ANALYTIC TORSION OF ARITHMETIC QUOTIENTS OF THE SYMMETRIC SPACE $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$

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Abstract. In this paper we define a regularized version of the analytic torsion for arithmetic quotients of the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. The definition is based on the study of the renormalized trace of the corresponding heat operators, which is defined as the geometric side of the Arthur trace formula applied to the heat operator.

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1. Introduction

In various papers [BV], [MaM], [MP4] the Ray-Singer analytic torsion [RS] has been used to study the growth of torsion in the cohomology of cocompact arithmetic groups. Since many important arithmetic groups are not cocompact, it is very desirable to extend these results to the noncompact case. There exist some results for hyperbolic 3-manifolds. In [PR], Pfaff and Raimbault obtained upper and lower bounds for the growth of torsion in the cohomology of congruence subgroups of Bianchi groups if the local system varies. In [Ra1], [Ra2], J. Raimbault has studied the case of sequences (Γ_i) of congruence subgroups of Bianchi groups such that $\operatorname{vol}(\Gamma_i\backslash\mathbb{H}^3)\to\infty$ as $i\to\infty$.

The approach in the cocompact case relies on the equality of analytic torsion and Reidemeister torsion of the corresponding locally symmetric manifolds. We briefly recall the definition of the Ray-Singer analytic torsion. Let X be a compact Riemannian manifold of dimension n and $\rho\colon\pi_1(X)\to\operatorname{GL}(V)$ a finite dimensional representation of its fundamental group. Let $E_\rho\to X$ be the flat vector bundle associated with ρ. Choose a Hermitian fiber metric in E_ρ. Let $\Delta_p(\rho)$ be the Laplace operator on E_ρ-valued p-forms with respect to the metrics on X and in E_ρ. It is an elliptic differential operator, which is formally self-adjoint and non-negative. Let $h_p(\rho):=\dim\ker\Delta_p(\rho)$. Using the trace of the heat operator $e^{-t\Delta_p(\rho)}$, the zeta function $\zeta_p(s;\rho)$ of $\Delta_p(\rho)$ can be defined by

$$\zeta_p(s;\rho):=\frac{1}{\Gamma(s)}\int_0^\infty\left(\operatorname{Tr}\left(e^{-t\Delta_p(\rho)}\right)-h_p(\rho)\right)t^{s-1}\,dt. \tag{1.1}$$

The integral converges for $\operatorname{Re}(s)>n/2$ and admits a meromorphic extension to the whole complex plane, which is holomorphic at $s=0$. Then the Ray-Singer analytic torsion $T_X(\rho)\in\mathbb{R}^+$ is defined by

$$\log T_X(\rho)=\frac{1}{2}\sum_{p=1}^d(-1)^p p\frac{d}{ds}\zeta_p(s;\rho)\big|_{s=0}. \tag{1.2}$$

The analytic torsion has a topological counterpart. This is the Reidemeister torsion $\tau_X(\rho)$, which is defined in terms of a smooth triangulation of X [RS], [Mu5]. It is known that for unimodular representations ρ (meaning that $|\det\rho(\gamma)|=1$ for all $\gamma\in\pi_1(X)$) one has the equality $T_X(\rho)=\tau_X(\rho)$ [Ch], [Mu4], [Mu5]. In the general case of a non-unimodular representation the equality does not hold, but the defect can be described [BZ]. This equality has the following interesting consequence. Assume that the space of the representation ρ contains a lattice which is invariant under $\pi_1(X)$. Let $\mathcal{M}$ be the associated local system of free $\mathbb{Z}$-modules. Let $H^p(X,\mathcal{M})_{\text{tors}}$ be the torsion subgroup of $H^p(X,\mathcal{M})$. Then

$$T_X(\rho)=R\cdot\prod_{p=0}^d|H^p(X,\mathcal{M})_{\text{tors}}|^{(-1)^{p+1}}, \tag{1.3}$$

where R is the so called "regulator", defined in terms of the free part of the cohomology $H^p(X,\mathcal{M})$ (see [BV], [MP4]). In particular, if ρ is acyclic, i.e., $H^*(X,E_\rho)=0$, then $R=1$.

The equality (1.3) is the starting point for the application of the analytic torsion to the study of the torsion in the cohomology of cocompact arithmetic groups.

The definition of the analytic torsion (1.2) obviously depends on the compactness of the underlying manifold. Without this assumption, the heat operator $e^{-t\Delta_p(\rho)}$ is, in general, not a trace class operator. If one attempts to generalize the above method to non-cocompact arithmetic groups, the first problem is to define an appropriate regularized trace of the heat operators. For hyperbolic manifolds of finite volume one can proceed as in Melrose [Me] to define the regularized trace by means of the renormalized trace of the heat kernel. This method has been used in [CV], [PR], [MP1], [MP3], [MP4]. One uses an appropriate height function to truncate the hyperbolic manifold X at height $T > 0$. This amounts to cut off the cusps at sufficiently high level $T > T_0$. Then one integrates the point wise trace of the heat kernel over the truncated manifold $X(T)$. This integral has an asymptotic expansion in $\log T$. The constant term is defined to be the renormalized trace of the heat operator.

The purpose of the present paper is to start the investigation of the case of finite volume locally symmetric spaces of any rank by defining a regularized analytic torsion for arithmetic quotients associated to split forms of type A_n over $\mathbb{Q}$. In the higher rank case we proceed in the same way as in the case of hyperbolic manifolds. The first problem is to define the truncation in the right way. For this we can build on Arthur's work. The definition of the truncation operator is an important issue in Arthur's trace formula [Ar1], which we will use for our purpose. To this end we need to switch to the adelic framework.

Now we will describe the approach in more detail. For simplicity assume that G is a connected semisimple algebraic group defined over $\mathbb{Q}$. Assume that $G(\mathbb{R})$ is not compact. Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$. Put $\widetilde{X} = G(\mathbb{R})/K_\infty$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ the ring of finite adeles. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup. We consider the adelic quotient

$$X(K_f) = G(\mathbb{Q})\backslash(\widetilde{X} \times G(\mathbb{A}_f))/K_f. \tag{1.4}$$

This is the adelic version of a locally symmetric space. In fact, $X(K_f)$ is the disjoint union of finitely many locally symmetric spaces $\Gamma_i\backslash\widetilde{X}$, $i = 1, \dots, l$, (see section 3). If G is simply connected, then by strong approximation we

$$X(K_f) = \Gamma\backslash\widetilde{X},$$

where $\Gamma = (G(\mathbb{R}) \times K_f) \cap G(\mathbb{Q})$. We will assume that K_f is neat, that is, the eigenvalues of any element in Γ generate a torsion free subgroup in $\mathbb{C}^\times$, so that $X(K_f)$ is a manifold. Let $\nu\colon K_\infty \to \mathrm{GL}(V_\nu)$ be a finite dimensional unitary representation. It induces a homogeneous Hermitian vector bundle $\widetilde{E}_\nu$ over $\widetilde{X}$, which is equipped with the canonical connection ∇^ν. Being homogeneous, $\widetilde{E}_\nu$ can be pushed down to a locally homogeneous Hermitian vector bundle over each component $\Gamma_i\backslash\widetilde{X}$ of $X(K_f)$. Their disjoint union is a Hermitian vector bundle E_ν over $X(K_f)$. Let $\widetilde{\Delta}_\nu$ (resp. Δ_ν) be the associated Bochner-Laplace operator acting in the space of smooth section of $\widetilde{E}_\nu$ (resp. E_ν). Let $e^{-t\widetilde{\Delta}_\nu}$ (resp. $e^{-t\Delta_\nu}$), $t > 0$, be the heat semigroup generated by $\widetilde{\Delta}_\nu$ (resp. Δ_ν). Since $\widetilde{\Delta}_\nu$ commutes with the action of $G(\mathbb{R})$, it follows that $e^{-t\widetilde{\Delta}_\nu}$ is a convolution operator with kernel given by a smooth map $H_t^\nu\colon G(\mathbb{R}) \to \mathrm{End}(V_\nu)$. Let $h_t^\nu(g) = \operatorname{tr} H_t^\nu(g)$, $g \in G(\mathbb{R})$. In fact, h_t^ν belongs to Harish-Chandra's Schwartz space $\mathcal{C}(G(\mathbb{R}))$. Let χ_{K_f} be the characteristic function of K_f in $G(\mathbb{A}_f)$. We define the function $\phi_t^\nu \in C^\infty(G(\mathbb{A}))$ by

$$\phi_t^\nu(g_\infty g_f) = h_t^\nu(g_\infty)\chi_{K_f}(g_f), \quad g_\infty \in G(\mathbb{R}),\ g_f \in G(\mathbb{A}_f).$$

In fact, ϕ_t^ν belongs to $\mathcal{C}(G(\mathbb{A}); K_f)$, the adelic version of the Schwartz space (see section 4 for its definition). If $X(K_f)$ is compact, then one has

$$\operatorname{Tr}\left(e^{-t\Delta_\nu}\right) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{\gamma \in G(\mathbb{Q})} \phi_t^\nu(x^{-1}\gamma x)\, dx. \tag{1.5}$$

This is our starting point for defining the renormalized trace in the noncompact case. We fix a minimal Levi subgroup M_0 of G. If $M \subseteq G$ is a Levi subgroup containing M_0, let A_M be the split component of the center of M. Let $\mathfrak{a}_0 := \mathfrak{a}_{M_0}$ be the Lie algebra of $A_{M_0}(\mathbb{R})$. Let J_{geo} be the geometric side of the Arthur trace formula introduced in [Ar1]; see also [Ar10] for an introduction to the trace formula. For $f \in C_c^\infty(G(\mathbb{A}))$, Arthur defines $J_{\mathrm{geo}}(f)$ as the value at a point $T_0 \in \mathfrak{a}_0$, specified in [Ar3, Lemma 1.1], of a polynomial $J^T(f)$ on $\mathfrak{a}_0$. In fact, by [FL1, Theorem 7.1], $J^T(f)$ is defined for all $f \in \mathcal{C}(G(\mathbb{A}); K_f)$. Furthermore, we use an appropriate height function to truncate $G(\mathbb{A})$. For $T \in \mathfrak{a}_0$ let $G(\mathbb{A})_{\le T}$ be obtained by truncating $G(\mathbb{A})$ at level T (see (4.34)). This is a compact subset of $G(\mathbb{A})$. By [FL1, Theorem 7.1] it follows that for sufficiently regular $T \in \mathfrak{a}_0$ we have

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})_{\le T}} \sum_{\gamma \in G(\mathbb{Q})} \phi_t^\nu(x^{-1}\gamma x)\, dx = J^T(\phi_t^\nu) + O\left(e^{-c\|T\|}\right). \tag{1.6}$$

Since $J^T(\phi_t^\nu)$ is a polynomial in T, we get an asymptotic expansion in T of the truncated integral. Under additional assumption on G, which are satisfied for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, the point $T_0 \in \mathfrak{a}_0$, determined by [Ar1, Lemma 1.1], is equal to 0. Thus in this case $J_{\mathrm{geo}}(\phi_t^\nu)$ is the constant term of the polynomial $J^T(\phi_t^\nu)$. This leads to our definition of the regularized trace

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_\nu}\right) := J_{\mathrm{geo}}(\phi_t^\nu). \tag{1.7}$$

In general, $J_{\mathrm{geo}}(\phi_t^\nu)$ is not the constant term of the polynomial $J^T(\phi_t^\nu)$. Nevertheless, we prefer this definition, because of its independence on the choice of the minimal parabolic subgroup P_0.

The next goal is to determine the asymptotic behavior of $\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_\nu}\right)$ as $t \to 0$ and $t \to \infty$, respectively. To this end we use the Arthur trace formula. Currently we are only able to deal with these problems for the groups $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$. For $N \in \mathbb{N}$ let $K(N) \subset G(\mathbb{A}_f)$ be the principal congruence subgroup of level N. Recall that $K(N)$ is neat for $N \ge 3$. Our first main result is the following proposition.

Theorem 1.1. *Let $G = \mathrm{GL}(n)$ or $\mathrm{SL}(n)$. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup. Assume that K_f is contained in $K(N)$ for some $N \ge 3$. Let ν be finite dimensional* *unitary representation of K_∞ and let Δ_ν be the associated Bochner-Laplace operator. Let $d = \dim X(K_f)$. As $t \to +0$, there is an asymptotic expansion*

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_\nu}\right) \sim t^{-d/2}\sum_{j=0}^{\infty} a_j(\nu)t^j + t^{-(d-1)/2}\sum_{j=0}^{\infty}\sum_{i=0}^{r_j} b_{ij}(\nu)t^{j/2}(\log t)^i. \tag{1.8}$$

Moreover $r_j \leq n-1$ for all $j \in \mathbb{N}_0$.

For hyperbolic manifolds a similar result was proved in [Mu6].

To study the large time behavior we restrict attention to twisted Laplace operators, which are relevant for studying the analytic torsion with coefficients in local systems. Let $\tau\colon G(\mathbb{R}) \to \mathrm{GL}(V_\tau)$ be a finite dimensional complex representation. Let $\Gamma_i\backslash\widetilde{X}$, $i = 1, \dots, l$, be the components of $X(K_f)$. The restriction of τ to Γ_i induces a flat vector bundle $E_{\tau,i}$ over $\Gamma_i\backslash\widetilde{X}$. The disjoint union is a flat vector bundle E_τ over $X(K_f)$. By [MM] it is isomorphic to the locally homogeneous vector bundle associated to $\tau|_{K_\infty}$. It can be equipped with a fiber metric induced from the homogeneous bundle. Let $\Delta_p(\tau)$ be the corresponding twisted Laplace operator on p-forms with values in E_τ. Let $\mathrm{Ad}_{\mathfrak{p}}\colon K_\infty \to \mathrm{GL}(\mathfrak{p})$ be the adjoint representation of K_∞ on $\mathfrak{p}$, where $\mathfrak{p} = \mathfrak{k}^\perp$, and $\nu_p(\tau) = \Lambda^p \mathrm{Ad}_{\mathfrak{p}}^* \otimes \tau$. Up to a vector bundle endomorphism, $\Delta_p(\tau)$ equals the Bochner-Laplace operator $\Delta_{\nu_p(\tau)}$. So $\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ is well defined. Let θ be the Cartan involution of $G(\mathbb{R})$ with respect to K_∞. Put $\tau_\theta := \tau \circ \theta$. The large time behavior of the regularized trace is described by the following proposition.

Theorem 1.2. *Let $G = \mathrm{GL}(n)$ or $\mathrm{SL}(n)$. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup which is contained in $K(N)$ for some $N \geq 3$. Let τ be finite dimensional representation of $G(\mathbb{R})$. Assume that $\tau \not\cong \tau_\theta$. Then we have*

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) = O(e^{-ct}) \tag{1.9}$$

as $t \to \infty$ for all $p = 0, \dots, d$.

The proof is an immediate consequence of Proposition 13.3 together with the trace formula. Without the assumption $\tau \not\cong \tau_\theta$ the behavior of $\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ as $t \to \infty$ is more complicated and it is definitely not exponentially decreasing. This condition is also relevant in [BV]. It implies that the representation τ is strongly acyclic [BV, Lemma 4.1], which is a necessary condition to establish the main results of [BV]. It is a very challenging problem to eliminate this condition. We also note that the condition $\tau \not\cong \tau_\theta$ implies the vanishing theorem of Borel-Wallach for the cohomology of a cocompact lattice in a semisimple Lie group [BW, Theorem 6.7, Ch. VII].

By Theorems 1.1 and 1.2 we can define the zeta function of $\Delta_p(\tau)$ as in (13.35), using the regularized trace of $e^{-t\Delta_p(\tau)}$ in place of the usual trace. The corresponding Mellin transform converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > d/2$ and admits a meromorphic extension to the whole complex plane. Because of the presence of the log-terms in the expansion (1.8), the zeta function may have a pole at $s = 0$. Let $f(s)$ be a meromorphic function on $\mathbb{C}$. For $s_0 \in \mathbb{C}$ let $f(s) = \sum_{k \geq k_0} a_k(s-s_0)^k$ be the

Laurent expansion of f at s_0. Put $\mathrm{FP}_{s=s_0} f(s) := a_0$. Now we define the analytic torsion $T_{X(K_f)}(\tau) \in \mathbb{C} \setminus \{0\}$ by

$$\log T_{X(K_f)}(\tau) = \frac{1}{2} \sum_{p=0}^{d} (-1)^p p \left(\mathrm{FP}_{s=0} \frac{\zeta_p(s;\tau)}{s} \right). \tag{1.10}$$

In the case of $G = \mathrm{GL}(3)$ we are able to determine the coefficients of the log-terms. This shows that the zeta functions definitely have a pole at $s = 0$. However, the combination $\sum_{p=1}^{5} (-1)^p p \zeta_p(s;\tau)$ turns out to be holomorphic at $s = 0$ and we can define the logarithm of the analytic torsion by

$$\log T_{X(K_f)}(\tau) = \frac{d}{ds} \left(\frac{1}{2} \sum_{p=1}^{5} (-1)^p p \zeta_p(s;\tau) \right) \bigg|_{s=0}.$$

Let $\{K_f(N)\}_{N\in\mathbb{N}}$ be the family of principal congruence subgroups of $\mathrm{GL}(n, \mathbb{A}_f)$, and $X(N) := X(K_f(N))$, $N \in \mathbb{N}$. The next problem is to study the limiting behavior of $\log T_{X(N)}(\tau)/\operatorname{vol}(X(N))$ as $N \to \infty$ which we do in subsequent work. In consideration of the results for the cocompact case in [BV], one can expect a different behavior of $\log T_{X(N)}(\tau)/\operatorname{vol}(X(N))$ in the limit $N \to \infty$ for different n. More precisely, the fundamental rank $\operatorname{rank} G(\mathbb{R}) - \operatorname{rank} K_\infty$ determined in [BV] whether the limit vanishes, which it does unless if the rank equals 1. In our case of $\mathrm{SL}_n(\mathbb{R})$, the fundamental rank is 1 precisely when $n = 3$ or $n = 4$.

An even more difficult problem is the question if there is a combinatorial counterpart of $T_{X(K_f)}(\tau)$ as there is in the compact case.

Now we briefly explain our method to prove Theorems 1.1 and 1.2. To determine the asymptotic behavior of the regularized trace as $t \to +0$, we use the geometric side of trace formula. The first step is to show that ϕ_t^ν can be replaced by a compactly supported function $\widetilde{\phi}_t^\nu \in C_c^\infty(G(\mathbb{A}))$ without changing the asymptotic behavior. Next we use the coarse geometric expansion of the geometric side, which expresses $J_{\mathrm{geo}}(f)$, $f \in C_c^\infty(G(\mathbb{A}))$, as a sum of distributions $J_{\mathfrak{o}}(f)$ associated to semisimple conjugacy classes of $G(\mathbb{Q})$. Let $J_{\mathrm{unip}}(f)$ be the distribution associated to the class of 1. If the support of $\widetilde{\phi}_t^\nu$ is a sufficiently small neighborhood of 1, it follows that

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_\nu}\right) := J_{\mathrm{unip}}(\widetilde{\phi}_t^\nu) + O\left(e^{-c/t}\right) \tag{1.11}$$

as $t \to +0$. To analyze $J_{\mathrm{unip}}(\widetilde{\phi}_t^\nu)$, we use the fine geometric expansion [Ar4] which expresses $J_{\mathrm{unip}}(\widetilde{\phi}_t^\nu)$ in terms of weighted orbital integrals. If the real rank of $G(\mathbb{R})$ is one, the weighted orbital integrals are rather simple and the weight factors are explicitly known (see [Wa]). In order to deal with the weighted orbital integrals in the higher rank case, we need to restrict to the groups $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$. In this case all unipotent orbits are Richardson, which simplifies the analysis considerably. We are only interested in the situation over the field $\mathbb{R}$. Let M be a Levi subgroup of G. Let $\mathcal{U}_M$ be the unipotent variety in M and $\mathcal{V} \in (\mathcal{U}_M)$ a conjugacy class. Let U be an $M(\mathbb{R})$ conjugacy class in $\mathcal{V}(\mathbb{R})$. There exists a standard parabolic subgroup $Q = LN \in \mathcal{F}$ and a constant $c > 0$ such that for every $\mathrm{O}(n)$-conjugation invariant function $f \in C_c^\infty(G(\mathbb{R}))$ the weighted orbital integral $J_M(U, f)$ is given by

$$J_M(U,f) = c \int_{N(\mathbb{R})} f(n) w_{M,\mathcal{V}}(n)\, dn, \tag{1.12}$$

where $w_{M,\mathcal{V}}(n)$ is a certain weight function. The main problem is now to determine the structure of the weight function. For $G(\mathbb{R}) = \mathrm{SO}_0(n,1)$ the weighted orbital integral is of the same form with weight function $w(n) = \log \|\log n\|$, where the inner log is the isomorphism $\log\colon N \to \mathfrak{n}$. This fact has been exploited in [Mu6] in order to establish the asymptotic expansion of the regularized trace in the case of hyperbolic manifolds of finite volume. It turns out that $w_{M,\mathcal{V}}$ has a similar behavior with respect to scaling. Note that the map $x \mapsto X = x - \mathrm{id}$ defines a bijection between the variety of unipotent elements in $G(\mathbb{R})$ and the nilpotent cone in the Lie algebra $\mathfrak{g}(\mathbb{R})$. For $s \in \mathbb{R}$ let $x_s := \mathrm{id} + s(x - \mathrm{id})$. Let $x \in \mathcal{V}^G(\mathbb{R})$ such that $w_{M,\mathcal{V}}(x)$ is defined. Then by Proposition 7.1, $w_{M,\mathcal{V}}(x_s)$ is well-defined for every $s > 0$ and $s \mapsto w_{M,\mathcal{V}}(x_s)$ is a polynomial in $\log s$ of degree at most $\dim \mathfrak{a}_M^G$. Inserting a standard parametrix for the heat kernel into (1.12) and using the structure of $w_{M,\mathcal{V}}$, we obtain Theorem 1.1. To eliminate the assumption that K_f is contained in some $K(N)$ with $N \geq 3$, we would have to consider orbital integrals associated to classes of finite order. For GL(2) and GL(3) we discuss this issue in section 15.

To prove Theorem 1.2, we use the spectral side of the trace formula. Let $\phi_t^{\tau,p}$ be the function in $\mathcal{C}(G(\mathbb{A}); K_f)$, which is defined in the same way as ϕ_t^ν in terms of the kernel of the heat operator on the universal covering. Then by the trace formula

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) = J_{\mathrm{spec}}(\phi_t^{\tau,p}).$$

The key input to deal with the spectral side is the refinement of the spectral expansion of the Arthur trace formula established in [FLM1] (see Theorem 5.1). For $f \in \mathcal{C}(G(\mathbb{A}))$ we have

$$J_{\mathrm{spec}}(f) = \sum_{[M]} J_{\mathrm{spec},M}(f),$$

where $[M]$ runs over the conjugacy classes of Levi subgroups of G and $J_{\mathrm{spec},M}(f)$ is a distribution associated to M. The distribution associated to G is $\mathrm{Tr}\, R_{\mathrm{dis}}(f)$, where R_{dis} denotes the restriction of the regular representation of $G(\mathbb{A})$ in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ to the discrete subspace. For a proper Levi subgroup M of G, $J_{\mathrm{spec},M}(f)$ is an integral whose main ingredient are logarithmic derivatives of intertwining operators. Using our assumption that $\tau \neq \tau_\theta$, we obtain $\dim \ker \Delta_p(\tau) = 0$. Then it follows as in the compact case that there exists $c > 0$ such that

$$\mathrm{Tr}\, R_{\mathrm{dis}}(\phi_t^{\tau,p}) = O(e^{-ct}), \quad \text{as } t \to \infty.$$

For a proper Levi subgroup M, the determination of the asymptotic behavior of $J_{\mathrm{spec},M}(\phi_t^{\tau,p})$ as $t \to \infty$ relies on two conjectural properties, one global and one local, of the intertwining operators. The global property is a uniform estimate on the winding number of the normalizing factors of the intertwining operators in the co-rank one case. For $\mathrm{GL}(n)$ and

SL(n), this property follows from known, but delicate, properties of the Rankin-Selberg L-functions [FLM2]. The local property is concerned with the estimation of logarithmic derivatives of normalized local intertwining operators, which are uniform in π. For GL(n) the pertinent estimates have been established in [MS, Proposition 0.2]. They are a consequence of a weak version of the Ramanujan conjecture. The case of SL(n) can be reduced to GL(n) in the same way as in the proof of [FLM2, Lemma 5.14]. Let $\theta\colon G \to G$ be the Cartan involution and let $\tau_\theta := \tau\circ\theta$. Using these estimations, it follows that for $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$, a proper Levi subgroup M of G and a finite dimensional representation τ of $G(\mathbb{R})$ such that $\tau \not\cong \tau_\theta$, one has $J_{\text{spec},M}(\phi_t^{\tau,p}) = O(e^{-ct})$ as $t \to \infty$. Putting everything together, we obtain Theorem 1.2.

We end this introduction with some remarks on the possible extension of the our results to other groups G. First of all, Theorem 1.2 depends on the estimations of logarithmic derivatives of global normalizing factors and normalized local intertwining operators. Using functoriality, T. Finis and E. Lapid [FL2] have recently established similar estimates of the logarithmic derivatives of global normalizing factors associated to intertwining operators for the following reductive groups over number fields: inner forms of GL(n), quasi-split classical groups and their similitude groups, and the exceptional groups G_2. One can expect that the estimates of the logarithmic derivatives of the normalized local intertwining operators can be established by the same methods. This would lead to an extension of Theorem 1.2 to these groups. It remains to deal with the unipotent orbital integrals for the groups above.

The paper is organized as follows. In section 2 we fix notations and recall some basic facts. In section 3 we introduce the locally symmetric manifolds as adelic quotients. In section 4 we compare two different methods of truncation. One of them is based on the truncation of kernels of integral operators which leads to the geometric side of the trace formula. The other one consists in the truncation of the underlying manifold, which is the basis for the renormalization of the trace of the heat operator. In section 5 we recall the spectral side of the Arthur trace formula. In section 6 we are assuming that $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$. We discuss the unipotent contribution to the trace formula and derive a simplified formula for the weighted orbital integral. Section 7 is devoted to the study of the weight functions for the groups GL(n) and SL(n). The main result is Proposition 7.1, which is the key result that enables us to determine the asymptotic behavior as $t \to +0$ of the corresponding orbital integrals. Examples of low rank are discussed in section 8. These are cases where the weight function is given explicitly. In section 9 we collect some basic facts concerning Bochner-Laplace operators. The regularized trace of the corresponding heat operators is introduced in section 11. The definition is based on section 4, which deals with truncation. In section 10 we establish some estimates of the heat kernel for Bochner-Laplace operators on the symmetric space $\widetilde{X}$. Combined with the analysis of the weight functions in section (7), the estimations are used in section (12) to prove Theorem 1.1. In section (13) we first use the spectral side of the Arthur trace formula to establish Theorem 1.2, which concerns the large time asymptotic behavior of the regularized trace of the heat operators. This finally enables us to define the regularized analytic torsion.

In section 14 we assume that $G = \mathrm{GL}(3)$. Using the explicit form of the weight functions described in section 8, we determine the coefficients of the possible poles at $s = 0$ of the zeta functions. It turns out that the combination of the zeta functions, which is used to define the analytic torsion, is holomorphic at $s = 0$. In the final section 15 we consider for $G = \mathrm{GL}(2)$ or $G = \mathrm{GL}(3)$ an arbitrary subgroup K_f of $G(\hat{\mathbb{Z}})$ and study the additional weighted orbital integrals that arise in this case.

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2. Preliminaries

Let G be a reductive algebraic group defined over $\mathbb{Q}$. We fix a minimal parabolic subgroup P_0 of G defined over $\mathbb{Q}$ and a Levi decomposition $P_0 = M_0 \cdot N_0$, both defined over $\mathbb{Q}$. Let $\mathcal{F}$ be the set of parabolic subgroups of G which contain M_0 and are defined over $\mathbb{Q}$. Let $\mathcal{L}$ be the set of subgroups of G which contain M_0 and are Levi components of groups in $\mathcal{F}$. For any $P \in \mathcal{F}$ we write

$$P = M_P N_P,$$

where N_P is the unipotent radical of P and M_P belongs to $\mathcal{L}$.

Let $M \in \mathcal{L}$. Denote by A_M the $\mathbb{Q}$-split component of the center of M. Put $A_P = A_{M_P}$. Let $L \in \mathcal{L}$ and assume that L contains M. Then L is a reductive group defined over $\mathbb{Q}$ and M is a Levi subgroup of L. We shall denote the set of Levi subgroups of L which contain M by $\mathcal{L}^L(M)$. We also write $\mathcal{F}^L(M)$ for the set of parabolic subgroups of L, defined over $\mathbb{Q}$, which contain M, and $\mathcal{P}^L(M)$ for the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. Each of these three sets is finite. If $L = G$, we shall usually denote these sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$.

Let $X(M)_{\mathbb{Q}}$ be the group of characters of M which are defined over $\mathbb{Q}$. Put

$$\mathfrak{a}_M := \mathrm{Hom}(X(M)_{\mathbb{Q}}, \mathbb{R}). \tag{2.13}$$

This is a real vector space whose dimension equals that of A_M. Its dual space is

$$\mathfrak{a}_M^* = X(M)_{\mathbb{Q}} \otimes \mathbb{R}.$$

We shall write,

$$\mathfrak{a}_P = \mathfrak{a}_{M_P},\ A_0 = A_{M_0} \quad \text{and} \quad \mathfrak{a}_0 = \mathfrak{a}_{M_0}. \tag{2.14}$$

For $M \in \mathcal{L}$ let $A_M(\mathbb{R})^0$ be the connected component of the identity of the group $A_M(\mathbb{R})$. Let $W_0 = N_{\mathbf{G}(\mathbb{Q})}(A_0)/M_0$ be the Weyl group of (G, A_0), where $N_{G(\mathbb{Q})}(H)$ is the normalizer of H in $G(\mathbb{Q})$. For any $s \in W_0$ we choose a representative $w_s \in G(\mathbb{Q})$. Note that W_0 acts on $\mathcal{L}$ by $sM = w_s M w_s^{-1}$. For $M \in \mathcal{L}$ let $W(M) = N_{\mathbf{G}(\mathbb{Q})}(M)/M$, which can be identified with a subgroup of W_0.

