the trivial representation of K. By [HC2, Theorem 10] the set $\Pi_2(\mathrm{GL}_m(F), K)$ of square-integrable representations π of $\mathrm{GL}_m(F)$ with $[\pi|_K : \mathbf{1}] \geq 1$ is a compact subset of the space $\Pi_2(\mathrm{GL}_m(F))$ of square-integrable representations of $\mathrm{GL}_m(F)$. By the definition of the topology in $\Pi_2(\mathrm{GL}_m(F))$ [HC2, §2], the set $\Pi_2(\mathrm{GL}_m(F), K)$ decomposes into a finite number of orbits under the canonical action of $i\mathbb{R}$ on $\Pi_2(\mathrm{GL}_m(F))$ given by $\pi \mapsto \pi[it]$. Since the conductor remains unchanged under twists by unramified characters, the lemma follows. $\square$

3. Estimation of the Langlands parameters

For the unramified places, the Langlands parameters of local components of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ have been estimated by Luo, Rudnick and Sarnak [LRS]. The main purpose of this section is to extend the estimations of [LRS] first to ramified places and then to local components of automorphic representations in the discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$ in general. To deal with cuspidal automorphic representations we follow the method of [LRS] which uses properties of the Rankin-Selberg L-functions. First note that any local component of a cuspidal automorphic representation of GL_n is generic [Sk]. Let F be a local field. By [JS3], any irreducible generic representation π of $\mathrm{GL}_n(F)$ is equivalent to a fully induced representation

$$\pi = I_P^G(\sigma, \mathbf{s}),$$

where P is a standard parabolic subgroup of type $(n_1, ..., n_r)$, $\mathbf{s} = (s_1, ..., s_r) \in \mathbb{R}^r$ satisfies $s_1 \geq s_2 \geq \cdots \geq s_r$ and σ is a square integrable representation of $M_P(F)$. We shall refer to $\mathbf{s}$ as the (continuous) Langlands parameters of π. We also note that an irreducible induced representation $I_P^G(\sigma, \mathbf{s})$, σ square-integrable and $\mathbf{s} \in \mathbb{R}^r$, is unitary, only if it is equivalent to its hermitian dual representation $I_P^G(\sigma, -\mathbf{s})$. By [KZ, Theorem 7] this implies that there exists a $w \in W(a_M)$ of order 2 so that

$$(\sigma_1[s_1] \otimes \cdots \sigma_r[s_r])^w = \sigma_1[-s_1] \otimes \cdots \otimes \sigma_r[-s_r].$$

Hence we have

$$\{\sigma_j[s_j]\} = \{\sigma_k[-s_k]\}. \tag{3.1}$$

Moreover by the classification of the unitary dual of $\mathrm{GL}_n(F)$ it follows that

$$|\operatorname{Re}(s_i)| < 1/2, \quad i = 1, ..., r.$$

The key result that we need about the local Rankin-Selberg L-functions is the following lemma.

Lemma 3.1. *Let π be an irreducible unitary generic representation of $\mathrm{GL}_n(F)$, and let $(s_1, ..., s_r) \in \mathbb{R}^r$ be the Langlands parameters of π. Then $L(s, \pi \times \widetilde{\pi})$ has a pole at the point*

$$s_0 = 2 \max_j |s_j|.$$

Proof. For the proof we need to describe the L-factors in more detail. Let $\pi \cong I_P^G(\sigma, \mathbf{s})$ be a fully induced representation with $\sigma = \otimes_i \sigma_i$ for discrete series representations σ_i of $\mathrm{GL}_{n_i}(F)$ and Langlands parameters $\mathbf{s} = (s_1, ..., s_r)$ satisfying the above conditions. Then by the multiplicativity of the local L-factors [Sh6] we get

$$L(s, \pi \times \widetilde{\pi}) = \prod_{i,j=1}^{r} L(s + s_i - s_j, \sigma_i \times \widetilde{\sigma}_j). \tag{3.2}$$

If F is non-Archimedean, then this is Proposition 9.4 of [JPS]. If F is Archimedean, (3.2) follows from the Langlands classification (see §2 of [Sh6]). This reduces the description of the L-factors to the case of square-integrable representations. We distinguish three cases according to the type of the field F.

1. $F = \mathbb{R}$.

The Rankin-Selberg local L-factors are defined in terms of L-factors attached to semisimple representations of the Weil group $W_{\mathbb{R}}$ by means of the Langlands correspondence [L3]. If τ is a semisimple representation of $W_{\mathbb{R}}$ of degree n and $\pi(\tau)$ is the associated irreducible admissible representation of $\mathrm{GL}_n(\mathbb{R})$, then

$$L(s, \pi(\tau)) = L(s, \tau).$$

Furthermore, if

$$\tau = \bigoplus_{1 \le j \le m} \tau_j$$

is the decomposition into irreducible representations of $W_{\mathbb{R}}$, then

$$L(s,\tau) = \prod_j L(s,\tau_j).$$

If τ' is another semisimple representation of $W_{\mathbb{R}}$ of degree n' and $\pi(\tau')$ is the associated irreducible admissible representation of $\mathrm{GL}_{n'}(\mathbb{R})$, then the Rankin-Selberg local L-factor is given by

$$L(s,\pi(\tau)\times\pi(\tau')) = L(s,\tau\otimes\tau').$$

This reduces the computation of the L-factors to the case of irreducible representations of the Weil group.

The irreducible representations of the Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$ are either 1 or 2 dimensional. The associated representations of $\mathrm{GL}_m(\mathbb{R})$, $m = 1, 2$, are square-integrable and all square-integrable representations are obtained in this way. Note that $\mathrm{GL}_m(\mathbb{R})$ does not have square-integrable representations if $m \ge 3$. To describe the L-factors, we define Gamma factors by

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Suppose that τ is a two-dimensional irreducible representation of $W_{\mathbb{R}}$. Then

$$\tau = \mathrm{ind}_{\mathbb{C}^*}^{W_{\mathbb{R}}}\theta,$$

where θ is a (not necessarily unitary) character of $W_{\mathbb{C}} = \mathbb{C}^*$. Thus there exist $t \in \mathbb{C}$ and $k \in \mathbb{Z}$ such that

$$\theta(z) = |z|^t (z/\overline{z})^{k/2}, \quad z \in \mathbb{C}^*.$$

Then the L-factor is defined as

$$L(s,\tau) = \Gamma_{\mathbb{C}}(s + t + |k|/2).$$

The one-dimensional irreducible representations of $W_{\mathbb{R}}$ are of the form

$$\psi_{\epsilon,t} : (z,\sigma) \in W_{\mathbb{R}} \to \mathrm{sign}^{\epsilon}(\sigma)|z|^t$$

where $\epsilon = 0, 1$ and "sign" is the sign character of the Galois group. The L-factor of $\psi_{\epsilon,t}$ is given by

$$L(s,\psi_{\epsilon,t}) = \Gamma_{\mathbb{R}}(s + t + \epsilon).$$

Next we have to consider the tensor products of irreducible representations.

If $\tau = \mathrm{ind}_{\mathbb{C}^*}^{W_\mathbb{R}}\theta$ and $\tau' = \mathrm{ind}_{\mathbb{C}^*}^{W_\mathbb{R}}\theta'$ are two-dimensional representations of $W_\mathbb{R}$, then we have

$$\begin{aligned}\tau \otimes \tau' &= \mathrm{ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\theta \otimes \mathrm{ind}_{\mathbb{C}^*}^{W_\mathbb{R}}\theta'|_{\mathbb{C}^*}) \\ &= \mathrm{ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\theta \otimes \theta') \oplus \mathrm{ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\theta \otimes \theta'^\sigma),\end{aligned}$$

where σ is the nontrivial element of the Galois group. Suppose that $\theta'(z) = |z|^{t'}(z/\overline{z})^{k'/2}$. Then we get

$$\begin{aligned}&L(s, \tau \otimes \tau') \\ &\quad = \Gamma_\mathbb{C}(s + t + t' + |k + k'|/2)\Gamma_\mathbb{C}(s + t + t' + |k - k'|/2).\end{aligned}$$

Similarly, if $\psi = \psi_{\epsilon,t'}$ is a one-dimensional representation, then we have

$$\tau \otimes \psi = \mathrm{ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\theta \otimes \psi|_{\mathbb{C}^*}),$$

and therefore, we get

$$L(s, \tau \otimes \psi) = \Gamma_\mathbb{C}(s + t + t' + |k|/2).$$

Finally, if $\psi = \psi_{\epsilon,t}$ and $\psi_{\epsilon',t'}$ are two one-dimensional representations of $W_\mathbb{R}$, then

$$L(s, \psi \otimes \psi') = \Gamma_\mathbb{R}(s + t + t' + \widetilde{\epsilon}),$$

where $0 \le \widetilde{\epsilon} \le 1$ and $\widetilde{\epsilon} = \epsilon + \epsilon'$ mod 2.

