A SELBERG TRACE FORMULA FOR NON-UNITARY TWISTS

WERNER MÜLLER

Abstract. Let $X = \Gamma\backslash G/K$ be a compact locally symmetric space. In this paper we establish a version of the Selberg trace formula for non-unitary representations of the lattice Γ. On the spectral side appears the spectrum of the "flat Laplacian" $\Delta^{\#}$, acting in the space of sections of the associated flat bundle. In general, this is a non-self-adjoint operator.

1. Introduction

Let G be a connected real semisimple Lie group with finite center and of non-compact type. Let K be a maximal compact subgroup of G. Then $S = G/K$ is a Riemannian symmetric space of nonpositive curvature. We fix an invariant metric on S which we normalize using the Killing form. Let $\Gamma \subset G$ be a discrete subgroup such that $\Gamma\backslash G$ is compact. We assume that Γ is torsion free. Then Γ acts properly discontinuously and fixed point free on S, and $X = \Gamma\backslash S$ is a compact locally symmetric manifold.

Let $\chi\colon \Gamma \to \mathrm{GL}(V_\chi)$ be a finite-dimensional unitary representation. Denote by $E_\chi \to \Gamma\backslash S$ the associated flat vector bundle. It is equipped with a canonical Hermitian fiber metric h^χ and a compatible flat connection ∇^χ. Let $d_\chi\colon C^\infty(X, E_\chi) \to \Lambda^1(X, E_\chi)$ be the associated exterior derivative and let δ_χ be the formal adjoint of d_χ with respect to the inner products in $C^\infty(X, E_\chi)$ and $C^\infty(X, T^*(X) \otimes E_\chi)$, respectively, induced by the invariant metric on S and the fiber metric h^χ in E_χ. Let $\Delta_\chi = \delta_\chi d_\chi$ be the associated Laplace operator. Then Δ_χ is a second order elliptic, formally self-adjoint, nonnegative differential operator. In this setting, the Selberg trace formula is an equality which expresses the trace of certain integral operators, which are functions of the Laplacian Δ_χ, in geometric terms associated to the conjugacy classes of Γ.

The trace formula has many applications. Of particular interest for the present paper are applications to Ruelle and Selberg zeta functions. Especially the analytic continuation and the functional equation of twisted Ruelle and Selberg zeta functions rely on the twisted Selberg trace formula [BO], [Se2]. Also spectral invariants of locally symmetric spaces such as analytic torsion and eta invariants can be studied with the help of the trace formula (see [Fr], [Mil], [MS1], [MS2]). So far, these applications are restricted to unitary

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representations of Γ and it this very desirable to extend the scope of the trace formula so that all finite-dimensional representations are covered. This is the main goal of this paper.

To begin with we recall the trace formula for a unitary representation χ (see [Se1], [Se2]). Let $\operatorname{spec}(\Delta_\chi)$ be the spectrum of Δ_χ. It consists of a sequence $0 \leq \lambda_1 < \lambda_2 < \cdots$ of eigenvalues of finite multiplicities. Denote by $m(\lambda_k)$ the multiplicity of λ_k. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be even and assume that the Fourier transform $\hat{\varphi}$ of φ belongs to $C_c^\infty(\mathbb{R})$. Then $\varphi((\Delta_\chi)^{1/2})$ is a trace class operator.

We note that in the scalar case it this convenient to introduce a shift of the spectrum of Δ_χ which is given by the lower bound c of the continuous spectrum of the Laplacian $\tilde{\Delta}$ on S. Then one considers the operator $\varphi\left((\Delta_\chi - c)^{1/2}\right)$ in place of $\varphi((\Delta_\chi)^{1/2})$. However this is not necessary at this stage. It only plays a role in the explicit expression of the trace formula (see (1.4)). Moreover for operators on vector bundles like the Laplacian on differential forms there is no appropriate choice of a shift of the spectrum. This will become clear in the discussions of section 6. Especially (6.20) shows that in general, there is no choice of a shift of the spectrum which leads to a simple formula for $\Theta_{\sigma,\lambda}(\operatorname{tr} h_\varphi)$ holding simultaneously for all σ. Therefore for the following discussion we prefer to work with $\varphi((\Delta_\chi)^{1/2})$. However everything that we say here holds for $\varphi\left((\Delta_\chi - c)^{1/2}\right)$ as well.

First observe that the trace of $\varphi((\Delta_\chi)^{1/2})$ is given by

$$\operatorname{Tr} \varphi((\Delta_\chi)^{1/2}) = \sum_{\lambda \in \operatorname{spec}(\Delta_\chi)} m(\lambda) \varphi(\lambda^{1/2}). \tag{1.1}$$

Let $\tilde{\Delta}$ be the Laplacian of S, and let h_φ be the convolution kernel of the invariant integral operator $\varphi(\tilde{\Delta}^{1/2})$. It belongs to the space $C_c^\infty(G//K)$ of K-bi-invariant compactly supported smooth functions on G. Given $\gamma \in \Gamma$ let $\{\gamma\}_\Gamma$ denote its Γ-conjugacy class. Furthermore, let G_γ and Γ_γ denote the centralizer of γ in G and Γ, respectively. Then the first version of the trace formula is the following identity.

$$\begin{aligned} \sum_{\lambda \in \operatorname{spec}(\Delta_\chi)} m(\lambda) \varphi(\lambda^{1/2}) &= \operatorname{vol}(\Gamma \backslash S) \dim V_\chi h_\varphi(e) \\ &\quad + \sum_{\{\gamma\}_\Gamma \neq e} \operatorname{tr} \chi(\gamma) \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} h_\varphi(g^{-1} \gamma g)\, d\dot{g}. \end{aligned} \tag{1.2}$$

To make this formula more explicit, one can use the Plancherel formula to express h_φ in terms of φ. Furthermore, the orbital integrals

$$I(g; \varphi) = \int_{G_\gamma \backslash G} h_\varphi(g^{-1} \gamma g)\, d\dot{g}$$

are invariant distributions and therefore, one can use Harish-Chandra's Fourier inversion formula to compute them (see [DKV, §4]). In the higher rank case this is rather complicated and no closed formula is available. In the rank one case, however, the situation is much better. There is a simple formula expressing the orbital integrals in terms of characters which leads to an explicit form of the trace formula [Wa, Theorem 6.7].

To extend the Selberg trace formula to all finite-dimensional representations of Γ, we first note that, because h_φ has compact support, the sum on the right hand side of (1.2) is finite and therefore, it is well defined for all finite-dimensional representations χ. The question is what is the appropriate operator which replaces the Laplacian on the left hand side. In general there is no Hermitian metric on E_χ which is compatible with the flat connection ∇^χ. A special case has been studied by Fay [Fa]. He considered the analytic torsion $T_M(\chi)$ of a Riemann surface $M = \Gamma\backslash\mathbb{H}$ of genus $g > 1$ and a unitary character $\chi \in \operatorname{Hom}(\Gamma, S^1)$ and established the analytic continuation of $T_M(\chi)$ to all characters $\chi \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$. To this end he introduced a non-self-adjoint Laplacian. We use a similar approach in the general case. The operator that replaces Δ_χ is the "flat Laplacian" $\Delta^\#_\chi$ which is defined as follows. Let $*\colon \Lambda^p(T^*X) \to \Lambda^{n-p}(T^*X)$ be the Hodge star operator associated to the Riemannian metric of X. Extend $*$ to an operator $*_\chi$ in $\Lambda^p(T^*X) \otimes E_\chi$ by $*_\chi = * \otimes \operatorname{Id}_{E_\chi}$. Define $\delta^\#_\chi := (-1)^{n+1} *_\chi d_\chi *_\chi$. Then the flat Laplacian $\Delta^\#$ is defined as

$$\Delta^\#_\chi = \delta^\#_\chi d_\chi.$$

If χ is unitary, $\Delta^\#_\chi$ equals Δ_χ. For an arbitrary χ we pick any Hermitian fiber metric in E_χ and use it together with the Riemannian metric on X to introduce an inner product in $C^\infty(X, E_\chi)$. In general, $\Delta^\#_\chi$ is a not self-adjoint w.r.t. this inner product. However, if we define the corresponding Laplace operator Δ_χ as above by $\delta_\chi d_\chi$, where the formal adjoint δ_χ is taken w.r.t. to the inner product, then $\Delta^\#_\chi$ has the same principal symbol as Δ_χ. This implies that the operator $\Delta^\#_\chi$ has nice spectral properties. Its spectrum is discrete and contained in a translate of a positive cone $C \subset \mathbb{C}$ with $\mathbb{R}^+ \subset C$ (see [Sh]). If we assume for the moment that the origin does not belong to the spectrum, then it follows that an Agmon angle θ exists for $\Delta^\#_\chi$.

Now recall that φ is the inverse Fourier transform of an even function $\hat{\varphi} \in C^\infty_c(\mathbb{R})$. Thus φ can be continued analytically to an entire function which is the Fourier-Laplace transform of $\hat{\varphi}$ and is usually called a *Paley-Wiener function.* We denote the space of Paley-Wiener functions on $\mathbb{C}$ by $\mathcal{P}(\mathbb{C})$. For its precise definition we refer to the section following (2.16). So from now on we will view φ as an even Paley-Wiener function.

Using the existence of an Agmon angle, this permits us to define $\varphi\left((\Delta^\#_\chi)^{1/2}_\theta\right)$ by the usual functional calculus [Sh]. It is a trace class operator. Since φ is assumed to be even, $\varphi\left((\Delta^\#_\chi)^{1/2}_\theta\right)$ is independent of θ and we can delete θ from the notation. Lidskii's theorem [GK, Theorem 8.4] generalizes (1.1). As mentioned above, the spectrum $\operatorname{spec}(\Delta^\#_\chi)$ of $\Delta^\#_\chi$ is discrete and consists of eigenvalues only. For $\lambda \in \operatorname{spec}(\Delta^\#_\chi)$ let $m(\lambda)$ denote the *algebraic* multiplicity of λ, i.e., $m(\lambda)$ is the dimension of the root space which consists of all $f \in C^\infty(X, E_\chi)$ such that there is $N \in \mathbb{N}$ with $(\Delta^\#_\chi - \lambda \mathrm{I})^N f = 0$. Then by Lidskii's theorem we have

$$\operatorname{Tr} \varphi((\Delta^\#_\chi)^{1/2}) = \sum_{\lambda \in \operatorname{spec}(\Delta^\#_\chi)} m(\lambda)\varphi(\lambda^{1/2}). \tag{1.3}$$

The first version of our trace formula generalizes (1.2) with $\operatorname{Tr}\varphi\left((\Delta_\chi^\#)^{1/2}\right)$ on the left hand side.

Actually, we prove a more general result. Let τ be an irreducible representation of K and $E_\tau \to \Gamma\backslash S$ the associated locally homogeneous vector bundle, equipped with its canonical invariant connection ∇^τ. Let $\nabla = \nabla^{\tau,\chi}$ be the product connection in $E_\tau \otimes E_\chi$, and let $\Delta^\#_{\tau,\chi} = -\operatorname{Tr}(\nabla^2)$ be the corresponding connection Laplacian. Then for φ as above $\varphi\left((\Delta^\#_{\tau,\chi})^{1/2}\right)$ is a trace class operator and we establish a trace formula for this operator which is similar to the scalar case.

If G has split rank one, we get an explicit version of the trace formula. To describe it we need to introduce some notation. Let $G = KAN$ be an Iwasawa decomposition of G. Then $\dim A = 1$. Let $\mathfrak{a}$ be the Lie algebra of A. The restriction of the Killing form to $\mathfrak{a}^*$ defines an inner product on $\mathfrak{a}^*$. Let $|\rho|$ denote the norm of the half-sum ρ of positive roots of (G, A). Let $\gamma \in \Gamma \setminus \{e\}$. Then there is a unique closed geodesic τ_γ that corresponds to the Γ-conjugacy class $\{\gamma\}_\Gamma$ of γ. Denote by $l(\gamma)$ the length of τ_γ. Furthermore, let $\gamma_0 \in \Gamma$ be the unique primitive element such that $\gamma = \gamma_0^k$ for some $k \in \mathbb{N}$. Finally let $D(\gamma)$ be the discriminant of γ (see (6.2) for its definition). Let $\beta(\lambda)d\lambda$ be the Plancherel measure for spherical functions on G [Hel]. We can now state our main result in the scalar case. As remarked above, in the scalar case it is convenient to introduce a shift of the spectrum by the lower bound of the essential spectrum of the Laplacian $\tilde{\Delta}$ on S which in the present case equals $|\rho|^2$. To introduce this shift is suggested by (6.23) and (6.24), because it leads to the simplified formulas (6.25) and (6.26) for the spherical Fourier transform of the kernel of the operator $\varphi\left((\Delta_\chi - |\rho|^2)^{1/2}\right)$. Using these observations, we get our main result in the scalar case which is the following theorem.

Theorem 1.1. *Let φ be an even Paley-Wiener function and let $\widehat{\varphi} \in C_c^\infty(\mathbb{R})$ be the Fourier transform of $\varphi|_\mathbb{R}$. Then we have*

$$\begin{aligned}\sum_{\lambda\in\operatorname{spec}\left(\Delta_\chi^\#\right)} m(\lambda)\varphi\left((\lambda-|\rho|^2)^{1/2}\right) =& \dim(V_\chi)\frac{\operatorname{vol}(\Gamma\backslash S)}{2}\int_\mathbb{R}\varphi(\lambda)\beta(\lambda)\,d\lambda \\ &+ \sum_{\{\gamma\}_\Gamma\neq e}\operatorname{tr}\chi(\gamma)\frac{l(\gamma_0)}{D(\gamma)}\widehat{\varphi}(l(\gamma)).\end{aligned} \tag{1.4}$$

Note that for every $c > 0$ there are only finitely many conjugacy classes $\{\gamma\}_\Gamma$ with $l(\gamma) \le c$. Therefore the sum on the right hand side is finite. If χ is unitary, this is the trace formula established by Selberg [Se1], [Hej].

