ON THE SINGULARITIES OF RESIDUAL INTERTWINING OPERATORS

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ABSTRACT. Let G be a reductive algebraic group defined over $\mathbb{Q}$. Let P, P' be parabolic subgroups of G, defined over $\mathbb{Q}$, and let $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$. In this paper we study the intertwining operator $M_{P'|P}(t,\lambda)$, $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, acting in corresponding spaces of automorphic forms. One of the main results states that each matrix coefficient of $M_{P'|P}(t,\lambda)$ is a meromorphic function of order $\leq n+1$, where $n = \dim G$. Using this result, we further investigate the rank one intertwining operators, in particular, we study the distribution of their poles.

0. INTRODUCTION

The main purpose of this paper is to study the analytic properties of the intertwining operators arising in the theory of Eisenstein series [L1]. Our primary interest in this problem comes from the Arthur trace formula [A8]. The spectral side of the Arthur trace formula consists of a sum of distributions J_χ, which are indexed by a certain set of cuspidal data $\mathfrak{X}$, and by the main result of [A4], each distribution J_χ can be expressed in terms of generalized logarithmic derivatives of intertwining operators. This is the motivation for our investigation of the intertwining operators. Using the results of the present paper, we will try to tackle the problem of absolute convergence of the spectral side of the Arthur trace formula in a forthcoming paper.

To state our results, we have to introduce some notation. Let G be a reductive algebraic group defined over $\mathbb{Q}$, which for the introduction we take to be a connected semi-simple algebraic group. Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$. We denote by $G(\mathbb{A})$ the adèlized group, i.e., $G(\mathbb{A})$ is the restricted direct product of the groups $G(\mathbb{Q}_v)$ where v runs over all valuations of $\mathbb{Q}$. We fix a Levi component M_0 of a minimal parabolic subgroup P_0 of G, defined over $\mathbb{Q}$, and a maximal compact subgroup

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K of $G(\mathbb{A})$ which is admissible with respect to M_0 [A5]. Let P be a parabolic subgroup of G, defined over $\mathbb{Q}$, with unipotent radical N_P. Let M_P be the unique Levi component of P which contains M_0. We denote the split component of the center of M_P by A_P and its Lie algebra by $\mathfrak{a}_P$. Let $A^2(P)$ be the space of square integrable automorphic forms on

$$N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}).$$

This is the space of square integrable functions on this quotient which are K-finite and $Z(\mathfrak{g})$-finite, $Z(\mathfrak{g})$ being the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. For every $\phi \in A^2(P)$ and every $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ with $\mathrm{Re}(\lambda)$ in a certain chamber, the Eisenstein series $E(P,\phi,\lambda,x)$ is defined by the absolutely convergent series (1.5), and one of the main results of Langlands' theory is that $E(P,\phi,\lambda,x)$ can be continued analytically to a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$.

Let Q be another parabolic subgroup of G, defined over $\mathbb{Q}$, with Levi component M_Q, split component A_Q and corresponding Lie algebra $\mathfrak{a}_Q$. Let $W(\mathfrak{a}_P,\mathfrak{a}_Q)$ be the set of all linear isomorphisms from $\mathfrak{a}_P$ to $\mathfrak{a}_Q$ which are of the form $\mathrm{Ad}(g)$ for some $g \in G(\mathbb{Q})$. P and Q are said to be associated if $W(\mathfrak{a}_P,\mathfrak{a}_Q)$ is not empty. Suppose that this is the case. Then the theory of Eisenstein series associates to each $s \in W(\mathfrak{a}_P,\mathfrak{a}_Q)$ an intertwining operator

$$M_{Q|P}(s,\lambda): A^2(P) \to A^2(Q), \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}},$$

which for $\mathrm{Re}(\lambda)$ in a certain chamber can be defined by an absolutely convergent integral (see section 1). It is also part of Langlands' theory that each intertwining operator $M_{Q|P}(s,\lambda)$ admits an analytic continuation to a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. The intertwining operators play an important role in the harmonic analysis of the spaces $G(\mathbb{Q})\backslash G(\mathbb{A})$. In particular, generalized logarithmic derivatives of intertwining operators occur on the spectral side of the Arthur trace formula [A8].

Let $\Pi(K)$ be the set of equivalence classes of irreducible unitary representations of K. We fix $\sigma \in \Pi(K)$ and cuspidal data $\chi \in \mathfrak{X}$ (see section 1). This singles out a finite dimensional subspace $A^2(P,\chi,\sigma)$ of $A^2(P)$. The intertwining operator $M_{Q|P}(s,\lambda)$ maps $A^2(P,\chi,\sigma)$ to $A^2(Q,\chi,\sigma)$. We shall call a holomorphic function $f: \mathbb{C}^n \to \mathbb{C}^m$ to be an entire function of order $r \geq 0$, if there exist $C, c > 0$ such that

$$\| f(z) \| \leq C e^{c\|z\|^r}, \quad z \in \mathbb{C}^n.$$

Then our first main results are the following two theorems.

Theorem 0.1. *Let $n = \dim G(\mathbb{R})/K_\infty$. Let P, Q be associated parabolic subgroups and let $t \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$. Let $\chi \in \mathfrak{X}$ and $\sigma \in \Pi(K)$. Let $M_{Q|P}(t,\lambda)$ be the intertwining operator mapping the space $A^2(P,\chi,\sigma)$ to $A^2(Q,\chi,\sigma)$, $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. There exist an entire function $F(\lambda)$ of order $\leq n+2$ and an entire function $\widehat{M}_{Q|P}(t,\lambda)$ of order $\leq n+2$ with values in the linear operators from $A^2(P,\chi,\sigma)$ to $A^2(Q,\chi,\sigma)$ such that*

$$M_{Q|P}(t,\lambda) = \frac{\widehat{M}_{Q|P}(t,\lambda)}{F(\lambda)}, \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}.$$

Theorem 0.2. *Let $Q \supset P$ be parabolic subgroups. Let $\chi \in \mathfrak{X}$ and $\sigma \in \Pi(K)$. There exists an entire function $q(\lambda) \neq 0$ of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ of order $\leq n+2$ such that the following holds:*

1) *For each $\phi \in A^2(P,\chi,\sigma)$ and each $x \in G(\mathbb{A})$, the function defined by*

$$\widehat{E}^Q(P,\phi,\lambda,x) = q(\lambda)E^Q(P,\phi,\lambda,x)$$

is an entire function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ of order $\leq n+2$.

2) *For every compact subset W of*

$$N_Q(\mathbb{A})M_Q(\mathbb{Q})A_Q(\mathbb{R})^0\backslash G(\mathbb{A}),$$

there exist $C, c > 0$ such that

$$\sup_{x\in W} |\widehat{E}^Q(P,\phi,\lambda,x)| \leq C \parallel \phi \parallel e^{c\|\lambda\|^{n+2}}$$

for all $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ and all $\phi \in A^2(P,\chi,\sigma)$.

For SL(2), the statements of Theorem 0.1 and Theorem 0.2 were proved by Selberg [Se]. In [Mu1], we treated the case of the rank one cuspidal intertwining operators and rank one cuspidal Eisenstein series for general G. The proof given in [Mu1] is based on an extension of Colin de Verdiere's method of the analytic continuation of Eisenstein series for SL(2). For this method to work, the assumption that we are dealing with cuspidal automorphic forms, is essential and the proof cannot be extended to the noncuspidal case.

To prove Theorems 0.1 and 0.2 in general, we use Langlands' description of the residual spectrum in terms of iterated residues of cuspidal Eisenstein series. We procced in several steps. In section 2, we use results of [Mu1] and the factorization of the intertwining operator into a product of intertwining operators induced from rank one intertwining operators to show that each cuspidal intertwining operator is a meromorphic operator valued function of order $\leq n+2$. Then, in section 3, we use this result together with the inner product formula for truncated

Eisenstein series [L2] to establish Theorem 0.2 for all cuspidal Eisenstein series. Finally, in section 4, we use Langlands' description of the space $A^2(P,\chi,\sigma)$ in terms of iterared residues of cuspidal Eisenstein series to express the general intertwining operator and noncuspidal Eisenstein series by iterated residues of cuspidal Eisenstein series and intertwining operators. Combining everything, this leads to the proof of Theorems 0.1 and 0.2.

In section 5 we study the rank one intertwining operators more closely. Let $P = NM$ be a parabolic subgroup and suppose that the split component A_P has dimension one. This is a maximal parabolic subgroup. Then any parabolic subgroup Q which is associated to P is either conjugate to P or its opposite group $P^- = N^- M$. If P and P^- are conjugate, then we put

$$M(z) = M_{P|P}(-1, z\lambda_0), \quad z \in \mathbb{C},$$

where λ_0 is the normalized simple root of (P, A_P). Then $M(z)$ is an operator in $A^2(P,\chi,\sigma)$. In the opposite case, if P and P^- are not conjugate, we form a block matrix $M(z)$ from the intertwining operators $M_{P|P^-}(1, -z\lambda_0)$ and $M_{P^-|P}(1, z\lambda_0)$. The resulting operator acts then in $A^2(P,\chi,\sigma) \oplus A^2(P^-,\chi,\sigma)$. It follows from Theorem 0.1 that $\det M(z)$ is a meromorphic operator valued function of $z \in \mathbb{C}$ of order $\leq n+2$. By recent results of Franke [Fr], it follows that $\det M(z)$ has only finitely many poles in the half-plane $\mathrm{Re}(z) \geq 0$ and all these poles are real. Denote them by $u_1, ..., u_l$. Let m_j be the order of the pole u_j. Using these facts together with Hadamard's factorization theorem, we obtain the following product formula

$$\det M(z) = \det M(0) q^z \prod_{j=1}^{l} \left(\frac{z+u_j}{z-u_j} \right)^{m_j} \prod_{\eta} \frac{z+\overline{\eta}}{z-\eta}, \quad z \in \mathbb{C}, \tag{0.1}$$

where η runs over all poles, counted with their orders, of $\det M(z)$ in the half-plane $\mathrm{Re}(z) < 0$. For the cuspidal rank one intertwining operators, this formula was proved in [Mu1, Section 6]. As in [Mu1], formula (0.1) can be used to compute the logarithmic derivative of $\det M(z)$. The formula that we obtain in this way shows that $(d/dz)\log\det M(i\lambda)$, $\lambda \in \mathbb{R}$, is essentially of the form $\Sigma_- - \Sigma_+$ where $\Sigma_\mp$ is a sum over all poles of $\det M(z)$ in the half-plane $\mp \mathrm{Re}(z) > 0$. The point is that all terms in $\Sigma_\mp$ are positive and Σ_+ is a finite sum. In section 6 we estimate the number of poles of $\det M(z)$ in a circle of radius $R > 0$. The estimates obtained in this way are uniform in the Casimir eigenvalues of the Archimedean constituent σ_∞ of σ, and χ.

