SCATTERING THEORY FOR THE LAPLACIAN ON MANIFOLDS WITH BOUNDED CURVATURE

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Abstract. In this paper we study the behaviour of the continuous spectrum of the Laplacian on a complete Riemannian manifold of bounded curvature under perturbations of the metric. The perturbations that we consider are such that its covariant derivatives up to some order decay with some rate in the geodesic distance from a fixed point. Especially we impose no conditions on the injectivity radius. One of the main results are conditions on the rate of decay, depending on geometric properties of the underlying manifold, that guarantee the existence and completeness of the wave operators.

0. Introduction.

The basic objects of the time-dependent approach to scattering theory are the wave operators. They are attached to a pair H_0 and H of self-adjoint operators, acting in Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively, and a unitary operator $J\colon \mathcal{H}_0 \to \mathcal{H}$. Let $P_{\mathrm{ac}}(H_0)$ be the orthogonal projection onto the absolutely continuous subspace of H_0. Then the wave operators $W_\pm(H, H_0; J)$ are said to exist, if the strong limit

$$W_\pm(H, H_0; J) = \text{s-lim}_{t\to\pm\infty} e^{itH} J e^{-itH_0} P_{\mathrm{ac}}(H_0), \tag{0.1}$$

exists. If the wave operators exist and are complete, they give rise to a unitary equivalence of the absolutely continuous parts $H_{0,\mathrm{ac}}$ and H_{ac} of H_0 and H, respectively. In this case the scattering operator is defined by $S = W_+^* \circ W_-$. There exist several general techniques to establish the existence and completeness of the wave operators. We will use the Kato-Birman theory, especially the invariance principle. Scattering theory is intimately connected with quantum mechanics and there is a vast literature dealing with the existence and completeness of the wave operators in this case. For a comprehensive account of mathematical scattering theory we refer to [BW], [RS], [Ya].

In this paper we study scattering theory in the geometric context. The basic setup is as follows. Let (M, g) be a complete Riemannian manifold and let Δ_g be the Laplacian on functions attached to g. Since M is complete, Δ_g is an essentially self-adjoint operator in $L^2(M)$ [Cn]. If M is non-compact, then Δ_g may have a nonempty continuous spectrum. We will consider perturbations h of the metric g which decay with a certain rate in the geodesic distance from a fixed point. Especially the metrics will be quasi-isometric so that

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the Hilbert spaces $L^2(M, d\mu_g)$ and $L^2(M, d\mu_h)$ are equivalent. Let J be the corresponding identification operator. The main purpose of this paper is to study the conditions on the perturbation h which imply the existence and completeness of the wave operators $W_\pm(\Delta_h, \Delta_g; J)$.

Scattering theory for the Laplacian on manifolds has been studied in a number of cases. In particular, it has been studied for manifolds with a special structure at infinity. So, for example, the case of manifolds with asymptotically cylindrical ends has been treated in [Me3]. Asymptotically Euclidean spaces were studied in [Me2]. In [Mu1], [Mu3] generalizations of locally symmetric manifolds of finite volume and $\mathbb{Q}$-rank one have been considered. See [Me1] for an overview and a discussion of some of these examples.

Our goal is to study non-compactly supported perturbations of the metric on arbitrary complete Riemannian manifolds with some restrictions on the curvature. To this end we introduce a certain class of functions, called functions of moderate decay, which describe the rate of decay of the perturbation of a given metric. Let $\beta\colon [1,\infty) \to \mathbb{R}^+$ be a function of moderate decay (see Definition 1.4). Then two complete metrics g and h are said to be equivalent up to order $k \in \mathbb{N}$, if there exist $C > 0$ and $p \in M$ such that

$$(0.2) \qquad |g-h|_g(x) + \sum_{j=0}^{k-1} \left|(\nabla^g)^j(\nabla^g - \nabla^h)\right|_g(x) \le C\beta(1 + d_g(x,p)), \quad x \in M,$$

where $d_g(x,p)$ is the geodesic distance of x and p with respect to g, and ∇^g (resp. ∇^h) the Levi-Civita connection with respect to g (resp. h). Note that $\nabla^g - \nabla^h$ is a tensor and therefore, $(\nabla^g)^j(\nabla^g - \nabla^h)$ is a tensor field. Condition (0.2) turns out to be an equivalence relation in the set of complete metrics on M. We denote this equivalence relation by $g \sim^k_\beta h$. It implies, in particular, that the two metrics are quasi-isometric.

To develop scattering theory for the Laplacian we need to impose additional assumptions on the metrics. In this paper we restrict attention to the class of complete metrics with bounded sectional curvature. In some cases we will also demand that higher derivatives of the curvature tensor are bounded. The assumption that the metric has bounded sectional curvature allows us to control the behavior of the injectivity radius $\imath(x)$ sufficiently well. Let $\tilde{\imath}(x)$ be the modified injectivity radius, defined by (2.1) which is bounded from above by a constant that depends on the bound of the sectional curvature. Then one of our main results is the following theorem.

Theorem 0.1. *Assume g and h be complete metrics on M with bounded curvature up to order 2. Let β be a function of moderate decay. Suppose that $g \sim^2_\beta h$. Assume that there exist real numbers a, b satisfying*

i) $b \ge 1$ *and* $a + b = 2$,
ii) $\beta^{\frac{b}{3}} \in L^1(M)$,
iii) $\beta^{\frac{a}{3}} \tilde{\imath}^{-\frac{n(n+2)}{2}} \in L^\infty(M)$.

Then $e^{-t\Delta_g} - e^{-t\Delta_h}$ is a trace class operator.

By the invariance principle for wave operators [Ka], Theorem 0.1 implies that the wave operators $W_\pm(\Delta_h, \Delta_g; J)$ exist and are complete (see Theorem 7.1). To demonstrate this result, we discuss three examples in section 7, namely manifolds with cylindrical ends, manifolds with bounded geometry, and manifolds with cusps. Under additional assumptions on (M, g), the conditions on β can be relaxed. This is, for example, the case for manifolds with cusps and manifolds with cylindrical ends. In either case, the method of Enss can be used to prove the existence and completeness of the wave operators.

The time-independent approach to scattering theory is based on the study of the resolvent. An important problem in this context is the question of the existence of an analytic continuation of the resolvent as operator in appropriate weighted L^2-spaces. We study this problem in the geometric setting described above. Assuming that the resolvent $R_g(\lambda) = (\Delta_g - \lambda \operatorname{Id})^{-1}$, regarded as an operator in certain weighted L^2-spaces, admits a meromorphic continuation to some ramified covering of a domain in $\Omega \subset \mathbb{C}$, we show that the same is true for the resolvent of the perturbed Laplacian Δ_h under suitable decay conditions on $h-g$. See Theorem 8.4 for details. The existence of an analytic continuation of the resolvent has been studied for several classes of metrics with special structures at infinity. Examples are: manifolds which outside a compact set are isometric to a neighborhood of a cusp of a locally symmetric space of $\mathbb{Q}$-rank one [Mu2], asymptotically flat metrics [Me2], cylindrical end metrics [Me3], and metrics with asymptotically constant negative curvature [MM]. [Me1] contains a discussion of these examples. In [MV] and [Sm], the analytic continuation of the resolvent of the Laplacian on a non-compact Riemannian symmetric space has been studied. As shown in [MV], there is a certain analogy between spectral theory of the Laplacian on symmetric spaces and N-body quantum scattering.

The structure of the paper is as follows. In section 1 we introduce our class of functions of moderate decay and study some of its elementary properties. Then we set up the equivalence relation mentioned above and prove some facts about equivalent metrics. In section 2 we study the behavior of the injectivity radius on manifolds with bounded sectional curvature. Then we introduce and study weighted Sobolev spaces in section 3. In section 4 we show that certain functions of the Laplacian including the heat kernel and the resolvent extend to bounded operators in weighted L^2-spaces. Section 5 deals with the comparison of weighted Sobolev spaces with respect to equivalent metrics. Then we prove Theorem 0.1 in section 6. In section 7 we deal with the existence and completeness of wave operators. First we prove a general result which is based on Theorem 0.1 and we discuss some examples. Then we consider the case of a manifold with cusps and use the method of Enss to establish the existence and completeness of the wave operators under weaker assumptions on β. The final section 8 deals with the analytic continuation of the resolvent, regarded as operator in weighted L^2-spaces.

1. Equivalence of Riemannian Metrics.

Let M be an open, connected C^∞–manifold of dimension n and let $\mathcal{M} = \mathcal{M}(M)$ be the space of all complete Riemannian metrics on M. Eichhorn [Ei1] has shown that $\mathcal{M}$ can be endowed with a canonical topology given by a metrizable uniform structure. We briefly recall its definition.

For a given Riemannian metric g on M, denote by ∇^g the Levi–Civita connection of g and by $|\cdot|_g$ the norm induced by g in the fibers of $\oplus_{p,q\geq 0}(TM^{\otimes p}\otimes T^*M^{\otimes q})$. Let h be any other Riemannian metric on M. For $k\geq 0$ set

$$
{}^k|g-h|_g(x)=|g-h|_g(x)+\sum_{j=0}^{k-1}\left|(\nabla^g)^j(\nabla^g-\nabla^h)\right|_g(x),\quad x\in M. \tag{1.1}
$$

and

$$
{}^k\|g-h\ \|_g=\sup_{x\in M}{}^k|g-h|_g(x). \tag{1.2}
$$

Recall that two metrics g,h are said to be quasi-isometric if there exist $C_1,C_2>0$ such that

$$
C_1g(x)\leq h(x)\leq C_2g(x),\quad \text{for all } x\in M, \tag{1.3}
$$

in the sense of positive definite forms. We shall write $g\sim h$ for quasi–isometric metrics g and h. If g and h are quasi–isometric, then (1.3) implies that for all $p,q\geq 0$, there exist $A_{p,q},B_{p,q}>0$ such that for every tensor field T on M of bidegree (p,q), we have

$$
A_{p,q}|T|_g(x)\leq |T|_h(x)\leq B_{p,q}|T|_g(x),\quad x\in M. \tag{1.4}
$$

Put $\nabla:=\nabla^g$ and $\nabla':=\nabla^h$. Let $\nabla^{p,q}$ and $\nabla'^{p,q}$ be the canonical extension of ∇ and ∇', respectively, to the tensor bundle $T^{p,q}(M)$. Then for all $p,q\in\mathbb{N}$ there exists $C_{p,q}>0$ such that

$$
|\nabla^{p,q}-\nabla'^{p,q}|_g(x)\leq C_{p,q}|\nabla-\nabla'|_g(x),\quad x\in M. \tag{1.5}
$$

For $k\geq 1$ and $\delta>0$, set

$$
V_\delta=\{(g,g')\in\mathcal{M}\times\mathcal{M}\mid g\sim g' \text{ and } {}^k\|g-g'\ \|_g<\delta\}.
$$

It is proved in [Ei1], Proposition 2.1, that $\{V_\delta\}_{\delta>0}$ is a basis for a metrizable uniform structure on $\mathcal{M}$.

Lemma 1.1. *Let $g,h\in\mathcal{M}$. Assume that there exists a compact subset $K\subset M$ and $0<\delta<1$ such that $|g-h|_g(x)\leq\delta$ for all $x\in M\setminus K$. Then g and h are quasi-isometric.*

Proof: Let $x\in M\setminus K$. Choose geodesic coordinates w.r.t. g, centered at x. Then $g_{ij}(x)=\delta_{ij}$. Let $H=(h_{ij}(x))$ be the matrix representing $h(x)$ in these coordinates. Denote by $\|\cdot\|$ the supremum norm of linear maps in $\mathbb{R}^n$. Then by assumption, we have

$\| H - \mathrm{Id} \| \leq \delta < 1$. Hence the Neumann series for $H^{-1} = (\mathrm{Id} - (\mathrm{Id} - H))^{-1}$ converges in norm which implies that $\| H^{-1} \| \leq 1/(1-\delta)$. Thus for all $\xi \in \mathbb{R}^n$, we get

$$(1-\delta) \| \xi \|^2 \leq \big(\| H^{-1} \| \big)^{-1} \| \xi \|^2 \leq \langle H\xi, \xi \rangle \leq \| H \| \| \xi \|^2 \leq (1+\delta) \| \xi \|^2 .$$

This implies that

$$(1-\delta) g(x) \leq h(x) \leq (1+\delta) g(x), \quad \text{for all } x \in M \setminus K.$$

Since K is compact, it follows that g and h are quasi-isometric. □

We need two results from the proof of Proposition 2.1 in [Ei1] which we state as lemmas. For the convenience of the reader we repeat the proofs.

Lemma 1.2. *Let $g, h \in \mathcal{M}$ be quasi–isometric. For every $k \geq 0$, there exists a polynomial $P_k(X_1, ..., X_k)$, depending on the quasi-isometry constants, with nonnegative coefficients and vanishing constant term, such that*

$$^k|g-h|_h(x) \leq P_k\big(|g-h|_g(x), |\nabla^g - \nabla^h|_g(x), ..., |(\nabla^g)^{k-1}(\nabla^g - \nabla^h)|_g(x)\big), \quad x \in M.$$

Proof: From (1.4) follows that

$$|g-h|_h(x) \leq C_3 |g-h|_g(x) \tag{1.6}$$

and

$$|\nabla^g - \nabla^h|_h(x) \leq C_4 |\nabla^g - \nabla^h|_g(x), \quad x \in M. \tag{1.7}$$

This takes care of the first two terms in (1.1) and settles the question for $k = 0, 1$. Now we shall proceed by induction. Let $k \geq 2$ and suppose that the lemma holds for $l \leq k-1$. For each $p \geq 0$, we have

$$(\nabla^h)^p(\nabla^h - \nabla^g) = \nabla^g(\nabla^h)^{p-1}(\nabla^h - \nabla^g) + (\nabla^h - \nabla^g)(\nabla^h)^{p-1}(\nabla^h - \nabla^g). \tag{1.8}$$

Let $p \leq k$. Using (1.7), (1.5) and the induction hypothesis, we can estimate the pointwise h-norm of the second term on the right hand side of (1.8) in the desired way. To deal with the first term, we use the formula

$$\begin{aligned}(\nabla^g)^p(\nabla^h)^l(\nabla^h - \nabla^g) =& (\nabla^g)^{p+1}(\nabla^h)^{l-1}(\nabla^h - \nabla^g) \\ &+ (\nabla^g)^p(\nabla^h - \nabla^g)(\nabla^h)^{l-1}(\nabla^h - \nabla^g).\end{aligned} \tag{1.9}$$

Applying the Leibniz rule, we get

$$\begin{aligned}&\big|(\nabla^g)^p(\nabla^h - \nabla^g)(\nabla^h)^{l-1}(\nabla^h - \nabla^g)\big|_g(x) \\ &\qquad \leq C \sum_{i=0}^{p} \big|\big((\nabla^g)^i(\nabla^h - \nabla^g)\big)\big|_g(x) \cdot \big|\big((\nabla^g)^{p-i}(\nabla^h)^{l-1}(\nabla^h - \nabla^g)\big)\big|_g(x)\end{aligned}$$

for some $C > 0$ and all $x \in M$. Inserting (1.8) and iterating these formulas reduces everything to the induction hypothesis. □

Lemma 1.3. *Let $g_i \in \mathcal{M}$, $i = 1, 2, 3$, and suppose that $g_1 \sim g_2 \sim g_3$. For every $k \geq 0$, there exists a polynomial Q_k, depending on the quasi-isometry constants, in the variables ${}^i|g_1 - g_2|_{g_1}(x)$ and ${}^j|g_2 - g_3|_{g_2}(x)$, $\quad i, j = 0, \ldots, k$, with nonnegative coefficients and vanishing constant term, such that*

$$ {}^k|g_1 - g_3|_{g_1}(x) \leq Q_k({}^i|g_1 - g_2|_{g_1}(x),\ {}^j|g_2 - g_3|_{g_2}(x)), \quad x \in M. $$

If there exists $\delta < 1$ such that $\|g_1 - g_2\|_{g_1} \leq \delta$ and $\|g_2 - g_3\|_{g_2} \leq \delta$, the dependence on the quasi-isometry constants can be removed.

Proof: Since $g_1 \sim g_2$, it follows from (1.4) that

$$ |g_1 - g_3|_{g_1}(x) \leq |g_1 - g_2|_{g_1}(x) + C_1|g_2 - g_3|_{g_2}(x). $$

Set $\nabla_i = \nabla^{g_i}$, $i = 1, 2, 3$. By the same argument, we get

$$ |\nabla_1 - \nabla_3|_{g_1}(x) \leq |\nabla_1 - \nabla_2|_{g_1}(x) + C_2|\nabla_2 - \nabla_3|_{g_2}(x). $$

Thus, the lemma holds for $k = 0, 1$, and we can use induction to prove the lemma. First observe that for $p \geq 0$,

$$ \nabla_1^p(\nabla_1 - \nabla_3) = \nabla_1^p(\nabla_1 - \nabla_2) + \nabla_1^p(\nabla_2 - \nabla_3). $$

The pointwise g_1–norm of the first term on the right hand side gives already what we want. The second term can be written as

$$ \nabla_1^p(\nabla_2 - \nabla_3) = (\nabla_1 - \nabla_2)\nabla_1^{p-1}(\nabla_2 - \nabla_3) + \nabla_2\nabla_1^{p-1}(\nabla_2 - \nabla_3). $$

Iteration of this formula and application of the Leibniz rule reduces again everything to the induction hypothesis. The last statement again follows from Lemma 1.1. $\square$

To set up our equivalence relation in $\mathcal{M}$, we introduce an appropriate class of functions.

Definition 1.4. Let $\beta : [1, \infty) \to \mathbb{R}$ be a positive, continuous, non-increasing function. Then β is called a function of *moderate decay*, if it satisfies the following conditions

$$ \begin{aligned} &1) \quad \sup_{x \in [1,\infty)} x\beta(x) < \infty; \\ &2) \quad \exists\, C_\beta > 0 : \beta(x + y) \geq C_\beta\, \beta(x)\beta(y), \quad x, y \geq 1. \end{aligned} \tag{1.10} $$

Furthermore, β is called of *sub-exponential decay* if for any $c > 0$, $e^{cx}\beta(x) \to \infty$ as $x \to \infty$.

Remark 1. The class of functions which are of moderate or sub-exponential decay are closed under multiplication, and also under raising to positive powers. The function e^{-tx}, $t \geq 0$, is of moderate decay and the functions x^{-1} and $\exp(-x^\alpha)$, $0 < \alpha < 1$, are of sub-exponential decay. Thus the class of functions introduced in Definition 1.4 is not empty.

Next we establish some elementary properties of β.