To describe our method we restrict attention to the scalar case, i.e, we consider the operator $\Delta_\chi^\# - |\rho|^2$. Our method is based on the approach of Bunke and Olbrich [BO] to the Selberg trace formula in the unitary case. We consider the wave equation

$$\left(\frac{\partial^2}{\partial t^2} + \Delta_\chi^\# - |\rho|^2\right)u(t) = 0, \quad u(0) = f,\ u_t(0) = 0, \tag{1.5}$$

for any initial conditions $f \in C^\infty(X, E_\chi)$. Since the principal symbol of $\Delta_{\chi^\#} - |\rho|^2$ is given by $\sigma(x,\xi) = \| \xi \|^2 \operatorname{Id}_{E_x}$, the operator $L = \frac{\partial^2}{\partial t^2} + \Delta_\chi^\# - |\rho|^2$ is strictly hyperbolic in the sense of [Ta1, Chapt. IV, §3]. Therefore (1.5) has a unique solution $u(t; f)$. Let $\varphi \in \mathcal{P}(\mathbb{C})$ be even and let $\hat{\varphi} \in C_c^\infty(\mathbb{R})$ be the Fourier transform of $\varphi|_{\mathbb{R}}$. Then it follows that

$$\varphi\left((\Delta_\chi^\# - |\rho|^2)^{1/2}\right) f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}(t) u(t; f)\, dt. \tag{1.6}$$

Let $\tilde{u}(t; f)$ and $\tilde{f}$ denote the lift of $u(t, f)$ and f, respectively, to S which is a universal covering of X. Then the corresponding wave equation on S with initial conditions $u(0) = \tilde{f}$, $u_t(0) = 0$ is also strictly hyperbolic and by finite propagation speed it follows that it has a unique solution $u(t; \tilde{f})$. Thus we obtain $\tilde{u}(t, \tilde{x}; f) = u(t, \tilde{x}; \tilde{f})$. Since the lift of E_χ to S is trivial, the lifted operator $\tilde{\Delta}_\chi^\#$ takes the form $\tilde{\Delta}_\chi^\# = \tilde{\Delta} \otimes \operatorname{Id}_{V_\chi}$, where $\tilde{\Delta}$ is the Laplace operator on S. Let $h_\varphi \in C_c^\infty(G//K)$ be the kernel of the G-invariant integral operator $\varphi\left((\tilde{\Delta} - |\rho|^2)^{1/2}\right)$. Then it follows that the kernel $K_\varphi(x, y)$ of $\varphi\left((\Delta_\chi^\# - |\rho|^2)^{1/2}\right)$ is given by

$$K_\varphi(x, y) = \sum_{\gamma \in \Gamma} h_\varphi(g_1^{-1} \gamma g_2) \chi(\gamma), \tag{1.7}$$

where $x = \Gamma g_1 K$ and $y = \Gamma g_2 K$. The derivation of this formula is one of the key issues in the proof. The main steps are (3.17), (4.3) and (5.6). One can now proceed in the same way as in the case of a unitary representation χ and derive the twisted Selberg trace formula (1.4).

Besides unitary representations of Γ, there is a second class of representations of Γ for which the usual trace formula can be applied. These are representations which are the restriction to Γ of a finite-dimensional representation $\eta\colon G \to \operatorname{GL}(E)$. Let $E_\eta \to X$ be the flat vector bundle associated to $\eta|_\Gamma$. Then E_η is canonically isomorphic to the locally homogeneous vector bundle E_τ associated to the principal K-bundle $\Gamma\backslash G \to X$ via the representation $\tau = \eta|_K$. The bundle carries a canonical Hermitian fiber metric and the Laplacian in $C^\infty(X, E_\eta)$ with respect to this metric is closely related to the Casimir operator acting in $C^\infty(X, E_\tau)$. This brings us back to the usual framework of the Selberg trace formula for locally homogeneous vector bundles. Details will be discussed in section 7.

We also note that Petersson [Pe] started to develop a theory of vector-valued holomorphic automorphic forms.

Finally, let me point out two problems related to a possible extension of this work. First it would be interesting to treat also the finite volume case. The main problem is the continuous spectrum which I don't know how to deal with. Secondly, in the unitary case there is the representation theoretic framework for the Selberg trace formula (see [Wa]). It would be interesting to see if there is a representation theoretic approach which works in the nonunitary case.

The paper is organized as follows. In section 2 we collect a number of facts about spectral theory of elliptic operators with leading symbol of Laplace type and we develop some functional calculus for such operators. The kernels of the associated integral operators are studied in section 3. Especially, we prove (1.6) and (1.7). In section 4 we apply these results to the case of twisted Bochner-Laplace operators. In section 5 we turn to the locally symmetric case and we prove the first version of the trace formula which is Proposition 5.1. In section 6 we specialize to the case where G has split rank one and we prove Theorem 1.1. In the final section 7 we are concerned with representations of Γ which are the restriction of a representation of G.

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2. Functional calculus

In this section we develop the necessary facts of the functional calculus we are going to use in this paper.

Let X be a compact Riemannian manifold without boundary of dimension n and $E \to X$ a Hermitian vector bundle over X. Let ∇ be a covariant derivative in E which is compatible with the Hermitian metric. We denote by $C^\infty(X, E)$ the space of smooth sections of E, and by $L^2(X, E)$ the space of L^2-sections of E w.r.t. the metrics on X and E. Furthermore, for each $s \in \mathbb{R}$ we will denote by $H^s(X, E)$ the Sobolev space of order s of sections of E (see [Sh, I, §7]. Let

$$\Delta_E = \nabla^* \nabla$$

be the Bochner-Laplace operator associated to the connection ∇ and the Hermitian fiber metric. Then Δ_E is a second order elliptic differential operator which is essentially self-adjoint in $L^2(X, E)$. Its leading symbol $\sigma(\Delta_E)\colon \pi^*E \to \pi^*E$, where π is the projection of T^*X, is given by

$$\sigma(\Delta_E)(x,\xi) = \| \xi \|_x^2 \cdot \mathrm{Id}_{E_x}, \quad x \in X,\ \xi \in T_x^*X. \tag{2.1}$$

In this section we consider the class of elliptic operators

$$P\colon C^\infty(X, E) \to C^\infty(X, E)$$

which are the perturbation of Δ_E by a first order differential operator, i.e., we assume that

$$P = \Delta_E + D, \tag{2.2}$$

where $D\colon C^\infty(X, E) \to C^\infty(X, E)$ is a first order differential operator. Equivalently, one can say that P is an elliptic second order differential operator with leading symbol given by

$$\sigma(P)(x,\xi) = \| \xi \|_x^2 \cdot \mathrm{Id}_{E_x}. \tag{2.3}$$

For $I \subset [0, 2\pi]$ let

$$\Lambda_I = \{re^{i\theta} \colon 0 \le r < \infty,\ \theta \in I\}.$$

be the solid angle attached to I. The following lemma describes the structure of the spectrum of P.

Lemma 2.1. *For every $0 < \varepsilon < \pi/2$ there exists $R > 0$ such that the spectrum of P is contained in the set $B_R(0) \cup \Lambda_{[-\varepsilon,\varepsilon]}$. Moreover the spectrum of P is discrete.*

Proof. The first statement follows from [Sh, Theorem 9.3]. The discreteness of the spectrum follows from [Sh, Theorem 8.4]. $\square$

Though P is not self-adjoint in general, it still has nice spectral properties [Sh, Chapt. I, §8]. Given $\lambda_0 \in \operatorname{spec}(P)$, let Γ_{λ_0} be a small circle around λ_0 which contains no other points of $\operatorname{spec}(P)$. Put

$$\Pi_{\lambda_0} = \frac{i}{2\pi} \int_{\Gamma_{\lambda_0}} R_\lambda(P)\, d\lambda. \tag{2.4}$$

Then Π_{λ_0} is the projection onto the *root subspace* V_{λ_0}. This is a finite-dimensional subspace of $C^\infty(X, E)$ which is invariant under P and there exists $N \in \mathbb{N}$ such that $(P - \lambda_0 \mathrm{I})^N V_{\lambda_0} = 0$. Furthermore, there is a closed complementary subspace V'_{λ_0} to V_{λ_0} in $L^2(X, E)$ which is invariant under the closure $\bar{P}$ of P in L^2 and the restriction of $(\bar{P} - \lambda_0 \mathrm{I})$ to V'_{λ_0} has a bounded inverse. The *algebraic multiplicity* $m(\lambda_0)$ of λ_0 is defined as

$$m(\lambda_0) = \dim V_{\lambda_0}.$$

If $\lambda_1, \lambda_2 \in \operatorname{spec}(P)$ with $\lambda_1 \neq \lambda_2$, then the projections Π_{λ_1} and Π_{λ_2} are disjoint, i.e.,

$$\Pi_{\lambda_1}\Pi_{\lambda_2} = \Pi_{\lambda_2}\Pi_{\lambda_1} = 0.$$

Let $R_\lambda(\Delta_E)$ be the resolvent of Δ_E and let D be the first order differential operator occurring in (2.2). Since D is a first order operator, it follows from Sobolev space theory that $DR_\lambda(\Delta_E)$ is a compact operator in $L^2(X, E)$. This means that D is compact relative to Δ_E. Therefore by [Mk, I,§4, Theorem 4.3] the root vectors are complete. This means that $L^2(X, E)$ is the closure of the algebraic direct sum of finite-dimensional P-invariant subspaces V_k

$$L^2(X, E) = \overline{\bigoplus_{k \ge 1} V_k} \tag{2.5}$$

such that the restriction of P to V_k has a unique eigenvalue λ_k, for each k there exists $N_k \in \mathbb{N}$ such that $(P - \lambda_k \mathrm{I})^{N_k} V_k = 0$, and $|\lambda_k| \to \infty$. In general, the sum (2.5) is not a sum of mutually orthogonal subspaces. This generalizes the spectral decomposition of a self-adjoint operator. Here, of course, we are making use of the compactness of X. At the moment it is not clear to the author how to generalize (2.5) in the cofinite case.

Given $r > 0$, let

$$N(r, P) := \sum_{\lambda \in \operatorname{spec}(P),\ |\lambda| \le r} m(\lambda).$$

be the counting function of the eigenvalues of P, where eigenvalues are counted with their algebraic multiplicity. For the self-adjoint operator Δ_E we have Weyl's formula which describes the asymptotic behavior of the counting function as $r \to \infty$. Since P is a perturbation of Δ_E by a lower order differential operator, we may expect Weyl's formula to hold for P as well. This is indeed the case as the following lemma shows.

Lemma 2.2. *Let $n = \dim X$. We have*

$$N(r,P) = \frac{\operatorname{rk}(E)\operatorname{vol}(X)}{(4\pi)^{n/2}\Gamma(n/2+1)} r^{n/2} + o(r^{n/2}), \quad r \to \infty.$$

Proof. We note that the trace $\operatorname{Tr}(e^{-t\Delta_E})$ of the heat semigroup $e^{-t\Delta_E}$ has an asymptotic expansion of the form

$$\operatorname{Tr}(e^{-t\Delta_E}) \sim t^{-n/2} \sum_{k \geq 0} a_k t^k, \quad t \to +0$$

(see [Gi, Lemma 1.8.3]), and by [Gi, Lemma 4.1.4] the leading coefficient a_0 is given by $a_0 = (4\pi)^{-n/2}\operatorname{rk}(E)\operatorname{vol}(X)$. Let $N(r,\Delta_E)$ be the counting function of the eigenvalues of Δ_E. Using the Tauberian theorem (see [Sh, Chapt. II, §14]), we get

$$N(r,\Delta_E) = \frac{\operatorname{rk}(E)\operatorname{vol}(X)}{(4\pi)^{n/2}\Gamma(n/2+1)} r^{n/2} + o(r^{n/2}), \quad r \to \infty. \tag{2.6}$$

The lemma follows from [Mk, I, §8, Corollary 8.5]. □

Denote by $\operatorname{spec}(P)$ the spectrum of P. First we assume that $0 \notin \operatorname{spec}(P)$. It follows from Lemma 2.1 that there exists an Agmon angle θ for P and we can define the square root $P_\theta^{1/2}$ as in [Sh]. For the convenience of the reader we include some details. By Lemma 2.1 there exist $0 < \theta < 2\pi$ and $\varepsilon > 0$ such that

$$\operatorname{spec}(P) \cap \Lambda_{[\theta-\varepsilon,\theta+\varepsilon]} = \emptyset.$$

θ is called an Agmon angle for P. Since $\operatorname{spec}(P)$ is discrete and $0 \notin \sigma(P)$, there exists also $r_0 > 0$ such that

$$\operatorname{spec}(P) \cap \{z \in \mathbb{C} \colon |z| < 2r_0\} = \emptyset.$$

Define the contour $\Gamma = \Gamma_{\theta,r_0} \subset \mathbb{C}$ as the union of three curves $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 = \{re^{i\theta} \colon \infty > r \geq r_0\}, \quad \Gamma_2 = \{r_0 e^{i\alpha} \colon \theta \leq \alpha \leq \theta + 2\pi\},$$
$$\Gamma_3 = \{re^{i(\theta+2\pi)} \colon r_0 \leq r < \infty\}.$$

The curve Γ_{θ,r_0} is oriented as follows. On Γ_1, r runs from ∞ to r_0, Γ_2 is oriented clockwise, and on Γ_3, r runs from r_0 to ∞. Put

$$P_\theta^{-1/2} = \frac{i}{2\pi} \int_{\Gamma_{\theta,r_0}} \lambda^{-1/2}(P-\lambda)^{-1}\, d\lambda. \tag{2.7}$$

By [Sh, Corollary 9.2, Chapt. II, §9] we have $\| (P-\lambda)^{-1} \| \leq C|\lambda|^{-1}$ for $\lambda \in \Gamma_{\theta,r_0}$. Therefore the integral is absolutely convergent. Put

$$P_\theta^{1/2} = P \cdot P_\theta^{-1/2}.$$

Then $P_\theta^{1/2}$ satisfies $(P_\theta^{1/2})^2 = P$. If θ is fixed, we simply denote this operator by $P^{1/2}$. We recall [See], [Sh, Theorem 11.2] that $P^{1/2}$ is a classical pseudo-differential operator with principal symbol

$$\sigma(P^{1/2})(x,\xi) = \| \xi \|_x \cdot \mathrm{Id}_{E_x}. \tag{2.8}$$

In each coordinate chart, the complete symbol $q(x,\xi)$ of $P^{1/2}$ has an asymptotic expansion

$$q(x,\xi) \sim \sum_{j=0}^{\infty} q_{1-j}(x,\xi), \tag{2.9}$$

where $q_{1-j}(x,\xi)$ is a symbol of order $1-j$ (see [Sh, I,§1]) which is homogeneous in ξ of order $1-j$ and q_1 equals the principal symbol (2.8). The same holds for $\Delta_E^{1/2}$, i.e., in each coordinate chart, the complete symbol $\tilde{q}(x,\xi)$ of $\Delta_E^{1/2}$ has an asymptotic expansion of the form (2.9) and the principal symbol $\tilde{q}_1$ equals (2.8). Since the principal symbols of $P^{1/2}$ and $\Delta_E^{1/2}$ coincide, it follows that

$$P^{1/2} = \Delta_E^{1/2} + B, \tag{2.10}$$

where B is a pseudo-differential operator of order zero, i.e., in each coordinate chart, the complete symbol $b(x,\xi)$ of B has an asymptotic expansion of the form

$$b(x,\xi) \sim \sum_{j=0}^{\infty} b_{-j}(x,\xi). \tag{2.11}$$

Being a pseudo-differential operator of order zero, B extends to a bounded operator in $L^2(X,E)$ (see [Sh, Theorem 7.1]). Thus similarly to (2.2) we may regard $P^{1/2}$ as a perturbation of $\Delta_E^{1/2}$ by a pseudo-differential operator of order zero.

