ON FRIED'S CONJECTURE FOR COMPACT HYPERBOLIC MANIFOLDS

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Abstract. Fried's conjecture is concerned with the behavior of dynamical zeta functions at the origin. For compact hyperbolic manifolds, Fried proved that for an orthogonal acyclic representation of the fundamental group, the twisted Ruelle zeta function is holomorphic at $s = 0$ and its value at $s = 0$ equals the Ray-Singer analytic torsion. He also established a more general result for orthogonal representations, which are not acyclic. The purpose of the present paper is to extend Fried's result to arbitrary finite dimensional representations of the fundamental group. The Ray-Singer analytic torsion is replaced by the complex-valued torsion introduced by Cappell and Miller.

Contents

1. Introduction

Let X be a d-dimensional closed, oriented hyperbolic manifold. Then there exists a discrete torsion free subgroup $\Gamma \subset \mathrm{SO}_0(d,1)$ such that $X = \Gamma\backslash\mathbb{H}^d$, where $\mathbb{H}^d = \mathrm{SO}_0(d,1)/\mathrm{SO}(d)$ is the d-dimensional hyperbolic space. Every $\gamma \in \Gamma \setminus \{e\}$ is hyperbolic and the Γ-conjugacy class $[\gamma]$ corresponds to a unique closed geodesic τ_γ. Let $\ell(\gamma)$ denote the length of τ_γ. A conjugacy class is called prime if γ is not a non-trivial power of some other element of Γ.

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Let $\chi\colon \Gamma \to \mathrm{GL}(V_\chi)$ be a finite dimensional complex representation of Γ and let $s \in \mathbb{C}$. Then the Ruelle zeta function $R(s,\chi)$ is defined by the following Euler product

$$R(s,\chi) := \prod_{\substack{[\gamma]\neq e\\ [\gamma] \text{ prime}}} \det\left(\mathrm{Id} - \chi(\gamma)e^{-s\ell(\gamma)}\right). \tag{1.1}$$

The infinite product is absolutely convergent in a certain half plane $\mathrm{Re}(s) > C$ and admits a meromorphic extension to the entire complex plane [Fr2], [Sp1]. The Ruelle zeta is a dynamical zeta function associated to the geodesic flow on the unit sphere bundle $S(X)$ of X. There are formal analogies with the zeta functions in number theory such as the Artin L-function associated to a Galois representation. Analogues to the role of zeta functions in number theory, one expects that special values of the Ruelle zeta function provide a connection between the length spectrum of closed geodesics and geometric and topological invariants of the manifold.

In [Fr1], Fried has established such a connection. To explain his result we need to introduce some notation. Recall that a representation χ is called *acyclic*, if the cohomology $H^*(X,F_\chi)$ of X with coefficients in the flat bundle $F_\chi \to X$ associated to χ vanishes. Let χ be an orthogonal acyclic representation. Then F_χ is equipped with a canonical fibre metric which is compatible with the flat connection. Let $\Delta_{k,\chi}$ be the Laplacian acting in the space $\Lambda^k(X,F_\chi)$ of F_χ-valued k-forms. Regarded as operator in the space of L^2-forms, it is essentially self-adjoint with a discrete spectrum $\mathrm{Spec}(\Delta_{k,\chi})$ consisiting of eigenvalues λ of finite multiplicity $m(\lambda)$. Let $\zeta_k(s;\chi) = \sum_{\lambda\in\mathrm{Spec}(\Delta_{k,\chi})} m(\lambda)\lambda^{-s}$ be the spectral zeta function of $\Delta_{k,\chi}$ [Shb]. The series converges absolutely in the half plane $\mathrm{Re}(s) > d/2$ and admits a meromorphic extension to the complex plane, which is holomorphic at $s = 0$. Then the Ray-Singer analytic torsion $T^{RS}(X,\chi) \in \mathbb{R}^+$ is defined by

$$\log T^{RS}(X,\chi) := \frac{1}{2}\sum_{k=1}^{d}(-1)^k k \frac{d}{ds}\zeta_k(s;\chi)\big|_{s=0}, \tag{1.2}$$

[RS]. Now we can state the result of Fried [Fr1, Theorem 1]. He proved that for an acyclic unitary representation χ the Ruelle zeta function $R(s,\chi)$ is holomorphic at $s = 0$ and

$$|R(0,\chi)^\varepsilon| = T^{RS}(X,\chi)^2, \tag{1.3}$$

where $\varepsilon = (-1)^{d-1}$ and the absolute value can be removed if $d > 2$. If χ is not acyclic, but still orthogonal, $R(s,\chi)$ may have a pole or zero at $s = 0$. Fried [Fr1] has determined the order of $R(s,\chi)$ at $s = 0$ and the leading coefficient of the Laurant expansion around $s = 0$. Let $b_k(\chi) := \dim H^k(X,E_\chi)$. Assume that $d = 2n + 1$. Put

$$h = 2\sum_{k=0}^{n}(n+1-k)(-1)^k b_k(\chi).$$

Then by [Fr1, Theorem 3], the order of $R(s,\chi)$ at $s = 0$ is h and the leading term of the Laurent expansion of $R(s,\chi)$ at $s = 0$ is

$$C(\chi)\cdot T^{RS}(X,\chi)^2 s^h, \tag{1.4}$$

where $C(\chi)$ is a constant that depends on the Betti numbers $b_k(\chi)$. In [Fr4, p. 66] Fried conjectured that (1.3) holds for all compact locally symmetric manifolds X and acyclic orthogonal bundles over $S(X)$. This conjecure was recently proved by Shu Shen [Shu].

Let χ be a unitary acyclic representation of Γ. Let $\tau(X,\chi)$ be the Reidemeister torsion [RS], [Mu3]. It is defined in terms of a smooth triangulation of X. However, it is independent of the particular C^∞-triangulation. Since χ is acyclic, $\tau(X,\chi)$ is a topological invariant, i.e., it does not depend on the metrics on X and in F_ρ. By [Ch], [Mu2] we have $T^{RS}(X,\chi) = \tau(X,\chi)$. Assume that d is odd. Then (1.3) can be restated as

$$R(0,\chi) = \tau(X,\chi)^2. \tag{1.5}$$

This provides an interesting relation between the length spectrum of closed geodesics and a secondary topological invariant.

Another class of interesting representations arises in the following way. Let $G := \mathrm{SO}_0(d,1)$. Let ρ be a finite dimensional complex or real representation of G. Then $\rho|_\Gamma$ is a finite dimensional representation of Γ. In general, $\rho|_\Gamma$ is not an orthogonal representation. However, the flat vector bundle F_ρ associated with $\rho|_\Gamma$ can be equipped with a canonical fibre metric which allows the use of methods of harmonic analysis to study the Laplace operators $\Delta_{k,\rho}$. Put

$$R(s,\rho) := R(s,\rho|_\Gamma).$$

The behavior of $R(s,\rho)$ at $s=0$ has been studied by Wotzke [Wo]. Let $\theta\colon G\to G$ be the Cartan involution of G with respect to $K = \mathrm{SO}(d)$. Let $\rho_\theta := \rho\circ\theta$. Also denote by $T^{RS}(X,\rho)$ the analytic torsion of X with respect to $\rho|_\Gamma$ and an admissible metric in F_ρ. Assume that $\rho\not\cong\rho_\theta$. Then Wotzke [Wo] has proved that $R(s,\rho)$ is holomorphic at $s=0$ and

$$|R(0,\rho)| = T^{RS}(X,\rho)^2. \tag{1.6}$$

If $\rho\cong\rho_\theta$, then $R(s,\rho)$ may have a zero or a pole at $s=0$. Wotzke [Wo] has also determined the order of $R(s,\rho)$ at $s=0$ and the coefficient of the leading term of the Laurent expansion of $R(s,\rho)$ at $s=0$. As in (1.4) the main contribution to the coefficient is the analytic torsion.

Let $\tau(X,\rho)$ be the Reidemeister torsion [Mu3] of X with respect to $\rho|_\Gamma$. If $\rho\not\cong\rho_\theta$, the cohomology $H^*(X,F_\rho)$ vanishes [BW, Chapt. VII, Theorem 6.7]. Then $\tau(X,\rho)$ is independent of the metrics on X and in F_ρ. By [Mu3, Theorem 1] we have $T^{RS}(X,\rho) = \tau(X,\rho)$. Thus (1.6) can be restated as

$$|R(0,\rho)| = \tau(X,\rho)^2. \tag{1.7}$$

This equality has interestig consequences for arithmetic subgroups Γ. Assume that there exists a Γ-invariant lattice $M_\rho\subset V_\rho$. Let $\mathcal{M}_\rho\to X$ be the associated local system of free $\mathbb{Z}$-modules of finite rank. The cohomology $H^*(X,\mathcal{M}_\rho)$ is a finitely generated abelian group. If ρ_Γ is acyclic, $H^*(X,\mathcal{M}_\rho)$ is a finite abelian group. Denote by $|H^k(X,\mathcal{M}_\rho)|$ the order of $H^k(X,\mathcal{M}_\rho)|$. By [Ch, (1.4)], [BV, Sect. 2.2], $\tau(X,\rho)$ can be expressed in terms of $|H^k(X,\mathcal{M}_\rho)|$, $k=0,...,d$. Combined with (1.7) we get

$$|R(0,\rho)| = \prod_{k=0}^{d} |H^k(X,\mathcal{M}_\rho)|^{(-1)^{k+1}}. \tag{1.8}$$

This is another interesting realtion between the length spectrum of X and topological invariants of X.