For any $L \in \mathcal{L}(M)$ we identify $\mathfrak{a}_L^*$ with a subspace of $\mathfrak{a}_M^*$. We denote by $\mathfrak{a}_M^L$ the annihilator of $\mathfrak{a}_L^*$ in $\mathfrak{a}_M$. We set

$$\mathcal{L}_1(M) = \{L \in \mathcal{L}(M) : \dim \mathfrak{a}_M^L = 1\}$$

and

$$\mathcal{F}_1(M) = \bigcup_{L \in \mathcal{L}_1(M)} \mathcal{P}(L). \tag{2.15}$$

We shall denote the simple roots of (P, A_P) by Δ_P. They are elements of $X(A_P)_{\mathbb{Q}}$ and are canonically embedded in $\mathfrak{a}_P^*$. Let $\Sigma_P \subset \mathfrak{a}_P^*$ be the set of reduced roots of A_P on the Lie algebra of G. The set $\Delta_0 = \Delta_{P_0}$ is a base for a root system. In particular, for every $\alpha \in \Delta_P$ we have a co-root $\alpha^\vee \in \mathfrak{a}_{P_0}$.

Let P_1 and P_2 be parabolic subgroups with $P_1 \subset P_2$. Then $\mathfrak{a}_{P_2}^*$ is embedded into $\mathfrak{a}_{P_1}^*$, while $\mathfrak{a}_{P_2}$ is a natural quotient vector space of $\mathfrak{a}_{P_1}$. The group $M_{P_2} \cap P_1$ is a parabolic subgroup of M_{P_2}. Let $\Delta_{P_1}^{P_2}$ denote the set of simple roots of $(M_{P_2} \cap P_1, A_{P_1})$. It is a subset of Δ_{P_1}. For a parabolic subgroup P with $P_0 \subset P$ we write $\Delta_0^P := \Delta_{P_0}^P$.

Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) be the ring of adeles (resp. finite adeles) of $\mathbb{Q}$. We fix a maximal compact subgroup $\mathbf{K} = \prod_v K_v = K_\infty \cdot \mathbf{K}_f$ of $G(\mathbb{A}) = G(\mathbb{R}) \cdot G(\mathbb{A}_f)$. We assume that the maximal compact subgroup $\mathbf{K} \subset G(\mathbb{A})$ is admissible with respect to M_0 [Ar5, §1]. Let $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ be the homomorphism given by

$$e^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_{\mathbb{A}} = \prod_v |\chi(m_v)|_v \tag{2.16}$$

for any $\chi \in X(M)$ and denote by $M(\mathbb{A})^1 \subset M(\mathbb{A})$ the kernel of H_M. Then $M(\mathbb{A})$ is the direct product of $M(\mathbb{A})^1$ and $A_M(\mathbb{R})^0$, the component of 1 in $A_M(\mathbb{R})$. By the conditions on $\mathbf{K}$, we have $G(\mathbb{A}) = P(\mathbb{A})\mathbf{K}$. Hence any $x \in G(\mathbb{A})$ can be written as

$$nmak, \quad n \in N(\mathbb{A}),\ m \in M(\mathbb{A})^1,\ a \in A_M(\mathbb{R})^0,\ k \in \mathbf{K}.$$

Define $H_P \colon G(\mathbb{A}) \to \mathfrak{a}_P$ by

$$H_P(x) := H_M(a), \tag{2.17}$$

where $x = nmak$ as above. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G(\mathbb{R})$ and K_∞, respectively. Let θ be the Cartan involution of $G(\mathbb{R})$ with respect to K_∞. It induces a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We fix an invariant bi-linear form B on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. This choice defines a Casimir operator Ω on $G(\mathbb{R})$. Let $\Pi(G(\mathbb{R}))$ denote the set of equivalence classes of irreducible unitary representations of $G(\mathbb{R})$. We denote the Casimir eigenvalue of any $\pi \in \Pi(G(\mathbb{R}))$ by λ_π. Similarly, we obtain a Casimir operator Ω_{K_∞} on K_∞ and write λ_τ for the Casimir eigenvalue of a representation $\tau \in \Pi(K_\infty)$ (cf. [BG, §2.3]). The form B induces a Euclidean scalar product $(X, Y) = -B(X, \theta(Y))$ on $\mathfrak{g}$ and all its subspaces. For $\tau \in \Pi(K_\infty)$ we define $\|\tau\|$ as in [CD, §2.2]. Note that the restriction of the scalar product $(\cdot, \cdot)$ on $\mathfrak{g}$ to $\mathfrak{a}_0$ gives $\mathfrak{a}_0$ the structure of a Euclidean space. In particular, this fixes Haar measures on the spaces $\mathfrak{a}_M^L$ and their duals $(\mathfrak{a}_M^L)^*$. We follow Arthur in the corresponding normalization of Haar measures on the groups $M(A)$ ([Ar1, §1]).

Let $L^2_{\mathrm{disc}}(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))$ be the discrete part of $L^2(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))$, i.e., the closure of the sum of all irreducible subrepresentations of the regular representation of $M(\mathbb{A})$. We denote by $\Pi_{\mathrm{disc}}(M(\mathbb{A}))$ the countable set of equivalence classes of irreducible unitary representations of $M(\mathbb{A})$ which occur in the decomposition of the discrete subspace $L^2_{\mathrm{disc}}(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A}))$ into irreducible representations.

3. Arithmetic manifolds

Let G be a reductive algebraic group over $\mathbb{Q}$. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup. The double coset space $A_G(\mathbb{R})^0G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})K_f$ is known to be finite (see [Bo1, §5]). Let $x_1 = 1, x_2, \dots, x_l$ be a set of representatives in $G(\mathbb{A}_f)$ of the double cosets. Then the groups

$$\Gamma_i := \left(G(\mathbb{R}) \times x_iK_fx_i^{-1}\right) \cap G(\mathbb{Q}), \quad 1 \le i \le l,$$

are arithmetic subgroups of $G(\mathbb{R})$ and the action of $G(\mathbb{R})$ on the space of double cosets $A_G(\mathbb{R})^0G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$ induces the following decomposition into $G(\mathbb{R})$-orbits:

$$A_G(\mathbb{R})^0G(\mathbb{Q})\backslash G(\mathbb{A})/K_f \cong \bigsqcup_{i=1}^{l} \left(\Gamma_i\backslash G(\mathbb{R})^1\right), \tag{3.18}$$

where $G(\mathbb{R})^1 = G(\mathbb{R})/A_G(\mathbb{R})^0$. Thus we get an isomorphism of $G(\mathbb{R})$-modules

$$L^2(A_G(\mathbb{R})^0G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \cong \bigoplus_{i=1}^{l} L^2(\Gamma_i\backslash G(\mathbb{R})^1). \tag{3.19}$$

We note that, in general, $l > 1$. However, if G is semisimple, simply connected, and without any $\mathbb{Q}$-simple factors H for which $H(\mathbb{R})$ is compact, then by strong approximation we have

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_f \cong \Gamma\backslash G(\mathbb{R}),$$

where $\Gamma = (G(\mathbb{R}) \times K_f) \cap G(\mathbb{Q})$. In particular this is the case for $G = \mathrm{SL}(n)$. Let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup. Let

$$\widetilde{X} := G(\mathbb{R})^1/K_\infty \tag{3.20}$$

be the associated global Riemannian symmetric space. Given an open compact subgroup $K_f \subset G(\mathbb{A}_f)$, we define the arithmetic manifold $X(K_f)$ by

$$X(K_f) := G(\mathbb{Q})\backslash(\widetilde{X} \times G(\mathbb{A}_f))/K_f. \tag{3.21}$$

By (3.18) we have

$$X(K_f) = \bigsqcup_{i=1}^{l} \left(\Gamma_i\backslash\widetilde{X}\right), \tag{3.22}$$

where each component $\Gamma_i\backslash\widetilde{X}$ is a locally symmetric space. We will assume that K_f is neat. Then $X(K_f)$ is a locally symmetric manifold of finite volume.

Now consider $G = \mathrm{GL}(n)$ as algebraic group over $\mathbb{Q}$. Then $A_G(\mathbb{R})^0$ is the group of scalar matrices with a positive real scalar and $K_\infty = \mathrm{O}(n)$. Let $N = \prod_p p^{r_p}$, $r_p \ge 0$. Put

$$K_p(N) := \{k \in G(\mathbb{Z}_p) \colon k \equiv 1 \bmod p^{r_p}\mathbb{Z}_p\}$$

and

$$K(N) := \prod_{p<\infty} K_p(N).$$

Then $K(N)$ is an open compact subgroup of $G(\mathbb{A}_f)$ and

$$A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A})/K(N) \cong \bigsqcup_{i=1}^{\varphi(N)} \Gamma(N)\backslash SL(n,\mathbb{R}) \tag{3.23}$$

where $\varphi(N) = \#[(\mathbb{Z}/N\mathbb{Z})^*]$ (see [Ar6]). Hence we have

$$L^2(A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))^{K(N)} \cong \bigoplus_{i=1}^{\varphi(N)} L^2(\Gamma(N)\backslash \operatorname{SL}(n,\mathbb{R})) \tag{3.24}$$

as $\operatorname{SL}(n,\mathbb{R})$-modules. We have

$$\widetilde{X} = \operatorname{SL}(n,\mathbb{R})/\operatorname{SO}(n).$$

Let

$$X(N) := G(\mathbb{Q})\backslash(\widetilde{X} \times G(\mathbb{A}_f))/K(N). \tag{3.25}$$

Let $\nu\colon K_\infty \to \operatorname{GL}(V_\nu)$ be a finite dimensional unitary representation of K_∞. Let $\widetilde{E}_\nu$ be the associated homogeneous Hermitian vector bundle over $\widetilde{X}$. Over each component of $X(K_f)$, $\widetilde{E}$ induces a locally homogeneous Hermitian vector bundle $E_{i,\nu} \to \Gamma_i\backslash\widetilde{X}$. Let

$$E_\nu := \bigsqcup_{i=1}^{l} E_{i,\nu}. \tag{3.26}$$

Then E_ν is a vector bundle over $X(K_f)$, which is locally homogeneous.

4. Truncation and the geometric side of the trace formula

The Arthur trace formula is obtained by truncating the kernels of integral operators associated to functions in $C_c^\infty(G(\mathbb{A})^1)$. On the other hand, the regularization of the trace of heat operators is based on the truncation of the underlying locally symmetric space. In this section we compare the two methods. Let P_0 be the fixed minimal parabolic subgroup of G.

For $f \in C_c^\infty(G(\mathbb{A})^1)$ let

$$K_f(x,y) = \sum_{\gamma\in G(\mathbb{Q})} f(x^{-1}\gamma y).$$

This is the kernel of an integral operator. In general, $K_f(x,x)$ is not integrable over $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ and needs to be truncated to get an integrable function. To define the truncated kernel we need to introduce some notations.

Let $P = M_P N_P$ be a standard parabolic subgroup and let Q be a parabolic subgroup containing P. Let Δ_P^Q be the set of simple roots of $(M_Q \cap P, A_P)$. Similarly, we have the set of coroots $\Delta_0^\vee$ and, more generally and, the set $(\Delta_P^Q)^\vee$ which forms a basis of $\mathfrak{a}_P^Q := \mathfrak{a}_P \cap \mathfrak{a}_0^Q$. We denote the basis of $(\mathfrak{a}_P^Q)^*$ (resp. $\mathfrak{a}_P^Q$) dual to $(\Delta_P^Q)^\vee$ (resp. Δ_P^Q) by $\hat{\Delta}_P^Q$ (resp. $(\hat{\Delta}_P^Q)^\vee$. Let τ_P^Q and $\widehat{\tau}_P^Q$ denote the characteristic functions of the set

$$\{X \in \mathfrak{a}_0 \colon \langle \alpha, X \rangle > 0 \text{ for all } \alpha \in \Delta_P^Q\}$$

and

$$\{X \in \mathfrak{a}_0 \colon \langle \varpi, X \rangle > 0 \text{ for all } \varpi \in \hat{\Delta}_P^Q\},$$

respectively. If $Q = G$, we will suppress the superscript. Moreover we put $\tau_0 := \tau_0^G$ and $\hat{\tau}_0 := \widehat{\tau}_0^G$. Now we can define the truncated kernel. Put

$$K_f^P(x,y) := \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \sum_{\gamma \in P(\mathbb{Q})} f(x^{-1}\gamma n y)\, dn.$$

Let $H_P \colon G(\mathbb{A}) \to \mathfrak{a}_P$ be the map defined by (2.17). For any $T \in \mathfrak{a}_0^+$ define

$$k^T(x,f) := \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_f^P(\delta x, \delta x) \widehat{\tau}_P(H_P(\delta x) - T_P), \tag{4.27}$$

where T_P denotes the projection of T on $\mathfrak{a}_P$. Note that the term in (4.27) which corresponds to $P = G$ is $K_f(x,x)$. If $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is compact, there are no proper parabolic subgroups of G over $\mathbb{Q}$. Thus, in this case we have $k^T(x,f) = K_f(x,x)$, and the truncation operation is trivial. By [Ar4, Theorem 6.1] the integral

$$J^T(f) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k^T(x,f)\, dx \tag{4.28}$$

converges absolutely. This is the first step toward the trace formula. As shown by Hoffmann [Ho], $J^T(f)$ is defined for a larger class of functions f and $J^T(f)$ is a polynomial in $T \in \mathfrak{a}_0$ of degree at most $d_0 = \dim \mathfrak{a}_{P_0}^G$. There is a distinguished point $T_0 \in \mathfrak{a}_0$ specified by [Ar3, Lemma 1.1], and Arthur defines the distribution J on $G(\mathbb{A})^1$ by

$$J(f) := J^{T_0}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1). \tag{4.29}$$

This is the geometric side of the trace formula. To distinguish it from the spectral side, we will denote it by J_{geo}.

In [Ar1] Arthur has introduced the coarse geometric expansion of $J^T(f)$. To define it, one has to introduce an equivalence relation in $G(\mathbb{Q})$. Define two elements γ and γ' in $G(\mathbb{Q})$ to be equivalent, if the semisimple components γ_s and γ_s' of their Jordan decompositions are $G(\mathbb{Q})$-conjugate. Let $\mathcal{O}$ be the set of equivalence classes. Note that the set $\mathcal{O}$ is in obvious bijection with the semisimple conjugacy classes in $G(\mathbb{Q})$. Furthermore, in case $G = \mathrm{GL}(n)$, the Jordan decomposition is given by the Jordan normal form. For $\mathfrak{o} \in \mathcal{O}$ and $f \in C_c^\infty(G(\mathbb{A})^1)$ let

$$K_{\mathfrak{o}}^P(x,y) := \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \sum_{\gamma \in P(\mathbb{Q})\cap\mathfrak{o}} f(x^{-1}\gamma n y)\, dn.$$

Given $T \in \mathfrak{a}_0$ and $x \in G(\mathbb{A})^1$, let

$$k_{\mathfrak{o}}^T(x,f) := \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_{\mathfrak{o}}^P(\delta x, \delta x) \widehat{\tau}_P \left(H_P(\delta x) - T_P\right). \tag{4.30}$$

Let

$$J_{\mathfrak{o}}^T(f) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_{\mathfrak{o}}^T(x,f)\, dx. \tag{4.31}$$

The integral converges absolutely and one obtains an absolutely convergent expansion

$$J^T(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^T(f), \quad f \in C_c^\infty(G(\mathbb{A})^1). \tag{4.32}$$

This is the coarse geometric expansion, introduced in [Ar1]. In [FL1], Finis and Lapid have shown that the coarse geometric expansion (4.32) extends continuously to the space of Schwartz functions $\mathcal{C}(G(\mathbb{A})^1)$ which is defined as follows. For any compact open subgroup K_f of $G(\mathbb{A}_f)$ the space $G(\mathbb{A})^1/K_f$ is the countable disjoint union of copies of $G(\mathbb{R})^1 = G(\mathbb{R}) \cap G(\mathbb{A})^1$ and therefore, it is a differentiable manifold. Any element $X \in \mathcal{U}(\mathfrak{g}_\infty^1)$ of the universal enveloping algebra of the Lie algebra $\mathfrak{g}_\infty^1$ of $G(\mathbb{R})^1$ defines a left invariant differential operator $f \mapsto f * X$ on $G(\mathbb{A})^1/K_f$. Let $\mathcal{C}(G(\mathbb{A})^1; K_f)$ be the space of smooth right K_f-invariant functions on $G(\mathbb{A})^1$ which belong, together with all their derivatives, to $L^1(G(\mathbb{A})^1)$. The space $\mathcal{C}(G(\mathbb{A})^1; K_f)$ becomes a Fréchet space under the seminorms

$$\|f * X\|_{L^1(G(\mathbb{A})^1)}, \quad X \in \mathcal{U}(\mathfrak{g}_\infty^1).$$

Denote by $\mathcal{C}(G(\mathbb{A})^1)$ the union of the spaces $\mathcal{C}(G(\mathbb{A})^1; K_f)$ as K_f varies over the compact open subgroups of $G(\mathbb{A}_f)$ and endow $\mathcal{C}(G(\mathbb{A})^1)$ with the inductive limit topology. For $f \in \mathcal{C}(G(\mathbb{A})^1; K_f)$ and $\mathfrak{o} \in \mathcal{O}$ let $J^T(f)$ and $J_{\mathfrak{o}}^T(f)$ be defined by (4.28) and (4.31), respectively. By [FL1, Theorem 7.1], the integrals defining $J^T(f)$ and $J_{\mathfrak{o}}^T(F)$ are absolutely convergent and we have

$$J^T(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^T(f), \quad f \in \mathcal{C}(G(\mathbb{A})^1; K_f). \tag{4.33}$$

We shall now discuss how $J^T(f)$ is related the integral of the kernel over the truncated manifold, where the truncated manifold is defined by a certain height function. For $T \in \mathfrak{a}_0$ let

$$G(\mathbb{A})^1_{\leq T} = \{g \in G(\mathbb{A})^1 \colon \hat{\tau}_0(T - H_0(\gamma g)) = 1, \text{ for all } \gamma \in G(\mathbb{Q})\}. \tag{4.34}$$

Note that by definition, $G(\mathbb{A})^1_{\leq T}$ is $G(\mathbb{Q})$-invariant. Furthermore, for $T_1 \in \mathfrak{a}_0$ let

$$\mathfrak{S}_{T_1} = \{x \in G(\mathbb{A}) \colon \tau_0(H_0(x) - T_1) = 1\}$$

and more generally

$$\mathfrak{S}_{T_1}^P = \{x \in G(\mathbb{A}) \colon \tau_0^P(H_0(x) - T_1) = 1\}$$

for any $P \supset P_0$. Note that these sets are left $P_0(\mathbb{A})^1$-invariant. By reduction theory, there exists $T_1 \in \mathfrak{a}_0$ such that

$$P(\mathbb{Q})\mathfrak{S}_{T_1}^P = G(\mathbb{A})$$

for all $P \supset P_0$, in particular for $P = G$. We fix such T_1. Let

$$d(T) = \min_{\alpha\in\Delta_0} \langle \alpha, T\rangle. \tag{4.35}$$

There exists $d_0 > 0$, which depends only on G, P_0 and $\mathbf{K}$, such that for all $T \in \mathfrak{a}_0$ with $d(T) > d_0$ one has

$$G(\mathbb{A})^1_{\le T} \cap \mathfrak{S}_{T_1} = \{g \in G(A)^1 \colon \tau_0(H_0(g) - T_1)\hat{\tau}_0(T - H_0(g)) = 1\}.$$

For $f \in \mathcal{C}(G(\mathbb{A})^1)$ recall that

$$K_f(x, y) = \sum_{\gamma\in G(\mathbb{Q})} f(x^{-1}\gamma y) \tag{4.36}$$

The series converges absolutely and uniformly on compact subsets. Then the following theorem, which is an immediate consequence of [FL1, Theorem 7.1], establishes the relation between $J^T(f)$ and naive truncation.

Theorem 4.1. *For every open compact subgroup K_f of $G(\mathbb{A}_f)$ there exists $r \ge 0$ and a continuous seminorm μ on $\mathcal{C}(G(\mathbb{A})^1; K_f)$ such that*

$$\left| \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1_{\le T}} K_f(x, x)\, dx - J^T(f) \right| \le \mu(f)(1 + \|T\|)^r e^{-d(T)}$$

for all $f \in \mathcal{C}(G(\mathbb{A})^1; K_f)$ and $T \in \mathfrak{a}_0$ such that $d(T) > d_0$.

Proof. For $\mathfrak{o} \in \mathcal{O}$ let

$$K_{\mathfrak{o}}(x, y) := \sum_{\gamma\in\mathfrak{o}} f(x^{-1}\gamma y).$$

We have

$$K_f(x, y) = \sum_{\mathfrak{o}\in\mathcal{O}} K_{\mathfrak{o}}(x, y), \tag{4.37}$$

where the series converges absolutely. Using (4.33), we get

$$\left| \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1_{\le T}} K_f(x, x)\, dx - J^T(f) \right| \le \sum_{\mathfrak{o}\in\mathcal{O}} \left| \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1_{\le T}} K_{\mathfrak{o}}(x, x)\, dx - J^T_{\mathfrak{o}}(f) \right|.$$

and the theorem follows from [FL1, Theorem 7.1]. We note that for the case of compactly supported functions f this is due to Arthur (see [Ar1, §7]). $\square$

5. The non-invariant trace formula

Arthur's (non-invariant) trace formula is the equality

$$J_{\mathrm{geo}}(f) = J_{\mathrm{spec}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1), \tag{5.1}$$

of the geometric side $J_{\mathrm{geo}}(f)$ and the spectral side $J_{\mathrm{spec}}(f)$ of the trace formula. The geometric side has been described in the previous section. In this section we recall the definition of the spectral side, and in particular the refinement of the spectral expansion obtained in [FLM1]. Combining [FLM1] and [FL1], it follows that (5.1) extends continuously to $f \in \mathcal{C}(G(\mathbb{A})^1)$.

The main ingredient of the spectral side are logarithmic derivatives of intertwining operators. We briefly recall the structure of the intertwining operators.

Let $P \in \mathcal{P}(M)$. Let U_P be the unipotent radical of P. Recall that we denote by $\Sigma_P \subset \mathfrak{a}_P^*$ the set of reduced roots of A_M of the Lie algebra $\mathfrak{u}_P$ of U_P. Let Δ_P be the subset of simple roots of P, which is a basis for $(\mathfrak{a}_P^G)^*$. Write $\mathfrak{a}_{P,+}^*$ for the closure of the Weyl chamber of P, i.e.

$$\mathfrak{a}_{P,+}^* = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Sigma_P\} = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta_P\}.$$

Denote by δ_P the modulus function of $P(\mathbb{A})$. Let $\bar{\mathcal{A}}_2(P)$ be the Hilbert space completion of

$$\{\phi \in C^\infty(M(\mathbb{Q})U_P(\mathbb{A})\backslash G(\mathbb{A})) : \delta_P^{-\frac{1}{2}}\phi(\cdot x) \in L^2_{\mathrm{disc}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A})), \ \forall x \in G(\mathbb{A})\}$$

with respect to the inner product

$$(\phi_1, \phi_2) = \int_{A_M(\mathbb{R})^0 M(\mathbb{Q})\mathbf{U}_P(\mathbb{A})\backslash \mathbf{G}(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}\, dg.$$

Let $\alpha \in \Sigma_M$. We say that two parabolic subgroups $P, Q \in \mathcal{P}(M)$ are *adjacent* along α, and write $P|^\alpha Q$, if $\Sigma_P \cap -\Sigma_Q = \{\alpha\}$. Alternatively, P and Q are adjacent if the group $\langle P, Q\rangle$ generated by P and Q belongs to $\mathcal{F}_1(M)$ (see (2.15) for its definition). Any $R \in \mathcal{F}_1(\mathcal{M})$ is of the form $\langle P, Q\rangle$, where P, Q are the elements of $\mathcal{P}(M)$ contained in R. We have $P|^\alpha Q$ with $\alpha^\vee \in \Sigma_P^\vee \cap \mathfrak{a}_M^R$. Interchanging P and Q changes α to $-\alpha$.

For any $P \in \mathcal{P}(M)$ let $H_P\colon G(\mathbb{A}) \to \mathfrak{a}_P$ be the extension of H_M to a left $U_P(\mathbb{A})$-and right $\mathbf{K}$-invariant map. Denote by $\mathcal{A}^2(P)$ the dense subspace of $\bar{\mathcal{A}}^2(P)$ consisting of its $\mathbf{K}$- and $\mathfrak{z}$-finite vectors, where $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}$. That is, $\mathcal{A}^2(P)$ is the space of automorphic forms ϕ on $U_P(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A})$ such that $\delta_P^{-\frac{1}{2}}\phi(\cdot k)$ is a square-integrable automorphic form on $A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A})$ for all $k \in \mathbf{K}$. Let $\rho(P, \lambda)$, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, be the induced representation of $G(\mathbb{A})$ on $\bar{\mathcal{A}}^2(P)$ given by

$$(\rho(P, \lambda, y)\phi)(x) = \phi(xy)e^{\langle \lambda, H_P(xy) - H_P(x)\rangle}.$$

It is isomorphic to the induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\left(L^2_{\mathrm{disc}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A})) \otimes e^{\langle \lambda, H_M(\cdot)\rangle}\right).$$

For alternative descriptions see [Ar8, §1], [MW, I.2.17, I.2.18].

For $P, Q \in \mathcal{P}(M)$ let

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

be the standard *intertwining operator* [Ar9, §1], which is the meromorphic continuation in λ of the integral

$$[M_{Q|P}(\lambda)\phi](x) = \int_{U_Q(\mathbb{A}) \cap U_P(\mathbb{A}) \backslash U_Q(\mathbb{A})} \phi(nx) e^{\langle \lambda, H_P(nx) - H_Q(x) \rangle} \, dn, \quad \phi \in \mathcal{A}^2(P), \; x \in G(\mathbb{A}).$$

Given $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$, let $\mathcal{A}^2_\pi(P)$ be the space of all $\phi \in \mathcal{A}^2(P)$ for which the function $M(\mathbb{A}) \ni x \mapsto \delta_P^{-\frac{1}{2}} \phi(xg)$, $g \in G(\mathbb{A})$, belongs to the π-isotypic subspace of the space $L^2(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))$. For any $P \in \mathcal{P}(\mathcal{M})$ we have a canonical isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_\mathbb{C}, K_\infty)$-modules

$$j_P : \mathrm{Hom}(\pi, L^2(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))) \otimes \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi) \to \mathcal{A}^2_\pi(P).$$

If we fix a unitary structure on π and endow $\mathrm{Hom}(\pi, L^2(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A})))$ with the inner product $(A, B) = B^* A$ (which is a scalar operator on the space of π), the isomorphism j_P becomes an isometry.

Suppose that $P|^\alpha Q$. The operator $M_{Q|P}(\pi, s) := M_{Q|P}(s\varpi)|_{\mathcal{A}^2_\pi(P)}$, where $\varpi \in \mathfrak{a}^*_M$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$, admits a normalization by a global factor $n_\alpha(\pi, s)$ which is a meromorphic function in s. We may write

$$M_{Q|P}(\pi, s) \circ j_P = n_\alpha(\pi, s) \cdot j_Q \circ (\mathrm{Id} \otimes R_{Q|P}(\pi, s)) \tag{5.2}$$

where $R_{Q|P}(\pi, s) = \otimes_v R_{Q|P}(\pi_v, s)$ is the product of the locally defined normalized intertwining operators and $\pi = \otimes_v \pi_v$ [Ar9, §6], (cf. [Mu2, (2.17)]). In many cases, the normalizing factors can be expressed in terms automorphic L-functions [Sha1], [Sha2]. For example, let $G = \mathrm{GL}(n)$. Then the global normalizing factors n_α can be expressed in terms of Rankin-Selberg L-functions. The known properties of these functions are collected and analyzed in [Mu1, §§4,5]. Write $M \simeq \prod_{i=1}^r \mathrm{GL}(n_i)$, where the root α is trivial on $\prod_{i \geq 3} \mathrm{GL}(n_i)$, and let $\pi \simeq \otimes \pi_i$ with representations $\pi_i \in \Pi_{\mathrm{disc}}(\mathrm{GL}(n_i, \mathbb{A}))$. Let $L(s, \pi_1 \times \tilde{\pi}_2)$ be the completed Rankin-Selberg L-function associated to π_1 and π_2. It satisfies the functional equation

$$L(s, \pi_1 \times \tilde{\pi}_2) = \epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2) N(\pi_1 \times \tilde{\pi}_2)^{\frac{1}{2} - s} L(1 - s, \tilde{\pi}_1 \times \pi_2) \tag{5.3}$$

where $|\epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2)| = 1$ and $N(\pi_1 \times \tilde{\pi}_2) \in \mathbb{N}$ is the conductor. Then we have

$$n_\alpha(\pi, s) = \frac{L(s, \pi_1 \times \tilde{\pi}_2)}{\epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2) N(\pi_1 \times \tilde{\pi}_2)^{\frac{1}{2} - s} L(s + 1, \pi_1 \times \tilde{\pi}_2)}. \tag{5.4}$$

We now turn to the spectral side. Let $L \supset M$ be Levi subgroups in $\mathcal{L}$, $P \in \mathcal{P}(M)$, and let $m = \dim \mathfrak{a}_L^G$ be the co-rank of L in G. Denote by $\mathfrak{B}_{P,L}$ the set of m-tuples $\underline{\beta} = (\beta_1^\vee, \ldots, \beta_m^\vee)$ of elements of $\Sigma_P^\vee$ whose projections to $\mathfrak{a}_L$ form a basis for $\mathfrak{a}_L^G$. For any

$\underline{\beta}=(\beta_1^\vee,\dots,\beta_m^\vee)\in\mathfrak{B}_{P,L}$ let $\mathrm{vol}(\underline{\beta})$ be the co-volume in $\mathfrak{a}_L^G$ of the lattice spanned by $\underline{\beta}$ and let

$$\begin{aligned}\Xi_L(\underline{\beta})&=\{(Q_1,\dots,Q_m)\in\mathcal{F}_1(M)^m:\quad \beta_i^\vee\in\mathfrak{a}_M^{Q_i},\ i=1,\dots,m\}\\&=\{\langle P_1,P_1'\rangle,\dots,\langle P_m,P_m'\rangle):\quad P_i|^{\beta_i}P_i',\ i=1,\dots,m\}.\end{aligned}$$

For any smooth function f on $\mathfrak{a}_M^*$ and $\mu\in\mathfrak{a}_M^*$ denote by $D_\mu f$ the directional derivative of f along $\mu\in\mathfrak{a}_M^*$. For a pair $P_1|^\alpha P_2$ of adjacent parabolic subgroups in $\mathcal{P}(M)$ write

$$\delta_{P_1|P_2}(\lambda)=M_{P_2|P_1}(\lambda)D_\varpi M_{P_1|P_2}(\lambda):\mathcal{A}^2(P_2)\to\mathcal{A}^2(P_2),\tag{5.5}$$

where $\varpi\in\mathfrak{a}_M^*$ is such that $\langle\varpi,\alpha^\vee\rangle=1$. [1] Equivalently, writing $M_{P_1|P_2}(\lambda)=\Phi(\langle\lambda,\alpha^\vee\rangle)$ for a meromorphic function Φ of a single complex variable, we have

$$\delta_{P_1|P_2}(\lambda)=\Phi(\langle\lambda,\alpha^\vee\rangle)^{-1}\Phi'(\langle\lambda,\alpha^\vee\rangle).$$

For any m-tuple $\mathcal{X}=(Q_1,\dots,Q_m)\in\Xi_L(\underline{\beta})$ with $Q_i=\langle P_i,P_i'\rangle$, $P_i|^{\beta_i}P_i'$, denote by $\Delta_{\mathcal{X}}(P,\lambda)$ the expression

(5.6)
$$\frac{\mathrm{vol}(\underline{\beta})}{m!}M_{P_1'|P}(\lambda)^{-1}\delta_{P_1|P_1'}(\lambda)M_{P_1'|P_2'}(\lambda)\cdots\delta_{P_{m-1}|P_{m-1}'}(\lambda)M_{P_{m-1}'|P_m'}(\lambda)\delta_{P_m|P_m'}(\lambda)M_{P_m'|P}(\lambda).$$

In [FLM1, pp. 179-180] the authors defined a (purely combinatorial) map $\mathcal{X}_L:\mathfrak{B}_{P,L}\to\mathcal{F}_1(M)^m$ with the property that $\mathcal{X}_L(\underline{\beta})\in\Xi_L(\underline{\beta})$ for all $\underline{\beta}\in\mathfrak{B}_{P,L}$.[2]

For any $s\in W_M$ let L_s be the smallest Levi subgroup in $\mathcal{L}(M)$ containing w_s. We recall that $\mathfrak{a}_{L_s}=\{H\in\mathfrak{a}_M\mid sH=H\}$. Set

$$\iota_s=|\det(s-1)_{\mathfrak{a}_M^{L_s}}|^{-1}.$$

For $P\in\mathcal{F}(M_0)$ and $s\in W(M_P)$ let $M(P,s):\mathcal{A}^2(P)\to\mathcal{A}^2(P)$ be as in [Ar3, p. 1309]. $M(P,s)$ is a unitary operator which commutes with the operators $\rho(P,\lambda,h)$ for $\lambda\in\mathrm{i}\mathfrak{a}_{L_s}^*$. Finally, we can state the refined spectral expansion.