For $k \in \mathbb{Z}$ let D_k be the k-th discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with the same infinitesimal character as the k-dimensional representation. Then D_k is associated with the two-dimensional representation $\tau_k = \mathrm{ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\theta_k)$ of $W_\mathbb{R}$ where the character θ_k of $\mathbb{C}^*$ is defined by $\theta_k(z) = (z/\overline{z})^{k/2}$. For $\epsilon \in \{0, 1\}$ let ψ_ϵ the character of $\mathbb{R}^\times$, defined by $\psi_\epsilon(r) = (r/|r|)^\epsilon$. It corresponds to the character $\psi_{\epsilon,0}$ of $W_\mathbb{R}$. Using the above description of the L-factors, we get

$$\begin{aligned}L(s, D_{k_1} \times D_{k_2}) &= \Gamma_\mathbb{C}(s + |k_1 - k_2|/2) \cdot \Gamma_\mathbb{C}(s + |k_1 + k_2|/2) \\ L(s, D_k \times \psi_\epsilon) &= L(s, \psi_\epsilon \times D_k) = \Gamma_\mathbb{C}(s + |k|/2) \\ L(s, \psi_{\epsilon_1} \times \psi_{\epsilon_2}) &= \Gamma_\mathbb{R}(s + \epsilon_{1,2}),\end{aligned}$$

where $0 \le \epsilon_{1,2} \le 1$ with $\epsilon_{1,2} \equiv \epsilon_1 + \epsilon_2$ mod 2. Up to twists by unramified characters this exhausts all possibilities for the L-factors in the square-integrable case.

2. $F = \mathbb{C}$.

As in the real case, the local L-factors are defined in terms of the L-factors attached to representations of the Weil group $W_\mathbb{C}$ by means of the Langlands correspondence. The Weil group $W_\mathbb{C}$ is equal to $\mathbb{C}^*$.

Furthermore we note that $\mathrm{GL}_m(\mathbb{C})$ has square-integrable representations only if $m = 1$. For $r \in \mathbb{Z}$ let χ_r be the character of $\mathbb{C}^*$ defined by $\chi_r(z) = (z/\overline{z})^r$, $z \in \mathbb{C}^*$. Then it follows that

$$L(s, \chi_{r_1} \times \chi_{r_2}) = \Gamma_{\mathbb{C}}(s + |r_1 + r_2|/2).$$

Again up to twists by unramified characters, these are all possibilities for the L-factors in the square-integrable case.

3. *F non-Archimedean.*

Let π be a square-integrable representation of $\mathrm{GL}_m(F)$. By [BZ] there is a divisor $d|m$, a standard parabolic subgroup P of $\mathrm{GL}_m(F)$ of type $(d, \ldots, d)$, and an irreducible supercuspidal representation ρ of $\mathrm{GL}_d(F)$ so that π is the unique quasi-square-integrable component of the induced representation $I_P^G(\rho_1, \ldots, \rho_r)$, where $r = m/d$ and $\rho_j = \rho \otimes |\det|^{j-(r+1)/2}$, $j = 1, ..., r$. We will write $\pi = \Delta(r, \rho)$. The representation π is unitary (or equivalently square-integrable) if only if ρ is unitary. Moreover the contragredient of $\Delta(r, \rho)$ is given by $\widetilde{\Delta}(r, \rho) = \Delta(r, \widetilde{\rho})$. Let $\sigma_1 = \Delta(r_1, \rho_1)$ and $\sigma_2 = \Delta(r_2, \rho_2)$ be square integrable representation of $\mathrm{GL}_{m_1}(F)$ and $\mathrm{GL}_{m_2}(F)$, respectively. Then by Theorem 8.2 of [JPS] we have

$$L(s, \sigma_1 \times \widetilde{\sigma}_2) = \prod_{j=1}^{\min(m_1, m_2)} L(s + (m_1 + m_2)/2 - j, \rho_1 \times \widetilde{\rho}_2).$$

Thus the description of the Rankin-Selberg L-functions is reduced to the case of two supercuspidal representations. Let ρ_i, $i = 1, 2$, be supercuspidal representations of $\mathrm{GL}_{k_i}(F)$. By Proposition (8.1) of [JPS] we have $L(s, \rho_1 \times \rho_2) = 1$ if $k_1 > k_2$. Since $L(s, \rho_1 \times \rho_2) = L(s, \rho_2 \times \rho_1)$, the same holds for $k_1 < k_2$. Let $k_1 = k_2$. Then by Proposition (8.1) of [JPS], $L(s, \rho_1 \times \rho_2) = 1$, unless ρ_1 and ρ_2 are in the same twist class, i.e., there exists $t \in \mathbb{C}$ such that $\rho_2 \cong \rho_1[t]$. In this case we have

$$L(s, \rho_1 \times \widetilde{\rho}_1[t]) = L(s + t, \rho_1 \times \widetilde{\rho}_1) = \left(1 - q^{-a(t+s)}\right)^{-1},$$

where $a|k_1$ is the order of the cyclic group of unramified characters $\chi = |\det|^u$ such that $\rho_1 \otimes \chi \cong \rho_1$. Let $\sigma = \Delta(r, \rho)$. Then we get

$$\begin{aligned} L(s, \sigma \times \widetilde{\sigma}) &= \prod_{j=1}^{r} L(s + r - j, \rho \times \widetilde{\rho}) \\ &= \prod_{j=1}^{r} \left(1 - q^{-a(s+r-j)}\right)^{-1}, \end{aligned}$$

where a is the order of the cyclic group of unramified characters χ so that $\rho \otimes \chi$ is isomorphic to ρ.

From the above description of the local L-factors we conclude that they have the following two properties. Let π be a square-integrable representation of $\mathrm{GL}_m(F)$. Then $L(s, \pi \times \widetilde{\pi})$ has a pole at $s = 0$. Furthermore, if π_1 and π_2 are square-integrable representations of $\mathrm{GL}_{m_1}(F)$ and $\mathrm{GL}_{m_2}(F)$, respectively, then $L(s, \pi_1 \times \pi_2)$ has no zeros.

Now we are ready to prove the lemma. Let $\mathbf{s} = (s_1, ..., s_r)$ be the Langlands parameters of π. Let $1 \leq i \leq r$. Then it follows from (3.1) and (3.2) that $L(s, \pi \times \widetilde{\pi})$ contains the factor $L(s - 2s_i, \sigma_i \times \widetilde{\sigma}_i)$. Using the above properties of the L-factors in the square-integrable case, it follows that $L(s, \pi \times \widetilde{\pi})$ has a pole at $2s_i$. By (3.1), $-s_i$ occurs also in $\mathbf{s}$. Hence $L(s, \pi \times \widetilde{\pi})$ has a pole at $2|s_i|$. In particular, $L(s, \pi \times \widetilde{\pi})$ has a pole at $2 \max_i |s_i|$. $\square$

Next we recall some facts about ray class characters. Let E be a number field. Let $\mathfrak{q}$ be a nonzero integral ideal of E and denote by $C(\mathfrak{q})$ the wide ray class group of E modulo $\mathfrak{q}$. We note that the term "wide" means that no positivity condition has been imposed at the real places of E. Then a character of $C(\mathfrak{q})$ is unramified at all infinite places. Now recall that any character χ of $C(\mathfrak{q})$ can be identified with a character of the idele class group $C_E = I_E/E^\times$ which is trivial on the congruence subgroup $C_E^{\mathfrak{q}} = I_E^{\mathfrak{q}} E^\times / E^\times$ [Neu]. Then $\chi = \otimes_v \chi_v$ where χ_v is a character of $E_v^\times$ which is unramified at all places $v|\infty$ and all finite places $\mathfrak{p} \nmid \mathfrak{q}$. Furthermore for a finite place $\mathfrak{p} \nmid \mathfrak{q}$, the character $\chi_{\mathfrak{p}}$ is given by

$$\chi_{\mathfrak{p}}(\alpha) = \chi(\mathfrak{p})^{v_{\mathfrak{p}}(\alpha)}, \quad \alpha \in E_{\mathfrak{p}}^\times, \tag{3.3}$$

where $v_{\mathfrak{p}} : E_{\mathfrak{p}}^\times \to \mathbb{Z}$ is the $\mathfrak{p}$-adic valuation.

Let S be any finite set of finite places of E. For an integral ideal $\mathfrak{q}$ of E let $X_{\mathfrak{q}}$ denote the set of all wide ray class characters of conductor $\mathfrak{q}$ such that $\chi(\mathfrak{p}) = 1$ for all $\mathfrak{p} \in S$. Therefore we have $(\mathfrak{p}, \mathfrak{q}) = 1$ for all $\mathfrak{p} \in S$. By (3.3) it follows that

$$\chi_v = 1 \quad \text{for all } v \in S \cup S_\infty. \tag{3.4}$$

Let $X_{\mathfrak{q}}^*$ be the subset of $X_{\mathfrak{q}}$ consisting of all primitive characters. Set $X = \cup_{\mathfrak{q}} X_{\mathfrak{q}}$ and $X^* = \cup_{\mathfrak{q}} X_{\mathfrak{q}}^*$. For $\chi \in X_{\mathfrak{q}}^*$ and a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$, the partial Rankin-Selberg L-function $L_S(s, (\pi \otimes \chi) \times \widetilde{\pi})$ is defined to be

$$L_S(s, (\pi \otimes \chi) \times \widetilde{\pi}) = \prod_{v \notin S} L(s, (\pi_v \otimes \chi_v) \times \widetilde{\pi}_v).$$

We shall use the following result of Luo, Rudnick and Sarnak which is the main result of [LRS].