Let $R_\lambda(P^{1/2}) = (P^{1/2} - \lambda\,\mathrm{I})^{-1}$ and $R_\lambda(\Delta_E^{1/2}) = (\Delta_E^{1/2} - \lambda\,\mathrm{I})^{-1}$ be the resolvents of $P^{1/2}$ and $\Delta_E^{1/2}$, respectively. For $\lambda \notin \mathrm{spec}(\Delta_E^{1/2})$ we have the following equality

$$P^{1/2} - \lambda\,\mathrm{I} = (\mathrm{I} + BR_\lambda(\Delta_E^{1/2}))(\Delta_E^{1/2} - \lambda\,\mathrm{I}). \tag{2.12}$$

Since $\Delta_E^{1/2}$ is self-adjoint, the resolvent of $\Delta_E^{1/2}$ satisfies

$$\| R_\lambda(\Delta_E^{1/2}) \| \leq |\,\mathrm{Im}(\lambda)|^{-1} \tag{2.13}$$

[Ka, Chapt. V, §3.5]. Let $b = 2 \| B \|$. It follows from (2.13) that for $|\,\mathrm{Im}(\lambda)| \geq b$ we have $\| BR_\lambda(\Delta_E^{1/2}) \| \leq 1/2$. Thus in this range of λ the operator $\mathrm{I} + BR_\lambda(\Delta_E^{1/2})$ is invertible and

$$R_\lambda(P^{1/2}) = R_\lambda(\Delta_E^{1/2})(\mathrm{I} + BR_\lambda(\Delta_E^{1/2}))^{-1}.$$

Combined with (2.13) we get

$$\| R_\lambda(P^{1/2}) \| \leq 2|\,\mathrm{Im}(\lambda)|^{-1}, \quad |\,\mathrm{Im}(\lambda)| \geq b. \tag{2.14}$$

We can now summarize the spectral properties of $P^{1/2}$.

Lemma 2.3. *The resolvent of $P^{1/2}$ is compact. The spectrum of $P^{1/2}$ is discrete. There exist $b > 0$ and $c \in \mathbb{R}$ such that the spectrum of $P^{1/2}$ is contained in the domain*

$$\Omega_{b,c} = \left\{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > c, \;\; |\operatorname{Im}(\lambda)| < b\right\}. \tag{2.15}$$

Proof. Since $P^{1/2}$ is an elliptic pseudo-differential operator of order 1 on a closed manifold, its resolvent is compact and hence, its spectrum is discrete. The remaining statements are a consequence of (2.14). □

It follows from the spectral decomposition(2.5) that $P^{1/2}$ has a similar spectral decomposition with eigenvalues $\lambda^{1/2}$, $\lambda \in \operatorname{spec}(P)$, and $m(\lambda^{1/2}) = m(\lambda)$.

Now we introduce the functions which we will use for the functional calculus. Let $\mathcal{P}(\mathbb{C})$ be the space of Paley-Wiener functions on $\mathbb{C}$. Recall that

$$\mathcal{P}(\mathbb{C}) = \bigcup_{R>0} \mathcal{P}^R(\mathbb{C}) \tag{2.16}$$

with the inductive limit topology, where $\mathcal{P}^R(\mathbb{C})$ is the space of entire functions ϕ on $\mathbb{C}$ such that for every $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|\phi(\lambda)| \le C_N(1 + |\lambda|)^{-N} e^{R|\operatorname{Im}(\lambda)|}, \quad \lambda \in \mathbb{C}. \tag{2.17}$$

Given $h \in C_c^\infty((-R,R))$, let

$$\varphi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(r) e^{-ir\lambda}\, dr, \quad \lambda \in \mathbb{C},$$

be the Fourier-Laplace transform of h. Then φ satisfies (2.17) for every $N \in \mathbb{N}$, i.e., $\varphi \in \mathcal{P}^R(\mathbb{C})$. Conversely, by the Paley-Wiener theorem [Ho2, Theorem 7.3.1], every $\phi \in \mathcal{P}^R(\mathbb{C})$ is the Fourier-Laplace transform of a function in $C_c^\infty((-R,R))$.

First assume that $0 \notin \operatorname{spec}(P)$. For $b > 0$ and $d \in \mathbb{R}$ let $\Gamma = \Gamma_{b,d} \subset \mathbb{C}$ be the contour which is union of the two half-lines $L_{\pm b,d} = \{z \in \mathbb{C}\colon \operatorname{Im}(z) = \pm b,\; \operatorname{Re}(z) \ge d\}$ and the semi-circle $S = \{d + be^{i\theta}\colon \pi/2 \le \theta \le 3\pi/2\}$, oriented counterclockwise. By Lemma 2.3 there exist $b > 0$, $d \in \mathbb{R}$ such that $\operatorname{spec}(P^{1/2})$ is contained in the interior of $\Gamma_{b,d}$. For an even Paley-Wiener function $\varphi \in \mathcal{P}(\mathbb{C})$ put

$$\varphi(P^{1/2}) := \frac{i}{2\pi} \int_\Gamma \varphi(\lambda)(P^{1/2} - \lambda\,\mathrm{I})^{-1}\, d\lambda. \tag{2.18}$$

Note that by (2.17), $\varphi(\lambda)$ is rapidly decreasing in each strip $|\operatorname{Im}(\lambda)| < \delta$, $\delta > 0$. Therefore it follows from (2.14) that the integral is absolutely convergent.

Now assume that $0 \in \operatorname{spec}(P)$. Then we modify the definition of $\varphi(P^{1/2})$ as follows. Let Π_0 be the projection (2.4) onto the root space V_0 of the eigenvalue 0. We claim that Π_0 is a smoothing operator. This can be seen as follows. The range of Π_0 is a finite-dimensional subspace of $C^\infty(X,E)$. Therefore, for all $k,l \in \mathbb{N}$, the operator $P^k \Pi_0 P^l = P^{k+l}\Pi_0$ extends to a bounded operator in $L^2(X,E)$. Let $s \in \mathbb{R}$. Since P is a second order elliptic operator, the Sobolev space $H^s(X,E)$ is the completion of $C^\infty(X,E)$ with respect to the norm $\| (I+P)^{s/2} f \|$. Since for all $k,l\in\mathbb{N}$, $P^k\Pi_0P^l$ extends to a bounded operator in $L^2(X,E)$, it follows that for all $k,l\in\mathbb{N}$, Π_0 extends to bounded operator of $H^k(X,E)$ into $H^l(X,E)$ which by standard Sobolev space theory implies that Π_0 is a smoothing operator.

The complementary subspace V_0' of V_0 is invariant under P. Let $P_1 = P|_{V_0'}$. Then $0\notin \operatorname{spec}(P_1)$ and we can define $(P_1)^{-1/2}$ as above by formula (2.7) with P replaced by P_1. To derive the properties of $P_1^{1/2}$ it is convenient to introduce the auxiliary operator

$$\widehat{P} = (\mathrm{I} - \Pi_0)P \oplus \Pi_0. \tag{2.19}$$

Since Π_0 is a smoothing operator, $\widehat{P}$ is a pseudo-differential operator with the same principal symbol as P and $0\notin\operatorname{spec}(\widehat{P})$. So $\widehat{P}^{-1/2}$ can be defined by (2.7). By definition $\widehat{P}^{-1/2}|_{V_0'} = P_1^{-1/2}$. It follows that Lemma 2.3 holds for $P_1^{1/2}$ and we can define $\varphi(P_1^{1/2})$ by formula (2.18).

It remains to deal with contribution of $N = P\Pi_0$. The operator $N\colon V_0\to V_0$ is nilpotent, i.e, there exists $m\in\mathbb{N}$ such that $N^m=0$. In general, such an operator has no square root. Nevertheless we can define $\varphi(N^{1/2})$. We owe the following approach to one of the referees and we are very grateful to him for his help with this issue. Put

$$U(t;N) := \sum_{k=0}^{m} \frac{(-1)^k t^{2k}}{(2k)!} N^k, \tag{2.20}$$

where $m = \dim V_0$. It satisfies

$$\left(\frac{\partial^2}{\partial t^2} + N\right) U(t;N) = 0, \quad U(0;N) = \mathrm{I},\ \frac{\partial}{\partial t} U(t;N)|_{t=0} = 0. \tag{2.21}$$

Thus we may regard $U(t;N)$ as $\cos(tN^{1/2})$. Let $\varphi\in\mathcal{P}(\mathbb{C})$ be even and let $\hat\varphi$ be the Fourier transform of $\varphi|_{\mathbb{R}}$. Put

$$\varphi(N^{1/2}) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat\varphi(t) U(t;N)\, dt. \tag{2.22}$$

It follows from (2.20) that

$$\varphi(N^{1/2}) = \sum_{k=0}^{m} \frac{\varphi^{2k}(0)}{(2k)!} N^k. \tag{2.23}$$

Now we put

$$\varphi(P^{1/2}) := \varphi(P_1^{1/2})(\mathrm{I} - \Pi_0) + \varphi(N^{1/2})\Pi_0. \tag{2.24}$$

The relevant property of these operators is described by the following lemma.

Lemma 2.4. *$\varphi(P^{1/2})$ is an integral operator with a smooth kernel.*

Proof. First note that

$$\int_\Gamma \varphi(\lambda)(P_1-\lambda^2)(P_1^{1/2}-\lambda)^{-1}\,d\lambda=\int_\Gamma \varphi(\lambda)(P_1^{1/2}+\lambda)\,d\lambda=0.$$

This implies that for $k,l\in\mathbb{N}$ we have

$$P^k\varphi(P_1^{1/2})(\mathrm{I}-\Pi_0)P^l=\frac{i}{2\pi}\int_\Gamma \lambda^{2(k+l)}\varphi(\lambda)(P_1^{1/2}-\lambda)^{-1}\,d\lambda.$$

The function $\lambda\mapsto\lambda^{2(k+l)}\varphi(\lambda)$ is rapidly decreasing on $|\operatorname{Im}(\lambda)|=\pm b$. Hence the operator $P^k\varphi(P_1^{1/2})(\mathrm{I}-\Pi_0)P^l$ is a bounded operator in $L^2(X,E)$. Let $s\in\mathbb{R}$. Since P is elliptic of order 2, the Sobolev space $H^s(X,E)$ is the completion of $C^\infty(X,E)$ with respect to the norm $\|\,(\mathrm{I}+P)^{s/2}f\,\|$. Together with the above observation it follows that for all $s,r\in\mathbb{R}$, $\varphi(P_1^{1/2})(\mathrm{I}-\Pi_0)$ extends to a bounded operator from $H^s(X,E)$ to $H^r(X,E)$, which shows that $\varphi(P_1^{1/2})(\mathrm{I}-\Pi_0)$ is a smoothing operator.

Since Π_0 is a smoothing operator, it follows that N is a smoothing operator, and hence $\varphi(N^{1/2})\Pi_0$ also. Thus $\varphi(P^{1/2})$ is a smoothing operator and hence, an integral operator with a smooth kernel. □

In order to continue, we need to establish an auxiliary result about smoothing operators. Let

$$A\colon L^2(X,E)\to L^2(X,E)$$

be an integral operator with a smooth kernel $H\in C^\infty(X\times X,E\boxtimes E^*)$.

Proposition 2.5. *A is a trace class operator and*

$$\operatorname{Tr}(A)=\int_X \operatorname{tr}H(x,x)\,d\mu(x). \tag{2.25}$$

Proof. We generalize the proof of Theorem 1 in [La, Chapt VII, §1]. Let ∇^E be a Hermitian connection in E and let $\Delta_E=(\nabla^E)^*\nabla^E$ be the associated Bochner-Laplace operator. Then Δ_E is a second order elliptic operator which is essentially self-adjoint and non-negative. Its spectrum is discrete. Let $\{\phi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(X,E)$ consisting of eigensections of Δ_E with eigenvalues $0\le\lambda_1\le\lambda_2\le\cdots\to\infty$. In other word, they are the eigensections for the standard self-adjoint unperturbed operator on $C^\infty(X,E)$. We can expand H in the orthonormal basis as

$$H(x,y)=\sum_{i,j=1}^\infty a_{i,j}\phi_i(x)\otimes\phi_j^*(y), \tag{2.26}$$

where

$$a_{i,j}=\langle A\phi_i,\phi_j\rangle. \tag{2.27}$$

Since H is smooth, the coefficients $a_{i,j}$ are rapidly decreasing. Indeed, for every N we have

$$(1+\lambda_i+\lambda_j)^N a_{i,j}=\langle(\mathrm{I}+\Delta_E\otimes\mathrm{I}+\mathrm{I}\otimes\Delta_E)^N H,\phi_i\otimes\phi_j^*\rangle.$$

Hence for every $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$|a_{i,j}| \leq C_N (1 + \lambda_i + \lambda_j)^{-N}, \quad i, j \in \mathbb{N}.$$

This implies that the series (2.26) converges in the C^∞ topology. Let $P_{i,j}$ be the integral operator with kernel $\phi_i \otimes \phi_j^*$. Thus

$$P_{i,j}(\phi_k) = \begin{cases} 0, & k \neq j; \\ \phi_i, & k = j. \end{cases}$$

Let P_j be the orthogonal projection of $L^2(X, E)$ onto the 1-dimensional subspace $\mathbb{C}\phi_j$. Put

$$B = \sum_{i,j} a_{i,j} (1 + \lambda_j)^n P_{i,j}, \quad C = \sum_j (1 + \lambda_j)^{-n} P_j.$$

Then $A = BC$ and it follows from (2.6) that B and C are Hilbert-Schmidt operators. Thus A is a trace class operator. Furthermore, by (2.26) and (2.27) we get

$$\int_X \operatorname{tr} H(x,x)\, dx = \sum_{i,j=1}^\infty a_{i,j} \int_X \langle \phi_i(x), \phi_j(x) \rangle\, dx = \sum_{i=1}^\infty a_{i,i} = \operatorname{Tr}(A).$$

$\square$

Now we apply this result to $\varphi(P^{1/2})$. Let $K_\varphi(x,y)$ be the kernel of $\varphi(P^{1/2})$. Then by Proposition 2.5, $\varphi(P^{1/2})$ is a trace class operator and we have

$$\operatorname{Tr} \varphi(P^{1/2}) = \int_X \operatorname{tr} K_\varphi(x,x)\, d\mu(x). \tag{2.28}$$

By Lidskii's theorem [GK, Theorem 8.4] the trace is equal to the sum of the eigenvalues of $\varphi(P^{1/2})$, counted with their algebraic multiplicities. The eigenvalues of $\varphi(P^{1/2})$ and their algebraic multiplicities can be determined as follows. Given $N \in \mathbb{N}$, let Π_N denote the projection onto the direct sum of the root subspaces V_k, $k \leq N$, of P. As explained above, we have

$$P\Pi_N = \sum_{k=1}^N (\lambda_k \Pi_k + D_k),$$

where Π_k is the projection onto V_k and D_k is a nilpotent operator in V_k. Note that this is just the Jordan normal form of a linear operator in a finite-dimensional complex vector space. Then it follows from [Ka, I, (5.50)] that

$$\varphi(P^{1/2})\Pi_N = \sum_{k=1}^N (\varphi(\lambda_k^{1/2})\Pi_k + D_k'),$$

where D_k' is again a nilpotent operator in V_k. Thus $\varphi(P^{1/2})$ leaves the decomposition (2.5) invariant and the restriction of $\varphi(P^{1/2})$ to V_k has a unique eigenvalue $\varphi(\lambda_k^{1/2})$. Of course, some of the eigenvalues $\varphi(\lambda_k^{1/2})$ may coincide in which case the root space is the sum of the corresponding root spaces V_k. Now, applying Lidskii's theorem [GK, Theorem 8.4] and (2.28), we get the following proposition.