Theorem 5.1 ([FLM1])**.** *For any $h\in C_c^\infty(G(\mathbb{A})^1)$ the spectral side of Arthur's trace formula is given by*

$$J_{\mathrm{spec}}(h)=\sum_{[M]}J_{\mathrm{spec},M}(h),\tag{5.7}$$

M ranging over the conjugacy classes of Levi subgroups of G (represented by members of $\mathcal{L}$), where

$$J_{\mathrm{spec},M}(h)=\frac{1}{|W(M)|}\sum_{s\in W(M)}\iota_s\sum_{\underline{\beta}\in\mathfrak{B}_{P,L_s}}\int_{\mathrm{i}(\mathfrak{a}_{L_s}^G)^*}\mathrm{tr}(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\lambda)M(P,s)\rho(P,\lambda,h))\,d\lambda\tag{5.8}$$

[1] Note that this definition differs slightly from the definition of $\delta_{P_1|P_2}$ in [FLM1].

[2] The map $\mathcal{X}_L$ depends in fact on the additional choice of a vector $\underline{\mu}\in(\mathfrak{a}_M^*)^m$ which does not lie in an explicit finite set of hyperplanes. For our purposes, the precise definition of $\mathcal{X}_L$ is immaterial.

with $P \in \mathcal{P}(M)$ arbitrary. The operators are of trace class and the integrals are absolutely convergent with respect to the trace norm and define distributions on $\mathcal{C}(G(\mathbb{A})^1)$.

Note that the term corresponding to $M = G$ is $J_{\text{spec},G}(h) = \operatorname{tr} R_{\text{disc}}(h)$.

6. The unipotent contribution to the trace formula

In this section we assume that $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$ as algebraic groups over $\mathbb{Q}$, and we specialize to one of these groups at some points. The purpose is to analyze the unipotent contribution to the geometric side of the trace formula. The point of departure is the coarse geometric expansion of J_{geo} as a sum of distributions

$$J_{\text{geo}}(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1), \tag{6.1}$$

parametrized by the set $\mathcal{O}$ of semisimple conjugacy classes of $G(\mathbb{Q})$. The distribution $J_{\mathfrak{o}}(f)$ is the value at $T = 0$ of the polynomial $J_{\mathfrak{o}}^T(f)$ defined in [Ar1]. In particular, following Arthur, we write $J_{\text{unip}}(f)$ for the contribution corresponding to the class of $\{1\}$. Let $K(N)$ be a principal congruence subgroup of level $N \geq 3$. By [LM, Corollary 5.2] there exists a bi-K_∞-invariant compact neighborhood ω of $K(N)$ in $G(\mathbb{A})^1$ such that

$$J_{\text{geo}}(f) = J_{\text{unip}}(f). \tag{6.2}$$

for all $f \in C_c^\infty(G(\mathbb{A})^1)$ supported in ω. ([LM, Corollary 5.2] was only stated for $\mathrm{GL}(n)$, but its proof holds also for $\mathrm{SL}(n)$ without modification.) For our applications we can choose f such that (6.2) holds if we restrict to K_f with $K_f \subseteq K(N)$ for some $N \geq 3$. See § 15 for computations for $n = 2, 3$ regarding orbits of finite order which appear in the case $K_f = G(\hat{\mathbb{Z}})$.

To analyze $J_{\text{unip}}(f)$ we use Arthur's fundamental result ([Ar4, Corollaries 8.3 and 8.5]) to express $J_{\text{unip}}(f)$ in terms of weighted orbital integrals. To state the result we recall some facts about weighted orbital integrals. Let S be a finite set of places of $\mathbb{Q}$ containing ∞. Set

$$\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v, \quad \text{and} \quad G(\mathbb{Q}_S) = \prod_{v \in S} G(\mathbb{Q}_v).$$

Let

$$G(\mathbb{Q}_S)^1 = G(\mathbb{Q}_S) \cap G(\mathbb{A})^1$$

and write $C_c^\infty(G(\mathbb{Q}_S)^1)$ for the space of functions on $G(\mathbb{Q}_S)^1$ obtained by restriction of functions in $C_c^\infty(G(\mathbb{Q}_S))$ to $G(\mathbb{Q}_S)^1$. Further, let $\mathbb{A}^S = \prod_{v \notin S} \mathbb{Q}_v$ be the restricted product over all places outside of S, and define $G(\mathbb{A}^S)$ similarly as above.

6.1. **The fine geometric expansion.** Let $M \in \mathcal{L}$ and $\gamma \in M(\mathbb{Q}_S)$. The general weighted orbital integrals $J_M(\gamma, f)$ defined in ([Ar5]) are distributions on $G(\mathbb{Q}_S)$. Denote by H_γ the centralizer of γ in a subgroup H of G. If γ is such that $M_\gamma = G_\gamma$ then $J_M(\gamma, f)$ is given by an integral of the form

$$J_M(\gamma, f) = \left|D(\gamma)\right|^{1/2} \int_{G_\gamma(\mathbb{Q}_S)\backslash G(\mathbb{Q}_S)} f(x^{-1}\gamma x) v_M(x)\, dx, \tag{6.3}$$

where $D(\gamma)$ is defined in [Ar5, p. 231] and $v_M(x)$ is the weight function associated to the (G, M)-family $\{v_P(\lambda, x) \colon P \in \mathcal{P}(M)\}$ defined in [Ar5, p.230]. It is a left $M(\mathbb{A})$-invariant and right $\mathbf{K}$-invariant function on $G(\mathbb{A})$. In particular, in the case $M = G$ (in which $v_M \equiv 1$) we obtain the usual (invariant) orbital integral. Of course, implicit in $J_M(\gamma, f)$ is a choice of a Haar measure on $G_\gamma(\mathbb{Q}_S)$. When the condition $G_\gamma \subset M$ is not satisfied (for example, if γ is unipotent and $M \neq G$), the definition of $J_M(\gamma, f)$ is more complicated. It is obtained as a limit of a linear combination of integrals as above. For more details we refer to [Ar5], see also the description below. If γ belongs to the intersection of $M(\mathbb{Q}_S)$ with $G(\mathbb{Q}_S)^1$, one can obviously define the corresponding weighted orbital integral as a linear form on $C_c^\infty(G(\mathbb{Q}_S)^1)$. Note that $J_M(\gamma, f)$ depends only on the $M(\mathbb{Q}_S)$-conjugacy class of γ.

To state the fine expansion of $J_{\text{unip}}(f)$, we need to introduce a certain equivalence relation as defined in [Ar4, Ar7]. Let $\mathcal{U}_M$ denote the variety of unipotent elements in M and $\mathcal{U}_M(A)$ its A-points for any $\mathbb{Z}$-algebra A. We say that $u_1, u_2 \in \mathcal{U}_M(\mathbb{Q})$ are (M, S)-*equivalent* if they are conjugate by some element in $M(\mathbb{Q}_S)$, cf. [Ar4, §7]. We denote by $\left(\mathcal{U}_M(\mathbb{Q})\right)_{M,S}$ the set of (M, S)-equivalence classes in $\mathcal{U}_M(\mathbb{Q})$. We note that $\left(\mathcal{U}_M(\mathbb{Q})\right)_{M,S}$ is finite for any S but may get larger as S grows. We get an injective map

$$\left(\mathcal{U}_M(\mathbb{Q})\right)_{M,S} \longrightarrow \left(\mathcal{U}_M(\mathbb{Q}_S)\right)_{M(\mathbb{Q}_S)}$$

where $\left(\mathcal{U}_M(\mathbb{Q}_S)\right)_{M(\mathbb{Q}_S)}$ denotes the set of $M(\mathbb{Q}_S)$-conjugacy classes in $\mathcal{U}_M(\mathbb{Q}_S)$. Note that this last set is evidently the same as the direct product $\prod_{v\in S} \left(\mathcal{U}_M(\mathbb{Q}_v)\right)_{M(\mathbb{Q}_v)}$ over the $M(\mathbb{Q}_v)$-conjugacy classes in $\mathcal{U}_M(\mathbb{Q}_v)$, $v \in S$. In particular, we can identify an equivalence class $U \in \left(\mathcal{U}_M(\mathbb{Q})\right)_{M,S}$ with its image under the above map, that is, with a tuple $(U_v)_{v\in S}$ of $M(\mathbb{Q}_v)$-conjugacy classes in $\mathcal{U}_M(\mathbb{Q}_v)$.

By [Ar4, Theorem 8.1] we have

$$J_{\text{unip}}(f \otimes \mathbf{1}_{K^S}) = \text{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1) f(1) + \sum_{(M,U)} a^M(S, U) J_M(U, f), \tag{6.4}$$

where (M, U) runs over all pairs of Levi subgroups $M \in \mathcal{L}$ and $U \in \left(\mathcal{U}_M(\mathbb{Q})\right)_{M,S}$ with $(M, U) \neq (G, 1)$. Here $f \in C_c^\infty(G(\mathbb{Q}_S)^1)$, $\mathbf{1}_{K^S}$ is the characteristic function of the standard maximal compact of $G(\mathbb{A}^S)$, and $a^M(S, U)$ are certain constants which depend on the normalization of measures (but are usually not known explicitly). The distributions $J_M(U, f)$ can be written as weighted orbital integrals [Ar5, p. 256]. In our case the integrals simplify as we are going to see later.

6.2. **Unipotent conjugacy classes.** Let $(\mathcal{U}_M)$ denote the set of geometric (that is, over an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$) M-conjugacy classes in $\mathcal{U}_M$. Then $(\mathcal{U}_M)$ is a finite set. Each $\mathcal{V} \in (\mathcal{U}_M)$ is defined over $\mathbb{Q}$ and the set of $\mathbb{Q}$-points $\mathcal{V}(\mathbb{Q})$ is non-empty. More precisely, each $\mathcal{V} \in (\mathcal{U}_M)$ corresponds to a partition of n, and $\mathcal{V}(\mathbb{Q})$ contains the matrix with Jordan normal form corresponding to that partition. For any $\mathcal{V} \in (\mathcal{U}_M)$ the set $\mathcal{V}(\mathbb{Q})$ is closed under the (M,S)-equivalence relation and we write $(\mathcal{V}(\mathbb{Q}))_{M,S}$ for the finite set of (M,S)-equivalence classes in $\mathcal{V}(\mathbb{Q})$.

Remark 6.1. *If $G = \mathrm{SL}(n)$, there might be infinitely many $M(\mathbb{Q})$-conjugacy classes in $\mathcal{V}(\mathbb{Q})$ depending on the type of $\mathcal{V}(\mathbb{Q})$. This is in contrast to the case of $\mathrm{GL}(n)$, where the (finite) set of geometric unipotent conjugacy classes is in bijection with the set of rational unipotent conjugacy classes.*

Each class $\mathcal{V} \in (\mathcal{U}_M)$ is a Richardson class, that is, there exists a standard parabolic subgroup $Q = LV \in \mathcal{L}(M)$ such that $\mathcal{V}$ is induced from the trivial orbit in L to M, see [Hu, §5.5 Proposition]. Equivalently, the intersection of $\mathcal{V}$ with V is an open and dense subset of V. Note that every (M,S)-equivalence class $U \subseteq \mathcal{V}(\mathbb{Q})$ has a representative in $V(\mathbb{Q})$.

Now let $\mathcal{V} \in (\mathcal{U}_M)$ and $U = (U_v)_{v\in S} \in (\mathcal{V}(\mathbb{Q}_S))_{M(\mathbb{Q}_S)}$. Here we write $(\mathcal{V}(\mathbb{Q}_S))_{M(\mathbb{Q}_S)} = \prod_{v\in S} (\mathcal{V}(\mathbb{Q}_v))_{M(\mathbb{Q}_v)}$ for the set of $M(\mathbb{Q}_S)$-conjugacy classes in $\mathcal{V}(\mathbb{Q}_S)$. To understand the S-adic integral $J_M(U,f)$, we decompose it into a sum of products of integrals at ∞ and at the finite places $S_{\mathrm{fin}} = S\backslash\{\infty\}$. More precisely, for every pair of Levi subgroups $L_1, L_2 \in \mathcal{L}(M)$ there exists a coefficient $d_M^G(L_1,L_2) \in \mathbb{C}$ such that

$$J_M(U,f) = \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) J_M^{L_1}(U_\infty, f_{\infty,Q_1}) J_M^{L_2}(U_{\mathrm{fin}}, f_{\mathrm{fin},Q_2}) \tag{6.5}$$

(see [Ar3]) where $U_{\mathrm{fin}} = (U_v)_{v\in S_{\mathrm{fin}}} \in \big(\mathcal{V}(\mathbb{Q}_{S_{\mathrm{fin}}})\big)_{M(\mathbb{Q}_{S_{\mathrm{fin}}})}$, and $Q_1 = L_1V_1 \in \mathcal{P}(L_1))$, $Q_2 = L_2V_2 \in \mathcal{P}(L_2)$ are certain parabolic subgroups the exact choice of which does not matter to us. Moreover,

$$f_{\infty,Q_1}(m) = \delta_{Q_1}(m)^{1/2} \int_{K_\infty}\int_{V_1(\mathbb{R})} f(k^{-1}mvk)\, dk\, dv,\ m \in L_1(\mathbb{R}),$$

with f_{fin,Q_2} being defined analogously, and the coefficients $d_M^G(L_1,L_2)$ are independent of S and they vanish unless the natural map $\mathfrak{a}_M^{L_1} \oplus \mathfrak{a}_M^{L_2} \longrightarrow \mathfrak{a}_M^G$ is an isomorphism. In case the coefficient does not vanish, it depends on the chosen measures on $\mathfrak{a}_M^{L_1}$, $\mathfrak{a}_M^{L_2}$ and $\mathfrak{a}_M^G$. The distributions $J_M^L(U,F)$, for $L \in \mathcal{L}(M)$, any finite set S' of places of $\mathbb{Q}$, $F \in C^\infty(L(\mathbb{Q}_{S'}))$, and a unipotent conjugacy class $U \subseteq L(\mathbb{Q}_{S'})$, are defined similarly as the weighted orbital integrals $J_M^G = J_M$ but with L in place of G.

Our test function at the places in $v \in S_{\mathrm{fin}}$ is fixed once and for all so that the integrals at those places can be viewed as constant for our purposes. Hence we need to understand the integral at the Archimedean place. We therefore need to better understand the unipotent conjugacy classes over $\mathbb{R}$. If $G = \mathrm{GL}(n)$, the unipotent orbits in $\mathrm{GL}_n(\mathbb{R})$ and all its

Levi subgroups in $\mathcal{L}$ are easy to describe as they are in one-to-one correspondence with the geometric unipotent conjugacy classes and therefore classified by partitions of n. We assume for the moment that $G = \mathrm{SL}(n)$. Note that $\mathrm{SL}_n(\mathbb{R})$ is normalized by $\mathrm{GL}_n(\mathbb{R})$. Moreover, each $M \in \mathcal{L}$ is of the form $M = \bar{M} \cap \mathrm{SL}(n)$ for a unique Levi subgroup $\bar{M} \in \mathcal{L}^{\mathrm{GL}(n)}$. Then $\bar{K}_\infty^M := \mathrm{O}(n) \cap \bar{M}(\mathbb{R})$ is a maximal compact subgroup in $\bar{M}$, $\bar{K}_\infty^M \cap M(\mathbb{R}) = K_\infty^M$, and $\bar{M}(\mathbb{R})$ normalizes $M(\mathbb{R})$. In particular, it makes sense to speak of $\bar{K}_\infty^M$-conjugation invariant functions on $M(\mathbb{R})$.

Lemma 6.2. *Let $\mathcal{V} \in (\mathcal{U}_M)$. For any two equivalence classes $U_1, U_2 \in (\mathcal{V}(\mathbb{R}))_{M(\mathbb{R})}$ there exists $k \in \bar{K}_\infty^M$ with $k^{-1}U_1k = U_2$ and $(\mathcal{V}(\mathbb{R}))_{M(\mathbb{R})}$ consists of at most two equivalence classes. More precisely, if $U_1, U_2 \in (\mathcal{V}(\mathbb{R}))_{M(\mathbb{R})}$ are the two distinct classes, we have $k^{-1}U_1k = U_2$ with $k = \mathrm{diag}(-1, 1, \dots, 1) \in \bar{K}_\infty^M$.*

Proof. Let $u_1 \in U_1$ and $u_2 \in U_2$. In $\bar{M}(\mathbb{R})$, u_1 and u_2 are conjugate, that is, there exists some $g \in \bar{M}(\mathbb{R})$ with $g^{-1}u_1g = u_2$. Without loss of generality we can assume that $|\det g| = 1$. If $\det g = 1$, then $g \in \bar{M}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R}) = M(\mathbb{R})$ and $U_1 = U_2$. If $\det g = -1$, let $k = \mathrm{diag}(-1, 1, \dots, 1) \in \bar{K}_\infty^M$ and $g_1 = gk^{-1}$. Then $g_1 \in \bar{M}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R}) = M(\mathbb{R})$, and $U_2 = g^{-1}U_1g = k^{-1}g_1^{-1}U_1g_1k = k^{-1}U_1k$ as asserted. □

6.3. **Measures on conjugacy classes.**

Corollary 6.3. *If $\mathcal{V} \in (\mathcal{U}_M)$ and $f_\infty \in C_c(G(\mathbb{R}))$ is conjugation invariant under $\bar{K}_\infty$, then we have $J_M(U_1, f_\infty) = J_M(U_2, f_\infty)$ for all $U_1, U_2 \in (\mathcal{V}(\mathbb{R}))_{M(\mathbb{R})}$.*

Proof. By the previous lemma, there are at most two distinct classes in $(\mathcal{V}(\mathbb{R}))_{M(\mathbb{R})}$. If there is only one class in $(\mathcal{V}(\mathbb{R}))_{M(\mathbb{R})}$, there is nothing to show. If there are two distinct classes U_1, U_2, they are conjugate to each other via the element $k = \mathrm{diag}(-1, 1, \dots, 1) \in \bar{K}_\infty^M$. Let $u_1 \in U_1$ and $u_2 = k^{-1}u_1k \in U_2$. Then the centralizers of u_1 and u_2 in $G(\mathbb{R})$ are the same, since they are the same in $\mathrm{GL}_n(\mathbb{R})$ and k normalizes $G(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R})$ in $\mathrm{GL}_n(\mathbb{R})$. Hence the invariant measures on the $G(\mathbb{R})$-conjugacy classes of u_1 and u_2 coincide, in particular, $J_G(U_1, f_\infty) = J_G(U_2, f_\infty)$ for every $\bar{K}_\infty$-conjugation invariant $f_\infty \in C_c(G(\mathbb{R}))$.

The non-invariant measure defining $J_M(U_i, \cdot)$ can be written as the product of some weight function times the invariant measure. We shall see in the next section (see (7.9) and (7.12)) that the weight function is invariant under the action of $k = \mathrm{diag}(-1, 1, \dots, 1)$ so that the claim of the lemma follows for the weighted orbital integrals as well. □

The following corollary now is valid for $G = \mathrm{SL}(n)$ as well as $G = \mathrm{GL}(n)$. However, we shall only prove it for $\mathrm{SL}(n)$. For $G = \mathrm{GL}(n)$ the proof in fact is easier and was already given in [LM, Lemma 5.3] (see also [Ho, Proposition 5]).

Corollary 6.4. *For every $\mathcal{V} \in (\mathcal{U}_G)$ there exists a standard parabolic subgroup $P = LV$ and a constant $c > 0$ such that for every $\mathrm{O}(n)$-conjugation invariant function f_∞ and every*

$U \in (\mathcal{V}(\mathbb{R}))_{G(\mathbb{R})}$ the invariant orbital integral $J_G(U, f_\infty)$ can be written as

$$J_G(U, f_\infty) = c \int_{V(\mathbb{R})} f_\infty(v)\, dv,$$

where dv denotes the Haar measure on $V(\mathbb{R})$ normalized such that it coincides with the measure obtained from the Lebesgue measure on $\mathbb{R}^{\dim V}$ when $V(\mathbb{R})$ is identified with $\mathbb{R}^{\dim V}$ via its matrix coordinates.

Proof. Let $P = LV \in \mathcal{P}$ be a Richardson parabolic subgroup for $\mathcal{V}$ so that $\mathcal{V}(\mathbb{R}) \cap V(\mathbb{R})$ is dense in $V(\mathbb{R})$. We have $\mathcal{V}(\mathbb{R}) = \bigcup_{U \in (\mathcal{V}(\mathbb{R}))_{G(\mathbb{R})}} U$ and this union is disjoint. Then

$$\mathcal{V}(\mathbb{R}) \cap V(\mathbb{R}) = \bigcup_{U \in (\mathcal{V}(\mathbb{R}))_{G(\mathbb{R})}} U \cap V(\mathbb{R}) \tag{6.6}$$

is also a disjoint union which is dense in $V(\mathbb{R})$. For each $U \in (\mathcal{V}(\mathbb{R}))_{G(\mathbb{R})}$ we can pick a representative $u \in V(\mathbb{R})$. Then the orbit of u under $P(\mathbb{R})$ equals $U \cap V(\mathbb{R})$. Since f_∞ is $\mathrm{O}(n)$-conjugation invariant, we have, using Iwasawa decomposition for $G(\mathbb{R})$, that

$$J_G(U, f_\infty) = \int_{P_u(\mathbb{R}) \backslash P(\mathbb{R})} \delta_P(p)^{-1} f_\infty(p^{-1}up)\, dp.$$

It follows as in the proof of [LM, Lemma 5.3] that

$$C_c^\infty(V(\mathbb{R})) \ni h \longrightarrow \int_{P_u(\mathbb{R}) \backslash P(\mathbb{R})} \delta_P(p)^{-1} h(p^{-1}up)\, dp$$

is absolutely continuous with respect to the Haar measure on $V(\mathbb{R})$. Hence

$$C_c^\infty(V(\mathbb{R})) \ni h \longrightarrow \sum_u \int_{P_u(\mathbb{R}) \backslash P(\mathbb{R})} \delta_P(p)^{-1} h(p^{-1}up)\, dp \tag{6.7}$$

is also absolutely continuous with respect to the Haar measure on $V(\mathbb{R})$. Here $u \in V(\mathbb{R})$ runs over a set of representatives for the classes $U \in (\mathcal{V}(\mathbb{R}))_{G(\mathbb{R})}$. Since the right hand side of (6.6) is a disjoint union and dense in $V(\mathbb{R})$, the measure defined by the right hand side of (6.7) must be proportional to the Haar measure on $V(\mathbb{R})$. Hence there exists a constant $C > 0$ such that for every $f_\infty \in C_c^\infty(G(\mathbb{R}))$ we have

$$\sum_{U \in (\mathcal{V}(\mathbb{R}))_{G(\mathbb{R})}} J_G(U, f_\infty) = C \int_{V(\mathbb{R})} f_\infty(v)\, dv.$$

By Corollary 6.3 and our assumption on f_∞ we have $J_G(U_1, f_\infty) = J_G(U_2, f_\infty)$ for all $U_1, U_2 \in (\mathcal{V}(\mathbb{R}))_{G(\mathbb{R})}$. Hence

$$J_G(U, f_\infty) = \frac{C}{N} \int_{V(\mathbb{R})} f_\infty(v)\, dv$$

where N is the number of classes in $(\mathcal{V}(\mathbb{R}))_{G(\mathbb{R})}$. □

The weighted integral $J_M(U, f_\infty)$ from [Ar5, p. 256] can be written as an integral over $\mathrm{Ind}_M^G U$ against the invariant measure on $\mathrm{Ind}_M^G U$ weighted by a certain function. Hence using the corollary above it follows that the real orbital integral $J_M(U, f_\infty)$ simplifies for every $\mathrm{O}(n)$-conjugation invariant f_∞ to

$$J_M(U, f_\infty) = \int_{N(\mathbb{R})} f(x) w_{M,U}(x)\, dx \tag{6.8}$$

where $Q = LN \in \mathcal{P}$ is a Richardson parabolic for $\mathrm{Ind}_M^G U$, the unipotent orbit induced from M to G, and the weight function $w_{M,U}(x)$ is described in [Ar5, Lemma 5.4]. (See also [LM, §5].) It is a finite linear combination of functions of the form $\prod_{i=1}^r \log \|p_i(x)\|$ where p_i are polynomials on $N_Q(\mathbb{R})$ into an affine space, $i = 1, \ldots, r$ (not necessarily distinct) and

$$\|(y_1, \ldots, y_m)\|_v = |y_1|_v^2 + \cdots + |y_m|_v^2$$

(The fact that the product is over r terms is implicit in [Ar5] but follows from the proof.) For our purpose we need to describe the weight function in more detail which we shall do in the next section.

7. The weight function

In this section, again $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$. We are only interested in the situation over the field $\mathbb{R}$, but most of this section holds over $\mathbb{Q}_p$, $p < \infty$, as well. As before, $\mathcal{U}_M$ denotes the unipotent variety in M and $\mathcal{V} \in (\mathcal{U}_M)$ a geometric conjugacy class. We write

$$\mathbf{K}^G = \begin{cases} \mathrm{SO}(n) & \text{if } G = \mathrm{SL}(n), \\ \mathrm{O}(n) & \text{if } G = \mathrm{GL}(n), \end{cases}$$

and we write $\mathbf{K}$ for $\mathbf{K}^G$ if G is clear from the context.

We first recall the definition of the local weight functions from [Ar5]. Let $\mathcal{V} \in (\mathcal{U}_M)$ and $U \in (\mathcal{V}(\mathbb{R}))_{M(\mathbb{R})}$. We denote by $\mathcal{V}^G$ the conjugacy class in $(\mathcal{U}_G)$ induced from $\mathcal{V}$ along some parabolic subgroup in $\mathcal{P}(M)$ (they yield all the same induced class). Let $U^G \in \big(\mathcal{V}^G(\mathbb{R})\big)_{G(\mathbb{R})}$ be such that $U^G \cap M(\mathbb{R}) = U$. (As explained above there are at most two different elements in $\big(\mathcal{V}^G(\mathbb{R})\big)_{G(\mathbb{R})}$ which are conjugate by $\mathrm{diag}(-1, 1, \ldots, 1)$.)

Arthur [Ar5] defines a weight function $w_{M,U}$ on a dense open subset of $\mathcal{V}^G(\mathbb{R})$ such that the local weighted orbital integral $J_M(U, f) = J_M^G(U, f)$ can be defined as

$$J_M^G(U, f) = \int_{U^G} f(x) w_{M,U}(x)\, dx \tag{7.1}$$

for any $f \in C^\infty(G(\mathbb{R}))$ of almost compact support with dx denoting the invariant measure on U^G, cf. [Rao]. Let $Q = LN \in \mathcal{P}$ be a Richardson parabolic subgroup for $\mathcal{V}^G$. By the results of the previous section, if f is conjugation invariant under the group $\mathrm{O}(n)$, we can write the above as

$$J_M^G(U,f) = c\int_{N(\mathbb{R})} f(n) w_{M,U}(n)\,dn \tag{7.2}$$

for some constant $c>0$. The function $w_{M,U}$ actually only depends on $\mathcal{V}$ and not on the specific $U \in (\mathcal{V}(F))_{M(\mathbb{R})}$ so that we can also write $w_{M,\mathcal{V}}$ for this function. Note that the map $x \mapsto X = x - \mathrm{id}$ defines a bijection between the variety $\mathcal{U}_G(\mathbb{R})$ and the nilpotent cone in the Lie algebra $\mathfrak{g}(\mathbb{R})$. Moreover, if $x \in U^G$, then for any $s \in \mathbb{R}$, $s \neq 0$, the element

$$x_s := \mathrm{id} + s(x - \mathrm{id}) \tag{7.3}$$

is an element in $\mathcal{V}^G(\mathbb{R})$ (but not necessarily in U^G).