Theorem 3.2. *Given n, π, S as the above and any $\beta > 1 - 2/(n^2+1)$, there are infinitely many $\chi \in X^*$ such that*

$$L_S(\beta, (\pi \otimes \chi) \times \widetilde{\pi}) \neq 0.$$

Now we can establish our extension of Theorem 2 of [LRS].

Proposition 3.3. *Suppose that $\pi = \otimes_v \pi_v$ is a cuspidal automorphic representation of $GL_n(\mathbb{A})$. Let $s_v = (s_{1,v}, \ldots, s_{k,v}) \in \mathbb{R}^k$ be the Langlands parameters of the representation π_v. Then we have*

$$\max_j |s_{j,v}| < \frac{1}{2} - \frac{1}{n^2+1}.$$

Proof. We follow the proof of Theorem 2 in [LRS]. Let v be a place of E and set $S = \{v\}$. Let $\chi \in X^*$, where X^* is the set of ray class characters with respect to S which we defined above. The Rankin-Selberg L-function

$$L(s, (\pi \otimes \chi) \times \widetilde{\pi}) = L(s, (\pi_v \otimes \chi_v) \times \widetilde{\pi}_v) L_S(s, (\pi \otimes \chi) \times \widetilde{\pi}).$$

is holomorphic in the whole complex plane except for simple poles at $s = 1$ and $s = 0$ if $\pi \otimes \chi \cong \pi$. This follows from the work of Jacquet, Piateski-Shapiro, Shalika, Shahidi, Mœglin and Waldspurger [JPS], [JS1], [JS2], [MW], [Sh2]. Choosing the conductor of χ sufficiently large, we have $\pi \otimes \chi \not\cong \pi$. Thus by Theorem 3.2 we may choose $\chi \in X^*$ such that $L(s, (\pi \otimes \chi) \times \widetilde{\pi})$ is an entire function and by (3.4) we have $\chi_v = 1$. Suppose that $s_0 > 0$ is a pole of

$$L(s, \pi_v \times \widetilde{\pi}_v) = L(s, (\pi_v \otimes \chi_v) \times \widetilde{\pi}_v).$$

Then s_0 must be a zero of $L_S(s, (\pi \otimes \chi) \times \widetilde{\pi})$. Assume that $s_0 > 1 - 2(1+n^2)^{-1}$. Then by Theorem 3.2 there exists $\chi \in X^*$ with $L_S(s_0, (\pi \otimes \chi) \times \widetilde{\pi}) \neq 0$, $\chi_v = 1$, and $L(s, (\pi \otimes \chi) \times \widetilde{\pi})$ entire. Hence it follows that $s_0 < 1 - 2(1+n^2)^{-1}$. Together with Lemma 3.1 the proposition follows. □

We shall now establish a similar result for the local components of residual automorphic representations of $\mathrm{GL}_n(\mathbb{A})$. First we recall some facts about representations of GL_n over a local field F. Any irreducible unitary representation π of $\mathrm{GL}_n(F)$ is equivalent to a Langlands quotient $J_R^G(\tau, \mu)$. This is the unique irreducible quotient of an induced representation $I_R^G(\tau, \mu)$ where τ is a tempered representation of $M_R(F)$ and μ is a point in the positive chamber attached to R. A slight variant of this description is as follows. Recall that a tempered representation τ of

$\mathrm{GL}_m(F)$ can be described as follows. There exist a standard parabolic subgroup Q of type $(m_1, ..., m_p)$ and square-integrable representations δ_j of $\mathrm{GL}_{m_j}(F)$ so that τ is isomorphic to the full induced representation $I_P^{\mathrm{GL}_m}(\delta_1, ..., \delta_p)$. Hence by induction in stages, there exist a standard parabolic subgroup P of G of type $(n_1, ..., n_r)$, discrete series representations δ_i of $\mathrm{GL}_{m_i}(F)$ and real numbers $s_1 \geq s_2 \geq \cdots \geq s_r$ such that π is equivalent to the unique irreducible quotient

$$J_P^G(\delta_1[s_1] \otimes \cdots \otimes \delta_r[s_r]),$$

of the induced representation $I_P^G(\delta_1[s_1] \otimes \cdots \otimes \delta_r[s_r])$ [MW, I.2]. We call $s_1, ..., s_r$ the (continuous) Langlands parameters of π.

The residual spectrum for $\mathrm{GL}_n(\mathbb{A})$ has been determined by Mœglin and Waldspurger [MW]. Let $\pi = \otimes_v \pi_v$ be an irreducible automorphic representation in the residual spectrum of $\mathrm{GL}_n(\mathbb{A})$. By [MW], there is a divisor $k|n$, a standard parabolic subgroup P of type $(d, ..., d)$, and a cuspidal automorphic representation ξ of $\mathrm{GL}_d(\mathbb{A})$, $d = n/k$, so that the representation π is a quotient of the induced representation

$$I_{P(\mathbb{A})}^{G(\mathbb{A})}\left(\xi[(k-1)/2] \otimes \cdots \otimes \xi[-(k-1)/2]\right).$$

Lemma 3.4. *Let π and ξ be as above. Let v be a place of E and let $s_1 \geq \cdots \geq s_r$, be the Langlands parameters of ξ_v. Then the Langlands parameters of π_v are given by*

$$\left(\frac{k-1}{2} + s_1, ..., \frac{k-1}{2} + s_r, ..., -\frac{k-1}{2} + s_1, ..., -\frac{k-1}{2} + s_r\right).$$

Proof. Since ξ_v is a local component of a cuspidal automorphic representation ξ of $\mathrm{GL}_d(\mathbb{A})$, it is generic. Using induction in stages, it follows that there exist a standard parabolic subgroup R of GL_d of type $(n_1, ..., n_r)$, discrete series representations $\delta_{i,v}$ of $\mathrm{GL}_{n_i}(E_v)$ and real numbers $(s_1, ..., s_r)$ satisfying

$$s_1 \geq s_2 \geq \cdots \geq s_r; \quad |s_j| < 1/2, \ j = 1, ..., r. \tag{3.5}$$

such that ξ_v is isomorphic to the full induced representation

$$\xi_v \cong I_R^{\mathrm{GL}_d}(\delta_{1,v}[s_1] \otimes \cdots \otimes \delta_{r,v}[s_r]).$$

Let $Q = M_Q N_Q$ be the standard parabolic subgroup of GL_n whose Levi component M_Q is a product of k copies of M_R and let

$$\delta_v = (\delta_{1,v} \otimes \cdots \otimes \delta_{r,v}) \otimes \cdots \otimes (\delta_{1,v} \otimes \cdots \otimes \delta_{r,v}),$$

where $(\delta_{1,v}\otimes\cdots\otimes\delta_{r,v})$ occurs k times. Define

$$\mu(k,\mathbf{s})=\left(\frac{k-1}{2}+s_1,\ldots,\frac{k-1}{2}+s_r,\frac{k-3}{2}+s_1,\ldots,-\frac{k-1}{2}+s_r\right).$$

By induction in stages we have

$$I_P^G(\xi_v[(k-1)/2]\otimes\cdots\otimes\xi_v[-(k-1)/2])=I_Q^G(\delta_v,\mu(k,\mathbf{s})).$$

Furthermore, by (3.5) the coordinates of $\mu(k,\mathbf{s})$ are decreasing. Thus the induced representation $I_Q^G(\delta_v,\mu(k,\mathbf{s}))$ has a unique irreducible quotient which must be isomorphic to π_v. □

Next we recall a different method to parametrize irreducible unitary representations of $\mathrm{GL}_n(F)$. Let $d|n$ and $k=n/d$. Let P be the standard parabolic subgroup of type $(d,...,d)$. Let δ be a discrete series representation of $\mathrm{GL}_d(F)$ and let $a,b\in\mathbb{R}$ be such that $b-a\in\mathbb{N}$. Then the induced representation

$$I_P^G(\delta[b]\otimes\delta[b-1]\otimes\cdots\otimes\delta[a])$$

has a unique irreducible quotient which we denote by $J(\delta,a,b)$. Especially, if $a=-(k-1)/2$ and $b=(k-1)/2$, then we put

$$J(\delta,k):=J(\delta,a,b). \tag{3.6}$$

By Theorem D of [Ta] and [Vo], for every irreducible unitary representation of $\mathrm{GL}_n(F)$ there exist a standard parabolic subgroup P of type $(n_1,...,n_r)$, $k_i|n_i$, discrete series representations δ_i of $\mathrm{GL}_{d_i}(F)$, $d_i=n_i/k_i$, and real numbers $s_1,...,s_r$ with $|s_i|<1/2$, $i=1,...,r$, such that π is isomorphic to the fully induced representation:

$$\pi\cong I_P^G(J(\delta_1,k_1)[s_1]\otimes\cdots\otimes J(\delta_r,k_r)[s_r]). \tag{3.7}$$

Using this parametrization, we get the following analogue to Proposition 3.3 for local components of automorphic representations in the residual spectrum of $\mathrm{GL}_n(\mathbb{A})$.