Proposition 2.6. *Let $\varphi \in \mathcal{P}(\mathbb{C})$ be even and let $\hat{\varphi}$ be the Fourier transform of $\varphi|_{\mathbb{R}}$. Then we have*

$$\sum_{\lambda \in \operatorname{spec}(P)} m(\lambda)\varphi(\lambda^{1/2}) = \int_X \operatorname{tr} K_\varphi(x,x)\, dx. \tag{2.29}$$

By Lemma 2.2, the series on the left hand side is absolutely convergent.

Remark. Recall that $P^{1/2} = P_\theta^{1/2}$ depends on the choice of an Agmon angle θ, and so do the eigenvalues $\lambda_k^{1/2} = (\lambda_k)_\theta^{1/2}$. Let $0 < \theta < \theta' < 2\pi$ be two Agmon angles. Then it follows from Lemma 2.1 that there are only finitely many eigenvalues $\lambda_1, ..., \lambda_m$ of P which are contained in $\Lambda_{[\theta,\theta']}$. Therefore for $\lambda \in \operatorname{spec}(P)$ we have

$$(\lambda)_{\theta'}^{1/2} = \begin{cases} (\lambda)_\theta^{1/2}, & \text{if } \lambda \notin \{\lambda_1, ..., \lambda_m\}; \\ -(\lambda)_\theta^{1/2}, & \text{if } \lambda \in \{\lambda_1, ..., \lambda_m\}. \end{cases}$$

Since φ is even $\varphi\left((\lambda)_\theta^{1/2}\right)$ is independent of θ. This justifies the notation on the left hand side of (2.29). □

3. The kernel and the wave equation

In this section we give a description of the kernel K_φ of the smoothing operator $\varphi(P^{1/2})$ in terms of the solution of the wave equation. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} + Pu = 0, \quad u(0,x) = f(x),\ u_t(0,x) = 0. \tag{3.1}$$

Proposition 3.1. *For each $f \in C^\infty(X,E)$ there is a unique solution $u(t;f) \in C^\infty(\mathbb{R} \times X, E)$ of the wave equation (3.1) with initial condition f. Moreover for every $T > 0$ and $s \in \mathbb{R}$ there exists $C > 0$ such that for every $f \in C^\infty(X,E)$*

$$\| u(t,f) \|_s \leq C \| u(0,f) \|_s, \quad |t| \leq T, \tag{3.2}$$

where $\| \cdot \|_s$ denotes the s-Sobolev norm.

Proof. We proceed in the same way as in [Ta1, Chapt. IV, §§1,2] and replace (3.1) by a first order system. Let Δ_E be the Bochner-Laplace operator associated to the connection ∇^E in E. Put $\Lambda = (\Delta_E + \mathrm{Id})^{1/2}$ and

$$L := \begin{pmatrix} 0 & \Lambda \\ -P\Lambda^{-1} & 0 \end{pmatrix} : \begin{matrix} C^\infty(X,E) \\ \oplus \\ C^\infty(X,E) \end{matrix} \longrightarrow \begin{matrix} C^\infty(X,E) \\ \oplus \\ C^\infty(X,E) \end{matrix}. \tag{3.3}$$

Then L is a pseudo-differential operator of order 1. Let u be a solution of (3.1). Put

$$u_1 = \Lambda u, \quad u_2 = \frac{\partial}{\partial t} u.$$

Then (u_1, u_2) satisfies

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = L\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = \Lambda f,\ u_2(0) = 0. \tag{3.4}$$

On the other hand, let (u_1, u_2) be a solution of the initial value problem (3.4). Put $u = \Lambda^{-1}u_1$. Then u is a solution of (3.1). Thus it suffices to consider (3.4). By (2.3) it follows that $P = \Delta_E + D$ where D is a differential operator of order ≤ 1. Therefore we get

$$P\Lambda^{-1} = (\Delta_E + \mathrm{Id})\Lambda^{-1} + (D - \mathrm{Id})\Lambda^{-1} = \Lambda + B_1,$$

where B_1 is a pseudo-differential operator of order 0. Therefore

$$L + L^* = \begin{pmatrix} 0 & \Lambda \\ -\Lambda - B_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\Lambda - B_1^* \\ \Lambda & 0 \end{pmatrix} = -\begin{pmatrix} 0 & B_1^* \\ B_1 & 0 \end{pmatrix},$$

is a pseudo-differential operator of order zero. Hence (3.4) is a symmetric hyperbolic system in the sense of [Ta1, Chapt. IV, §2]. So we can proceed as in the proof of Theorem 2.3 in [Ta1, Chapt. IV, §2] to establish existence and uniqueness of solutions of (3.1). The estimation (3.2) follows from the proof using Gronwall's inequality. □

Proposition 3.2. *Let $\varphi \in \mathcal{P}(\mathbb{C})$ be even and let $\widehat{\varphi}$ be the Fourier transform of $\varphi|_{\mathbb{R}}$. Then for every $f \in C^\infty(X, E)$ we have*

$$\varphi(P^{1/2})f = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{\varphi}(t)u(t; f)\, dt. \tag{3.5}$$

Proof. First consider the case where $0 \notin \operatorname{spec}(P)$. Let $\Gamma \subset \mathbb{C}$ be as in (2.18). Let $c \geq 0$ be such that the spectrum of $P + c$ is contained in $\operatorname{Re}(z) > 0$. For $\sigma > 0$ define the operator $\cos(tP^{1/2})e^{-\sigma(P+c)}$ by the functional integral

$$\cos(tP^{1/2})e^{-\sigma(P+c)} = \frac{i}{2\pi}\int_\Gamma \cos(t\lambda)e^{-\sigma(\lambda^2+c)}(P^{1/2} - \lambda)^{-1}\, d\lambda.$$

By (2.14) the integral is absolutely convergent. For $f \in C^\infty(X, E)$ and $\sigma > 0$ put

$$u(t; \sigma, f) := \cos(tP^{1/2})e^{-\sigma(P+c)}f. \tag{3.6}$$

Then $u(t; \sigma, f)$ satisfies

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} + P\right)u(t; \sigma, f) &= \frac{i}{2\pi}\int_\Gamma \cos(t\lambda)e^{-\sigma(\lambda^2+c)}(P - \lambda^2)(P^{1/2} - \lambda)^{-1}\, d\lambda \\ &= \frac{i}{2\pi}\int_\Gamma \cos(t\lambda)e^{-\sigma(\lambda^2+c)}(P^{1/2} + \lambda)\, d\lambda = 0.\end{aligned}$$

and $u(0; \sigma, f) = e^{-\sigma(P+c)}f$. Thus $u(t; \sigma, f)$ is the unique solution of (3.1) with initial condition $e^{-\sigma(P+c)}f$. Then $u(t; f) - u(t; \sigma, f)$ is the solution of (3.1) with initial condition $f - e^{-(P+c)}f$. Hence by (3.2) we get for all $s \in \mathbb{R}$

$$\| u(t; f) - u(t; \sigma, f) \|_{H^s} \leq C \| f - e^{-\sigma(P+c)}f \|_{H^s}, \quad |t| \leq T. \tag{3.7}$$

Now note that for every $f \in C^\infty(X,E)$ we have

$$\lim_{\sigma\to 0} \| e^{-\sigma(P+c)} f - f \| = 0.$$

This follows from the parametrix construction. Hence we get

$$\| f - e^{-\sigma(P+c)} f \|_{H^s} = \| (P+c)^{s/2} f - e^{-\sigma(P+c)} (P+c)^{s/2} f \|_{L^2} \to 0$$

as $\sigma \to 0$. Combined with (3.7) we get

$$\lim_{\sigma\to 0} \| u(t;f) - u(t;\sigma,f) \|_{H^s} = 0. \tag{3.8}$$

Furthermore we have

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\varphi}(t) u(t;\sigma,f)\, dt &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\varphi}(t) \frac{i}{2\pi} \int_\Gamma \cos(t\lambda) e^{-\sigma(\lambda^2+c)} (P^{1/2}-\lambda)^{-1} f \, d\lambda\, dt \\
&= \frac{i}{2\pi} \int_\Gamma \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\varphi}(t) \cos(t\lambda)\, dt \right) e^{-\sigma(\lambda^2+c)} (P^{1/2}-\lambda)^{-1} f \, d\lambda \\
&= \frac{i}{2\pi} \int_\Gamma \varphi(\lambda) e^{-\sigma(\lambda^2+c)} (P^{1/2}-\lambda)^{-1} f \, d\lambda.
\end{aligned}$$

For $\sigma \to 0$, the right hand side converges to $\varphi(P^{1/2})f$. By (3.8) the left hand side converges to $(2\pi)^{-1/2} \int_{\mathbb{R}} \widehat{\varphi}(t) u(t;f)\, dt$.

Now assume that $0 \in \operatorname{spec}(P)$. Then we use the definition (2.24) of $\varphi(P^{1/2})$. Let $f \in C^\infty(X,E)$. Since Π_0 is a smoothing operator, we have $\Pi_0 f, (\mathrm{I}-\Pi_0)f \in C^\infty(X,E)$. Moreover by the uniqueness of the solution of 3.1 it follows that

$$u(t;f) = u(t;\Pi_0 f) + u(t;(\mathrm{I}-\Pi_0)f). \tag{3.9}$$

We use again the auxiliary operator $\widehat{P}$ defined by (2.19). Let $\hat{u}(t;(\mathrm{I}-\Pi_0)f)$ be the unique solution of the initial value problem (3.1) with respect to $\widehat{P}$. Since $0 \notin \operatorname{spec}(\widehat{P})$ it follows from the first part that

$$\varphi(\widehat{P}^{1/2})(\mathrm{I}-\Pi_0)f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\varphi}(t) \hat{u}(t;(\mathrm{I}-\Pi_0)f)\, dt. \tag{3.10}$$

Since by definition of $\widehat{P}$ and P_1, $\widehat{P}^{1/2}(\mathrm{I}-\Pi_0) = P_1^{1/2}(\mathrm{I}-\Pi_0)$, we get

$$\varphi(\widehat{P}^{1/2})(\mathrm{I}-\Pi_0)f = \varphi(P_1^{1/2})(\mathrm{I}-\Pi_0)f. \tag{3.11}$$

Next observe that $\Pi_0 \hat{u}(0;(\mathrm{I}-\Pi_0)f) = \Pi_0(\mathrm{I}-\Pi_0)f = 0$. By uniqueness of solutions of (3.1) we get $\Pi_0 \hat{u}(0;(\mathrm{I}-\Pi_0)f) = 0$. Using that $\widehat{P}(\mathrm{I}-\Pi_0) = P(\mathrm{I}-\Pi_0)$, it follows that $\hat{u}(t;(\mathrm{I}-\Pi_0)f) = u(t;(\mathrm{I}-\Pi_0)f)$. Together with (3.10) and (3.11) we get

$$\varphi(P_1^{1/2})(\mathrm{I}-\Pi_0)f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{\varphi}(t) u(t;(\mathrm{I}-\Pi_0)f)\, dt. \tag{3.12}$$

Next consider $u(t;\Pi_0 f)$. Observe that by (2.21) we have

$$u(t;\Pi_0 f) = U(t;N)\Pi_0 f.$$

Hence by (2.22) and (2.23) we get

$$\varphi(N^{1/2})\Pi_0 f = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{\varphi}(t)u(t,\Pi_0 f)\,dt. \tag{3.13}$$

Combining (3.9), (3.12) and (3.13), and using the definition of $\varphi(P^{1/2})$ by (2.24), we get (3.5). $\square$

Let $p\colon \widetilde{X} \to X$ be a universal covering of X which is fixed by the choice of a base point $x_0 \in X$. The fiber of p over x_0 is equal to the fundamental group $\Gamma := \pi_1(X, x_0)$ of X with the base point x_0. The group Γ acts properly and freely on $\widetilde{X}$ which leads to an identification of X with the quotient manifold $\Gamma\backslash\widetilde{X}$, when p is identified with the quotient map $\widetilde{X} \to \Gamma\backslash\widetilde{X}$. If conversely $\widetilde{X}$ is a simply connected manifold on which a group Γ acts acts properly and freely, as in the Introduction with $\widetilde{X} = G/K$, then, after a choice of a base point in $X := \Gamma\backslash\widetilde{X}$, the manifold $\widetilde{X}$ and the canonical projection $\widetilde{X} \to \Gamma\backslash\widetilde{X}$ are isomorphic to the universal covering described above.