The goal is to show the following in this section:

Proposition 7.1. *Let $P = MV \in \mathcal{F}(M)$. Let $x \in \mathcal{V}^G(\mathbb{R})$ be such that $w_{M,\mathcal{V}}(x)$ is defined. Then $w_{M,\mathcal{V}}(x_s)$ is also well-defined for every $s>0$, and as a function of s, it is a polynomial in $\log s$ of degree at most $\dim \mathfrak{a}_M^G$. Moreover, there are $k \in \mathbb{N}$, homogeneous polynomials $P_1, \dots, P_t : N(\mathbb{R}) \longrightarrow \mathbb{R}^k$, and coefficients $\alpha_I \in \mathbb{R}$ for each subset $I \subseteq \{1,\dots,t\}$ such that $w_{M,\mathcal{V}}(\mathrm{id} + X) = \sum_{I\subseteq\{1,\dots,t\}} \alpha_I \prod_{i\in I} \log\|P_i(X)\|$ for X in a dense subset of $N(\mathbb{R})$. Here $\|\cdot\|$ denotes the vector norm on $\mathbb{R}^k$.*

To prove this proposition, we follow along the lines of Arthur's construction of the weight functions. Let $\Phi(A_M, G)$ be the set of roots of (A_M, G), and let $\beta \in \Phi(A_M, G)$. Note that every root in $\Phi(A_M, G)$ is reduced in $\Phi(A_M, G)$, that is, if $\gamma \in \Phi(A_M, G)$, then $m\gamma \notin \Phi(A_M, G)$ for any integer $m \neq \pm 1$. Let $M_\beta \subseteq G$ be a Levi subgroup containing M such that $\mathfrak{a}_{M_\beta} = \{X \in \mathfrak{a}_M \mid \beta(X) = 0\}$. Let $P_\beta \in \mathcal{P}^{M_\beta}(M)$ be the unique parabolic subgroup of M_β such that the unique root of A_M on the unipotent radical of P_β equals β. Suppose that P, $P_1 \in \mathcal{P}(M)$, $P_1 = MN_1$, are such that $P \cap M_\beta = P_\beta$ and $P_1 \cap M_\beta = \overline{P_\beta}$ (the opposite parabolic), and write $P_\beta = L_\beta N_\beta$ for the Levi decomposition of P_β with $M \subseteq L_\beta$.

Suppose that $\pi = u\nu \in \mathcal{U}_M(\mathbb{R})\overline{N_\beta}(\mathbb{R})$. Then for any $a \in A_{M,\mathrm{reg}}$ there is a unique $n_\beta \in \overline{N_\beta}(\mathbb{R})$ such that

$$a\pi = n_\beta^{-1} a u n_\beta. \tag{7.4}$$

Note that n_β is independent of the A_{M_β}-part of a, that is, it only depends on a^β. Let $\mathrm{Wt}(\mathfrak{a}_M) \subseteq X(A_M)$ be the sublattice of all ϖ which are extremal weights for some finite dimensional representation of $G(\mathbb{R})$. Let $\varpi \in \mathrm{Wt}(\mathfrak{a}_M) \subseteq \mathfrak{a}_M^*$ be such that $\varpi(\beta^\vee) > 0$. Consider

$$v_P(\varpi, n_\beta) := e^{-\varpi(H_P(n_\beta))} = e^{-\varpi(H_{P_\beta}(n_\beta))} = v_{P_\beta}(\varpi, n_\beta)$$

as a function of $a \in A_M/A_{M_\beta} \simeq A_M^{M_\beta}$, and π as above.

Write $\nu = \mathrm{id} + X$ and $n_\beta = \mathrm{id} + Y_\beta$ with $X, Y_\beta \in \overline{\mathfrak{n}_\beta}(\mathbb{R})$. Note that $a^{-1} n_\beta a = \mathrm{id} + a^\beta Y_\beta$. Further note that $Y_\beta^2 = 0$ since $2\beta \notin \Phi(A_M, G)$. Hence

$$\begin{aligned}\pi = u + uX = a^{-1} n_\beta^{-1} a u n_\beta &= (\mathrm{id} + a^\beta Y_\beta)^{-1} u (\mathrm{id} + Y_\beta)\\ &= u + (1 - a^\beta) Y_\beta u + [u, Y_\beta] - a^\beta Y_\beta [u, Y_\beta]\end{aligned}$$

where $[u, Y_\beta] = u Y_\beta - Y_\beta u$ is again nilpotent and contained in the Lie algebra of $\overline{N_\beta}(\mathbb{R})$. Again, this implies that the term $Y_\beta [u, Y_\beta]$ vanishes since 2β is not a root of (A_M, G). Hence

$$u + uX = u + (1 - a^\beta) Y_\beta u + [u, Y_\beta].$$

Let $Q_0 = M_0 V_0$ be a semistandard minimal parabolic subgroup containing N_1, so in particular it contains $\overline{N_\beta}$ as well. Conjugating u by some element in $\mathbf{K}^M := \mathbf{K} \cap M(\mathbb{R})$ if necessary, we can assume that $u \in V_0(\mathbb{R}) \cap M(\mathbb{R})$. In particular, we can write $u = \mathrm{id} + X_0$ with X_0 a nilpotent matrix in the Lie algebra of $V_0(\mathbb{R}) \cap M(\mathbb{R})$. Then $[u, Y_\beta] = [X_0, Y_\beta]$. Hence the above equality becomes

$$uX = X + X_0 X = (1 - a^\beta) Y_\beta (\mathrm{id} + X_0) + [X_0, Y_\beta] = (1 - a^\beta) Y_\beta + X_0 Y_\beta - a^\beta Y_\beta X_0. \tag{7.5}$$

Q_0 determines a choice of positive reduced roots $\Phi_{Q_0}^+ := \Phi(A_0, Q_0)$. Then there exists $\beta' \in \Phi_{Q_0}^+$ with $\beta'_{|A_M} = -\beta$, and we denote by $\Psi_\beta \subseteq \Phi_{Q_0}^+$ the subset of all such β'. Let $\Phi_{Q_0}^{M,+} \subseteq \Phi_{Q_0}^+$ denote the subset of positive roots on M. Then $\Psi_\beta \cap \Phi_{Q_0}^{M,+} = \emptyset$.

We have a partial order $\prec_\beta$ on Ψ_β: If $\alpha_1, \alpha_2 \in \Psi_\beta$, then $\alpha_1 \prec_\beta \alpha_2$ if and only if there exists $\gamma \in \Phi_{Q_0}^+$ with $\alpha_2 = \alpha_1 + \gamma$. Note that then $\gamma \in \Phi_{Q_0}^{M,+}$. We define subsets of Ψ_β according to how much the elements fail to be minimal with respect to $\prec_\beta$: Let $\Psi_\beta^{(0)}$ be the set of all $\alpha \in \Psi_\beta$ which are minimal with respect to $\prec_\beta$. If $k \geq 0$ is a non-negative integer, let $\Psi_\beta^{(k+1)}$ be the set of all $\alpha \in \Psi_\beta \backslash \Psi_\beta^{(k)}$ which can be written as $\alpha = \alpha_1 + \gamma$ with $\alpha_1 \in \Psi_\beta^{(k)}$ and $\gamma \in \Phi_{Q_0}^{M,+}$. Note that $\Psi_\beta^{(k)} \neq \emptyset$ for only finitely many k. Moreover, $\alpha_1 \in \Psi_\beta^{(l)}$ implies that $\alpha_2 \prec_\beta \alpha_1$ for any $\alpha_2 \in \Psi_\beta^{(k)}$ with $k < l$.

To recover the matrix entries of Y_β from (7.5) we now proceed inductively over l by considering the matrix entries correspond to roots in $\Psi_\beta^{(l)}$. In the following we write

$$Z := X + X_0 X.$$

Note that

$$\pi = \mathrm{id} + X_0 + Z, \text{ so that } \pi_s = \mathrm{id} + s X_0 + s Z.$$

If $Y \in \mathfrak{g}(\mathbb{R})$ is a matrix in the Lie algebra, and $\alpha \in \Phi(A_0, G)$ we denote by $Y^{(\alpha)} \in \mathbb{R}$ the matrix entry of Y corresponding to α.

Lemma 7.2. *Let $l \geq 0$, and $\alpha \in \Psi_\beta^{(l)}$. There are rational polynomials $P_{\alpha,i}$, $0 \leq i \leq l$ in the variables $X_0^{(\gamma)}$ ($\gamma \in \Phi_{Q_0}^{M,+}$), $Z^{(\gamma)}$ ($\gamma \in \Psi_\beta$), and a^β such that for $a \neq 1$ the following holds*

- *as a polynomial of* Z *and* X_0, $P_{\alpha,i}(Z, X_0, a^\beta)$ *is homogeneous of degree* $(i+1)$ *in the matrix entries of* $Z+X_0$*, so in particular, for every* $s \in \mathbb{R}$ *we have* $P_{\alpha,i}(sZ, sX_0, a^\beta) = s^{i+1} P_{\alpha,i}(Z, X_0, a^\beta)$,
- *and*

$$Y_\beta^{(\alpha)} = \sum_{i=0}^{l} \frac{P_{\alpha,i}(Z, X_0, a^\beta)}{(1-a^\beta)^{i+1}}.$$

Proof. We prove the lemma by induction on l. If $l = 0$, then for any $\alpha \in \Psi_\beta^{(0)}$ we obtain from (7.5) that

$$Y_\beta^{(\alpha)} = \frac{Z^{(\alpha)}}{(1-a^\beta)}$$

so the assertion of the lemma is true for $l = 0$.

Now suppose that for some non-negative integer $l \geq 0$ we know that for every $0 \leq k \leq l$ and every $\alpha \in \Psi_\beta^{(k)}$ we have

$$Y_\beta^{(\alpha)} = \sum_{i=0}^{k} \frac{P_{\alpha,i}(Z, X_0, a^\beta)}{(1-a^\beta)^{i+1}} \tag{7.6}$$

with $P_{\alpha,i}$ polynomials satisfying the assertions of the lemma. Then for $\alpha \in \Psi_\beta^{(k+1)}$ the equation (7.5) gives

$$Z^{(\alpha)} = (1-a^\beta) Y_\beta^{(\alpha)} + (X_0 Y_\beta)^{(\alpha)} - a^\beta (Y_\beta X_0)^{(\alpha)}.$$

By definition of $\Psi_\beta^{(k+1)}$ we have

$$(X_0 Y_\beta)^{(\alpha)} - a^\beta (Y_\beta X_0)^{(\alpha)} = \sum_{\substack{\gamma \in \Psi_\beta^{(k)},\ \delta \in \Phi_{Q_0}^{M,+}: \\ \gamma+\delta=\alpha}} e_\alpha^{\gamma,\delta} Y_\beta^{(\gamma)} X_0^{(\delta)}$$

with

$$e_\alpha^{\gamma,\delta} = \begin{cases} 1 & \text{if } [E_\delta, E_\gamma] = E_{\delta+\gamma}, \\ -a^\beta & \text{if } [E_\delta, E_\gamma] = -E_{\delta+\gamma} \end{cases}$$

where E_γ, E_δ denote the elements of the standard Chevalley basis attached to our root system $\Phi_{Q_0}^+$. By the inductive assumption we can insert (7.6) for $Y_\beta^{(\gamma)}$ for every γ occurring in the sum. Dividing both sides of the so obtained equality by $(1-a^\beta)$ then yields the assertion of the lemma. $\square$

7.1. **Weight functions.** We now consider the function $v_{P_\beta}(\varpi, n_\beta) = e^{-\varpi(H_P(n_\beta))}$ for $\varpi \in \mathrm{Wt}(\mathfrak{a}_M)$. Note that H_P is invariant under left and right multiplication with elements of $\mathbf{K}^M$. There is an irreducible representation Λ_ϖ of G on a finite dimensional vector space V_ϖ, defined over $\mathbb{R}$, together with an extremal vector $\phi_\varpi \in V_\varpi(\mathbb{R})$ of weight ϖ and a norm $\|\cdot\|$ on $V_\varpi(\mathbb{R})$ such that $\|\phi_\varpi\| = 1$ and

$$v_{P_\beta}(\varpi, n) = \|\Lambda_\varpi(n^{-1})\phi_\varpi\|.$$

We can identify $V_\varpi(\mathbb{R})$ with $\mathbb{R}^{m_\varpi}$ for $m_\varpi = \dim V_\varpi$ and we can assume that the norm is of the form

$$\|(x_1, \dots, x_{m_\varpi})\| = \left(x_1^2 + \dots + x_{m_\varpi}^2\right)^{1/2}$$

for $(x_1, \dots, x_{m_\varpi}) \in \mathbb{R}^{m_\varpi}$. Recall that Λ_ϖ is an algebraic representation. Thus $\Lambda_\varpi \colon G \to \mathrm{GL}(V_\varpi)$ is a morphism of algebraic varieties. Hence as a function of $n_\beta \in \overline{N_\beta}(\mathbb{R})$ the function $v_{P_\beta}(\varpi, n_\beta)$ is the vector norm applied to a polynomial function from $\overline{\mathfrak{n}_\beta}(\mathbb{R})$ to $\mathbb{R}^{m_\varpi}$. Using the above lemma (and the notation therein), we therefore get that

$$v_{P_\beta}(\varpi, n_\beta)^2 = \sum_{\kappa \in R} \frac{f_\kappa(Z, X_0, a^\beta)}{(a^\beta - 1)^\kappa} \tag{7.7}$$

where κ runs over a finite set of integers $R \subseteq \mathbb{Z}$, and f_κ is the norm of some rational function that is homogeneous of degree κ in the matrix entries of X_0 and Z, and has a finite value at $a = 1$ so that we may write $f_\kappa(Z, X_0, 1)$. Here n_β, Z, X_0, and a are related as in (7.4). The function $f_\kappa(Z, X_0, a^\beta)$ in general depends on ϖ and if we want to make this dependence explicit we write $f_{\kappa,\varpi}(Z, X_0, a^\beta)$.

Now let $u \in U$, $U \subseteq M(\mathbb{R})$, and $U^G \subseteq G(\mathbb{R})$ be as before, and write $u = \mathrm{id} + X_0$ with X_0 nilpotent. Let $\kappa_0(\beta, X_0) \in R$ be the largest $\kappa \in R$ such that $f_\kappa(\cdot, \cdot, a^\beta)$ does not vanish identically on $\overline{\mathfrak{n}_\beta}(\mathbb{R}) \times \mathcal{N}$, where $\mathcal{N} \subseteq \mathfrak{m}(\mathbb{R})$ is the nilpotent orbit defined by $U = \mathrm{id} + \mathcal{N}$. Let $\rho(\beta, X_0)$ be the product of $1/(2\varpi(\beta^\vee))$ with $\kappa_0(\beta, X_0)$. It follows from [Ar5, p. 238] that $\kappa_0(\beta, X_0) \geq 0$ and $\rho(\beta, X_0) \geq 0$. The $\rho(\beta, X_0)$ is independent of ϖ as explained in [Ar5, p. 238] but $\kappa_0(\beta, X_0)$ in general depends on ϖ. If we want to emphasize this dependence, we write $\kappa_0(\beta, X_0) = \kappa_0(\varpi, \beta, X_0)$.

Recall the definition of the weight function $w_P(\lambda, a, \pi)$ from [Ar5, (3.6)]: Fix a parabolic subgroup $P_1 \in \mathcal{P}(M)$. Then for any other $P \in \mathcal{P}(M)$ and any P-dominant $\varpi \in \mathrm{Wt}(\mathfrak{a}_M)$ Arthur defines for $\pi = u\nu \in UN_1(\mathbb{R})$ the function

$$w_P(\varpi, a, \pi) = \left(\prod_{\beta \in \Phi_P \cap \Phi_{\overline{P_1}}} r_\beta(\varpi, u, a) \right) v_P(\varpi, n) \tag{7.8}$$

where $\Phi_P = \Phi(A_M, P)$ and $\Phi_{\overline{P_1}} = \Phi(A_M, \overline{P_1})$. Here u, ν, a, and $n \in N_1(\mathbb{R})$ are related by $a\pi = n^{-1}aun$. The function r_β is given by (see [Ar5, (3.4)])

$$r_\beta(\varpi, u, a) = |a^\beta - a^{-\beta}|^{\rho(\beta, X_0)\varpi(\beta^\vee)}.$$

It follows from this definition that

$$w_P(\varpi, a, k^{-1}\pi k) = w_P(\varpi, a, \pi) \tag{7.9}$$

for $k = \operatorname{diag}(-1, 1, \ldots, 1) \in \bar{\mathbf{K}}^M_\infty$.

If $\varpi_1, \ldots, \varpi_r \in \operatorname{Wt}(\mathfrak{a}_M)$ is a basis of $\mathfrak{a}_M^*$ consisting of P-dominant weights, and $\lambda = \lambda_1\varpi_1 + \ldots + \lambda_r\varpi_r \in \mathfrak{a}^*_{M,\mathbb{C}}$ with $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$, then

$$w_P(\lambda, a, \pi) = \prod_{i=1}^r w_P(\varpi_i, a, \pi)^{\lambda_i} \tag{7.10}$$

By [Ar5, Lemma 4.1] the limit

$$w_P(\lambda, \pi) := \lim_{a\to 1} w_P(\lambda, a, \pi)$$

exists and is non-zero for all π in an open and dense subset of $UN_1(\mathbb{R})$.

7.2. **The adjacent case.** For a unipotent element π recall the definition of π_s from (7.3). If P and P_1 are adjacent, the function $w_P(\lambda, \pi)$ has the following behavior when π is replaced by π_s:

Lemma 7.3. *Suppose P and P_1 are adjacent via $\beta \in \Phi(A_M, P)$. Then for all π in an open dense subset of $UN_1(\mathbb{R})$ we have*

$$w_P(\lambda, \pi_s) = s^{\rho(\beta, X_0)\lambda(\beta^\vee)} \cdot w_P(\lambda, \pi)$$

for all $s > 0$.

Proof. Note that $w_P(\lambda, a, \pi) = w_P(\lambda, a, k^{-1}\pi k)$ for any $k \in \mathbf{K}^M = \mathbf{K} \cap M(\mathbb{R})$ so that we can assume π to be of the form $\mathrm{id} + X_0 + Z$ with $X_0 \in \mathcal{N} \cap \mathfrak{v}_0(\mathbb{R})$, and $Z \in \mathfrak{n}_1(\mathbb{R})$ as before. Here $\mathfrak{v}_0$, resp. $\mathfrak{n}_1$, denotes the Lie algebra of V_0 (the unipotent radical of Q_0), resp. N_1 (the unipotent radical of P_1). Since P and P_1 are assumed to be adjacent along β, we have $\Phi_P \cap \Phi_{\bar{P}_1} = \{\beta\}$. Hence it follows from (7.8) that $w_P(\lambda, a, \pi) = r_\beta(\lambda, X_0, a)v_P(\lambda, n)$ where

$$r_\beta(\lambda, X_0, a) = |a^\beta - a^{-\beta}|^{\rho(\beta, X_0)\lambda(\beta^\vee)}$$

. for a with $a^\beta \neq 1$. We can write $n \in N_1$ as $n = n_\beta \tilde{n}$ with n_β in the unipotent radical of $\overline{P_\beta}$ and $\tilde{n}$ in the unipotent radical of $P_1 \cap P$. Then $v_P(\lambda, n) = v_{P_\beta}(\lambda, n_\beta)$ so that

$$w_P(\lambda, a, \pi) = r_\beta(\lambda, X_0, a)v_{P_\beta}(\lambda, n_\beta).$$

Hence, using (7.7) and (7.10),

$$\begin{aligned} w_P(\lambda,\pi)^2 &= \lim_{a\to 1}\prod_{i=1}^r \left(|a^\beta - a^{-\beta}|^{2\rho(\beta,X_0)\varpi_i(\beta^\vee)} \sum_{\kappa\in R} \frac{f_{\kappa,\varpi_i}(Z,X_0,a^\beta)}{(a^\beta-1)^\kappa}\right)^{\lambda_i} \\ &= \lim_{a\to 1}\prod_{i=1}^r \left(|a^\beta - a^{-\beta}|^{\kappa_0(\varpi_i,\beta,X_0)} \sum_{\kappa\in R} \frac{f_{\kappa,\varpi_i}(Z,X_0,a^\beta)}{(a^\beta-1)^\kappa}\right)^{\lambda_i} \\ &= \prod_{i=1}^r 2^{\lambda_i} f_{\kappa_0(\varpi_i,\beta,X_0),\varpi_i}(Z,X_0,1)^{\lambda_i}. \end{aligned}$$

Now $f_{\kappa_0(\varpi_i,\beta,X_0),\varpi_i}(Z,X_0,1)$ is non-zero for generic X_0, Z, and satisfies

$$f_{\kappa_0(\varpi_i,\beta,X_0),\varpi_i}(sZ,sX_0,1) = s^{\kappa_0(\varpi_i,\beta,X_0)} f_{\kappa_0(\varpi_i,\beta,X_0),\varpi_i}(Z,X_0,1).$$

Hence

$$w_P(\lambda,\pi_s)^2 = \prod_{i=1}^r s^{\lambda_i\kappa_0(\varpi_i,\beta,X_0)} \prod_{i=1}^r 2^{\lambda_i} f_{\kappa_0(\varpi_i,\beta,X_0),\varpi_i}(Z,X_0,1)^{\lambda_i} = s^{2\rho(\beta,X_0)\lambda(\beta^\vee)} w_P(\lambda,\pi)^2$$

since $\sum_{i=1}^r \lambda_i\kappa_0(\varpi_i,\beta,X_0) = 2\rho(\beta,X_0)\sum_{i=1}^r \lambda_i\varpi_i(\beta^\vee) = 2\rho(\beta,X_0)\lambda(\beta^\vee)$. Since $w_P(\lambda,\pi_s)$ is a real-valued function and continuous in s with $w_P(\lambda,\pi_1) = w_P(\lambda,\pi)$, we can take the square-root on both sides of the equation and obtain the assertion of the lemma. □

7.3. **The general case.** If $Q, Q' \in \mathcal{P}(M)$ are adjacent along some root $\beta \in \Phi(A_M, Q)$, we write $Q|_\beta Q'$.

Corollary 7.4. *Suppose that P and P_1 are not necessarily adjacent. Choose a minimal chain $P = Q_0|_{\beta_1}Q_1|_{\beta_2}\dots|_{\beta_t}Q_t = P_1$ of adjacent parabolic subgroups $Q_1,\dots,Q_t \in \mathcal{P}(M)$ from P to P_1. Then there exist rational numbers $\rho_1,\dots,\rho_t$ such that for all π in an open dense subset of $UN_1(\mathbb{R})$ we have*

$$w_P(\lambda,\pi_s) = s^{\rho_1\lambda(\beta_1^\vee)+\ldots+\rho_t\lambda(\beta_t^\vee)}\cdot w_P(\lambda,\pi)$$

for all $s>0$.

Proof. This follows by induction on t together with Lemma 7.3 and the proof of [Ar5, Lemma 4.1].

The case $t=1$ is covered in the last lemma. Let $P_1' := Q_1 = MN_1'$, and assume that the corollary is true for P replaced by P_1'. Let $\pi = u\nu \in UN_1(\mathbb{R})$, and $a \in A_{M,\text{reg}}$. Let $n \in N_1(\mathbb{R})$ be the unique element with $\pi = a^{-1}n^{-1}aun$. Write $n = m'n'k'$ with $m' \in M(\mathbb{R})$, $n' \in N_1'(\mathbb{R})$, and $k' \in \mathbf{K}$, and put $u' = (m')^{-1}um'$. Let $\pi' = a^{-1}(n')^{-1}au'n' \in UN_1'(\mathbb{R})$. Then by the proof of [Ar5, Lemma 4.1, p. 241], we have

$$w_P(\lambda,a,\pi) = w_P(\lambda,a,\pi')w_{P_1'}(\lambda,a,\pi).$$

Suppose that $\pi = \mathrm{id}+Y$. Then $\pi' = \mathrm{id}+k'Yk'^{-1}$ so that that the map $\pi\mapsto\pi'$ also maps π_s to π_s' for any s. We are further allowed to take the value at $a=1$ on both sides because of [Ar5, Lemma 4.1]. The assertion of the corollary therefore follows from Lemma 7.3 and the induction hypothesis. ☐

Arthur defines polynomials $W_P(\varpi, a, \pi) \in V_\varpi$ for a P-dominant weight $\varpi \in \mathrm{Wt}(\mathfrak{a}_M)$ and $(a, \pi) \in A_{M,\mathrm{reg}} \times UN_1(\mathbb{R})$ by

$$W_P(\varpi, a, \pi) = \left(\prod_{\beta \in \Phi_P \cap \Phi_{\overline{P_1}}} r_\beta(\varpi, u, a) \right) \Lambda_\varpi(n^{-1}) \phi_\varpi$$

so that $w_P(\varpi, a, \pi) = \|W_P(\varpi, a, \pi)\|$ and

$$w_P(\lambda, a, \pi) = \prod_{i=1}^{r} \|W_P(\varpi_i, a, \pi)\|^{\lambda_i}. \tag{7.11}$$

Here π, u, a, and n are related as explained after (7.8)

Corollary 7.5. *If $\varpi \in \mathrm{Wt}(\mathfrak{a}_M)$ is a P-dominant weight, the polynomial $W_P(\varpi, a, \pi)$ is defined on all of $A_M \times UN_1$ and does not vanish at $a = 1$. Moreover, there is a constant r_ϖ depending only on P and ϖ such that for all $\pi \in UN_1(\mathbb{R})$ and all $s > 0$ we have*

$$\|W_P(\varpi, 1, \pi_s)\| = s^{r_\varpi} \|W_P(\varpi, 1, \pi)\|.$$

Proof. All assertions except the homogeneity are subject of [Ar5, Corollary 4.3]. The homogeneity follows from the previous corollary and the definition of $W_P(\varpi, a, \pi)$. ☐

Let $\theta_P(\lambda) = v_P^{-1} \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee)$ for $\lambda \in \mathfrak{a}_M^*$, where v_P denotes the covolume of the lattice spanned by all $\alpha^\vee$, $\alpha \in \Delta_P$, in $\mathfrak{a}_M$. Then $\{w_P(\lambda, a, \pi)\}_{P \in \mathcal{P}(M)}$ defines a (G, M)-family, and one can attach a certain number to this family by defining

$$w_M(a, \pi) = \lim_{\lambda \to 0} \left(\sum_{P \in \mathcal{P}(M)} w_P(\lambda, a, \pi) \theta_P(\lambda)^{-1} \right) \tag{7.12}$$

as in [Ar5, §6]. By [Ar3, (6.5)] this can be computed by

$$w_M(a, \pi) = \frac{1}{r!} \sum_{P \in \mathcal{P}(M)} \left(\lim_{t \to 0} \frac{d^r}{dt^r} w_P(t\Lambda, a, \pi) \right) \theta_P(\Lambda)^{-1}$$

with $r = \dim \mathfrak{a}_M^G$ and $\Lambda \in \mathfrak{a}_M^*$ some fixed generic element. (Note that $w_M(a, \pi)$ is independent of the choice of Λ.) Using (7.11) we get (cf. [Ar5, Lemma 5.4])

$$w_M(a, \pi) = \sum_\Omega c_\Omega \prod_{(P,\varpi) \in \Omega} \log \|W_P(\varpi, a, \pi)\|$$

where Ω runs over all finite multisets consisting of elements in $\mathcal{P}(M) \times \mathrm{Wt}(\mathfrak{a}_M)$, and the $c_\Omega \in \mathbb{C}$ are suitable coefficients which vanish for all but finitely many Ω. Moreover, each Ω contains at most $r = \dim \mathfrak{a}_M^G$ many elements. Note that in a neighborhood of $a = 1$ this expression is well-defined for all π in an open dense subset of UN_1 because of Corollary 7.5. Hence we can evaluate $w_M(a,\pi)$ at $a=1$ by means of this expression. Moreover, Corollary 7.5 implies the following result.

Corollary 7.6. *For all π in an open dense subset of $UN_1(\mathbb{R})$ and all $s>0$*

$$w_M(1,\pi_s)=\sum_{\Omega}c_\Omega\prod_{(P,\varpi)\in\Omega}(r_\varpi\log s+\log\|W_P(\varpi,1,\pi)\|).$$

In particular, as a function of s, $w_M(1,\pi_s)$ is a polynomial in $\log s$ of degree at most $r=\dim\mathfrak{a}_M^G$.

If now $Q'\in\mathcal{F}(M)$ is an arbitrary subgroup, we can analogously define all the above functions with respect to the Levi component $M_{Q'}$ instead of G, in particular we can define the analogue of $w_M(1,\pi)$ which we denote by $w_M^{Q'}(1,\pi)$. The corollary then stays true for $w_M^{Q'}(1,\pi)$ with the necessary changes. Note that a priori $w_M^{Q'}(1,\pi)$ is defined for π in a dense open subset of $UN_1^{M_{Q'}}$ for $P_1^{M_{Q'}}=P_1\cap M_{Q'}$. However, we can trivially extend $w_M^{Q'}(1,\pi)$ to a dense open subset of U^G. By the first equation on [Ar5, p. 256] the weight function $w_{M,\mathcal{V}}$ from (7.1) can then be written as

$$w_{M,\mathcal{V}}(\pi)=\sum_{Q'\in\mathcal{F}(M)}w_M^{Q'}(1,\pi).$$

This together with Corollary 7.6 implies the Proposition 7.1.

7.4. **Convergence.**

Lemma 7.7. *Let $p_i:\mathbb{R}^k\longrightarrow\mathbb{R}$, $i=1,...,l$, be homogeneous polynomials and put*

$$\lambda(x):=\prod_{i=1}^{l}\big|\log|p_i(x)|\big|,\quad x\in\mathbb{R}^k. \tag{7.13}$$

Then for every $a>0$

$$\int_{\mathbb{R}^k}e^{-a(\log(1+\|x\|))^2}\lambda(x)\,dx<\infty. \tag{7.14}$$

Proof. Since $\prod_{i=1}^l\big|\log|p_i(x)|\big|\le\sum_{i=1}^l\big|\log|p_i(x)|\big|^l$, it suffices to consider the case $\lambda(x)=\big|\log|p(x)|\big|^l$ with $p:\mathbb{R}^k\longrightarrow\mathbb{R}$ a homogeneous polynomial of degree κ.

Further note that for every $M\in\mathbb{N}$ there exists $r_M>0$ such that $(\log(1+\|x\|))^2\ge(M/a)\log\|x\|$ for all x with $\|x\|\ge r_M$. In particular,

$$e^{-a(\log(1+\|x\|))^2}\le\|x\|^{-M} \tag{7.15}$$

for all x with $\|x\|\ge r_M$.