Proposition 3.5. *Let π_v be a local component of an automorphic representation in the residual spectrum of $\mathrm{GL}_n(\mathbb{A})$. There exist $k|n$, a parabolic subgroup P of type $(kn_1,\ldots,kn_r)$, discrete series representations $\delta_{i,v}$ of $\mathrm{GL}_{n_i}(E_v)$ and real numbers $s_1,...,s_r$ satisfying*

$$s_1\geq s_2\geq\cdots\geq s_r,\quad |s_i|<1/2-(1+n^2)^{-1},\quad i=1,...,r,$$

such that

$$\pi_v\cong I_P^G(J(\delta_{1,v},k)[s_1]\otimes\cdots\otimes J(\delta_{r,v},k)[s_r]).$$

Proof. By the proof of Lemma 3.4, π_v is equivalent to a Langlands quotient of the form

$$\pi_v \cong J_Q^G(\delta_v, \mu(k, \mathbf{s})),$$

where the parameters $\mathbf{s}$ satisfy (3.5). Set

$$b_i = \frac{k-1}{2} + s_i, \qquad a_i = -\frac{k-1}{2} + s_i, \quad i = 1, ..., r.$$

Suppose there exist $1 \leq i < j \leq r$ such that the triples $(\delta_{i,v}, a_i, b_i)$ and $(\delta_{j,v}, a_j, b_j)$ are linked in the sense of I.6.3 or I.7 in [MW]. Suppose that $s_i \geq s_j$. Then it follows from (2) and (3)(i) on p.622 or from (1) and (2) on p.624 of [MW] that $a_i \geq a_j + 1$ and $b_i \geq b_j + 1$. This implies $1 \leq |s_i - s_j|$ which contradicts (3.5). Hence the triples $(\delta_{i,v}, a_i, b_i)$ are pairwise not linked. Now observe that

$$J(\delta_{i,v}, a_i, b_i) = J(\delta_{i,v}, k)[s_i].$$

Let P be the standard parabolic subgroup with Levi component

$$\mathrm{GL}_{kn_1} \times \cdots \times \mathrm{GL}_{kn_r}.$$

Let $\widetilde{\delta}_v = \otimes_{i=1}^r \delta_{i,v}$ and set

$$J(\widetilde{\delta}_v, k) := \otimes_{i=1}^r J(\delta_{i,v}, k).$$

Then it follows from Proposition I.9 of [MW] that the induced representation

$$I_P^G(J(\widetilde{\delta}_v, k), \mathbf{s}) := I_P^G(J(\delta_{1,v}, k)[s_1] \otimes \cdots \otimes J(\delta_{r,v}, k)[s_r])$$

is irreducible. By Lemma I.8, (ii), of [MW], $I_P^G(J(\widetilde{\delta}_v, k), \mathbf{s})$ is a quotient of the induced representation $I_Q^G(\delta_v, \mu(k, \mathbf{s}))$. Since $I_P^G(J(\widetilde{\delta}_v, k), \mathbf{s})$ is irreducible, this quotient must be the Langlands quotient. Thus

$$\pi_v \cong J_Q^G(\delta_v, \mu(k, \mathbf{s})) \cong I_P^G(J(\delta_{1,v}, k)[s_1] \otimes \cdots \otimes J(\delta_{r,v}, k)[s_r]).$$

By construction, the s_i's are the Langlands parameters of a local component of a cuspidal automorphic representation. Therefore it follows from Proposition 3.3 that they satisfy

$$|s_i| < \frac{1}{2} - \frac{1}{n^2+1}, \quad i = 1, ..., r.$$

□

This result has an important consequence for the location of the poles of normalized intertwining operators (see Proposition 4.2).

4. Proof of the main results

In this section we prove Proposition 0.2 and Theorem 0.1. To this end we need some preparation.

Let M be a standard Levi subgroup of type $(n_1, ..., n_r)$ and let $P, P' \in \mathcal{P}(M)$. Given a place v of E, let $\Pi_{\text{disc}}(M(E_v))$ denote the set of all $\pi_v \in \Pi(M(E_v))$ which are local components of some automorphic representation π in the discrete spectrum of $M(\mathbb{A})$. Without loss of generality, we may assume that P is a standard parabolic subgroup. Let $\pi_v \in \Pi_{\text{disc}}(M(E_v))$. Then $\pi_v = \otimes_i \pi_{i,v}$ with $\pi_{i,v} \in \Pi_{\text{disc}}(\mathrm{GL}_{n_i}(E_v))$, $i = 1, ..., r$. By Proposition 3.5 there exist a standard parabolic subgroup R_i of GL_{n_i} with

$$M_{R_i} = \mathrm{GL}_{n_{i1}} \times \cdots \times \mathrm{GL}_{n_{im_i}},$$

$k_{ij}|n_{ij}$, discrete series representations δ_{ij} of $\mathrm{GL}_{d_{ij}}(E_v)$, $d_{ij} = n_{ij}/k_{ij}$, and $s_{ij} \in \mathbb{R}$ satisfying

$$s_{i1} \geq \cdots \geq s_{im_i}, \quad |s_{ij}| < \frac{1}{2} - \frac{1}{n^2+1}, \tag{4.1}$$

such that

$$\pi_{i,v} \cong I_{R_i}^{\mathrm{GL}_{n_i}} \left(J(\delta_{i1}, k_{i1})[s_{i1}] \otimes \cdots \otimes J(\delta_{im_i}, k_{im_i})[s_{im_i}] \right).$$

Set $R = \prod_i R_i$. Then R is a standard parabolic subgroup of M. Put $m = m_1 + \cdots + m_r$. We identify $\{(i,j) \mid i = 1, ..., r, \ j = 1, ..., m_i\}$ with $\{1, ..., m\}$ by

$$(i,j) \mapsto \sum_{k<i} m_k + j.$$

For $1 \leq l \leq m$ let (i,j) be the pair that corresponds to l. Put

$$\delta_l = \delta_{ij}, \quad k_l = k_{ij}, \quad s_l = s_{ij}.$$

Set

$$J_{\pi_v} = \otimes_{i=1}^m J(\delta_i, k_i), \quad \mathbf{s}_{\pi_v} = (s_1, ..., s_m). \tag{4.2}$$

Combing the above equivalences, we get

$$\pi_v \cong I_R^M(J_{\pi_v}, \mathbf{s}_{\pi_v}).$$

Let $P(R)$ and $P'(R)$ be the parabolic subgroups of GL_n with $P(R) \subset P$, $P'(R) \subset P'$, $P(R) \cap M = R$ and $P'(R) \cap M = R$. Then by induction in stages, the induced representation $I_P^G(\pi_v, \mathbf{s})$ is unitarily equivalent to the induced representation $I_{P(R)}^G(J_{\pi_v}, \mathbf{s} + \mathbf{s}_{\pi_v})$. The unitary map μ

which provides this equivalence, is given by the evaluation map as in [KS, p.31]. Here we identify $\mathbf{s} \in \mathbb{C}^r$ with an element in $\mathbb{C}^m$ by

$$(s_1, ..., s_r) \mapsto (s_1, ..., s_1, s_2, ..., s_2, ..., s_r, ..., s_r),$$

where s_i is repeated m_i times. [KS, p.31].