Lemma 1.5. *Let β be of moderate decay. Then there exist constants $C > 0$ and $c \geq 0$ such that*

$$ \beta(x) \geq Ce^{-cx}, \quad x \in [1, \infty). \tag{1.11} $$

Proof: Given $x \in [1, \infty)$, write x as $x = y + n$, where $y \in [1, 2)$ and $n \in \mathbb{N}$. Applying condition 2) of (1.10) repeatedly, we get

$$\beta(x) \geq \beta(y)(C_\beta \beta(1))^n. \tag{1.12}$$

By assumption, β is continuous. Hence there exists $C > 0$ such that $\beta(y) \geq C$ for $y \in [1, 2]$. Since β is non-increasing, it follows that $C_\beta \beta(1) \leq 1$. Thus there exists $c \geq 0$ such that $C_\beta \beta(1) = e^{-c}$. Together with (1.12) the claim follows. □

Thus for a function β of moderate decay there exist constants $c, C_1, C_2 > 0$ such that

$$C_1 e^{-cx} \leq \beta(x) \leq C_2 x^{-1}, \quad x \geq 1.$$

Lemma 1.6. *Let β be a function of moderate decay. Then for all $x, y, q \in M$, we have*

$$C_\beta \, \beta\big(1 + d(x, y)\big) \leq \frac{\beta\big(1 + d(x, q)\big)}{\beta\big(1 + d(y, q)\big)} \leq \frac{1}{C_\beta \beta\big(1 + d(x, y)\big)}. \tag{1.13}$$

Moreover, for every $q' \in M$ there exists a constant $C > 0$, depending only on q and q', such that

$$C^{-1} \, \beta\big(1 + d(x, q')\big) \leq \beta\big(1 + d(x, q)\big) \leq C \, \beta\big(1 + d(x, q')\big). \tag{1.14}$$

Proof: Since β is non-increasing, it follows from (1.10) that

$$\begin{aligned}
\frac{\beta\big(1 + d(x, q)\big)}{\beta\big(1 + d(y, q)\big)} &\leq \frac{\beta\big(1 + d(x, q)\big)}{\beta\big(1 + d(x, q) + 1 + d(x, y)\big)} \\
&\leq \frac{\beta\big(1 + d(x, q)\big)}{C_\beta \, \beta\big(1 + d(x, q)\big)\beta\big(1 + d(x, y)\big)} \\
&= \frac{1}{C_\beta \beta\big(1 + d(x, y)\big)}.
\end{aligned}$$

Switching the roles of x and y, we obtain the other inequality in (1.13). Furthermore, switching the roles of x and q and putting $y = q'$ in (1.13) gives (1.14). □

Lemma 1.7. *Let β be a function of moderate decay. Let $g, h \in \mathcal{M}$, $q \in M$, and suppose that*

$$|g - h|_g(x) \leq \beta(1 + d_g(x, q)), \quad x \in M. \tag{1.15}$$

Then g and h are quasi-isometric and there exist constants $C_1, C_2 > 0$ such that

$$C_1 d_g(x, y) \leq d_h(x, y) \leq C_2 d_g(x, y), \quad x, y \in M, \tag{1.16}$$

and

$$C_1 \beta(1 + d_g(x, q)) \leq \beta(1 + d_h(x, q)) \leq C_2 \beta(1 + d_g(x, q)), \quad x \in M. \tag{1.17}$$

Proof: Let $0 < \delta < 1$. From condition 1) of (1.10) follows that there exists r_0 such that $\beta(1+r) \leq \delta$ for $r \geq r_0$. Thus by Lemma 1.1, g and h are quasi-isometric and this implies (1.16). To prove the second part, we first note that it follows from the proof of Lemma 1.1 that

$$d_h(x,q) \leq (1+\beta(1+d_g(x,q))d_g(x,q), \quad d_g(x,q) \geq r_0.$$

Moreover, by condition 1) of (1.10) there exists $C > 0$ such that

$$\beta(1+d_g(x,q))d_g(x,q) \leq C, \quad x \in M.$$

Then using (1.10), (1.16) and the assumption that β is non-increasing, we get

$$\beta(1+d_h(x,q)) \geq \beta(1+(1+\beta(1+d_g(x,q))d_g(x,q)) \geq C_\beta \beta(C)\beta(1+d_g(x,q)).$$

Switching the roles of g and h, we obtain the other inequality. □

Let $k \geq 0$, and consider the following relation for metrics $g, h \in \mathcal{M}$:

(1.18) *There exist* $q \in M$ and $C > 0$ *such that for all* $x \in M$ *we have*

$$^k|g-h|_g(x) \leq C\beta(1+d_g(x,q)).$$

Proposition 1.8. *The relation (1.18) defines an equivalence relation in* $\mathcal{M}$.

Proof: Let $g, h \in \mathcal{M}$ and suppose that (1.18) holds. Then by Lemma 1.7, g, h are quasi–isometric. Then Lemma 1.2 combined with (1.17) implies that

$$^k|g-h|_h(x) \leq C_3\beta(1+d_g(x,y)) \leq C_4\beta(1+d_h(x,q)).$$

Thus the relation (1.18) is symmetric. The transitivity follows from Lemma 1.3 and (1.17). By Lemma 1.6, the relation is independent of q. □

This justifies the following definition.

Definition 1.9. Let β be a function of moderate decay. Two metrics $g, h \in \mathcal{M}$ are said to be β–equivalent up to order k if (1.18) holds. In this case we write $g \sim_\beta^k h$.

Example. Let (M, g) be a complete Riemannian manifold which is Euclidean at infinity, that is, there exists a compact subset $K \subset M$ such that $(M \setminus K, g)$ is isometric to $\mathbb{R}^n \setminus B_r(0)$ for some $r > 0$, where $\mathbb{R}^n$ is equipped with its standard metric. Let $\beta(r) = r^{-a}$, $a > 1$, and let h be a complete Riemannian metric on M such that $h \sim_\beta^k g$ for some $k \in \mathbb{N}$. Then $h|_{M\setminus K}$ may be regarded as metric on $\mathbb{R}^n \setminus B_r(0)$ and if h_{ij} are the components of $h|_{M\setminus K}$ with respect to the standard coordinates $x_1, ..., x_n \in \mathbb{R}^n$, then the condition $h \sim_\beta^k g$ is equivalent to

$$\left| \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} (h_{ij}(x) - \delta_{ij}) \right| \leq C(1+ \| x \|)^{-a} \tag{1.19}$$

for all multi-indices α with $|\alpha| \leq k$ and all $x \in \mathbb{R}^n \setminus B_r(0)$. Such metrics are called asymptotically Euclidean.

To simplify notation, we will write $\beta(x)$ in place of $\beta(1+d_g(x,q))$. If $g \sim_\beta^k h$, it follows from Lemma 1.7, that we may use both d_g and d_h in (1.18).

Next we show that the β-equivalence can also be defined in a different manner. Namely we have the following proposition.

Proposition 1.10. *Let $k \geq 0$ and let $g, h \in \mathcal{M}$. Then $g \sim_\beta^k h$ holds if and only if there exists $C_1 > 0$ such that*

$$\sum_{i=0}^{k} \left|(\nabla^g)^i (g-h)\right|_g(x) \leq C_1 \beta(x), \quad x \in M.$$

Proof: Let $g, h \in \mathcal{M}$. The lemma holds obviously for $k=0$. Let $k \geq 1$. Recall that $\nabla^g g = 0$ and $\nabla^h h = 0$. Using this fact, we get

$$(\nabla^g)^k (g-h) = -(\nabla^g)^k h = -(\nabla^g)^{k-1}(\nabla^g - \nabla^h) h$$

Using the Leibniz rule it follows that there exists $C_1 > 0$ such that

$$\left|(\nabla^g)^k (g-h)\right|_g(x) \leq C_1 \sum_{i=0}^{k-1} \left|(\nabla^g)^i (\nabla^g - \nabla^h)\right|_g(x) \cdot \left|(\nabla^g)^{k-1-i}(h)\right|_g(x), \quad x \in M. \tag{1.20}$$

Suppose that ${}^k|g-h|_g(x) \leq C\beta(x)$, $x \in M$, for some constant $C > 0$. Then $|h|_g(x) \leq C'$ for some constant $C' > 0$. By induction it follows from (1.5) and (1.20) that

$$\sum_{i=0}^{k} \left|(\nabla^g)^i (g-h)\right|_g(x) \leq C_2 \beta(x), \quad x \in M, \tag{1.21}$$

for some constant $C_2 > 0$, depending on C and k.

Now assume that (1.21) holds. We observe that for any smooth vector fields X, Y, Z, the following formula holds

$$\begin{aligned} h\big((\nabla_X^g - \nabla_X^h) Y, Z\big) = \frac{1}{2} \big\{ \nabla_X^g (g-h)(Y,Z) \,+\, \nabla_Y^g (g-h)(X,Z) \\ - \nabla_Z^g (g-h)(X,Y) \big\}. \end{aligned} \tag{1.22}$$

From this formula we get

$$|\nabla^h - \nabla^g|_h \leq C\, |\nabla^g (g-h)|_h.$$

Taking covariant derivatives of (1.22) and using induction, we obtain

$${}^k|h-g|_h(x) \leq C \sum_{i=0}^{k} \left|(\nabla^g)^i (g-h)\right|_h(x).$$

By (1.4) and (1.21), we get

$${}^k|h-g|_h(x) \leq C\, \beta(x),$$

and Lemma 1.2 implies that

$$^k|g-h|_g(x) \le C_1\,\beta(x), \quad x \in M,$$

for some constant $C_1 > 0$. $\square$

Thus, we may define β–equivalence also by requiring that (1.21) holds for some constant C_1. It follows from the previous proposition that this gives rise to an equivalence relation.

Finally, we study the behavior of the curvature tensor and its covariant derivatives under β–equivalence. Given $g \in \mathcal{M}$, denote by R^g the curvature tensor of g.

Lemma 1.11. *Let $k \ge 2$ and let $g, h \in \mathcal{M}$. Suppose that $g \sim_\beta^k h$. Then there exists $C_k > 0$ such that*

$$|(\nabla^g)^i(R^g - R^h)|_g(x) \le C_k\beta(x),\ x \in M,\ i = 0, \dots, k-2.$$

Proof: Set $\nabla = \nabla^g$, $\nabla' = \nabla^h$. We define the exterior differential

$$d^\nabla : C^\infty(\Lambda^p(T^*M) \otimes TM) \to C^\infty(\Lambda^{p+1}(T^*M) \otimes TM)$$

associated with ∇ by the following formula

$$\begin{aligned}(d^\nabla\alpha)(X_0, ..., X_p) = &\sum_{i=0}^{p}(-1)^i\nabla_{X_i}\big(\alpha(X_0, ..., \widehat{X}_i, ..., X_p)\big)\\ &- \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_0, ..., \widehat{X}_i, ..., \widehat{X}_j, ..., X_p).\end{aligned}$$

Then, regarded as operators $C^\infty(TM) \to C^\infty(\Lambda^2(T^*M) \otimes TM)$, we have

$$R^\nabla = d^\nabla \circ d^\nabla,$$

and a corresponding formula holds for ∇'. Set $A = \nabla' - \nabla$ and let X, Y be smooth vector fields on M. Then we have [Be, p. 25]

$$\begin{aligned}R^{\nabla'}(X,Y) \ - \ R^\nabla(X,Y) &= \nabla_X\big(A(Y)\big) - \nabla_Y\big(A(X)\big) - A([X,Y])\\ &\qquad - A(X) \circ A(Y) + A(Y) \circ A(X)\\ &= (\nabla A)(X,Y) - (\nabla A)(Y,X)\\ &\qquad - A(X) \circ A(Y) + A(Y) \circ A(X).\end{aligned}$$

Differentiating this equality and using induction gives the desired result. $\square$

Recall that a Riemannian manifold (M, g) is said to have bounded curvature of order k, if the covariant derivatives $\nabla^i R$, $0 \le i \le k$, of the curvature tensor R are uniformly bounded on M, i.e., there exists $C > 0$ such that $|\nabla^i R|(x) \le C$, $x \in M$, $0 \le i \le k$.

Corollary 1.12. *Let $k \ge 2$ and let $g, h \in \mathcal{M}$. Suppose that $g \sim_\beta^k h$. Then*

1) *(M, g) has bounded curvature of order $k-2$ if and only if (M, h) has bounded curvature of order $k-2$.*

2) *The sectional curvature of* (M, g) *is bounded from below (resp. from above) if and only if the sectional curvature of* (M, h) *is bounded from below (resp. above).*
3) *The Ricci curvature of* (M, g) *is bounded from below (resp. from above) if and only if the Ricci curvature of* (M, h) *is bounded from below (resp. above).*

2. Injectivity radius and bounded curvature.

In this section we establish some properties of the injectivity radius on a manifold with bounded sectional curvature. Let (M, g) be a complete, n–dimensional Riemannian manifold with bounded sectional curvature, say $|K_M| \leq K$. Let $p \in M$. Recall that the injectivity radius $i(p)$ at p equals the minimal distance from p to its cut locus $C(p)$ (see [CE], [Kl]). Also note that $i(p)$ is a continuous function of $p \in M$ [Kl, Proposition 2.1.10].

Proposition 2.1. *Let* h *be another complete Riemannian metric on* M *with bounded sectional curvature* $|K_M^h| \leq K$ *and assume that* g *and* h *are equivalent. Given* $p \in M$*, let* $i_g(p)$ *and* $i_h(p)$ *denote the injectivity radii at* p *with respect to* g *and* h*, respectively. Then there exist constants* $c, c' > 0$ *such that*

$$i_h(p) \geq \min\{c i_g(p), c'\}, \quad p \in M.$$

Proof: Since g and h are assumed to be equivalent, there exists $\varepsilon > 0$ such that

$$e^{-\varepsilon} g \leq h \leq e^{\varepsilon} g.$$

Let $x \in M$ and suppose that $i_h(x) < \min\left\{e^{-2\epsilon}\pi/(2\sqrt{K}), e^{-\epsilon} i_g(x)/2\right\}$. It follows from [CE, Corollary 1.30] that distinct conjugate points along a geodesic (with respect to h) have distance $\geq \pi/\sqrt{K}$. Therefore, by [CE, Lemma 5.6], there exists a closed geodesic loop γ^h at x with respect to the metric h, with

$$h\text{-length}(\gamma^h) < \min\left\{e^{-2\epsilon}\pi/\sqrt{K}, e^{-\epsilon} i_g(x)\right\}.$$

Hence, we have

$$g\text{-length}(\gamma^h) < \min\left\{e^{-\epsilon}\pi/\sqrt{K},\ i_g(x)\right\}.$$

In particular, g-length$(\gamma^h) < \pi/\sqrt{K}$. Let $r_{\max}(x)$ be the maximal rank radius of $\exp_x$ with respect to g. Then we obtain g–length$(\gamma^h) < \pi/\sqrt{K} \leq r_{\max}(x)$. By [BK], Proposition 2.2.2, there exists a unique g-geodesic loop $\widetilde{\gamma}\colon [0,1] \longrightarrow M$ at x with g–length$(\widetilde{\gamma}) < r_{\max}(x)$, which is obtained from γ^h by a length decreasing homotopy $H\colon [0,1] \times [0,1] \longrightarrow M$ (cf. [BK], 2.1.2). Hence, we have

$$g\text{-length}(\widetilde{\gamma}) \leq g\text{-length}(\gamma^h) < \min\left\{e^{-\epsilon}\frac{\pi}{\sqrt{K}},\ i_g(x)\right\}.$$

Since h-length$(H(\cdot, s)) \leq e^{\epsilon} g$-length$(\gamma^h) < 2\pi/\sqrt{K}$ for $s \in [0,1]$, it follows from [Kl, Lemma 2.6.4], that g-length$(\widetilde{\gamma}) > 0$. Parameterize $\widetilde{\gamma}$ by g–arc length. Then either $\widetilde{\gamma}(t)$ or $\widetilde{\gamma}(\text{length}(\widetilde{\gamma}) - t)$ belongs to the cut locus of x for some $t \leq \frac{1}{2} g$-length$(\widetilde{\gamma})$. Therefore $i_g(x) < i_g(x)$, a contradiction. □

Let β be a function of moderate decay. Suppose that $g \sim^0_\beta h$. Then by Lemma 1.7, g and h are quasi-isometric. Therefore, if h has bounded sectional curvature, then Proposition 2.1 can be applied to g, h. For $x \in M$ set

$$\tilde{\imath}(x) := \min\left\{\frac{\pi}{12\sqrt{K}},\ i(x)\right\}. \tag{2.1}$$

Then it follows, that under the assumptions of Proposition 2.1, there exists $c_2 > 0$ such that

$$\tilde{\imath}_h(p) \geq c_2 \tilde{\imath}_g(p), \quad p \in M.$$

Next recall the Bishop–Günther inequalities [Gra, Theorem 3.17], [Gro, Lemma 5.3], which give estimates of the volume of small balls from above and below.

Lemma 2.2. *For $r \leq \tilde{\imath}(x_0)$,*

$$\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^r \left(\frac{\sin t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt \leq \operatorname{Vol}(B_r(x_0)) \leq \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^r \left(\frac{\sinh t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt.$$

We note that the inequality on the right hand side holds for all $r \in \mathbb{R}_+$. In particular

$$\operatorname{Vol}(B_r(x_0)) = O\left(e^{(n-1)\sqrt{K}r}\right) \tag{2.2}$$

as $r \longrightarrow \infty$.

It is also important to know the maximal possible decay of the injectivity radius.

Lemma 2.3. *There exists a constant $C > 0$, depending only on K, such that*

$$\tilde{\imath}(x) \geq C\, \tilde{\imath}(p)^n e^{-(n-1)\sqrt{K}\, d(x,p)} \tag{2.3}$$

for all $x, p \in M$.

Proof: Let $p \in M$ and fix r, r_0, s, with $r_0 + 2s < \pi/\sqrt{K}$, $r_0 \leq \pi/4\sqrt{K}$. By [CGT, Theorem 4.7] we get

$$\tilde{\imath}(x) \geq \frac{r_0}{2} \cdot \frac{1}{1 + \left(V^K_{r_0+s}/\operatorname{Vol}(B_r(p))\right)\left(V^K_{d(x,p)+r}/V^K_s\right)}, \tag{2.4}$$

where V^K_s denotes the volume of a ball of radius s in the n–dimensional hyperbolic space of curvature $-K$. Set $r_0 = s = \frac{\pi}{5\sqrt{K}}$, $r = \tilde{\imath}(p)$ and apply Lemma 2.2 to estimate $\operatorname{Vol}\left(B_{\tilde{\imath}(p)}(p)\right)$ from below. Then (2.4) implies

$$\begin{aligned}\tilde{\imath}(x) &\geq C_1\, \tilde{\imath}(p)^n e^{-(n-1)\sqrt{K}(d(x,p)+\tilde{\imath}(p))}\\ &\geq C\, \tilde{\imath}(p)^n e^{-(n-1)\sqrt{K}d(x,p)}.\end{aligned}$$

□

Corollary 2.4. *Given $p \in M$, there exists a constant $C = C(p) > 0$ such that*

$$\tilde{\imath}(x) \geq Ce^{-(n-1)\sqrt{K}\, d(x,p)}, \quad x \in M.$$

□

Lemma 2.5. *There exists a constant C, depending only on K, such that for each $x, y \in M$ we have the inequality*

$$\tilde{\imath}(y) \geq C\tilde{\imath}(x)e^{-\frac{(n-1)\pi}{12}\frac{d(x,y)}{\tilde{\imath}(x)}}. \tag{2.5}$$

Proof: Let $\lambda = \max\{1, \frac{\pi^2}{144K i(x)^2}\}$. Then the injectivity radius i_λ at x with respect to λg is given by

$$i_\lambda(x) = \lambda^{\frac{1}{2}} i(x) = \begin{cases} i(x) & ,\text{if } i(x) > \frac{\pi}{12\sqrt{K}};\\ \frac{\pi}{12\sqrt{K}} & ,\text{if } i(x) \leq \frac{\pi}{12\sqrt{K}}.\end{cases}$$

Since $\lambda^{-1} \leq 1$, the sectional curvature $K_M^{\lambda g}$ with respect to λg also satisfies $|K_M^{\lambda g}| \leq K$.

Let $r = \frac{\pi}{\sqrt{K}}$, $r_0 = s = \frac{r}{12} = \frac{\pi}{12\sqrt{K}}$ and set $d = \lambda^{\frac{1}{2}} d_g(x,y)$. Then d is the distance between x and y with respect to λg.