These facts will be important for the estimation of the generalized logarithmic derivatives of the intertwining operators occurring in the Arthur trace formula which we will consider in a forthcoming paper. Theorems 0.1 and 0.2 may also have applications to the study of automorphic L-functions. In [GS] and [Ra], for example, Theorems 0.1 and 0.2 have been established in special cases and were used to prove that certain automorphic L-functions are bounded in vertical strips.

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1. Preliminaries

We shall follow partially the notation introduced by Arthur [A1] -[A4].

1.1. Let G be a reductive algebraic group defined over $\mathbb{Q}$. As in [A4], [A5], we shall fix a subgroup M_0 of G, defined over $\mathbb{Q}$, which is a Levi component of some minimal parabolic subgroup of G defined over $\mathbb{Q}$. In this paper, a *parabolic subgroup* will mean a parabolic subgroup of G, defined over $\mathbb{Q}$, which contains M_0. A *Levi subgroup* of G will mean a subgroup of G which contains M_0 and is the Levi component of some parabolic subgroup of G. It is a reductive subgroup of G which is defined over $\mathbb{Q}$. If $M \subset L$ are Levi subgroups, we denote the set of Levi subgroups of L which contain M by $\mathcal{L}^L(M)$. Furthermore, let $\mathcal{F}^L(M)$ denote the set of parabolic subgroups of L defined over $\mathbb{Q}$ which contain M, and let $\mathcal{P}^L(M)$ be the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. If $L = G$, we shall denote these sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$. Suppose that $P \in \mathcal{F}^L(M)$. Then

$$P = N_P M_P,$$

where N_P is the unipotent radical of P and M_P is the unique Levi component of P which contains M.

Suppose that $M \subset M_1 \subset L$ are Levi subgroups of G. If $Q \in \mathcal{P}^L(M_1)$ and $R \in \mathcal{P}^{M_1}(M)$, there is a unique group $Q(R) \in \mathcal{P}^L(M)$ which is contained in Q and whose intersection with M_1 is R. Notice that $M_{Q(R)} = M_R$.

Let A_P be the split component of the center of M_P. A_P is defined over $\mathbb{Q}$. Let $X(M_P)_{\mathbb{Q}}$ be the group of characters of M_P defined over $\mathbb{Q}$. Then

$$\mathfrak{a}_P = \operatorname{Hom}(X(M_P)_{\mathbb{Q}}, \mathbb{R})$$

is a real vector space whose dimension equals that of A_P. Its dual space is

$$\mathfrak{a}_P^* = X(M_P)_{\mathbb{Q}} \otimes \mathbb{R}.$$

We shall often denote A_P, $\mathfrak{a}_P$ and $\mathfrak{a}_P^*$ by A_M, $\mathfrak{a}_M$ and $\mathfrak{a}_M^*$, respectively, since they depend only on M. Also, we shall write $A_0 = A_{M_0}$, $\mathfrak{a}_0 = \mathfrak{a}_{M_0}$ and $\mathfrak{a}_0^* = \mathfrak{a}_{M_0}^*$.
Let $P \in \mathcal{F}(M_0)$. We shall denote the roots of (P, A_P) by Φ_P, and the simple roots by Δ_P. They are elements in $X(A_P)_{\mathbb{Q}}$ and are canonically embedded in $\mathfrak{a}_P^*$.
Let P and Q be groups in $\mathcal{F}(M_0)$ with $P \subset Q$. Then there is a canonical surjection $\mathfrak{a}_P \to \mathfrak{a}_Q$ and a canonical injection $\mathfrak{a}_Q^* \hookrightarrow \mathfrak{a}_P^*$. The kernel of the first map will be denoted by $\mathfrak{a}_P^Q$. Then $\mathfrak{a}_P^Q$ is dual to $\mathfrak{a}_P^*/\mathfrak{a}_Q^*$.
The group $M_Q \cap P$ is a parabolic subgroup of M_Q with unipotent radical

$$N_P^Q = N_P \cap M_Q.$$

Let Δ_P^Q be the set of simple roots of $(M_Q \cap P, A_P)$. Then Δ_P^Q is a subset of Δ_P. We may identify $\mathfrak{a}_Q$ with the subspace

$$\{H \in \mathfrak{a}_P \mid \alpha(H) = 0,\ \alpha \in \Delta_P^Q\}.$$

Furthermore, to Δ_P^Q one can associate a basis $\{\alpha^\vee \mid \alpha \in \Delta_P^Q\}$ of $\mathfrak{a}_P^Q$ and $\hat{\Delta}_P^Q$ is defined to be the corresponding dual basis of $(\mathfrak{a}_P^Q)^*$ [A2]. Then Δ_P^Q and $\hat{\Delta}_P^Q$ are naturally embedded subsets of $\mathfrak{a}_0^*$. Let

$$\mathfrak{a}_P^+ = \{H \in \mathfrak{a}_P \mid \alpha(H) > 0 \quad \text{for all} \quad \alpha \in \Delta_P\},$$

and

$$(\mathfrak{a}_P^*)^+ = \{\Lambda \in \mathfrak{a}_P^* \mid \Lambda(\alpha^\vee) > 0 \quad \text{for all} \quad \alpha \in \Delta_P\}.$$

We shall denote the restricted Weyl group of (G, A_0) by W_0. It acts on $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ in the usual way. For every $s \in W_0$ we shall fix a representative w_s in the intersection of $G(\mathbb{Q})$ with the normalizer of A_0. w_s is determined modulo $M_0(\mathbb{Q})$. If P_1 and P_2 are parabolic subgroups, let $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ denote the set of distinct isomorphisms from $\mathfrak{a}_{P_1}$ onto $\mathfrak{a}_{P_2}$ obtained by restricting elements of W_0 to $\mathfrak{a}_{P_1}$. P_1 and P_2 are said to be *associated* if $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ is not empty.

1.2. We fix an embedding of G into Gl_n, defined over $\mathbb{Q}$. For a given place v of $\mathbb{Q}$, let $G(\mathbb{Q}_v)$ be the group of $\mathbb{Q}_v$-rational points of G. Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$ and let $G(\mathbb{A})$ be the corresponding adèles -valued group. If f stands for the set of finite places of $\mathbb{Q}$ and $\mathbb{A}_f$ is the corresponding ring of finite adèles, then

$$G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f).$$

For any prime p, let

$$G(\mathbb{Z}_p) = Gl_n(\mathbb{Z}_p) \cap G(\mathbb{Q}_p).$$

This is an open compact subgroup of $G(\mathbb{Q}_p)$. We shall fix a maximal compact subgroup

$$K = \prod_v K_v$$

of $G(\mathbb{A})$ which is admissible relative to M_0 in the sense of [A5]. For any such K the following properties hold:

1) For almost all primes p, one has $K_p = G(\mathbb{Z}_p)$.
2) For every finite p, K_p is a special maximal compact subgroup. This implies that $G(\mathbb{Q}_p) = P_0(\mathbb{Q}_p) \cdot K_p$ for all $P \in \mathcal{F}(M_0)$.
3) The Lie algebras of $K_{\mathbb{R}}$ and $A_0(\mathbb{R})$ are orthogonal with respect to the Killing form.

Given $P \in \mathcal{F}(M_0)$, let

$$M_P(\mathbb{A})^1 = \bigcap_{\chi \in X(M_P)_{\mathbb{Q}}} \ker(|\chi|).$$

This is a closed subgroup of $M_P(\mathbb{A})$, and $M_P(\mathbb{A})$ is the direct product of $M_P(\mathbb{A})^1$ and $A_P(\mathbb{R})^0$. By the assumptions on K, $G(\mathbb{A}) = P(\mathbb{A})K$. Therefore, any $x \in G(\mathbb{A})$ can be written as

$$x = namk,\ n \in N_P(\mathbb{A}),\ m \in M(\mathbb{A})^1,\ a \in A_P(\mathbb{R})^0,\ k \in K. \tag{1.1}$$

Let

$$H_P \colon G(\mathbb{A}) \to \mathfrak{a}_P$$

be the associated height function as defined in [A2]. Then $H_P(x)$ is the image of $a \in A_P(\mathbb{R})^0$ in the decomposition (1.1) with respect to the isomorphism $A_P(\mathbb{R})^0 \xrightarrow{\sim} \mathfrak{a}_P$.

We shall fix a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}_0$ which is invariant under the action of the Weyl group of (G, A_0). On each space $\mathfrak{a}_P^Q$, $P \subset Q$, we take as Haar measure the Euclidean measure associated to the restriction of $\|\cdot\|$ to $\mathfrak{a}_P^Q$. We then normalize the Haar measures on K, $G(\mathbb{A})$, $N_P(\mathbb{A})$, $M_P(\mathbb{A})$, $A_P(\mathbb{R})^0$, $M_P(\mathbb{A})^1$, etc. as in [A2].

1.3. By $\Pi(G(\mathbb{A}))$ (resp. $\Pi(M_P(\mathbb{A}))$, $\Pi(M_P(\mathbb{A})^1)$, $\Pi(K)$, etc.) we shall denote the set of equivalence classes of irreducible unitary representations of $G(\mathbb{A})$ (resp. $M_P(\mathbb{A})$, $M_P(\mathbb{A})^1$, K, etc.). We note that the representations $\Pi(M_P(\mathbb{A})^1)$ can be naturally identified with the orbits of $i\mathfrak{a}_P^*$ under the action of

$$\pi \to \pi_\lambda = e^{\lambda(H_P(\cdot))}\pi, \quad \pi \in \Pi(M_P(\mathbb{A})), \quad \lambda \in i\mathfrak{a}_P^*.$$

1.4. Let $P = NM$ be a parabolic subgroup and let ϕ be a measurable, locally integrable function on $N(\mathbb{Q})\backslash G(\mathbb{A})$. Then the constant term ϕ_P of ϕ along P is defined by

$$\phi_P(g) = \int_{\mathbb{N}(\mathbb{Q})\backslash N(\mathbb{A})} \phi(ng)\, dn. \tag{1.2}$$

This is a measurable, locally integrable function on $N(\mathbb{A})\backslash G(\mathbb{A})$.