Lemma 4.1. *Let μ be the unitary equivalence of the induced representations as above. Then we have*

$$\mu \circ R_{P'|P}(\pi_v, \mathbf{s}) = R_{P'(R)|P(R)}(J_{\pi_v}, \mathbf{s} + \mathbf{s}_{\pi_v}) \circ \mu. \tag{4.3}$$

Proof. First consider the unnormalized intertwining operators. Recall that the unnormalized intertwining operators are defined by integrals which are absolutely convergent in a certain shifted chamber [KS, Theorem 6.6], [Sh2]. If we compare these integrals in their range of convergence as in [KS, p.31], it follows immediately that (4.3) holds for the unnormalized intertwining operators. So it remains to consider the normalizing factors. If we apply Lemma 3.4 to each component $\pi_{i,v}$ of π_v, it follows that π_v is equivalent to a Langlands quotient of the form $J_Q^M(\delta_v, \mu(\mathbf{k}, \mathbf{s}_{\pi_v}))$, where $\delta_v = \otimes_{i=1}^m \delta_i$ and $\mu(\mathbf{k}, \mathbf{s}_{\pi_v}) = (\mu(k_1, \mathbf{s}_{\pi_{1,v}}), ..., \mu(k_r, \mathbf{s}_{\pi_{r,v}}))$. Hence by (2.3) in [A7] we get

$$r_{P'|P}(\pi_v, \mathbf{s}) = r_{P'(Q)|P(Q)}(\delta_v, \mathbf{s} + \mu(\mathbf{k}, \mathbf{s}_{\pi_v})).$$

Let S_i be the standard parabolic subgroup of M_{R_i} with Levi component

$$(\mathrm{GL}_{d_{i1}} \times \cdots \times \mathrm{GL}_{d_{i1}}) \times \cdots \times (\mathrm{GL}_{d_{ip_i}} \times \cdots \times \mathrm{GL}_{d_{ip_i}}),$$

where each factor $\mathrm{GL}_{d_{ij}}$ occurs k_i times. Let $S = \prod_i S_i$. Then M_S and M_Q are conjugate. Let $w \in S_m$ be the element which conjugates M_S into M_Q. Referring again to (2.3) in [A7], we get

$$r_{P'(R)|P(R)}(J_{\pi_v}, \mathbf{s} + \mathbf{s}_{\pi_v}) = r_{P'(R(S))|P(R(S))}(w\delta_v, \mathbf{s} + w\mu(\mathbf{k}, \mathbf{s}_{\pi_v})).$$

By (r.6) of [A8, p.172] it follows that

$$r_{P'(R)|P(R)}(J_{\pi_v}, \mathbf{s} + \mathbf{s}_{\pi_v}) = r_{P'(R(S)^w)|P(R(S)^w)}(\delta_v, \mathbf{s} + \mu(\mathbf{k}, \mathbf{s}_{\pi_v})),$$

where $R(S)^w = w^{-1}R(S)w$. Finally note that $P(Q)$, $P'(Q)$, $P(R(S)^w)$ and $P'(R(S)^w)$ have the same Levi component and the reduced roots satisfy

$$\Sigma^r_{\overline{P'(R(S)^w)}} \cap \Sigma^r_{P(R(S)^w)} = \Sigma^r_{\overline{P'(Q)}} \cap \Sigma^r_{P(Q)}.$$

Using the product formula (r.1) in [A8, p.171], it follows that

$$r_{P'(R(S)^w)|P(R(S)^w)}(\delta_v, \mathbf{s} + \mu(\mathbf{k}, \mathbf{s}_{\pi_v})) = r_{P'(Q)|P(Q)}(\delta_v, \mathbf{s} + \mu(\mathbf{k}, \mathbf{s}_{\pi_v})).$$

Combining the above equations, we get

$$r_{P'|P}(\pi_v, \mathbf{s}) = r_{P'(R)|P(R)}(J_{\pi_v}, \mathbf{s} + \mathbf{s}_{\pi_v}),$$

and this finishes the proof of the lemma. □

We say that $R_{Q|P}(\pi_v, \mathbf{s})$ has a pole at $\mathbf{s}_0 \in \mathbb{C}^r$, if $R_{Q|P}(\pi_v, \mathbf{s})$ has a matrix coefficient with a pole at $\mathbf{s}_0$. Otherwise, $R_{Q|P}(\pi_v, \mathbf{s})$ is called holomorphic in $\mathbf{s}_0$.

Proposition 4.2. *Let $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ be a standard Levi subgroup of GL_n and let $P, P' \in \mathcal{P}(M)$. For every place v of E and for all $\pi_v \in \Pi_{\mathrm{disc}}(M(E_v))$, the normalized intertwining operator $R_{P'|P}(\pi_v, \mathbf{s})$ is holomorphic in the domain*

$$\{\mathbf{s} \in \mathbb{C}^r \mid \mathrm{Re}(s_i - s_j) > -2/(1+n^2),\ 1 \leq 1 < j \leq r\}.$$

Proof. Using Lemma 4.1 we immediately reduce to the consideration of the corresponding problem for $R_{P'(R)|P(R)}(J_{\pi_v}, \mathbf{s} + \mathbf{s}_\pi)$, regarded as function of $\mathbf{s} \in \mathbb{C}^r$. By Proposition I.10 of [MW], the intertwining operator $R_{P'(R)|P(R)}(J_{\pi_v}, \mathbf{s})$ is holomorphic in the domain of all $\mathbf{s} \in \mathbb{C}^m$ satisfying $\mathrm{Re}(s_i - s_j) > -1$, $1 \leq i < j \leq m$. Furthermore by (4.1) the absolute value of all components of $\mathbf{s}_\pi$ is bounded by $1/2 - (1+n^2)^{-1}$. Combining these observations the claimed result follows.

□

Proof of Proposition 0.2:

Let M be a standard Levi subgroup of type $(n_1, ..., n_r)$ and let $P, Q \in \mathcal{P}(M)$. We distinguish between the Archimedean and non-Archimedean case.

Case 1: $v|\infty$.

Let K_v be the standard maximal compact subgroup of $\mathrm{GL}_n(E_v)$. Given $\pi_v \in \Pi(M(E_v))$ and $\sigma_v \in \Pi(K_v)$, denote by $R_{Q|P}(\pi_v, \mathbf{s})_{\sigma_v}$ the restriction of $R_{Q|P}(\pi_v, \mathbf{s})$ to the σ_v-isotypical subspace $\mathcal{H}_P(\pi_v)_{\sigma_v}$ of the Hilbert space $\mathcal{H}_P(\pi_v)$ of the induced representation. If π_v belongs to $\Pi_{\mathrm{disc}}(M(E_v))$, then by Proposition 4.2, $R_{Q|P}(\pi_v, \mathbf{s})_{\sigma_v}$ has no poles in the domain of all $\mathbf{s} \in \mathbb{C}^r$ satisfying $|\mathrm{Re}(s_i)| < (1+n^2)^{-1}$, $i = 1, ..., r$. Using the factorization of normalized intertwining operators and Corollary A.3 it follows that there exist constants $C > 0$ and $k \in \mathbb{N}$ such that

$$\| D_{\mathbf{u}} R_{Q|P}(\pi_v, i\mathbf{u})_\sigma \| \leq C\,(1 + \| \sigma_v \|)^k$$

for all $\pi_v \in \Pi(M(E_v))$, $\sigma_v \in \Pi(K_v)$ and $\mathbf{u} \in \mathbb{R}^r$. This proves (0.3).

Case 2: $v < \infty$.

Using the properties of the normalized intertwining operators [A7, Theorem2.1], one can factorize $R_{P'|P}(\pi_v, \mathbf{s})$ in a product of normalized intertwining operators associated to maximal parabolic subgroups. Thus we immediately reduce to the case where P is maximal and $P' = \overline{P}$.

Consider a matrix coefficient $(R_{\overline{P}|P}(\pi_v,s)v_1,v_2)$, where $\| v_1 \|=\| v_2 \|=1$. By Theorem 2.1 of [A7], there is a rational function $f(z)$ of one complex variable z such that

$$f(q^{-s}) = (R_{\overline{P}|P}(\pi_v,s)v_1,v_2), \quad s\in\mathbb{C}. \tag{4.4}$$

We shall now investigate the properties of the rational function f. By Proposition I.10 of [MW] we know that $(R_{\overline{P}|P}(\pi_v,s)v_1,v_2)$ is holomorphic in the half-plane $\mathrm{Re}(s)>0$. Hence $f(z)$ is holomorphic in the punctured disc $0<|z|<1$. Moreover by unitarity of $R_{\overline{P}|P}(\pi_v,it)$, $t\in\mathbb{R}$, we have $|(R_{\overline{P}|P}(\pi_v,it)v_1,v_2)|\le 1$, $t\in\mathbb{R}$, and hence $|f(z)|\le 1$ for $|z|=1$. To determine the behaviour of f at $z=0$ we observe that the unnormalized intertwining operator $J_{\overline{P}|P}(\pi_v,s)$ is defined by an integral which is absolutely and uniformly convergent in some half-plane $\mathrm{Re}(s)\ge c$. Especially, $J_{\overline{P}|P}(\pi_v,s)$ is uniformly bounded for $\mathrm{Re}(s)\gg 0$. The normalizing factor $r_{\overline{P}|P}(\pi_v,s)$ is given by (2.1). It follows from the expressions (2.5) and (2.6) for the L-factors and the epsilon factor, that there exist polynomials $P(z)$ and $Q(z)$ with $P(0)=Q(0)=1$, a constant $a\in\mathbb{C}$ and $m\in\mathbb{Z}$ such that

$$r_{\overline{P}|P}(\pi_v,s)=aq^{ms}\frac{P(q^{-s})}{Q(q^{-s})}, \quad s\in\mathbb{C}.$$