Let $V_s(y)$ be the volume of the geodesic ball of radius s and center y with respect to λg and let V_s^K denote the volume of a ball of radius s in the n-dimensional simply connected space of constant curvature $-K$. Then by [CGT, Theorem 4.3] we get

$$i_\lambda(y) \geq \frac{r_0}{2}\frac{1}{1+\frac{V^K_{r_0+s}}{V_s(y)}} \geq \frac{r_0}{4}\frac{V_s(y)}{V^K_{r_0+s}}. \tag{2.6}$$

Now, [CGT, Proposition 4.1, i)] states that

$$\frac{V_s(y)}{V_s^K} \geq \frac{V_{d+s}(y)}{V^K_{d+s}}.$$

Together with (2.6) this gives

$$i_\lambda(y) \geq \frac{r_0}{4}\frac{V_{d+s}(y)V_s^K}{V^K_{d+s}V^K_{r_0+s}}.$$

From the definition of d it follows that, with respect to the metric λg, the ball of radius $d+s$ around y contains the ball of radius s around x. Hence $V_{d+s}(y) \geq V_s(x)$. Since

$s = \frac{\pi}{12\sqrt{K}} = \tilde{\imath}_\lambda(x)$, it follows from Lemma 2.2 that there exists $c > 0$ such that $V_s(x) \geq c$ for all $x \in M$. Hence, we get

$$\begin{aligned} i_\lambda(y) &\geq \frac{r_0}{4} \frac{V_s(x) V_s^K}{V_{d+s}^K V_{r_0+s}^K} \geq C \frac{V_s^K}{V_{d+s}^K} \geq C e^{-(n-1)\sqrt{K} d} \\ &\geq C e^{-(n-1)\max\{\sqrt{K}, \frac{\pi}{12 i(x)}\} d(x,y)} = C e^{-\frac{(n-1)\pi}{12} \frac{d(x,y)}{\tilde{\imath}(x)}}, \end{aligned}$$

for some constant $C > 0$. Now the lemma follows by dividing both sides of this inequality by $\lambda^{\frac{1}{2}}$. $\square$

We can now establish the following basic result about the existence of uniformly locally finite coverings on manifolds with bounded curvature.

Theorem 2.6. *Assume that M is non-compact. Let h be a continuous real valued function on M such that*

i) $\forall x : 0 < h(x) \leq \tilde{\imath}(x)$.
ii) *There exists constants $C_1, C_2 > 0$ such that*

$$h(x) \geq C_1 h(x_0) e^{-C_2 \frac{d(x,x_0)}{h(x_0)}}$$

for all $x, x_0 \in M$.

Then for each $a \geq 1$, there exists a sequence $\{x_i\}_{i=0}^\infty \subset M$ and a constant $C_3 < \infty$, depending only on K, a, C_1 and C_2 such that

1)
$$\bigcup_{i=0}^\infty B_{h(x_i)}(x_i) = M.$$
2) $\forall i \in \mathbb{N} : \#\{j \mid B_{ah(x_i)}(x_i) \cap B_{ah(x_j)}(x_j) \neq \emptyset\} \leq C_3$.

Proof: Let $x_0 \in M$. For $k \in \mathbb{N}$ define recursively

$$m(k) = \min\{m \in \mathbb{N} \mid B_m(x_0) \setminus \cup_{i<k} B_{h(x_i)}(x_i) \neq \emptyset\}$$

and pick $x_k \in B_{m(k)} \setminus \cup_{i<k} B_{h(x_i)}(x_i)$. In this way we get a sequence $\{x_i\}_{i=0}^\infty$ of points of M. From the construction it follows that this sequence satisfies the following condition:

$$\forall i, j \in \mathbb{N} : \ d(x_i, x_j) \geq \min\{h(x_i), h(x_j)\}. \tag{2.7}$$

Let $m \in \mathbb{N}$. Then by ii), there exists $c > 0$ such that $h(x) \geq c$ for all $x \in B_m(x_0)$. Hence it follows from (2.7) that $d(x_i, x_j) \geq c$ if $x_i, x_j \in B_m(x_0)$. Since $B_m(x_0)$ is compact, this implies that only finitely many of the x_i's, say $x_1, ..., x_{r_m}$, are contained in $B_m(x_0)$. Hence

$$B_m(x_0) \subset \bigcup_{i=0}^{r_m} B_{h(x_i)}(x_i)$$

which implies that

$$M = \bigcup_{i=0}^{\infty} B_{h(x_i)}(x_i).$$

It remains to prove 2). Let $a \geq 1$. For $j \in \mathbb{N}$ put $B_j = B_{ah(x_j)}(x_j)$. Let $i \in \mathbb{N}$ be given and put

$$\Omega_i = \{x_j \mid B_i \cap B_j \neq \emptyset\}.$$

Since h is bounded from above, Ω_i is contained in a compact subset Y of M. By ii) there exists $c > 0$ such that $h(x) \geq c$ for all $x \in Y$. Using (2.7), it follows that Ω_i is a discrete subset of Y and hence, Ω_i is a finite set. Let $x_{j_1} \in \Omega_i$ be such that

$$h(x_{j_1}) = \max\{h(x_j) \mid x_j \in \Omega_i\}.$$

Since $B_i \cap B_{j_1} \neq \emptyset$, it follows that $B_i \subset B_{3ah(x_{j_1})}(x_{j_1})$ which in turn implies that

$$B_{\frac{h(x_j)}{2}}(x_j) \subset B_{(4a+1)h(x_{j_1})}(x_{j_1})$$

for all $x_j \in \Omega_i$. Therefore by ii) we get

$$h(x_j) \geq C_1 h(x_{j_1}) e^{-C_2 \frac{d(x_{j_1}, x_j)}{h(x_{j_1})}} \geq C_1 h(x_{j_1}) e^{-4aC_2}.$$

Thus there exists $C_4 > 0$ such that

$$h(x_j) \geq C_4 h(x_{j_1}) \tag{2.8}$$

for all $x_j \in \Omega_i$. Obviously $C_4 \leq 1$. Hence by i), we obtain

$$\frac{C_4 h(x_{j_1})}{2} \leq \frac{\tilde{\imath}(x_{j_1})}{2} \leq \frac{\pi}{24\sqrt{K}}. \tag{2.9}$$

Moreover, by (2.7) and (2.8) we have $d(x_k, x_l) \geq C_4 h(x_{j_1})$ for all $x_k, x_l \in \Omega_i$. Therefore, the balls $B_{\frac{C_4}{2}h(x_{j_1})}(x_j)$, $x_j \in \Omega_i$, are pairwise disjoint. Using Lemma 2.2, we get

$$\#\{x_j \mid B_i \cap B_j \neq \emptyset\} \leq \frac{\int_0^{(4a+1)h(x_{j_1})} \left(\frac{\sinh t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt}{\int_0^{\frac{C_4 h(x_{j_1})}{2}} \left(\frac{\sin t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt}. \tag{2.10}$$

There exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\sinh t\sqrt{K} \leq c_1 t, \quad 0 \leq t \leq \frac{(4a+1)\pi}{12C_4\sqrt{K}};$$

$$\sin t\sqrt{K} \geq c_2 t, \quad 0 \leq t \leq \frac{\pi}{24\sqrt{K}}.$$

Hence by (2.9), it follows that the right hand side of (2.10) is bounded by $\frac{c_2}{c_1}\left(\frac{(8a+2)c_1}{C_4 c_2}\right)^n$. This proves the lemma. $\square$

Finally we will define and estimate some global invariants of (M, g).

Definition 2.7. Let $s > 0$. For $s > \varepsilon \geq 0$, let $\kappa_\varepsilon(M, g; s) \in \mathbb{N} \cup \{\infty\}$ be the smallest number such that there exists a sequence $\{x_i\}_{i=1}^{\infty}$ such that $\{B_{s-\varepsilon}(x_i)\}_{i=1}^{\infty}$ is an open covering of M and

$$\sup_{x \in M} \#\{i \in \mathbb{N} \mid x \in B_{3s+\varepsilon}(x_i)\} \leq \kappa_\varepsilon(M, g; s). \tag{2.11}$$

Further, let $\kappa(M, g; s) = \kappa_0(M, g, s)$. Put $\kappa(M, g, 0) = 1$.

Lemma 2.8. *$\kappa_\varepsilon(M, g; s)$ is finite for all $s > \varepsilon$. Moreover, there exist constants $C, c > 0$, which depend only on K, such that for $s > \frac{2\pi}{\sqrt{K}} + \varepsilon$, we have*

$$\kappa_\varepsilon(M, g; s) \leq Ce^{cs}.$$

Proof: We may proceed as in the proof of Theorem 2.6 and construct a sequence $\{x_i\}_{i=1}^{\infty} \subset M$ such that $d(x_i, x_j) \geq s - \varepsilon$ for all $i, j \in \mathbb{N}$ and $\{B_{s-\varepsilon}(x_i)\}_{i=1}^{\infty}$ is a covering of M. Let $x \in M$. If $x \in B_{3s+\varepsilon}(x_i)$, it follows that $B_{\frac{s-\varepsilon}{2}}(x_i) \subset B_{5s}(x)$. Moreover, $B_{\frac{s-\varepsilon}{2}}(x_i) \cap B_{\frac{s-\varepsilon}{2}}(x_j) = \emptyset$ if $i \neq j$. Hence, we get

$$\#\{i \mid x \in B_{3s+\varepsilon}(x_i)\} \leq \frac{\operatorname{Vol}(B_{5s}(x))}{\min_i \operatorname{Vol}(B_{\frac{s-\varepsilon}{2}}(x_i))}. \tag{2.12}$$

Next observe that for any x_i with $d(x, x_i) \leq 5s$ we have $B_{5s}(x) \subset B_{10s}(x_i)$. Moreover, by Lemma 5.3 of [Gro], we have

$$\frac{\operatorname{Vol}(B_{10s}(x_i))}{\operatorname{Vol}(B_{\frac{s-\varepsilon}{2}}(x_i))} \leq \frac{\int_0^{10s} \left(\sinh t\sqrt{K}\right)^{n-1} \, dt}{\int_0^{\frac{s-\varepsilon}{2}} \left(\sinh t\sqrt{K}\right)^{n-1} \, dt}.$$

Then combined with (2.12) we obtain

$$\#\{i \mid x \in B_{3s+\varepsilon}(x_i)\} \leq \frac{\int_0^{10s} \left(\sinh t\sqrt{K}\right)^{n-1} \, dt}{\int_0^{\frac{s-\varepsilon}{2}} \left(\sinh t\sqrt{K}\right)^{n-1} \, dt}.$$

If $(s - \varepsilon)/2 \geq \pi/\sqrt{K}$, the right hand side can be estimated by Ce^{cs} for certain constants $C, c > 0$ depending on K. $\square$

3. Weighted Sobolev Spaces

In this section we introduce weighted Sobolev spaces on manifolds with bounded curvature.

Let (M, g) be a Riemannian manifold. Let ∇ be the Levi-Civita connection of g and let $\Delta = d^*d$ be the Laplacian on functions with respect to g. Let ξ be a positive, measurable function on M, which is finite a.e. Given $m \in N_0$, and $p \in N$, we define the weighted L^p-space $L^p_\xi(M, TM^{\otimes m})$ by

$$L^p_\xi(M, TM^{\otimes m}) = \left\{\varphi \in L^p_{\text{loc}}(M, TM^{\otimes m}) \mid \xi^{1/p}\varphi \in L^p(M, TM^{\otimes m})\right\}.$$

Then for $k \in \mathbb{N}$ we define the weighted Sobolev space $W^{p,k}_\xi(M)$ by

$$W^{p,k}_\xi(M) = \Big\{ f \in L^p_\xi(M) \mid \nabla^m f \in L^p_\xi(M, TM^{\otimes m}) \text{ for all } m = 1, ..., k \Big\}, \tag{3.1}$$

where ∇ is applied iteratively in the distributional sense and the norm of $f \in W^{p,k}_\xi(M)$ is given by

$$\| f \|_{W^{p,k}_\xi} = \left(\sum_{i=0}^{k} \int_M |\nabla^i f(x)|^p_g \xi(x) \, dv_g(x) \right)^{1/p}. \tag{3.2}$$

Then $W^{p,k}_\xi(M)$ is a Banach space. In this paper we will only consider the case $p = 2$. To simplify notation we shall write $W^k_\xi(M)$ in place of $W^{2,k}_\xi(M)$. The closure of $C^\infty_0(M)$ in $W^k_\xi(M)$ will be denoted by $W^k_{0,\xi}(M)$. We shall write $W^k(M)$ for $W^k_1(M)$ and $W^k_0(M)$ for $W^k_{0,1}(M)$. Since 0 is not a weight, this cannot lead to any confusion. Note that $W^k_\xi(M)$ and $W^k_{0,\xi}(M)$ are Hilbert spaces. The weighted Sobolev space $H^l_\xi(M)$ is defined for even integers l. Let $k \in \mathbb{N}$. Then

$$H^{2k}_\xi(M) = \Big\{ f \in L^2_\xi(M) \mid \Delta^l f \in L^2_\xi(M) \text{ for all } \ l = 1, ..., k \Big\}. \tag{3.3}$$

The norm is given by

$$\| f \|^2_{H^{2k}_\xi} = \sum_{j=0}^{k} \int_M |\Delta^j f(x)|^2 \xi(x) \, dv_g(x).$$

As an equivalent norm one can use the norm defined by

$$\| f \|_{H^{2k}_\xi} = \| (\Delta + \mathrm{Id})^k f \|_{L^2_\xi}. \tag{3.4}$$

The closure of $C^\infty_0(M)$ in $H^{2k}_\xi(M)$ will be denoted by $H^{2k}_{0,\xi}(M)$. If $\xi \equiv 1$, the Sobolev space $H^{2k}_\xi(M)$ will be denoted by $H^{2k}(M)$ and $H^{2k}_{0,\xi}$ by $H^{2k}_0(M)$. Note that the Laplacian Δ induces a bounded operator

$$\Delta_\xi \colon H^2_\xi(M) \to L^2_\xi(M) \tag{3.5}$$

which is defined in the obvious way.

Next we establish some elementary properties of weighted Sobolev spaces.

Lemma 3.1. *Assume that ξ is continuous. Let $p, k \in \mathbb{N}$. Then $C^\infty(M) \cap W^{p,k}_\xi(M)$ is dense in $W^{p,k}_\xi(M)$ and $C^\infty(M) \cap H^{2k}_\xi(M)$ is dense in $H^{2k}_\xi(M)$.*

Proof: We proceed as in the proof of Theorem 1 in [Ma, 1.1.5]. Let $\{U_i \colon i \in I\}$ be a locally finite covering of M such that for each $i \in I$ there exists an open subset V_i with $\overline{U}_i \subset V_i$ and V_i is diffeomorphic to the unit ball in $\mathbb{R}^n$. Let $\{\varphi_i \colon i \in I\}$ be an associated partition of unity. Let $u \in W^{p,k}_\xi(M)$ and let $\varepsilon \in (0, 1/2)$. For each $i \in I$ let $u_i = \varphi_i u$. Then u_i belongs to $W^{p,k}_\xi(M)$ with $\operatorname{supp} u_i \subset U_i$. Since ξ is continuous, it follows that $u_i \in W^{p,k}(U_i)$ and

$\operatorname{supp} u_i$ is contained in the interior of U_i. Hence there exists a mollification $g_i \in C_c^\infty(U_i)$ of u_i such that

$$\| g_i - u_i \|_{W^{p,k}} \leq \frac{\varepsilon^i}{\max_{x \in \overline{U}_i} \xi(x)}.$$

[Ev, Section 5.3]. Then

$$\| g_i - u_i \|_{W_\xi^{p,k}} \leq \varepsilon^i.$$

Clearly $g = \sum_i g_i$ belongs to $C^\infty(M)$. Let $\omega \subset M$ be a relatively compact open subset. Then we have

$$u|_\omega = \sum_i u_i|_\omega,$$

and the sum is finite. Hence

$$\| g - u \|_{W_\xi^{p,k}(\omega)} \leq \sum_i \| g_i - u_i \|_{W_\xi^{p,k}} \leq \varepsilon(1-\varepsilon)^{-1} \leq 2\varepsilon.$$

This implies that $\| u \|_{W_\xi^{p,k}(\omega)} \leq \| u \|_{W^{p,k}} + 2\epsilon$ for all relatively compact open subsets $\omega \subset M$. Hence by the theorem of Beppo-Levi, we have $g \in C^\infty \cap W_\xi^{p,k}(M)$ and

$$\| g - u \|_{W_\xi^{p,k}} \leq 2\varepsilon.$$

The proof that $C^\infty(M) \cap H_\xi^{2k}(M)$ is dense in $H_\xi^{2k}(M)$ is similar. $\square$

Therefore we can use the following alternative definition of the Sobolev spaces. Let $C_k^\infty(M)$ denote the space of all $f \in C^\infty(M)$ such that $|\nabla^j f| \in L_\xi^p(M)$ for $j = 0, ..., k$. Then $W_\xi^{p,k}(M)$ is the completion of $C_k^\infty(M)$ with respect to the norm (3.2). Similarly let $\widetilde{C}_k^\infty(M)$ the space of all $f \in C^\infty(M)$ such that $(\Delta + \mathrm{Id})^k f \in L_\xi^2(M)$. Then $H_\xi^{2k}(M)$ is the completion of $\widetilde{C}_k^\infty(M)$ with respect to the norm (3.4). This implies that we can define $H_\xi^s(M)$ for all $s \in \mathbb{R}$. Let $(\Delta + \mathrm{Id})^{s/2}$ be defined by the spectral theorem. Let $\widetilde{C}_s^\infty(M)$ be the space of all $f \in C^\infty(M)$ such that $(\Delta + \mathrm{Id})^{s/2} f \in L_\xi^2(M)$. Let $H_\xi^s(M)$ be the completion of $\widetilde{C}_s^\infty(M)$ with respect to the norm

$$\| f \|_{H_\xi^s(M)} := \| (\Delta + \mathrm{Id})^{s/2} f \|_{L_\xi^2}.$$

In general the Sobolev spaces $W_\xi^k(M)$ and $W_{0,\xi}^k(M)$ (resp. $H_\xi^{2k}(M)$ and $H_{0,\xi}^{2k}(M)$) will not coincide. If (M, g) is complete and $\xi \equiv 1$, the following is known [Sa] .

Lemma 3.2. *Assume that (M, g) is complete. Then for all $k \in N$ we have*

$$W^k(M) = W_0^k(M), \quad H^{2k}(M) = H_0^{2k}(M), \quad \text{and} \quad W^{2k}(M) = H^{2k}(M).$$

Proof: For the proof we refer to [Sa]. The fact that $C_0^\infty(M)$ is dense in $H^{2k}(M)$ is an immediate consequence of [Cn]. Indeed by [Cn], $(\Delta + \mathrm{Id})^k$ is essentially self-adjoint on $C_0^\infty(M)$ for all $k \in \mathbb{N}$. Thus

$$\overline{(\Delta + \mathrm{Id})^k(C_c^\infty(M))} = L^2(M). \tag{3.6}$$

Let $f \in H^{2k}(M)$. Then $(\Delta + \mathrm{Id})^k f \in L^2(M)$ and hence, by (3.6) there exists a sequence $\{\varphi_j\} \subset C_c^\infty(M)$ such that

$$\| f - \varphi_j \|_{H^{2k}} = \| (\Delta + \mathrm{Id})^k (f - \varphi_j) \|_{L^2} \to 0$$

as $j \to \infty$. □

Under additional assumptions on ξ, similar results hold for weighted Sobolev spaces [Sa]. In general the following weaker results hold.