We decompose the integral in (7.14) into a sum of integrals over certain subsets of $\mathbb{R}^k$ similarly as in [Ar5, §7]. For each $m\in\mathbb{N}_0$ let

$$B_m=\{x\in\mathbb{R}^k\mid 2^m\le\|x\|\le2^{m+1}\}$$

and let $B = \{x \in \mathbb{R}^k \mid \|x\| \leq 2\}$. For $\epsilon > 0$ set

$$\Gamma_\epsilon = \{x \in B \mid |p(x)| < \epsilon\}, \text{ and } \Gamma_{m,\epsilon} = \{x \in B_m \mid |p(x)| < \epsilon\}.$$

By [Ar5, (7.1)] there are constants $C, t > 0$ such that for every $\epsilon > 0$ we have

$$\int_{\Gamma_\epsilon} \lambda(x)\, dx \leq C\epsilon^t. \tag{7.16}$$

For $m \geq 0$ we can therefore compute, using that p is homogeneous of degree κ, that

$$\begin{aligned}\int_{\Gamma_{m,\epsilon}} \lambda(x)\, dx &\leq 2^{mk} \int_{\Gamma_{2^{-m\kappa}\epsilon}} \lambda(2^m y)\, dy \leq c_1 2^{mk} m^l \int_{\Gamma_{2^{-m\kappa}\epsilon}} (1+\lambda(y))\, dy \leq c_2 2^{mk} m^l 2^{-m\kappa t_1} \epsilon^{t_1} \\ &\leq c_3 2^{m(k+l)} \epsilon^{t_1}\end{aligned}$$

where $c_1, c_2, c_3, t_1 > 0$ are suitable constants independent of m and ϵ and the second last inequality follows from (7.16). Fix $\epsilon > 0$ and let $Z_\epsilon = \{x \in \mathbb{R}^k \mid |p(x)| < \epsilon\}$.

Then, using (7.15) for every $M \in \mathbb{N}$ there exist $c_4, r_M > 0$ such that

$$\begin{aligned}&\int_{Z_\epsilon} e^{-a(\log(1+\|x\|))^2} \lambda(x)\ dx \\ \leq &\int_{\Gamma_\epsilon} \lambda(x)\, dx + \sum_{m \leq \log_2 r_M} c_4 \int_{\Gamma_{m,\epsilon}} \lambda(x)\, dx + \sum_{m \geq \log_2 r_M} 2^{-mM} \int_{\Gamma_{m,\epsilon}} \lambda(x)\, dx \\ \leq &C\epsilon^t + c_5 \epsilon^{t_1} \left(1 + \sum_{m \geq \log_2 r_M} 2^{-mM} 2^{m(k+l)}\right).\end{aligned}$$

This is finite if we choose M sufficiently large.

Now on $\mathbb{R}^k \backslash Z_\epsilon$ the polynomial $p(x)$ is bounded away from 0 so that $\log|p(x)|$ is bounded from below on $\mathbb{R}^k \backslash Z_\epsilon$. Since $p(x)$ is of degree κ, there is a constant $A > 0$ such that $|p(x)| \leq A(1+\|x\|)^\kappa$ for all $x \in \mathbb{R}^k$. Choose M again sufficiently large and let r_M be as in (7.15). Then for some suitable constant $A_1 > 0$ we get

$$\int_{\mathbb{R}^k \backslash Z_\epsilon} e^{-a(\log(1+\|x\|))^2} \lambda(x)\, dx \leq A_1 \int_{\mathbb{R}^k \backslash Z_\epsilon} \|x\|^{-M} \left(1 + (\log(1+\|x\|))^l\right)\, dx,$$

which is finite if M was chosen sufficiently large.

□

8. Examples for weight functions in low rank

8.1. $G = \mathrm{GL}(2)$**.** There are two unipotent conjugacy classes in GL(2), the trivial class for which Richardson parabolic subgroup equals G, and the regular unipotent conjugacy class with Richardson parabolic equal to the minimal parabolic subgroup $P_0 = M_0 U_0$. The archimedean orbital integrals appearing in the fine expansion of J_{unip} are $J_G(1, f_\infty)$, $J_G(u_0, f_\infty)$, and $J_{M_0}(1, f_\infty)$, where $u_0 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ represents the regular class in $G(\mathbb{Q})$. The first two integrals are unweighted, and the last integral $J_{M_0}(1, f_0)$ is up to a normalization of Haar measure equal to

$$\int_{U_0(\mathbb{R})} f_\infty\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right) \log|x|\, dx,$$

see, e.g., [Gel].

8.2. $G = \mathrm{GL}(3)$. There are three unipotent conjugacy class in GL(3): The trivial, the regular, and the subregular class. Let

$$u(x, y, z) = \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}$$

for $x, y, z \in \mathbb{R}$. Then $u(0, 1, 0)$ is a representative for the subregular class, and $u(1, 0, 1)$ a representative for the regular class. Let M_1 be the Levi subgroup corresponding to the partition $(2, 1)$ of 3. Every other corank-1 Levi subgroup in $\mathcal{L}$ is conjugate to M_1 by some Weyl group element so that it suffices to consider $J_M(u, f_\infty)$ for $M \in \{M_0, M_1, G\}$. The integrals $J_G(1, f_\infty)$, $J_G(u(0, 1, 0), f_\infty)$, and $J_G(u(1, 0, 1), f_\infty)$ are all unweighted. For the other cases we get (up to normalization of the measures)

$$J_{M_1}(1, f_\infty) = \int_{U_1(\mathbb{R})} f_\infty(u(0, y, z)) \log(y^2 + z^2)\, du(0, y, z), \tag{8.1}$$

$$J_{M_1}(u(1, 0, 0), f_\infty) = \int_{U_0(\mathbb{R})} f_\infty(u(x, y, z)) \log|xz|\, du(x, y, z), \tag{8.2}$$

and

$$J_{M_0}(1, f_\infty) = \int_{U_0(\mathbb{R})} f_\infty(u(x, y, z)) \left(\log|x| \log|z| + (\log|x|)^2 + (\log|z|)^2\right) du(x, y, z), \tag{8.3}$$

cf. [Fli, p. 67, Lemma 4].

9. Bochner Laplace operators

In this section we summarize some basic facts about Bochner-Laplace operators on global Riemannian symmetric spaces. For simplicity we assume that G is semisimple and $G(\mathbb{R})$ is of noncompact type. Then $G(\mathbb{R})$ is a semisimple real Lie group of noncompact type. Let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup and

$$\widetilde{X} = G(\mathbb{R})/K_\infty$$

the associated Riemannian symmetric space. Let $\Gamma \subset G(\mathbb{R})$ be a torsion free lattice and let $X = \Gamma\backslash\widetilde{X}$. Let ν be a finite-dimensional unitary representation of K_∞ on $(V_\nu, \langle\cdot,\cdot\rangle_\nu)$. Let

$$\widetilde{E}_\nu := G(\mathbb{R}) \times_\nu V_\nu$$

be the associated homogeneous vector bundle over $\tilde{X}$. Then $\langle\cdot,\cdot\rangle_\nu$ induces a $G(\mathbb{R})$-invariant metric $\tilde{h}_\nu$ on $\tilde{E}_\nu$. Let $\widetilde{\nabla}^\nu$ be the connection on $\tilde{E}_\nu$ induced by the canonical connection on the principal K_∞-fibre bundle $G(\mathbb{R})\to G(\mathbb{R})/K_\infty$. Then $\widetilde{\nabla}^\nu$ is $G(\mathbb{R})$-invariant. Let

$$E_\nu := \Gamma\backslash\widetilde{E}_\nu$$

be the associated locally homogeneous vector bundle over X. Since $\tilde{h}_\nu$ and $\widetilde{\nabla}^\nu$ are $G(\mathbb{R})$-invariant, they push down to a metric h_ν and a connection ∇^ν on E_ν. Let $C^\infty(\widetilde{X},\widetilde{E}_\nu)$ resp. $C^\infty(X,E_\nu)$ denote the space of smooth sections of $\widetilde{E}_\nu$, resp. E_ν. Let

$$\begin{aligned} C^\infty(G(\mathbb{R}),\nu) := \{f\colon G(\mathbb{R})\to V_\nu\colon f\in C^\infty,\ f(gk) =&\nu(k^{-1})f(g),\\ &\forall g\in G(\mathbb{R}),\ \forall k\in K_\infty\}, \end{aligned} \tag{9.1}$$

Let $L^2(G(\mathbb{R}),\nu)$ be the corresponding L^2-space. There is a canonical isomorphism

$$\widetilde{A}\colon C^\infty(\widetilde{X},\widetilde{E}_\nu)\cong C^\infty(G(\mathbb{R}),\nu) \tag{9.2}$$

(see [Mia, p. 4]). $\widetilde{A}$ extends to an isometry of the corresponding L^2-spaces. Let

$$C^\infty(\Gamma\backslash G(\mathbb{R}),\nu) := \{f\in C^\infty(G(\mathbb{R}),\nu)\colon f(\gamma g)=f(g)\,\forall g\in G(\mathbb{R}),\forall\gamma\in\Gamma\} \tag{9.3}$$

and let $L^2(\Gamma\backslash G(\mathbb{R}),\nu)$ be the corresponding L^2-space. The isomorphism (9.2) descends to isomorphisms

$$A\colon C^\infty(X,E_\nu)\cong C^\infty(\Gamma\backslash G(\mathbb{R}),\nu),\quad L^2(X,E_\nu)\cong L^2(\Gamma\backslash G(\mathbb{R}),\nu). \tag{9.4}$$

Let $\widetilde{\Delta}_\nu=\widetilde{\nabla^\nu}^*\widetilde{\nabla}^\nu$ be the Bochner-Laplace operator of $\widetilde{E}_\nu$. This is a $G(\mathbb{R})$-invariant second order elliptic differential operator whose principal symbol is given by

$$\sigma_{\widetilde{\Delta}_\nu}(x,\xi)=\|\xi\|_x^2\cdot\mathrm{Id}_{E_{\nu,x}},\quad x\in\widetilde{X},\ \xi\in T_x^*(\widetilde{X}).$$

Since $\widetilde{X}$ is complete, $\widetilde{\Delta}_\nu$ with domain the smooth compactly supported sections is essentially self-adjoint [LM, p. 155]. Its self-adjoint extension will be denoted by $\widetilde{\Delta}_\nu$ too. Let $\Omega\in\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ and $\Omega_{K_\infty}\in\mathcal{Z}(\mathfrak{k})$ be the Casimir operators of $\mathfrak{g}$ and $\mathfrak{k}$, respectively, where the latter is defined with respect to the restriction of the normalized Killing form of $\mathfrak{g}$ to $\mathfrak{k}$. Let $C^\infty(G(\mathbb{R}),V_\nu)$ be the space of smooth V_ν-valued functions on $G(\mathbb{R})$ and R the right regular representation of $G(\mathbb{R})$ in in $C^\infty(G(\mathbb{R}),V_\nu)$. Let $R(\Omega)$ be the differential operator induced by Ω. Since $\mathrm{Ad}(g)\Omega=\Omega$, $g\in G(\mathbb{R})$, it follows that $R(\Omega)$ preserves the subspace $C^\infty(G(\mathbb{R}),\nu)$. Then with respect to the isomorphism (9.2) we have

$$\widetilde{\Delta}_\nu=-R(\Omega)+\nu(\Omega_{K_\infty}), \tag{9.5}$$

(see [Mia, Proposition 1.1]). Let $e^{-t\widetilde{\Delta}_\nu}$, $t>0$, be the heat semigroup generated by $\widetilde{\Delta}_\nu$. It commutes with the action of $G(\mathbb{R})$. With respect to the isomorphism (9.2) we may regard $e^{-t\widetilde{\Delta}_\nu}$ as bounded operator in $L^2(G(\mathbb{R}),\nu)$, which commutes with the action of $G(\mathbb{R})$. Hence it is a convolution operator, i.e., there exists a smooth map

$$H_t^\nu\colon G(\mathbb{R})\to\mathrm{End}(V_\nu) \tag{9.6}$$

such that

$$(e^{-t\widetilde{\Delta}_\nu}\phi)(g) = \int_{G(\mathbb{R})} H_t^\nu(g^{-1}g')(\phi(g'))\,dg', \quad \phi \in L^2(G(\mathbb{R}),\nu).$$

The kernel H_t^ν satisfies

$$H_t^\nu(k^{-1}gk') = \nu(k)^{-1}\circ H_t^\nu(g)\circ\nu(k'),\ \forall k,k'\in K, \forall g\in G. \tag{9.7}$$

For $q>0$ let $\mathscr{C}^q(G(\mathbb{R}))$ be Harish-Chandra's L^q-Schwartz space. We briefly recall its definition. Let Ξ and $\|\cdot\|$ be the functions on $G(\mathbb{R})$ used to define Harish-Chandra's Schwartz space $\mathcal{C}(G(\mathbb{R}))$ (see [Wal, 7.1.2]). Furthermore, for $Y\in\mathcal{U}(\mathfrak{g}_\mathbb{C})$ denote by $L(Y)$ (resp. $R(Y)$) the associated left (resp. right) invariant differential operator on $G(\mathbb{R})$. Then $\mathscr{C}^q(G(\mathbb{R}))$ consists of all $f\in C^\infty(G(\mathbb{R}))$ such that

$$\sup_{x\in G(\mathbb{R})}(1+\|x\|)^m\Xi(x)^{-2/q}|L(Y_1)R(Y_2)f(x)|<\infty$$

for all $m\ge 0$ and $Y_1,Y_2\in\mathcal{U}(\mathfrak{g}_\mathbb{C})$. Note that $\mathscr{C}^2(G(\mathbb{R}))$ equals Harish-Chandra's Schwartz space $\mathcal{C}(G(\mathbb{R}))$. Proceeding as in the proof of [BM, Proposition 2.4] it follows that H_t^ν belongs to $(\mathscr{C}^q(G(\mathbb{R}))\otimes\operatorname{End}(V_\nu))^{K_\infty\times K_\infty}$ for all $q>0$.

Let π be a unitary representation of $G(\mathbb{R})$ on a Hilbert space $\mathcal{H}_\pi$. Define a bounded operator on $\mathcal{H}_\pi\otimes V_\nu$ by

$$\tilde{\pi}(H_t^\nu(g)) := \int_{G(\mathbb{R})}\pi(g)\otimes H_t^\nu(g)\,dg. \tag{9.8}$$

Then relative to the splitting

$$\mathcal{H}_\pi\otimes V_\nu = (\mathcal{H}_\pi\otimes V_\nu)^{K_\infty}\oplus\left((\mathcal{H}_\pi\otimes V_\nu)^{K_\infty}\right)^\perp,$$

$\tilde{\pi}(H_t^\nu)$ has the form

$$\begin{pmatrix}\pi(H_t^\nu) & 0\\ 0 & 0\end{pmatrix},$$

where $\pi(H_t^\nu)$ acts on $(\mathcal{H}_\pi\otimes V_\nu)^{K_\infty}$. Assume that π is irreducible. Let $\pi(\Omega)$ be the Casimir eigenvalue of π. Then as in [BM, Corollary 2.2] it follows from (9.5) that

$$\pi(H_t^\nu) = e^{t(\pi(\Omega)-\nu(\Omega_{K_\infty}))}\operatorname{Id}, \tag{9.9}$$

where Id is the identity on $(\mathcal{H}_\pi\otimes V_\nu)^{K_\infty}$. Put

$$h_t^\nu(g) := \operatorname{tr} H_t^\nu(g), \quad g\in G(\mathbb{R}). \tag{9.10}$$

Then $h_t^\nu\in\mathscr{C}^q(G(\mathbb{R}))$ for all $q>0$. In particular, h_t^ν belongs to $\mathscr{C}^2(G(\mathbb{R}))$, which equals Harish-Chandra's Schwartz space $\mathcal{C}(G(\mathbb{R}))$. Let π be a unitary representation of $G(\mathbb{R})$. Put

$$\pi(h_t^\nu) = \int_{G(\mathbb{R})} h_t^\nu(g)\pi(g)\,dg.$$

Assume that $\pi(H_t^\nu)$ is a trace class operator. Then it follows as in [BM, Lemma 3.3] that $\pi(h_t^\nu)$ is a trace class operator and

$$\operatorname{Tr}\pi(h_t^\nu)=\operatorname{Tr}\pi(H_t^\nu). \tag{9.11}$$

Now assume that π is a unitary admissible representation. Let $A:\mathcal{H}_\pi\to\mathcal{H}_\pi$ be a bounded operator which is an intertwining operator for $\pi|_K$. Then $A\circ\pi(h_t^\nu)$ is again a finite rank operator. Define an operator $\tilde{A}$ on $\mathcal{H}_\pi\otimes V_\nu$ by $\tilde{A}:=A\otimes\operatorname{Id}$. Then by the same argument as in [BM, Lemma 5.1] one has

$$\operatorname{Tr}\left(\tilde{A}\circ\tilde{\pi}(H_t^\nu)\right)=\operatorname{Tr}\left(A\circ\pi(h_t^\nu)\right). \tag{9.12}$$

Together with (9.9) we obtain

$$\operatorname{Tr}\left(A\circ\pi(h_t^\nu)\right)=e^{t(\pi(\Omega)-\nu(\Omega_{K_\infty}))}\operatorname{Tr}\left(\tilde{A}|_{(\mathcal{H}_\pi\otimes V_\nu)^K}\right). \tag{9.13}$$

10. Heat kernel estimates

Let the notation be as in the previous section. In this section we prove some estimations for the function h_t^ν defined by (9.10). Let $\widetilde{K}^\nu(t,x,y)$ be the kernel of $e^{-t\widetilde{\Delta}_\nu}$. Observe that $\widetilde{K}^\nu(t,x,y)\in\operatorname{Hom}((\widetilde{E}_\nu)_y,(\widetilde{E}_\nu)_x)$. Denote by $|\widetilde{K}^\nu(t,x,y)|$ the norm of this homomorphism. Furthermore, let $r(x,y)$ denote the geodesic distance of $x,y\in\widetilde{X}$.

Proposition 10.1. *Let $d=\dim\widetilde{X}$. For every $T>0$ there exists $C>0$ such that we have*

$$|\widetilde{K}^\nu(t,x,y)|\le Ct^{-d/2}exp\left(-\frac{r^2(x,y)}{4t}\right)$$

for all $0<t\le T$ and $x,y\in\widetilde{X}$.

Proof. If ν is irreducible, this is proved in [Mu1, Proposition 3.2]. However, the proof does not make any use of the irreducibility of ν. So it extends without any change to the case of finite-dimensional representations. □

Let $x_0:=eK_\infty\in\widetilde{X}$ be the base point. For $g\in G(\mathbb{R})$ and $x\in\widetilde{X}$ let $L_g\colon\widetilde{E}_x\to\widetilde{E}_{gx}$ be the isomorphism induced by the left translation. The kernel $\widetilde{K}^\nu$ is related to the convolution kernel $H_t^\nu\colon G(\mathbb{R})\to\operatorname{End}(V_\nu)$ by

$$H_t^\nu(g_1^{-1}g_2)=L_{g_1}^{-1}\circ\widetilde{K}^\nu(t,g_1x_0,g_2x_0)\circ L_{g_2},\quad g_1,g_2\in G(\mathbb{R}). \tag{10.1}$$

Thus we get

$$h_t^\nu(g):=\operatorname{tr}H_t^\nu(g)=\operatorname{tr}(\widetilde{K}^\nu(t,x_0,gx_0)\circ L_g),\quad g\in G(\mathbb{R}). \tag{10.2}$$

Using Proposition 10.1 and the fact that L_g is an isometry, we obtain the following corollary.

Corollary 10.2. *Let $d=\dim\widetilde{X}$. For all $T>0$ there exists $C>0$ such that we have*

$$|h_t^\nu(g)|\leq Ct^{-d/2}\exp\left(-\frac{r^2(gx_0,x_0)}{4t}\right)$$

for all $0<t\leq T$ and $g\in G(\mathbb{R})$.

Next we turn to the asymptotic expansion of the heat kernel. For $x_0\in\widetilde{X}$ and $\mathbf{x}\in T_{x_0}\widetilde{X}$ let $d_{\mathbf{x}}\exp_{x_0}$ be the differential of the exponential map $\exp_{x_0}\colon T_{x_0}\widetilde{X}\to\widetilde{X}$ at the point $\mathbf{x}$. It is a map from $T_{x_0}\widetilde{X}$ to $T_x\widetilde{X}$, where $x=\exp_{x_0}(\mathbf{x})$. Let

$$j_{x_0}(\mathbf{x}):=|\det(d_{\mathbf{x}}\exp_{x_0})| \tag{10.3}$$

be the Jacobian, taken with respect to the inner products in the tangent spaces. We use $\exp_{x_0}$ to introduce normal coordinates centered at x_0. Let $g_{ij}(\mathbf{x})$ denote the components of the metric tensor in these coordinates. Then on has

$$j_{x_0}(\mathbf{x})=|\det(g_{ij}(\mathbf{x}))|^{1/2} \tag{10.4}$$

(see [BGV, (1.22)]). Given $y\in\widetilde{X}$ and $\mathbf{x}\in T_y\widetilde{X}$, let $x=\exp_y(\mathbf{x})$. Put

$$j(x;y):=j_y(\mathbf{x}). \tag{10.5}$$

Let $\varepsilon>0$ be sufficiently small. Let $\psi\in C^\infty(\mathbb{R})$ with $\psi(u)=1$ for $u<\varepsilon$ and $\psi(u)=0$ for $u>2\varepsilon$.

Proposition 10.3. *Let $d=\dim\widetilde{X}$. Let (ν,V_ν) be a finite-dimensional unitary representation of K_∞. There exist smooth sections $\Phi_i^\nu\in C^\infty(\widetilde{X}\times\widetilde{X},\widetilde{E}_\nu\boxtimes\widetilde{E}_\nu^*)$, $i\in\mathbb{N}_0$, such that for every $N\in\mathbb{N}$*

$$\widetilde{K}^\nu(t,x,y)=(4\pi t)^{-d/2}\psi(d(x,y))\exp\left(-\frac{r^2(x,y)}{4t}\right)\sum_{i=0}^N\Phi_i^\nu(x,y)j(x;y)^{-1/2}t^i + O(t^{N+1-d/2}), \tag{10.6}$$

uniformly for $0<t\leq 1$. Moreover the leading term $\Phi_0^\nu(x,y)$ is equal to the parallel transport $\tau(x,y)\colon(\widetilde{E}_\nu)_y\to(\widetilde{E}_\nu)_x$ with respect to the connection ∇^ν along the unique geodesic joining x and y.

Proof. Let $\Gamma\subset G$ be a co-compact torsion free lattice. It exists by [Bo]. Let $X=\Gamma\backslash\widetilde{X}$ and $E_\nu=\Gamma\backslash\widetilde{E}_\nu$. As in [Do, Sect 3], the proof can be reduced to the compact case, which follows from [BGV, Theorem 2.30]. □

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Iwasawa decomposition. We recall that the mapping

$$\varphi\colon\mathfrak{p}\times K_\infty\to G(\mathbb{R}),$$

defined by $\varphi(Y,k)=\exp(Y)\cdot k$ is a diffeomorphism [He, Ch. VI, Theorem 1.1]. Thus each $g\in G(\mathbb{R})$ can be uniquely written as

$$g=\exp(Y(g))\cdot k(g),\quad Y(g)\in\mathfrak{p},\ k(g)\in K_\infty. \tag{10.7}$$

Using (10.2) and Proposition 10.3, we obtain the following corollary.

Corollary 10.4. *There exist* $a_i^\nu \in C^\infty(G(\mathbb{R}))$, $i \in \mathbb{N}_0$, *such that for every* $N \in \mathbb{N}$ *we have*

$$h_t^\nu(g) = (4\pi t)^{-d/2}\psi(d(gx_0, x_0)) \exp\left(-\frac{r^2(gx_0,x_0)}{4t}\right)\sum_{i=0}^{N} a_i^\nu(g)t^i + O(t^{N+1-d/2}) \tag{10.8}$$

which holds for $0 < t \leq 1$. *Moreover the leading coefficient* a_0^ν *is given by*

$$a_0^\nu(g) = \operatorname{tr}(\nu(k(g))) \cdot j(x_0, gx_0)^{-1/2}. \tag{10.9}$$

Proof. By (10.1) we have

$$H_t^\nu(g) = \widetilde{K}^\nu(t, x_0, gx_0) \circ L_g, \quad g \in G(\mathbb{R}).$$

Put

$$a_i^\nu(g) := \operatorname{tr}(\Phi_i^\nu(x_0, gx_0) \circ L_g) \cdot j(x_0, gx_0)^{-1/2}, \quad g \in G(\mathbb{R}). \tag{10.10}$$

Then (10.8) follows immediately from (10.6) and the definition of h_t^ν. To prove the second statement, we recall that $\Phi_0^\nu(x,y)$ is the parallel transport $\tau(x,y)$ with respect to the canonical connection of $\widetilde{E}_\nu$ along the geodesic connecting x and y. Let $g = \exp(Y) \cdot k$, $Y \in \mathfrak{p}$, $k \in K_\infty$. Then the geodesic connecting x_0 and gx_0 is the curve $\gamma(t) = \exp(tY)x_0$, $t \in [0,1]$ (see [He, Ch. IV, Theorem 3.3]). The parallel transport along $\gamma(t)$ equals $L_{\exp(Y)}$. Thus $\Phi_0^\nu(x_0, gx_0) = L_{\exp(Y)}^{-1}$. Hence we get.

$$\Phi_0^\nu(x_0, gx_0) \circ L_g = L_k = \nu(k).$$

Together with (10.10) the claim follows. □

11. Regularized traces

Let G be a reductive algebraic group over $\mathbb{Q}$. For simplicity, we assume that the center Z_G splits over $\mathbb{Q}$, i.e., we have $Z_G = A_G$. Let

$$G(\mathbb{R})^1 = G(\mathbb{A})^1 \cap G(\mathbb{R}).$$

Then $G(\mathbb{R})^1$ is semisimple and

$$G(\mathbb{R}) = G(\mathbb{R})^1 \cdot A_G(\mathbb{R})^0. \tag{11.1}$$

Let $K_\infty \subset G(\mathbb{R})^1$ be a maximal compact subgroup and let K_∞^0 be the connected component of the identity. Let

$$\widetilde{X} = G(\mathbb{R})^1/K_\infty^0$$

be the associated Riemannian symmetric space. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. For simplicity we assume that Γ is torsion free. Note that $\Gamma \subset G(\mathbb{R})^1$. Let $X = \Gamma\backslash\widetilde{X}$ be the associated locally symmetric manifold. Let $\nu \colon K_\infty \to \mathrm{GL}(V_\nu)$ an irreducible unitary representation of K_∞ and let $E_\nu \to X$ be the associated locally homogeneous vector bundle over X. Let Δ_ν be the corresponding Bochner-Laplace operator acting in $C^\infty(X, E_\nu)$. Our goal is to define a regularized trace of the heat operator $e^{-t\Delta_\nu}$.

Recall that $e^{-t\Delta_\nu}$ is an integral operator with a smooth kernel $K_\nu(t,x,y)$. By definition, $K_\nu(t,x,y) \in \operatorname{Hom}((E_\sigma)_y,(E_\sigma)_x)$. Especially, $K(t,x,x)$ is an endomorphism of $(E_\sigma)_x$. Let $\operatorname{tr} K_\nu(t,x,x)$ be the trace of this endomorphism. If X is compact, then we have

$$\operatorname{Tr}\left(e^{-t\Delta_\nu}\right) = \int_X \operatorname{tr} K_\nu(t,x,x)\,dx. \tag{11.2}$$

To begin with we rewrite (11.2). Let $\widetilde{\Delta}_\nu$ be the Bochner-Laplace operator on the universal covering $\widetilde{X}$. Let $H_t^\nu\colon G(\mathbb{R})^1 \to \operatorname{End}(V_\nu)$ be the convolution kernel of $e^{-t\widetilde{\Delta}_\nu}$ and

$$h_t^\nu(g) := \operatorname{tr} H_t^\nu(g). \tag{11.3}$$

Assume that $\Gamma\backslash G(\mathbb{R})^1$ is compact. Then one has

$$\operatorname{Tr}\left(e^{-t\Delta_\nu}\right) = \int\limits_{\Gamma\backslash G(\mathbb{R})^1} \sum_{\gamma\in\Gamma} h_t^\nu(g^{-1}\gamma g)\,dg. \tag{11.4}$$

For the proof see [MP2, (3.13)]. Let R_Γ denote the right regular representation of $G(\mathbb{R})^1$ on $L^2(\Gamma\backslash G(\mathbb{R})^1)$. Recall that for any $f \in C_c^\infty(G(\mathbb{R})^1)$, $R_\Gamma(f)$ is the integral operator with kernel $\sum_{\gamma\in\Gamma} f(g_1^{-1}\gamma g_2)$. Thus the right hand side of (11.4) equals $\operatorname{Tr} R_\Gamma(h_t^\nu)$ and (11.2) can be rewritten as

$$\operatorname{Tr}\left(e^{-t\Delta_\nu}\right) = \operatorname{Tr} R_\Gamma(h_t^\nu). \tag{11.5}$$

If X is not compact, we choose an appropriate height function h on X, that is, a function which measures how far out in the cusp a point is. For $Y > 0$ let $X_Y = \{x \in X\colon h(x) \leq Y\}$. If $X = \Gamma\backslash\tilde{X}$ with $\Gamma \subseteq G(\mathbb{Q})$ a congruence subgroup, there is a canonical choice of height function: Recall the function $H_0 = H_{P_0} : G(\mathbb{R})^1 \longrightarrow \mathfrak{a}_0$ and fix a norm $\|\cdot\|$ on $\mathfrak{a}_0$. Then $X \ni x \mapsto \max_{\gamma\in\Gamma} \|H_0(\gamma x)\|$ defines a height function on X. Assume that X_Y is compact. Then the integral

$$\int_{X_Y} \operatorname{tr} K_\nu(t,x,x)\,dx \tag{11.6}$$

is well defined. Suppose that this integral has an asymptotic expansion in Y. Then it is natural to define the regularized trace $\operatorname{Tr}_{\text{reg}}(e^{-t\Delta_\nu})$ as the finite part of the integral as $Y \to \infty$. For hyperbolic manifolds this has been carried out in [MP1]. The regularized trace defined in this way depends of course on the choice of the height function. To choose the height function, we pass to the adelic setting. In fact, we will not define it explicitly. Instead we use directly the truncated manifold.