1.5. Let P be a parabolic subgroup. Then we denote by $A^2(P)$ the space of automorphic forms on $N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A})$ which are square integrable on $M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1 \times K$. This is the space of functions

$$\phi\colon N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}) \to \mathbb{C}$$

which satisfy the following conditions:

i) For any $x \in G(\mathbb{A})$, the function $m \mapsto \phi(mx)$, $m \in M(\mathbb{A})$, is $Z(\mathfrak{m}_\mathbb{C})$-finite.

ii) The span of the set of functions $\phi_k(x) = \phi(xk)$, $x \in G(\mathbb{A})$, indexed by $k \in K$, is a finite dimensional space.

iii)

$$\| \phi \|^2 = \int_K \int_{M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1} |\phi(mk)|^2 \mathrm{dm}\,\mathrm{dk} < \infty.$$

Furthermore, an automorphic form $\phi \in A^2(P)$ is called cuspidal, if the following additional condition holds:

iv) For all standard parabolic subgroups $Q \subsetneqq P$, $\phi_Q = 0$.

The subspace of all cuspidal automorphic forms in $A^2(P)$ will be denoted by $A^2_{\text{cusp}}(P)$.

1.6. The space $A^2(P)$ has several natural decompositions into invariant subspaces. Let $L^2_{\text{disc}}(M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash M_P(\mathbb{A}))$ be the largest closed subspace of $L^2(M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash M_P(\mathbb{A}))$ on which the regular representation of $M_P(\mathbb{A})$ decomposes discretely. Then each $\phi \in A^2(P)$ has the property that for any $x \in G(\mathbb{A})$, the function

$$\phi_x(m) = \phi(mx), \quad m \in M_P(\mathbb{A}), \tag{1.3}$$

belongs to $L^2_{\text{disc}}(M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash M_P(\mathbb{A}))$. Let $\Pi^0(M_P(\mathbb{A}))$ be the set of all $\pi \in \Pi(M_P(\mathbb{A}))$ which are trivial on $A_P(\mathbb{R})^0$. For any $\pi \in \Pi^0(M_P(\mathbb{A}))$, let $L^2(M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash M_P(\mathbb{A}))(\pi)$ be the isotypical component of π, and let $A^2(P)(\pi)$ be the subspace of all $\phi \in A^2(P)$ such that for each $x \in G(\mathbb{A})$, the function ϕ_x, defined by (1.3), belongs to $L^2(M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash M_P(\mathbb{A}))(\pi)$. Then we get the following direct sum decomposition

$$A^2(P) = \bigoplus_{\pi\in\Pi^0(M_P(\mathbb{A}))} A^2(P)(\pi).$$

Another decomposition can be described in terms of the cuspidal data which means that one keeps track of the source of an iterated residue of a cuspidal Eisenstein series. In [A2], Arthur introduced a decomposition

$$L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1) = \bigoplus_{\chi\in\mathfrak{X}} L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)_\chi$$

indexed by a certain set $\mathfrak{X}$. We recall that $\mathfrak{X}$ is the set of W_0 conjugacy classes of pairs (M_B, r_B), where B is a parabolic subgroup and r_B is an irreducible cuspidal automorphic representation of $M_B(\mathbb{A})^1$. Given $\chi \in \mathfrak{X}$, let $A^2(P,\chi)$ be the closed subspace of $A^2(P)$ consisting of those ϕ such that for all $x \in G(\mathbb{A})$, the function ϕ_x, defined by (1.3), belongs to $L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)_\chi$. Then

$$A^2(P) = \bigoplus_{\chi\in\mathfrak{X}} A^2(P,\chi).$$

For our purpose it is important to single out a finite dimensional subspace of automorphic forms. To do so we fix a finite dimensional unitary representation σ of K on a complex vector space V. Then we denote by $A^2(P,\chi,\sigma)$ the subspace of all $\phi \in A^2(P,\chi)$ such that for all $x \in G(\mathbb{A})$, the function

$$\phi_x(k) = \phi(xk), \quad k \in K,$$

belongs to the isotypical subspace $L^2(K)(\sigma)$ of σ in $L^2(K)$. Then $A^2(P,\chi,\sigma)$ is a finite dimensional subspace of $A^2(P,\chi)$.

To any $\chi \in \mathfrak{X}$, there corresponds a class $\mathcal{P}_\chi$ of associated parabolic subgroups. Suppose that $P \in \mathcal{P}_\chi$. Then $L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)_\chi$ is a subspace of the space of cusp forms $L^2_{\text{cusp}}(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)$. It is known that

$$L^2_{\text{cusp}}(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1) = \bigoplus_\rho V_\rho,$$

where ρ ranges over all irreducible unitary representations of $M_P(\mathbb{A})^1$ and each V_ρ is an $M_P(\mathbb{A})^1$- invariant subspace of $L^2_{\text{cusp}}(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)$, isomorphic under the action of $M_P(\mathbb{A})^1$ to a finite number of copies of ρ. Then

$$L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)_\chi = L^2_{\text{cusp}}(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)_\chi = \bigoplus_{\{\rho \mid (M_P,\rho)\in\chi\}} V_\rho.$$

Therefore, if $P \in \mathcal{P}_\chi$ then $A^2(P,\chi)$ consists only of cusp functions. So, let $A^2_{\text{cusp}}(P,\chi)$ denote the subspace of cusp functions in $A^2(P,\chi)$, and similarly by $A^2_{\text{cusp}}(P,\chi,\sigma)$ we denote the subspace of cusp functions in $A^2(P,\chi,\sigma)$. Then it follows from the above remark that

$$A^2_{\text{cusp}}(P,\chi,\sigma) = \begin{cases} A^2(P,\chi,\sigma) & ; \quad \text{if } P \in \mathcal{P}_\chi, \\ \{0\} & ; \quad \text{if } P \notin \mathcal{P}_\chi. \end{cases} \tag{1.4}$$

1.7. Next we recall some facts about Eisenstein series [L1]. Let $\phi \in A^2(P)$, $x \in G(\mathbb{A})$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ with $\operatorname{Re}(\lambda) \in (\mathfrak{a}^*_P)^+ + \rho_P$. Then the series

$$E(\phi,\lambda,x) = \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\gamma x) e^{(\lambda+\rho_P)(H_P(\gamma x))} \tag{1.5}$$

converges absolutely. This is the Eisenstein series attached to ϕ. As a function of λ, $E(\phi,\lambda,x)$ has an analytic continuation to a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ [L1]. If it is necessary to indicate that the Eisenstein series is associated to P, we will denote it by $E(P,\phi,\lambda,x)$.

Let P' be any other parabolic subgroup which is associated to P. Each $s \in W(\mathfrak{a}_P,\mathfrak{a}_{P'})$ has a representative w_s in $G(\mathbb{Q})$. Given $t \in W(\mathfrak{a}_P,\mathfrak{a}_{P'})$, $\phi \in A^2(P)$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ with $\operatorname{Re}(\lambda) \in (\mathfrak{a}^*_P)^+ + \rho_P$, define $M_{P'|P}(t,\lambda)\phi$ to be

$$(M_{P'|P}(t,\lambda)\phi)(x) = e^{-(t\lambda+\rho_{P'})(H_{P'}(x))} \int_{N'(\mathbb{A})\cap w_t N(\mathbb{A}) w_t^{-1}\backslash N'(\mathbb{A})} \phi(w_t^{-1}n'x) e^{(\lambda+\rho_P)(H_P(w_t^{-1}n'x))}\, dn'. \tag{1.6}$$

The integral is absolutely convergent for λ as above and admits an analytic continuation to a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ with values in the space of linear operators from $A^2(P)$ to $A^2(P')$. This is the global intertwining operator

$$M_{P'|P}(t,\lambda)\colon A^2(P) \to A^2(P').$$

If $\chi \in \mathfrak{X}$ and $\sigma \in \Pi(K)$, then $M_{P'|P}(t,\lambda)$ maps $A^2(P,\chi,\sigma)$ to $A^2(P',\chi,\sigma)$. The main functional equations are

$$\begin{aligned} E(\phi,\lambda,x) &= E(M_{P'|P}(t,\lambda)\phi, t\lambda, x)\\ M_{P_2|P}(ts,\lambda) &= M_{P_2|P_1}(t,s\lambda)M_{P_1|P}(s,\lambda). \end{aligned} \tag{1.7}$$

for $t \in W(\mathfrak{a}_1,\mathfrak{a}_2)$ and $s \in W(\mathfrak{a},\mathfrak{a}_1)$.
Furthermore, suppose that ϕ belongs to $A^2_{\text{cusp}}(P)$. Then the constant term $E_{P'}(P,\phi,\lambda,x)$ of $E(P,\phi,\lambda,x)$ along P' is given by

$$E_{P'}(P,\phi,\lambda,x) = \sum_{t\in W(\mathfrak{a}_P,\mathfrak{a}_{P'})} e^{(t\lambda+\rho_{P'})(H_{P'}(x))}\big(M_{P'|P}(t,\lambda)\phi\big)(x) \tag{1.8}$$

[MoWa].

2. Cuspidal intertwining operators

Let $P \in \mathcal{P}(M)$, $P' \in \mathcal{P}(M')$ and $s \in W(\mathfrak{a}_P,\mathfrak{a}_{P'})$ and we assume that $P \in \mathcal{P}_\chi$. Then

$$L^2_{\text{cusp}}(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)_\chi \neq \{0\}.$$

Our main purpose in this section is to study the intertwining operator

$$M_{P'|P}(s,\lambda)\colon A^2_{\text{cusp}}(P,\chi,\sigma) \to A^2_{\text{cusp}}(P',\chi,\sigma),$$

for $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$.
First assume that the relative rank of P is equal to one, that is $\dim \mathfrak{a}_P^G = 1$. Then any element of $\mathcal{P}(M)$ is conjugate either to P or to the opposite group $P^- = N^-M$. Moreover, P and P^- are conjugate if and only if $-1 \in W(\mathfrak{a}_P)$. Let $P_1,...,P_r$, $P_1 = P$, be a set of representatives for $\mathcal{P}(M)/W(\mathfrak{a}_P)$. Then $r = 1$ or $r = 2$. Set $\mathcal{P} = \mathcal{P}(M)$ and put

$$A^2_{\text{cusp}}(\mathcal{P},\chi,\sigma) = \bigoplus_{j=1}^r A^2_{\text{cusp}}(P_j,\chi,\sigma). \tag{2.1}$$

Let α_i be the unique simple root of (P_i,A_i), $i = 1,...,r$, and put $\lambda_i = \alpha_i/|\alpha_i|$. If $r = 2$, then $\lambda_1 = -\lambda_2$ and we put $\lambda = \lambda_1$.