The integer m is given by (2.7) and it follows from (2.11) that there exists $c\ge 0$, which depends on the choice of a nontrivial continuous character of E_v^+, such that $-c\le m$ for all $\pi_v\in\Pi(M(E_v))$. Thus by the maximum principle it follows that for $0<|z|\le 1$ we have

$$|f(z)|\le\begin{cases}1 & : \ m\ge 0\\ |z|^m & : \ m<0.\end{cases} \tag{4.5}$$

Now assume that $\pi_v\in\Pi_{\mathrm{disc}}(M(E_v))$. Then by Proposition 4.2, $f(z)$ is actually holomorphic for $|z|<q^{2/(1+n^2)}$. Set

$$\delta=\min\{2,q^{2/(1+n^2)}\}.$$

Note that $\delta>1$. Let $\rho_1,...,\rho_r$ be the poles of f, where each pole is counted with its multiplicity. Let $-l$ be the order of f at infinity. Set

$$g(z)=f(z)z^{-l}\prod_j\frac{z-\rho_j}{1-\overline{\rho}_j z}.$$

Since $|\rho_j|>\delta$, $j=1,...,r$, the rational function $g(z)$ is holomorphic for $|z|>1$, bounded for $|z|\ge 1$ and satisfies $|g(z)|=|f(z)|\le 1$ for

$|z|=1$. Thus $|g(z)|\le 1$ for $|z|\ge 1$ and hence,

$$|f(z)|\le |z|^l\prod_{j=1}^r\left|\frac{1-\overline{\rho}_j z}{z-\rho_j}\right| = |z|^l\prod_{j=1}^r\left|\frac{\overline{z}-1/\rho_j}{1-z/\rho_j}\right|,\quad |z|\ge 1.$$

For $1\le |z|<(1+\delta)/2$ the right hand side is bounded by $\left(\frac{1+\delta}{2}\right)^l\left(\frac{7}{\delta-1}\right)^r$. Together with (4.5) it follows that there exists $C>0$, which is independent of π_v, such that in the annulus $2/(1+\delta)<|z|<(1+\delta)/2$ we have

$$|f(z)|\le C\left(\frac{1+\delta}{2}\right)^l\left(\frac{7}{\delta-1}\right)^r.$$

Using Cauchy's formula we obtain a similar bound for any derivative of f. By (4.4) this leads to a bound for any derivative of $(R_{\overline{P}|P}(\pi_v,s)v_1,v_2)$ in a strip $|\operatorname{Re}(s)|<\varepsilon$ for some $\varepsilon>0$. To complete the proof we need to verify that for a given open compact subgroup K_v of $\mathrm{GL}_n(E_v)$, the numbers r and l are bounded independently of π_v if $\pi_v^{K_v\cap M(E_v)}\ne 0$.

First consider r. By Theorem 2.2.2 of [Sh2, p.323] there exists a polynomial $p(z)$ with $p(0)=1$ such that $p(q^{-s})J_{\overline{P}|P}(\pi_v,s)$ is holomorphic on $\mathbb{C}$. Moreover the degree of p is bounded independently of π_v. Using the definition of the normalizing factors (2.1), it follows immediately that there exists a polynomial $\widetilde{p}(z)$ whose degree is bounded independently of π_v such that $\widetilde{p}(q^{-s})R_{\overline{P}|P}(\pi_v,s)$ is holomorphic on $\mathbb{C}$. This proves that r is bounded independently of π_v.

To estimate l, we fix an open compact subgroup K_v of $\mathrm{GL}_n(E_v)$. Our goal is now to estimate the order at ∞ of any matrix coefficient of $R_{\overline{P}|P}(\pi_v,s)_{K_v}$, regarded as a function of $z=q^{-s}$. Write π_v as Langlands quotient $\pi_v=J_R^M(\delta_v,\mu)$ where R is a parabolic subgroup of M, δ_v a square-integrable representation of $M_R(E_v)$ and $\mu\in(\mathfrak{a}_R^*/\mathfrak{a}_M^*)_{\mathbb{C}}$ with $\operatorname{Re}(\mu)$ in the chamber attached to R.Then

$$R_{\overline{P}|P}(\pi_v,s)=R_{\overline{P(R)}|P(R)}(\delta_v,s+\mu)$$

with respect to the identifications described in [A7, p.30]. Here s is identified with a point in $(\mathfrak{a}_R^*/\mathfrak{a}_G^*)_{\mathbb{C}}$ with repsect to the canonical embedding $\mathfrak{a}_M^*\subset\mathfrak{a}_R^*$. Using again the factorization of normalized intertwining operators we reduce to the case of a square-integrable representation. Let $\mathbf{1}$ denote the trivial representation of K_v. By the same reasoning as in the proof of Lemma 2.2 we get

$$[I_P^G(\delta_v,s)|_{K_v}:\mathbf{1}]\le \#(\mathrm{GL}_n(\mathcal{O}_v)/K_v)[\delta_v|_{K_v\cap M(E_v)}:\mathbf{1}].$$

By [HC2, Theorem 10] the set $\Pi_2(M(E_v),K_v)$ of square-integrable representations of $M(E_v)$ with $[\delta|_{K_v\cap M(E_v)}:\mathbf{1}]\ge 1$ is a compact subset of the space $\Pi_2(M(E_v))$ of square-integrable representations of $M(E_v)$. Under the canonical action of $i\mathfrak{a}_M^*$, the set $\Pi_2(M(E_v), K_v)$ decomposes into a finite number of orbits. In this way our problem is finally reduced to the consideration of the matrix coefficients of $R_{\overline{P}|P}(\pi_v, s)_{K_v}$ for a finite number of representations π_v. This implies the claimed bound for l. $\square$

Proof of Theorem 0.1:

Recall from §2 that at finite places the normalization of the local intertwining operators differs from the normalization used in [Mu4]. Let $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$, $Q, P \in \mathcal{P}(M)$ and v a finite place of E. Let $\widetilde{r}_{Q|P}(\pi_v, \mathbf{s})$, $\mathbf{s} \in \mathbb{C}^r$, be the normalizing factor used in [Mu4] and let

$$\widetilde{R}_{Q|P}(\pi_v, \mathbf{s}) = \widetilde{r}_{Q|P}(\pi_v, \mathbf{s})^{-1} J_{Q|P}(\pi_v, \mathbf{s})$$

be the corresponding local normalized intertwining operator. Then it follows from Lemma 2.1 together with (2.11) and Lemma 2.2 that for every multi-index $\alpha \in \mathbb{N}_0^r$ there exists $C > 0$ such that

$$\| D_{\mathbf{u}}^{\alpha} \widetilde{R}_{Q|P}(\pi_v, \mathbf{u})_{K_v} \| \leq C \sum_{|\beta| \leq |\alpha|} \| D_{\mathbf{u}}^{\beta} R_{Q|P}(\pi_v, \mathbf{u})_{K_v} \|$$

for all $\mathbf{u} \in \mathbb{R}^r$ and all $\pi_v \in \Pi_{\mathrm{disc}}(M(E_v))$. Hence Proposition 0.2 holds also with respect to $\widetilde{R}_{Q|P}(\pi_v, \mathbf{s})$. Together with Theorem 0.1 of [Mu4] we obtain Theorem 0.1 of the present paper. $\square$

Remark. As the proof of Proposition 0.2 shows, the estimations (0.2) and (0.3) hold for all generic representations π_v of $M(E_v)$ whose Langlands parameters $s_1, \ldots, s_r$ satisfy a non-trivial bound of the form $|s_i| < 1/2 - \varepsilon$, where $\varepsilon > 0$ is independent of π_v. We note that this assumption is really necessary and can not be removed in general. Especially, as the following example shows, the estimations can not be expected to be uniform in all $\pi_v \in \Pi(M(E_v))$.

Example.

Let $G = \mathrm{GL}_4(\mathbb{R})$ and P the standard parabolic subgroup with $M_P = \mathrm{GL}_2 \times \mathrm{GL}_2$. Consider the representation $I_P^G(\sigma \times \sigma, \mathbf{s})$ where σ is the spherical principal series representation induced from the character $\mu = (\mu, -\mu)$, μ real and $0 \leq \mu \leq 1/2$ of the Borel subgroup of $\mathrm{GL}_2(\mathbb{R})$. We may assume that $\mathbf{s} = (s, -s)$ with s real. Then

$$I_P^G(\sigma \times \sigma, \mathbf{s}) = I_B^G(\mu + s, -\mu + s, \mu - s, \mu - s).$$

For fixed $0 \leq \mu < 1/2$ this representation is irreducible for $0 \leq |s| < 1/2 - \mu$ and reducible for $|s| = 1/2 - \mu$ [Sp]. The intertwining operator $R_{\bar{P}|P}(\sigma \times \sigma, \mathbf{s})$ is therefore well defined on the interval $0 \leq |s| < 1/2 - \mu$ and has a pole for $-s = 1/2 - \mu$. □

The example shows that the poles of the normalized intertwining operator can be arbitrary close to the imaginary axis. Thus, we can not expect to have uniform bounds of the derivatives of the normalized intertwining operators along the imaginary axis for all unitary π.