For arithmetic subgroups $\Gamma \subset G$, representations of G with Γ-invariant lattices in the corresponding representation space exist. See [BV], [MaM].

The main purpose of this paper is to extend the above results about the behaviour of the Ruelle zeta function at $s=0$ to every finite dimensional representation χ of Γ. To this end we use a complex version $T^{\mathbb{C}}(X,\chi)$ of the analytic torsion, which was introduced by Cappell and Miller [CM]. It is defined in terms of the flat Laplacians $\Delta^\sharp_{k,\chi}$, $k=0,...,d$, which are obtained by coupling the Laplacian Δ_k on k-forms to the flat bundle F_χ (see section 2 for its definition). In general, the flat Laplacian $\Delta^\sharp_{k,\chi}$ is not self-adjoint. However, its principal symbol equals the principal symbol of a Laplace type operator. Therefore, it has good spectral properties which allows to carry over most of the results from the self-adjoint case. The Cappell-Miller torsion $T^{\mathbb{C}}(X,\chi)$ is defined as an element of the determinant line

$$T^{\mathbb{C}}(X,\chi) \in \det H^*(X,F_\chi) \otimes (\det H_*(X,F_\chi))^*.$$

For an acyclic representation $T^{\mathbb{C}}(X,\chi)$ is a complex number and

$$|T^{\mathbb{C}}(X,\chi)| = T^{RS}(X,\chi)^2, \tag{1.9}$$

where $T^{\mathbb{C}}(X,\chi)$ is the Ray-Singer analytic torsion with respect to any choice of a fibre metric in F_χ. Since χ is acyclic, $T^{RS}(X,\chi)$ is independent of the choice of the metric in F_χ.

Let V_0^k be the generalized eigenspace of $\Delta^\sharp_{k,\chi}$, $k=0,...,d$, with generalized eigenvalue 0. Let $d^{*,\sharp}_\chi$ be the coupling of the codifferential $d^*_\chi\colon \Lambda^*(X)\to\Lambda^*(X)$ to the flat bundle F_χ. Then $(V_0^*, d_\chi, d^{*,\sharp}_\chi)$ is a double complex in the sense of [CM, §6]. Let

$$T_0(X,\chi) \in \det H^*(X,F_\chi) \otimes (\det H_*(X,F_\chi))^*. \tag{1.10}$$

be its torsion [CM, §6]. We note that $T^{\mathbb{C}}(X,\chi)$ and $T_0(X,\chi)$ are both non-zero elements of the determinant line $\det H^*(X,F_\chi)\otimes(\det H_*(X,F_\chi))^*$. Hence there exists $\lambda\in\mathbb{C}$ with $T^{\mathbb{C}}(X,\chi)=\lambda T_0(X,\chi)$. Set

$$\frac{T^{\mathbb{C}}(X,\chi)}{T_0(X,\chi)} := \lambda.$$

Put

$$h_k := \dim V_0^k, \quad k=0,...,d. \tag{1.11}$$

Furthermore, let $d = 2n+1$ and put

$$h := \sum_{k=0}^{n} (d+1-2k)(-1)^k h_k \tag{1.12}$$

and

$$C(d,\chi) := \prod_{k=0}^{d-1} \prod_{p=k}^{d-1} (2(n-p))^{(-1)^k h_k}. \tag{1.13}$$

Then our main result is the following theorem.

Theorem 1.1. *Let χ be a finite dimensional complex representation of Γ. Let h be defined by* (1.12)*. Then the order of the singularity of $R(s,\chi)$ at $s=0$ is h and*

$$\lim_{s\to 0} s^{-h} R(s,\chi) = C(d,\chi) \cdot \frac{T^{\mathbb{C}}(X,\chi)}{T_0(X,\chi)}. \tag{1.14}$$

Now choose a triangulation of X. Let $\tau_{\mathrm{comb}}(X,\chi) \in \det H^*(X,F_\chi) \otimes (\det H_*(X,F_\chi))^*$ be the combinatorial torsion defined Cappell and Miller [CM, Sect. 9]. It is independent of the choice of the triangulation. By [CM, Theorem 10.1] we have $T^{\mathbb{C}}(X,\chi) = \tau_{\mathrm{comb}}(X,\chi)$. Thus we can restate Theorem 1.1 as

$$\lim_{s\to 0} s^{-h} R(s,\chi) = C(d,\chi) \cdot \frac{\tau_{\mathrm{comb}}(X,\chi)}{T_0(X,\chi)}. \tag{1.15}$$

If χ is acyclic, then $T^{\mathbb{C}}(X,\chi)$, $T_0(X,\chi)$ and $\tau_{\mathrm{comb}}(X,\chi)$ are complex numbers and on the right hand side of (1.14) and (1.15) appear quotients of complex numbers.

Now we apply Theorem 1.1 to representations of Γ which are restrictions of representations of G. Then we get

Corollary 1.2. *Let $\rho \in \mathrm{Rep}(G)$ and assume that $\rho \not\cong \rho_\theta$. Then $R(s,\rho)$ is holomorphic at $s=0$ and*

$$R(0,\rho) = C(d,\rho) \cdot \frac{T^{\mathbb{C}}(X,\rho)}{T_0(X,\rho)}. \tag{1.16}$$

Using (1.6) and (1.9), it follows that

$$|T_0(X,\rho)| = C(d,\rho). \tag{1.17}$$

Let $d=3$. Then $\mathbb{H}^3 \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. For $m \in \mathbb{N}$ let $\rho_m\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(S^m(\mathbb{C}^2))$ be the m-th symmetric power of the standard representation of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$. For a compact, oriented hyperbolic 3-manifold $X = \Gamma\backslash\mathbb{H}^3$ and the representations ρ_m, $m \in \mathbb{N}$, Corollary 1.2 was proved by J. Park [Pa, (5.5)]. He also determined the constant $C(3,\rho_m)$ and $|T_0(X,\rho_m)|$. By [Pa, Prop. 5.1] we have

$$\begin{aligned} h_0 &= 1 \quad \text{and} \quad |T_0(X,\rho_m)| = 2, \quad \text{if } m \text{ is even} \\ h_0 &= 0 \quad \text{and} \quad T_0(X,\rho_m) = 1, \quad \text{if } m \text{ is odd.} \end{aligned} \tag{1.18}$$

Moreover $C(3,\rho_m)=(-4)^{h_0}$. The order h of $R(s,\rho_m)$ at $s=0$ is zero. Thus by (1.12) we have $h_1=2h_0$. Note that ρ_m is acyclic. Let Δ_{k,ρ_m} be the usual Laplacian in $\Lambda^k(X,F_{\rho_m})$ with respect to the admissible metric in F_{ρ_m}. Then for $m\in\mathbb{N}$ even we have

$$\ker\Delta_{k,\rho_m}=0,\quad \ker\Delta^\sharp_{k,\rho_m}\neq 0,\quad k=0,..., \tag{1.19}$$

which shows that for acyclic representations χ, in general, the flat Laplacian $\Delta^\sharp_{k,\chi}$ need not be invertible.

Again we can replace $T^{\mathbb{C}}(X,\rho)$ in (1.16) by the combinatorial torsion $\tau_{\mathrm{comb}}(X,\rho)$. However, in the present case we can replace the combinatorial torsion by the complex Reidemeister torsion. Since G is a connected semisimple Lie group and ρ a representation of G, it follows from [Mu3, Lemma 4.3] that ρ is actually a representation in $\mathrm{SL}(n,\mathbb{C})$. This implies that the complex Reidemeister torsion $\tau^{\mathbb{C}}(X,\rho)\in\mathbb{C}^*/\{\pm1\}$ can be defined as the usual Reidemeister torsion $\tau(X,\rho)$ [RS, Definition 1.1], where the absolute value of the determinant is deleted. In particular, we have

$$|\tau^{\mathbb{C}}(X,\rho)|=\tau(X,\rho). \tag{1.20}$$

Using (1.16) we get

$$R(0,\rho)=C(d,\rho)\cdot\frac{\tau^{\mathbb{C}}(X,\rho)^2}{T_0(X,\rho)}. \tag{1.21}$$

Another case to which Theorem 1.1 can be applied are deformations of unitary acyclic representations. Let $\mathrm{Rep}(\Gamma,\mathbb{C}^n)$ be the set of all n-dimensional complex representations of Γ equipped with usual topology. Let $\mathrm{Rep}_0^u(\Gamma,\mathbb{C}^n)\subset\mathrm{Rep}(\Gamma,\mathbb{C}^n)$ be the subset of all unitary acyclic representations (see section 6.2). By [FN, Theorem 1.1], we have $\mathrm{Rep}_0^u(\Gamma,\mathbb{C}^n)\neq\emptyset$. There exists a neighborhood V of $\mathrm{Rep}_0^u(\Gamma,\mathbb{C}^n)$ in $\mathrm{Rep}(\Gamma,\mathbb{C}^n)$ such that $\Delta^\sharp_\chi$ is invertible for all $\chi\in V$. Using Theorem 1.1, we get

Proposition 1.3. *Let $\chi\in V$. Then $R(s,\chi)$ is regular at $s=0$ and*

$$R(0,\chi)=T^{\mathbb{C}}(X,\chi).$$

This proposition was first proved by P. Spilioti [Sp3] using the odd signature operator [BK1]. She also discussed the relation with the refined analytic torsion.