Lemma 3.3. *For all $k \in \mathbb{N}$, the natural inclusion $W_\xi^{2k}(M) \hookrightarrow H_\xi^{2k}(M)$ is bounded.*

Proof: Let $k \in \mathbb{N}$. Let $f \in W_\xi^{2k}(M)$. Then we have $\nabla^j f \in L_\xi^2(M)$ for $j = 0, ..., 2k$. Recall that

$$\Delta = -\operatorname{Tr}(\nabla^2 f) \tag{3.7}$$

and $\nabla \operatorname{Tr} = 0$. Hence it follows that there exists $C > 0$ such that

$$|\Delta^j f|(x) \leq C |\nabla^{2j} f|_g(x)$$

for all $j = 0, ..., k$ and $x \in M$. This implies $\Delta^j f \in L_\xi^2(M)$ for $j = 0, ..., k$, and

$$\| f \|_{H_\xi^{2k}} \leq C \| f \|_{W_\xi^{2k}} .$$

□

In order to deal with the inclusion in the other direction, we need some preparation. Let $B_s \subset \mathbb{R}^n$ denote the ball of radius $s > 0$ around the origin in $\mathbb{R}^n$. Given $m \in \mathbb{N}$ and $r, K, \lambda > 0$, denote by $\mathcal{E}ll^m(r, K, \lambda)$ the set of elliptic differential operators

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

of order m in B_r such that the coefficients of P satisfy:

(1) $a_\alpha \in C^m(B_r)$.
(2) $\sum_{|\alpha|<m} \| a_\alpha \|_{C^0(B_r)} \leq K$, $\sum_{|\alpha|=m} \| a_\alpha \|_{C^1(B_r)} \leq K$.
(3) $\lambda^{-1} \| \xi \|^m \leq \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha \leq \lambda \| \xi \|^m$ for all $\xi \in \mathbb{R}^n$ and $x \in B_r$.

Given an open subset $\Omega \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, $W^k(\Omega)$ is the usual Sobolev space.

Lemma 3.4. *Let $K, \lambda > 0$ be given. There exists $r_0 = r_0(K, \lambda) > 0$ and $C = C(\lambda) > 0$ such that for all $r \leq r_0$, $P \in \mathcal{E}ll^m(r, K, \lambda)$ and $x_0 \in B_r$:*

$$\| u \|_{W^m(B_r)} \leq C \left(\| Pu \|_{L^2(B_r)} + \| u \|_{L^2(B_r)} \right)$$

for all $u \in C_0^\infty(B_r)$

Proof: Let $1 \geq r > 0$ and let $P \in \mathcal{E}ll^m(r, K, \lambda)$. Put

$$P_0 = \sum_{|\alpha|=m} a_\alpha(0) D^\alpha.$$

By Lemma 17.1.2 of [H] there exists $C_1 > 0$ which depends only on λ such that for all $u \in C_0^\infty(B_r)$:

$$\| u \|_{W^m(B_r)} \leq C \left(\| P_0 u \|_{L^2(B_r)} + \| u \|_{L^2(B_r)} \right).$$

Now $Pu = P_0 u + (P - P_0)u$. Thus

$$\| u \|_{W^m(B_r)} \leq C \left(\| Pu \|_{L^2(B_r)} + \| (P - P_0)u \|_{L^2(B_r)} + \| u \|_{L^2(B_r)} \right). \tag{3.8}$$

Next observe that

$$(P - P_0)u = \sum_{|\alpha|=m} (a_\alpha(x) - a_\alpha(0)) D^\alpha u + \sum_{|\alpha|<m} a_\alpha(x) D^\alpha u.$$

Hence by 2):

$$\begin{aligned} \| (P - P_0)u \|_{L^2(B_r)} \leq & \, r \sum_{|\alpha|=m} \| a_\alpha \|_{C^1(B_r)} \| u \|_{W^m(B_r)} \\ & + \sum_{|\alpha|<m} \| a_\alpha \|_{C^0(B_r)} \| u \|_{W^{m-1}(B_r)} \\ & \leq K \left(r \| u \|_{W^m(B_r)} + \| u \|_{W^{m-1}(B_r)} \right). \end{aligned} \tag{3.9}$$

By the Poincaré inequality there exists $C_2 > 0$ which is independent of $r \leq 1$ such that for all $u \in C_0^\infty(B_r)$:

$$\| u \|_{W^{m-1}(B_r)} \leq r\, C_2 \| u \|_{W^m(B_r)}.$$

Using this inequality, it follows from (3.9) that

$$\| (P - P_0)u \|_{L^2(B_r)} \leq r\, C(K) \| u \|_{W^m(B_r)}.$$

Together with (3.8) we get

$$(1 - rCC(K)) \| u \|_{W^m(B_r)} \leq C \left(\| Pu \|_{L^2(B_r)} + \| u \|_{L^2(B_r)} \right).$$

Set

$$r_0 = \min\{1, \frac{1}{2CC(K)}\}.$$

Then it follows that for all $r \leq r_0$ and $u \in C_0^\infty(B_r)$:

$$\| u \|_{W^m(B_r)} \leq 2C \left(\| Pu \|_{L^2(B_r)} + \| u \|_{L^2(B_r)} \right).$$

□

Lemma 3.5. *Let $k \geq 1$ be even. Assume that M has bounded curvature of order k. Let $K > 0$ be such that $\sup_{x \in M} |\nabla^l R(x)| \leq K$, $l = 0, ..., 2k$. There exist constants $r_0 = r_0(K) > 0$ and $C = C(K) > 0$ such that for all $x_0 \in M$ and $r \leq \min\{r_0, \tilde{\imath}(x_0)\}$ one has*

$$\| u \|_{W^{2k}(B_r(x_0))} \leq C \| u \|_{H^{2k}(B_r(x_0))}$$

for all $u \in C_0^\infty(B_r(x_0))$.

Proof: By [Ei2, Corollary 2.6 and 2.7] there exists a constant $C_1 > 0$, which depends only on K, such that for every $x_0 \in M$, every $r \leq \tilde{\imath}(x_0)$, and all $i, j, k = 1, ..., n$, one has

$$\sup_{x \in B_r(x_0)} |D^\alpha g_{ij}(x)| \leq C_1, \;\; |\alpha| \leq 2k, \qquad \sup_{x \in B_r(x_0)} |D^\beta \Gamma^i_{jk}(x)| \leq C_1, \;\; |\beta| \leq 2k-1, \tag{3.10}$$

where the g_{ij} and Γ^i_{jk} denote the coefficients of g and ∇, respectively, with respect to normal coordinates on the geodesic ball $B_r(x_0)$ of radius r with center x_0.

Let $x_0 \in M$ and $r \leq \tilde{\imath}(x_0)$. Let $B_r \subset T_{x_0}M$ denote the ball of radius r around the origin. Let $W^{2k}(B_r)$ be the Sobolev space with respect to the flat connection. Then it follows from (3.10) that there exists $C_2 = C_2(K) > 0$ such that

$$C_2^{-1} \parallel u \circ \exp_{x_0} \parallel_{W^{2k}(B_r)} \leq \parallel u \parallel_{W^{2k}(B_r(x_0))} \leq C_2 \parallel u \circ \exp_{x_0} \parallel_{W^{2k}(B_r)} \tag{3.11}$$

for all $x_0 \in M$, $r \leq \tilde{\imath}(x_0)$, and $u \in C_c^\infty(B_r(x_0))$. Let $\tilde{g}$ be the metric on B_r which is the pull-back of $g \restriction B_r(x_0)$ with respect to $\exp_{x_0} \colon B_r \to B_r(x_0)$. Let $\widetilde{\Delta}$ be the Laplacian on B_r with respect to $\tilde{g}$. Then by (3.11) it is sufficient to show that there exists $C_3 = C_3(K) > 0$ such that

$$\parallel f \parallel_{W^{2k}(B_r)} \leq C_3 \parallel (\widetilde{\Delta} + \mathrm{Id})^k f \parallel_{L^2(B_r)} \tag{3.12}$$

for all $x_0 \in M$, $r \leq \tilde{\imath}(x_0)$, and $f \in C_0^\infty(B_r)$. Set $P = (\widetilde{\Delta} + \mathrm{Id})^k$. By (3.10) there exists $C_4 > 0$, which depends only on K, such that $P \in \mathcal{E}ll^{2k}(r, 1, C_4)$. Then by Lemma 3.4, there exist $r_0 > 0$ and $C_3 > 0$ such that (3.12) holds for all $x_0 \in M$ and $r \leq \min\{r_0, \tilde{\imath}(x_0)\}$. This completes the proof of the lemma. $\square$

Lemma 3.6. *Let $k \in \mathbb{N}$ be even. Suppose that (M, g) has bounded curvature of order $2k$. Let $\beta : M \to \mathbb{R}^+$ be a function of moderate decay. Then there exists a canonical bounded inclusions*

$$H^k_{\beta \tilde{\imath}^{-2kn}}(M) \hookrightarrow W^k_\beta(M) \quad \text{and} \quad H^k_\beta(M) \hookrightarrow W^k_{\beta \tilde{\imath}^{2kn}}(M). \tag{3.13}$$

Proof: By Theorem 2.6, there exists a covering $\{B_{\frac{\tilde{\imath}}{2^k}(x_i)}(x_i)\}_{i=1}^\infty$ of M by balls and a constant $C > 0$ such that

$$\forall x \in M : \#\{x_i \mid x \in B_{\tilde{\imath}(x_i)}(x_i)\} \leq C. \tag{3.14}$$

Let $\varphi \in C^\infty(\mathbb{R})$ be such that $\varphi = 1$ on $[0, 1]$ and $\varphi = 0$ on $[2, \infty)$. For $x \in M$ and $1 \leq j \leq k$, we define

$$\varphi_{j,x}(y) = \begin{cases} \varphi(2^j \frac{d(x,y)}{\tilde{\imath}(x)}), & y \in B_{\tilde{\imath}(x)}(x); \\ 0, & \text{otherwise.} \end{cases} \tag{3.15}$$

Then $\varphi_{j,x} \in C_0^\infty(M)$. Let $f \in H^k(M)$. Using Lemma 3.1, it follows that $\varphi_{j,x} f \in H^k(B_{\tilde{\imath}(x)}(x))$. Then by Lemma 3.5 we get $\varphi_{j,x} f \in W^k(B_{\tilde{\imath}(x)}(x))$, and by the Leibniz rule there is $C > 0$ such that

$$\left|\nabla^j(\varphi_{k,x} f)\right|_g(y) \leq C \sum_{p=0}^{j} \left|\nabla^p \varphi_{k,x}\right|_g(y) \cdot \left|\nabla^{j-p} f)\right|_g(y), \quad y \in M,\ j = 0, ..., k.$$

By estimating the supremum-norm of the derivatives of $\varphi_{k,x}$ and using Lemma 3.5 , we get

$$\begin{aligned}\|\varphi_{k,x} f\|_{W^k} &\leq C\|f\|_{W^k\left(B_{\frac{\tilde{\imath}}{2^{k-1}}(x)}(x)\right)} + C'\sum_{p=1}^{k}\binom{k}{p}\tilde{\imath}^{-p}(x)\|\varphi_{k-1,x}f\|_{W^{k-p}} \\ &\leq C\|f\|_{H^k\left(B_{\frac{\tilde{\imath}}{2^{k-1}}(x)}(x)\right)} + C''\sum_{p=1}^{k}\binom{k}{p}\tilde{\imath}^{-p}(x)\|\varphi_{k-1,x}f\|_{H^{k-p}}.\end{aligned} \tag{3.16}$$

By induction, this yields

$$\|\varphi_{k,x_i} f\|_{W^k} \leq C\tilde{\imath}^{-k}(x_i)\|f\|_{H^k(B_{\tilde{\imath}(x_i)}(x_i))}. \tag{3.17}$$

Let $f \in H^k_\beta$. By Lemma 1.6, Lemma 3.5, (3.14) and (3.17) we get

$$\begin{aligned}\|f\|_{W^k_{\tilde{\beta}}} \leq C\sum_{i=1}^{\infty}\beta^{\frac{1}{2}}(x_i)\|\varphi_{k,x_i}f\|_{W^k}\tilde{\imath}^k(x_i) &\leq C\sum_{i=1}^{\infty}\beta^{\frac{1}{2}}(x_i)\|\varphi_{k,x_i}f\|_{H^k} \\ &\leq C\sum_{i=1}^{\infty}\beta^{\frac{1}{2}}(x_i)\tilde{\imath}^{-k}(x_i)\|f\|_{H^k(B_{\tilde{\imath}(x_i)}(x_i))}.\end{aligned} \tag{3.18}$$

By (2.3) there exists a constant $C_1 > 0$ such that

$$\tilde{\imath}(x_i)^{-k}\tilde{\imath}(x)^{kn} \leq C_1$$

for all $i \in \mathbb{N}$ and $x \in B_{\tilde{\imath}(x_i)}(x_i)$. This implies

$$\sum_{i=1}^{\infty}\beta^{\frac{1}{2}}(x_i)\tilde{\imath}^{-k}(x_i)\|f\|_{H^k(B_{\tilde{\imath}(x_i)}(x_i))} \leq C_2 \parallel f \parallel_{H^k_{\tilde{\imath}^{-2kn}\beta}},$$

which together with (3.18) gives the first inclusion. The proof of the second inclusion is analogous. □

Remark 2. Lemma 3.6 is not optimal. Under additional assumptions on β one can show that $W^{2k}_\beta(M) = H^{2k}_\beta(M)$ [Sa].

4. Functions of the Laplacian.

Assume that (M, g) is complete. Then $\Delta\colon C_c^\infty(M) \to L^2(M)$ is essentially self-adjoint and functions $f(\sqrt{\Delta})$ can be defined by the spectral theorem for unbounded self-adjoint operators by

$$f(\sqrt{\Delta}) = \int_0^\infty f(\lambda)dE_\lambda,$$

where dE_λ is the projection spectral measure associated with $\sqrt{\Delta}$. Let $f \in L^1(\mathbb{R})$ be even and let

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\cos(\lambda x)\, dx.$$

Then $f(\sqrt{\Delta})$ can also be defined by

$$f(\sqrt{\Delta}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)\cos(\lambda\sqrt{\Delta})\, d\lambda. \tag{4.1}$$

This representation has been used in [CGT] to study the kernel of $f(\sqrt{\Delta})$. We will use (4.1) to study $f(\sqrt{\Delta})$ as operator in weighted L^2-spaces. To this end we need to study $\cos(\lambda\sqrt{\Delta})$ as operator in $L^2_\beta(M)$. Given $s > 0$, let $\kappa(M, g, s)$ be the constant introduced in Definition 2.7.

Theorem 4.1. *Assume that (M, g) has bounded curvature. Let β be a function of moderate decay. Then $\cos(s\sqrt{\Delta})$ extends to a bounded operator in $L^2_\beta(M)$ for all $s \in \mathbb{R}$ and there exist $C, c > 0$ such that*

$$\| \cos(s\sqrt{\Delta})\|_{L^2_\beta, L^2_\beta} \leq Ce^{c|s|}, \quad s \in \mathbb{R}. \tag{4.2}$$

Moreover $\cos(s\sqrt{\Delta})\colon L^2_\beta(M) \to L^2_\beta(M)$ is strongly continuous in s.

Proof: Let $s > 0$. Choose a sequence $\{x_k\}_{k=1}^\infty \subset M$ which minimizes $\kappa(M, g; s)$. For $k \in \mathbb{N}$ let P_k denote the multiplication by the characteristic function of $B_s(x_k) \setminus \bigcup_{i=0}^{k-1} B_s(x_j)$. Then each P_k is an orthogonal projection in $L^2(M)$ and $L^2_\beta(M)$, respectively. Moreover the projections satisfy $P_k P_{k'} = 0$ for $k \neq k'$ and $\sum_{k=1}^\infty P_k = 1$, where the series is strongly convergent. Obviously the image of P_k consists of functions with support in $B_s(x_k)$. Now recall that $\cos(t\sqrt{\Delta})$ has unit propagation speed [CGT, p.19], i.e.,

$$\operatorname{supp}\cos(s\sqrt{\Delta})\delta_x \subset \overline{B_{|t|}(x)}$$

for all $x \in M$ and $t \in \mathbb{R}$. Let $f \in L^2(M)$. Then it follows that

$$\operatorname{supp}\cos(s\sqrt{\Delta})P_k f \subset B_{2s}(x_k)$$

and

$$\operatorname{supp}\cos(s\sqrt{\Delta})\left((1 - \chi_{B_{3s}(x_k)})f\right) \subset M - B_{2s}(x_k).$$

Hence

$$\begin{aligned}
\| \cos(s\sqrt{\Delta})f \|_\beta^2 &= \sum_{k=1}^\infty \langle \cos(s\sqrt{\Delta})P_k f, \cos(s\sqrt{\Delta})f\rangle_\beta \\
&= \sum_{k=1}^\infty \langle \cos(s\sqrt{\Delta})P_k f, \cos(s\sqrt{\Delta})(\chi_{B_{3s}(x_k)} f)\rangle_\beta.
\end{aligned} \tag{4.3}$$

Now observe that the norm of $\cos(s\sqrt{\Delta})$ as an operator in $L^2(M)$ is bounded by 1. This implies

$$\begin{aligned}
&\left|\langle \cos(s\sqrt{\Delta})P_k f, \cos(s\sqrt{\Delta})(\chi_{B_{3s}(x_k)} f)\rangle_\beta\right| \\
&\qquad\qquad \leq \sup_{y \in B_{3s}(x_k)} \beta(y) \parallel P_k f \parallel_{L^2} \cdot \parallel \chi_{B_{3s}(x_k)} f \parallel_{L^2} .
\end{aligned}$$

To estimate the right hand side, we write

$$\sup_{y\in B_{3s}(x_k)} \beta(y) \parallel P_k f \parallel^2_{L^2} = \int_M |P_k f(x)|^2 \sup_{y\in B_{3s}(x_k)} \left(\frac{\beta(y)}{\beta(x)}\right) \beta(x)\, dx.$$

Since the support of $P_k f$ is contained in $B_s(x_k)$, we can use (1.13) to estimate the right hand side. This gives

$$\sup_{y\in B_{3s}(x_k)} \beta(y) \parallel P_k f \parallel^2_{L^2} \leq C_\beta^{-1} \frac{1}{\beta(1+4s)} \parallel P_k f \parallel^2_{L^2_\beta}.$$

A similar inequality holds with respect to $\parallel \chi_{B_{3s}(x_k)} f \parallel_{L^2}$. Putting the estimations together, we get

$$\begin{aligned} &\left|\langle \cos(s\sqrt{\Delta}) P_k f, \cos(s\sqrt{\Delta})(\chi_{B_{3s}(x_k)} f)\rangle_\beta\right| \\ &\qquad\qquad \leq C_\beta^{-1} \frac{1}{\beta(1+6s)} \parallel P_k f \parallel_{L^2_\beta} \cdot \parallel \chi_{B_{3s}(x_k)} f \parallel_{L^2_\beta}. \end{aligned} \tag{4.4}$$

Now recall that by Lemma 2.8 we have $\kappa(M,g;s) < \infty$. Hence we get

$$\sum_{k=1}^\infty \|\chi_{B_{3s}(x_k)} f\|^2_{L^2_\beta} \leq \kappa(M,g;s)\|f\|^2_{L^2_\beta} < \infty.$$

Together with 4.3 and (4.4) we obtain

$$\begin{aligned} \parallel \cos(s\sqrt{\Delta}) f \parallel^2_{L^2_\beta} &\leq C_\beta^{-1} \frac{1}{\beta(1+6s)} \parallel f \parallel_{L^2_\beta} \sum_{k=1}^\infty \parallel \chi_{B_{3s}(x_k)} f \parallel_{L^2_\beta} \\ &\leq C_\beta^{-1} \frac{1}{\beta(1+6s)} \kappa(M,g,s)^{1/2} \parallel f \parallel^2_{L^2_\beta}. \end{aligned} \tag{4.5}$$

Recall that by (1.10) we have $\beta(x) \leq C(1+d(x,p))^{-1}$, $x \in M$. Therefore $L^2(M) \subset L^2_\beta(M)$, and $L^2(M)$ is a dense subspace of $L^2_\beta(M)$. This implies that $\cos(s\sqrt{\Delta})$ extends to a bounded operator in $L^2_\beta(M)$. Moreover by (1.11) and Lemma 2.8 it follows that there exist constants $C, c > 0$ such that

$$\parallel \cos(s\sqrt{\Delta}) \parallel^2_{L^2_\beta, L^2_\beta} \leq Ce^{cs}, \quad s \in [0,\infty).$$

Since $\cos(-s\sqrt{\Delta}) = \cos(s\sqrt{\Delta})$, this extends to all $s \in \mathbb{R}$ such that (4.2) holds. The strong continuity is a consequence of the local bound of the norm and the strong continuity on the dense subspace $L^2(M) \subseteq L^2_\beta(M)$. $\square$

Using Theorem 4.1, we can study $f(\sqrt{\Delta})$ as an operator in $L^2_\beta(M)$. Given $c \geq 0$, let

$$\mathcal{F}^1(c) = \left\{ f \in L^1(\mathbb{R}) \colon \int_{-\infty}^\infty |\hat{f}(\lambda)| e^{c|\lambda|} d\lambda < \infty \right\}.$$

Lemma 4.2. *Assume (M,g) has bounded curvature and let β be a function of moderate decay. Then there exists a constant $c = c(M,g,\beta)$, such that for all even functions $f \in \mathcal{F}^1(c)$, the operator $f(\sqrt{\Delta})$ extends to a bounded operator in $L^2_\beta(M)$. Moreover, there exists a constant $C_1 = C_1(M,g,\beta) > 0$ such that*

$$\|f(\sqrt{\Delta})\|_{L^2_\beta, L^2_\beta} \leq C_1 \|\hat{f}\|_{L^1_{e^{c|\cdot|}}} \tag{4.6}$$

for all f as above. If $\kappa(M,g;s)$ is at most sub-exponentially increasing, then $c(M,g,\beta) > 0$ can be chosen arbitrarily.