Appendix A

by Erez M. Lapid

Let G be the real points of a connected reductive group defined over $\mathbb{R}$. Let K be a maximal compact subgroup of G and let $P = M_P N_P$ be a parabolic subgroup of G with its Levi decomposition. Write $M = M_P = {}^0MA_M$ in the usual way. Let σ be an irreducible unitary representation of ${}^0M$ acting on a Hilbert space H_σ and let H_σ^∞ be its smooth part. We denote by I_σ^∞ the space of smooth functions $f : K \to H_\sigma^\infty$ such that $f(mk) = \sigma(m)f(k)$ for any $m \in K_M = M \cap K$ and $k \in K$ with the inner product

$$\langle f_1, f_2 \rangle = \int_K (f_1(k), f_2(k))_{H_\sigma} \, dk$$

We denote the Lie algebra of A_M by $\mathfrak{a}_M$. Let P' be another parabolic subgroup of G containing M as its Levi part. For any $\nu \in \mathfrak{a}_{M,\mathbb{C}}^*$, let $J_{P'|P}(\nu)$ be the usual intertwining operator on I_σ^∞ ([Wal2, Chapter 10]) and let $R_{P'|P}(\sigma, \nu) = r_{P'|P}(\sigma, \nu)^{-1} J_{P'|P}(\nu)$ be the normalized intertwining operator (cf. [A7]). Finally, for any irreducible representation γ of K we denote by $I_\sigma^G(\gamma) = I_\sigma(\gamma)$ the γ-isotypic part of I_σ^∞. We also denote by $\|\gamma\|$ the norm of the highest weight of γ.

The purpose of this appendix is to give a bound for the matrix coefficients of the operator $R_{P'|P}(\sigma, \nu)$ on any K-type near the unitary axis. By factoring $R_{P'|P}(\sigma, \nu)$ it is enough to consider the "basic" case where P, P' are adjacent – say along the root α. In this case the operator $J_{P'|P}(\sigma, \nu)$ depends only on (ν, α) and will be written as $J_{P'|P}(s)$ for $(\nu, \alpha) = 4s(\rho_P, \alpha)$. Similarly for $R_{P'|P}(\sigma, s)$.

It follows from [Wal2, Lemma 10.1.11, Theorem 10.1.6, 10.1.13] that the poles of $J_{P'|P}(s)$ (counted with multiplicities) are contained in $\bigcup_{i=1}^r (\rho_i - \mathbb{N})$ for some complex numbers $\rho_1, \ldots, \rho_r$. By the nature of the normalization factors we may enlarge the set $\{\rho_i\}$ to assume that the same holds for $R_{P'|P}(\sigma,s)$ as well. Let

$$M_\sigma^+ = \max\{0, \operatorname{Re}\rho : \rho \text{ is a pole of } R_{P'|P}(\sigma,s)\}$$

and for any $\gamma \in \widehat{K}$ set

$$M_{\sigma,\gamma}^- = \max\{0, -\operatorname{Re}(\rho) : \rho \text{ is a pole of } R_{P'|P}(\sigma,s)\big|_{I_\sigma(\gamma)}\}.$$

Finally, let

$$\delta = \min\{\frac{1}{2}, |\operatorname{Re}(\rho)| : \rho \text{ is a pole of } R_{P'|P}(\sigma,s)\}.$$

Lemma A.1. *For any $\gamma \in \widehat{K}$ and any unit vectors $\varphi_1, \varphi_2 \in I_\sigma(\gamma)$ and any $\epsilon > 0$ we have*

$$\big|(R_{P'|P}(\sigma,s)\varphi_1,\varphi_2)\big| \le \big[(M_\sigma^+ + M_{\sigma,\gamma}^- + 1)/\epsilon\big]^r$$

in the strip $|\operatorname{Re}(s)| < \delta - \epsilon$.

Proof. Let $f(s) = (R_{P'|P}(\sigma,s)\varphi_1,\varphi_2)$. It is a rational function of s ([A7]). We also have $|f(s)| \le 1$ for $s \in i\mathbb{R}$ since $R_{P'|P}(\sigma,s)$ is unitary there. Define

$$g(s) = f(s) \times \prod_{i=1}^{r} \prod_{j=\lceil \operatorname{Re}\rho_i+\delta\rceil}^{\lfloor \operatorname{Re}\rho_i + M_{\sigma,\gamma}^-\rfloor} \frac{s-\rho_i+j}{s+\overline{\rho_i}-j}.$$

Then $g(s)$ is holomorphic (and rational) for $\operatorname{Re}(s) \le 0$ and $|g(s)| = |f(s)| \le 1$ on $i\mathbb{R}$. Thus $|g(s)| \le 1$ for $\operatorname{Re}(s) \le 0$. It follows that in this region

$$|f(s)| \le \prod_{i=1}^{r} \prod_{j=\lceil \operatorname{Re}\rho_i+\delta\rceil}^{\lfloor \operatorname{Re}\rho_i + M_{\sigma,\gamma}^-\rfloor} \left|\frac{s+\overline{\rho_i}-j}{s-\rho_i+j}\right| \le \prod_{i=1}^{r} \prod_{j=\lceil \operatorname{Re}\rho_i+\delta\rceil}^{\lfloor \operatorname{Re}\rho_i + M_{\sigma,\gamma}^-\rfloor} \left|\frac{\operatorname{Re}(s+\overline{\rho_i}-j)}{\operatorname{Re}(s-\rho_i+j)}\right|.$$

Then for $0 \ge \operatorname{Re} s \ge -\delta+\epsilon$ each factor is bounded by

$$\begin{aligned}\prod_{j=\lceil \operatorname{Re}\rho_i+\delta\rceil}^{\lfloor \operatorname{Re}\rho_i + M_{\sigma,\gamma}^-\rfloor} \frac{-\operatorname{Re}s - \operatorname{Re}\rho_i + j}{\operatorname{Re}s - \operatorname{Re}\rho_i + j} &< \prod_j \frac{\operatorname{Re}s - \operatorname{Re}\rho_i + j + 1}{\operatorname{Re}s - \operatorname{Re}\rho_i + j} \\ &< \frac{M_{\sigma,\gamma}^- + 1}{\epsilon}.\end{aligned}$$

Similarly, one shows that for $0 \le \operatorname{Re} s < \delta - \epsilon$

$$|f(s)| < \left[\frac{M_\sigma^+ + 1}{\epsilon}\right]^r.$$

□

The following Proposition will be proved below.

Proposition A.2. *There exists a constant c depending only on G such that*

$$M_\sigma^+ \le c, \quad r \le c \quad M_{\sigma,\gamma}^- \le c(1+\|\gamma\|) \tag{1.1}$$

for all unitary σ.

By Cauchy's formula, Lemma A.1 and Proposition A.2 will imply the following.

Corollary A.3. *For any differential operator $D(s)$ with constants coefficients there exist constants c', k' (depending only on G) such that*

$$\|D(s)R_{P'|P}(\sigma,s)_{I_\sigma(\gamma)}\| \le c'\left(\frac{1+\|\gamma\|}{\delta}\right)^{k'} \tag{1.2}$$

for all $\gamma \in \widehat{K}$ and $s \in i\mathbb{R}$.

Remark A.4. *The example in §4 emphasizes that the dependence on δ is essential if σ is not tempered. This is already important in order to lift the K-finiteness assumption in the absolute convergence of the contribution of an individual cuspidal datum. This point was overlooked in* [A4] *(cf., p. 1329]). More precisely, the property* [A5, (7.6)] *holds only for tempered representations. We mention that it follows from* [KS, Theorem 16.2] *that for all σ tempered we have $\delta > \delta_0$ where $\delta_0 > 0$ depends only on G.*

We will now prove Proposition A.2. We first deal with the first part of (1.1). More precisely, we have

Lemma A.5. *There exists $s_0 \in \mathbb{R}$, depending only on G, such that $J_{P'|P}(\sigma,s)$ converges and $r_{P'|P}(\sigma,s)$ is holomorphic and non-zero for all* $\mathrm{Re}(s) > s_0$. *In particular, $M_\sigma^+ \le s_0$.*

Proof. Let (Q,τ,λ) be Langlands data for σ, i.e., Q is a parabolic subgroup of M with Levi subgroup L, τ is a tempered representation of L and λ is a real parameter in the positive Weyl chamber of $\mathfrak{a}_Q^*$ and σ is the irreducible quotient of the standard module defined by Q and λ. By [Wal1, 5.5.2, 5.5.3], or [BW, Ch. XI, Theorem 3.3] $\|\lambda\|$ is bounded in terms of G only. Moreover identifying I_σ^∞ with a quotient of I_τ^∞, we may identify $J_{P'|P}(\sigma,s)$ with $J_{QN_{P'}|QN_P}(\tau,s+\lambda)$ on the quotient space (cf. [A7, p. 30] or §4). Moreover, we have $r_{P'|P}(\sigma,s) = r_{QN_{P'}|QN_P}(\tau,s+\lambda)$. By factoring $J_{QN_{P'}|QN_P}(\tau,s+\lambda)$ and $r_{QN_{P'}|QN_P}(\tau,s+\lambda)$ the Lemma easily reduces to the tempered case. Similarly, we reduce to the square-integrable case. For σ square-integrable we can take $s_0 = 0$ ([A7]). □

The same argument reduces the second statement of (1.1) to the square-integrable case. This case follows from [KS, Theorem 16.2] and the compatibility of the normalization factors with Artin's factors ([A7]).