2. Coupling differential operators to a flat bundle

We recall a construction of the flat extension of a differential operator introduced in [CM]. Let X be a smooth manifold and E_1 and E_2 complex vector bundles over X. Let

$$D\colon C^\infty(X,E_1)\to C^\infty(X,E_2)$$

be a differential operator. Let $F\to X$ be a flat vector bundle. Then there is a canonically operator

$$D^\sharp_F\colon C^\infty(X,E_1\otimes F)\to C^\infty(X,E_2\otimes F)$$

associated to D, which is defined as follows. Let $U \subset X$ be an open subset such that $F|_U$ is trivial. Let $s_1, \dots, s_k \in C^\infty(U, F|_U)$ be a local frame field of flat sections. Every section φ of $(E_1 \otimes F)|_U$ can be written as

$$\varphi = \sum_{i=1}^{k} \psi_i \otimes s_i$$

for some sections $\psi_1, ..., \psi_k \in C^\infty(U, E_1|U)$. Then define

$$D_F^\sharp|_U \colon C^\infty(U, (E_1 \otimes F)|_U) \to C^\infty(U, (E_2 \otimes F)|_U)$$

by

$$(D_F^\sharp|_U)(\varphi) := \sum_{i=1}^{k} D(\psi_i) \otimes s_i.$$

Let $s'_1, ..., s'_k$ be another local frame field of flat sections of $F|_U$. Then $s_i = \sum_{j=1}^{k} f_{ij} s'_j$, $i = 1, ..., k$, with $f_{ij} \in C^\infty(U)$, and it follows that the transition functions f_{ij} are constant. Since D is linear, $(D_F^\sharp|_U)(\varphi)$ is independent of the choice of the local frame field of flat sections and therefore, $D_F^\sharp$ is globally well defined. Let $\sigma(D)$ be the principal symbol of D. Then the principal symbol $\sigma(D_F^\sharp)$ of $D_F^\sharp$ is given by $\sigma(D_F^\sharp) = \sigma(D) \otimes \mathrm{Id}_F$. Thus if D is elliptic, then $D_F^\sharp$ is also an elliptic differential operator.

As an example consider a Riemannian manifold X and the Laplace operator Δ_p on p-forms. Let F be a flat bundle over X. Denote by $\Lambda^p(X, F)$ the space of smooth F-valued p-forms, i.e., $\Lambda^p(X, F) = C^\infty(X, \Lambda^p T^*(X) \otimes F)$. By the construction above we obtain the flat Laplacian $\Delta_{p,F}^\sharp \colon \Lambda^p(X, F) \to \Lambda^p(X, F)$. If the flat bundle is fixed, we will denote the flat Laplacian simply by $\Delta_p^\sharp$. The flat Laplacian can be also described as the usual Laplacian. Let $d_F \colon \Lambda^{p-1}(X, F) \to \Lambda^p(X, F)$ be the exterior derivative defined as above. Let $\star \colon \Lambda^p(X) \to \Lambda^{n-p}(X)$ denote the Hodge $\star$-operator. Then the flat extension

$$d_F^{*;\sharp} \colon \Lambda^p(X, F) \to \Lambda^{p-1}(X, F)$$

of the co-differential d^* is given by

$$d_F^{*,\sharp} = (-1)^{np+n+1} (\star \otimes \mathrm{Id}_F) \circ d_F \circ (\star \otimes \mathrm{Id}_F).$$

Then $d_F^{*,\sharp}$ satisfies $d_F^{*;\sharp} \circ d_F^{*,\sharp} = 0$ and we have

$$\Delta_F^\sharp = (d_F + d_F^{*;\sharp})^2.$$

If we choose a Hermitian fibre metric on F, we can define the usual Laplace operator Δ_F in $\Lambda^p(X, F)$, which is defined by

$$\Delta_F = (d_F + d_F^*)^2 = d_F d_F^* + d_F^* d_F,$$

which is formally self-adjoint. Now note that $d_F^{*,\sharp} = d_F^* + B$, where B is a smooth homomorphism of vector bundles. Thus it follows that $\Delta_F^\sharp = \Delta_F + (B d_F + d_F B)$. Thus the principal symbol $\sigma(\Delta_F^\sharp)(x,\xi)$ of $\Delta_F^\sharp$ is given by

$$\sigma(\Delta_F^\sharp)(x,\xi)=\|\xi\|_x^2\,\mathrm{Id}_{\Lambda^p T_x^*(X)\otimes F_x},\quad x\in X,\ \xi\in F_x. \tag{2.1}$$

More generally, let $E\to X$ be a Hermitian vector bundle over X. Let ∇ be a covariant derivative in E which is compatible with the Hermitian metric. We denote by $C^\infty(X,E)$ the space of smooth sections of E. Let

$$\Delta_E=\nabla^*\nabla$$

be the Bochner-Laplace operator associated to the connection ∇ and the Hermitian fiber metric. Then Δ_E is a second order elliptic differential operator. Its leading symbol $\sigma(\Delta_E)\colon \pi^*E\to\pi^*E$, where π is the projection of T^*X, is given by

$$\sigma(\Delta_E)(x,\xi)=\parallel \xi\parallel_x^2\cdot\mathrm{Id}_{E_x},\quad x\in X,\ \xi\in T_x^*X. \tag{2.2}$$

Let $F\to X$ be a flat vector bundle and

$$\Delta_{E\otimes F}^\sharp\colon C^\infty(X,E\otimes F)\to C^\infty(X,E\otimes F)$$

the coupling of Δ_E to F. Then the principal symbol of $\Delta_{E\otimes F}^\sharp$ is given by

$$\sigma(\Delta_{E\otimes F}^\sharp)(x,\xi)=\|\xi\|_x^2\cdot\mathrm{Id}_{E_x\otimes F_x}\,. \tag{2.3}$$

3. Regularized determinants and analytic torsion

Let Δ_E be as above. Let

$$P\colon C^\infty(X,E)\to C^\infty(X,E)$$

be an elliptic second order differential operator which is a perturbation of Δ_E by a first order differential operator, i.e.,

$$P=\Delta_E+D, \tag{3.4}$$

where $D\colon C^\infty(X,E)\to C^\infty(X,E)$ is a first oder differential operator. This implies that P is an elliptic second order differential operator with leading symbol $\sigma(P)(x,\xi)$ given by

$$\sigma(P)(x,\xi):=\|\xi\|_x^2\cdot\mathrm{Id}_{E_x}\,. \tag{3.5}$$

Though P is not self-adjoint in general, it still has nice spectral properties [Shb, Chapt. I, §8]. We recall the basic facts. For $I\subset[0,2\pi]$ let

$$\Lambda_I=\{re^{i\theta}\colon 0\le r<\infty,\ \theta\in I\}. \tag{3.6}$$

The following lemma describes the structure of the spectrum of P.