Let $\widetilde{E} = p^*E$, and $\widetilde{P}\colon C^\infty(\widetilde{X}, \widetilde{E}) \to C^\infty(\widetilde{X}, \widetilde{E})$ the lift of P to $\widetilde{X}$. Let $\widetilde{u}(t,\tilde{x}, f)$ and $\widetilde{f}$ be the pull back to $\widetilde{X}$ of $u(t,x;f)$ and f, respectively. Then $\widetilde{u}(t,f)$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} + \widetilde{P}\right)\widetilde{u}(t;f) = 0, \quad \widetilde{u}(0;f) = \widetilde{f},\ \widetilde{u}_t(0,f) = 0. \tag{3.14}$$

By (2.2) we have $\widetilde{P} = \widetilde{\Delta}_E + \widetilde{D}$, where $\widetilde{D}$ is a differential operator of order ≤ 1. Then it follows from energy estimates as in [Ta2, Chapt. 2, §8] that solutions of $(\frac{\partial^2}{\partial t^2} + \widetilde{P})u = 0$ have finite propagation speed. This implies that for every $\psi \in C^\infty(\widetilde{X}, \widetilde{E})$ the wave equation

$$\left(\frac{\partial^2}{\partial t^2} + \widetilde{P}\right)u(t;\psi) = 0, \quad u(0;\psi) = \psi,\ u_t(0;\psi) = 0,$$

has a unique solution. Hence we get

$$\widetilde{u}(t;f) = u(t,\widetilde{f}). \tag{3.15}$$

Let $d(x,y)$ denote the geodesic distance of $x,y \in \widetilde{X}$. For $\delta > 0$ let

$$U_\delta = \{(x,y) \in \widetilde{X} \times \widetilde{X} \colon d(x,y) < \delta\}.$$

Proposition 3.3. *There exist $\delta > 0$ and $H_\varphi \in C^\infty(\widetilde{X}\times\widetilde{X}, \operatorname{Hom}(\widetilde{E},\widetilde{E}))$ with $\operatorname{supp} H_\varphi \subset U_\delta$ such that for all $\psi \in C^\infty(\widetilde{X}, \widetilde{E})$ we have*

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{\varphi}(t)u(t,\widetilde{x};\psi)\,dt = \int_{\widetilde{X}} H_\varphi(\widetilde{x},\widetilde{y})(\psi(\widetilde{y}))\,d\widetilde{y}.$$

Proof. Suppose that $\operatorname{supp}\widehat{\varphi} \subset [-T,T]$. Let $V \subset \widetilde{X}$ be an open relatively compact subset. For $r > 0$ let

$$V_r = \{y \in V \colon d(y,V) < r\}.$$

Let $\chi \in C_c^\infty(V_{2T})$ such that $\chi = 1$ on V_T. By finite propagation speed, we have

$$u(t,\widetilde{x};\psi) = u(t,\widetilde{x};\chi\psi), \quad \widetilde{x} \in V,\ |t| < T,$$

for all $\psi \in C^\infty(\widetilde{X},\widetilde{E})$. Thus we are reduced to the case of a compact manifold and the proof follows from Lemma 2.4 and Proposition 3.2. ☐

Using Proposition 3.3 together with (3.15) and Proposition 3.2, we obtain

$$\varphi(P^{1/2})f(\widetilde{x}) = \int_{\widetilde{X}} H_\varphi(\widetilde{x},\widetilde{y})(\widetilde{f}(\widetilde{y}))\, d\widetilde{y} \tag{3.16}$$

for all $f \in C^\infty(X,E)$. Let $F \subset \widetilde{X}$ be a fundamental domain for the action of the fundamental group Γ on $\widetilde{X}$. Given $\gamma \in \Gamma$, let $R_\gamma\colon \widetilde{E} \to \widetilde{E}$ be the induced bundle map. Thus for each $\widetilde{y} \in \widetilde{X}$, we have a linear isomorphism $R_\gamma\colon \widetilde{E}_{\widetilde{y}} \to \widetilde{E}_{\gamma\widetilde{y}}$. Note that $\widetilde{f}$ satisfies

$$\widetilde{f}(\gamma\widetilde{y}) = R_\gamma(\widetilde{f}(\widetilde{y})), \quad \gamma \in \Gamma.$$

Then we get

$$\begin{aligned}\int_{\widetilde{X}} H_\varphi(\widetilde{x},\widetilde{y})(\widetilde{f}(\widetilde{y}))\, d\widetilde{y} &= \sum_{\gamma\in\Gamma} \int_{\gamma F} H_\varphi(\widetilde{x},\widetilde{y})(\widetilde{f}(\widetilde{y}))\, d\widetilde{y}\\ &= \sum_{\gamma\in\Gamma} \int_{F} H_\varphi(\widetilde{x},\gamma\widetilde{y})(\widetilde{f}(\gamma\widetilde{y}))\, d\widetilde{y}\\ &= \int_F \left(\sum_{\gamma\in\Gamma} H_\varphi(\widetilde{x},\gamma\widetilde{y}) \circ R_\gamma\right)(\widetilde{f}(\widetilde{y}))\, d\widetilde{y}.\end{aligned}$$

Combining this expression with (3.16), it follows that the kernel K_φ of $\varphi(P^{1/2})$ is given by

$$K_\varphi(x,y) = \sum_{\gamma\in\Gamma} H_\varphi(\widetilde{x},\gamma\widetilde{y}) \circ R_\gamma, \tag{3.17}$$

where $\widetilde{x}$ and $\widetilde{y}$ are any lifts of x and y to the fundamental domain F. So by Proposition 2.6 we get

Proposition 3.4. *Let $\varphi \in \mathcal{P}(\mathbb{C})$ be even. Then we have*

$$\sum_{\lambda\in\operatorname{spec}(P)} m(\lambda)\varphi(\lambda^{1/2}) = \sum_{\gamma\in\Gamma}\int_F \operatorname{tr}(H_\varphi(\widetilde{x},\gamma\widetilde{x}) \circ R_\gamma)\, d\widetilde{x}.$$

Note that the sum on the right is finite.

4. The twisted Bochner-Laplace operator

Let $E \to X$ be a complex vector bundle with covariant derivative ∇. Define the invariant second covariant derivative ∇^2 by

$$\nabla^2_{U,V} = \nabla_U \nabla_V - \nabla_{\nabla_U V},$$

where U, V are any two vector fields on X. Then the connection Laplacian $\Delta^\#$ is defined by

$$\Delta^\# = -\operatorname{Tr}(\nabla^2).$$

Let $(e_1, ..., e_n)$ be a local frame field. Then

$$\Delta^\# = -\sum_j \nabla^2_{e_j,e_j}.$$

This formula implies that the principal symbol of $\Delta^\#$ is given by

$$\sigma(\Delta^\#)(x,\xi) = \| \xi \|^2_x \operatorname{Id}_{E_x}.$$

Thus the results of the previous section can be applied to $\Delta^\#$.

Assume that E is equipped with a Hermitian fiber metric and ∇ is compatible with the metric. Then it follows that

$$\nabla^* \nabla = -\operatorname{Tr} \nabla^2, \tag{4.1}$$

[LM, p.154], i.e., the connection Laplacian equals the Bochner-Laplace operator $\Delta_E = \nabla^* \nabla$.

Now let $\rho \colon \Gamma \to \operatorname{GL}(V)$ be a finite-dimensional complex representation of $\Gamma = \pi_1(X)$. Let $F \to X$ be the associated flat vector bundle with connection ∇^F. Let E be a Hermitian vector bundle over X with Hermitian connection ∇^E. We equip $E \otimes F$ with the product connection $\nabla^{E\otimes F}$, which is defined by

$$\nabla^{E\otimes F}_Y = \nabla^E_Y \otimes 1 + 1 \otimes \nabla^F_Y,$$

for $Y \in C^\infty(X, TX)$. Let $\Delta^\#_{E,\rho}$ be the connection Laplacian associated to $\nabla^{E\otimes F}$. Locally it can be described as follows. Let $U \subset X$ be an open subset such that $F|_U$ is trivial. Then $(E \otimes F)|_U$ is isomorphic to the direct sum of $m = \operatorname{rank}(F)$ copies of $E|_U$:

$$(E \otimes F)|_U \cong \oplus_{i=1}^m E|_U.$$

Let $e_1, ..., e_m$ be a basis of flat sections of $F|_U$. Then each $\varphi \in C^\infty(U, (E \otimes F)|_U)$ can be written as

$$\varphi = \sum_{j=1}^m \varphi_j \otimes e_j,$$

where $\varphi_i \in C^\infty(U, E|_U)$, $i = 1, ..., m$. Then

$$\nabla^{E\otimes F}_Y(\varphi) = \sum_j (\nabla^E_Y \varphi_j) \otimes e_j.$$

Let $\Delta_E=(\nabla^E)^*\nabla^E$ be the Bochner-Laplace operator associated to ∇^E. Using (4.1), we get

$$\Delta^\#_{E,\rho}\varphi=\sum_j(\Delta_E\varphi_j)\otimes e_j. \tag{4.2}$$

Let $\widetilde{E}$ and $\widetilde{F}$ be the pullback to $\widetilde{X}$ of E and F, respectively. Then $\widetilde{F}\cong\widetilde{X}\times V$ and

$$C^\infty(\widetilde{X},\widetilde{E}\otimes\widetilde{F})\cong C^\infty(\widetilde{X},\widetilde{E})\otimes V.$$

It follows from (4.2) that with respect to this isomorphism, the lift $\widetilde{\Delta}^\#_{E,\rho}$ of $\Delta^\#_{E,\rho}$ to $\widetilde{X}$ takes the form

$$\widetilde{\Delta}^\#_{E,\rho}=\widetilde{\Delta}_E\otimes\operatorname{Id},$$

where $\widetilde{\Delta}_E$ is the lift of Δ_E to $\widetilde{X}$. Let $\psi\in C^\infty_c(\widetilde{X},\widetilde{E})\otimes V$. Then the unique solution of the wave equation

$$\left(\frac{\partial^2}{\partial t^2}+\widetilde{\Delta}^\#_{E,\rho}\right)u(t;\psi)=0,\quad u(0;\psi)=\psi,\ u_t(0,\psi)=0,$$

is given by

$$u(t;\psi)=\left(\cos\left(t(\widetilde{\Delta}_E)^{1/2}\right)\otimes\operatorname{Id}\right)\psi.$$

Let φ be as above and let $k_\varphi(\widetilde{x},\widetilde{y})$ be the kernel of

$$\varphi\left((\widetilde{\Delta}_E)^{1/2}\right)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{\varphi}(t)\cos\left(t(\widetilde{\Delta}_E)^{1/2}\right)\,dt.$$

Then the kernel H_φ of Proposition 3.3 is given by $H_\varphi(\widetilde{x},\widetilde{y})=k_\varphi(\widetilde{x},\widetilde{y})\otimes\operatorname{Id}$. Let $R_\gamma\colon\widetilde{E}_{\widetilde{y}}\to\widetilde{E}_{\gamma\widetilde{y}}$ be the canonical isomorphism. Then it follows from (3.17) that the kernel of the operator $\varphi\big((\Delta^\#_{E,\rho})^{1/2}\big)$ is given by

$$K_\varphi(x,y)=\sum_{\gamma\in\Gamma}k_\varphi(\widetilde{x},\gamma\widetilde{y})\circ(R_\gamma\otimes\rho(\gamma)). \tag{4.3}$$

Combined with (3.4) we get

Proposition 4.1. *Let F_ρ be a flat vector bundle over X, associated to a finite-dimensional complex representation $\rho\colon\pi_1(X)\to\operatorname{GL}(V)$. Let $\Delta^\#_{E,\rho}$ be the twisted connection Laplacian acting in $C^\infty(X,E\otimes F_\rho)$. Let $\varphi\in\mathcal{P}(\mathbb{C})$ be even and denote by $k_\varphi(\widetilde{x},\widetilde{y})$ the kernel of $\varphi\left((\widetilde{\Delta}_E)^{1/2}\right)$. Then we have*

$$\sum_{\lambda\in\operatorname{spec}(\Delta^\#_{E,\rho})}m(\lambda)\varphi(\lambda^{1/2})=\sum_{\gamma\in\Gamma}\operatorname{tr}\rho(\gamma)\int_F\operatorname{tr}\left(k_\varphi(\widetilde{x},\gamma\widetilde{x})\circ R_\gamma\right)\,d\widetilde{x}. \tag{4.4}$$

5. Locally symmetric spaces

In this section we specialize to the case where X is a locally symmetric manifold. We recall some basic facts about harmonic analysis on symmetric spaces. This is much in the spirit of Selberg's original approach [Se1]. However, we consider only the Casimir operator and not the full algebra of invariant differential operators.

Let G be a connected semisimple real Lie group of non-compact type with finite center. Let $K \subset G$ be a maximal compact subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Let

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k} \tag{5.1}$$

be the Cartan decomposition. Put $S = G/K$. This is a Riemannian symmetric space of non-positive curvature. The invariant metric is obtained by translation of the restriction of the Killing form to $\mathfrak{p} \cong T_e(G/K)$. Let $\Gamma \subset G$ be a discrete, torsion free, cocompact subgroup. Then Γ acts freely on S by isometries and $X = \Gamma\backslash S$ is a compact locally symmetric manifold.

Let $\tau\colon K \to \mathrm{GL}(V_\tau)$ be a finite-dimensional unitary representation of K, and let

$$\widetilde{E}_\tau = (G \times V_\tau)/K \to G/K$$

be the associated homogeneous vector bundle, where K acts on the right as usual by

$$(g, v)k = (gk, \tau(k^{-1})v), \quad g \in G,\ k \in K,\ v \in V_\tau.$$

Let

$$C^\infty(G;\tau) := \left\{ f\colon G \to V_\tau \mid f \in C^\infty,\ f(gk) = \tau(k^{-1})f(g),\ g \in G,\ k \in K \right\}. \tag{5.2}$$

Similarly, by $C_c^\infty(G;\tau)$ we denote the subspace of $C^\infty(G;\tau)$ of compactly supported functions and by $L^2(G;\tau)$ the completion of $C_c^\infty(G;\tau)$ with respect to the inner product

$$\langle f_1, f_2 \rangle = \int_{G/K} \langle f_1(g), f_2(g) \rangle \, d\dot{g}.$$

There is a canonical isomorphism

$$C^\infty(S, \widetilde{E}_\tau) \cong C^\infty(G;\tau). \tag{5.3}$$

[Mi, p.4]. Similarly, there are isomorphisms $C_c^\infty(S, \widetilde{E}_\tau) \cong C_c^\infty(G;\tau)$ and $L^2(S, \widetilde{E}_\tau) \cong L^2(G;\tau)$.