If $r = 1$, then $W(\mathfrak{a}_P) = \{\pm 1\}$. In this case, we set

$$M(z) = M_{P|P}(-1, z\lambda), \quad z \in \mathbb{C}. \tag{2.2}$$

If $r = 2$, then $W(\mathfrak{a}_{P_i}) = \{1\}$, $i = 1,2$. Then we put

$$M(z)=\begin{pmatrix} 0 & M_{P_1|P_2}(1,-z\lambda) \\ M_{P_2|P_1}(1,z\lambda) & 0 \end{pmatrix}, \quad z\in\mathbb{C}. \tag{2.3}$$

In either case, $M(z)$ defines a linear operator

$$M(z)\colon A^2_{\mathrm{cusp}}(\mathcal{P},\chi,\sigma)\to A^2_{\mathrm{cusp}}(\mathcal{P},\chi,\sigma)$$

which is a meromorphic function of $z\in\mathbb{C}$. Moreover, by (1.7) it satisfies the functional equation

$$M(z)M(-z)=\mathrm{Id},\quad z\in\mathbb{C}. \tag{2.4}$$

We also have

$$M(z)^*=M(\overline{z}),\quad z\in\mathbb{C}. \tag{2.5}$$

The poles of $M(z)$ in the half-plane $0\le \mathrm{Re}(z)$ are all simple and contained in the interval $(0,\|\rho_P\|]$.
Given $\phi\in A^2(\mathcal{P},\chi,\sigma)$, $\phi=\{\phi_i\mid i=1,\cdots,r\}$, put

$$E(\phi,z)=\sum_{i=1}^r E(P_i,\phi_i,z\lambda_i),\quad z\in\mathbb{C}.$$

Then the functional equation (1.7) for the Eisenstein series takes the following form

$$E(\phi,z)=E(M(z)\phi,-z),\quad z\in\mathbb{C}. \tag{2.6}$$

The poles of $E(\phi,z)$ coincide with the poles of $M(z)$ (cf. [HC, Chapter IV, Theorem 7], for the analogous statement in the real case). Furthermore, it follows from (1.8) that the constant term of $E(\phi,z)$ along P_i, $i=1,\cdots,r$, is given by

$$\begin{aligned} E_{P_i}(\phi,z,x)&=e^{(z\lambda_i+\rho_i)(H_i(x))}\phi_i(x)\\ &\qquad+e^{(-z\lambda_i+\rho_i)(H_i(x))}(M(z)\phi)_i(x),\quad x\in G(\mathbb{A}),\end{aligned}$$

and if $P'\notin\mathcal{P}$, then $E_{P'}(\phi,z)=0$.
In [Mu1] we investigated the rank one cuspidal intertwining operators in the context of real Lie groups. In particular, it follows from the results of [Mu1, Section 5] that all matrix coefficients of any cuspidal rank one intertwining operator are meromorphic functions of finite order.

We need to extend this result to the adèlic setting. For this purpose we use the well known correspondence between automorphic forms on the adèlic group $G(\mathbb{A})$ and on the real reductive group $G(\mathbb{R})$ [BJ].
Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. Then the double coset space

$$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})K_f$$

is known to be finite [Bo1, Section 5]. Let $C \subset G(A_f)$ be a set of representatives of the double cosets. Then the groups

$$\Gamma_c = (G(\mathbb{R}) \times cK_f c^{-1}) \cap G(\mathbb{Q}), \quad c \in C, \tag{2.7}$$

are arithmetic subgroups of $G(\mathbb{R})$, and the action of $G(\mathbb{R})$ on the space of double cosets $G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$ induces the following decomposition into $G(\mathbb{R})$-orbits

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_f \cong \bigsqcup_{c\in C} (\Gamma_c\backslash G(\mathbb{R})). \tag{2.8}$$

Now let

$$\varphi\colon G(\mathbb{Q})Z_G(\mathbb{R})^0\backslash G(\mathbb{A}) \to \mathbb{C}$$

be an automorphic form [MoWa] which is K_f-invariant. Given $c \in C$, let φ_c be the function on $G(\mathbb{R})$ which is defined by

$$\varphi_c(x) = \varphi(cx), \quad x \in G(\mathbb{R}).$$

Then φ_c is Γ_c-invariant and

$$\varphi_c\colon \Gamma_c Z_G(\mathbb{R})^0\backslash G(\mathbb{R}) \to \mathbb{C}$$

is an automorphic form on $G(\mathbb{R})$. As proved by Borel and Jacquet [BJ], the map $\phi \longmapsto (\phi_c)_{c\in C}$ induces an isomorphism

$$A(G(\mathbb{Q})Z_G(\mathbb{R})^0\backslash G(\mathbb{A}))^{K_f} \cong \bigoplus_{c\in C} A(\Gamma_c Z_G(\mathbb{R})^0\backslash G(\mathbb{R})) \tag{2.9}$$

of the space of K_f-invariant automorphic forms on $G(\mathbb{A})$ and the direct sum of the spaces of Γ_c-automorphic forms on $G(\mathbb{R})$. In particular, this correspondence may be applied to Eisenstein series.
The following computations were kindly communicated to us by W. Hoffmann. Let P be a standard parabolic subgroup and suppose

that $\phi \in A^2(P)$ is right K_f-invariant. Then we have to compute $E(P, \phi, \lambda, cx)$ as a function of $x \in G(\mathbb{R})$. Since N is unipotent, it satisfies strong approximation [Bo1], i.e., we have that for any compact open subgroup U of $N(\mathbb{A}_f)$,

$$N(\mathbb{A}) = N(\mathbb{Q})N(\mathbb{R})U \tag{2.10}$$

and therefore,

$$N(\mathbb{Q})\backslash N(\mathbb{A}) \cong (N(\mathbb{Q}) \cap (N(\mathbb{R}) \times U))\backslash(N(\mathbb{R}) \times U).$$

Given $\delta \in G(\mathbb{Q})$ and $c \in C$, let $\phi_{\delta c}$ be the function on $G(\mathbb{R})$, defined by $\phi_{\delta c}(x) = \phi(\delta c x),\ x \in G(\mathbb{R})$. Let ${}^\delta P = \delta P \delta^{-1}$. Then an easy calculation using (2.10) shows that $\phi_{\delta c}$ is a square integrable automorphic form

$$\phi_{\delta c} \colon \delta^{-1} N(\mathbb{R}) A(\mathbb{R})^0 \delta(\Gamma_c \cap {}^\delta P(\mathbb{R}))\backslash G(\mathbb{R}) \to \mathbb{C}.$$

Let $E(\Gamma_c, {}^\delta P(\mathbb{R}), \phi_{\delta c}, \lambda, x)$ be the Eisenstein series on $\Gamma_c\backslash G(\mathbb{R})$ associated to these data. Using that

$$H_P(\delta x) = Ad(\delta) H_{{}^\delta P}(x) + H_P(\delta),$$

we get

$$\begin{aligned} &E(P, \phi, \lambda, cx) \\ &\quad = \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})/\Gamma_c} e^{(\lambda+\rho_P)(H_P(\delta c))} E(\Gamma_c, {}^\delta P(\mathbb{R}), \phi_{\delta c}, {}^\delta\lambda, x), \end{aligned} \tag{2.11}$$

where ${}^\delta\lambda = \lambda \circ Ad(\delta)$.
Next we compare the constant terms. Let P' be any other parabolic group and let φ be an automorphic form on $G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$. Then the constant term $\varphi_{P'}$ is defined by (1.2). Given $c \in C, \varphi_c$ is an automorphic form on $\Gamma_c\backslash G(\mathbb{R})$ and the constant term $(\varphi_c)_{P'}$ of φ_c along P' is defined as

$$(\varphi_c)_{P'}(x) = \int_{\Gamma_c \cap N'(\mathbb{R})\backslash N'(\mathbb{R})} \varphi_c(n_\infty x)\, dn_\infty.$$

To compare the constant terms we use (2.10) where we choose U to be $cK_f c^{-1} \cap N(\mathbb{A}_f)$. Then we obtain

$$\varphi_{P'}(cx) = (\varphi_c)_{P'}(x), \quad x \in G(\mathbb{R}). \tag{2.12}$$

If we apply this equality to (2.11), we get

$$\begin{aligned}&E_{P'}(P,\phi,\lambda,cx)\\&\quad=\sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})/\Gamma_c} e^{(\lambda+\rho_P)(H_P(\delta c))}E_{P'}(\Gamma_c,{}^{\delta}P(\mathbb{R}),\phi_{\delta c},{}^{\delta}\lambda,x)\end{aligned}\tag{2.13}$$

for all $x\in G(\mathbb{R})$. Now assume that the automorphic form ϕ belongs to $A^2_{\mathrm{cusp}}(P,\chi,\sigma)$. Then there exists a compact open subgroup $K_f\subset G(\mathbb{A}_f)$ such that each $\phi\in A^2_{\mathrm{cusp}}(P,\chi,\sigma)$ is right K_f-invariant. Being a cuspidal automorphic function, the constant term $E_{P'}(P,\phi,\lambda,x)$ is given by (1.8). Furthermore, by (2.12) each $\phi_{\delta c}$ is a cuspidal automorphic function on $\Gamma_c\backslash G(\mathbb{R})$. To describe the constant terms on the right hand side of (2.13), we consider an arithmetic subgroup Γ of $G(\mathbb{R})$ and a parabolic subgroup P. Let $A^2_{\mathrm{cusp}}(\Gamma,P)$ be the space of cuspidal automorphic forms on $(\Gamma\cap P(\mathbb{R}))N(\mathbb{R})A(\mathbb{R})^0\backslash G(\mathbb{R})$ [L1]. Recall that this is the subspace of $L^2_{\mathrm{cusp}}((\Gamma\cap P(\mathbb{R}))N(\mathbb{R})A(\mathbb{R})^0\backslash G(\mathbb{R}))$ consisting of all ϕ such that

1) ϕ is right K_∞-finite.
2) For each $x\in G(\mathbb{R})$, the function $m\to\phi(mx)$ of $m\in M(\mathbb{R})$ is $Z(\mathfrak{m}_{\mathbb{C}})$-finite.