Proof: By Theorem 4.1 there exist constants $C, c > 0$, depending on (M,g,β), such that

$$\|\cos(\sqrt{\Delta})\|_{L^2_\beta, L^2_\beta} \leq C e^{c|s|},$$

for all $s \in \mathbb{R}$. Let $\varphi \in L^2(M)$. Using (4.1), it follows that

$$\|f(\sqrt{\Delta})\varphi\|_{L^2_\beta} \leq \frac{C}{\sqrt{2\pi}} \|\hat{f}\|_{L^1_{e^{C|\cdot|}}} \|\varphi\|_{L^2_\beta}. \tag{4.7}$$

Since $L^2(M)$ is dense in $L^2_\beta(M)$, it follows from (4.7) that $f(\sqrt{\Delta})$ extends to a bounded operator in $L^2_\beta(M)$. The last statement is obvious. □

Remark 3. It is not difficult to see, that (4.1) is in fact strongly convergent in L^2_β.

Corollary 4.3. *Suppose that (M,g) has bounded curvature and let β be a function of moderate decay. Then the following holds:*

a) For every $t > 0$, the heat operator $e^{-t\Delta}$ extends to bounded operator in $L^2_\beta(M)$. Its norm is uniformly bounded in t on compact intervals of $\mathbb{R}^+$.

b) In the region $\{\lambda \in \mathbb{C}\colon \operatorname{Re}(\sqrt{-\lambda}) > c(M,g,\beta)\}$ the resolvent $(\Delta - \lambda)^{-1}$ extends to a bounded operator in $L^2_\beta(M)$. The function of $\lambda \mapsto (\Delta - \lambda)^{-1}$ is locally bounded and holomorphic on this domain.

c) If β is of sub-exponential decay and $\kappa(M,g;s)$ is at most sub-exponentially increasing for $s > s_0$, then $(\Delta - \lambda)^{-1} : L^2_\beta(M) \mapsto L^2_\beta(M)$ is defined and bounded for all $\lambda \in \mathbb{C} \setminus [0,\infty)$.

Proof: This follows from Lemma 4.2 and

$$\int_{-\infty}^{\infty} e^{-tx^2} \cos(xy)dx = \sqrt{\frac{\pi}{t}} e^{-\frac{y^2}{4t}}, \quad \int_{-\infty}^{\infty} \frac{1}{\lambda + x^2} \cos(xy)dx = \frac{\pi}{\sqrt{\lambda}} e^{-\sqrt{\lambda}|y|}. \tag{4.8}$$

□

Let β be of moderate decay. There is a canonical pairing $(\cdot,\cdot)$ between $L^2_\beta(M)$ and $L^2_{\beta^{-1}}(M)$ given by

$$(f,g) = \int_M f(x)g(x)\, dx, \quad f \in L^2_\beta(M),\ g \in L^2_{\beta^{-1}}(M).$$

This pairing is non-degenerate so that $L^2_{\beta^{-1}}(M)$ is canonically isomorphic to the dual of $L^2_\beta(M)$. Moreover, we have the following inclusions

$$L^2_{\beta^{-1}}(M) \subset L^2(M) \subset L^2_\beta(M).$$

By duality it follows that Theorem 4.1, Lemma 4.2 and Corollary 4.3 also hold w.r.t. β^{-1}. Especially, it follows that $f(\sqrt{\Delta})$ defined on $L^2_{\beta^{-1}}(M)$ is the restriction of $f(\sqrt{\Delta})_{|L^2}$. Moreover, we have the identity

$$f(\sqrt{\Delta})_{|L^2_{\beta^{-1}}} = \left(\bar{f}(\sqrt{\Delta})_{|L^2_\beta}\right)^*. \tag{4.9}$$

Lemma 4.4. *Let β be a function of moderate decay. If λ and $\bar{\lambda}$ satisfy condition b) of Corollary 4.3, then*

$$H^2_\beta(M) = (\Delta - \lambda)^{-1}(L^2_\beta(M)).$$

Proof: First note that $C_0^\infty(M)$ is dense in $L^2_\beta(M)$. Indeed $C_0^\infty(M)$ is dense in $L^2(M)$, and $L^2(M)$ is dense in $L^2_\beta(M)$. Let $f = (\Delta - \lambda)^{-1}g$, $g \in L^2_\beta(M)$. Then there exists a sequence $\{\varphi_i\}_{i\in\mathbb{N}} \subset C_0^\infty(M)$ which converges to g in $L^2_\beta(M)$ and $(\Delta-\lambda)^{-1}\varphi_i$ converges to f in $L^2(M)$. Let $\varphi \in C_0^\infty(M)$. Then

$$\langle f, \Delta\varphi\rangle = \lim_{i\to\infty}\langle(\Delta-\lambda)^{-1}\varphi_i, \Delta\varphi\rangle = \lim_{i\to\infty}\langle\varphi_i + \lambda(\Delta-\lambda)^{-1}\varphi_i, \varphi\rangle = \langle g + \lambda f, \varphi\rangle.$$

Thus $\Delta f = g + \lambda f \in L^2_\beta(M)$ and hence $f \in H^2_\beta(M)$. Now suppose that $f \in H^2_\beta(M)$ and set $g = (\Delta-\lambda)f$. Then $g \in L^2_\beta(M)$ and we need to show that $f = (\Delta-\lambda)^{-1}g$. Let $\varphi \in C_0^\infty(M)$. By definition of $(\Delta-\lambda)^{-1}g$, there exists a sequence $\{g_i\}_{i\in\mathbb{N}} \subset L^2(M)$ such that $(\Delta-\lambda)^{-1}g_i$ converges to $(\Delta-\lambda)^{-1}g$ in $L^2_\beta(M)$ as $i \to \infty$. Using this fact, we get

$$\langle(\Delta-\lambda)^{-1}g, \varphi\rangle = \langle g, (\Delta-\bar{\lambda})^{-1}\varphi\rangle = \langle(\Delta-\lambda)f, (\Delta-\bar{\lambda})^{-1}\varphi\rangle. \tag{4.10}$$

Now observe that $(\Delta-\bar{\lambda})^{-1}\varphi$ belongs to $H^2(M)$. By Lemma 3.1, there exists a sequence $\{\varphi_i\}_{i\in\mathbb{N}} \subset C_0^\infty(M)$ which converges to $(\Delta-\bar{\lambda})^{-1}\varphi$ in $H^2(M)$. Thus

$$\langle(\Delta-\lambda)f, (\Delta-\bar{\lambda})^{-1}\varphi\rangle = \lim_{i\to\infty}\langle(\Delta-\lambda)f, \varphi_i\rangle = \langle f, (\Delta-\bar{\lambda})\varphi_i\rangle = \langle f, \varphi\rangle.$$

Together with (4.10) this implies that $f = (\Delta-\lambda)^{-1}g$. □

Lemma 4.5. *Let β be a function of moderate decay. Then $(\Delta+\lambda)(C_0^\infty(M))$ is dense in $L^2_\beta(M)$ for every $\lambda \in \mathbb{R}^+$.*

Proof: As in (3.6) it follows from the essential self-adjointness of $\Delta + \lambda\,\mathrm{Id}$ that $(\Delta + \lambda)(C_c^\infty(M))$ is dense in $L^2(M)$. Moreover since β is monotonically decreasing, we have that $L^2(M) \subset L^2_\beta(M)$ is dense and $\| f \|_\beta \leq C \| f \|$ for $f \in L^2(M)$. This implies that $(\Delta+\lambda)(C_c^\infty(M))$ is also dense in $L^2_\beta(M)$. □

Corollary 4.6. *Let β be of moderate decay. Then $C_0^\infty(M)$ is dense in $H^2_\beta(M)$.*

Proof: Let $f \in H^2_\beta(M)$. Let $\lambda \gg 0$. By Lemma 4.4 there exists $g \in L^2_\beta(M)$ such that $f = (\Delta + \lambda)^{-1} g$. By Lemma 4.5 there exists a sequence $\{\varphi_i\}_{i\in\mathbb{N}} \subset C_c^\infty(M)$ such that $(\Delta + \lambda)\varphi_i$ converges to g in $L^2_\beta(M)$ as $i \to \infty$. Thus $\varphi_i \to f$ in $L^2_\beta(M)$ and $(\Delta + \lambda)\varphi_i$ converges to $g = (\Delta + \lambda)f$ as $i \to \infty$. This implies that φ_i converges to f in $H^2_\beta(M)$. $\square$

5. Equivalent Metrics and Sobolev Spaces.

In this section we study the dependence of the Sobolev spaces on the metric. We will prove, that if $g \sim^k_\beta h$ for an appropriate β, then the Sobolev spaces defined with respect to g and h are equivalent up to order k. We assume that all metrics have bounded sectional curvature. To indicate the dependence of the corresponding Sobolev space on the Riemannian metric g, we will write $W^k_\xi(M; g)$ and $H^{2k}_\xi(M; g)$, respectively.

Lemma 5.1. *Let β be of moderate decay. Assume that $g \sim^k_\beta h$. Then the Sobolev spaces $W^k_\xi(M; g)$ and $W^k_\xi(M; h)$ are equivalent.*

Proof: First note that by Lemma 1.7 the metrics g and h are quasi-isometric. This implies that $L^2_\xi(M, g)$ and $L^2_\xi(M; h)$ are equivalent. So the statement of the lemma holds for $k = 0$. Let $f \in C^\infty(M)$. Let $k \in \mathbb{N}$. By induction we will prove that for $l \le k$ there exists $C_l > 0$ such that for $a, b \in \mathbb{N}_0$, $a + b = l$,

$$\big|(\nabla^g)^a(\nabla^h)^b f\big|_h(x) \le C_l \sum_{i=0}^{a+b} \big|(\nabla^g)^i f|_g(x), \quad x \in M. \tag{5.1}$$

Let $l = 1$. Since on functions the connections equal d, (5.1) follows from quasi-isometry of g and h.

Next suppose that (5.1) holds for $1 \le l < k$. To establish (5.1) for $l + 1$, we proceed by induction with respect to a. Let $a, b \in N_0$ with $a + b = l + 1$. We may assume that $a < l + 1$. Using

$$(\nabla^g)^a(\nabla^h)^b f = (\nabla^g)^a(\nabla^h - \nabla^g)(\nabla^h)^{b-1} f + (\nabla^g)^{a+1}(\nabla^h)^{b-1} f, \tag{5.2}$$

and $g \sim^k_\beta h$, it follows that (5.1) holds for $l + 1$.

Especially, putting $a = 0$ we get

$$|(\nabla^h)^l f|_h(x) \le C_l \sum_{i=0}^{l} |(\nabla^g)^i f|_g(x), \quad x \in M,\ l \le k. \tag{5.3}$$

Suppose that $f \in C^\infty(M) \cap W^k_\xi(M; g)$. Then (5.3) implies that $f \in C^\infty(M) \cap W^k_\xi(M; h)$ and

$$\parallel f \parallel_{W^k_\xi(M;h)} \le C \parallel f \parallel_{W^k_\xi(M;g)} .$$

By Lemma 3.1, $C^\infty(M) \cap W^k_\xi(M; g)$ is dense in $W^k_\xi(M; g)$. Therefore this inequality holds for all $f \in W^k_\xi(M, g)$. By symmetry, a similar inequality holds with the roles of g and h interchanged. This concludes the proof. $\square$

Next we compare the Sobolev spaces $H^{2k}_\xi(M;g)$ and $H^{2k}_\xi(M;h)$. Let Δ_g denote the Laplace operator with respect to the metric g. Recall, that

$$\Delta_g = (\nabla^g)^*\nabla^g,$$

and that the formal adjoint $(\nabla^g)^*$ of ∇^g is given by

$$(\nabla^g)^* = -\operatorname{Tr}(g^{-1}\nabla^g), \tag{5.4}$$

where $g^{-1} : T^*M \to TM$ is the isomorphism induced by the metric and $\operatorname{Tr}\colon T^*M \otimes TM \to \mathbb{R}$ denotes the contraction. Since contraction commutes with covariant differentiation and $\nabla^g g^{-1} = 0$, we get the well known formula

$$\Delta = -\operatorname{Tr}(g^{-1}\nabla^2).$$

This can be iterated. For $\omega_1 \otimes \cdots \otimes \omega_k \in (T^*M)^{\otimes k}$ define

$$g_j^{-1}(\omega_1 \otimes \cdots \otimes \omega_k) := \omega_1 \otimes \cdots \otimes \omega_{j-1} \otimes g^{-1}(\omega_j) \otimes \omega_{j+1} \otimes \cdots \otimes \omega_k,$$

and let $\operatorname{Tr}_{i,j}(g_j^{-1})$ denote g_j^{-1} followed by the contraction of the i-th and j-th component. Using that contraction commutes with covariant differentiation and $\nabla^g g^{-1} = 0$, we get

$$\Delta_g^k = (-1)^k \operatorname{Tr}_{1,2}(g_2^{-1}) \circ \cdots \circ \operatorname{Tr}_{2k-1,2k}(g_{2k}^{-1})(\nabla^g)^{2k}. \tag{5.5}$$

In a more traditional notation this means

$$\Delta_g^k f = (-1)^k \sum_{i_1,\dots,i_k} f_{;i_1 i_1 i_2 i_2 \dots i_k i_k}.$$

For short notation we will write

$$\operatorname{Tr}\left((g^{-1})^{\otimes k}\right) := \operatorname{Tr}_{1,2}(g_2^{-1}) \circ \cdots \circ \operatorname{Tr}_{2k-1,2k}(g_{2k}^{-1}).$$

Lemma 5.2. *Assume that $g \sim^{2k}_\beta h$. Then for each l, $0 \le l \le 2k$ and j, $0 \le j \le 2l$, there exist sections $\xi^g_{jl}, \xi^h_{jl} \in C^\infty(\operatorname{Hom}((T^*M)^{\otimes j}, \mathbb{R}))$ such that*

$$\Delta_g^l - \Delta_h^l = \sum_{j=0}^{2l} \xi^g_{jl} \circ (\nabla^g)^j = \sum_{j=0}^{2l} \xi^h_{jl} \circ (\nabla^h)^j \tag{5.6}$$

and there exists $C > 0$ such that for $0 \le p \le l$

$$|(\nabla^g)^p \xi^g_{jl}|_g(x) \le C\beta(x), \quad |(\nabla^h)^p \xi^h_{jl}|_h(x) \le C\beta(x),\ x \in M. \tag{5.7}$$

Proof: Using (5.5) we get

$$\begin{aligned}(-1)^l(\Delta_g^l - \Delta_h^l) &= \operatorname{Tr}\left((g^{-1})^{\otimes l}\right)(\nabla^g)^{2l} - \operatorname{Tr}\left((h^{-1})^{\otimes l}\right)(\nabla^h)^{2l} \\ &= \operatorname{Tr}\left((g^{-1})^{\otimes l}\right)\left((\nabla^g)^{2l} - (\nabla^h)^{2l}\right) \\ &\quad + \left(\operatorname{Tr}\left((g^{-1})^{\otimes l}\right) - \operatorname{Tr}\left((h^{-1})^{\otimes l}\right)\right)(\nabla^h)^{2l}.\end{aligned} \tag{5.8}$$

First consider the second term. Note that there exists $C > 0$ such that

$$\left|(\nabla^g)^p\left(\operatorname{Tr}\left((g^{-1})^{\otimes l}\right) - \operatorname{Tr}\left((h^{-1})^{\otimes l}\right)\right)\right|_g(x) \le C|(\nabla^g)^p(g-h)|_g(x). \tag{5.9}$$

Since $g \sim_\beta^{2k} h$, the right hand side is bounded by $C_1\beta(x)$. By symmetry, the same estimation holds with respect to h.

To deal with the first term on the right hand side of (5.8), we use

$$(\nabla^g)^j - (\nabla^h)^j = (\nabla^g)^{j-1}(\nabla^g - \nabla^h) + \left((\nabla^g)^{j-1} - (\nabla^h)^{j-1}\right)\nabla^h$$

and proceed by induction with respect to j. □

Corollary 5.3. *Let β be of moderate decay. Assume that $\beta\tilde{\imath}^{-2kn}$ is bounded, $g \sim_\beta^{2k} h$ and (M,g) and (M,h) have both bounded curvature of order $2k$. Then $H_\rho^{2k}(M,g)$ and $H_\rho^{2k}(M,h)$ are equivalent for all functions ρ of moderate decay.*

Proof: Let $f \in C^\infty(M) \cap H_\rho^{2k}(M;g)$. Using Lemma 3.6 and Lemma 5.1 we get

$$\| f \|_{H_\rho^{2k}(M;g)} \geq C_1 \| f \|_{W^{2k}_{\tilde{\imath}^{4k}\rho}(M;g)} \geq C_2 \| f \|_{W^{2k}_{\tilde{\imath}^{4kn}\rho}(M;h)} \geq C_3 \| f \|_{W^{2k}_{\beta^2\rho}(M;h)} .$$

By Lemma 5.2 it follows that $f \in C^\infty(M) \cap H_\rho^{2k}(M,h)$ and there exists a constant $C > 0$, which is independent of f, such that

$$\| f \|_{H_\rho^{2k}(M;h)} \leq C \| f \|_{H_\rho^{2k}(M;g)} .$$

By symmetry, a similar inequality holds with g and h interchanged. □

6. Trace class estimates

Let (M,g) be an n-dimensional Riemannian manifold with bounded sectional curvature, $|K_M| < K$. Let $e^{-t\Delta_g}(x,y)$ denote the heat kernel of Δ_g. Let $0 < a_1 < a_2 < \infty$. Let $\tilde{\imath}$ be the modified injectivity radius defined by (2.1). It follows from [CGT, Proposition 1.3], that there exist $C_1, c_1 > 0$ such that

$$e^{-t\Delta_g}(x,y) \leq C_1 \tilde{\imath}(x)^{-\frac{n}{2}} \tilde{\imath}(y)^{-\frac{n}{2}} e^{-c_1 d^2(x,y)}, \quad t \in [a_1, a_2]. \tag{6.1}$$

Let $c < c_1$. Then by (6.1) and (2.3) there exists $C > 0$ such that

$$e^{-t\Delta_g}(x,y) \leq C \tilde{\imath}(x)^{-\frac{n(n+1)}{2}} e^{-cd^2(x,y)}, \quad t \in [a_1, a_2]. \tag{6.2}$$

Lemma 6.1. *Let β be a function of moderate decay. Assume that there exist real numbers a, b such that*

i) $a + b = 2$,
ii) $\beta^b \in L^1(M)$,
iii) $\beta^a \tilde{\imath}^{-\frac{n(n+1)}{2}} \in L^\infty(M)$.