Let ∇^τ be the canonical G-invariant connection on $\widetilde{E}_\tau$. It is defined by

$$\nabla^\tau_{g_* Y} f(gK) = \frac{d}{dt}\bigg|_{t=0} (g\exp(tY))^{-1} f(g\exp(tY)K),$$

where $f \in C^\infty(G;\tau)$ and $Y \in \mathfrak{p}$. Let $\widetilde{\Delta}_\tau$ be the associated Bochner-Laplace operator. Then $\widetilde{\Delta}_\tau$ is G-invariant, i.e., $\widetilde{\Delta}_\tau$ commutes with the right action of G on $C^\infty(S, \widetilde{E}_\tau)$. Let $\Omega \in Z(\mathfrak{g}_\mathbb{C})$ and $\Omega_K \in Z(\mathfrak{k}_\mathbb{C})$ be the Casimir elements of G and K, respectively. Assume that τ is irreducible. Let R denote the right regular representation of G on $C^\infty(G;\tau)$. Then with respect to (5.3), we have

$$\widetilde{\Delta}_\tau = -R(\Omega) + \lambda_\tau \operatorname{Id}, \tag{5.4}$$

where $\lambda_\tau = \tau(\Omega_K)$ is the Casimir eigenvalue of τ [Mi, Proposition 1.1]. We note that $\lambda_\tau \geq 0$.

Let $\varphi \in \mathcal{P}(\mathbb{C})$ be even. Then $\varphi(\widetilde{\Delta}_\tau^{1/2})$ is a G-invariant integral operator. Therefore its kernel k_φ satisfies

$$k_\varphi(g\widetilde{x}, g\widetilde{y}) = k_\varphi(\widetilde{x}, \widetilde{y}), \quad g \in G.$$

In the scalar case this is a point-pair invariant considered originally by Selberg [Se1]. With respect to the isomorphism (5.3) it can be identified with a compactly supported C^∞-function

$$h_\varphi \colon G \to \operatorname{End}(V_\tau),$$

which satisfies

$$h_\varphi(k_1 g k_2) = \tau(k_1) \circ h_\varphi(g) \circ \tau(k_2), \quad k_1, k_2 \in K.$$

Then $\varphi(\widetilde{\Delta}_\tau^{1/2})$ acts by convolution

$$\left(\varphi(\widetilde{\Delta}_\tau^{1/2}) f\right)(g_1) = \int_G h_\varphi(g_1^{-1} g_2)(f(g_2))\, dg_2. \tag{5.5}$$

Let

$$E_\tau = \Gamma \backslash \widetilde{E}_\tau$$

be the locally homogeneous vector bundle over $\Gamma\backslash S$ induced by $\widetilde{E}_\tau$. Let $\chi \colon \Gamma \to \operatorname{GL}(V_\chi)$ be a finite-dimensional complex representation and let F_χ be the associated flat vector bundle over $\Gamma\backslash S$. Let $\Delta^\#_{\tau,\chi}$ be the twisted connection Laplacian acting in $C^\infty(\Gamma\backslash S, E_\tau \otimes F_\chi)$. Then it follows from (4.3) that the kernel K_φ of $\varphi(\widetilde{\Delta}_\tau^{1/2})$ is given by

$$K_\varphi(g_1 K, g_2 K) = \sum_{\gamma \in \Gamma} h_\varphi(g_1^{-1} \gamma g_2) \otimes \chi(\gamma). \tag{5.6}$$

For the unitary case compare [Se1, (2.2)]. By Proposition 4.1 we get

$$\sum_{\lambda \in \operatorname{spec}(\Delta^\#_{E,\rho})} m(\lambda) \varphi(\lambda^{1/2}) = \sum_{\gamma \in \Gamma} \operatorname{tr} \chi(\gamma) \int_{\Gamma\backslash G} \operatorname{tr} h_\varphi(g^{-1}\gamma g)\, d\dot{g}. \tag{5.7}$$

We now proceed in the usual way, grouping terms together into conjugacy classes. Given $\gamma \in \Gamma$, denote by $\{\gamma\}_\Gamma$, Γ_γ, and G_γ the Γ-conjugacy class of γ, the centralizer of γ in Γ, and the centralizer of γ in G, respectively. With the conjugacy class $\{e\}_\Gamma$ separated from the others as usual, we get a first version of the trace formula.

Proposition 5.1. *For all even $\varphi \in \mathcal{P}(\mathbb{C})$ we have*

$$\sum_{\lambda\in\operatorname{spec}(\Delta^{\#}_{E,\rho})} m(\lambda)\varphi(\lambda^{1/2}) = \dim(V_\chi)\operatorname{vol}(\Gamma\backslash S)\operatorname{tr} h_\varphi(e) \tag{5.8}$$
$$+ \sum_{\{\gamma\}_\Gamma\neq e} \operatorname{tr}\chi(\gamma)\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G}\operatorname{tr} h_\varphi(g^{-1}\gamma g)\,d\dot{g}.$$

In order to make this formula more explicit, one needs to express the kernel h_φ in terms of φ, and to evaluate the orbital integrals on the right hand side. The kernel h_φ can be determined using Harish-Chandra's Plancherel formula. The orbital integrals can be computed using the Fourier inversion formula. However, both formulae are pretty complicated in the higher rank case. A sufficiently explicit formula can be obtained in the rank one case which we discuss in the next section.

6. The rank one case

Let G and K be as above. We introduce some notation following [Wa]. Let $G = KAN$ be an Iwasawa decomposition of G (see [Hel]). Then A is a maximal vector subgroup of G and N is a maximal unipotent subgroup of G. In this section we assume that G has split rank one, i.e., $\dim A = 1$. Let M be the centralizer of A in K. We set $P = MAN$. Then P is a parabolic subgroup of G. Since G has split rank 1, every proper parabolic subgroup of G is conjugate to P.

Denote by $\widehat{G}$ and $\widehat{M}$ the set of equivalence classes of irreducible unitary representations of G and M, respectively. For $\pi \in \widehat{G}$ we denote by $\mathcal{H}_\pi$ the Hilbert space in which π operates.

Let $\mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebras of A and N, respectively. Choose $H \in \mathfrak{a}$ such that $\operatorname{ad}(H)|_{\mathfrak{n}}$ has eigenvalues 1 and possibly 2. Then $\mathfrak{a} = \mathbb{R}H$. For $t \in R$ we set $a_t = \exp(tH)$ and $\log a_t = t$. Let $A^+ = \{a_t \colon t > 0\}$.

Let ρ be the half-sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$. Its norm $|\rho|$ with respect to the normalized Killing form is given as follows. Let p and q be the dimensions of the eigenspaces of $\operatorname{ad}(H)|_{\mathfrak{n}}$ with eigenvalues 1 and 2, respectively. Then $p > 0$ and $0 \le q < p$. Then

$$|\rho| = \frac{1}{2}(p + 2q). \tag{6.1}$$

For $\sigma \in \widehat{M}$ and $\lambda \in \mathbb{R}$ let $\pi_{\sigma,\lambda}$ be the unitarily induced representation from P to G which is defined as in [Wa, p. 177]. Let $\Theta_{\sigma,\lambda}$ denote the character of $\pi_{\sigma,\lambda}$.

If $\gamma \in \Gamma$, $\gamma \neq e$, then there exists $g \in G$ such that $g\gamma g^{-1} \in MA^+$. Thus there are $m_\gamma \in M$ and $a_\gamma \in A^+$ such that $g\gamma g^{-1} = m_\gamma a_\gamma$. By [Wa, Lemma 6.6], a_γ depends only on γ and m_γ is determined by γ up to conjugacy in M. Let

$$l(\gamma) = \log a_\gamma.$$

Then $l(\gamma)$ is the length of the unique closed geodesic of $\Gamma\backslash S$ determined by $\{\gamma\}_\Gamma$. Furthermore, by the above remark

$$D(\gamma) := e^{-l(\gamma)|\rho|} \big| \det \left(\operatorname{Ad}(m_\gamma a_\gamma)|_{\mathfrak{n}} - \operatorname{Id}\right) \big| \tag{6.2}$$

is well defined. Let

$$u(\gamma) = \operatorname{vol}(G_{m_\gamma a_\gamma}/A).$$

Let $h \in C_c^\infty(G)$ be K-finite. Then by [Wa, pp. 177-178] (correcting a misprint) we have

$$\int_{G_\gamma\backslash G} h(g\gamma g^{-1})\, d\dot{g} = \frac{1}{2\pi} \frac{1}{u(\gamma)D(\gamma)} \sum_{\sigma\in\widehat{M}} \overline{\operatorname{tr}\sigma(\gamma)} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(h)\cdot e^{-il(\gamma)\lambda}\, d\lambda. \tag{6.3}$$

Since h is K-finite, $\Theta_{\sigma,\lambda}(h) \neq 0$ only for finitely many σ. Thus the sum over $\sigma \in \widehat{M}$ is finite. The volume factors in (5.8) are computed as follows. Since G has rank one, Γ_γ is infinite cyclic [DKV, Proposition 5.16]. Thus there is $\gamma_0 \in \Gamma_\gamma$ such that γ_0 generates Γ_γ and $\gamma = \gamma_0^{n(\gamma)}$ for some integer $n(\gamma) \geq 1$. Then

$$\frac{\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)}{u(\gamma)} = l(\gamma_0). \tag{6.4}$$

Inserting (6.3) and (6.4) into (5.8), we get the following form of the trace formula in the rank one case.

Proposition 6.1. *Let $\varphi \in \mathcal{P}(\mathbb{C})$ be even and let $\hat{\varphi}$ be the Fourier transform of $\varphi|_{\mathbb{R}}$. Then*

$$\begin{aligned} \operatorname{Tr}\varphi\left((\Delta^{\#}_{\tau,\chi})^{1/2}\right) =& \dim(V_\chi)\operatorname{vol}(\Gamma\backslash S)\operatorname{tr} h_\varphi(e) \\ &+ \sum_{\{\gamma\}_\Gamma\neq e} \operatorname{tr}\chi(\gamma)\frac{1}{2\pi}\frac{l(\gamma_0)}{D(\gamma)} \sum_{\sigma\in\widehat{M}} \overline{\operatorname{tr}\sigma(\gamma)} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(h_\varphi)\cdot e^{-il(\gamma)\lambda}\, d\lambda. \end{aligned} \tag{6.5}$$

We note that in the self-adjoint case the Selberg trace formula is stated in this form in [Wa, Theorem 6.7]. The right hand side of (6.5) is still not in an explicit form. First of all we can use the Plancherel formula [Kn] to express $\operatorname{tr} h_\varphi(e)$ in terms of characters. In this way we are reduced to the computation of the characters Θ_π, $\pi \in \widehat{G}$, evaluated on $\operatorname{tr} h_\varphi$.

To this end we use the theory of τ-spherical functions and the spherical transform for homogeneous vector bundles [Ca]. We will assume that K is multiplicity free in G, i.e., we require that

$$\forall\tau \in \widehat{K},\ \forall\pi \in \widehat{G}\colon [\pi|_K : \tau] \leq 1. \tag{6.6}$$

For each $\tau \in \widehat{K}$ let $I_{0,\tau}(G)$ be the convolution algebra consisting of all $f \in C_c^\infty(G)$ which are K-central and invariant under convolution with $d_\tau\bar{\chi}_\tau$, where d_τ and χ_τ are the dimension and the character of τ, respectively (see [Ca, §2]). If τ is the trivial representation, this is the usual convolution algebra $C_c^\infty(G/\!/K)$ of bi-K-invariant smooth compactly supported functions on G. Condition (6.6) implies that for all $\tau \in \widehat{K}$ the convolution algebra $I_{0,\tau}(G)$ is commutative [Ca, Proposition 2.2]. This simplifies the theory of the τ-spherical functions. We will use (6.6) in the computations following (6.19).

By [Ko] condition (6.6) is satisfied for $G = SO_0(n,1)$ and $G = SU(n,1)$. Let

$$\widehat{G}(\tau) = \left\{ \pi \in \widehat{G} \colon [\pi|_K : \tau] = 1 \right\}.$$

Then for each $\pi \in \widehat{G}(\tau)$ we can identify the τ-isotypical subspace $\mathcal{H}_\pi(\tau)$ of τ in $\mathcal{H}_\pi$ with V_τ. Let P_τ be the orthogonal projection of $\mathcal{H}_\pi$ onto $\mathcal{H}_\pi(\tau)$. Define the τ-spherical function Φ^π_τ on G by

$$\Phi^\pi_\tau(g) := P_\tau \pi(g) P_\tau, \quad g \in G.$$

Then Φ^π_τ is a C^∞-map

$$\Phi^\pi_\tau \colon G \to \operatorname{End}(V_\tau)$$

which satisfies

$$\begin{aligned} \Phi^\pi_\tau(g)^* &= \Phi^\pi_\tau(g^{-1}) \\ \Phi^\pi_\tau(k_1 g k_2) &= \tau(k_1)\Phi^\pi_\tau(g)\tau(k_2), \quad g \in G,\ k_1, k_2 \in K. \end{aligned} \tag{6.7}$$

Let $v \in V_\tau$ and set

$$f^\pi_{\tau,v}(g) = \Phi^\pi_\tau(g^{-1})(v). \tag{6.8}$$

Then $f^\pi_{\tau,v} \in C^\infty(G;\tau)$ and it follows from (5.4) that

$$\widetilde{\Delta}_\tau f^\pi_{\tau,v} = (-\pi(\Omega) + \lambda_\tau) f^\pi_{\tau,v}. \tag{6.9}$$

The functions $f^\pi_{\tau,v}$ generalize the usual spherical functions [Hel] which correspond to the trivial representation $\tau = 1$. Indeed $\widehat{G}(1)$ is the set of all $\pi \in \widehat{G}$ which have a nonzero K-invariant vector $v \in \mathcal{H}_\pi$. These are exactly the principal series representations $\pi_\lambda := \pi_{1,\lambda}$, $\lambda \in \mathbb{R}$, which are induced from the trivial representation of M. The subspace $\mathcal{H}^K_\lambda$ of K-invariant vectors in the Hilbert space $\mathcal{H}_\lambda$ of the representation π_λ has dimension one. So let $v \in \mathcal{H}_\lambda$ with $\| v \| = 1$. Set

$$\phi_\lambda(g) = \langle \pi_\lambda(g) v, v \rangle, \quad g \in G,\ \lambda \in \mathbb{R}. \tag{6.10}$$

Then ϕ_λ is a smooth bi-K-invariant function on G which is an eigenfunction of the Casimir operator. It corresponds to a smooth K-invariant function on $S = G/K$ which is an eigenfunction of the Laplace operator $\tilde{\Delta}$ on S. In the higher rank case they are eigenfunctions of the whole algebra of invariant differential operators on S. These are the eigenfunctions considered by Selberg [Se1, p. 53]. See also (6.28), where (6.10) is used.