Let P_1 and P_2 be two associated parabolic subgroups. For each $t\in W(\mathfrak{a}_{P_1},\mathfrak{a}_{P_2})$, there is an associated intertwining operator

$$C^{\Gamma}_{P_2|P_1}(t,\lambda)\colon A^2_{\mathrm{cusp}}(\Gamma,P_1)\to A^2_{\mathrm{cusp}}(\Gamma,P_2)\tag{2.14}$$

which is a meromorphic function of $\lambda\in\mathfrak{a}^*_{P_1,\mathbb{C}}$ [L1]. Then the constant terms of the Eisenstein series on the right hand side of (2.13) are given by

$$\begin{aligned}&E_{P'}(\Gamma_c,{}^{\delta}P(\mathbb{R}),\phi_{\delta c},{}^{\delta}\lambda,x)\\&\quad=\sum_{w\in W(\mathfrak{a}_{\delta P},\mathfrak{a}_{P'})} e^{(w^{\delta}\lambda+\rho_{P'})(H_{P'}(x))}\left(C^{\Gamma_c}_{P'|{}^{\delta}P}(w,{}^{\delta}\lambda)\phi_{\delta c}\right)(x).\end{aligned}\tag{2.15}$$

Next observe that for each $w\in W(\mathfrak{a}_{\delta P},\mathfrak{a}_{P'})$ there exists $t\in W(\mathfrak{a}_P,\mathfrak{a}_{P'})$ such that $w=t\circ Ad(\delta^{-1})$. Hence, the right hand side of (2.15) can be written as a sum over $W(\mathfrak{a}_P,\mathfrak{a}_{P'})$. Collecting all terms on the right hand side of (2.13) that correspond to a given $t\in W(\mathfrak{a}_P,\mathfrak{a}_{P'})$, gives

$$\begin{aligned}&(M_{P'|P}(t,\lambda)\phi)(cx)\\&\quad=\sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})/\Gamma_c} e^{(\lambda+\rho_P)(H_P(\delta c))}\left(C^{\Gamma_c}_{P'|{}^{\delta}P}({}^{\delta}t,{}^{\delta}\lambda)\phi_{\delta c}\right)(x).\end{aligned}\tag{2.16}$$

The intertwining operators (2.14) are not exactly those that we considered in [Mu1], but they are closely related to these operators. To describe the relation, let Γ and P be as above. Let $\sigma_\infty\colon K_\infty \to \mathrm{GL}(V_\sigma)$ be an irreducible unitary representation of K_∞ and let $\psi\colon Z(\mathfrak{m}_\mathbb{C}) \to \mathbb{C}$ be a character. Then we denote by $\mathcal{A}^2_{\text{cusp}}(\Gamma, P, \psi, \sigma_\infty)$ the subspace of all $\phi \in \mathcal{A}^2_{\text{cusp}}(\Gamma, P)$ such that

1) For each $x \in G(\mathbb{R})$, the function $k \to \phi(xk)$ of $k \in K_\infty$ is contained in the σ_∞-isotypical subspace of $L^2(K_\infty)$.
2) For each $x \in G(\mathbb{R})$, the function $m \to \phi(mx)$ of $m \in M(\mathbb{R})$ is an eigenfunction of $Z(\mathfrak{m}_\mathbb{C})$ with character ψ.

This is a finite-dimensional space of automorphic forms on $\Gamma\backslash G(\mathbb{R})$ and $\mathcal{A}^2_{\text{cusp}}(\Gamma, P)$ is the direct sum of all such subspaces. The intertwining operator (2.14) preserves this decomposition, i.e., for all σ_∞ and ψ, (2.14) induces an operator

$$C^\Gamma_{P_2|P_1}(t, \lambda)\colon \mathcal{A}^2_{\text{cusp}}(\Gamma, P_1, \psi, \sigma_\infty) \to \mathcal{A}^2_{\text{cusp}}(\Gamma, P_2, {}^t\psi, \sigma_\infty).$$

Now let σ_∞ be the Archimedean component of σ. Since ϕ belongs to $\mathcal{A}^2_{\text{cusp}}(P, \chi, \sigma)$, there exists a character $\psi : Z(\mathfrak{m}_\mathbb{C}) \to \mathbb{C}$ such that $\phi_{\delta c} \in \mathcal{A}^2_{\text{cusp}}(\Gamma_c, \delta_P, \psi, \sigma_\infty)$. Next we introduce the corresponding space of vector valued automorphic forms. For $P = NM$, let $K^M_\infty = M(\mathbb{R}) \cap K$. Set

$$X_M = A_P(\mathbb{R})^0\backslash M(\mathbb{R})/K^M_\infty.$$

Let $\Gamma_M \subset M(\mathbb{R})$ be the image of $\Gamma \cap P(\mathbb{R})$ under the canonical projection $P(\mathbb{R}) \to P(\mathbb{R})/N(\mathbb{R}) \cong M(\mathbb{R})$. Then Γ_M is an arithmetic subgroup of $M(\mathbb{R})$. Let $L^2_{\text{cusp}}(\Gamma_M A_P(\mathbb{R})^0\backslash M(\mathbb{R}), \psi, \sigma_\infty)$ be the space of all cuspidal functions

$$\varphi\colon \Gamma_M A_P(\mathbb{R})^0\backslash M(\mathbb{R}) \to V_\sigma$$

which satisfy:

1) $\varphi(mk) = \sigma_\infty(k)^{-1}\varphi(m)$ for all $m \in M(\mathbb{R})$ and $k \in K^M_\infty$.
2) $D\varphi = \psi(D)\varphi$ for all $D \in Z(\mathfrak{m}_\mathbb{C})$.

Then there is a canonical isomorphism

$$\mathcal{A}^2_{\text{cusp}}(\Gamma, P, \psi, \sigma_\infty) \cong L^2_{\text{cusp}}(\Gamma_M A_P(\mathbb{R})^0\backslash M(\mathbb{R}), \psi, \sigma_\infty) \otimes V^*_\sigma,$$

and with respect to this isomorphism, the intertwining operator (2.14) takes the form

$$C^\Gamma_{P_2|P_1}(t, \lambda) = c^\Gamma_{P_2|P_1}(t, \lambda) \otimes \mathrm{Id}_{V_\sigma}$$

where $c^{\Gamma}_{P_2|P_1}(t,\lambda)$ is the corresponding intertwining operator in the space of vector valued automorphic forms. The $c^{\Gamma}_{P_2|P_1}(t,\lambda)$ are the intertwining operators which have been studied in [Mu1]. Now we return to formula (2.16). Suppose that P is a parabolic subgroup of relative rank one and that P' is associated to P. Observe that the set $P(\mathbb{Q})\backslash G(\mathbb{Q})/\Gamma_c$ parametrizes conjugacy classes of parabolic subgroups mod Γ_c. In [Mu1, p.484] we have put together the intertwining operators, associated to representatives of Γ_c-conjugacy classes of parabolic subgroups, into a block matrix called $C_c(s)$, $s \in \mathbb{C}$. The resulting operator acts on a space of cuspidal automorphic forms which was denoted in [Mu1] by $\mathcal{E}_{\text{cus}}(\sigma_\infty,\mathcal{O})$. Here $\mathcal{O}$ stands for an orbit type of associated characters ψ. From the definition of $C_c(s)$ follows that each of the operators $c^{\Gamma_c}_{P'|^{\delta}P}({}^{\delta}t,{}^{\delta}\lambda)$ occurs as an entry of the block matrix $C_c(\lambda)$, $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. Therefore, we can apply the results of [Mu1]. It follows from [Mu1, Theorem 4.30] and [Mu1, Section 5] that for all $\phi,\psi \in \mathcal{E}_{cus}(\sigma,\mathcal{O})$, $\langle C_c(\lambda)\phi,\psi\rangle$ is a meromorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ of order $\leq n+2$, where $n = \dim G(\mathbb{R})/K_\infty$. Combined with (2.16) we obtain the following theorem.

Theorem 2.1. *Let $n = \dim G(\mathbb{R})/K_\infty$. Then for all elements ϕ and ψ of $A^2_{\text{cusp}}(\mathcal{P},\chi,\sigma)$, the function $\langle M(z)\phi,\psi\rangle$ is a meromorphic function of $z \in \mathbb{C}$ of order $\leq n+2$.*

Remark. Theorem 2.1 can also be proved directly without using the expression of the adèlic intertwining operators in terms of the intertwining operators for the real group. Since $G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f$ is the disjoint union of locally symmetric spaces $\Gamma_c\backslash X$, the cut-off Laplacian Δ_T, introduced in [Mu1], can also be defined in the adèlic setting (see [DH] for details). Once Δ_T exists, one can proceed as in [Mu1].

Now we shall extend Theorem 2.1 to cuspidal intertwining operators of any rank. Let $P \in \mathcal{P}(M)$, $P_1 \in \mathcal{P}(M_1)$. Suppose that P and P_1 are associated and let $t \in W(\mathfrak{a}_P,\mathfrak{a}_{P_1})$. Let $w_t \in G(\mathbb{Q})$ be a representative of t. Then $M_1 = w_t M w_t^{-1}$ and $tP = w_t P w_t^{-1}$ is a parabolic subgroup which belongs to $\mathcal{P}(M_1)$. The restriction of t to $\mathfrak{a}_P \subset \mathfrak{a}_0$ defines an element in $W(\mathfrak{a}_P,\mathfrak{a}_{tP})$. Let

$$t\colon A^2(P) \to A^2(tP)$$

be the linear operator defined by

$$(t\phi)(x) = \phi(w_t^{-1}x), \quad x \in G(\mathbb{A}). \tag{2.17}$$

By Lemma 1.1 of [A5] there exists $T_0 \in \mathfrak{a}_0$ such that

$$H_{P_0}(w_t^{-1}) = T_0 - t^{-1}T_0.$$

Then by (1.4) and (1.5) in [A4], one has

$$tM_{P_1|P}(s,\lambda) = M_{tP_1|P}(ts,\lambda)e^{-(s\lambda+\rho_{P_1})(T_0-t^{-1}T_0)}, \tag{2.18}$$

and

$$M_{P_1|P}(s,\lambda)t^{-1} = M_{P_1|P}(st^{-1},t\lambda)e^{(\lambda+\rho_P)(T_0-t^{-1}T_0)} \tag{2.19}$$

for $s \in W(\mathfrak{a}_P,\mathfrak{a}_{P_1})$.