Lemma 3.1. *For every $0<\varepsilon<\pi/2$ there exists $R>0$ such that the spectrum of P is contained in the set $B_R(0)\cup\Lambda_{[-\varepsilon,\varepsilon]}$. Moreover the spectrum of P is discrete.*

Proof. The first statement follows from [Shb, Theorem 9.3]. The discreteness of the spectrum follows from [Shb, Theorem 8.4]. ☐

For $\lambda \in \mathbb{C} \setminus \operatorname{spec}(P)$ let $R_\lambda(P) := (P - \lambda \operatorname{Id})^{-1}$ be the resolvent. Given $\lambda_0 \in \operatorname{spec}(P)$, let Γ_{λ_0} be a small circle around λ_0 which contains no other points of $\operatorname{spec}(P)$. Put

$$\Pi_{\lambda_0} = \frac{i}{2\pi} \int_{\Gamma_{\lambda_0}} R_\lambda(P)\, d\lambda. \tag{3.7}$$

Then Π_{λ_0} is the projection onto the *root subspace* V_{λ_0}. This is a finite-dimensional subspace of $C^\infty(X, E)$ which is invariant under P and there exists $N \in \mathbb{N}$ such that $(P - \lambda_0 \mathrm{I})^N V_{\lambda_0} = 0$. Furthermore, there is a closed complementary subspace V'_{λ_0} to V_{λ_0} in $L^2(X, E)$ which is invariant under the closure $\bar{P}$ of P in L^2 and the restriction of $(\bar{P} - \lambda_0 \mathrm{I})$ to V'_{λ_0} has a bounded inverse. The *algebraic multiplicity* $m(\lambda_0)$ of λ_0 is defined as

$$m(\lambda_0) := \dim V_{\lambda_0}.$$

Moreover $L^2(X, E)$ is the closure of the algebraic direct sum of finite-dimensional P-invariant subspaces V_k

$$L^2(X, E) = \overline{\bigoplus_{k \geq 1} V_k} \tag{3.8}$$

such that the restriction of P to V_k has a unique eigenvalue λ_k, for each k there exists $N_k \in \mathbb{N}$ such that $(P - \lambda_k \mathrm{I})^{N_k} V_k = 0$, and $|\lambda_k| \to \infty$. In general, the sum (3.8) is not a sum of mutually orthogonal subspaces. See [Mu1, Sect. 2] for details.

Recall that an angle $\theta \in [0, 2\pi)$ is called an *Agmon angle* for P, if there exists $\varepsilon > 0$ such that

$$\operatorname{spec}(P) \cap \Lambda_{[\theta - \varepsilon, \theta + \varepsilon]} = \emptyset. \tag{3.9}$$

By Lemma 3.1 it is clear that an Agmon angle always exists for P. Assume that P is invertible. Choose an Agmon angle for P. Define the complex power P_θ^{-s}, $s \in \mathbb{C}$, as in [Shb, §10]. For $\operatorname{Re}(s) > n/2$, the complex power P_θ^{-s} is a trace class operator and the zeta function $\zeta_\theta(s, P)$ of P is defined by

$$\zeta_\theta(s, P) := \operatorname{Tr}(P_\theta^{-s}), \quad \operatorname{Re}(s) > \frac{n}{2}. \tag{3.10}$$

The zeta function admits a meromorphic extension to the entire complex plane which is holomorphic at $s = 0$ [Shb, Theorem 13.1]. Let $R_\theta := \{\rho e^{i\theta} \colon \rho \in \mathbb{R}^+\}$. Denote by $\log_\theta(\lambda)$ the branch of the logarithm in $\mathbb{C} \setminus R_\theta$ with $\theta < \operatorname{Im} \log_\theta < \theta + 2\pi$. We enumerate the eigenvalues of P such that

$$\operatorname{Re}(\lambda_1) \leq \operatorname{Re}(\lambda_2) \leq \cdots \leq \operatorname{Re}(\lambda_k) \leq \cdots.$$

By Lidskii's theorem [GK, Theorem 8.4] if follows that for $\operatorname{Re}(s) > n/2$ we have

$$\zeta_\theta(s, P) = \operatorname{Tr}(P_\theta^{-s}) = \sum_{k=1}^\infty m(\lambda_k) (\lambda_k)_\theta^{-s}, \tag{3.11}$$

where $(\lambda_k)_\theta^{-s} = e^{-s\log_\theta(\lambda_k)}$. We will need a different description of the zeta functionThe zeta function in terms of the heat operator e^{-tP}, which can be defined using the functional calculus developed in [Mu1, Sect. 2] by

$$e^{-tP} := \frac{i}{2\pi}\int_\Gamma e^{-t\lambda^2}(P^{1/2}-\lambda)^{-1}d\lambda, \tag{3.12}$$

where $\Gamma \subset \mathbb{C}$ is the same contour as in [Mu1, (2.18)]. As in [Mu1, Lemma 2.4] one can show that e^{-tP} is an integral operator with a smooth kernel. By [Mu1, Prop. 2.5] it follows that e^{-tP} is a trace class operator. Using Lidskii's theorem as above we get

$$\operatorname{Tr}(e^{-tP}) = \sum_{k=1}^\infty m(\lambda_k)e^{-t\lambda_k}. \tag{3.13}$$

The absolute convergence of the right hand side follows from Weyl's law [Mu1, Lemma 2.2]. Assume that there exists $\delta > 0$ such that $\operatorname{Re}(\lambda_k) \geq \delta$ for all $k \in \mathbb{N}$. Then by (3.13) and Weyl's law it follows that there exist $C, c > 0$ such that

$$|\operatorname{Tr}(e^{-tP})| \leq Ce^{-ct} \tag{3.14}$$

for $t \geq 1$. Since $\operatorname{spec}(P)$ is contained in the half plane $\operatorname{Re}(s) > 0$, we can choose the Agmon angle as $\theta = \pi$. Using the asymptotic expansion of $\operatorname{Tr}(e^{-tP})$ as $t \to 0$, it follows from (3.11) and (3.14) that

$$\zeta(s,P) = \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}(e^{-tP})t^{s-1}dt$$

for $\operatorname{Re}(s) > n/2$.

Then the regularized determinant of P is defined by

$$\det_\theta(P) := \exp\left(-\frac{d}{ds}\zeta_\theta(s,P)\Big|_{s=0}\right). \tag{3.15}$$

As shown in [BK1, 3.10], $\det_\theta(P)$ is independent of θ. Therefore we will denote the regularized determinant simply by $\det(P)$.

Assume that the vector bundle E is $\mathbb{Z}/2\mathbb{Z}$-graded, i.e., $E = E^+ \oplus E^-$ and P preserves the grading, i.e., assume that with respect to the decomposition

$$C^\infty(Y,E) = C^+(Y,E^+) \oplus C^\infty(Y,E^-)$$

P takes the form

$$P = \begin{pmatrix} P^+ & 0 \\ 0 & P^- \end{pmatrix}.$$

Then we define the graded determinant $\det_{\mathrm{gr}}(P)$ of P by

$$\det_{\mathrm{gr}}(P) = \frac{\det(P^+)}{\det(P^-)}. \tag{3.16}$$

Next we introduce the analytic torsion defined in terms of the non-selfadjoint operators $\Delta^\sharp_{p,\chi}$. We use the definition given in [CM, section 8]. Recall that the principal symbol of $\Delta^\sharp_{p,\chi}$ is given by (2.3). Therefore, $\Delta^\sharp_{p,\chi}$ satisfies the assumptions of section 3.

Let $r>0$ be such that $\operatorname{Re}(\lambda)\neq r$ for all generalized eigenvalues λ of $\Delta^\sharp_{p,\chi}$. Let $\Pi_{p,r}$ be the spectral projection on the span of the generalized eigenvectors with eigenvalues with real part less than r. Let $\Delta^\sharp_{p,\chi,r}:=(1-\Pi_{p,r})\Delta^\sharp_{p,\chi}$. Let $S(p,\chi,r)$ be the set of all nonzero generalized eigenvalues with real part less than r. Furthermore, let V^p_0 be the generalized eigenspace of $\Delta^\sharp_{p,\chi}$ with generalized eigenvalues 0. Then $(V^*_0,d,d^{*,\sharp})$ is double complex in the sense of [CM]. Let

$$T_0(X,\chi)\in(\det H^*(X,F_\chi))\otimes(\det H_*(X,F_\chi))^* \tag{3.17}$$

be the torsion of the double complex. Then the Cappell-Miller torsion is defined by

$$T^{\mathbb{C}}(X,\chi):=\prod_{p=1}^d\det(\Delta^\sharp_{p,\chi,r})^{(-1)^{p+1}p}\cdot\prod_{p=1}^d\Big(\prod_{\lambda\in S(p,\chi,r)}\lambda^{m(\lambda)}\Big)^{(-1)^{p+1}p}\cdot T_0(X,\chi), \tag{3.18}$$

where $m(\lambda)$ denotes the algebraic multiplicity of λ. Let $\Pi_{k,0}$ be the spectral projection on the generalized eigenspace of $\Delta^\sharp_{k,\chi}$ with generalized eigenvalue 0. Let

$$(\Delta^\sharp_{k,\chi})':=(\operatorname{Id}-\Pi_{k,0})\Delta^\sharp_{k,\chi}.$$

If we choose an Agmon angle we can also write

$$T^{\mathbb{C}}(X,\chi)=\prod_{k=1}^d\left[\det(\Delta^\sharp_{k,\chi})'\right]^{(-1)^{k+1}k}\cdot T_0(X,\chi). \tag{3.19}$$

If χ is acyclic, i.e., $H^*(X,E_\chi)=0$, then $T_0(X,\chi)$ and $T^{\mathbb{C}}(X,\chi)$ are complex numbers.

4. Twisted Ruelle and Selberg zeta functions

In this section we consider compact oriented hyperbolic manifolds of odd dimension $d=2n+1$ and we recall some basic facts about Ruelle and Selberg type zeta functions.