Let $K_f \subset G(\mathbb{A}_f)$ be a neat open compact subgroup. Let $X(K_f)$ be the arithmetic manifold defined by (3.21) and let $E_\nu \to X(K_f)$ be the locally homogeneous vector bundle defined by (3.26). By (3.19) we have

$$L^2(X(K_f),E_\nu) = \bigoplus_{i=1}^{l} L^2(\Gamma_i\backslash\widetilde{X},E_{i,\nu}).$$

Using (11.1) we extend h_t^ν to a C^∞-function on $G(\mathbb{R})$ by

$$\widetilde{h}_t^\nu(g_\infty z) = h_t^\nu(g_\infty), \quad g_\infty \in G(\mathbb{R})^1, \ z \in A_G(\mathbb{R})^0. \tag{11.7}$$

Let $\mathbf{1}_{K_f}$ denote the characteristic function of K_f in $G(\mathbb{A}_f)$. We normalize the characteristic function by

$$\chi_{K_f} := \frac{\mathbf{1}_{K_f}}{\operatorname{vol}(K_f)}. \tag{11.8}$$

Define $\phi_t^\nu \in C^\infty(G(\mathbb{A}))$ by

$$\phi_t^\nu(g_\infty g_f) = \widetilde{h}_t^\nu(g_\infty)\chi_{K_f}(g_f) \tag{11.9}$$

for $g_\infty \in G(\mathbb{R})$, $g_f \in G(\mathbb{A}_f)$. Let R denote the right regular representation of $G(\mathbb{A})$ on $L^2(A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))$, and let Π_{K_f} denote the orthogonal projection of $L^2(A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))$ onto $L^2(A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f}$. Using the isomorphism (3.19) of $G(\mathbb{R})$-modules, it follows that

$$R(\phi_t^\nu) = \left[\bigoplus_{i=1}^{l} R_{\Gamma_i}(h_t^\nu)\right] \circ \Pi_{K_f}. \tag{11.10}$$

Let $\mathcal{C}(G(\mathbb{R})^1)$ be Harish-Chandra's Schwartz space (see [Wal, 7.1.2]). As explained in section 9, we have $h_t^\nu \in \mathcal{C}(G(\mathbb{R})^1)$ for all $t > 0$. This implies $\phi_t^\nu \in \mathcal{C}(G(\mathbb{A}), K_f)$. Thus by [FL1, Theorem 7.1], $J^T(\phi_t^\nu)$ is defined for all $T \in \mathfrak{a}_0$. Let $G(\mathbb{A})^1_{\le T}$ be defined by (4.34) and for $T \in \mathfrak{a}_0$ let $d(T)$ be defined by (4.35). Let $C \subset \mathfrak{a}_0^+$ be a positive cone for which there exists $c > 0$ such that

$$d(T) \ge c\|T\|, \quad \text{for all } T \in C.$$

Then by Theorem 4.1 it follows that for every $t > 0$ we have

$$\int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1_{\le T}} \sum_{\gamma \in G(\mathbb{Q})} \phi_t^\nu(x^{-1}\gamma x)\, dx = J^T(\phi_t^\nu) + O\left(e^{-\|T\|/2}\right) \tag{11.11}$$

for all $T \in C$ with $\|T\| > d_0/c$, where $d_0 > 0$ is as in Theorem 4.1 and the implied constant in the remainder term depends on t. Now by [FL1, Theorem 7.1], $J^T(\phi_t^\nu)$ is a polynomial in T. Therefore (11.11) suggests to define the regularized trace of $e^{-t\Delta_\nu}$ as the constant term of this polynomial. In order to relate it to the trace formula, we need an additional assumption. Let $T_0 \in \mathfrak{a}_0$ be the unique point determined by [Ar3, Lemma 1.1]. Then the distribution J_{geo} is defined by

$$J_{\text{geo}}(f) := J^{T_0}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1).$$

and by [FL1, Theorem 7.1], J_{geo} extends continuously to $\mathcal{C}(G(\mathbb{A})^1)$. As proved in [Ar3, p. 19], J_{geo} depends only on the choice of a maximal compact subgroup K of $G(\mathbb{A})$ and M_0, but is independent of the choice of the minimal parabolic subgroup P_0 with Levi component M_0. Let W_0 be the Weyl group of (G, A_0). For $s \in W_0$ let $w_s \in G(\mathbb{Q})$ be a representative of s. As shown in [Ar3, p. 10], w_s belongs to $KM_0(\mathbb{A})$ for all $s \in W_0$. Now assume that each $s \in W_0$ has a representative in $G(\mathbb{Q}) \cap K$. Then it follows from [Ar3, Lemma 1.1] that

$sT_0 = T_0$ for all $s \in W_0$, and therefore $T_0 = 0$. Thus in this case, the constant term of $J^T(f)$ equals $J_{\text{geo}}(f)$. Let $G = \mathrm{GL}(n)$. Then A_0 consists of diagonal matrices and W_0 is equal to the symmetric group S_n which acts by permutations. A permutation matrix P_π is a $n \times n$-matrix where in row i the entry $\pi(i)$ is equal to 1 and all other entries are equal to 0. Such a matrix belongs to $\mathrm{GL}(n, \mathbb{Q}) \cap K$. The case $G = \mathrm{SL}(n)$ is similar. Thus for $G = \mathrm{GL}(n)$ and $G = \mathrm{SL}(n)$ the constant term of $J^T(f)$ equals $J_{\text{geo}}(f)$ and the above discussion suggests to define the regularized trace to be $J_{\text{geo}}(\phi_t^\nu)$. In general $T_0 \neq 0$ and therefore, $J_{\text{geo}}(\phi_t^\nu)$ is not the constant term of the polynomial $J_{\text{geo}}^T(\phi_t^\nu)$. Nevertheless we choose $J_{\text{geo}}(\phi_t^\nu)$ as the definition of the regularized trace in general, because of its independence on the choice of the minimal parabolic subgroup P_0.

Definition 11.1. *Let G be a reductive algebraic group over $\mathbb{Q}$ with $\mathbb{Q}$-split center. The regularized trace of $e^{-t\Delta_\nu}$ is defined to be*

$$\mathrm{Tr}_{\text{reg}}\left(e^{-t\Delta_\nu}\right) := J_{\text{geo}}(\phi_t^\nu).$$

12. The asymptotic expansion of the regularized trace for GL(n) and SL(n)

It is well known that on a compact manifold, the trace of the heat operator $\mathrm{Tr}\left(e^{-t\Delta}\right)$ of a generalized Laplacian Δ admits an asymptotic expansion as $t \to 0$ (see [BGV]). We wish to establish a similar result for the regularized trace $\mathrm{Tr}_{\text{reg}}(e^{-\Delta_\nu})$ introduced in section 11. For technical reasons we have to restrict to the group $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$.

12.1. **Auxiliary results.** We fix an open compact subgroup $K_f \subset G(\mathbb{A}_f)$. Let $\phi_t^\nu \in C^\infty(G(\mathbb{A})^1)$ be defined by (11.9). To begin with we replace ϕ_t^ν by a compactly supported function. Let $0 < a < b$. Let $h \in C^\infty(\mathbb{R})$ such that $h(u) = 1$ if $|u| \leq a$, and $h(u) = 0$, if $|u| \geq b$. Let $d(x, y)$ denote the geodesic distance of $x, y \in \widetilde{X}$. Put

$$r(g_\infty) := d(g_\infty K_\infty, K_\infty).$$

Let $\varphi \in C_c^\infty(G(\mathbb{R}))$ be defined by

$$\varphi(g_\infty) := h(r(g_\infty)).$$

Define $\tilde{\phi}_t^\nu \in C^\infty(G(\mathbb{A}))$ by

$$\tilde{\phi}_t^\nu(g_\infty g_f) := \varphi(g_\infty) h_t^\nu(g_\infty) \chi_{K_f}(g_f). \tag{12.1}$$

for $g_\infty \in G(\mathbb{R})$ and $g_f \in G(\mathbb{A}_f)$. Then the restriction of $\tilde{\phi}_t^\nu$ to $G(\mathbb{A})^1$ belongs to $C_c^\infty(G(\mathbb{A})^1)$.

Proposition 12.1. *There exist $C, c > 0$ such that*

$$|J_{\text{geo}}(\phi_t^\nu) - J_{\text{geo}}(\tilde{\phi}_t^\nu)| \leq C e^{-c/t}$$

for $0 < t \leq 1$.

Proof. Let $J_{\text{spec}}(\Phi)$, $\Phi \in C_c^\infty(G(\mathbb{A})^1)$, be the spectral side of the trace formula. By [FLM1], $J_{\text{spec}}(\Phi)$ converges absolutely for $\Phi \in \mathcal{C}(G(\mathbb{A})^1; K_f)$ and by the trace formula we have

$$J_{\text{geo}}(\Phi) = J_{\text{spec}}(\Phi), \quad \Phi \in \mathcal{C}(G(\mathbb{A})^1; K_f). \tag{12.2}$$

Put $\psi_t^\nu := \phi_t^\nu - \tilde{\phi}_t^\nu$ and $f := 1 - \varphi$. Let Ω (resp. Ω_{K_∞}) denote the Casimir operator of $G(\mathbb{R})$ (resp. K_∞). Let

$$\Delta_G = -\Omega + 2\Omega_{K_\infty}.$$

By the proof of Theorem 3 of [FLM1] (see [FLM1, Sect. 5]), it follows that there exists $k \in \mathbb{N}$ such that

$$|J_{\text{spec}}(\phi_t^\nu) - J_{\text{spec}}(\tilde{\phi}_t^\nu)| = |J_{\text{spec}}(\psi_t^\nu)| \leq C \|(\mathrm{Id} + \Delta_G)^k(\psi_t^\nu)\|_{L^1(G(\mathbb{A})^1)}$$

for some $C > 0$. Now note that by definition

$$\psi_t^\nu(g_\infty g_f) = f(g_\infty) h_t^\nu(g_\infty) \chi_{K_f}(g_f).$$

Hence

$$\|(\mathrm{Id} + \Delta_G)^k(\psi_t^\nu)\|_{L^1(G(\mathbb{A})^1)} = \mathrm{vol}(K_f) \|(\mathrm{Id} + \Delta_G)^k(f h_t^\nu)\|_{L^1(G(\mathbb{R})^1)}.$$

Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$ and let $Y_1, \dots, Y_r$ be an orthonormal basis of $\mathfrak{g}$. Then $\Delta_G = -\sum_i Y_i^2$. Denote by ∇ the canonical connection on $G(\mathbb{R})$. Then it follows that there exists $C_1 > 0$ such that

$$|(\mathrm{Id} + \Delta_G)^k F(g)| \leq C \sum_{l=0}^k \|\nabla^l F(g)\|, \quad g \in G(\mathbb{R}),$$

for all $F \in C^\infty(G(\mathbb{R}))$. Let $m = \dim G(\mathbb{R})$. and $j \in \mathbb{N}$. By [Mu1, Proposition 2.1], for every $j \in \mathbb{N}$ there exist $C_2, c > 0$ such that

$$\|\nabla^j h_t^\nu(g)\| \leq C_2 t^{-(m+j)/2} e^{-cr^2(g)/t}, \quad g \in G(\mathbb{R}), \tag{12.3}$$

for all $0 < t \leq 1$. Since f vanishes in neighborhood of $1 \in G(\mathbb{R})$, it follows that there exist $C_4, c_4, c_5 > 0$ such that

$$\sum_{l=0}^k \|\nabla^l (f h_t^\nu)(g)\| \leq C_4 e^{-c_4/t} e^{-c_5 r^2(g)}$$

for all $g \in G(\mathbb{R})$ and $0 < t \leq 1$. Since $G(\mathbb{R}) \ni g \mapsto e^{-c_5 r^2(g)}$ is integrable on $G(\mathbb{R})$, we obtain

$$\|(\mathrm{Id} + \Delta_G)^k (f h_t^\nu)\|_{L^1(G(\mathbb{R})^1)} \leq C_5 e^{-c_4/t}$$

for some constant $C_5 > 0$, which completes the proof. □

Let $K(N) \subset G(\mathbb{A}_f)$ be the principal congruence subgroup of level $N \in \mathbb{N}$. From now on we assume that K_f is contained in $K(N)$ for some $N \geq 3$. By Proposition 12.1 it suffices to show that $J_{\text{geo}}(\widetilde{\phi_t^\nu})$ admits an asymptotic expansion as $t \to 0$. Now by our assumption on K_f it follows that if the support of f is a sufficiently small neighborhood of 0, then by (6.2) we have

$$J_{\text{geo}}(\widetilde{\phi_t^\nu}) = J_{\text{unip}}(\widetilde{\phi_t^\nu}).$$

Now we use the fine geometric expansion (6.4) by which we express $J_{\mathrm{unip}}(\widetilde{\phi}_t^\nu)$ in terms of a finite linear combination of weighted orbital integrals. Next we use (6.5). Since at the finite places our test function is fixed, we are reduced to the consideration of real weighted orbital integrals in some Levi subgroup of G. Since the weighted orbital integrals J_M^L for $L \subseteq G$ a semistandard Levi subgroup can be treated analogously to the case of $L = G$ (they can be reduced to weighted orbital integrals of the form $J_{M'}^{\mathrm{GL}(m)}$ or $J_{M'}^{\mathrm{SL}(m)}$ for suitable m and M'), it suffices to consider the weighted orbital integrals for $L = G$. Now observe that the kernel $H_t^\nu \colon G(\mathbb{R})^1 \to \mathrm{End}(V_\sigma)$ of $e^{-t\widetilde{\Delta}_\nu}$ satisfies

$$H_t^\nu(k^{-1}gk') = \nu(k)^{-1} \circ H_t^\nu(g) \circ \nu(k'), \quad \forall k, k' \in K, \forall\, g \in G(\mathbb{R})^1.$$

(see [MP4, §3]). Therefore the function $h_t^\nu = \operatorname{tr} H_t^\nu$ is invariant under conjugation by $k_\infty \in K_\infty$. Define F_t^ν by

$$F_t^\nu(g) = \varphi(g) h_t^\nu(g), \quad g \in G(\mathbb{R}).$$

Then by (7.2) the orbital integral that we need to consider is given by

$$J_M^G(U, F_t^\nu) = c \int_{N(\mathbb{R})} F_t^\nu(n) w_M(n)\, dn, \tag{12.4}$$

where $M \in \mathcal{L}$, $U \in (\mathcal{U}_M(\mathbb{R}))_{M(\mathbb{R})}$, $w_M(n) := w_{M,U}(n)$ is the weight function described in section 7 and N is the unipotent radical of some parabolic subgroup $Q \in \mathcal{F}$. Our goal is to determine the asymptotic behavior of the integral on the right hand side as $t \to 0^+$. To study this integral, we identify $N(\mathbb{R})$ with its Lie algebra $\mathfrak{n}$ via the map $n \in N(\mathbb{R}) \mapsto n - \mathrm{Id}$. Furthermore $\mathfrak{n} \cong \mathbb{R}^k$ for some $k \in \mathbb{N}$. Let

$$x \in \mathbb{R}^k \mapsto n(x) \in N(\mathbb{R})$$

be the inverse map. With respect to the isomorphism the invariant measure dn is identified with Lebesgue measure dx in $\mathbb{R}^k$. Thus (12.4) equals

$$\int_{\mathbb{R}^k} F_t^\nu(n(x)) w_M(n(x))\, dx. \tag{12.5}$$

To determine the asymptotic behavior of this integral as $t \to 0^+$, we will use the asymptotic expansion (10.8) of h_t^ν. To this end we need to estimate the function

$$r(x) := r(n(x)x_0, x_0), \quad x \in \mathbb{R}^k, \tag{12.6}$$

where $x_0 = eK_\infty \in \widetilde{X}$. Note that $\widetilde{X}$ is a Hadamard manifold of nonpositive curvature and the orbit $N(\mathbb{R})x_0$ is a horosphere in $\widetilde{X}$. Then it follows by [HH, Theorem 4.6] that there exist constants $C, c > 0$ such that

$$r(x) \ge C \operatorname{arcsinh}(c\|x\|), \quad x \in \mathbb{R}^k. \tag{12.7}$$

Now note that

$$\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right).$$

Thus we get

$$r(x) \ge C \ln\left(1 + \|x\|\right), \quad x \in \mathbb{R}^k. \tag{12.8}$$

We also need the Taylor expansion of $r(x)^2$ at $x = 0$. This is described by the following lemma.

Lemma 12.2. *We have*

$$r(x)^2 = \frac{1}{4}\|x\|^2 + O(\|x\|^3)$$

as $x \to 0$.

Proof. Let $H = (H_1, \ldots, H_n) \in \mathfrak{a}$, $H_1 + \ldots, H_n = 0$, with $n(x) \in K_\infty e^H K_\infty$. Now note that $r(e^H x_0, x_0) = \|H\|$ (see [BH, Corollary 10.42]). Thus it follows that $r(x)^2 = \|H\|^2$. Moreover,

$$n + \|x\|^2 = \operatorname{tr}\left(n(x)^t n(x)\right) = \operatorname{tr} e^{2H} = n + 4\|H\|^2 + O(\|H\|^3).$$

If $x \to 0$, then also $H \to 0$ so that this equation implies $\|H\|^3 = O(\|x\|^3)$ for small x. Hence $r(x)^2 = \|H\|^2 = \frac{1}{4}\|x\|^2 + O(\|x\|^3)$ as $x \to 0$. $\square$

12.2. **Asymptotics for $t \to 0$.** We can now turn to the estimation of the weighted orbital integral (12.5). For $\varepsilon > 0$ let $B(\varepsilon) \subset \mathbb{R}^k$ denote the ball of radius ε centered at the origin and let $U(\varepsilon) = \mathbb{R}^k \setminus B(\varepsilon)$. Let ψ be the function occurring in (10.8). Choose $\varepsilon > 0$ so small such that $\varphi(n(x)) = 1$ for $x \in B(\varepsilon)$ and $\operatorname{supp} \psi(n(\cdot)) \subset B(\varepsilon)$. Let $0 < t \leq 1$. By Corollary 10.2 we have

$$\left| \int_{U(\varepsilon)} \varphi(n(x)) h_t^\nu(n(x)) w_M(n(x))\, dx \right| \leq C t^{-d/2} \int_{U(\varepsilon)} \exp\left(-\frac{r^2(x)}{4t}\right) |w_M(n(x))|\, dx$$

for some absolute constant $C > 0$. Using the lower bound 12.8 for $r(x)$, and the result on the weight function from Proposition 7.1 we can find $c, C_1, C_2 > 0$ such that

$$\begin{aligned} t^{-d/2} \int_{U(\varepsilon)} \exp\left(-\frac{r^2(x)}{4t}\right) |w_M(n(x))|\, dx \\ \leq C_1 \exp\left(-\frac{c(\varepsilon)}{t}\right) \int_{\mathbb{R}^k} \exp\left(-c\left(\log(1 + \|x\|)\right)^2\right) \lambda(x)\, dx \end{aligned}$$

where $c(\varepsilon) = C_2 \log(1 + \varepsilon)$ and $\lambda : \mathbb{R}^k \longrightarrow \mathbb{C}$ is a function of the form (7.13). By (7.14) of Lemma 7.7 the last integral is bounded by a constant so that we finally obtain

$$\left| \int_{U(\varepsilon)} \varphi(n(x)) h_t^\nu(n(x)) w_M(n(x))\, dx \right| \leq C_3 \exp\left(-\frac{c(\varepsilon)}{t}\right) \tag{12.9}$$

for some absolute constant $C_3 > 0$. To deal with the integral over $B(\varepsilon)$, we use (10.8), which gives

$$\begin{aligned} \int_{B(\varepsilon)} h_t^\nu(n(x)) w_M(n(x))\, dx =& t^{-d/2} \sum_{i=0}^N t^i \int_{B(\varepsilon)} \exp\left(-\frac{r^2(x)}{4t}\right) a_i^\nu(x) w_M(n(x))\, dx \\ &+ O(t^{N+1-d/2}) \end{aligned} \tag{12.10}$$

for $0 < t \le 1$, where $a_i^\nu \in C_c^\infty(B(\varepsilon))$. Each of the integrals is of the form

$$\int_{B(\varepsilon)} \exp\left(-\frac{r^2(x)}{4t}\right) f(x) w_M(\mathrm{Id} + x)\, dx, \tag{12.11}$$

where $f \in C_c^\infty(B(\varepsilon))$. We expand $r^2(x)$ and $f(x)$ in their Taylor series at 0. Let $N \in \mathbb{N}$, $N \ge 3$. By Lemma 12.2 we have

$$r^2(x) = a\|x\|^2 + \psi_N(x), \quad \psi_N(x) = \sum_{3 \le |\alpha| \le N} a_\alpha x^\alpha + R_N(x), \tag{12.12}$$

where

$$R_N(x) = \sum_{|\alpha| = N+1} \frac{D^\alpha r^2(\theta x)}{\alpha!} x^\alpha \tag{12.13}$$

for $x \in B(\varepsilon)$ and some $0 \le \theta \le 1$. Now we change variables by $x \mapsto \sqrt{t} x$. Then (12.11) equals

$$t^{k/2} \int_{B(\varepsilon t^{-1/2})} \exp(-a\|x\|^2) \exp\left(-\frac{\psi_N(t^{1/2}x)}{t}\right) f(t^{1/2}x) w_M(1, \mathrm{Id} + t^{1/2}x) \tag{12.14}$$

Now observe that by (12.12) we have

$$t^{-1}\psi_N(t^{1/2}x) = \sum_{3 \le |\alpha| \le N} a_\alpha t^{|\alpha|/2 - 1} x^\alpha + t^{-1} R_N(t^{1/2}x)$$

and

$$t^{-1} R_N(t^{1/2}x) = t^{(N-1)/2} \sum_{|\alpha| = N+1} \frac{D^\alpha r^2(t^{1/2}\theta x)}{\alpha!} x^\alpha.$$

There is $C > 0$ such that

$$|D^\alpha r^2(t^{1/2}\theta x)| \le C$$

for all $x \in B(t^{-1/2}\varepsilon)$, $0 < t \le 1$, and $0 \le \theta \le 1$. Hence it follows that there is $C_1 > 0$ such that for all $0 < t \le 1$

$$|t^{-1} R_N(t^{1/2}x)| \le C_1 t^{(N-1)/2} \|x\|^{N+1}, \quad x \in B(t^{-1/2}\varepsilon).$$

Using the Taylor expansion of $\exp(u)$, we get for $n \ge 3$

$$\exp\left(-\frac{\psi_N(t^{1/2}x)}{t}\right) = \sum_{j=0}^{N} t^{j/2} p_j(x) + R_N(t, x), \tag{12.15}$$

where $p_j(x)$ is a polynomial of degree $\le N^2$ and the remainder term satisfies

$$|R_N(t, x)| \le C_2 t^{(N-1)/2} (1 + \|x\|)^{N^2} \tag{12.16}$$

for some constant $C_2 > 0$, $0 < t \le 1$ and $x \in B(t^{-1/2}\varepsilon)$. Similarly, using the Taylor expansion of $f(x)$ we get

$$f(t^{1/2}x) = \sum_{|\alpha| \le N} b_\alpha t^{|\alpha|/2} x^\alpha + Q_N(t, x) \tag{12.17}$$

with

$$|Q_N(t,x)| \leq C_3 t^{(N+1)/2}(1+\|x\|)^{N+1}$$

for $0 < t \leq 1$ and $x \in B(t^{-1/2}\varepsilon)$. Using (12.14), (12.15), (12.17), and Proposition 7.1, it follows that

$$\int_{B(\varepsilon)} \exp\left(-\frac{r^2(x)}{4t}\right) f(x) w_M(\mathrm{Id}+x)\, dx = t^{k/2} \sum_{j=0}^{N} \sum_{i=0}^{r_j} a_{ij}(t)(\log t)^i t^{j/2} + \phi_N(t), \tag{12.18}$$

where each $a_{ij}(t)$ is of the form

$$\int_{B(t^{-1/2}\varepsilon)} e^{-a\|x\|^2} p(x) \prod_{l=1}^{h} \big| \log |p_l(x)| \big|\, dx,$$

if $i < r_j$, or

$$\int_{B(t^{-1/2}\varepsilon)} e^{-a\|x\|^2} p(x)\, dx,$$

if $i = r_j$, with homogeneous polynomials $p(x)$, $p_1(x), ..., p_h(x)$. The fact that for $i = r_j$ no logarithm appears in the integral for $a_{ij}(t)$ can easily be seen by changing variables from x to $t^{1/2}x$ in the integral on the left hand side of (12.18) and collecting all $\log t$ terms coming from the weight function.

Finally, $\phi_N(t)$ satisfies

$$|\phi_N(t)| \leq C t^{(N+k+1)/2} \int_{B(t^{-1/2}\varepsilon)} e^{-a\|x\|^2} (1+\|x\|)^{N^2} \prod_{i=1}^{m} \big\| \log |p_i(x)| \big\|\, dx.$$

Let $U(r) = \mathbb{R}^k \setminus B(r)$. Since $\log(1+\|x\|) \leq \|x\|$ for all x, it follows from Lemma 7.7 that

$$\Big| \int_{U(t^{-1/2}\varepsilon)} e^{-a\|x\|^2} p(x) \prod_{l=1}^{h} \big| \log |p_l(x)| \big|\, dx \Big| \leq C e^{-a\varepsilon^2/(2t)},$$

for $0 < t \leq 1$. Thus there are constants $c_{ij} \in \mathbb{R}$ and $c > 0$ such that

$$a_{ij}(t) = c_{ij} + O(e^{-c/t})$$

for $0 < t \leq 1$. By the considerations above, for each pair (i,j), $0 \leq i \leq r_j$, there exist homogeneous polynomials $p, p_1, \ldots, p_h$, such that

$$c_{ij} = \begin{cases} \int_{\mathbb{R}^k} e^{-a\|x\|^2} p(x) \prod_{l=1}^{h} \big| \log |p_l(x)| \big|\, dx, & \text{if } i < r_j, \\ \\ \int_{\mathbb{R}^k} e^{-a\|x\|^2} p(x)\, dx, & \text{if } i = r_j. \end{cases} \tag{12.19}$$

In the same way we get

$$|\phi_N(t)| \leq C t^{(N+k+1)/2}, \quad 0 < t \leq 1.$$

Putting everything together, we get

Proposition 12.3. *Let $M \in \mathcal{L}$, $M \neq G$. For every $N \in \mathbb{N}$, $N \geq 3$, there is an expansion*

$$J_M(u, \widetilde{\phi}_t^\nu) = t^{-(d-k)/2} \sum_{j=0}^{N} \sum_{i=0}^{r_j} c_{ij}(\nu) t^{j/2} (\log t)^i + O(t^{(N-d+k+1)/2}) \tag{12.20}$$

as $t \to 0^+$.

Now we come to the first term in (6.4), where $f = \widetilde{\phi}_t^\nu$. Then we have to determine the asymptotic behavior of $h_t^\nu(1)$ as $t \to +0$. Let $\Gamma' \subset G(\mathbb{R})$ be a cocompact torsion free lattice. Such a lattice exists by [Bo1]. Let $X' = \Gamma' \backslash \widetilde{X}$ and $E'_\nu \to X'$ the locally homogeneous vector bundle associated to ν. Let $\Delta_{X',\nu}$ be the corresponding Bochner-Laplace operator. The kernel of $e^{-t\Delta_{X',\nu}}$, regarded as operator in $L^2(\Gamma \backslash G(\mathbb{R}), \nu)$, is given by

$$K^\nu(t, g_1, g_2) := \sum_{\gamma \in \Gamma'} H_t^\nu(g_1^{-1} \gamma g_2).$$

Hence

$$\begin{aligned} \operatorname{Tr}\left(e^{-t\Delta_{X',\nu}}\right) &= \int_{\Gamma \backslash G(\mathbb{R})} \operatorname{tr} K^\nu(t, g, g)\, dg = \int_{\Gamma \backslash G(\mathbb{R})} \sum_{\gamma \in \Gamma'} h_t^\nu(g^{-1} \gamma g)\, dg \\ &= \operatorname{vol}(\Gamma' \backslash G(\mathbb{R})) h_t^\nu(1) + \int_{\Gamma' \backslash G(\mathbb{R})} \sum_{\gamma \in \Gamma' \backslash \{1\}} h_t^\nu(g^{-1} \gamma g)\, dg. \end{aligned}$$

As in [MP2, (5.10)], the last term on the right can be estimated by $C_1 e^{-c_1/t}$ for $0 < t \leq 1$ and some constants $C_1, c_1 > 0$. Thus we get

$$h_t^\nu(1) = \frac{1}{\operatorname{vol}(\Gamma' \backslash G(\mathbb{R})} \operatorname{Tr}\left(e^{-t\Delta_{X',\nu}}\right) + O(e^{-c_1/t})$$

for $0 < t \leq 1$. Now the trace of the heat operator on a compact manifold has an asymptotic expansion as $t \to +0$ (see [Gi]). Hence, it follows that there is an asymptotic expansion

$$h_t^\nu(1) \sim \sum_{j=0}^{\infty} a_j t^{-d/2+j}$$

as $t \to +0$. Combined with Proposition 12.3 we obtain Theorem 1.1.

13. The analytic torsion

In this section we assume that $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$. We consider the case $G = \mathrm{SL}(n)$. The case $G = \mathrm{GL}(n)$ is similar. We choose $K_\infty = \mathrm{SO}(n)$ as maximal compact subgroup of $G(\mathbb{R}) = \mathrm{SL}(n, \mathbb{R})$. Then $\widetilde{X} = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$ and $X(K_f) = \Gamma \backslash \widetilde{X}$, where $\Gamma = (G(\mathbb{R}) \times K_f) \cap G(\mathbb{Q})$.