To continue the proof of Proposition A.2 we suppress for the moment the assumption that P, P' are adjacent and set $\Sigma(P'|P) = \Sigma(P)\cap\Sigma(\overline{P'})$ where $\overline{P'}$ is the parabolic opposite to P' and $\Sigma(P) = \Sigma(P, A_M)$ be the set of reduced roots of A_M in P.

The main assertion is the following.

Lemma A.6. *There exists a constant d (depending only on G) such that for any $\gamma \in \widehat{K}$ $J_{P'|P}(\nu)$ is holomorphic and injective on $I_\sigma(\gamma)$ in the domain*

$$\{\nu \in \mathfrak{a}^*_{M,\mathbb{C}} : \mathrm{Re}(\nu, \alpha) > d(1 + \|\gamma\|) \text{ for all } \alpha \in \Sigma(P'|P)\}.$$

The last inequality of (1.1) then follows from Lemma A.5, Lemma A.6 and the relation

$$R_{P|P'}(\sigma, -s)R_{P'|P}(\sigma, s) = \mathrm{id}\,.$$

It remains to prove Lemma A.6. Clearly we may assume, by passing to the derived group, that G is semisimple. We first need some more notation. Let $P_0 = {}^0M_0A_0N_0$ be a minimal parabolic subgroup of G, contained in P, so that ${}^0M_0$ is compact. Let $\mathfrak{t}$ be a maximal abelian subalgebra of ${}^0\mathfrak{m}$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}_0$ (a direct sum with respect to the Killing form). Then $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ and the real vector space $\mathfrak{h}_R$ spanned by the co-roots is $i\mathfrak{t} + \mathfrak{a}_0$ ([Wal1, 2.2.5]). The Weyl group $W = W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ acts on $\mathfrak{h}_R$ as well as on $\mathfrak{h}^*_\mathbb{C}$. We identify the characters of the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$ as W-orbits of $\mathfrak{h}^*_\mathbb{C}$ via the Harish-Chandra isomorphism. A similar discussion applies to M. We denote by χ_σ the infinitesimal character of σ.

For any (finite dimensional) irreducible representation σ' of ${}^0M_0$ and $\mu \in \mathfrak{a}^*_{0,\mathbb{C}}$ we denote by $\pi_{\sigma',\mu} = \pi^G_{\sigma',\mu}$ the corresponding principal series representation on G. Its infinitesimal character is (the W-orbit) of $\chi_{\sigma'} + \mu$ where $\chi_{\sigma'} \in i\mathfrak{t}^*$ (the infinitesimal character of σ') is the translate of the highest weight of σ' by the half-sum of positive roots in $\mathfrak{m}_0$.

Lemma A.7. *There exists a constant c depending only on G such that any unitary representation σ of M can be embedded (infinitesimally) as a subrepresentation of a (non-unitary) $\pi^M_{\sigma',\mu}$ and $\|\mathrm{Re}\,\mu\| \le c(1 + \|\gamma\|)$ whenever $\mathrm{Hom}_{K_M}(\gamma, \sigma) \neq 0$.*

Proof. Suppose first that σ is square-integrable. Using the Casselman subrepresentation Theorem (e.g. [Wal1, Ch. 4] or [Kn, Theorem 8.37]) we may embed σ in some principal series $\pi^M_{\sigma',\mu}$. By comparing infinitesimal characters we infer that $\mu \in \mathfrak{a}_0^*$ and

$$\|\chi_\sigma\|^2 = \|\mu\|^2 + \|\chi_{\sigma'}\|^2 \geq \|\mu\|^2.$$

On the other hand by [Wa2, p.398], (cf. [Wal2, p. 258]) the square of the norm of any K-type of σ is bounded below, up to a fixed additive constant, by $\|\chi_\sigma\|^2$. The lemma follows in this case.

To treat the general case we use the Langlands classification Theorem to imbed σ in $S(\tau, \lambda)$ where Q is a parabolic subgroup with Levi subgroup L, τ is a square- integrable representation of L, λ is in the closed negative Weyl chamber of $\mathfrak{a}_Q^*$ and $S(\tau, \lambda)$ is the corresponding induced representation. As in the proof of Lemma A.5 we have $\|\lambda\| < C$ independently of σ. All K-types of $S(\tau, \lambda)$ (and hence, of σ) contain a K_L-type of τ in their restriction to K_L. Hence, by induction in stages, the Lemma reduces to the square integrable case. $\square$

We will now reduce Lemma A.6 to the case where P is a minimal parabolic of G.

Imbed σ in $\pi^M_{\sigma',\mu}$ as in the Lemma and suppose that $\mathrm{Re}(\mu + \nu, \beta) > 0$ for all $\beta \in \Sigma(P_0N_{P'}|P_0N_P)$. Then $J_{P'|P}(\pi_{\sigma',\mu}, \nu)$ can be identified with $J_{P_0N_{P'}|P_0N_P}(\sigma', \mu + \nu)$ and it is given by an absolutely convergent integral. Its restriction to I^∞_σ is $J_{P'|P}(\sigma, \nu)$. Thus, in that region the injectivity of $J_{P'|P}(\sigma, \nu)$ on $I_\sigma(\gamma)$ follows from that of $J_{P_0N_{P'}|P_0N_P}(\sigma', \mu+\nu)$. We note that the restriction to A_M defines a bijection $\alpha \leftrightarrow \alpha'$ between $\Sigma(P_0N_{P'}|P_0N_P)$ and $\Sigma(P'|P)$, and we have $(\nu, \alpha) = (\nu, \alpha')$. The reduction follows.

By factoring $J_{P'|P}$ as a product of "basic" intertwining operators we may also assume that P' is adjacent to P. Let $Q = LV$ be the parabolic subgroup generated by P and P'. Then L has rank one and it follows from the argument of [Wal2, 10.4.5] that $J_{P'|P}(\sigma, \gamma)$ is injective on $I_\sigma(\gamma)$ if and only if $J^L_{P'\cap L|P\cap L}(\sigma, \nu^L)$ is injective on $I^L_\sigma(\gamma')$ for all $\gamma' \in \widehat{K_L}$ which occur in the restriction of γ. We observe that $\|\gamma'\| \leq \|\gamma\|$ for such γ'. Hence, we reduce to the case where G is of rank one, P is minimal and $P' = \bar{P}$. Once again we can assume that G is semisimple as well. From now on we assume that this is the case.

For $\mathrm{Re}(s) > 0$ the representation $\pi_{\sigma,s\alpha}$ is of finite length and its Langlands quotient is given by the image of $J_{P'|P}(\sigma, s\alpha)$. Thus, $J_{P'|P}(\sigma, s\alpha)$ is not injective on $I_\sigma(\gamma)$ if and only if γ occurs in one of the subquotients of $\pi_{\sigma,s\alpha}$ other than the Langlands quotient. Assume that this is the case and let π' be any such subquotient. Then by [Wal1, Corollary 5.5.3] the Langlands parameter of π' is smaller than that of π. Thus, either π' is square-integrable or π' can be imbedded in $\pi_{\sigma',s'}$ with $0 \leq \mathrm{Re}(s') < \mathrm{Re}(s)$. In the first case, the infinitesimal character of π' is in $\mathfrak{h}_R^*$, i.e., $s \in \mathbb{R}$, and by ([Wa2, p. 398])

$$C + \|\gamma\|^2 \geq \|\chi_{\pi'}\|^2 = \|\chi_\sigma\|^2 + s^2\|\alpha\|^2 \geq s^2\|\alpha\|^2$$

for a certain constant C. It follows that s is bounded by a constant multiple of $\|\gamma\|$. In the second case, we have

$$\chi_{\sigma'} + s'\alpha = w(\chi_\sigma + s\alpha) \tag{1.3}$$

for some $w \in W$. Write $w\alpha = \xi\alpha + \beta$ with $\xi \in \mathbb{R}$ and $\beta \in i\mathfrak{t}^*$. If $\beta = 0$ then $\xi = \pm 1$, w stabilizes $i\mathfrak{t}^*$ and we obtain $s = \pm s'$ – a contradiction. Thus, $\beta \neq 0$. Projecting (1.3) onto $i\mathfrak{t}^*$ we obtain

$$\chi_{\sigma'} = (w\chi_\sigma)_{i\mathfrak{t}^*} + s\beta.$$

On the other hand, since γ occurs $\pi_{\sigma,s\alpha}$, σ occurs in the restriction of γ to ${}^0M = K_M$ and hence $\|\sigma\| \leq \|\gamma\|$. Similarly, $\|\sigma'\| \leq \|\gamma\|$. Once again, it follows that $|s|$ is bounded by a constant multiple of $1 + \|\gamma\|$. This concludes the proof of Lemma A.6.

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