Let M_β the operator of multiplication by β. Then for every $p \in \mathbb{N}_0$, the operator $M_\beta \Delta_g^p e^{-t\Delta_g}$ is Hilbert-Schmidt. For t in a compact interval in $\mathbb{R}^+$, the Hilbert-Schmidt norm is bounded.

Proof: we have

$$M_\beta \Delta^p e^{-t\Delta} = \left(M_\beta e^{-\frac{t}{2}\Delta}\right)\left(\Delta^p e^{-\frac{t}{2}\Delta}\right). \tag{6.3}$$

Note that the operator norm of $\Delta^p e^{-\frac{t}{2}\Delta}$ is bounded on compact subsets of $\mathbb{R}^+$. Hence we may assume that $p = 0$. By Corollary 4.3, 1), it follows that $e^{-t\Delta}$ extends to a bounded operator in $L^2_{\beta^b}(M)$ and its norm is uniformly bounded for $0 < a \le t \le b$. The condition $\beta^b \in L^1(M)$ implies that $1 \in L^2_{\beta^b}$. Hence $e^{-t\Delta}1 \in L^2_{\beta^b}(M)$. Let $e^{-t\Delta}(x,y)$ be the kernel of $e^{-t\Delta}$. Then

$$\langle 1, e^{-t\Delta}1\rangle_{L^2_{\beta^b}} = \int_M \int_M \beta^b(x) e^{-t\Delta_g}(x,y)dydx.$$

The integral converges since $e^{-t\Delta}(x,y)$ is positive. Thus we get

$$\begin{aligned}\int_M\int_M |\beta(x)e^{-t\Delta_g}(x,y)|^2 dydx &= \int_M\int_M \beta^2(x)\left(e^{-t\Delta_g}(x,y)\right)^2 dydx\\ &\le \sup_{z,w\in M} |\beta^a(z)e^{-t\Delta_g}(z,w)| \int_M\int_M \beta^b(x)e^{-t\Delta_g}(x,y)dydx\\ &\le C\sup_{z\in M}|\beta^a(z)\tilde{\imath}^{-\frac{n(n+1)}{2}}(z)| \int_M \beta^b(x)\left(e^{-t\Delta}(1)\right)(x)dx\\ &\le C_1 \parallel e^{-t\Delta}(1) \parallel_{L^2_{\beta^b}}.\end{aligned}$$

This proves the lemma. □

Lemma 6.2. *Assume β is a function of moderate decay and that there exist real numbers a, b such that*

i) $b \ge 1$ *and* $a + b = 2$,
ii) $\beta^{\frac{b}{3}} \in L^1(M)$,
iii) $\beta^{\frac{a}{3}}\tilde{\imath}^{-\frac{n(n+2)}{2}} \in L^\infty(M)$.

Let M_β be the operator of multiplication by β. Then the operator $M_{\tilde{\imath}^{-2n}}M_\beta\Delta^p e^{-t\Delta}$ is a trace-class operator for $p \in \mathbb{N}$. For t in a compact interval, the trace-class norm is bounded.

Proof: We decompose the operator as

$$M_{\tilde{\imath}^{-2n}}M_\beta\Delta^p e^{-t\Delta} = \left\{M_{\tilde{\imath}^{-2n}}M_\beta e^{-\frac{t}{2}\Delta}M_{\beta^{-\frac{1}{3}}}\right\}\cdot\left\{M_{\beta^{\frac{1}{3}}}\Delta^p e^{-\frac{t}{2}\Delta^2}\right\}. \tag{6.4}$$

Since β is non-increasing and $\beta(x) \le 1/2$ outside a compact set, it follows that $\beta^{\frac{1}{3}} \le C\beta^{\frac{b}{3}}$ for $b \ge 1$. Hence by ii) we get $\beta^{\frac{1}{3}} \in L^1(M)$. Moreover by iii) it follows that $\beta^{\frac{a}{3}}\tilde{\imath}^{-\frac{n(n+1)}{2}} \in L^\infty(M)$. Hence by Lemma 6.1, the second factor on the right hand side of (6.4) is a Hilbert-Schmidt operator and its Hilbert-Schmidt norm is bounded for t in a compact interval in $\mathbb{R}^+$. It remains to show that the first factor is Hilbert-Schmidt and that the Hilbert-Schmidt norm is bounded on compact intervals. By iii) we have

$$\beta^a\tilde{\imath}^{-\frac{n(n+1)}{2}-2n} \in L^\infty(M).$$

Using this observation together with (6.2), we get

$$\int_M \int_M |\tilde{\imath}^{-2n}(x)\beta(x)e^{-t\Delta}(x,y)\beta^{-\frac{1}{3}}(y)|^2 dxdy \\ \leq C \sup_{z\in M} |\tilde{\imath}^{-\frac{n(n+1)}{2}-2n}(z)\beta^a(z)| \int_M \int_M \beta^b(x)e^{-t\Delta}(x,y)\beta^{-\frac{2}{3}}(y)dxdy. \tag{6.5}$$

Now observe that by ii), $\beta^{-\frac{2}{3}}$ belongs to $L^2_{\beta^{\frac{b+4}{3}}}(M)$. Since $\beta^{\frac{b+4}{3}} \leq C\beta^{\frac{b}{3}}$, it follows from ii) that $\beta^{\frac{b+4}{3}}$ is integrable. Hence by Corollary 4.3, $e^{-t\Delta}$ extends to a bounded operator in $L^2_{\beta^{\frac{b+4}{3}}}(M)$. Therefore $\int_M e^{-\Delta}(x,y)\beta^{-\frac{2}{3}}(y)dy \in L^2_{\beta^{\frac{b+4}{3}}}$, and the norm is uniformly bounded for t in a compact interval of $\mathbb{R}^+$. Next note that $\beta^b \in L^2_{\beta^{-\frac{b+4}{3}}}$. Hence

$$\int_M \int_M \beta^b(x)e^{-t\Delta}(x,y)\beta^{-\frac{2}{3}}(y)dxdy = \left\langle \beta^b, e^{-t\Delta}\beta^{-\frac{2}{3}} \right\rangle < \infty. \tag{6.6}$$

This implies the lemma. $\square$

Lemma 6.3. *Let β be a function of moderate decay, satisfying the conditions of Lemma 6.2. Let g, h be two complete metrics on M such that $g \sim^2_\beta h$. Let Δ_g and Δ_h be the Laplacians of g and h, respectively. Then*

$$(\Delta_g - \Delta_h)e^{-t\Delta_g} \quad \text{and} \quad e^{-t\Delta_g}(\Delta_g - \Delta_h)$$

are trace class operators, and the trace norm is uniformly bounded for t in a compact subset of $(0,\infty)$.

Proof: We decompose $e^{-t\Delta_g}$ as

$$e^{-t\Delta_g} = \left(e^{-\frac{t}{2}\Delta_g}M_{\beta^{-\frac{1}{3}}}\right) \cdot \left(M_{\beta^{\frac{1}{3}}}e^{-\frac{t}{2}\Delta_g}\right). \tag{6.7}$$

By Lemma 6.1, the second factor is a Hilbert-Schmidt operator and it suffices to show that $(\Delta_g - \Delta_h)e^{-t\Delta_g}M_{\beta^{-\frac{1}{3}}}$ is Hilbert-Schmidt and that the Hilbert-Schmidt norm is bounded for t in a compact interval. Using Lemma 5.2 and Lemma 3.6,it follows that the Hilbert-Schmidt norm can be estimated by

$$\begin{aligned}\| (\Delta_g - \Delta_h) e^{-t\Delta_g} M_{\beta^{-\frac{1}{3}}} \|_2^2 &\leq C \sum_{i=0}^{2} \int_M \int_M |(\nabla^g)^i e^{-t\Delta_g}(x,y) \beta^{-\frac{1}{3}}(y)|_g^2 \beta^2(x)\, dx dy \\ &= C \int_M \| e^{-t\Delta_g}(\cdot, y)\beta^{-\frac{1}{3}}(y) \|_{W^2_{\beta^2}}^2 \; dy \\ &\leq C_1 \int_M \| e^{-t\Delta_g}(\cdot, y)\beta^{-\frac{1}{3}}(y) \|_{H^2_{\beta^2 \tilde{\imath}^{-4n}}}^2 \; dy \\ &\leq C_2 \sum_{q=0}^{1} \int_M \|\beta(\cdot) \tilde{\imath}^{-2n}(\cdot) \Delta_g^q e^{-t\Delta_g}(\cdot, y) \beta^{-\frac{1}{3}}(y)\|_2^2 dy \\ &= C_2 \sum_{q=0}^{1} \|M_\beta M_{\tilde{\imath}^{-2n}} \Delta_g^q e^{-t\Delta_g} M_{\beta^{-\frac{1}{3}}}\|_2^2.\end{aligned}$$

By Lemma 6.2 the right hand side is finite and bounded for t in a compact interval of $\mathbb{R}^+$. To prove that $e^{-t\Delta_g}(\Delta_g - \Delta_h)$ is a trace class operator, it suffices to establish it for its adjoint $(\Delta_g - (\Delta_h)^{*_g})e^{-t\Delta_g}$ with respect to g. By (5.6) and (5.4) we have

$$\Delta_g - (\Delta_h)^{*_g} = (\xi_{01}^g)^{*_g} + (\nabla^g)^{*_g} \circ (\xi_{11}^g)^{*_g} + \left[(\nabla^g)^{*_g}\right]^2 \circ (\xi_{21}^g)^{*_g}. \tag{6.8}$$

Using (5.4) and (5.7), it follows that there exist $\eta_j \in C^\infty(\mathrm{Hom}((T^*M)^{\otimes j}, \mathbb{R}))$ such that

$$\Delta_g - (\Delta_h)^{*_g} = \eta_0 + \eta_1 \circ \nabla^g + \eta_2 \circ (\nabla^g)^2 \tag{6.9}$$

and these section satisfy

$$|\eta_j|_g(x) \leq C\beta(x), \quad 0 \leq j \leq 2,\ x \in M. \tag{6.10}$$

Using (6.9) and (6.10) we can proceed as above and prove that $(\Delta_g - (\Delta_h)^{*_g})e^{-t\Delta_g}$ is a trace class operator. □

We are now ready to prove Theorem 0.1. We note that for equivalent metrics, the Hilbert spaces $L^2(M, g)$ and $L^2(M, h)$ are equivalent. Hence we may regard $e^{-t\Delta_h}$ as bounded operator in $L^2(M, g)$.

Proof of Theorem 0.1: By Duhamel's principle we have

$$\begin{aligned} e^{-t\Delta_g} - e^{-t\Delta_h} &= \int_0^t e^{-s\Delta_g}(\Delta_h - \Delta_g) e^{-(t-s)\Delta_h}\, ds \\ &= \int_0^{t/2} e^{-s\Delta_g}(\Delta_h - \Delta_g) e^{-(t-s)\Delta_h}\, ds \\ &\quad + \int_{t/2}^t e^{-s\Delta_g}(\Delta_h - \Delta_g) e^{-(t-s)\Delta_h}\, ds. \end{aligned} \tag{6.11}$$

The integrals converge in the strong operator topology. By Lemma 6.3 the first integral is a trace class operator. In order to prove that the second integral is a trace class operator, it is sufficient to prove, that its adjoint with respect to h is of the trace class. This adjoint can be written as the strong integral

$$\int_{\frac{t}{2}}^{t} \left(e^{-(t-s)\Delta_g}\right)^{*_h} \left(\Delta_h - (\Delta_g)^{*_h}\right) e^{-s\Delta_h} ds. \tag{6.12}$$

Since $(e^{-(t-s)\Delta_g})^{*_h}$ is uniformly bounded in s, it follows again from Lemma 6.3 that (6.12) is a trace class operator. □

7. Existence and completeness of wave operators

In this section we study the wave operators associated to (Δ_g, Δ_h) for equivalent metrics g and h. Let $J\colon L^2(M, d\mu_g) \to L^2(M, d\mu_h)$ be the identification operator.

Theorem 7.1. *Let g and h be two complete metrics of bounded curvature on M which satisfy the assumptions of Theorem 0.1. Let $P_{ac}(\Delta_g)$ be the orthogonal projection onto the absolutely continuous subspace of Δ_g. Then the strong wave operators*

$$W_{\pm}(\Delta_h, \Delta_g; J) = s - \lim_{t \to \pm\infty} e^{it\Delta_h} J e^{-it\Delta_g} P_{ac}(\Delta_g)$$

exist and are complete. In particular, the absolutely continuous parts of Δ_g and Δ_h are unitarily equivalent.

Proof: By Theorem 0.1, $e^{-t\Delta_g} - e^{-t\Delta_h}$ is trace class. Then the existence and completeness of the wave operators follows from the invariance principle of Birman and Kato [Ka, Chapter X, Theorem 4.7]. □

Examples. We give some examples to demonstrate Theorem 0.1:

1) Let M be a manifold with cylindrical ends. Then $\tilde{\imath}$ is bounded from below, and we may take $b = 2$, $a = 0$. The condition $\beta^{\frac{2}{3}} \in L^1(M)$ is satisfied for $\beta(t) = t^{-\frac{3}{2}-\varepsilon}$ for any $\varepsilon > 0$. We note that scattering theory on manifolds with asymptotically cylindrical ends has been studied by Melrose [Me3], [Me1, Section 7]. A metric with "asymptotically cylindrical ends" in the sense of [Me3] means a perturbation of a metric with cylindrical ends such that all derivatives of the perturbation decay exponentially. This is a much stronger condition than the decay condition we impose. On the other hand, it gives the existence of an analytic continuation of the resolvent and the existence of generalized eigenfunctions.

2) More generally, let M be a manifold with bounded geometry of order 2. (i.e. there is a lower bound for the injectivity radius and the covariant derivatives of the curvature of order ≤ 2 are bounded). Then we may choose $x_0 \in M$ arbitrary and let $\beta(t) \leq \mathrm{vol}(B_t(x_0))^{-\frac{3}{2}-\varepsilon}$ for any $\varepsilon > 0$. To see this we first notice that if M is non-compact, the volume of such a manifold is infinite. This follows from Günther's inequality because we may find infinitely many disjoint balls of the same radius. Let $a(r) := \frac{\partial}{\partial r}\operatorname{vol}(B_r(x_0))$. Then $\int_0^1 a(r)\beta(1+r)^{\frac{2}{3}}dr < \infty$ and

$$\int_1^\infty a(r)\beta(1+r)^{\frac{2}{3}}dr \le \int_1^\infty a(r)\beta(r)^{\frac{2}{3}}dr \le \int_1^\infty a(r)\left(\int_0^r a(s)ds\right)^{-1-\frac{2\varepsilon}{3}} dr$$
$$= \int_{\operatorname{vol}(B_1(x_0))}^\infty t^{-1-\frac{2\varepsilon}{3}}dt < \infty.$$

3) Let M be a Riemannian manifold with cusps in the sense of [Mu1]. Assume that M has bounded curvature. Then the injectivity radius is exponentially decreasing in the distance and the volume of M is finite. Thus we may take $b = 1$. It follows $a = 1$, and we may take $\beta(t) = e^{-(\frac{n(n+1)}{2}+4n)ct}$, where c is chosen such that $\tilde{\imath}(x) \ge Ce^{-cd(x,q)}$. Scattering theory on manifolds with asymptotically cusp metrics has also been studied by Melrose [Me1, Section 8]. Again the decay conditions are stronger than in our case.

□

The assumptions on β in Theorem 7.1 that guarantee the existence of the wave operators are not optimal. Under additional assumptions on (M, g), the conditions on β can be relaxed. For example, let (M, g) be a complete manifold which is Euclidean at infinity and let h be a metric on M which satisfies (1.19), that is (M, h) is an asymptotically Euclidean manifold. Then Cotta-Ramusino, Krüger, and Schrader [CKS] proved that the wave operators $W_\pm(\Delta_g, \Delta_h)$ exist. The condition (1.19) is weaker than the assumption which is necessary in Theorem 7.1 in this case. The proof is based on Enss's method [Si], which applies to this scattering system. An abstract version of Enss's method has been developed by Amrein, Pearson and Wollenberg [APW], [BW, 16,IV,§15]. This method can be applied in cases where the structure of the continuous spectrum of the "free Hamiltonian" is sufficiently well known. To explain this in more detail we need to introduce some notation.

Let $C_\infty(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$ that vanish at infinity. For any closed countable subset $I \subset \mathbb{R}$ let $C_\infty(\mathbb{R} - I)$ of all functions $f \in C_\infty(\mathbb{R})$ satisfying $f(x) = 0$ for $x \in I$. A subset $\mathcal{A}_I$ of the space $C(\mathbb{R})$ of all bounded continuous functions on $\mathbb{R}$ is called multiplicative generating for $C_\infty(\mathbb{R} - I)$, if the linear span of the set

$$\{f \mid f = hg,\ h \in \mathcal{A}_I,\ g \in C_c^\infty(\mathbb{R} - I)\}$$

is dense in $C_\infty(\mathbb{R} - I)$ with respect to the norm $\| f \| = \sup_{x\in\mathbb{R}} |f(x)|$. The main result of [APW] can be stated as follows.

Theorem 7.2. *Let H and H_0 be two self-adjoint operators in a Hilbert space $\mathcal{H}$. Let $R_H(\lambda)$ and $R_{H_0}(\lambda)$ denote the resolvents of H and H_0, respectively. Assume that there exist self-adjoint operators P_+ and P_- in $\mathcal{H}$ and a set $\mathcal{A}_I$ of multiplicative generating functions with respect to some closed countable subset $I \subset \mathbb{R}$ satisfying the following properties*

(1) $P_{\mathrm{ac}}(H_0) = P_+ + P_-$ *and* $\operatorname{s-lim}_{t\to\pm\infty} e^{itH_0}P_\mp e^{-itH_0}P_{\mathrm{ac}}(H_0) = 0$.
(2) $(\mathrm{Id} - P_{\mathrm{ac}}(H_0))\alpha(H_0)$ *is compact for all* $\alpha \in \mathcal{A}_I$.

(3) *$R_H(i) - R_{H_0}(i)$ is compact.*
(4) $\int_0^{\pm\infty} \| (R_H(i) - R_{H_0}(i))e^{-itH_0}\alpha(H_0)P_\pm \| \; dt < \infty$ *for all* $\alpha \in \mathcal{A}_I$.

Then the wave operators $W_\pm(H, H_0)$ exist and are complete. Moreover H and H_0 have no singularly continuous spectrum and each eigenvalue of H and H_0 in $\mathbb{R} - I$ is of finite multiplicity. These eigenvalues accumulate at most at points of $I \cup \{\pm\infty\}$.

For the proof see Corollary 19 in [BW, 16,IV,§15].