We need a more general class of Ruelle zeta functions than the one defined by (4.6). To begin with we fix some notation. Let $G=\mathrm{SO}_0(d,1)$ and $K=\mathrm{SO}(d)$. Then G/K equipped with the normalized invariant metric is isometric to the d-dimensional hyperbolic space $\mathbb{H}^d$. Let $G=KAN$ be the standard Iwasawa decompositon. Let M be the centralizer of A in K. Then $M\cong\mathrm{SO}(d-1)$. Denote by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{n}$, and $\mathfrak{a}$ the Lie algebras of G, K, M, N, and A, respectively. Let $W(A)\cong\mathbb{Z}/2\mathbb{Z}$ be the Weyl group of $(\mathfrak{g},\mathfrak{a})$.

Let $\Gamma\subset G$ be a discrete, torsion free, cocompact subgroup. Then Γ acts fixed point free on $\mathbb{H}^d$. The quotient $X=\Gamma\backslash\mathbb{H}^n$ is a closed, oriented hyperbolic manifold and each such manifold is of this form. Given $\gamma\in\Gamma$, we denote by $[\gamma]$ the Γ-conjugacy class of γ. The set of all conjugacy classes of Γ will be denoted by $C(\Gamma)$. Let $\gamma \neq 1$. Then there exist $g \in G$, $m_\gamma \in M$, and $a_\gamma \in A^+$ such that

$$g\gamma g^{-1} = m_\gamma a_\gamma. \tag{4.1}$$

By [Wa, Lemma 6.6], a_γ depends only on γ and m_γ is determined up to conjugacy in M. By definition there exists $\ell(\gamma) > 0$ such that

$$a_\gamma = \exp\left(\ell(\gamma)H\right). \tag{4.2}$$

Then $\ell(\gamma)$ is the length of the unique closed geodesic in X that corresponds to the conjugacy class $[\gamma]$. An element $\gamma \in \Gamma - \{e\}$ is called primitive, if it can not be written as $\gamma = \gamma_0^k$ for some $\gamma_0 \in \Gamma$ and $k > 1$. For every $\gamma \in \Gamma - \{e\}$ there exist a unique primitive element $\gamma_0 \in \Gamma$ and $n_\Gamma(\gamma) \in \mathbb{N}$ such that $\gamma = \gamma_0^{n_\Gamma(\gamma)}$. We recall that for $R > 0$ we have

$$\#\left\{[\gamma] \in C(\Gamma) \colon \ell(\gamma) \leq R\right\} \ll e^{(n-1)R} \tag{4.3}$$

[BO, (1.31)]. We also need the following auxiliary lemma.

Lemma 4.1. *Let $\chi\colon \Gamma \to \operatorname{GL}(V)$ be a finite dimensional representation of Γ. There exist $C, c > 0$ such that*

$$|\operatorname{tr}(\chi(\gamma))| \leq Ce^{c\ell(\gamma)}, \quad \forall \gamma \in \Gamma - \{e\}. \tag{4.4}$$

For the proof see [Sp1, Lemma 3.3]. Let $\theta\colon \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution with respect to $\mathfrak{k}$. Let $\bar{\mathfrak{n}} = \theta\mathfrak{n}$ be the negative root space. Let $\chi\colon \Gamma \to \operatorname{GL}(V)$ be a finite dimensional complex representation. For $\sigma \in \widehat{M}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) \gg 0$ the twisted Selberg zeta function is defined by

$$Z(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e\\ [\gamma]\text{ prime}}} \prod_{k=0}^{\infty} \det\left(1 - \left(\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k\left(\operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}}\right)\right) e^{-(s+\|\rho\|)\ell(\gamma)}\right), \tag{4.5}$$

where $[\gamma]$ runs over the primitive Γ-conjugacy classes and $S^k\left(\operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}}\right)$ denotes the k-th symmetric power of the adjoint map $\operatorname{Ad}(m_\gamma a_\gamma)$ restricted to $\bar{\mathfrak{n}}$. It follows from (4.3) and (4.4) that there exists $C > 0$ such that the product converges absolutely and uniformly on campact subsets of the half-plane $\operatorname{Re}(s) > C$. See [Sp1, Prop 3.4]. In the same way the twisted Ruelle zeta function $R(s;\sigma,\chi)$ is defined by

$$R(s;\sigma,\chi) := \prod_{\substack{[\gamma]\neq e\\ [\gamma]\text{ prime}}} \det\left(1 - \left(\chi(\gamma) \otimes \sigma(m_\gamma)\right) e^{-(s+|\rho|)\ell(\gamma)}\right). \tag{4.6}$$

By [Sp1, Prop. 3.5] the product converges absolutely and uniformly in some half-plane $\operatorname{Re}(s) > C$. Furthermore, $Z(s;\sigma,\chi)$ and $R(s;\sigma,\chi)$ admit meromorphic extensions to the entire complex plane [Sp1] and satisfy functional equations [Sp2]. For unitary representations χ, these results were proved by Bunke and Olbrich [BO]. The main technical tool is the Selberg trace formula. For the extension to the non-unitary case the Selberg trace formula is replaced by a Selberg trace formula for non-unitary twists, developed in [Mu1]. The proofs are similar except that on has to deal with non-self-adjoint operators.

There are also expressions of the zeta functions in terms of determinants of certain elliptic operators. To explain the formulas we need to recall the definition of the relevant differential operators. Given $\tau \in \widehat{K}$, let $\widetilde{E}_\tau \to \widetilde{X}$ be the homogeneous vector bundle associated to τ and let $E_\tau := \Gamma\backslash\widetilde{E}_\tau$ be the corresponding locally homogeneous vector bundle over X. Denote by $C^\infty(X, E_\tau)$ the space of smooth sections of E_τ. There is a canonical isomorphism

$$C^\infty(X, E_\tau) \cong (C^\infty(\Gamma\backslash G) \otimes V_\tau)^K \tag{4.7}$$

[Mia, §1]. Let $\Omega \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the Casimir element and denote by R_Γ the right regular representation of G in $C^\infty(\Gamma\backslash G)$. Then $R_\Gamma(\Omega)$ acts on the right hand side of (4.7) and via this isomorphism, defines an operator in $C^\infty(X, E_\tau)$. We denote the operator induced by $-R_\Gamma(\Omega)$ by A_τ.

Denote by $\widetilde{\nabla}^\tau$ the canonical connection in $\widetilde{E}_\tau$ and let ∇^τ be the induced connection in E_τ. Let $\Delta_\tau := (\nabla^\tau)^*\nabla^\tau$ be the associated Bochner-Laplace operator acting in $C^\infty(X, E_\tau)$. Let $\Omega_K \in \mathcal{Z}(\mathfrak{k}_\mathbb{C})$ be the Casimir element of K. Assume that τ is irreducible. Let $\lambda_\tau := \tau(\Omega_K)$ denote the Casimir eigenvalue of τ. Then we have

$$A_\tau := \Delta_\tau - \lambda_\tau \operatorname{Id}. \tag{4.8}$$

[Mia, §1]. Thus A_τ is a formally self-adjoint second order elliptic differential operator. Let $F_\chi \to X$ be the flat vector bundle defined by χ. Let

$$A^\sharp_{\tau,\chi} \colon C^\infty(X, E_\tau \otimes F_\chi) \to C^\infty(X, E_\tau \otimes F_\chi)$$

be the coupling of A_τ to F_χ.

Denote by $R(K)$ and $R(M)$ the representation rings of K and M, respectively. Let $i \colon M \to K$ be the inclusion and $i^* \colon R(K) \to R(M)$ the induced map of the representation rings. The Weyl group $W(A)$ acts on $R(M)$ in the canonical way. Let $R^\pm(M)$ denote the ±1-eigenspaces of the non-trivial element $w \in W(A)$. Let $\sigma \in R(M)$. It follows from the proof of Proposition 1.1 in [BO] that there exist $m_\tau(\sigma) \in \{-1, 0, 1\}$, depending on $\tau \in \widehat{K}$, which are equal to zero except for finitely many $\tau \in \widetilde{K}$, such that

$$\sigma = \sum_{\tau\in\widetilde{K}} m_\tau(\sigma) i^*(\tau), \tag{4.9}$$

if $\sigma \in R^+(M)$, and

$$\sigma + w\sigma = \sum_{\tau\in\widetilde{K}} m_\tau(\sigma) i^*(\tau), \tag{4.10}$$

if $\sigma \neq w\sigma$. Let

$$E(\sigma) := \bigoplus_{\substack{\tau\in\widetilde{K} \\ m_\tau(\sigma)=\neq 0}} E_\tau. \tag{4.11}$$

Then $E(\sigma)$ has a grading

$$E(\sigma) = E^+(\sigma) \oplus E^-(\sigma)$$

defined by the sign of $m_\tau(\sigma)$. Let $\sigma \in \widehat{M}$. Denote by ν_σ the highest weight of σ. Let $\mathfrak{b}$ be the standard Cartan subalgebra of $\mathfrak{m}$ [MP1, Sect. 2]. Let $\rho_{\mathfrak{m}}$ be the half-sum of positive roots of $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$. Put

$$c(\sigma) := -\|\rho\|^2 - \|\rho_{\mathfrak{m}}\|^2 + \|\nu_\sigma + \rho_{\mathfrak{m}}\|^2. \tag{4.12}$$