Now we return to the general case. Let $u(t,x; f^\pi_{\tau,v})$ be the unique solution of

$$\left(\frac{\partial^2}{\partial t^2} + \widetilde{\Delta}_\tau \right) u(t) = 0, \quad u(0) = f^\pi_{\tau,v},\ u_t(0) = 0.$$

Lemma 6.2. *For $t \in \mathbb{R}$, $\tau \in \widehat{K}$ and $\pi \in \widehat{G}(\tau)$, we have $-\pi(\Omega) + \lambda_\tau \geq 0$ and*

$$u(t,x; f^\pi_{\tau,v}) = \cos\left(t\sqrt{-\pi(\Omega) + \lambda_\tau} \right) f^\pi_{\tau,v}(x).$$

Proof. Let $\langle\cdot,\cdot\rangle$ be the Killing form on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ (resp. $\mathfrak{k}$) is positive (resp. negative) definite. Let $X_1,...,X_d\in\mathfrak{p}$ and $Y_1,...,Y_m\in\mathfrak{k}$ be bases of $\mathfrak{p}$ and $\mathfrak{k}$, respectively, such that $\langle X_i,X_j\rangle=\delta_{ij}$, $\langle Y_i,Y_j\rangle=-\delta_{ij}$. Then $\Omega=\sum_i X_i^2-\sum_j Y_j^2$ and $\Omega_K=-\sum_j Y_j^2$. Let $v\in\mathcal{H}_\pi(\tau)$, $\parallel v\parallel=1$. Then we get

$$-\pi(\Omega)+\lambda_\tau=-\langle\pi(\Omega)v,v\rangle+\lambda_\tau=\sum_i\parallel\pi(X_i)v\parallel^2\geq 0,$$

which proves the first statement. For the second statement, we note that by definition, we have

$$\frac{\partial^2}{\partial t^2}u(t,x;f^\pi_{\tau,v})=-\widetilde{\Delta}_\tau u(t,x;f^\pi_{\tau,v}). \tag{6.11}$$

Fix $x_0\in S$. Let $\chi\in C_c^\infty(S)$ be such that

$$\chi(y)=\begin{cases}1, & y\in B_{2t}(x);\\ 0, & y\in S\setminus B_{3t}(x).\end{cases}$$

Then by finite propagation speed we have

$$u(t,x;f^\pi_{\tau,v})=u(t,x;\chi f^\pi_{\tau,v}),\quad x\in B_t(x_0).$$

Since $\chi f^\pi_{\tau,v}\in C_c^\infty(S,\widetilde{E}_\tau)$, we have

$$u(t,x;f^\pi_{\tau,v})=\left(\cos\left(t(\widetilde{\Delta}_\tau)^{1/2}\right)(\chi f^\pi_{\tau,v})\right)(x),\quad x\in B_t(x_0).$$

Using that $\widetilde{\Delta}_\tau$ commutes with $\cos\left(t(\widetilde{\Delta}_\tau)^{1/2}\right)$, and finite propagation speed, we get

$$\widetilde{\Delta}_\tau u(t,x;f^\pi_{\tau,v})=u(t,x;\widetilde{\Delta}_\tau f^\pi_{\tau,v}).$$

By (6.9) it follows that

$$u(t,x;\widetilde{\Delta}_\tau f^\pi_{\tau,v})=-(-\pi(\Omega)+\lambda_\tau)u(t,x;f^\pi_{\tau,v}).$$

Combined with (6.11) it follows that for every $x\in S$, $u(t,x;f^\pi_{\tau,v})$ satisfies the following differential equation in t

$$\left(\frac{d^2}{dt^2}-\pi(\Omega)+\lambda_\tau\right)u(t,x;f^\pi_{\tau,v})=0,\quad u(0,x;f^\pi_{\tau,v})=f^\pi_{\tau,v}(x),\ u_t(0,x;\phi_\lambda)=0.$$

This implies the claimed equality. □

Let $\varphi\in\mathcal{P}(\mathbb{C})$ be even and let $\widehat{\varphi}$ be the Fourier transform of $\varphi|_\mathbb{R}$. Since the kernel of the integral operator $\varphi((\widetilde{\Delta}_\tau)^{1/2})$ is given by $h_\varphi\in C_c^\infty(G)$, $\varphi((\widetilde{\Delta}_\tau)^{1/2})(f^\pi_{\tau,v})$ is well defined and it follows from Lemma 6.2 that

$$\begin{aligned}\varphi((\widetilde{\Delta}_\tau)^{1/2})(f^\pi_{\tau,v})=\frac{1}{\sqrt{2\pi}}\int_\mathbb{R}\widehat{\varphi}(t)u(t;f^\pi_{\tau,v})\,dt&=\frac{1}{\sqrt{2\pi}}\int_\mathbb{R}\widehat{\varphi}(t)\cos(t\sqrt{-\pi(\Omega)+\lambda_\tau})f^\pi_{\tau,v}\,dt\\&=\varphi\left(\sqrt{-\pi(\Omega)+\lambda_\tau}\right)f^\pi_{\tau,v}.\end{aligned}$$

If we rewrite this equality in terms of the kernel h_φ and use the definition of $f^\pi_{\tau,v}$, we get

$$\int_G h_\varphi(g^{-1}g_1)\Phi^\pi_\tau(g_1^{-1})v\, dg_1 = \varphi\left(\sqrt{-\pi(\Omega)+\lambda_\tau}\right)\Phi^\pi_\tau(g^{-1})v. \tag{6.12}$$

Let $d_\tau := \dim V_\tau$. Putting $g=1$ and taking the trace of both sides, we get

$$\int_G \mathrm{Tr}[h_\varphi(g)\Phi^\pi_\tau(g^{-1})]\, dg = d_\tau\varphi\left(\sqrt{-\pi(\Omega)+\lambda_\tau}\right).$$

We continue by rewriting the left hand side. To this end put

$$\phi^\pi_\tau(g) := \mathrm{tr}\,\Phi^\pi_\tau(g), \quad g\in G.$$

Note that ϕ^π_τ satisfies $\phi^\pi_\tau(g)=\phi^\pi_\tau(g^{-1})$. Using the Schur orthogonality relations (see [Kn, Chapt. I, §5]), we get

$$\Phi^\pi_\tau(g) = d_\tau\int_K \mathrm{Tr}[\tau(k^{-1})\Phi^\pi_\tau(g)]\tau(k)\, dk = d_\tau\int_K \phi^\pi_\tau(k^{-1}g)\tau(k)\, dk. \tag{6.13}$$

Using (6.13), we get

$$\begin{aligned}\int_G \mathrm{Tr}[h_\varphi(g)\Phi^\pi_\tau(g^{-1})]\, dg &= d_\tau\int_G\int_K \phi^\pi_\tau(k^{-1}g^{-1})\,\mathrm{Tr}[h_\varphi(g)\tau(k)]\, dkdg\\ &= d_\tau\int_K\int_G \phi^\pi_\tau(gk)\,\mathrm{tr}\,h_\varphi(gk)\, dgdk\\ &= d_\tau\int_G \mathrm{tr}\,h_\varphi(g)\phi^\pi_\tau(g)\, dg.\end{aligned} \tag{6.14}$$

Together with (6.12) we obtain

$$\int_G \mathrm{tr}\,h_\varphi(g)\phi^\pi_\tau(g)\, dg = \varphi\left(\sqrt{-\pi(\Omega)+\lambda_\tau}\right). \tag{6.15}$$

Now let $\tau'\in\widehat{K}$ be any other representation which occurs in $\pi|_K$. Repeating the argument used in (6.14), we get

$$\int_G \mathrm{tr}\,h_\varphi(g)\phi^\pi_{\tau'}(g)\, dg = \int_G \mathrm{Tr}\left[\left(\int_K \phi^\pi_{\tau'}(k^{-1}g^{-1})\tau(k)\, dk\right)h_\varphi(g)\right]\, dg.$$

Again by the Schur orthogonality relations, we have

$$\int_K \phi^\pi_{\tau'}(k^{-1}g^{-1})\tau(k)\, dk = 0,$$

if $\tau'\ncong\tau$. Hence we get

$$\int_G \mathrm{tr}\,h_\varphi(g)\phi^\pi_{\tau'}(g)\, dg = 0, \quad \tau'\in\widehat{K},\ \tau'\ncong\tau. \tag{6.16}$$

Choose an orthonormal basis of $\mathcal{H}_\pi$ which is adapted to the decomposition of $\pi|_K$ into irreducible representations of K. Then it follows from (6.16) that

$$\begin{aligned}\Theta_\pi(\operatorname{tr} h_\varphi) &= \operatorname{Tr}\left[\int_G \operatorname{tr} h_\varphi(g)\pi(g)\,dg\right] = \sum_{\tau'}\int_G \operatorname{tr} h_\varphi(g)\phi^\pi_{\tau'}(g)\,dg \\ &= \int_G \operatorname{tr} h_\varphi(g)\phi^\pi_\tau(g)\,dg.\end{aligned} \tag{6.17}$$

Combined with (6.15) we obtain the following lemma.

Proposition 6.3. *Let $\varphi \in \mathcal{P}(\mathbb{C})$ be even and let $\hat\varphi$ be the Fourier transform of $\varphi|_\mathbb{R}$. Let h_φ be the kernel of $\varphi((\widetilde{\Delta}_\tau)^{1/2})$. Then for all $\pi \in \widehat{G}(\tau)$ we have*

$$\Theta_\pi(\operatorname{tr} h_\varphi) = \varphi\left(\sqrt{-\pi(\Omega)+\lambda_\tau}\right).$$

Since G has split rank one, the tempered dual of G (which is the support of the Plancherel measure) is the union of the unitarily induced representations $\pi_{\sigma,\lambda}$, $\sigma \in \widehat{M}$, $\lambda \in \mathbb{R}$, and the discrete series, where the latter exists only if $\operatorname{rank} G = \operatorname{rank} K$. First consider the induced representation $\pi_{\sigma,\lambda}$. Let $T \subset M$ be a maximal torus and $\mathfrak{t}$ the Lie algebra of T. Let $\Lambda_\sigma \in i\mathfrak{t}$ be the infinitesimal character of $\sigma \in \widehat{M}$ and ρ_M the half-sum of positive roots of (M,T). Then by [Kn, Proposition 8.22]

$$\pi_{\sigma,\lambda}(\Omega) = -\lambda^2 - |\rho|^2 + |\Lambda_\sigma + \rho_M|^2 - |\rho_M|^2, \tag{6.18}$$

where $|\rho|$ is given by (6.1). Let $\tau \in \widehat{K}$. By Frobenius reciprocity [Kn, p.208] we have

$$[\pi_{\sigma,\lambda}|_K : \tau] = [\tau|_M : \sigma], \quad \sigma \in \widehat{M}. \tag{6.19}$$

Since we are assuming that K is multiplicity free in G, it follows that $[\tau|_K : \sigma] \le 1$. Let

$$\widehat{M}(\tau) = \{\sigma \in \widehat{M} \colon [\tau|_M : \sigma] = 1\}.$$

Then by (6.19) it follows that $\pi_{\sigma,\lambda} \in \widehat{G}(\tau)$ if and only if $\sigma \in \widehat{M}(\tau)$, and by Proposition 6.3 we get

$$\Theta_{\sigma,\lambda}(\operatorname{tr} h_\varphi) = \varphi\left(\sqrt{\lambda^2 + |\rho|^2 + |\rho_M|^2 - |\Lambda_\sigma + \rho_M|^2 + \lambda_\tau}\right), \quad \sigma \in \hat{M}(\tau),\ \lambda \in \mathbb{R}. \tag{6.20}$$

Now suppose that $\operatorname{rank} G = \operatorname{rank} K$. Then G has a non-empty discrete series. Let $H \subset G$ be a compact Cartan subgroup with Lie algebra $\mathfrak{h}$. Let $L \subset i\mathfrak{h}$ be the lattice of all $\mu \in i\mathfrak{h}$ such that $\xi_\mu(\exp Y) = e^{\mu(Y)}$, $Y \in \mathfrak{h}_\mathbb{C}$ exists. Let $L' \subset L$ be the subset of regular elements. According to Harish-Chandra the discrete series of G is parametrized by L', i.e., for each $\mu \in L'$ there is a discrete series representation π_μ. Moreover $\pi_\mu \cong \pi_{\mu'}$ iff there exists $w \in W$ such that $\mu = w\mu'$, and each discrete series representation is of the form π_μ for some $\mu \in L'$. Then by [Ar, (6.8)] we have

$$\pi_\mu(\Omega) = |\mu+\rho|^2 - |\rho|^2, \quad \mu \in L'. \tag{6.21}$$

So Proposition 6.3 gives in this case

$$\Theta_{\pi_\mu}(\operatorname{tr} h_\varphi) = \varphi\left(\sqrt{|\mu+\rho|^2 - |\rho|^2 + \lambda_\tau}\right), \quad \mu \in L',\ \pi_\mu \in \widehat{G}(\tau). \tag{6.22}$$

Using the Plancherel formula, (6.20) and (6.22), we get an explicit form of the trace formula (6.5).

Now we consider the case where $\tau_0 = 1$ is the trivial representation. Then we always have $[\pi|_K \colon \tau_0] \le 1$. Indeed, if $[\pi|_K \colon \tau_0] > 0$, then π is a principal series representation $\pi_{1,\lambda}$, where 1 denotes the trivial representation of M and it follows from Frobenius reciprocity [Kn, p. 208] that $[\pi|_K \colon \tau_0] = 1$. Then h_φ is a function which belongs to the space $C_c^\infty(G//K)$ of bi-K-invariant, smooth, compactly supported functions on G. Let $c(\lambda)$ be Harish-Chandra's c-function. Then the Plancherel measure for the spherical Fourier transform is given by $|c(\lambda)|^{-2} d\lambda$, and the Plancherel formula for spherical functions (see [Hel]) gives

$$h_\varphi(e) = \frac{1}{2}\int_{\mathbb{R}} \varphi\left(\sqrt{\lambda^2 + |\rho|^2}\right) |c(\lambda)|^{-2}\, d\lambda. \tag{6.23}$$

Furthermore, note that by Frobenius reciprocity $\widehat{M}(1)$ consists only of the trivial representation 1 of M. We denote the character of the induced representation $\pi_\lambda := \pi_{1,\lambda}$ by Θ_λ. By (6.20) we have

$$\Theta_\lambda(h_\varphi) = \varphi\left(\sqrt{\lambda^2 + |\rho|^2}\right), \quad \lambda \in \mathbb{R}. \tag{6.24}$$

Formulas (6.23) and (6.24) suggest to shift the spectrum by $|\rho|^2$, i.e., to replace $\Delta_\chi^\#$ by $\Delta_\chi^\# - |\rho|^2$. So let $h_{\varphi,\rho}$ be the kernel of $\varphi\left((\tilde{\Delta} - |\rho|^2)^{1/2}\right)$. As above, we get

$$h_{\varphi,\rho}(e) = \frac{1}{2}\int_{\mathbb{R}} \varphi(\lambda) |c(\lambda)|^{-2}\, d\lambda \tag{6.25}$$

and

$$\Theta_\lambda(h_{\varphi,\rho}) = \varphi(\lambda), \quad \lambda \in \mathbb{R}. \tag{6.26}$$

Inserting (6.25) and (6.26) into the trace formula for $\varphi\left((\tilde{\Delta} - |\rho|^2)^{1/2}\right)$ which is analogous to (6.5) we obtain Theorem 1.1.