Thus as far as the order of the matrix coefficients of $M_{P_1|P}(s,\lambda)$ are concerned, we can always assume that P_1 and P have the same Levi component M, that is $P, P_1 \in \mathcal{P}(M)$. Let $\mathcal{C}$ be the set of Weyl chambers in $\mathfrak{a}_P$. Then the Weyl group $W(\mathfrak{a}_P)$ acts both on $\mathcal{P}(M)$ and $\mathcal{C}$, and

$$\mathcal{P}(M)/W(\mathfrak{a}_P) \cong \mathcal{C}/W(\mathfrak{a}_P).$$

In particular, for each chamber $C \in \mathcal{C}$, we get a parabolic group $P_C = N_C M$. Set $\Phi(C) := \Phi_{P_C}$ and $\Delta(C) := \Delta_{P_C}$. Let $\mathcal{C} = \{C_1,\dots C_r\}$ and set $P_i = P_{C_i}$, $i=1,\dots,r$. This set contains a set of representatives for $\mathcal{P}(M)/W(\mathfrak{a}_P)$. Let $1 \le i,j \le r$ and $t \in W(\mathfrak{a}_P)$ be given. Then there exists k, $1 \le k \le r$, such that $tC_i = C_k$. By the functional equation (1.7), we have

$$M_{P_j|P_i}(t,\lambda) = M_{P_j|P_k}(1,t\lambda)M_{P_k|P_i}(t,\lambda). \tag{2.20}$$

First consider $M_{P_k|P_i}(t,\lambda)$. Since $tC_i = C_k$ we have $P_k = tP_i$. Hence, by (2.19) it follows that

$$M_{P_k|P_i}(t,\lambda) = M_{P_k|P_k}(1,t\lambda)te^{(\lambda+\rho_P)(T_0-t^{-1}T_0)}.$$

But $M_{P_k|P_k}(1,\lambda) = \mathrm{Id}$. Thus

$$M_{P_k|P_i}(t,\lambda) = e^{(\lambda+\rho_P)(T_0-t^{-1}T_0)}t.$$

Therefore, there exist $C, c > 0$ such that for all $\phi \in A^2_{\mathrm{cusp}}(P_i,\chi,\sigma)$, $\psi \in A^2_{\mathrm{cusp}}(P_k,\chi,\sigma)$ and $t \in W(\mathfrak{a}_P)$, one has

$$|\langle M_{P_k|P_i}(t,\lambda)\phi,\psi\rangle| \le C \parallel \phi \parallel \parallel \psi \parallel e^{c\|\lambda\|}, \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}. \tag{2.21}$$

Definition 2.2. *Let V be a finite dimensional complex vector space with norm $\|\cdot\|$ and let $\alpha \in \mathbb{R}_+$. An entire function*

$$f\colon \mathbb{C}^N \to V$$

is said to be of order α, if there exist $C, c > 0$ such that

$$\| f(\Lambda) \| \leq C \exp(c \| \Lambda \|^{\alpha}), \quad \Lambda \in \mathbb{C}^N.$$

Then (2.21) can be reformulated as follows:

Proposition 2.3. *Let $1 \leq i, j \leq r$ and $t \in W(\mathfrak{a}_P)$ be such that $t(C_i) = C_j$. Then $M_{P_j|P_i}(t, \lambda)$ is an entire operator valued function of order 1.*

If remains to consider $M_{P_j|P_i}(1, \lambda)$. To this end we proceed as in [HC, Chapter V, 8]. First we recall some notation. Two chambers C_1 and C_2 are called adjacent if $C_1 \neq C_2$ and $\mathrm{Cl}(C_1) \cap \mathrm{Cl}(C_2)$ contains a semi-regular element. Let $C \in \mathcal{C}$ be given and let $\alpha \in \Delta(C)$. Then there exists a unique chamber C^α which is adjacent to C and $-\alpha \in \Delta(C^\alpha)$. Given $C_i, C_j \in \mathcal{C}$, there exists a sequence of chambers $C_{i_1}, \dots, C_{i_N}$ such that

1) $C_{i_1} = C_i$, $C_{i_N} = C_j$.
2) C_{i_k} and $C_{i_{k+1}}$ are adjacent for all k, $1 \leq k \leq N$.
3) If $\alpha_k \in \Delta(C_{i_k})$ is such that $C_{i_{k+1}} = C_{i_k}^{\alpha_k}$, $1 \leq k \leq N$, then $\alpha_k \in \Phi(C_l)$ for $l \leq k$ and $-\alpha_k \in \Phi(C_l)$ for $l > k$.

Using the functional equation (1.7), it follows by induction that

$$\begin{aligned} M_{P_j|P_i}(1, \lambda) = M_{P_{i_N}|P_{i_{N-1}}}(1, \lambda) \cdots \\ \cdots M_{P_{i_3}|P_{i_2}}(1, \lambda) M_{P_{i_2}|P_{i_1}}(1, \lambda), \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}. \end{aligned} \tag{2.22}$$

Hence we need only to consider $M_{P_j|P_i}(1, \lambda)$ for adjacent chambers C_j and C_i. Assume that $i = 1$ and $j = 2$, so that $C_2 = C_1^\alpha$, where $\alpha \in \Delta(C_1)$. Then $P_1 = N_1 \cdot M$ and $P_2 = N_2 \cdot M$. Define $(P', A') = (P_1, A_P)_{\{\alpha\}}$. Then also $(P', A') = (P_2, A_P)_{\{\alpha\}}$. Thus $P_1, P_2 \subset P'$ and $M \subset M'$. Let $P_i^* = M' \cap P_i$, $i = 1, 2$. Then P_i^* are parabolic subgroups of M' and $M_{P_i^*} = M$. Moreover, we have

$$N_i = N_i^* N' \quad \text{and} \quad A = A^*. \tag{2.23}$$

Let $\sigma_{M'}$ be the restriction of σ to $K^{M'} = M'(\mathbb{A}) \cap K$. Given $\chi \in \mathfrak{X}$, let $A^2_{\mathrm{cusp}}(P_i^*, \chi, \sigma_{M'})$ be the corresponding space of automorphic forms on $N_i^*(\mathbb{A}) M^*(\mathbb{Q}) A^*(\mathbb{R})^0 \backslash M'(\mathbb{A})$, and for $\lambda^* \in \mathfrak{a}^*_{P_1^*,\mathbb{C}}$, let

$$M_{P_2^*|P_1^*}(1, \lambda^*)\colon A^2_{\mathrm{cusp}}(P_1^*, \chi, \sigma_{M'}) \to A^2_{\mathrm{cusp}}(P_2^*, \chi, \sigma_{M'}), \tag{2.24}$$

be the associated intertwining operator. Let $\phi \in A^2_{\mathrm{cusp}}(P_i,\chi,\sigma)$. Then for any $k \in K$, the function

$$\phi_k(m') = \phi(m'k), \quad m' \in M'(\mathbb{A}),$$

belongs to $A^2_{\mathrm{cusp}}(P_i^*,\chi,\sigma_{M'})$, and using the definition (1.6) of the intertwining operator combined with (2.23), it follows that

$$(M_{P_2|P_1}(1,\lambda)\phi)_k = M_{P_2^*|P_1^*}(1,\lambda^*)(\phi_k), \tag{2.25}$$

where λ^* is the projection of λ onto $(\mathfrak{a}_{M_1^*}^{M'})^*_{\mathbb{C}} = (\mathfrak{a}_M^{M'})^*_{\mathbb{C}}$.
Since P_1^* and P_2^* are parabolic subgroups of M' of relative rank one, it follows from Theorem 2.1 that there exist an entire function $F_{21}(z)$ of order $\leq n_1 + 2$, $n_1 = \dim M'(\mathbb{R})/K^{M'}$, and an entire function $\widehat{M}_{P_2^*|P_1^*}(z)$, of order $\leq n_1+2$, with values in the linear operators from $A^2_{\mathrm{cusp}}(P_1^*,\chi,\sigma_{M'})$ to $A^2_{\mathrm{cusp}}(P_2^*,\chi,\sigma_{M'})$ such that

$$M_{P_2^*|P_1^*}(1,\lambda^*) = \frac{\widehat{M}_{P_2^*|P_1^*}(\lambda^*)}{F_{21}(\lambda^*)}, \quad \lambda^* \in \mathfrak{a}^*_{P_1^*,\mathbb{C}}.$$

Given $\phi \in A^2_{\mathrm{cusp}}(P_1,\chi,\sigma)$, $k \in K$ and $\lambda \in (\mathfrak{a}_M^{M'})^*_{\mathbb{C}}$, put

$$(\widehat{M}_{P_2|P_1}(\lambda)\phi)(mk) = \left(\widehat{M}_{P_2^*|P_1^*}(\lambda^*)\phi_k\right)(m) \quad m \in M'(\mathbb{A}), \tag{2.26}$$

where again λ^* is the projection of λ onto $(\mathfrak{a}_M^{M'})^*_{\mathbb{C}}$. Then $\widehat{M}_{P_2|P_1}(\lambda)\phi$ can be identified with an element of $A^2_{\mathrm{cusp}}(P_2,\chi,\sigma)$. In this way we obtain a holomorphic function of linear operators

$$\widehat{M}_{P_2|P_1}(\lambda)\colon A^2_{\mathrm{cusp}}(P_1,\chi,\sigma) \to A^2_{\mathrm{cusp}}(P_2,\chi,\sigma), \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}},$$

such that

$$M_{P_2|P_1}(1,\lambda) = \frac{\widehat{M}_{P_2|P_1}(\lambda)}{F_{21}(\lambda)}, \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}.$$