We define the operator $A^\sharp_\chi(\sigma)$ acting in $C^\infty(X, E(\sigma) \otimes F_\chi)$ by

$$A^\sharp_\chi(\sigma) := \bigoplus_{\substack{\tau \in \widetilde{K} \\ m_\tau(\sigma) \neq 0}} A^\sharp_{\tau,\chi} + c(\sigma). \tag{4.13}$$

For $p \in \{0, ..., d-1\}$ let σ_p, be the standard representation of $M = \mathrm{SO}(d-1)$ on $\Lambda^p \mathbb{R}^{d-1} \otimes \mathbb{C}$. Put

$$A^\sharp_\chi(\sigma_p \otimes \sigma) := \bigoplus_{[\sigma'] \in \widehat{M}/W} \; \bigoplus_{i=1}^{[(\sigma_p \otimes \sigma) \colon \sigma']} A^\sharp_\chi(\sigma'), \tag{4.14}$$

Recall that $A^\sharp_\chi(\sigma_p \otimes \sigma)$ acts in the space of sections of a graded vector bundle. Then by [Sp2, Prop. 1.7] we have the following determinant formula.

Proposition 4.2. *For every $\sigma \in \widehat{M}$ one has*

$$\begin{aligned} R(s; \sigma, \chi) = \prod_{p=0}^{d-1} & \det\nolimits_{\mathrm{gr}} \left(A^\sharp_\chi(\sigma_p \otimes \sigma) + (s + n - p)^2 \right)^{(-1)^p} \\ & \cdot \exp\left(-\frac{2\pi(n+1)\dim(V_\chi)\dim(V_\sigma)\operatorname{vol}(X)}{\operatorname{vol}(S^d)} s \right), \end{aligned} \tag{4.15}$$

if σ is Weyl-invariant, and

$$\begin{aligned} R(s; \sigma, \chi) R(s; w\sigma, \chi) = \prod_{p=0}^{d-1} & \det\nolimits_{\mathrm{gr}} \left(A^\sharp_\chi(\sigma_p \otimes \sigma) + (s + n - p)^2 \right)^{(-1)^p} \\ & \cdot \exp\left(-\frac{4\pi(n+1)\dim(V_\chi)\dim(V_\sigma)\operatorname{vol}(X)}{\operatorname{vol}(S^d)} s \right), \end{aligned} \tag{4.16}$$

otherwise. Here $\operatorname{vol}(S^d)$ denotes the volume of the d-dimensional unit sphere.

For unitary χ this was proved in [BO, Prop. 4.6].

5. Proof of the main theorem

To prove Theorem 1.1 we apply Proposition 4.2 for the case $\sigma = 1$. Let $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$ be the standard Cartan subalgebra of $\mathfrak{g}$. Let $e_1, ..., e_{n+1} \in \mathfrak{h}^*_{\mathbb{C}}$ be the standard basis [MP1, Sect. 2]. Thus $e_2, ..., e_{n+1}$ is a basis of $\mathfrak{b}$. Then $\rho_{\mathfrak{m}} = \sum_{j=2}^{n+1} (n+1-j) e_j$ and

$$\nu_p = \begin{cases} e_2 + \cdots + e_{p+1}, & \text{if } p \leq n, \\ e_2 + \cdots + e_{2n+1-p}, & \text{if } p > n, \end{cases}$$

[Kn, Chap. IV, §7]. Moreover, $|\rho| = n$. An explicit computation shows that $c(\sigma_p) = -(n-p)^2$. Let λ_p be the p-th exterior power of the standard representation of $\mathrm{SO}(d)$. Then for $p = 0, ..., d-1$ we have $i^*(\lambda_p) = \sigma_p + \sigma_{p-1}$. Put $\tau_p := \sum_{k=0}^{p}(-1)^k\lambda_{p-k}$. Then it follows that $i^*(\tau_p) = \sigma_p$, $p = 0, ..., d-1$. Using (4.13) and (4.14), we obtain

$$A_\chi^\sharp(\sigma_p) + (n-p)^2 = \bigoplus_{k=0}^{p} A_{\lambda_k,\chi}^\sharp. \tag{5.1}$$

Now recall that $A_{\lambda_k,\chi}^\sharp$ is the coupling of A_{λ_k} to F_χ. Furthermore, with respect to the isomorphism (4.7), A_{λ_k} corresponds to the action of $-R_\Gamma(\Omega)$ on $(C^\infty(\Gamma\backslash G) \otimes \Lambda^k\mathbb{C}^d)$. By the Lemma of Kuga, this operator corresponds to the Laplacian Δ_k on $\Lambda^k(X)$. Let $\Delta_{k,\chi}^\sharp$ be the coupling of Δ_k to F_χ. Then by (5.1) we get

$$A_\chi^\sharp(\sigma_p) + (n-p)^2 = \bigoplus_{k=0}^{p} \Delta_{k,\chi}^\sharp. \tag{5.2}$$

Using (4.15) we obtain

$$\begin{aligned} R(s;\chi) &= \prod_{p=0}^{d-1} \det\nolimits_{\mathrm{gr}}(A_\chi^\sharp(\sigma_p) + (s+n-p)^2)^{(-1)^p} \\ &= \prod_{p=0}^{d-1}\prod_{k=0}^{p} \det(\Delta_{p-k,\chi}^\sharp + s(s+2(n-p)))^{(-1)^{p+k}} \\ &= \prod_{k=0}^{d-1}\prod_{p=k}^{d-1} \det(\Delta_{k,\chi}^\sharp + s(s+2(n-p)))^{(-1)^{k}}. \end{aligned} \tag{5.3}$$

Let h_k be the dimension of the generalized eigenspace of $\Delta_{k,\chi}^\sharp$ with eigenvalue zero.

Lemma 5.1. *We have $h_p = h_{d-p}$ for $p = 0, ..., d$.*

Proof. Let $\star\colon \Lambda^p(X, F_\chi) \to \Lambda^{d-p}(X, F_\chi)$ be the extension of the Hodge $\star$-star operator, which acts locally as $\star(\omega \otimes f) = (\star\omega) \otimes f$, where ω is a usual p-form and f a local section of F_χ. Since $\star\Delta_p = \Delta_{d-p}\star$, it follows from the definition of the Laplacians coupled to F_χ that $\star\Delta_{p,\chi}^\sharp = \Delta_{d-p,\chi}^\sharp\star$. It follows that for every $k \in \mathbb{N}$ we have $\star(\Delta_{p,\chi}^\sharp)^k = (\Delta_{d-p,\chi}^\sharp)^k\star$.This proves the lemma. ∎

Denote by h the order of the singularity of $R(s,\chi)$ at $s = 0$. Using (5.3) and Lemma 5.1 it follows that

$$h = \sum_{k=0}^{d-1}(d+1-k)(-1)^k h_k = \sum_{k=0}^{n}(d+1-2k)(-1)^k h_k. \tag{5.4}$$

Let $\Pi_{k,0}$ be the spectral projection on the generalize eigenspace of $\Delta_{k,\chi}^\sharp$ with eigenvalue 0. Let $(\Delta_{k,\chi}^\sharp)' := (\mathrm{Id} - \Pi_{k,0})\Delta_{k,\chi}^\sharp$. We note that for $s \in \mathbb{C}$, $|s| \ll 1$, there is a common

Agmon angle for the operator $(\Delta^\sharp_{k,\chi})'+s(s+2(n-p))$. Therefore, in order to study the limit of $\det((\Delta^\sharp_{k,\chi})'+s(s+2(n-p)))$ as $s\to 0$, we can use one and the same Agmon angle.