As example, we consider a manifold X with cusps as defined in [Mu1]. For simplicity we assume that X has a single cusp. Then X is a complete Riemannian manifold of dimension $n+1$ that admits a decomposition

$$X = M \cup_Y Z$$

in a compact Riemannian manifold M with boundary Y and a half-cylinder $Z = [1, \infty) \times Y$, and M and Z are glued along their common boundary Y. The metric g on X is such that its restriction to Z is given by

$$g^Z = u^{-2}(du^2 + g^Y), \tag{7.1}$$

where g^Y denotes the metric of Y. The metric g is the fixed background metric and we consider perturbations h of g. As free Hamiltonian H_0 we are taking a modification of the Laplacian Δ_g which is defined as follows. We regard Y as a hypersurface in X that separates X into M and Z. Let $C_0^\infty(X-Y)$ be the subspace of all $f \in C_c^\infty(X)$ that vanish in a neighborhood of Y. Let Δ_0 denote Friedrichs's extension of

$$\Delta_g \colon C_0^\infty(X-Y) \to L^2(X).$$

To begin with we need to study the spectrum of Δ_0. With respect to the decomposition $L^2(X) = L^2(M) \oplus L^2(Z)$ we have

$$\Delta_0 = \Delta_{M,0} \oplus \Delta_{Z,0}, \tag{7.2}$$

where $\Delta_{M,0}$ and $\Delta_{Z,0}$ are the Dirichlet Laplacians on M and Z, respectively. Since M is compact, $\Delta_{M,0}$ has pure point spectrum. Let

$$L_0^2(Z) := \left\{ f \in L^2(Z) \colon \int_Y f(u,y)\, dy = 0 \text{ for almost all } u \in [1,\infty) \right\}.$$

The orthogonal complement $L_0^2(Z)^\perp$ of $L_0^2(Z)$ in $L^2(Z)$ consists of functions which are independent of $y \in Y$ and therefore, can be identified with $L^2([1,\infty), u^{-(n+1)}du)$. The decomposition

$$L^2(Z) = L_0^2(Z) \oplus L_0^2(Z)^\perp \tag{7.3}$$

is invariant under $\Delta_{Z,0}$.

Lemma 7.3. *The restriction of $\Delta_{Z,0}$ to $L_0^2(Z)$ has a compact resolvent. In particular, $\Delta_{Z,0}$ has pure point spectrum.*

Proof: Let Δ_Y be the Laplacian of Y. Let $\{\phi_j\}_{j=0}^{\infty}$ be an orthonormal basis of eigenfunctions of Δ_Y with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Let $f \in C_c^{\infty}(Z) \cap L_0^2(Z)$. Then f has an expansion of the form

$$f(u, y) = \sum_{k=1}^{\infty} a_k(u)\phi_k(y),$$

where the series converges in the C^{∞}-topology. Let $b > 1$ and put $Z_b = [b, \infty) \times Y$. Let $C = \lambda_1^{-1}$. Then we have

$$\| f \|^2_{L^2(Z_b)} = \sum_{k=1}^{\infty} \int_b^{\infty} |a_k(u)|^2 \frac{du}{u^{n+1}} \leq \frac{C}{b^2} \sum_{k=1}^{\infty} \lambda_k \int_b^{\infty} |a_k(u)|^2 \frac{du}{u^{n-1}}. \tag{7.4}$$

Now observe that the Laplacian Δ_Z with respect to the metric (7.1) equals

$$-u^2 \frac{\partial^2}{\partial u^2} + (n-1)u\frac{\partial}{\partial u} + u^2 \Delta_Y. \tag{7.5}$$

Moreover, since $a_k \in C_c^{\infty}((1, \infty))$, we have

$$\int_1^{\infty} \left(-u^2 a_k''(u) + (n-1)u a_k'(u)\right) \overline{a_k(u)} \frac{du}{u^{n+1}} = \int_1^{\infty} |a_k'(u)|^2 u^{1-n}\, du \geq 0.$$

This together with (7.4) implies

$$\| f \|^2_{L^2(Z_b)} \leq \frac{C}{b^2} \langle \Delta_Z f, f \rangle_{L^2(Z)} = \frac{C}{b^2} \| \nabla f \|^2_{L^2(Z)} \leq \frac{C}{b^2} \| f \|^2_{H^1(Z)} . \tag{7.6}$$

Let $H_0^1(Z) := H^1(Z) \cap L_0^2(Z)$. By continuity, (7.4) holds for all $f \in H_0^1(Z)$. By Rellich's lemma, the embedding

$$i_b \colon H^1(Z - Z_b) \cap L_0^2(Z - Z_b) \to L^2(Z)$$

is compact. It follows from (7.6) that as $b \to \infty$, i_b converges strongly to the embedding

$$i \colon H_0^1(Z) \to L^2(Z).$$

Hence i is compact which implies the lemma. □

Let

$$D_0 := -u^2 \frac{d^2}{du^2} + (n-1)u\frac{d}{du} \colon C_c^{\infty}((1, \infty)) \to L^2([1, \infty), u^{-(n+1)} du)$$

and let L_0 be the self-adjoint extension of D_0 with respect to Dirichlet boundary conditions at 1. By (7.5), the restriction of $\Delta_{Z,0}$ to $L_0^2(Z)^{\perp} \cong L^2([1, \infty), u^{-(n+1)} du)$ is equivalent to L_0. The spectrum of L_0 is absolutely continuous and equals $[n^2/4, \infty)$. Thus we get the following lemma.

Lemma 7.4. *The spectrum of Δ_0 is the union of a pure point point spectrum and an absolutely continuous spectrum. The point spectrum consists of eigenvalues of finite multiplicity $0 < \lambda_1 < \lambda_2 < \cdots \to \infty$. The absolutely continuous spectrum is equal to $[n^2/4, \infty)$ and the absolutely continuous part $\Delta_{0,\mathrm{ac}}$ of Δ_0 is equivalent to L_0.*

Let $\varepsilon > 0$ and let $\beta(t) = e^{-\varepsilon t}$. Let h be a complete metric on X. We put

$$H := \Delta_h \quad \text{and} \quad H_0 := \Delta_0.$$

Since H and H_0 are positive operators, we can replace i by -1 in Theorem 7.2. So let

$$R_g := (\Delta_g + \mathrm{Id})^{-1}, \;\; R_h := (\Delta_h + \mathrm{Id})^{-1} \text{ and } R_0 := (\Delta_0 + \mathrm{Id})^{-1}. \tag{7.7}$$

First we have the following lemma.

Lemma 7.5. *Suppose that $h \sim_\beta^2 g$. Then $R_h - R_0$ is a compact operator.*

Proof: First we show that $R_g - R_0$ is a compact operator. Let $f_1, f_2 \in C^\infty(\mathbb{R}^+)$ be such that $f_1(u) = 1$ for $u \geq 3$, $f_1(u) = 0$ for $u \leq 2$, and $f_2(u) = 1$ for $u \geq 2$, $f_2(u) = 0$ for $u \leq 1$. Put $\phi(u, y) = f_1(u)$ and $\psi(u, y) = f_2(u)$, $(u, y) \in \mathbb{R}^+ \times Y$. Let M_ϕ and M_ψ denote the multiplication operator by ϕ and ψ, respectively. By [Mu1, Remark 4.29], $R_g - M_\phi(L_0+1)^{-1}M_\psi$ is a compact operator. So it suffices to show that $R_0 - M_\phi(L_0+1)^{-1}M_\psi$ is compact. By (7.2) we have

$$R_0 = (\Delta_{M,0} + 1)^{-1} \oplus (\Delta_{Z,0} + 1)^{-1}. \tag{7.8}$$

Since M is compact, $(\Delta_{M,0} + 1)^{-1}$ is compact. By Lemma 7.3 it suffices to prove that

$$(L_0 + 1)^{-1} - M_\phi(L_0 + 1)^{-1}M_\psi = M_{1-\phi}(L_0 + 1)^{-1}M_\psi + (L_0 + 1)^{-1}M_{1-\psi}$$

is compact as operator in$L^2([1, \infty), u^{-(n+1)}du)$. To this end consider the kernel $g(u, u')$ of $(L_0 + 1)^{-1}$. It equals

$$g(u, u') = \frac{(uu')^{n/2}}{\sqrt{n^2/4 + 1}} \begin{cases} (u'/u)^{\sqrt{n^2/4+1}} - (uu')^{-\sqrt{n^2/4+1}}, & u > u'; \\ (u/u')^{\sqrt{n^2/4+1}} - (uu')^{-\sqrt{n^2/4+1}}, & u' > u. \end{cases} \tag{7.9}$$

From this formula follows that $g(u, u')$ is bounded on $[1, 3] \times [1, \infty)$ and $[1, \infty) \times [1, 3]$ and therefore, the kernels of the operators $M_{1-\phi}(L_0 + 1)^{-1}M_\psi$ and $(L_0 + 1)^{-1}M_{1-\psi}$ are square integrable w.r.t. the measure $u^{-(n+1)}du\, (u')^{-(n+1)}du'$. Hence $(L_0 + 1)^{-1} - M_\phi(L_0 + 1)^{-1}M_\psi$ is compact.

So in order to prove the lemma, it suffices to show that $R_h - R_g$ is compact. We have

$$R_h - R_g = -R_g(\Delta_h - \Delta_g)R_h. \tag{7.10}$$

By Lemma 5.2 we have

$$\Delta_h - \Delta_g = \sum_{j=0}^{2} \xi_j \circ (\nabla^h)^j \tag{7.11}$$

and ξ_j satisfies

$$|\xi_j(x)| \leq Ce^{-\varepsilon d(x,x_0)}, \quad x \in X. \tag{7.12}$$

Now $R_h\colon L^2(X) \to W^2(X)$ is continuous. Therefore by (7.11) and (7.12) it follows that

$$(\Delta_h - \Delta_g)R_h\colon L^2(X) \to L^2(X)$$

is a bounded operator. Using again that $R_g - R_0$ is compact, it follows from (7.10) that it suffices to show that $R_0(\Delta_h - \Delta_g)R_h$ is a compact operator.

For $a > 1$ let

$$X_a = M \cup_Y ([1,a] \times Y).$$

Denote by χ_a the characteristic function of X_a in X. We claim that $R_0\chi_a$ is a compact operator. Let $\chi_{[1,a]}$ be the characteristic function of the interval $[1,a]$ in $[1,\infty)$. Using (7.8) and (7.9), this follows in the same way as above. Let $M_{(1-\chi_a)\beta}$ denote the multiplication operator by $(1-\chi_a)\beta$. By (7.11) and (7.12), we get

$$\begin{aligned} &\| R_0(1-\chi_a)(\Delta_h - \Delta_g)R_h \| \\ &\leq C\left(\sum_{j=0}^{2} \| (\nabla^h)^j R_h \|\right) \cdot \| R_0 \| \cdot \| M_{(1-\chi_a)\beta} \| . \end{aligned} \tag{7.13}$$

Let $Z_a = [a,\infty) \times Y$. Then

$$\| M_{(1-\chi_a)\beta} \| \leq \sup_{x\in Z_a} \beta(x) = \sup_{x\in Z_a} e^{-\varepsilon d(x,x_0)}.$$

Now observe that there exists $C_1 > 0$ such that for all $(u,y) \in Z_a$ we have

$$d((u,y),x_0) \geq d((u,y),(1,y)) - C = \log u - C_1.$$

Hence together with (7.13) we get

$$\| R_0(1-\chi_a)(\Delta_h - \Delta_g)R_h \| \leq C_2 a^{-\varepsilon}.$$

Thus $R_0(\Delta_h - \Delta_g)R_h$ can be approximated in the operator norm by compact operators and hence, is a compact operator. $\square$

Next we construct self-adjoint projections $P_\pm$ which satisfy the conditions of Theorem 7.2. Let

$$e(u,\lambda) := u^{n/2+i\lambda} - u^{n/2-i\lambda}, \quad u \in [1,\infty),\ \lambda \in \mathbb{R}. \tag{7.14}$$

Then $e(u,\lambda)$ satisfies

$$D_0 e(u,\lambda) = (n^2/4 + \lambda^2)e(u,\lambda), \quad e(1,\lambda) = 0.$$

Thus $e(u,\lambda)$ is the generalized eigenfunction for L_0. For $\varphi \in C_c^\infty(1,\infty))$ set

$$\hat{\varphi}(\lambda) := \frac{1}{2\pi}\int_1^\infty e(u,\lambda)\varphi(u)\,\frac{du}{u^{n+1}}.$$

The map $\varphi \mapsto \hat{\varphi}$ extends to an isometry

$$F\colon L^2([1,\infty), u^{-(n+1)}du) \to L^2(\mathbb{R}^+)$$

such that

$$F \circ L_0 \circ F^* = \widetilde{L}_0,$$

where $\widetilde{L}_0$ is the multiplication operator by $(n^2/4 + \lambda^2)$. Let

$$U\colon L^2(\mathbb{R}^+) \to L^2([n^2/4,\infty))$$

be defined by

$$(Uf)(\lambda) = \frac{f(\sqrt{\lambda - n^2/4})}{\sqrt{2}(\lambda - n^2/4)^{1/4}}.$$

Then U is an isometry such that $U \circ \widetilde{L}_0 \circ U^* = \widehat{L}_0$, where $\widehat{L}_0$ is the multiplication operator by λ. Thus $U \circ F$ provides the spectral resolution of $L_0 = \Delta_{0,\mathrm{ac}}$. Let

$$J\colon L^2([n^2/4, \infty)) \to L^2(\mathbb{R})$$

denote the inclusion, let $\mathcal{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transform, and let $\chi_\pm$ denote the characteristic function of $[0, \infty)$ and $(-\infty, 0]$, respectively. Set

$$\widetilde{P}_\pm := J^* \mathcal{F} \chi_\pm \mathcal{F}^* J.$$

Then $\widetilde{P}_+ + \widetilde{P}_-$ is the identity of $L^2([n^2/4, \infty))$. Let $A = -id/du$, regarded as self-adjoint operator in $L^2(\mathbb{R})$. Then

$$\widetilde{P}_\pm e^{-it\widehat{L}_0} = J^* \mathcal{F} \chi_\pm e^{-itA} \mathcal{F}^* J.$$

Let $f \in L^2(\mathbb{R})$. Using the Fourier transformation, it follows that $(e^{-itA} f)(u) = f(u - t)$. Thus we get

$$\| \chi_\pm e^{-itA} f \|^2 = \pm \int_{-t}^{\pm\infty} |f(u)|^2 \, du \to 0$$

as $t \to \mp\infty$. Hence we get

$$\text{s-lim}_{t \to \pm\infty} e^{it\widehat{L}_0} \widetilde{P}_\mp e^{-it\widehat{L}_0} = 0. \tag{7.15}$$

Now put

$$P_\pm := F^* U^* \widetilde{P}_\pm U F$$

on $L^2([1, \infty), u^{-(n+1)} du)$ and set $P_\pm := 0$ on the orthogonal complement of $L^2_0(Z)^\perp = L^2([1, \infty), u^{-(n+1)} du)$ in $L^2(X)$. Then $P_\pm$ are self-adjoint projections that satisfy

$$P_+ + P_- = P_{\mathrm{ac}}(\Delta_0).$$

Furthermore we have

$$e^{itH_0} P_\pm e^{-it\Delta_0} P_{\mathrm{ac}}(\Delta_0) = F^* U^* e^{it\widehat{L}_0} \widetilde{P}_\pm e^{-it\widehat{L}_0} U F.$$

So it follows from (7.15) that

$$\text{s-lim}_{t \to \pm\infty} e^{itH_0} P_\mp e^{-itH_0} P_{\mathrm{ac}}(H_0) = 0.$$

Thus condition (1) of Theorem 7.2 is satisfied. Let $I = \{n^2/4\}$. and put

$$\mathcal{A}_I := C_c^\infty(\mathbb{R} - I).$$

Then it is clear that $\mathcal{A}_I$ is multiplicative generating for $C_\infty(\mathbb{R} - I)$. By Lemma 7.4, Δ_0 has pure point spectrum in the subspace $(\mathrm{Id} - P_{\mathrm{ac}}(\Delta_0)) L^2(X)$ consisting of eigenvalues of finite multiplicity with no finite points of accumulation. Let $\alpha \in \mathcal{A}_I$. Then $(\mathrm{Id} - P_{\mathrm{ac}}(\Delta_0))\alpha(\Delta_0)$ is a finite rank operator. This is condition (2) of Theorem 7.2. Condition (3) holds by Lemma 7.5. It remains to verify condition (4).

Given $t > 0$, let χ_t be the characteristic function of $[e^t, \infty) \times Y$ in X. Let $\delta > 0$. We have

$$
\begin{aligned}
&\| (R_h - R_0) e^{it\Delta_0} \alpha(\Delta_0) P_\pm \| \\
&\quad \le \| R_h - R_0 \| \cdot \| (1 - \chi_{\delta t}) e^{it\Delta_0} \alpha(\Delta_0) P_\pm \| \\
&\quad\quad + \| (R_h - R_0)\chi_{\delta t} \| \cdot \| \alpha(\Delta_0) \| .
\end{aligned}
\tag{7.16}
$$

We will prove that for each $\alpha \in C_c^\infty(\mathbb{R} - \{n^2/4\})$ there exists $\delta > 0$ such that the right hand side is an integrable function of $t \in \mathbb{R}^+$. To estimate the first term on the right hand side we need the following auxiliary result.

Lemma 7.6. *Let $a \in \mathbb{R}$ and let $f \in C_c^\infty(\mathbb{R} - \{a\})$. Let $\varepsilon > 0$ such that $f(\lambda^2 + a) = 0$ for $|\lambda| < \varepsilon$. Then for every $m \in \mathbb{N}$ there exists $C > 0$ such that for $t \in \mathbb{R} - \{0\}$ and $|u| < \varepsilon|t|/2$ one has*

$$
\left| \int_0^\infty e^{2iu\lambda + it\lambda^2} f(\lambda^2 + a)\, d\lambda \right| \le C|t|^{-m}.
$$

Proof: Let $t \neq 0$ and set $x = u/t$. Then the left hand side of the inequality equals

$$
\begin{aligned}
&\left| \int_0^\infty e^{it(\lambda+x)^2} f(\lambda^2 + a)\, d\lambda \right| \\
&\qquad = (2t)^{-m} \left| \int_0^\infty e^{it(\lambda+x)^2} \left(\frac{1}{\lambda + x} \frac{d}{d\lambda} - \frac{1}{(\lambda+x)^2} \right)^m f(\lambda^2 + a)\, d\lambda \right| .
\end{aligned}
$$

Now assume that $|u| < \varepsilon|t|/2$. Then $|x| < \varepsilon/2$. On the other hand, we have $f(\lambda^2 + a) = 0$ for $|\lambda| < \varepsilon$. Thus if $f(\lambda^2 + a) \neq 0$, then we have $|\lambda + x| \ge |\lambda| - |x| > \varepsilon/2$. Hence the right hand side can be estimated by $C|t|^{-m}$. $\square$

Let $\varphi \in L^2([1, \infty), u^{-(n+1)} du) = P_{\mathrm{ac}}(\Delta_0)(L^2(X))$. Then

$$
\begin{aligned}
&\big(e^{-it\Delta_0} \alpha(\Delta_0) \varphi \big)(u) \\
&\qquad = \frac{1}{2\pi} \int_0^\infty e(u, n/2 - i\lambda) e^{-it(\lambda^2 + n^2/4)} \alpha(\lambda^2 + n^2/4)(F\varphi)(\lambda)\, d\lambda.
\end{aligned}
\tag{7.17}
$$

Let $v \in C_c^\infty((1, \infty))$. Put $\varphi = P_+ v$ and $w = \mathcal{F}^* J U F v$. Then $w \in L^1(\mathbb{R})$ and $F\varphi = U^* J^* \mathcal{F}(\chi_+ w)$. Using the definition of U, J and $\mathcal{F}$, we get

$$
(F\varphi)(\lambda) = \sqrt{2\lambda} \int_0^\infty e^{-is(\lambda^2 + n^2/4)} w(s)\, ds.
$$

Assume that $t > 0$. If we insert this expression into the right hand side of (7.17) and switch the order of integration, we obtain

$$
\begin{aligned}
&\big(e^{-it\Delta_0} \alpha(\Delta_0) P_+ v \big)(u) \\
&\quad = \frac{1}{\sqrt{2}\pi} \int_0^\infty w(s) \int_0^\infty e(u, n/2 - i\lambda) e^{-i(t+s)(\lambda^2 + n^2/4)} \alpha(\lambda^2 + n^2/4) \sqrt{\lambda}\, d\lambda\, ds.
\end{aligned}
\tag{7.18}
$$

Now there exists $\varepsilon > 0$ such that $\alpha(\lambda^2+n^2/4) = 0$ for $|\lambda| < \varepsilon$. Assume that $|\log(u)| < \varepsilon t/2$. Using the definition (7.14) of $e(u,\lambda)$ and Lemma 7.6, it follows that there exists $C > 0$ such that

$$\left| e^{-it\Delta_0}\alpha(\Delta_0)P_+v(u)\right|^2 \leq C \parallel w \parallel^2 u^n t^{-3} \leq C \parallel v \parallel^2 u^n t^{-3}. \tag{7.19}$$

Thus for every $\alpha \in C_c^\infty(\mathbb{R} - \{n^2/4\})$ there exist $C > 0$ and $\delta > 0$ such that for $t > \delta^{-1}$ one has

$$\parallel (1-\chi_{\delta t})e^{-it\Delta_0}\alpha(\Delta_0)P_+ \parallel \leq C\, t^{-3}\int_1^{e^{\delta t}} \frac{du}{u} = C\delta t^{-2}.$$

Similarly one can show that

$$\parallel (1-\chi_{\delta t})e^{-it\Delta_0}\alpha(\Delta_0)P_- \parallel \leq C\, t^{-3}\int_1^{e^{\delta t}} \frac{du}{u} = C\delta t^{-2}, \quad t > \delta^{-1}.$$

Hence for this choice of δ, the first term on the right hand side of (7.16) is an integrable function of $t \in \mathbb{R}^+$.