13.1. **The Hodge-Laplace operator and heat kernels.** Let τ be an irreducible finite-dimensional representation of $G(\mathbb{R})$ on V_τ. Let E_τ be the flat vector bundle over X associated to the restriction of τ to Γ. Let $\widetilde{E}^\tau$ be the homogeneous vector bundle associated to $\tau|_{K_\infty}$ and let $E^\tau := \Gamma\backslash\widetilde{E}^\tau$. There is a canonical isomorphism

$$E^\tau \cong E_\tau \tag{13.1}$$

[MM, Proposition 3.1]. By [MM, Lemma 3.1], there exists a positive definite inner product $\langle\cdot,\cdot\rangle$ on V_τ such that

(1) $\langle\tau(Y)u, v\rangle = -\langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{k}$, $u, v \in V_\tau$
(2) $\langle\tau(Y)u, v\rangle = \langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{p}$, $u, v \in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\tau|_{K_\infty}$ is unitary with respect to this inner product, it induces a metric on E^τ, and by (13.1) on E_τ, which we also call admissible. Let $\Lambda^p(E_\tau) = \Lambda^p T^*(X) \otimes E_\tau$. Let

$$\nu_p(\tau) := \Lambda^p \operatorname{Ad}^* \otimes\tau :\ K_\infty \to \operatorname{GL}(\Lambda^p\mathfrak{p}^* \otimes V_\tau). \tag{13.2}$$

Then by (13.1) there is a canonical isomorphism

$$\Lambda^p(E_\tau) \cong \Gamma\backslash(G(\mathbb{R}) \times_{\nu_p(\tau)} (\Lambda^p\mathfrak{p}^* \otimes V_\tau)) \tag{13.3}$$

of locally homogeneous vector bundles. Let $\Lambda^p(X, E_\tau)$ be the space the smooth E_τ-valued p-forms on X. The isomorphism (13.3) induces an isomorphism

$$\Lambda^p(X, E_\tau) \cong C^\infty(\Gamma\backslash G(\mathbb{R}), \nu_p(\tau)), \tag{13.4}$$

where the latter space is defined as in (9.3). A corresponding isomorphism also holds for the spaces of L^2-sections. Let $\Delta_p(\tau)$ be the Hodge-Laplacian on $\Lambda^p(X, E_\tau)$ with respect to the admissible metric in E_τ. Let R_Γ denote the right regular representation of $G(\mathbb{R})$ in $L^2(\Gamma\backslash G(\mathbb{R}))$. By [MM, (6.9)] it follows that with respect to the isomorphism (13.4) one has

$$\Delta_p(\tau) = -R_\Gamma(\Omega) + \tau(\Omega)\operatorname{Id}. \tag{13.5}$$

Let $\widetilde{E}_\tau \to \widetilde{X}$ be the lift of E_τ to $\widetilde{X}$. There is a canonical isomorphism

$$\Lambda^p(\widetilde{X}, \widetilde{E}_\tau) \cong C^\infty(G(\mathbb{R}), \nu_p(\tau)). \tag{13.6}$$

Let $\widetilde{\Delta}_p(\tau)$ be the lift of $\Delta_p(\tau)$ to $\widetilde{X}$. Then again it follows from [MM, (6.9)] that with respect to the isomorphism (13.6) we have

$$\widetilde{\Delta}_p(\tau) = -R(\Omega) + \tau(\Omega)\operatorname{Id}. \tag{13.7}$$

Let $e^{-t\widetilde{\Delta}_p(\tau)}$ be the corresponding heat semigroup. Regarded as an operator in $L^2(G(\mathbb{R}), \nu_p(\tau))$, it is a convolution operator with kernel

$$H_t^{\tau,p}\colon G(\mathbb{R}) \to \operatorname{End}(\Lambda^p\mathfrak{p}^* \otimes V_\tau) \tag{13.8}$$

which belongs to $C^\infty \cap L^2$ and satisfies the covariance property

$$H_t^{\tau,p}(k^{-1}gk') = \nu_p(\tau)(k)^{-1} H_t^{\tau,p}(g)\nu_p(\tau)(k') \tag{13.9}$$

with respect to the representation (13.2). Moreover, for all $q > 0$ we have

$$H_t^{\tau,p} \in (\mathcal{C}^q(G(\mathbb{R})) \otimes \operatorname{End}(\Lambda^p \mathfrak{p}^* \otimes V_\tau))^{K_\infty \times K_\infty}, \tag{13.10}$$

where $\mathcal{C}^q(G(\mathbb{R}))$ denotes Harish-Chandra's L^q-Schwartz space (see [MP2, Sect. 4]). We note that the kernel $H_t^{\tau,p}$ can be expressed in terms of the kernel $H_t^{\nu_p(\tau)}$ of the heat semi-group $e^{-t\widetilde{\Delta}_{\nu_p(\tau)}}$ associated to the Bochner-Laplace operator $\widetilde{\Delta}_{\nu_p(\tau)}$ acting in $C^\infty(\widetilde{X}, \widetilde{E}_{\nu_p(\tau)})$. For $p = 0, \dots, n$ put

$$E_p(\tau) := \nu_p(\tau)(\Omega_{K_\infty}),$$

which we regard as an endomorphism of $\Lambda^p \mathfrak{p}^* \otimes V_\tau$. It defines an endomorphism of $\Lambda^p T^*(X) \otimes E_\tau$. By (9.5) and (13.7) we have

$$\widetilde{\Delta}_p(\tau) = \widetilde{\Delta}_{\nu_p(\tau)} + \tau(\Omega) \operatorname{Id} - E_p(\tau).$$

Let $\nu_p(\tau) = \oplus_{\sigma \in \Pi(K_\infty)} m(\sigma)\sigma$ be the decomposition of $\nu_p(\tau)$ into irreducible representations. This induces a corresponding decomposition of the homogeneous vector bundle

$$\widetilde{E}_{\nu_p(\tau)} = \bigoplus_{\sigma \in \Pi(K_\infty)} m(\sigma) \widetilde{E}_\sigma. \tag{13.11}$$

With respect to this decomposition we have

$$E_p(\tau) = \bigoplus_{\sigma \in \Pi(K_\infty)} m(\sigma)\sigma(\Omega_{K_\infty}) \operatorname{Id}_{V_\sigma},$$

where $\sigma(\Omega_{K_\infty})$ is the Casimir eigenvalue of σ and V_σ the corresponding representation space. Let $\widetilde{\Delta}_\sigma$ be the Bochner-Laplace operator associated to σ. By (13.11) we get a corresponding decomposition of $C^\infty(\widetilde{X}, \widetilde{E}_{\nu_p(\tau)})$ and with respect to this decomposition we have

$$\widetilde{\Delta}_{\nu_p(\tau)} = \bigoplus_{\sigma \in \Pi(K_\infty)} m(\sigma) \widetilde{\Delta}_\sigma.$$

This shows that $\widetilde{\Delta}_{\nu_p(\tau)}$ commutes with $E_p(\tau)$. Hence we get

$$H_t^{\tau,p} = e^{-t(\tau(\Omega) - E_p(\tau))} \circ H_t^{\nu_p(\tau)}. \tag{13.12}$$

Let $h_t^{\tau,p} \in C^\infty(G(\mathbb{R}))$ be defined by

$$h_t^{\tau,p}(g) = \operatorname{tr} H_t^{\tau,p}(g), \quad g \in G(\mathbb{R}). \tag{13.13}$$

Then by (13.12) we get

$$h_t^{\tau,p} = e^{t(\tau(\Omega) - \operatorname{tr} E_p(\tau))} h_t^{\nu_p(\tau)}, \tag{13.14}$$

where $h_t^{\nu_p(\tau)} = \operatorname{tr} H_t^{\nu_p(\tau)}$. As in (11.9) we define $\phi_t^{\tau,p} \in C^\infty(G(\mathbb{A}))$ by

$$\phi_t^{\tau,p}(g_\infty g_f) := h_t^{\tau,p}(g_\infty)\chi_{K_f}(g_f) \tag{13.15}$$

for $g_\infty \in G(\mathbb{R})$ and $g_f \in G(\mathbb{A}_f)$. Following Definition 11.1, we define the regularized trace of $e^{-t\Delta_p(\tau)}$ by

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) := J_{\mathrm{geo}}(\phi_t^{\tau,p}). \tag{13.16}$$

13.2. **Decay for the continuous spectrum.** The next goal is to determine the asymptotic behavior of $\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ as $t \to \infty$ and $t \to 0^+$. To study the asymptotic behavior as $t \to \infty$ we use the trace formula (5.1). By Theorem 5.1, J_{spec} is a distribution on $\mathcal{C}(G(\mathbb{A}); K_f)$ and by [FL1, Theorem 7.1], J_{geo} is continuous on $\mathcal{C}(G(\mathbb{A}); K_f)$. This implies that (5.1) holds for $\phi_t^{\tau,p}$ and we have

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) = J_{\mathrm{spec}}(\phi_t^{\tau,p}). \tag{13.17}$$

Now we apply Theorem 5.1 to study the asymptotic behavior as $t \to \infty$ of the right hand side. Let $M \in \mathcal{L}$ and $P \in \mathcal{P}(M)$. Recall that $L^2_{\mathrm{dis}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))$ splits as the completed direct sum of its π-isotypic components for $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$. We have a corresponding decomposition of $\bar{\mathcal{A}}^2(P)$ as a direct sum of Hilbert spaces $\hat{\oplus}_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))}\bar{\mathcal{A}}^2_\pi(P)$. Similarly, we have the algebraic direct sum decomposition

$$\mathcal{A}^2(P) = \bigoplus_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))} \mathcal{A}^2_\pi(P),$$

where $\mathcal{A}^2_\pi(P)$ is the $\mathbf{K}$-finite part of $\bar{\mathcal{A}}^2_\pi(P)$. For $\sigma \in \widehat{K_\infty}$ let $\mathcal{A}^2_\pi(P)^\sigma$ be the σ-isotypic subspace. Then $\mathcal{A}^2_\pi(P)$ decomposes as

$$\mathcal{A}^2_\pi(P) = \bigoplus_{\sigma\in\widehat{K_\infty}} \mathcal{A}^2_\pi(P)^\sigma.$$

Let $\mathcal{A}^2_\pi(P)^{K_f}$ be the subspace of K_f-invariant functions in $\mathcal{A}^2_\pi(P)$, and for any $\sigma \in \widehat{K_\infty}$ let $\mathcal{A}^2_\pi(P)^{K_f,\sigma}$ be the σ-isotypic subspace of $\mathcal{A}^2_\pi(P)^{K_f}$. Recall that $\mathcal{A}^2_\pi(P)^{K_f,\sigma}$ is finite dimensional. Let $M_{Q|P}(\pi,\lambda)$ denote the restriction of $M_{Q|P}(\lambda)$ to $\mathcal{A}^2_\pi(P)$. Recall that the operator $\Delta_{\mathcal{X}}(P,\lambda)$, which appears in the formula (5.8), is defined by (5.6). Its definition involves the intertwining operators $M_{Q|P}(\lambda)$. If we replace $M_{Q|P}(\lambda)$ by its restriction $M_{Q|P}(\pi,\lambda)$ to $\mathcal{A}^2_\pi(P)$, we obtain the restriction $\Delta_{\mathcal{X}}(P,\pi,\lambda)$ of $\Delta_{\mathcal{X}}(P,\lambda)$ to $\mathcal{A}^2_\pi(P)$. Similarly, let $\rho_\pi(P,\lambda)$ be the induced representation in $\bar{\mathcal{A}}^2_\pi(P)$. Fix $\beta \in \mathfrak{B}_{P,L_s}$ and $s \in W(M)$. Then for the integral on the right of (5.8) with $h = \phi_t^{\tau,p}$ we get

$$\sum_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))} \int_{i(\mathfrak{a}^G_{L_s})^*} \operatorname{Tr}\left(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\lambda) M(P,\pi,s)\rho_\pi(P,\lambda,\phi_t^{\tau,p})\right)\, d\lambda. \tag{13.18}$$

Let $P, Q \in \mathcal{P}(M)$ and $\nu \in \Pi(K_\infty)$. Denote by $\widetilde{M}_{Q|P}(\pi,\nu,\lambda)$ the restriction of

$$M_{Q|P}(\pi,\lambda)\otimes \mathrm{Id}\colon \mathcal{A}^2_\pi(P)\otimes V_\nu \to \mathcal{A}^2_\pi(P)\otimes V_\nu$$

to $(\mathcal{A}^2_\pi(P)^{K_f}\otimes V_\nu)^{K_\infty}$. Denote by $\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)$ and $\widetilde{M}(P,\pi,\nu,s)$ the corresponding restrictions. Let $m(\pi)$ denote the multiplicity with which π occurs in the regular representation of $M(\mathbb{A})$ in $L^2_{\text{dis}}(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A}))$. Then

$$\rho_\pi(P,\lambda)\cong\oplus_{i=1}^{m(\pi)}\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi,\lambda). \tag{13.19}$$

Fix positive restricted roots of $\mathfrak{a}_P$ and let $\rho_{\mathfrak{a}_P}$ denote the corresponding half-sum of these roots. For $\xi\in\Pi(M(\mathbb{R}))$ and $\lambda\in\mathfrak{a}_P^*$ let

$$\pi_{\xi,\lambda}:=\operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\xi\otimes e^{i\lambda})$$

be the unitary induced representation. Let $\xi(\Omega_M)$ be the Casimir eigenvalue of ξ. Define a constant $c(\xi)$ by

$$c(\xi):=-\langle\rho_{\mathfrak{a}_P},\rho_{\mathfrak{a}_P}\rangle+\xi(\Omega_M). \tag{13.20}$$

Then for $\lambda\in\mathfrak{a}_P^*$ one has

$$\pi_{\xi,\lambda}(\Omega)=-\|\lambda\|^2+c(\xi) \tag{13.21}$$

(see [Kn, Theorem 8.22]). Let

$$\mathcal{T}:=\{\nu\in\Pi(K_\infty)\colon[\nu_p(\tau)\colon\nu]\neq0\}. \tag{13.22}$$

Using (13.12), (13.19) and (9.13), it follows that (13.18) is equal to

(13.23)
$$\sum_{\pi\in\Pi_{\text{dis}}(M(\mathbb{A}))}\sum_{\nu\in\mathcal{T}}e^{-t(\tau(\Omega)-c(\pi_\infty))}\int_{i(\mathfrak{a}_{L_s}^G)^*}e^{-t\|\lambda\|^2}\operatorname{Tr}\left(\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)\widetilde{M}(P,\pi,\nu,s)\right)\,d\lambda.$$

In order to estimate (13.23) from above, we need the following two preparatory results.

Lemma 13.1. *Let $(\tau,V_\tau)\in\operatorname{Rep}(G(\mathbb{R}))$. Assume that $\tau\not\cong\tau_\theta$. Let $P=MAN$ be a proper parabolic subgroup of G and let $K_\infty^M=M(\mathbb{R})\cap K_\infty$. Let $\xi\in\widehat{M(\mathbb{R})}$ and assume that $\dim(W_\xi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau)^{K_\infty^M}\neq0$. Then one has*

$$\tau(\Omega)-c(\xi)>0.$$

Proof. Let $\xi\in\widehat{M(\mathbb{R})}$ with $\dim(W_\xi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau)^{K_\infty^M}\neq0$. Assume that $\tau(\Omega)-c(\xi)\le0$. Then by (13.21) there exists $\lambda_0\in\mathfrak{a}^*$ such that

$$\pi_{\xi,\lambda_0}(\Omega)=\tau(\Omega).$$

By Frobenius reciprocity we have

$$\dim\left(W_\xi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^{K_\infty^M}=\dim\left(\mathcal{H}_{\xi,\lambda_0}\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^{K_\infty}.$$

Combined with our assumption and [BW, Proposition II,3.1] it follows that

$$\dim H^p(\mathfrak{g},K_\infty;\mathcal{H}_{\xi,\lambda_0,K_\infty}\otimes V_\tau)\neq0,$$

where $\mathcal{H}_{\xi,\lambda_0,K_\infty}$ denotes the subspace of K_∞-finite vectors of $\mathcal{H}_{\xi,\lambda_0}$. Since $\tau \neq \tau_\theta$, this is a contradiction to the first statement of [BW, Proposition II. 6.12]. Thus it follows that

$$\tau(\Omega) - c(\xi) > 0$$

for all $\xi \in \widehat{M(\mathbb{R})}$ satisfying $\dim(W_\xi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^{K_\infty^M} \neq 0$. □

Lemma 13.2. *For every $R \geq 0$, the number of $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$ with $\lambda_{\pi_\infty} \geq -R$ and $\mathcal{A}^2_\pi(P)^{K_f,\nu} \neq 0$ for some $\nu \in \mathcal{T}$ is finite.*

Proof. To prove the lemma, it suffices to show that for every $R \geq 0$ we have

$$\sum_{\substack{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})) \\ -\lambda_{\pi_\infty} \leq R}} \dim(\mathcal{A}^2_\pi(P)^{K_f,\nu}) < \infty. \tag{13.24}$$

By passing to a subgroup of finite index, we may assume that $K_f = \prod_{p<\infty} K_p$. Let $K_{M,f} = K_f \cap M(\mathbb{A}_f)$ and $K_{M.\infty} = K_\infty \cap M(\mathbb{R})$. For $\pi \in \Pi(M(\mathbb{A}))$ and $\tau \in \Pi(K_{M,\infty})$ let $\mathcal{H}_{\pi_\infty}(\tau)$ denote the τ-isotypical subspace of the representation space $\mathcal{H}_{\pi_\infty}$. Arguing as in the proof of Proposition 3.5 in [Mu1], it follows that in order to establish (13.24), it suffices to show that for every $\tau \in \Pi(K_{M,\infty})$

$$\sum_{\substack{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})) \\ -\lambda_{\pi_\infty} \leq R}} \dim(\mathcal{H}_{\pi_f}^{K_{M,f}}) \cdot \dim(\mathcal{H}_{\pi_\infty}(\tau)) < \infty.$$

Let $\Gamma_M \subset M(\mathbb{R})$ be an arithmetic subgroup. Let $\Omega_{M(\mathbb{R})^1}$ be the Casimir element of $M(\mathbb{R})^1$ and let A_τ be the differential operator in $C^\infty(\Gamma_M \backslash M(\mathbb{R})^1; \tau)$ which is induced by $-\Omega_{M(\mathbb{R})^1}$. Let $\bar{A}_\tau$ be its self-adjoint extension of A_τ in L^2. Proceeding as in the proof of Lemma 3.2 of [Mu1], it follows that it suffices to show that for every $R \geq 0$, the number of eigenvalues λ_i of $\bar{A}_\tau$ (counted with multiplicities), satisfying $\lambda_i \leq R$ is finite. Let Δ_τ be the Bochner-Laplace operator and let Λ_τ be the Casimir eigenvalue of τ. Then $\Delta_\tau = A_\tau + \Lambda_\tau \operatorname{Id}$. Since $\Delta_\tau \geq 0$ and by [Mu3], the counting function of the eigenvalues has a polynomial bound, the lemma follows. □

Now we can begin with the estimation of (13.23). Using that $M(P,\pi,s)$ is unitary, it follows that (13.23) can be estimated by

$$\begin{split} \sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))} \sum_{\nu \in \mathcal{T}} \dim\left(\mathcal{A}^2_\pi(P)^{K_f,\nu}\right) \\ \cdot e^{-t(\tau(\Omega) - c(\pi_\infty))} \int_{i(\mathfrak{a}^G_{L_s})^*} e^{-t\|\lambda\|^2} \|\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)\| \, d\lambda. \end{split} \tag{13.25}$$

For $\pi \in \Pi(M(\mathbb{A}))$ denote by λ_{π_∞} the Casimir eigenvalue of the restriction of π_∞ to $M(\mathbb{R})^1$. Given $\lambda > 0$, let

$$\Pi_{\mathrm{dis}}(M(\mathbb{A}); \lambda) := \left\{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})) \colon |\lambda_{\pi_\infty}| \leq \lambda\right\}.$$

Let $d=\dim M(\mathbb{R})^1/K_\infty^M$. As in [Mu1, Proposition 3.5] it follows that for every $\nu\in\Pi(K_\infty)$ there exists $C>0$ such that

$$\sum_{\pi\in\Pi_{\operatorname{dis}}(M(\mathbb{A});\lambda)}\dim\mathcal{A}^2_\pi(P)^{K_f,\nu}\le C(1+\lambda^{d/2}) \tag{13.26}$$

for all $\lambda\ge 0$. Next we estimate the integral in (13.25). Let $\underline{\beta}=(\beta_1^\vee,\dots,\beta_m^\vee)$ and $\mathcal{X}_{L_s}(\underline{\beta})=(Q_1,\dots,Q_m)\in\Xi_{L_s}(\underline{\beta})$ with with $Q_i=\langle P_i,P_i'\rangle$, $P_i|^{\beta_i}P_i'$, $i=1,\dots,m$. Using the definition (5.6) of $\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)$, it follows that we can bound the integral by a constant multiple of

$$\dim(\nu)\int_{i(\mathfrak{a}^G_{L_s})^*}e^{-t\|\lambda\|^2}\prod_{i=1}^m\left\|\delta_{P_i|P_i'}(\lambda)\Big|_{\mathcal{A}^2_\pi(P_i')^{K_f,\nu}}\right\|\,d\lambda. \tag{13.27}$$

We introduce new coordinates $s_i:=\langle\lambda,\beta_i^\vee\rangle$, $i=1,\dots,m$, on $(\mathfrak{a}^G_{L_s,\mathbb{C}})^*$. Using (5.2), we can write

$$\delta_{P_i|P_i'}(\lambda)=\frac{n'_{\beta_i}(\pi,s_i)}{n_{\beta_i}(\pi,s_i)}+j_{P_i'}\circ(\operatorname{Id}\otimes R_{P_i|P_i'}(\pi,s_i)^{-1}R'_{P_i|P_i'}(\pi,s_i))\circ j_{P_i'}^{-1}. \tag{13.28}$$

Put

$$\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}}=\bigoplus_{\nu\in\mathcal{T}}\mathcal{A}^2_\pi(P)^{K_f,\nu},$$

where $\mathcal{T}$ is defined by (13.22). It follows from [Mu2, Theorem 5.3] that there exist $N,k\in\mathbb{N}$ and $C>0$ such that

$$\int_{i\mathbb{R}}\left|\frac{n'_{\beta_i}(\pi,s)}{n_{\beta_i}(\pi,s)}\right|(1+|s|^2)^{-k}\,ds\le C(1+\lambda^2_{\pi_\infty})^N,\ i=1,\dots,m, \tag{13.29}$$

for all $\pi\in\Pi_{\operatorname{dis}}(M(\mathbb{A}))$ with $\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}}\neq 0$. Furthermore, for $G=\operatorname{GL}(n)$ it follows from [MS, Proposition 0.2] that there exist $k,C>0$ such that

$$\int_{i\mathbb{R}}\left\|R_{P_i|P_i'}(\pi,s)^{-1}R'_{P_i|P_i'}(\pi,s)\Big|_{\mathcal{A}^2_\pi(P_i')^{K_f,\nu}}\right\|(1+|s|^2)^{-k}\,ds\le C,\ i=1,\dots,m, \tag{13.30}$$

for all $\nu\in\mathcal{T}$ and $\pi\in\Pi_{\operatorname{dis}}(M(\mathbb{A}))$ with $\mathcal{A}^2_\pi(P)^{K_f,\nu}\neq 0$. To show that (13.30) also holds for $G=\operatorname{SL}(n)$, we proceed as in the proof of [FLM2, Lemma 5.14]. Combining (13.28), (13.29) and (13.30), it follows that for $t\ge 1$ we have

$$\int_{i(\mathfrak{a}^G_{L_s})^*}e^{-t\|\lambda\|^2}\prod_{i=1}^m\left\|\delta_{P_i|P_i'}(\lambda)\Big|_{\mathcal{A}^2_\pi(P_i')^{K_f,\nu}}\right\|\,d\lambda\ll(1+\lambda^2_{\pi_\infty})^{mN}$$

for all $\pi\in\Pi_{\operatorname{dis}}(M(\mathbb{A}))$ with $\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}}\neq 0$. Thus (13.25) can be estimated by a constant multiple of

$$\sum_{\pi\in\Pi_{\operatorname{dis}}(M(\mathbb{A}))}\sum_{\nu\in\mathcal{T}}\dim\left(\mathcal{A}^2_\pi(P)^{K_f,\nu}\right)(1+\lambda^2_{\pi_\infty})^{mN}e^{-t(\tau(\Omega)-c(\pi_\infty))}. \tag{13.31}$$

First assume that M is a proper Levi subgroup. Note that by (13.20) one has

$$\tau(\Omega)-c(\pi_\infty)=\tau(\Omega)+\|\rho_{\mathfrak{a}}\|^2-\lambda_{\pi_\infty}. \tag{13.32}$$

Together with Lemma 13.2, it follows that there exists $\lambda_0 > 0$ such that

$$\tau(\Omega) - c(\pi_\infty) \geq |\lambda_{\pi_\infty}|/2$$

for all $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$ with $\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}} \neq 0$ and $|\lambda_{\pi_\infty}| \geq \lambda_0$. Decompose the sum over π in (13.31) in two summands $\Sigma_1(t)$ and $\Sigma_2(t)$, where in $\Sigma_1(t)$ the summation runs over all π with $|\lambda_{\pi_\infty}| \leq \lambda_0$. Using (13.26), it follows that for $t \geq 1$

$$\Sigma_2(t) \ll e^{-t|\lambda_0|/2}.$$

Since $\Sigma_1(t)$ is a finite sum by Lemma 13.2, both in π and ν, it follows from Lemma 13.1 that there exists $c > 0$ such that

$$\Sigma_1(t) \ll e^{-ct}$$

for $t \geq 1$. Putting everything together it follows that for every $\tau \in \mathrm{Rep}(G(\mathbb{R}))$ such that $\tau \ncong \tau_\theta$ and every proper Levi subgroup M of G there exists $c > 0$ such that

$$J_{\mathrm{spec},M}(\phi_t^{\tau,p}) = O(e^{-ct}) \tag{13.33}$$

for $t \geq 1$.

Now consider the case $M = G$. Then $c(\pi_\infty) = \pi_\infty(\Omega)$ and we need to show that

$$\tau(\Omega) - \pi_\infty(\Omega) > 0 \tag{13.34}$$

for all $\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}))$ with $\dim \mathcal{H}_\pi^{K_f,\mathcal{T}} \neq 0$. This follows from [BV, Lemma 4.1], and we can proceed as in the case $M \neq G$ to prove that

$$J_{\mathrm{spec},G}(\phi_t^{\tau,p}) = O(e^{-ct})$$

for $t \geq 1$. Combined with (13.33) we obtain

Proposition 13.3. *Let $\tau \in \mathrm{Rep}(G(\mathbb{R}))$. Assume that $\tau \ncong \tau_\theta$. Then there exists $c > 0$ such that*

$$J_{\mathrm{spec}}(\phi_t^{\tau,p}) = O\left(e^{-ct}\right)$$

for all $t \geq 1$ and $p = 0, \dots, n$.

13.3. **Definition of analytic torsion.** Applying the trace formula (5.1), we get

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) = O(e^{-ct}), \quad \text{as } t \to \infty,$$

which is the proof of Theorem 1.2. Using (13.16), (13.14) and Theorem 1.1, it follows that as $t \to +0$, there is an asymptotic expansion of the form

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) \sim t^{-d/2}\sum_{j=0}^\infty a_j t^j + t^{-(d-1)/2}\sum_{j=0}^\infty \sum_{i=0}^{r_j} b_{ij} t^{j/2}(\log t)^i.$$

Thus the corresponding zeta function $\zeta_p(s;\tau)$, defined by the Mellin transform

$$\zeta_p(s;\tau) := \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) t^{s-1}\, dt. \tag{13.35}$$

is holomorphic in the half-plane $\mathrm{Re}(s) > d/2$ and admits a meromorphic extension to the whole complex plane. It may have a pole at $s = 0$. Let $f(s)$ be a meromorphic function on $\mathbb{C}$. For $s_0 \in \mathbb{C}$ let

$$f(s) = \sum_{k \geq k_0} a_k (s - s_0)^k$$

be the Laurent expansion of f at s_0. Put $\mathrm{FP}_{s=s_0} := a_0$. Now we define the analytic torsion $T_X(\tau) \in \mathbb{R}^+$ by

$$\log T_X(\tau) = \frac{1}{2} \sum_{p=0}^{d} (-1)^p p \left(\mathrm{FP}_{s=0} \frac{\zeta_p(s;\tau)}{s} \right). \tag{13.36}$$

Put

$$K(t,\tau) := \sum_{p=1}^{d} (-1)^p p \operatorname{Tr}_{\mathrm{reg}} \left(e^{-t\Delta_p(\tau)} \right). \tag{13.37}$$

Then $K(t,\tau) = O(e^{-ct})$ as $t \to \infty$ and the Mellin transform

$$\int_0^\infty K(t,\tau) t^{s-1} dt$$

converges absolutely and uniformly on compact subsets of $\mathrm{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$. Moreover, by (13.36) we have

$$\log T_X(\tau) = \mathrm{FP}_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^\infty K(t,\tau) t^{s-1} dt \right). \tag{13.38}$$

Let

$$\phi_t^\tau := \sum_{p=1}^{d} (-1)^p p \phi_t^{\tau,p} \quad \text{and} \quad k_t^\tau := \sum_{p=1}^{d} (-1)^p p h_t^{\tau,p}.$$

Then by (13.16) we have

$$K(t,\tau) = J_{\mathrm{spec}}(\phi_t^\tau). \tag{13.39}$$

For $\pi \in \Pi(G(\mathbb{R}))$ let Θ_π be the global character. Then we get

$$J_{\mathrm{spec},G}(\phi_t^\tau) = \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}))} m(\pi) \dim \left(\mathcal{H}_{\pi_f}^{K_f} \right) \Theta_{\pi_\infty}(k_t^\tau). \tag{13.40}$$

For $n \in \mathbb{N}$, $n \geq 2$, let $\delta_n := \mathrm{rank}_{\mathbb{C}} \mathrm{SL}(n) - \mathrm{rank}_{\mathbb{C}} \mathrm{SO}(n)$ be the fundamental rank of $\mathrm{SL}(n)$.

Lemma 13.4. *For $G = \mathrm{GL}(n)$ and $n \geq 5$ we have $J_{\mathrm{spec},G}(\phi_t^\tau) = 0$.*

Proof. Let Q be a standard cuspidal parabolic subgroup of $G(\mathbb{R})$. Let $Q = M_Q A_Q N_Q$ be the Langlands decomposition of Q. Let (ξ, W_ξ) be a discrete series representation of M_Q and let $\nu \in \mathfrak{a}_{Q,\mathbb{C}}^*$. Let $\pi_{\xi,\nu}$ be the induced representation. By [MP2, Proposition 4.1] we have $\Theta_{\xi,\nu}(k_t^\tau) = 0$, if $\dim \mathfrak{a}_Q \geq 2$. If $\delta_n \geq 2$, it follows that $\dim \mathfrak{a}_Q \geq 2$ for every cuspidal parabolic subgroup Q of $G(\mathbb{R})$. Thus $\Theta_{\xi,\nu}(k_t^\tau) = 0$ for all cuspidal parabolic subgroups Q and pairs (ξ, ν) as above. Now observe that for $\mathrm{GL}(n)$ the R-group is trivial. Therefore, it follows from [De, section 2.2] that the Grothendieck group of all admissible representations of $G(\mathbb{R})$ is generated by the induced representations $\pi_{\xi,\nu}$ as above. Hence $\Theta_\pi(k_t^\tau)=0$ for all $\pi\in\Pi(G(\mathbb{R}))$. If $n\geq 5$, then $\delta_n\geq 2$ and the lemma follows from (13.40). □

Remark 13.5. *If Γ is cocompact and $n\geq 5$, then it follows that $T_X(\tau)=1$. In the noncompact case this need not be true. In* [MP1] *the case of finite volume hyperbolic manifolds has been studied. It has been shown that in even dimensions, the renormalized analytic torsion has a simple expression, but is not trivial. This includes the case of* SL(2).