If $p\neq n$, we get

$$\begin{aligned}\lim_{s\to 0} s^{-h_k}\det(\Delta^\sharp_{k,\chi}+s(s+2(n-p))) &= \lim_{s\to 0}\Big[\det((\Delta^\sharp_{k,\chi})'+s(s+2(n-p)) \\ &\qquad\qquad \cdot\frac{(s(s+2(n-p)))^{h_k}}{s^{h_k}}\Big] \\ &= (2(n-p))^{h_k}\cdot\det((\Delta^\sharp_{k,\chi})').\end{aligned}\tag{5.5}$$

For $p=n$ we get a similar formula

$$\lim_{s\to 0} s^{-2h_k}\det(\Delta^\sharp_{k,\chi}+s^2)=\lim_{s\to 0} q\det((\Delta^\sharp_{k,\chi})'+s^2)=\det((\Delta^\sharp_{k,\chi})').\tag{5.6}$$

Let

$$C(d,\chi):=\prod_{k=0}^{d-1}\prod_{p=k}^{d-1}(2(n-p))^{(-1)^k h_k}.\tag{5.7}$$

Using (5.4), (5.5) and (5.6) we get

(5.8)

$$\begin{aligned}\lim_{s\to 0} s^{-h}R(s;\chi) &= \prod_{k=0}^{d-1}\prod_{\substack{p=k\\ p\neq n}}^{d-1}\lim_{s\to 0}\left[s^{-h_k}\det(\Delta^\sharp_{k,\chi}+s(s+2(n-p)))\right]^{(-1)^k} \\ &\qquad\cdot\prod_{k=0}^{n}\lim_{s\to 0}\left[s^{-2h_k}\det(\Delta^\sharp_{k,\chi}+s^2)\right]^{(-1)^k} \\ &= C(d,\chi)\cdot\prod_{k=0}^{d-1}\det((\Delta^\sharp_{k,\chi})')^{(d-k)(-1)^k}=C(d,\chi)\cdot\prod_{k=1}^{d}\det((\Delta^\sharp_{k,\chi})')^{k(-1)^{k+1}}.\end{aligned}$$

For the last equality we used that $\Delta^\sharp_{k,\chi}\cong\Delta^\sharp_{d-k,\chi}$. Let $T_0(X,\chi)$ be the torsion (3.17) of the double complex $(V_0^*,d,d^{*,\sharp}$ and $T^{\mathbb{C}}(X,\chi)$ the Cappell-Miller torsion defined by (3.18). We note that $T^{\mathbb{C}}(X,\chi)$ and $T_0(X,\chi)$ are both non-zero elements of the determinant line $\det H^*(X,F_\chi)\otimes(\det H_*(X,F_\chi))^*$. Hence there exists $\lambda\in\mathbb{C}$ with $T^{\mathbb{C}}(X,\chi)=\lambda T_0(X,\chi)$. Set

$$\frac{T^{\mathbb{C}}(X,\chi)}{T_0(X,\chi)}:=\lambda.$$

If we combine this convention with the definition of the Cappell-Miller torsion (3.19), then (5.8) implies Theorem 1.1.

6. Acyclic representations

In this section we assume that χ is acyclic. Then $T^{\mathbb{C}}(X,\chi)$, $T_0(X,\chi)$ and $\tau_{\mathrm{comb}}(X,\chi)$ are complex numbers and the right hand side of (1.14) is the quotient of the two complex numbers. Besides the Cappell-Miller torsion we need another version of a complex analytic torsion for arbitrary flat vector bundles F_χ. This is the *refined analytic torsion* $T^{ran}(X,\chi)\in \det(H^*(X,F_\chi)$ introduced by Braverman and Kappeler [BK2]. The definition is based on the consideration of the odd signature operator B_χ [BK1, 2.1]. It is defined as follows. Let

$$\alpha\colon \Lambda^*(X,F_\chi)\to\Lambda^*(X,F_\chi)$$

be the chirality operator defined by

$$\alpha(\omega):=i^{n+1}(-1)^{k(k+1)/2}\star\omega,\quad \omega\in\Lambda^k(M,F_\chi).$$

Let ∇_χ be the flat connection in F_χ. Then the odd signature operator is defined as

$$B_\chi:=\alpha\nabla_\chi+\nabla_\chi\alpha. \tag{6.1}$$

It leaves the even subspace $\Lambda^{\mathrm{ev}}(X,F_\chi)$ invariant. Let $B_{\mathrm{ev},\chi}$ be the restriction of B_χ to $\Lambda^{\mathrm{ev}}(X,F_\chi)$. Then $T^{\mathrm{ran}}(X,\chi)\in\det(H^*(X,F_\chi))$ is defined in terms of $B_{\mathrm{ev},\chi}$. If χ is acyclic, then $T^{\mathrm{ran}}(X,\chi)$ is a complex number. In [BK3], Braverman and Kappeler determined the relation between the Cappell-Miller torsion and the refined analytic torsion. Let $\eta(B)$ be the eta-invariant of $B_{\mathrm{ev},\chi}$. In general, B_χ is not self-adjoint and therefore, $\eta(B)$ is in general not real. Furthermore, let η_0 be the eta-invariant of the trivial line bundle. Then by Proposition 4.2 and Theorem 5.1 of [BK3] it follows that

$$T^{\mathbb{C}}(X,\chi)=\pm T^{\mathrm{ran}}(X,\chi)^2\cdot e^{-2\pi i(\eta(B)-\dim(\chi)\eta_0)}. \tag{6.2}$$

On the other hand, it follows from [BK2, Theorem 1.9] that

$$|T^{\mathrm{ran}}(X,\chi)|=T^{RS}(X,\chi)\cdot e^{\pi\,\mathrm{Im}(\eta(B))}. \tag{6.3}$$

Combining (6.2) and (6.3), we obtain (1.9).

6.1. **Restriction of repesentations of the underlying Lie group.**

The first case that we consider are representations which are restictions to Γ of representations of G.

Let $\rho\colon G\to\mathrm{GL}(V_\rho)$ be a finite dimensional real (resp. complex) representation of G. Denote by $F_\rho\to X$ the flat vector bundle associated to $\rho|_\Gamma$. Let $\widetilde{E}_\rho\to G/K$ be the homogeneous vector bundle associated to $\rho|_K$. By [MM, Part I, Prop. 3.3] there is a canonical isomorphism

$$F_\rho\cong\Gamma\backslash\widetilde{E}_\rho. \tag{6.4}$$

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. By [MM, Part I, Lemma 3.1], there exists an inner product $\langle\cdot,\cdot\rangle$ in V_ρ such that

(1) $\langle\tau(Y)u,v\rangle=-\langle u,\tau(Y)v\rangle$ for all $Y\in\mathfrak{k}$, $u,v\in V_\tau$
(2) $\langle\tau(Y)u,v\rangle=\langle u,\tau(Y)v\rangle$ for all $Y\in\mathfrak{p}$, $u,v\in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\rho|_K$ is unitary with respect to this inner product, it induces a metric in $\Gamma\backslash\widetilde{E}_\tau$ and by (6.4) also in F_ρ. Denote by $T^{RS}(X,\rho)$ the Ray-Singer anaytic torsion of (X,F_ρ) with respect to metric on X and the metric in F_ρ. Denote by $\theta\colon G\to G$ the Cartan involution. Let $\rho_\theta:=\rho\circ\theta$. Assume that $\rho\not\cong\rho_\theta$. Then $H^*(X,F_\rho)=0$, i.e., $\rho|_\Gamma$ is acyclic. In this case $T^{RS}(X,\rho)$ is independent of the metrics on X and in F_ρ [Mu3, Corollary 2.7]. Let $R^\rho(s):=R(s,\rho)R(s,\rho_\theta)$. Then by [Wo, Theorem 8.13] $R^\rho(s)$ is holomorphic at $s=0$ and

$$R^\rho(0)=T^{RS}(X,\rho)^4. \tag{6.5}$$

Furthermore, from the discussions in [Wo, Sect. 9.1] follows that both $R(s,\rho)$ and $R(s,\rho_\theta)$ are holomorphic at $s=0$ and $R(0,\rho_\theta)=\overline{R(0,\rho)}$. Thus it follows that

$$|R(0,\rho)|=T^{RS}(X,\rho)^2, \tag{6.6}$$

which is (6.6). Hence $R(s,\rho)$ is regular at $s=0$ and $R(0,\rho)\neq 0$. Applying Theorem 1.1 we obtain Corollary 1.2.

Next we briefly recall the definition of the Reidemeister torsion [RS]. We work with vector spaces over $\mathbb{C}$. Let V be $\mathbb{C}$-vector space of dimension m. Let $v=(v_1,\dots,v_m)$ and $w=(w_1,\dots,w_m)$ be two basis of V. Let $T=(t_{ij})$ be the matrix of the change of basis from v to W, i.e., $w_i=\sum_j t_{ij}v_j$. Put $[W/v]:=\det(T)$. Let

$$C^*\colon C^0\xrightarrow{\delta_0}C^1\xrightarrow{\delta_1}\cdots\xrightarrow{\delta_{n-2}}C^{n-1}\xrightarrow{\delta_{n-1}}C^n$$

be a cochain complex of finite dimensional complex vector spaces. Let $Z_q=\ker(\delta_q)$ and $B_q:=\operatorname{Im}(\delta_{q-1})\subset C^q$. Let c_q (resp. h_q) be a preferred base of C_q (resp. $H^q(C^*)$). Choose a basis b_q for B_q, $q=0,\dots,n$, and let $\tilde b_{q+1}$ be an independent set in C_q such that $\delta_q(\tilde b_{q+1})=b_{q+1}$, and let $\tilde h_q$ be an independent set in Z_q which represents the base h_q of $H^q(C^*)$. Then $(b_q,\tilde h_q,\tilde b_{q+1})$ is a basis of C^q and $[b_q,\tilde h_q,\tilde b_{q+1}/c_q]$ depends only on b_q,h_q and b_{q+1}. Therefore, we denote it by $[b_q,h_q,b_{q+1}/c_q]$. Then the complex Reidemeister torsion $\tau^{\mathbb{C}}(C_*)\in\mathbb{C}$ of the chain complex C_* is defined by

$$\tau^{\mathbb{C}}(C^*):=\prod_{q=0}^n[b_q,h_q,b_{q+1}/c_q]^{(-1)^q}. \tag{6.7}$$