Now consider the second term on the right hand side of (7.16). We have

$$\parallel (R_h - R_0)\chi_{\delta t} \parallel \leq \parallel (R_h - R_g)\chi_{\delta t} \parallel + \parallel (R_g - R_{\Delta_0})\chi_{\delta t} \parallel . \tag{7.20}$$

Let $M_{\chi_{\delta t}\beta}$ denote the multiplication operator by $\chi_{\delta t}\beta$. By (7.10) - (7.12) we get

$$\begin{aligned} \parallel (R_h - R_g)\chi_{\delta t} \parallel &\leq \parallel R_g \parallel \cdot \parallel \chi_{\delta t}(\Delta_h - \Delta_g)R_h \parallel \\ &\leq C \parallel M_{\chi_{\delta t}\beta} \parallel \left(\sum_{j=0}^{2} \parallel (\nabla^h)^j R_h \parallel\right) \leq C_1 e^{-\varepsilon\delta t}. \end{aligned} \tag{7.21}$$

It remains to estimate the second term on the right of (7.20). Let $\psi \in C^\infty(\mathbb{R})$ such that $f(u) = 0$, if $u \leq 2$, and $f(u) = 1$, if $u \geq 3$. Define $f \in C^\infty(Z)$ by $f(u,y) = \psi(u)$ and extend f by zero to a smooth function on X. Then we have

$$R_g - R_0 = (f-1)R_0 - R_g((\Delta_g + \mathrm{Id})(fR_0) - \mathrm{Id}).$$

Observe that

$$(\Delta_g + \mathrm{Id})(fR_0) - \mathrm{Id} = f - 1 + 2\nabla f \cdot \nabla R_0 + \Delta f \cdot R_0.$$

Moreover note that $(f-1)\chi_{\delta t} = 0$ if $t \gg 0$. Thus

$$(R_g - R_0)\cdot\chi_{\delta t} = (f-1)\cdot R_0\cdot\chi_{\delta t} - R_g(2\nabla f\cdot\nabla R_0\cdot\chi_{\delta t} + \Delta f\cdot R_0\cdot\chi_{\delta t}) \tag{7.22}$$

for $t \gg 0$. It follows from (7.2) that $R_0 \cdot \chi_{\delta t}$ acts in $L^2(Z)$ and preserves the decomposition (7.3). Moreover $\parallel R_0 \cdot \chi_{\delta t}|_{L_0^2(Z)} \parallel = \parallel \chi_{\delta t}\cdot R_0|_{L_0^2(Z)} \parallel$. Let $\varphi \in L_0^2(Z)$. Then $R_0\varphi \in L_0^2(Z) \cap H^2(Z)$ and by (7.4) we obtain

$$\parallel \chi_{\delta t} R_0 \varphi \parallel \leq C e^{-2\delta t} \parallel R_0\varphi \parallel_1 \leq C e^{-2\delta t} \parallel \varphi \parallel . \tag{7.23}$$

On the orthogonal complement $L^2_0(Z)^\perp$, the kernel of R_0 is given by (7.9). Let $h \in C^\infty_c(Z)$. Then it follows from (7.9) that

$$\| h \cdot R_0 \cdot \chi_{\delta t}|_{L^2_0(Z)^\perp} \| \leq Ce^{-\delta t\sqrt{n^2/4+1}} \leq Ce^{-\delta t}. \tag{7.24}$$

Combining (7.23) and (7.24) we obtain

$$\| h \cdot R_0 \cdot \chi_{\delta t} \| \leq Ce^{-\delta t}.$$

Similar estimations hold for ∇R_0. This proves that the second term on the right hand side of (7.16) is an integrable function of $t \in \mathbb{R}^+$. This is condition (4) of Theorem 7.2. Summarizing we have proved the following theorem.

Theorem 7.7. *Let (X, g) be a manifold with cusps and let Δ_0 be defined by (7.2). Let $\varepsilon > 0$ and put $\beta(u) = e^{-\varepsilon u}$, $u \in \mathbb{R}$. Let h be a complete metric on X such that $h \sim^2_\beta g$. Then we have*

1. *The wave operators $W_\pm(\Delta_h, \Delta_0; J)$ exist and are complete.*
2. *Δ_h has no singularly continuous spectrum.*

Corollary 7.8. *Let g and h be as above. Then the wave operators $W_\pm(\Delta_h, \Delta_g; J)$ exist and are complete.*

This is a considerable improvement of the result that we get from Theorem 7.1 in this case.

Remark. Other cases of complete manifolds (M, g) with a sufficiently explicit structure at infinity can be treated in the same way. This includes, for example, manifolds with cylindrical ends and asymptotically Euclidean manifolds.

8. β-Equivalence and Analytic Continuations of the Resolvent

In this section we study the existence of an analytic continuation of the resolvent in weighted L^2-spaces. Provided that such a continuation exists, we are able to study the behavior of the absolutely continuous spectrum under perturbation in more detail. The method is a modification of the method used in [Mu2].

Definition 8.1. Let $\mathcal{B}$ be a Banach space, $\Omega \subset \mathbb{C}$ a domain and $F : \Omega \mapsto \mathcal{B}$ a meromorphic function. Let Σ be a Riemann surface and let $\pi : \Sigma \to \mathbb{C}$ be a ramified covering. A meromorphic continuation of F to Σ is a meromorphic function $\tilde{F} : \Sigma \to \mathcal{B}$ such that

a) There exists $\tilde{\Omega} \subseteq \Sigma$ such that $\pi : \tilde{\Omega} \to \Omega$ is biholomorphic.
b) $F \circ \pi = \tilde{F}$ on $\tilde{\Omega}$.

Definition 8.2. Let δ be a function of moderate decay and let $p \in \mathbb{N}$. By $H^{-p}_{\delta^{-1}}$ we denote the dual space of H^p_δ, with respect to the extension of the L^2-pairing.

Lemma 8.3. *Let $\zeta(u)$ be a non-increasing continuous function on $[1,\infty)$ with $\zeta(u)\to 0$ as $u\to\infty$ and let δ be a weight function. Then the canonical inclusion $j\colon L^2_{\delta\zeta^{-1}}(M)\to H^{-2}_{\delta}(M)$ is compact.*

Proof: It is enough to prove, that the adjoint $j^*: H^2_{\delta^{-1}}(M)\to L^2_{\delta^{-1}\zeta}(M)$ is compact. For $k\in\mathbb{N}$ let

$$\Omega_k=\{x\in M\mid \zeta(1+d(x,x_0))\geq 1/k\}.$$

Then each Ω_k is a compact subset of M. Let P_k be the multiplication operator by the characteristic function of Ω_k. By Rellich's lemma, j^*P_k is compact. For $f\in H^2_{\delta^{-1}}(M)$ we have

$$\int_{M-\Omega_k}|f(x)|^2\delta^{-1}(x)\zeta(x)\,dx\leq\frac{1}{k}\parallel f\parallel^2_{H^2_{\delta^{-1}}}.$$

Thus j^*P_k converges to j^* in the operator topology. Hence j^* is compact. □

Let δ,ρ be functions of moderate decay. Then $L^2_{\delta^{-1}}(M)\subset L^2(M)$ and $H^2(M)\subset H^2_\rho(M)$. Thus for $\lambda\in\mathbb{C}-[0,\infty)$, the resolvent $(\Delta-\lambda)^{-1}\colon L^2(M)\to H^2(M)$ may be regarded as a bounded operator

$$(\Delta-\lambda)^{-1}\colon L^2_{\delta^{-1}}(M)\to H^2_\rho(M).$$

Denote by $\mathcal{L}(L^2_{\delta^{-1}}(M),H^2_\rho(M))$ the Banach space of all bounded operators from $L^2_{\delta^{-1}}(M)$ into $H^2_\rho(M)$, equipped with the strong operator norm.

Theorem 8.4. *Let g,h be complete Riemannian metrics on M with bounded curvature of order 2. Let β, δ, ζ and ρ be functions of moderate decay on M such that*

$$\beta^2(x)\leq C\tilde{\imath}_g^{4n}(x)\rho(x)\delta(x)\zeta(x),\quad x\in M,\tag{8.1}$$

and $g\sim^2_\beta h$. Let $\Omega\subset\mathbb{C}-[0,\infty)$ be open. Assume that there is a Riemann surface Σ and a covering $\Sigma\to\Omega$ such that the operator valued function

$$\lambda\in\Omega\mapsto(\Delta_g-\lambda)^{-1}\in\mathcal{L}(L^2_{\delta^{-1}}(M,g),H^2_\rho(M,g))$$

admits an analytic continuation to a meromorphic function

$$\lambda\in\Sigma\to R_g(\lambda)\in\mathcal{L}(L^2_{\delta^{-1}}(M,g),H^2_\rho(M,g))$$

with finite rank residues. Then

$$\lambda\in\Omega\mapsto(\Delta_h-\lambda)^{-1}\in\mathcal{L}(L^2_{\delta^{-1}}(M,h),H^2_\rho(M,h))$$

also admits a meromorphic continuation to Σ with finite rank residues.

Proof: By assumption, $\beta\tilde{\imath}^{-2n}$ is bounded. Hence by Corollary 5.3, $H^2(M,g)$ and $H^2(M,h)$ are equivalent and therefore, by duality, $H^{-2}(M,g)$ and $H^{-2}(M,h)$ are also equivalent. Let $\lambda\in\mathbb{C}-[0,\infty)$. Then

$$K(\lambda):=(\Delta_g-\lambda)^{-1}(\Delta_h-\Delta_g)$$

is a bounded operator in $L^2(M)$. Moreover $\mathrm{Id}+K(\lambda)=(\Delta_g-\lambda)^{-1}(\Delta_h-\lambda)$ has a bounded inverse in $L^2(M)$ which is given by

$$(\mathrm{Id}+K(\lambda))^{-1}=(\Delta_h-\lambda)^{-1}(\Delta_g-\lambda).$$

Thus for $\lambda \in \mathbb{C} - [0, \infty)$ we have

$$(\Delta_h - \lambda)^{-1} = (\mathrm{Id} + K(\lambda))^{-1}(\Delta_g - \lambda)^{-1}. \tag{8.2}$$

By Corollary 4.3 there exists $\lambda \in \mathbb{C} - [0, \infty)$ such that $(\Delta_h - \lambda)^{-1}$ extends to a bounded operator in $L^2_\rho(M)$. By Lemma 4.4 it follows that $(\Delta_h - \lambda)^{-1}$ maps $L^2_\rho(M)$ into $H^2_\rho(M)$. Moreover by definition $\Delta_g - \lambda$ is a bounded operator of $H^2_\rho(M)$ to $L^2_\rho(M)$. Hence $(\mathrm{Id} + K(\lambda))^{-1}$ extends to a bounded operator in $H^2_\rho(M)$. Let $\mu \in \Omega$. Then

$$\begin{aligned} \mathrm{Id} + K(\mu) &= (\mathrm{Id} + K(\lambda)) - \{K(\lambda) - K(\mu)\} \\ &= (\mathrm{Id} + K(\lambda)) - (\lambda - \mu)(\Delta_g - \mu)^{-1}(\Delta_g - \lambda)^{-1}(\Delta_h - \Delta_g). \end{aligned} \tag{8.3}$$

By Corollary 4.3 we may choose λ such that $(\Delta_g - \lambda)^{-1}$ extends to a bounded operator in $L^2_{\delta\zeta}(M)$. By duality, and Lemma 4.4, it defines a bounded operator

$$(\Delta_g - \lambda)^{-1} \colon L^2_{\delta^{-1}\zeta^{-1}}(M) \to H^2_{\delta^{-1}\zeta^{-1}}(M).$$

Using Lemma 3.6, Lemma 5.2 and the assumption on β, it follows that the operator $(\Delta_g - \lambda)^{-1}(\Delta_h - \Delta_g)$ is the composition of the following chain of bounded operators

$$\begin{aligned} &H^2_\rho(M) \to W^2_{\tilde{\imath}^{4n}\rho}(M) \xrightarrow{\Delta_h - \Delta_g} L^2_{\beta^{-2}\tilde{\imath}^{4n}\rho}(M) \to \\ &\qquad L^2_{\delta^{-1}\zeta^{-1}}(M) \xrightarrow{(\Delta_g - \lambda)^{-1}} H^2_{\delta^{-1}\zeta^{-1}}(M) \xrightarrow{j} L^2_{\delta^{-1}}(M). \end{aligned} \tag{8.4}$$

By Lemma 8.3, the inclusion j is a compact. Hence

$$(\Delta_g - \lambda)^{-1}(\Delta_h - \Delta_g) \colon H^2_\rho(M) \to L^2_{\delta^{-1}}(M)$$

is compact operator. Set

$$H_\lambda(\mu) = (\lambda - \mu) R_g(\mu) \circ (\Delta_g - \lambda)^{-1}(\Delta_h - \Delta_g), \quad \mu \in \Sigma. \tag{8.5}$$

Then $H_\lambda(\mu)$, $\mu \in \Sigma$, is a meromorphic family of compact operators and

$$\mathrm{Id} + K(\mu) = (\mathrm{Id} + K(\lambda)) \left\{ \mathrm{Id} - (\mathrm{Id} + K(\lambda))^{-1} H_\lambda(\mu) \right\}. \tag{8.6}$$

It then follows from [St], that $(\mathrm{Id} + K(\mu))^{-1}$ exists except for on a discrete set and is meromorphic in μ. Thus, we may define

$$R_h(\mu) = (\mathrm{Id} + K(\mu))^{-1} \circ R_g(\mu). \tag{8.7}$$

By (8.2) this is the desired meromorphic continuation of the resolvent $(\Delta_h - \lambda)^{-1}$. □

Examples.

1) Let M be a surface with cusps. Here by a cusp we mean a half-cylinder $[a, \infty) \times S^1$, $a > 0$, equipped with the Poincaré metric $y^{-2}(dx^2 + dy^2)$, and M is a surface with a complete metric g which in the complement of compact set is isometric to the disjoint union of finitely many cusps. Let $c > 0$ and let $x_0 \in M$. Set

$$\delta(x) := e^{-cd(x, x_0)}, \quad x \in M, \tag{8.8}$$

and $\rho = \zeta = \delta$. Then δ, ρ, and ζ are functions of moderate decay. Let

$$\Omega = \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1/2, \;\; s \notin (1/2, 1]\}.$$

We consider the resolvent $R_g(s) = (\Delta_g - s(1-s))^{-1}$ as a function of $s \in \Omega$. Then it follows from [Mu2, Theorem 1] that $R(s)$ admits an analytic continuation to a meromorphic function on $\mathbb{C}$ with values in $\mathcal{L}(L^2_{\delta^{-1}}(M), L^2_\delta(M))$. Using the same method, one can show that the $R_g(s)$ takes values in $H^2_\delta(M)$. Now observe that the injectivity radius satisfies $\imath(x) \sim e^{-d(x,x_0)}$. Let $\epsilon > 0$ and set $\beta(x) = e^{-(4+\epsilon)d(x,x_0)}$. Choose the constant $c > 0$ in (8.8) such that $c < \epsilon/4$. Then β is a function of moderate decay which satisfies (8.1) with respect to our choice of the functions δ, ρ, and ζ. Now note that the metric g has bounded curvature of all orders. Let h be complete metric on M with bounded curvature of order 2 which satisfies $g \sim^2_\beta h$. Then it follows from Theorem 8.4 that the resolvent $R_h(s) = (\Delta_h - s(1-s))^{-1}$, $s \in \Omega$, also admits a meromorphic extension to $\mathbb{C}$ with values in $\mathcal{L}(L^2_{\delta^{-1}}(M), H^2_\delta(M))$. We think that the condition on β can be weakened.

2) Let M be a manifold with a cylindrical end. This means that M is a complete Riemannian manifold that admits a decomposition $M = M_0 \cup_Y (\mathbb{R}^+ \times Y)$ into a compact manifold M_0 with boundary Y and a half-cylinder $(\mathbb{R}^+ \times Y)$ which is glued to M_0 along the common boundary Y. The restriction of the metric g of M to the half-cylinder is assumed to be the product metric. Then g is a metric with bounded geometry, that is, g has bounded curvature of all orders and the injectivity radius has a positive lower bound. Let Δ_Y be the Laplacian of Y and let $0 = \mu_1 < \mu_2 \le \mu_3 \le \cdots$ be the eigenvalues of Δ_Y. Let $\Sigma \to \mathbb{C}$ be the Riemann surface to which the square roots $\lambda \mapsto \sqrt{\lambda - \mu_j}$, $j \in \mathbb{N}$, extend holomorphically. Define δ, ρ, and ζ as in example 1. Then it follows as in [Mu2, Theorem 5] that the resolvent $(\Delta_g - \lambda)^{-1}$ extends from $\mathbb{C} - [0,\infty)$ to a meromorphic function $\lambda \in \Sigma \mapsto R_g(\lambda)$ with values in $\mathcal{L}(L^2_{\delta^{-1}}(M), H^2_\delta(M))$. Now let $\epsilon > 0$, $x_0 \in M$, and set

$$\beta(x) = e^{-\epsilon d(x,x_0)}, \quad x \in M.$$

Choose c in the definition of δ such that $c < \epsilon/2$. Then β satisfies (8.1) with respect to our choice of the functions δ, ρ, and ζ. Let h be a complete metric on M with bounded curvature of order 2, and suppose that $g \sim^2_\beta h$. Then it follows from Theorem 8.4 that the resolvent $(\Delta_h - \lambda)^{-1}$ also admits a extension from $\mathbb{C} - [0,\infty)$ to a meromorphic function $\lambda \in \Sigma \mapsto R_g(\lambda)$ with values in $\mathcal{L}(L^2_{\delta^{-1}}(M), H^2_\delta(M))$.

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