□

Finally, we indicate the connection with [Se1]. Let $v \in \mathcal{H}_\lambda^K$ with $\| v \| = 1$ and let ϕ_λ be the spherical function (6.10) associated with v. Let $f \in C_c^\infty(G//K)$. Since f is bi-K-invariant, we get

$$\Theta_\lambda(f) = \operatorname{Tr} \pi_\lambda(f) = \int_G \langle \pi_\lambda(g)v, v\rangle f(g)\, d\lambda. \tag{6.27}$$

Using (6.10) we get

$$\Theta_\lambda(f) = \int_G \phi_\lambda(g) f(g)\, dg. \tag{6.28}$$

The right hand side is the spherical Fourier transform $\mathcal{F}(f)(\lambda)$ of f. Using the Cartan decomposition $G = KA^+K$ and $A^+ \cong (1,\infty)$, the bi-K-invariant functions f and ϕ_λ can be identified with functions on $(1,\infty)$. Choosing a standard integral representation for the spherical function, we end up with the classical Selberg transform.

7. Restrictions of representations of G

In this section we consider representations of Γ which are the restriction of a finite-dimensional complex representation $\eta\colon G \to \mathrm{GL}(E)$ of G. For such representations there exists another approach to the Selberg trace formula.

Denote the flat bundle associated to $\eta|_\Gamma$ by E_η. There is a different description of E_η as follows. Let $E_\tau = \Gamma\backslash\widetilde{E}_\tau$ be the locally homogeneous vector bundle associated to the restriction τ of η to K. Then there is a canonical isomorphism

$$E_\eta \cong E_\tau \tag{7.1}$$

[MM, Proposition 3.1]. Note that the space of C^∞-sections of E_τ can be identified with the space $\left(C^\infty(\Gamma\backslash G)\otimes E\right)^K$ of K-invariant vectors in $(C^\infty(\Gamma\backslash G)\otimes E)$, where K acts by $k \mapsto R(k)\otimes\eta(k)$, $k\in K$. Thus there is a canonical isomorphism

$$\phi\colon C^\infty(X,E_\eta) \cong \left(C^\infty(\Gamma\backslash G)\otimes E\right)^K. \tag{7.2}$$

Let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition. By [MM, Lemma 3.1] there exists a Hermitian inner product $\langle\cdot,\cdot\rangle_E$ in E which satisfies the following properties.

$$\begin{aligned} \langle\eta(Y)u,v\rangle_E &= -\langle u,\eta(Y)v\rangle_E, \quad \text{for } Y\in\mathfrak{k},\ u,v\in E;\\ \langle\eta(Y)u,v\rangle_E &= \langle u,\eta(Y)v\rangle_E, \quad \text{for } Y\in\mathfrak{p},\ u,v\in E. \end{aligned}$$

In particular, $\langle\cdot,\cdot\rangle_E$ is K-invariant. Therefore, it defines a G-invariant Hermitian fiber metric in $\widetilde{E}_\tau$ which descends to a fiber metric in E_τ. By (7.1) it corresponds to a fiber metric in E_η. Let $\Delta_\eta = (\nabla^\eta)^*\nabla^\eta$ be the associated Laplacian in $C^\infty(X,E_\eta)$. It is a formally self-adjoint operator. Its spectral decomposition can be determined as follows. By Kuga's lemma [MM, (6.9)] we have

$$\Delta_\eta = -R(\Omega) + \eta(\Omega)\,\mathrm{Id}. \tag{7.3}$$

Assume that η is absolutely irreducible. Then there is a scalar $\lambda_\eta \ge 0$ such that

$$\eta(\Omega) = \lambda_\eta\,\mathrm{Id}.$$

Let R_Γ be the right regular representation of G in $L^2(\Gamma\backslash G)$. Let

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus}_{\pi,\in\widehat{G}} m_\Gamma(\pi)\mathcal{H}_\pi \tag{7.4}$$

be the decomposition of R_Γ into irreducible subrepresentations, where $\mathcal{H}_\pi$ denotes the Hilbert space of the representation π. Denote by $(\mathcal{H}_\pi\otimes E)^K$ the space of K invariant vectors of $\mathcal{H}_\pi \otimes E$, where the action of K is given by $k \mapsto \pi(k) \otimes \eta(k)$. By (7.1) and (7.4) we get

$$L^2(X, E_\eta) \cong (L^2(\Gamma\backslash G) \otimes E)^K \cong \widehat{\bigoplus}_{\pi \in \widehat{G}} m_\Gamma(\pi)(\mathcal{H}_\pi \otimes E)^K. \tag{7.5}$$

For $\pi \in \widehat{G}$ let

$$\lambda_\pi = \pi(\Omega)$$

be the Casimir eigenvalue of π. Then $R(\Omega)$ acts in $(\mathcal{H}_\pi \otimes E)^K$ by λ_π. By (7.3) it follows that w.r.t. the isomorphism (7.5), Δ_η acts in $(\mathcal{H}_\pi \otimes E)^K$ as $(-\lambda_\pi + \lambda_\eta)\,\mathrm{Id}$. Thus (7.5) is the eigenspace decomposition of Δ_η.

Let $\varphi \in \mathcal{P}(\mathbb{C})$ be even. By Lemma 2.4 $\varphi\left((\Delta_\eta)^{1/2}\right)$ is a smoothing operator. So it is a trace class operator. It acts in $(\mathcal{H}_\pi \otimes E)^K$ by $\varphi\left((-\lambda_\pi + \lambda_\eta)^{1/2}\right)$. Then it follows from (7.5) that

$$\operatorname{Tr} \varphi\left((\Delta_\eta)^{1/2}\right) = \sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \dim(\mathcal{H}_\pi \otimes E)^K \varphi\left((-\lambda_\pi + \lambda_\eta)^{1/2}\right). \tag{7.6}$$

To derive the trace formula, we can proceed as in section 5. The lift $\widetilde{\Delta}_\eta$ of Δ_η to S is a G-invariant elliptic differential operator which is symmetric and non-negative. Let $h_{\eta,\varphi}\colon \Gamma\backslash G \to \operatorname{End}(E)$ be the kernel of $\varphi\big((\widetilde{\Delta}_\eta)^{1/2}\big)$. Applying Proposition 5.1 with $\chi = 1$ and (7.6), we get

$$\begin{aligned}\sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \dim(\mathcal{H}_\pi \otimes E)^K \varphi\left((-\lambda_\pi + \lambda_\eta)^{1/2}\right) &= \operatorname{vol}(\Gamma\backslash S) \operatorname{tr} h_{\eta,\varphi}(e) \\ &+ \sum_{\{\gamma\}_\Gamma \neq e} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} \operatorname{tr} h_{\eta,\varphi}(g^{-1}\gamma g)\, d\dot{g}.\end{aligned} \tag{7.7}$$

Remark. Let $\chi = \eta|_\Gamma$. Then we also have the trace formula of Proposition 5.1 with $\tau = 1$. The two formulas are, of course, different, since the operators are different. In the present case, the advantage is that we can work with self-adjoint operators. On the other hand, the formula (5.8) is more suitable for applications to Ruelle- and Selberg zeta functions. □

If the split rank of G is 1, we can use (6.3) to express the orbital integrals in terms of characters. This gives

Proposition 7.1. *Assume that the split rank of G is 1. Let $\eta\colon G \to \mathrm{GL}(E)$ be an absolutely irreducible finite-dimensional complex representation of G. Let $\varphi \in \mathcal{P}(\mathbb{C})$ be even. Then with the same notation as above we have*

$$\begin{aligned}\sum_{\pi \in \widehat{G}} m_\Gamma(\pi) \dim(\mathcal{H}_\pi \otimes E)^K \varphi\left((-\lambda_\pi + \lambda_\eta)^{1/2}\right) &= \operatorname{vol}(\Gamma\backslash S) \operatorname{tr} h_{\eta,\varphi}(e) \\ &+ \sum_{\{\gamma\}_\Gamma \neq e} \frac{1}{2\pi} \frac{l(\gamma_0)}{D(\gamma)} \sum_{\sigma \in \widehat{M}} \overline{\operatorname{tr} \sigma(\gamma)} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(h_{\eta,\varphi}) \cdot e^{-il(\gamma)\lambda}\, d\lambda.\end{aligned} \tag{7.8}$$

The characters $\Theta_{\sigma,\lambda}(h_{\eta,\varphi})$ can be computed by the method explained in section 6.

So there are two classes of finite-dimensional representations of Γ for which we can work with self-adjoint operators and apply the usual Selberg trace formula. These are unitary representations and restrictions of rational representations of G. In general, not every representation of Γ belongs to one of these classes. However, if $\mathrm{rank}(G) \geq 2$, the superrigidity theorem of Margulis [Ma, Chapt. VII, §5] implies that a general representation of Γ is not too far from a representation which is either unitary or the restriction of a rational representation. See [BW, p. 245] for more details.

References

[Ar] Arthur, J., *Harmonic analysis of tempered distributions on semisimple Lie groups of real rank one.* Dissertation, Yale University, New Haven, CT, 1970. Math. Surveys Monogr., 31, Representation theory and harmonic analysis on semisimple Lie groups, 13–100, Amer. Math. Soc., Providence, RI, 1989.

[BW] Borel, A., Wallach, N., *Continuous cohomology, discrete subgroups, and representations of reductive groups.* Second edition. Amer. Math. Soc., Providence, RI, 2000.

[BO] Bunke, U., Olbrich, M. *Selberg zeta and theta functions.* Akademie Verlag, Berlin, 1995.

[Ca] Camporesi, R., *The spherical transform for homogeneous vector bundles over Riemannian symmetric spaces.* J. Lie Theory **7** (1997), no. 1, 29–60.

[DKV] Duistermaat, J. J., Kolk, J. A. C.; Varadarajan, V. S. *Spectra of compact locally symmetric manifolds of negative curvature.* Invent. Math. 52 (1979), no. 1, 27–93.

[Fa] Fay, J., *Analytic torsion and Prym differentials.* In: Riemann surfaces and related topics, pp. 107–122, Ann. of Math. Stud., **97**, Princeton Univ. Press, Princeton, N.J., 1981.

[Fr] Fried, D., *Analytic torsion and closed geodesics on hyperbolic manifolds.* Invent. Math. **84** (1986), 523–540.

[Gi] Gilkey, P.B., *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* Second edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[GK] Gohberg, I.C., Krein, M.G., *Introduction to the theory of linear nonselfadjoint operators.* Translations of Mathematical Monographs, Vol. 18 American Mathematical Society, Providence, R.I. 1969.

[Hej] Hejhal, Dennis A. *The Selberg trace formula for* $\mathrm{PSL}(2,\mathbb{R})$. Vol. I. Lecture Notes in Mathematics, Vol. **548**, Springer-Verlag, Berlin-New York, 1976.

[Hel] Helgason,S., *Groups and geometric analysis*, Mathematical Surveys and Monographs, vol. 83, American Mathematical Society, Providence, RI, 2000.

[Ho1] Hörmander, L., *The spectral function of an elliptic operator.* Acta Math. **121** (1968), 193–218.

[Ho2] Hörmander, L., *The analysis of linear partial differential operators.* I. Second edition. Grundlehren der Mathematischen Wissenschaften 256. Springer-Verlag, Berlin, 1990.

[Ka] Kato, T., *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1966.

[Kn] Knapp, A.W., *Representation theory of semisimple groups*, Princeton University Press, Princeton, New Jersey 1986.

[Ko] Koornwinder, Tom H., *A note on the multiplicity free reduction of certain orthogonal and unitary groups.* Nederl. Akad. Wetensch. Indag. Math. 44 (1982).

[La] Lang, S., $\mathrm{SL}_2(\mathbb{R})$, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1975.

[LM] Lawson, H.B., Michelson, M.-L., *Spin geometry*, Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989.

[Ma] Margulis, G.A., *Discrete subgroups of semisimple Lie groups,* Springer-Verlag, Berlin, 1991.

[Mk] Markus, A.S., *Introduction to the spectral theory of polynomial operator pencils*, Translation of Math. Monogr. Vol. **71**, Providence, RI, 1988.

[MM] Matsushima, Y., Murakami, S., *On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds.* Ann. of Math. (2) 78 1963 365–416.

[Mi] Miatello, R., *The Minakshisundaram-Pleijel coefficients for the vector-valued heat kernel on compact locally symmetric spaces of negative curvature.* Trans. Amer. Math. Soc. **260** (1980), no. 1, 1–33.

[Mil] Millson, J., *Closed geodesics and the η-invariant.* Ann. of Math. (2) **108** (1978), 1–39.

[MS1] Moscovici, H., Stanton, R., *Eta invariants of Dirac operators on locally symmetric manifolds*, Invent. Math. 95 (1989), 629–666.

[MS2] Moscovici, H., Stanton, R., *R-torsion and zeta functions for locally symmetric manifolds.* Invent. Math. **105** (1991), 185–216.

[Pe] Petersson, H., *On a certain kind of zetafuchsian functions*, In: Report of an international colloquium on zeta-functions, Bombay 1956, 297–298.

[See] Seeley, R. T., *Complex powers of an elliptic operator.* In: Proc. Sympos. Pure Math., Vol. **10**, pp. 288–307, Amer. Math. Soc., Providence, R.I., 1967.

[Se1] Selberg, A., *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series.* J. Indian Math. Soc. (N.S.) **20** (1956), 47–87.

[Se2] Selberg, A., *Harmonic analysis*, in "Collected Papers", Vol. I, Springer-Verlag, Berlin-Heidelberg-New York (1989), 626–674.

[Sh] Shubin, M. A., *Pseudodifferential operators and spectral theory.* Springer-Verlag, Berlin, 1987.

[Ta1] Taylor, M., *Pseudo-differential operators*, Princeton University Press, Princeton, NJ, 1981.

[Ta2] Taylor, M., *Partial differential equations. I. Basic theory.* Applied Mathematical Sciences, 115. Springer-Verlag, New York, 1996.

[Wa] Wallach, N., *On the Selberg trace formula in the case of compact quotient*, Bull. Amer. Math. Soc. **82** (1976), 171 –195.

Universität Bonn, Mathematisches Institut, Beringstrasse 1, D – 53115 Bonn, Germany

E-mail address: `mueller@math.uni-bonn.de`