Let K be a C^∞-triangulation of X and $\widetilde K$ the lift of K to a triangulation of the univeral covering $\mathbb{H}^d$ of X. Then $C^q(\widetilde K,\mathbb{C})$ is a module over the complex group algebra $\mathbb{C}[\Gamma]$. Now recall that ρ is the restriction of a representation of G. Since G is a connected semisimple Lie group, it follows from [Mu3, Lemma 4.3] that ρ is a representation of Γ in $\mathrm{SL}(N,\mathbb{C})$. Let

$$C^q(K,\rho):=C^q(\widetilde K,\mathbb{C})\otimes_{C[\Gamma]}\mathbb{C}^N$$

the twisted cochain group and

$$C^*(K,\rho)\colon 0\to C^0(K,\rho)\xrightarrow{\partial_\rho}C^1(K,\rho)\xrightarrow{\partial_\rho}\cdots\xrightarrow{\partial_\rho}C^d(K,\rho)\to 0$$

the corresponding cochain complex. Let $e_1, ..., e_{r_q}$ be a preferres basis of $C^q(\tilde{K}, \mathbb{C})$ as a $\mathbb{C}[\Gamma]$-module consisting of the duals of lifts of q-simpexes and let $v_1, ..., v_N$ be a basis of $\mathbb{C}_N$. Then $\{e_i \otimes v_j \colon i = 1, ..., r_q, \quad j = 1, ..., N\}$ is a preferres basis of $C^q(K, \rho)$. Now consider the complex-valued Reidemeister torsion $\tau^{\mathbb{C}}(C^*(K, \rho))$. Since ρ is a representation in $\mathrm{SL}(N, \mathbb{C})$, a different choice of the preferred basis $\{e_i\}$ leads at most a to sign change of $\tau^{\mathbb{C}}(X, \rho)$. If v' is a different basis of $\mathbb{C}^N$, then $\tau^{\mathbb{C}}(X, \rho)$ changes by $[v'/v]^{\chi(X)}$. Hence, if $\chi(X) = 0$, $\tau^{\mathbb{C}}(C^*(K, \rho))$ is well defined as an element of $\mathbb{C}^*/\{\pm 1\}$. It depends only on the choice of the basis h_q of $H^q(X, F_\rho))$. Since every two smooth triangulations of X admit a common subdivision, it follows from [Mi] that $\tau^{\mathbb{C}}(C^*(K, \rho))$ is independent of the smooth triangulation K. Put

$$\tau^{\mathbb{C}}(X, \rho) := \tau^{\mathbb{C}}(C^*(K, \rho)). \tag{6.8}$$

This is the complex-valued Reidemeister torsion of X and ρ. If $\rho \ncong \rho_\theta$, then $H^*(X, F_\rho) = 0$. Thus in this case $\tau^{\mathbb{C}}(X, \rho)$ is a combinatorial invariant. It follows from property A, satisfied by $\tau_{comb}(X, \rho)$ [CM, 6.2], that

$$\tau_{\mathrm{comb}}(X, \rho) = \tau^{\mathbb{C}}(X, \rho)^2. \tag{6.9}$$

This implies (1.21).

6.2. **Deformations of acyclic unitary representations.** Let $\mathrm{Rep}(\Gamma, \mathbb{C}^n)$ be the set of all n-dimensional complex representations of Γ. It is well known that $\mathrm{Rep}(\Gamma, \mathbb{C}^n)$ has a natural structure of a complex algebraic variety [BK1, 13.6]. Recall that $\chi \in \mathrm{Rep}(\Gamma, \mathbb{C}^n)$ is called acyclic, if $H^*(X, F_\chi) = 0$, where $F_\chi \to X$ is the flat vector bundle associated to χ. Denote by $\mathrm{Rep}_0(\Gamma, \mathbb{C}^n) \subset \mathrm{Rep}(\Gamma, \mathbb{C}^n)$ the subset of all acyclic representations. A representation $\chi \in \mathrm{Rep}(\Gamma, \mathbb{C}^n)$ is called unitary, if there exists a Hermitian scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ which is preserved by all maps $\chi(\gamma)$, $\gamma \in \Gamma$. Let $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^n) \subset \mathrm{Rep}_0(\Gamma, \mathbb{C}^n)$ be the subset of all unitary acyclic representations. By [FN, Theorem 1.1] we get

Proposition 6.1. *For every compact hyperbolic manifold* $\Gamma \backslash \mathbb{H}^d$, *we have* $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^n) \neq \emptyset$.

Now let $\chi \in \mathrm{Rep}_0^u(\Gamma, \mathbb{C}^n)$. For such a representation the flat Laplacian $\Delta^\sharp_{k,\chi}$ equals the usual Laplace operator $\Delta_{k,\chi}$ and $T^{\mathbb{C}}(X, \chi) = T^{RS}(X, \chi)^2$. Moreover, $h_k = 0$, $k = 0, ..., d$, which implies $h = 0$ and $T_0(X, \chi) = 1$. Thus $R(s, \chi)$ is regular at $s = 0$ and from Theorem (1.1) we recover Fried's result [Fr1]

$$R(0, \chi) = T^{RS}(X, \chi)^2. \tag{6.10}$$

We equip $\mathrm{Rep}(\Gamma, \mathbb{C}^n)$ with the topology obtained from its structure as complex algebraic variety. The complement of the singular set is a complex manifold. Let $W \subset \mathrm{Rep}(\Gamma, \mathbb{C}^n)$ be the connected component of $\mathrm{Rep}(\Gamma, \mathbb{C}^n)$ which contains $\mathrm{Rep}_0^u(\Gamma, \mathbb{C}^n)$. Let $\chi_0 \in W$ be a unitary acyclic representation and let E_0 be the associated flat vector bundle. By [GM, Prop. 4.5] every vector bundles E_χ, $\chi \in W$, is isomorphic to E_0. Thus the flat connection on E_χ, which is induced by the trivial connection on $\widetilde{X} \times \mathbb{C}^n$, corresponds to a flat connection ∇_χ on E_0. Now recall that

$$\Delta^\sharp_\chi = (d_\chi + d_\chi^{*,\sharp})^2,$$

where

$$d_\chi^{*,\sharp}\big|_{\Lambda^p(X,E_\chi)} = (-1)^p(\star\otimes\mathrm{Id})d_\chi(\star\otimes\mathrm{Id}).$$

Via the isomorphism $E_\chi \cong E_0$, the operator $d_\chi + d_\chi^{*,\sharp}$ corresponds to the operator

$$D_\chi^\sharp := \nabla_\chi + \nabla_\chi^{*,\sharp}\colon \Lambda^*(X,E_0)\to\Lambda^*(X,E_0),$$

where

$$\nabla_\chi^{*,\sharp} = (-1)^p(\star\otimes\mathrm{Id})\nabla_\chi(\star\otimes\mathrm{Id}).$$

Let ∇_0 be the unitary flat connection on E_0. Let $\mathcal{C}(E_0)$ denote the space of connections on E_0. Recall that $\mathcal{C}(E_0)$ can be identified with $\Lambda^1(X,\mathrm{End}(E_0))$ by associating to a connection $\nabla\in\mathcal{C}(E_0)$ the 1-form $\nabla-\nabla_0\in\Lambda^1(X,\mathrm{End}(E_0))$. We equipp $\mathcal{C}(E_0)$ with the C^0-topology defined by the sup-norm $\|\omega\|_{\sup} := \max_{x\in X}|\omega(x)|$, $\omega\in\Lambda^1(X,\mathrm{End}(E_0))$, where $|\cdot|$ denotes the natural norm on $\Lambda^1(T^*X)\otimes E_0$. Since E_0 is acyclic, $D_0 := \nabla_0+\nabla_0^*$ is invertible. If $\|\nabla_\chi-\nabla_0\|\ll 1$ it follows as in [BK1, Prop 6.8] that $D_\chi^\sharp$ is invertible and hence $\Delta_\chi^\sharp = (D_\chi^\sharp)^2$ is invertible too. Thus we get

Lemma 6.2. *There exists an open neighborhood $V\subset W$ of $\mathrm{Rep}_0^u(\Gamma,\mathbb{C}^n)$ such that $\Delta_\chi^\sharp$ is invertible for all $\chi\in V$.*

Let $\chi\in V$. Then we have $h_k = \dim(\Delta_{k,\chi}^\sharp) = 0$, $k=0,\dots,d$, and therefore the order h of $R(s,\chi)$ at $s=0$ vanishes. Also $C=1$ and $T_0(X,\chi)=1$. Thus by Theorem 1.1 we obtain Proposition 1.3.

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