Let $\phi \in A^2_{\mathrm{cusp}}(P_1,\chi,\sigma)$ and $\psi \in A^2_{\mathrm{cusp}}(P_2,\chi,\sigma)$. Then by (2.26), we have

$$\langle \widehat{M}_{P_2|P_1}(\lambda)\phi, \psi\rangle = \int_K \langle \widehat{M}_{P_2^*|P_1^*}(\lambda^*)\phi_k, \psi_k\rangle \, dk.$$

Hence, we get

$$
\begin{aligned}
&|\langle \widehat{M}_{P_2|P_1}(\lambda)\phi,\psi\rangle| \\
&\quad \le \| \widehat{M}_{P_2^*|P_1^*}(\lambda^*) \| \sqrt{\int_K \| \phi_k \|^2 \, dk} \sqrt{\int_K \| \psi_k \|^2 \, dk} \\
&\quad \le \| \widehat{M}_{P_2^*|P_1^*}(\lambda^*) \| \| \phi \| \| \psi \| .
\end{aligned}
$$

Since $\widehat{M}_{P_2^*|P_1^*}(\lambda^*)$ is of order $\le n_1+2$, where $n_1 \le n$, it follows that $\widehat{M}_{P_2|P_1}(\lambda)$ is an entire operator valued function of order $\le n_1+2$. Now observe that for any i,j, the length N of the chain of adjacent chambers connecting C_i and C_j is bounded by the number of chambers. Therefore the above estimates combined with (2.22) and Proposition 2.3 imply the following theorem

Theorem 2.4. *Let $n = \dim G(\mathbb{R})/K_\infty$. Let M be a Levi subgroup. For all $P_1, P_2 \in \mathcal{P}(M)$ and $t \in W(\mathfrak{a}_M)$, there exist an entire function $F_{P_1,P_2}(t,\lambda)$ of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ of order $\le n+2$ and an entire operator valued function.*

$$
\widehat{M}_{P_2|P_1}(t,\lambda)\colon A^2_{\mathrm{cusp}}(P_1,\chi,\sigma) \to A^2_{\mathrm{cusp}}(P_2,\chi,\sigma), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},
$$

of order $\le n+2$ such that

$$
M_{P_2|P_1}(t,\lambda) = \frac{\widehat{M}_{P_2|P_1}(t,\lambda)}{F_{P_1,P_2}(t,\lambda)}, \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}.
$$

3. Cuspidal Eisenstein series

In this section we study Eisenstein series $E(P,\phi,\lambda,x)$ attached to cuspidal automorphic forms ϕ and our main purpose is to show that $\lambda \mapsto E(P,\phi,\lambda,x)$ is a meromorphic function of finite order.
First we recall the definition of the truncation operator Λ^T introduced by Arthur [A7]. Fix $P_0 \in \mathcal{P}(M_0)$. Let $T \in \mathfrak{a}_0^+$ be sufficiently regular which means that $\alpha(T) \gg 0$ for all $\alpha \in \Delta_{P_0}$. For any such T one has the truncation operator Λ^T which acts on continuous functions h on $G(\mathbb{Q})\backslash G(\mathbb{A})$. Let $\hat{\tau}_P$ be the characteristic function of

$$
\{H \in \mathfrak{a}_0 \mid \omega(H) > 0,\ \omega \in \hat{\Delta}_P\}.
$$

Then $\Lambda^T h$ is defined as

$$\Lambda^T h(x) = \sum_{\{P \mid P \supset P_0\}} (-1)^{\dim(A_P/A_G)}$$
$$\sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} h(n\gamma x)\hat{\tau}_P(H_P(\gamma x) - T)\, dn.$$

Let P be a standard parabolic subgroup, i.e., $P_0 \subset P$. Fix $\chi \in \mathfrak{X}$ and $\sigma \in \Pi(K)$ as above. Let $\phi \in A^2_{\text{cusp}}(P, \chi, \sigma)$. Then the truncated Eisenstein series $\Lambda^T E(\phi, \lambda, x)$ is square integrable and Langlands [L2] has established the following inner product formula for truncated Eisenstein series.

$$\langle \Lambda^T E(\phi, \lambda),\ \Lambda^T E(\phi, \lambda')\rangle = \sum_{P_1 \supset P_0} \sum_{t,t' \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})} \operatorname{Vol}(\mathfrak{a}_{P_1/L_{P_1}})$$
$$\times \frac{\langle M_{P_1|P}(t, \lambda)\phi,\ M_{P_1|P}(t', \lambda')\phi\rangle}{\prod_{\alpha \in \Delta_{P_1}} (t\lambda + t'\overline{\lambda'})(\alpha^\vee)} e^{(t\lambda + t'\overline{\lambda'})(T)}. \tag{3.1}$$

where $L_{P_1} \subset \mathfrak{a}_{P_1}$ is the lattice generated by the co-roots $\{\alpha^\vee \mid \alpha \in \Delta_{P_1}\}$. By Theorem 2.4 there exists an entire function $p(\lambda) \neq 0$ on $\mathfrak{a}^*_{P,\mathbb{C}}$ of order $\leq n + 2$ such that for all $P_1 \supset P_0$ and $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$, the operator valued function

$$\widetilde{M}_{P_1|P}(t, \lambda) = p(\lambda) M_{P_1|P}(t, \lambda)$$

is an entire function of order $\leq n + 2$. Given $P_1 \supset P_0$ and $t, t' \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$, let

$$\mathcal{D}_{P_1,t,t'} = \{\lambda \in \mathfrak{a}^*_{P,\mathbb{C}} \mid \textstyle\prod_{\alpha \in \Delta_{P_1}} \langle t'\overline{\lambda} + t\lambda, \alpha\rangle = 0\}. \tag{3.2}$$

Put

$$\mathcal{D} = \bigcup_{P_1 \supset P_0} \bigcup_{t,t' \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})} \mathcal{D}_{P_1,t,t'}. \tag{3.3}$$

This is the union of finitely many real subspaces $V_1, \dots, V_m$ of $\mathfrak{a}^*_{P,\mathbb{C}}$ of codimension 1 or 2. Let $L_1, \dots, L_r$ be those subspaces V_i which are complex subspaces of $\mathfrak{a}^*_{P,\mathbb{C}}$. In particular we have $\operatorname{codim}_{\mathbb{R}} L_i = 2$, $1 \leq i \leq r$. Moreover, let $L_{r+1}, \dots, L_k$ be the intersections of all pairs (V_i, V_j), $1 \leq i, j \leq m$, $i \neq j$, such that $V_i \cap V_j$ is a complex subspaces of $\mathfrak{a}^*_{P,\mathbb{C}}$ of codimension 1. For each i, $1 \leq i \leq k$, let $\alpha_i \in \mathfrak{a}^*_P$ be an defining equation for L_i:

$$L_i = \{\lambda \in \mathfrak{a}^*_{P,\mathbb{C}} \mid \alpha_i(\lambda) = 0\},$$

and let m_i be the minimum of all $m \in \mathbb{N}$ such that $\alpha_i(\lambda)^m E(\phi, \lambda)$ is holomorphic near L_i for all $\phi \in A^2_{\rm cusp}(P, \chi, \sigma)$. Put

$$q(\lambda) = p(\lambda) \prod_{j=1}^{k} \alpha_j(\lambda)^{m_j}. \tag{3.4}$$

Then $q(\lambda)$ is entire of order $\leq n + 2$. Put

$$\widehat{E}(\phi, \lambda, x) = q(\lambda) E(\phi, \lambda, x). \tag{3.5}$$

Let $U \subset \mathbb{C}$ be an open subset and let $\mathcal{F}$ be a Fréchet space. We recall that a function $f : U \to \mathcal{F}$ is called holomorphic, if for each $z \in U$, the function $z' \mapsto (z' - z)^{-1}(f(z') - f(z))$ has a limit in $\mathcal{F}$ if z' tends to z. An equivalent condition is that for each continuous linear functional $\ell : \mathcal{F} \to \mathbb{C}$, the function $\ell \circ f$ is holomorphic. If $U \subset \mathbb{C}^N$ is open, then $f : U \to \mathcal{F}$ is called holomorphic, if f is holomorphic in each variable $z_1, ..., z_N$ of $\mathbb{C}^N$.

Lemma 3.1. *For every $\phi \in A^2_{\rm cusp}(P, \chi, \sigma)$, $\lambda \mapsto \Lambda^T \widehat{E}(\phi, \lambda) \in L^2$ is a holomorphic function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$.*

Proof. For all $P_1 \supset P_0$ and $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$ put

$$\widehat{M}_{P_1|P}(t, \lambda) = q(\lambda) M_{P_1|P}(t, \lambda). \tag{3.6}$$

Then $\widehat{M}_{P_1|P}(t, \lambda)$ is an entire function of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ and it follows from (3.1) that

$$\begin{aligned} \| \Lambda^T \widehat{E}(\phi, \lambda) \|^2 = & \sum_{P_1 \supset P_0} \sum_{t,t' \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})} \mathrm{Vol}(\mathfrak{a}_{P_1}/L_{P_1}) \\ & \times \frac{\langle \widehat{M}_{P_1|P}(t, \lambda)\phi,\ \widehat{M}_{P_1|P}(t', \lambda)\phi \rangle}{\prod_{\alpha \in \Delta_{P_1}} (t\lambda + t'\overline{\lambda})(\alpha^\vee)}\, e^{(t\lambda + t'\overline{\lambda})(T)}. \end{aligned} \tag{3.7}$$

By (3.7), $\lambda \mapsto \Lambda^T \widehat{E}(\phi, \lambda)$ is holomorphic on the open dense subset $\mathfrak{a}^*_{P,\mathbb{C}} \setminus \mathcal{D}$.