14. The case $G=\mathrm{GL}(3)$

If $G=\mathrm{GL}(3)$, the weight functions are explicitly given by (8.1)- (8.3). Using the explicit form of the weight function, we can extract more precise information about the pole at $s=0$. To this end we need to show that the coefficients $c_{ij}(\nu)$ in (12.20) with $j=d-k$ and $i=1,\dots,r_{d-k}$ vanish for the corresponding orbital integrals. In our case $d=5$. Now consider the first integral (8.1). Then $k=2$ and the weight function is $\log(y^2+z^2)$. Thus the highest power with which $\log t$ occurs in the asymptotic expansion of (8.1) is 1. This means that $c_{13}(\nu)$ is the only coefficient that we need to consider. It is of the form (12.19). We are in the case $i=r_j$. Hence

$$c_{13}(\nu)=\int_{\mathbb{R}^2}e^{-a\|x\|^2}p(x)\,dx,$$

where $p(x)$ is a homogeneous polynomial. Moreover, from its construction it follows that $p(x)$ is odd, i.e., $p(-x)=-p(x)$. Hence $c_{13}(\nu)=0$. Thus the asymptotic expansion of the first integral has the form

$$J_{M_1}(1,h_t^\nu)\sim t^{-3/2}\sum_{j=0}^\infty a_j(\nu)t^{j/2}+t^{-3/2}\sum_{k=0}^\infty b_k(\nu)t^{k/2}\log t, \tag{14.1}$$

as $t\to+0$, and $b_3(\nu)=0$.

Now consider the second integral (8.2). Then $k=3$. By (8.2) we need only to consider $c_{12}(\nu)$, which we denote by $c_2(\nu)$. Let $p_1(x)$ and $p_2(x)$ be the polynomials occurring on the right hand side of (12.15) and $a_j^\nu(g)$ the coefficients on the right hand side of (10.8). If we collect all possible contributions, we get

$$\begin{aligned}c_2(\nu)=&a_0^\nu(1)\int_{\mathbb{R}^3}p_2(x)e^{-\|x\|^2}\,dx+\sum_{i=1}^3\frac{\partial}{\partial x_i}a_0^\nu(n(x))\big|_{x=0}\int_{\mathbb{R}^3}x_ip_1(x)e^{-\|x\|^2}\,dx\\&+\sum_{i,j=1}^3\frac{\partial^2}{\partial x_i\partial x_j}a_0^\nu(n(x))\big|_{x=0}\int_{R^3}x_ix_je^{-\|x\|^2}dx+a_1^\nu(1)\int_{\mathbb{R}^3}e^{-\|x\|^2}\,dx.\end{aligned} \tag{14.2}$$

By definition we have

$$p_1(x)=\sum_{|\alpha|=3}D^\alpha r^2(x)\big|_{x=0}x^\alpha.$$

Now recall that for $g \in \mathrm{SL}(n, \mathbb{R})$ the distance $r(g(x_0), x_0)$ is given as follows. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the positive definite matrix $g^\top \cdot g$. Then

$$r(g(x_0), x_0)^2 = \sum_{i=1}^{n} (\log \lambda_i)^2.$$

An explicit computation shows that

$$r^2(x_1, x_2, 0) = 2\log^2\left(1 + \frac{x_1^2 + x_2^2}{2} + \sqrt{x_1^2 + x_2^2 + \frac{(x_1^2 + x_2^2)^2}{4}}\right).$$

Thus $r(x_1, x_2, 0)$ is even in x_1 and x_2. The same holds for $r(x_1, 0, x_3)$ and $r(0, x_2, x_3)$. This implies that for $\alpha \neq (1,1,1)$ we have $D^\alpha r^2(x)\big|_{x=0} = 0$. Finally note that

$$\int_{\mathbb{R}^3} x_i x_1 x_2 x_3 e^{-\|x\|^2} dx = 0, \quad \text{and} \quad \int_{\mathbb{R}^3} x_i x_j e^{-\|x\|^2} dx = 0, \; i \neq j.$$

Thus (14.2) is reduced to

$$\begin{aligned} c_2(\nu) =& a_0^\nu(1) \int_{\mathbb{R}^3} p_2(x) e^{-\|x\|^2} \, dx + a_1^\nu(1) \int_{\mathbb{R}^3} e^{-\|x\|^2} \, dx \\ &+ \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} a_0^\nu(n(x))\big|_{x=0} \int_{\mathbb{R}^3} x_i^2 e^{-\|x\|^2} dx. \end{aligned} \tag{14.3}$$

Thus for the second integral we get an asymptotic expansion of the form

$$J_{M_1}(u(1,0,0), h_t^\nu) \sim t^{-1} \sum_{j=0}^{\infty} a_j(\nu) t^{j/2} + t^{-1} \sum_{k=0}^{\infty} c_k(\nu) t^{k/2} \log t \tag{14.4}$$

with $c_2(\nu)$ given by (14.3). Finally consider the integral (8.3). Again $k = 3$. By (8.3) we only need to consider $c_{12}(\nu)$ and $c_{22}(\nu)$. By the same considerations as in the previous case, it follows that $c_{22}(\nu) = c_2(\nu)$. Furthermore, $c_{12}(\nu)$ has the same form as $c_2(\nu)$, except that the integrals contain in addition some factors $\log|x_i|$ for $i = 1, 2, 3$. Thus we obtain

$$J_{M_0}(1, h_t^\nu) \sim t^{-1} \sum_{j=0}^{\infty} a_j(\nu) t^{j/2} + t^{-1} \sum_{k=0}^{\infty} c_{1k}(\nu) t^{k/2} \log t + t^{-1} \sum_{l=0}^{\infty} c_{2l}(\nu) t^{l/2} (\log t)^2, \tag{14.5}$$

with $c_{22}(\nu) = c_2(\nu)$, where $c_2(\nu)$ is given by (14.3), and $c_{12}(\nu)$ is given by a similar formula as described above. Now we specialize ν to $\nu_p(\tau)$, which is defined by (13.2).

Lemma 14.1. *Let (τ, V_τ) be a finite dimensional representation of $G(\mathbb{R})$. We have*

$$\sum_{p=1}^{5} (-1)^p p \cdot a_0^{\nu_p(\tau)}(1) = 0, \quad \sum_{p=1}^{5} (-1)^p p \cdot a_1^{\nu_p(\tau)}(1) = 0.$$

Proof. By (10.9) we have $a_0^{\nu_p(\tau)} = \dim(\Lambda^p \mathfrak{p}^* \otimes V_\tau) = \binom{5}{p} \cdot \dim V_\tau$. Now observe that $\sum_{p=1}^{5} (-1)^p p \binom{5}{p} = 0$. This proves the first statement. For the second statement we note that by (10.10) we have $a_1^{\nu_p(\tau)}(1) = \operatorname{tr}(\phi_1^{\nu_p(\tau)}(x_0, x_0))$, and by (10.6), $\phi_1^{\nu_p(\tau)}(x_0, x_0)$ is the second coefficient of the asymptotic expansion as $t \to +0$ of $\operatorname{tr} K^{\nu_p(\tau)}(t, x_0, x_0)$. Using the known structure of the coefficient, we get

$$a_1^{\nu_p(\tau)}(1) = -\frac{R \cdot \dim(\tau)}{6} \left\{ \binom{5}{p} - 6 \binom{3}{p-1} \right\}, \tag{14.6}$$

where R is the scalar curvature (which is constant) and it is understood that $\binom{m}{p} = 0$, if $p < 0$ or $p > m$. For $\tau = 1$, this follows from [Gi, Theorem 4.1.7, (b)]. It is easy to extend this to the twisted case. Using (14.6), the second statement follows. □

Lemma 14.2. *For every finite dimensional representation (τ, V_τ) of $G(\mathbb{R})$ we have*

$$\frac{\partial^2}{\partial x_i^2}\bigg|_{x=0} \left(\sum_{p=1}^{5} (-1)^p p \, a_0^{\nu_p(\tau)}(n(x)) \right) = 0.$$

for $i = 1, 2, 3$.

Proof. We consider the derivative with respect to x_1. Let

$$n_1(u) = \begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\frac{\partial^2}{\partial x_1^2} a_0^{\nu_p(\tau)}(n(x))\big|_{x=0} = \frac{\partial^2}{\partial u^2} a_0^{\nu_p(\tau)}(n_1(u))\big|_{u=0}.$$

By (10.9) we have

$$a_0^{\nu_p(\tau)}(n_1(u)) = \operatorname{tr}(\nu_p(\tau)(k(u)) \cdot j(x_0, n_1(u)x_0),$$

where $k(u) := k(n_1(u)) \in \mathrm{SO}(3)$ is determined by (10.7). Furthermore, by (13.2) we have

$$\operatorname{tr}(\nu_p(\tau)(k(u)) = \operatorname{tr}(\Lambda^p \operatorname{Ad}_{\mathfrak{p}}^*(k(u)) \cdot \operatorname{tr}(\tau(k(u)).$$

Let

$$S := \left\{ A \in \operatorname{Mat}_3(\mathbb{R}) \colon A = A^t, \ \operatorname{tr}(A) = 0 \right\},$$

equipped with the inner product

$$\langle Y_1, Y_2 \rangle = \operatorname{Tr}(Y_1 Y_2), \quad Y_1, Y_2 \in S.$$

Then $\mathfrak{p} \cong S$ as inner product spaces. Moreover, the adjoint representation $\operatorname{Ad}_{\mathfrak{p}}$ of $\mathrm{SO}(3)$ on S is given by

$$\operatorname{Ad}_{\mathfrak{p}}(k)Y = k \cdot Y \cdot k^*, \quad k \in \mathrm{SO}(3), \ Y \in S. \tag{14.7}$$

With respect to this isomorphism, $k(u)$ is determined as follows. Let $A(u) := n_1(u)n_1(u)^*$. Then $A(u) = A(u)^t$ and $A(u) > 0$. Let $S(u) = A(u)^{-1/2}$. Then $k(u) = S(u) \cdot n_1(u)$. Note that $k(u)$ is a block diagonal matrix of the form

$$\begin{pmatrix} r(\theta) & 0 \\ 0 & 1 \end{pmatrix},$$

where $r(\theta) \in \mathrm{SO}(2)$ is the rotation by the angle θ. Let

$$Y_1 = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then with respect to (14.7) we have $\mathrm{Ad}_{\mathfrak{p}}(k(u))(Y_1) = Y_1$. Let $S_1 := \mathbb{R}Y_1$ and $S_0 = S_1^{\perp}$. Then the decomposition $S = S_0 \oplus S_1$ is invariant under $\mathrm{Ad}_{\mathfrak{p}}(k(u))$ and $\mathrm{Ad}_{\mathfrak{p}}(k(u))|_{S_1} = \mathrm{Id}$. Let $T(u) := \mathrm{Ad}_{\mathfrak{p}}(k(u))|_{S_0}$. Then we have

$$\sum_{p=1}^{5}(-1)^p p \ \operatorname{tr}(\Lambda^p \mathrm{Ad}_{\mathfrak{p}}^*(k(u))) = \sum_{p=0}^{5}(-1)^p \operatorname{tr}(\Lambda^p T(u)) = \det(\mathrm{Id} - T(u)). \tag{14.8}$$

For $\lambda \in \mathbb{C}$ let

$$f(\lambda, u) := \det(\lambda \,\mathrm{Id} - T(u)), \quad \lambda \in \mathbb{C},\ u \in \mathbb{R}. \tag{14.9}$$

Recall that $T(u)$ is unitary. So every eigenvalue μ of $T(u)$ satisfies $|\mu| = 1$. Assume that $|\lambda| \neq 1$. Then $f(\lambda, u) \neq 0$ for all $u \in \mathbb{R}$ and

$$\frac{\partial}{\partial u} \log f(\lambda, u) = -\operatorname{tr}(T'(u)(\lambda \,\mathrm{Id} - T(u))^{-1}), \tag{14.10}$$

where $T'(u) = \frac{d}{du}T(u)$. Note that $f(\lambda, 0) = \det(\lambda \,\mathrm{Id} - T(0)) = (\lambda - 1)^4$. Thus

$$\frac{\partial}{\partial u} f(\lambda, u)\big|_{u=0} = -(\lambda - 1)^3 \operatorname{tr}(T'(0)).$$

Since $T(u)$ is orthogonal, it follows that $\operatorname{tr}(T'(0)) = 0$, and therefore

$$\frac{\partial}{\partial u} f(\lambda, u)\big|_{u=0} = 0. \tag{14.11}$$

Using (14.10), we get

$$\begin{aligned} \frac{\partial^2}{\partial u^2} f(\lambda, u) &= -\frac{\partial}{\partial u} f(\lambda, u) \cdot \operatorname{tr}(T'(u)(\lambda \,\mathrm{Id} - T(u))^{-1}) \\ &\quad - f(\lambda, u) \operatorname{tr}(T''(u)(\lambda \,\mathrm{Id} - T(u))^{-1}) \\ &\quad - f(\lambda, u) \operatorname{tr}(T'(u)(\lambda \,\mathrm{Id} - T(u))^{-1} T'(u)(\lambda \,\mathrm{Id} - T(u))^{-1}). \end{aligned} \tag{14.12}$$

Using (14.11), we obtain

$$\frac{\partial^2}{\partial u^2} f(\lambda, u)\big|_{u=0} = -(\lambda - 1)^3 \operatorname{tr}(T''(0)) - (\lambda - 1)^2 \operatorname{tr}(T'(0)^2).$$

Since $f(\lambda, u)$ is a polynomial in λ, it follows that this equality holds for all $\lambda \in \mathbb{C}$. In particular, we get

$$\frac{\partial^2}{\partial u^2} f(1, u)\big|_{u=0} = 0.$$

Combined with (14.8) and the definition of $f(\lambda, u)$, the statement follows for $i = 1$. The proof of the other cases is similar. $\square$

Using (14.4), (14.5) and Lemmas 14.1 and 14.2, it follows that

$$\sum_{p=1}^{5}(-1)^p p\, \zeta_p(s;\tau)$$

is holomorphic at $s=0$. Thus in this case we can define $\log T_{X(K_f)}(\tau)$ by

$$\log T_{X(K_f)}(\tau) = \frac{1}{2}\frac{d}{ds}\left(\sum_{p=1}^{5}(-1)^p p\, \zeta_p(s;\tau)\right)\bigg|_{s=0}.$$

15. Example: Classes of finite order for GL(2) and GL(3)

In order to remove the assumption that $\Gamma \subseteq \Gamma(N)$ for some $N \geq 3$, we need to understand distributions $J_{\mathfrak{o}}$ appearing in the coarse geometric expansion of the trace formula for which the equivalence classes $\mathfrak{o}$ which are not necessarily unipotent. Let K_f be an arbitrary subgroup of $G(\widehat{\mathbb{Z}})$ of finite index and let $f = f_\infty \cdot 1_{K_f} \in C_c^\infty(G(\mathbb{A})^1)$ with $f_\infty \in C_c^\infty(G(\mathbb{R})^1)$ and $1_{K_f} \in C_c^\infty(G(\mathbb{A}_f))$ the characteristic function of K_f. In this situation, more than just the unipotent orbit may contribute non-trivially to the coarse geometric expansion. The equivalence classes $\mathfrak{o} \in \mathcal{O}$ are in bijection with semisimple orbits in $G(\mathbb{Q})$. Hence there is a canonical bijection between $\mathcal{O}$ and monic polynomial of degree n with rational coefficients and non-vanishing constant term if $G = \mathrm{GL}(n)$ by sending the semisimple conjugacy class to its characteristic polynomial. We may therefore speak of the characteristic polynomial and the eigenvalues of a class $\mathfrak{o}$. The following lemma is explained in the proof of [LM, Lemma 5.1].

Lemma 15.1. *We can choose a K_∞-bi-invariant neighborhood $\omega \subseteq G(\mathbb{R})^1$ of K_∞ such that if $\mathfrak{o} \in \mathcal{O}$ is such that there exists $f_\infty \in C_c^\infty(G(\mathbb{R})^1)$ supported in ω with $J_{\mathfrak{o}}(f_\infty \cdot 1_{K_f}) \neq 0$, then the eigenvalues of $\mathfrak{o}$ are all roots of unity (over some algebraic closure of $\mathbb{Q}$).*

Let $\mathcal{O}_1$ denote the set of all $\mathfrak{o} \in \mathcal{O}$ whose eigenvalues (in some algebraic closure of $\mathbb{Q}$) are all roots of unity. Note that this set is finite. By the preceding lemma we can choose a bi-K_∞-invariant $f_\infty \in C_c^\infty(G(\mathbb{R})^1)$ with $f(1) = 1$ and

$$J_{\mathrm{geo}}(f_\infty \cdot 1_{K_f}) = \sum_{\mathfrak{o} \in \mathcal{O}_1} J_{\mathfrak{o}}(f_\infty \cdot 1_{K_f}).$$

Let $\mathfrak{o} \in \mathcal{O}_1$, and let $\sigma \in G(\mathbb{Q}) \cap \mathfrak{o}$ be a semisimple representative for $\mathfrak{o}$. Then σ is in $G(\mathbb{R})$ conjugate to some element σ_∞ in $\mathrm{O}(n)$. For each $\mathfrak{o}$ and $f \in C_c^\infty(G(\mathbb{A})^1)$ we have the fine expansion

$$J_{\mathfrak{o}}(f) = \sum_{(M,\gamma)} a^M(\gamma, S) J_M(\gamma, f),$$

where S is a sufficiently large finite set of places of $\mathbb{Q}$ with $\infty \in S$, $a^M(\gamma, S)$ are certain global coefficients as defined in [Ar7], (M,γ) runs over all pairs of Levi subgroups M containing M_0 and γ over representatives of the $M(\mathbb{Q})$-conjugacy classes in $M(\mathbb{Q}) \cap \mathfrak{o}$, and

$J_M(\gamma, f)$ are S-adic weighted orbital integrals. Since the (finite) set $\mathcal{O}_1$ and the set S are fixed in our setting, the value of the coefficients $a^M(\gamma, S)$ is not relevant for us.

15.1. **Orbits of finite order for** $\mathrm{GL}(2)$**.** If $G = \mathrm{GL}(2)$, then each $\mathfrak{o} \in \mathcal{O}_1$ is represented by one of the following semisimple elements:

$$\sigma_0^{\pm} = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_i^{\pm} = \pm \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}, \quad i = 2, 3,$$

with $\theta_2 = \pi/2$ and $\theta_3 = \pi/3$. We accordingly write $\mathfrak{o}_i$ or $\mathfrak{o}_i^{\pm}$ for the associated equivalence classes. Note that $\sigma_1, \sigma_2^{\pm}, \sigma_3^{\pm}$ are all regular semisimple so that the associated equivalence class is in fact equal to the conjugacy class of the respective element. (In fact, $\mathfrak{o}_2^+ = \mathfrak{o}_2^-$, but we keep the superscript to make notation more uniform.) Moreover, since we assume that our test function f is K_∞-invariant, $J_{\mathrm{unip}}(f) = J_{\mathfrak{o}_0^-}(f)$ for $\mathfrak{o}_0^-$ the class attached to σ_0^-. Hence we only need to consider the regular elements.

The element σ_1 is the only of the remaining elements which is split over $\mathbb{R}$. Since it is regular, the distribution $\mathfrak{o}_1$ is of a simple form, namely,

$$J_{\mathfrak{o}_1}(f) = \int_{U_0(\mathbb{A})} f(u^{-1}\sigma_1 u) v_T(u)\, du$$

for every bi-K_∞-invariant $f \in C_c^\infty(G(\mathbb{A})^1)$. It follows that if $f = f_\infty \cdot 1_{K_f}$, then

$$J_{\mathfrak{o}_1}(f) = a_1 \int_{\mathbb{R}} f_\infty(u(x)) \log(1 + x^2)\, dx + a_2 \int_{\mathbb{R}} f_\infty(u(x))\, dx$$

where $u(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, and $a_1, a_2 \in \mathbb{R}$ are suitable constants depending only on K_f. Using Taylor expansion of $\log(1 + x^2)$ around $x = 0$, we get, as in § 12, that for every $N > 0$,

$$J_{\mathfrak{o}_1}(\tilde{\phi}_t^\nu) = t^{-(d-1)/2} \sum_{k=0}^{N} c_k t^{k/2} + O_N(t^{(N-d+1)/2})$$

for suitable coefficients c_k.

The remaining classes are regular elliptic and non-split over $\mathbb{R}$. In particular, for each $i \in \{2, 3\}$ we have

$$J_{\mathfrak{o}_i^{\pm}}(f) = a_i \int_{G_{\sigma_i}(\mathbb{R}) \backslash G(\mathbb{R})} f_\infty(g^{-1}\sigma_i^{\pm} g)\, dg$$

for a suitable constant $a_i \in \mathbb{R}$ again depending only on K_f and $\mathfrak{o}_i^{\pm}$. We have for $i = 2, 3$ that $G_{\sigma_i^{\pm}}(\mathbb{R}) = Z(\mathbb{R})K_\infty$ so that using KAK decomposition we get

$$J_{\mathfrak{o}_i^{\pm}}(f) = a_i \int_0^\infty f_\infty(a^{-1}\sigma_i^{\pm} a) \sinh(2X)\, dX$$

where $a = e^X$. We can write

$$\pm \begin{pmatrix} \cos\theta_i & e^{-2X}\sin\theta_i \\ -e^{-2X}\sin\theta_i & \cos\theta_i \end{pmatrix} = a^{-1}\sigma_i^{\pm} a = k_1 \begin{pmatrix} e^Y & 0 \\ 0 & e^{-Y} \end{pmatrix} k_2$$

with suitable $k_1, k_2 \in K_\infty$ and $Y \geq 0$. Hence

$$\sinh(Y) = \alpha_i \sinh(2X)$$

with $\alpha_i = \sqrt{2}\sin\theta_i$. Hence

$$Y^2 = 4\alpha_i^2 X^2 + O_{\mathfrak{o}_i^\pm}(X^4)$$

around 0. Since $r(a^{-1}\sigma_i^\pm a) = \|(Y, -Y)\|$, we therefore get

$$r^2(a^{-1}\sigma_i^\pm a) = 8\alpha_i^2 X^2 + O_{\mathfrak{o}_i^\pm}(X^4).$$

Using the Taylor expansion of $\sinh(2X)$, we get, as in § 12, that for any N,

$$J_{\mathfrak{o}_i^\pm}(\tilde{\phi}_t^\nu) = t^{-(d-2)/2} \sum_{k=0}^N c_k t^{k/2} + O_{N,\mathfrak{o}_i^\pm}(t^{(N-d+2)/2})$$

as $t \to 0^+$ for suitable coefficients c_k depending on $\mathfrak{o}_i^\pm$.

15.2. **Orbits of finite order for** GL(3)**.** For $G = \mathrm{GL}(3)$ the real weighted orbital integrals associated to $G(\mathbb{R})$-conjugacy classes of elements in the classes in $\mathcal{O}_1$ can have a more complicated form. Each $\mathfrak{o} \in \mathcal{O}_1$ has a semisimple element which in $G(\mathbb{R})$ is conjugate to one of the following matrices:

$$\sigma_0^\pm = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_1^\pm = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \sigma_i^{\pm,\pm} = \pm \begin{pmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \quad i = 2, 3,$$

with θ_2 and θ_3 as for GL(2). Again, we write $\mathfrak{o}_i^\pm$, $i = 0, 1$, and $J_{\mathfrak{o}_i^{\pm,\pm}}$, $i = 2, 3$, for the associated equivalence classes. We already understand the distributions $J_{\mathfrak{o}_0^\pm}$.

If $i = 2, 3$, the only semi-standard Levi subgroups of $G(\mathbb{R})$ containing $\sigma_i^{\pm,\pm}$ are $M(\mathbb{R}) := \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R})$ (diagonally embedded in $G(\mathbb{R})$) and $G(\mathbb{R})$ itself. Let $P(\mathbb{R})$ be the standard parabolic subgroup of $G(\mathbb{R})$ with Levi component $M(\mathbb{R})$. Moreover, $\mathfrak{o}_i^{\pm,\pm}$ in fact equals the conjugacy class of $\sigma_i^{\pm,\pm}$. Hence we need to understand the real weighted orbital integrals $J_M(\sigma_i^{\pm,\pm}, \tilde{\phi}_{t,\infty}^\nu)$ and $J_G(\sigma_i^{\pm,\pm}, \tilde{\phi}_{t,\infty}^\nu)$. The latter integral equals

$$\begin{aligned} J_G(\sigma_i^{\pm,\pm}, \tilde{\phi}_{t,\infty}^\nu) &= \int_{M(\mathbb{R})_{\sigma_i^{\pm,\pm}} \backslash M(\mathbb{R})} \int_{U(\mathbb{R})} \tilde{\phi}_{t,\infty}^\nu(u^{-1}m^{-1}\sigma_i^{\pm,\pm} mu)\, du\, dm \\ &= \int_{M(\mathbb{R})_{\sigma_i^{\pm,\pm}} \backslash M(\mathbb{R})} \int_{U(\mathbb{R})} \tilde{\phi}_{t,\infty}^\nu(m^{-1}\sigma_i^{\pm,\pm} mu)\, du\, dm \end{aligned}$$

where we used the $O(n)$-conjugation invariance of $\tilde{\phi}_{t,\infty}^\nu$ and that the centralizer of $\sigma_i^{\pm,\pm}$ in $G(\mathbb{R})$ and $M(\mathbb{R})$ coincide. Hence for $t \to 0$ we get an asymptotic expansion

$$J_G(\sigma_i^{\pm,\pm}, \tilde{\phi}_{t,\infty}^\nu) = t^{-(d-4)/2} \sum_{k=0}^N C_k t^{k/2} + O_N(t^{(N-d+4)/2})$$

for any $N>0$ where C_k are certain coefficients depending on $\mathfrak{o}_i^{\pm,\pm}$.

The other weighted orbital integral is of the form

$$J_M(\sigma_i^{\pm,\pm},\tilde{\phi}_{t,\infty}^\nu)=\int_{M(\mathbb{R})_{\sigma_i^{\pm,\pm}}\backslash M(\mathbb{R})}\int_{U(\mathbb{R})}\tilde{\phi}_{t,\infty}^\nu(u^{-1}m^{-1}\sigma_i^{\pm,\pm}mu)v_M(u)\,du\,dm$$

where the weight function is given by

$$v_M(u)=\log(1+x^2+y^2)$$

for $u=\begin{pmatrix}1&0&x\\0&1&y\\0&0&1\end{pmatrix}\in U(\mathbb{R})$. A change of variables therefore gives

$$J_M(\sigma_i^{\pm,\pm},\tilde{\phi}_{t,\infty}^\nu)=\int_{M(\mathbb{R})_{\sigma_i^{\pm,\pm}}\backslash M(\mathbb{R})}\int_{U(\mathbb{R})}\tilde{\phi}_{t,\infty}^\nu(m^{-1}\sigma_i^{\pm,\pm}mu)\log\left(1+\|w\|^2\right)\,du\,dm$$

where

$$w=\left(id-m^{-1}\sigma_i^{\pm,\pm}m\right)^{-1}\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2.$$

Using the series expansion of log around 1, this integral also has an asymptotic expansion in t as $t\to 0$. Altogether, we get

$$J_{\mathfrak{o}_i^{\pm,\pm}}(\tilde{\phi}_t^\nu)=t^{-(d-4)/2}\sum_{k=0}^N B_k t^{k/2}+O_N(t^{(N-d+4)/2})$$

for suitable constants B_k and any $N>0$.

The remaining two classes $\mathfrak{o}_1^\pm$ contain more elements than just the conjugates of $\sigma_1^\pm$. Hence we have more orbital integrals to consider. Let $u_1=\left(\begin{smallmatrix}1&1&0\\0&1&0\\0&0&1\end{smallmatrix}\right)$. Then we need to consider the weighted orbital integrals $J_{M_0}(\sigma_1^\pm,f_\infty)$, $J_L(\sigma_1^\pm,f_\infty)$, and $J_L(\sigma_1^\pm u_1,f_\infty)$, $L=M,G$, for $f_\infty=\tilde{\phi}_{t,\infty}^\nu$. For the invariant integrals we get

$$J_G(\sigma_1^\pm,f_\infty)=\int_{U(\mathbb{R})}f_\infty(u^{-1}\sigma_1^\pm u)\,du=c_1\int_{U(\mathbb{R})}f_\infty(\sigma_1^\pm u)\,du,$$

and similarly, after a change of variables,

$$J_G(\sigma_1^\pm u_1,f_\infty)=c_2\int_{U_0(\mathbb{R})}f_\infty(\sigma_1^\pm u)\,du.$$

Here $c_1,c_2>0$ are suitable constants. The weighted orbital integrals can also be written as integrals over $U_0(\mathbb{R})$, but against a non-invariant measure. For $J_{M_0}(\sigma_1^\pm,f_\infty)$ it involves a weight function of the form $\log(1+x^2+y^2)$ as above and a linear function in $\log|a|$ if we write $u=\left(\begin{smallmatrix}1&a&0\\0&1&0\\0&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0&x\\0&1&y\\0&0&1\end{smallmatrix}\right)$. Similarly, $J_M(\sigma_1^\pm u_1,f_\infty)$ equals an integral over $U_0(\mathbb{R})$ against $\log(1+x^2+y^2)$ times the invariant measure, and $J_M(\sigma_i^\pm,f_\infty)$ equals the integral over $U(\mathbb{R})$

against $\log(1 + x^2 + y^2)$ times the invariant measure on $U(\mathbb{R})$. Proceeding similarly as before, one can then show that

$$J_{\mathfrak{o}_1^\pm}(\tilde{\phi}_t^\nu) = t^{-(d-3)/2} \sum_{k=0}^{N} C_k t^{k/2} + t^{-(d-3)/2} \sum_{k=0}^{N} B_k t^{k/2} \log t + O_N(t^{(N-d+3)/2})$$

for suitable constants C_k, B_k and any N.

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