Now let $\lambda_0 \in \mathcal{D}$ and suppose that $\Lambda^T \widehat{E}(\phi, \lambda)$ is singular at λ_0. Fix P_1 and $t, t' \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$ such that $\lambda_0 \in \mathcal{D}_{P_1,t,t'}$. As proved by Langlands [L1], the singularities of $E(\phi, \lambda)$ lie along hyperplanes defined by root equations. By (3.5) the same holds for $\widehat{E}(\phi,\lambda)$ and also for $\Lambda^T\widehat{E}(\phi,\lambda)$. Therefore there exist $\beta\in\Delta_P$ and $c\in\mathbb{C}$, such that

$$\mathcal{H}=\{\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}\mid\lambda(\beta^\vee)=c\}$$

is a singular hyperplane of $\Lambda^T\widehat{E}(\phi,\lambda)$ and $\lambda_0\in\mathcal{H}$. Since $\Lambda^T\widehat{E}(\phi,\lambda)$ is holomorphic in $\mathfrak{a}^*_{P,\mathbb{C}}\setminus\mathcal{D}$, it follows that $\mathcal{H}\subset\mathcal{D}$. We may assume that $\mathcal{H}\subset\mathcal{D}_{P_1,t,t'}$. Moreover, by the definition (3.4) of $q(\lambda)$, there exists a unique root $\alpha_0\in\Delta_{P_1}$ such that

$$\mathcal{H}\subset\{\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}\mid\langle t\lambda+t'\overline{\lambda},\alpha_0\rangle=0\}$$

and this subspace has real codimension 1. This implies that we may choose the singular point $\lambda_0\in\mathcal{H}$ of $\Lambda^T\widehat{E}(\phi,\lambda)$ such that

$$\begin{aligned}&\langle t\lambda_0+t'\overline{\lambda}_0,\alpha_0\rangle=0\\&\langle t\lambda_0+t'\overline{\lambda}_0,\alpha\rangle\neq0\quad\text{for all }\ \alpha\neq\alpha_0.\end{aligned}\tag{3.8}$$

Let $\nu\in\mathfrak{a}^*_{P,\mathbb{C}}$ be a unit vector which is orthogonal to the subspace

$$\{\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}\mid\langle t\lambda+t'\overline{\lambda},\alpha_0\rangle=0\}$$

and consider the line

$$\lambda(u)=u\nu+\lambda_0,\quad u\in\mathbb{R}.$$

For $\epsilon\ll1$, we have

$$\mathcal{D}\cap\{\lambda(u)\mid|u|<\epsilon\}=\{\lambda_0\}.$$

Since $E(\phi,\lambda)$ is a meromorphic function whose singularities lie along hyperplanes, there exists $m\geq1$ such that the function

$$F(\phi,\lambda)=(\langle\lambda,\beta\rangle-c)^m\Lambda^T\widehat{E}(\phi,\lambda)\tag{3.9}$$

is holomorphic in a neighborhood of λ_0 and $F(\phi,\lambda_0)\not\equiv0$. We simply have to choose λ_0 such that it is not contained in the intersection of any two singular hyperplanes. By assumption we have $\langle\lambda_0,\beta\rangle=c$. Hence

$$\langle\lambda(u),\beta\rangle-c=u\langle\nu,\beta\rangle+\langle\lambda_0,\beta\rangle-c=u\langle\nu,\beta\rangle,$$

and it follows from (3.9) that there exists $a\in L^2(G(\mathbb{Q})Z_G(\mathbb{R})^0\backslash G(\mathbb{A}))$ such that

$$\Lambda^T \widehat{E}(\phi, \lambda(u)) = \frac{a}{u^m} + O(u^{-m+1})$$

as $u \to 0$. Therefore there exists $c > 0$ such that

$$\| \Lambda^T \widehat{E}(\phi, \lambda(u)) \|^2 = \frac{c}{u^{2m}} + O(u^{-2m+1})$$

as $u \to 0$. On the other hand, since $\langle t\lambda_0 + t'\overline{\lambda}_0, \alpha_0\rangle = 0$, we have

$$\langle t\lambda(u) + t'\overline{\lambda(u)}, \alpha_0\rangle = u\langle t\nu + t'\overline{\nu}, \alpha_0\rangle,$$

and by our choice of ν, $\langle t\nu + t'\overline{\nu}, \alpha_0\rangle \neq 0$. By (3.8), it follows that the right hand side of (3.7) is $O(u^{-1})$ as $u \to 0$. But the left hand side is of order $O(u^{-2m})$, $m \geq 1$, as $u \to 0$. Therefore, we get a contradiction which proves that $\Lambda^T \widehat{E}(\phi, \lambda)$ is entire. □

Proposition 3.2. *There exist $C, c > 0$ such that for all $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ and all $\phi \in A^2_{\mathrm{cusp}}(P, \chi, \sigma)$, we have*

$$\| \Lambda^T \widehat{E}(\phi, \lambda) \| \leq C \| \phi \| e^{c\|\lambda\|^{n+2}}.$$

Proof. Put

$$h(\lambda) = \prod_{P_1 \supset P_0} \prod_{t,t' \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})} \prod_{\alpha \in \Delta_{P_1}} \langle t\lambda + t'\overline{\lambda}, \alpha\rangle.$$

Then it follows from (3.6), (3.7) and Theorem 2.4 that there exist $C, c > 0$ such that

$$|h(\lambda)|^2 \| \Lambda^T \widehat{E}(\phi, \lambda) \|^2 \leq C \| \phi \| \exp(c \| \lambda \|^{n+2}), \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}, \tag{3.10}$$

for all $\phi \in A^2_{\mathrm{cusp}}(P, \chi, \sigma)$. We have to show that (3.10) implies a similar estimate for $\| \Lambda^T \widehat{E}(\phi, \lambda) \|$. This is a consequence of the repeated application of the following lemma. □

Lemma 3.3. *Let $g(z)$ be an analytic function of $z \in \mathbb{C}^n$. Let $t, t' \in GL_n(\mathbb{R})$ and $v \in \mathbb{R}^n$. Suppose that there exist constants $C, c > 0$ and $m \in \mathbb{N}$ such that*

$$|\langle tz + t'\overline{z}, v\rangle| \, |g(z)|^2 \leq C e^{c\|z\|^m}$$

for all $z \in \mathbb{C}^n$. Then there exist $C_1, c_1 > 0$ such that

$$|g(z)| \leq C_1 e^{c_1\|z\|^m}, \quad z \in \mathbb{C}^n.$$

Proof. To begin with, we reduce the proof to a similar estimate for a function of one complex variable. Since t, t' and v are real, there exist $f_1, f_2 \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ such that

$$\langle tz + t'\overline{z}, v\rangle = f_1(\operatorname{Re}(z)) + i f_2(\operatorname{Im}(z)). \tag{3.11}$$

Suppose that $f_1 \neq 0$. The case $f_2 \neq 0$ can be treated in the same way. Let

$$f_1(z) = \sum_{i=1}^{n} a_i z_i.$$

We may assume that $a_n \neq 0$. Let $w \in \mathbb{C}^{n-1}$ be fixed and put

$$a = -\sum_{i=1}^{n-1} \frac{a_i}{a_n} \operatorname{Re}(w_i).$$

Then for $z \in \mathbb{C}$, we have

$$\begin{aligned} &|\operatorname{Re}(z) - a|\ |g(w,z)|^2 \leq \\ &\quad \leq |a_n|^{-1} |\langle t(w,z) + t'\overline{(w,z)}, v\rangle|\ |g(w,z)|^2 \\ &\quad \leq \frac{C}{|a_n|} e^{2^m c\|w\|^m} e^{2^m c|z|^m}, \end{aligned} \tag{3.12}$$

and it is sufficient to prove that a similar estimate holds for $g(w,z)$. Thus we may consider an analytic function $f(z)$ of $z \in \mathbb{C}$ which satisfies

$$|\operatorname{Re}(z) - a|\ |f(z)|^2 \leq C e^{c|z|^m}, \quad z \in \mathbb{C}, \tag{3.13}$$

for certain constants $C, c > 0$ and $m \in \mathbb{N}$. Then our goal is to estimate $|f(z)|$. By (3.13) it follows that

$$|f(z)| \leq C e^{c|z|^m} \text{ for } |\operatorname{Re}(z) - a| \geq 1.$$

Therefore, it suffices to estimate $|f(z)|$ for $|\operatorname{Re}(z) - a| \leq 1$. For $z = x + iy$ and $|x - a| < 1$, let $\Gamma_z \subset \mathbb{C}$ be the rectangle with edges $a-2+i(y-1)$, $a+2+i(y-1)$, $a+2+i(y+1)$ and $a-2+i(y+1)$. Then we have

$$f(z) = (2\pi i)^{-1} \int_{\Gamma_z} f(\xi)(\xi - z)^{-1}\, d\xi.$$

Furthermore, for $\xi \in \Gamma_z$ we have

$$|f(\xi)(\xi - z)^{-1}| \le C e^{c|\xi|^m/2} |\operatorname{Re}(\xi) - a|^{-1/2} |\xi - z|^{-1}.$$

Since $1 \le |\xi - z| \le 4$, there exist $C_1, c' > 0$, which are independent of z and a, such that

$$C e^{c|\xi|^m/2} |\xi - z|^{-1} \le C_1 e^{c'|z|^m}.$$

Hence

$$|f(z)| \le C_1 (2\pi)^{-1} e^{c'|z|^m} \int_{\Gamma_z} |\operatorname{Re}(\xi) - a|^{-1/2} |d\xi|.$$

The last integral is bounded by a constant which is independent of z and a. Therefore, there exists $C_2 > 0$, independent of z and a, such that

$$|f(z)| \le C_2 e^{c'|z|^m}. \tag{3.14}$$

Now we apply (3.14) to the function $f(z) = g(w, z)$ with the initial estimate (3.12). This leads to

$$\begin{aligned} |g(w,z)| &\le C_1 e^{c_1(|a|^m + \|w\|^m + |z|^m)} \\ &\le C_2 e^{c_2 \|(w,z)\|^m} \end{aligned}$$

with constants $C_2, c_2 > 0$ which depend only on C, c, t, t' and v.

□

Corollary 3.4. *For every compact subset $W \subset G(\mathbb{Q}) Z_G(\mathbb{R})^0 \backslash G(\mathbb{A})$ there exist $C, c > 0$ such that*

$$\sup_{x \in W} |\widehat{E}(\phi, \lambda, x)| \le C \parallel \phi \parallel e^{c\|\lambda\|^{n+2}}$$

for all $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ and $\phi \in A^2_{\mathrm{cusp}}(P, \chi, \sigma)$.

Proof. Let $W \subset G(\mathbb{Q}) Z_G(\mathbb{R})^0 \backslash G(\mathbb{A})$ be any compact subset. We may assume that W is K-invariant. There exists a truncation parameter $T \in \mathfrak{a}^*_0$ such that

$$\Lambda^T \widehat{E}(\phi, \lambda, x) = \widehat{E}(\phi, \lambda, x) \tag{3.15}$$

for all $x \in W$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. Furthermore, there exists an open compact subgroup K_f of $G(\mathbb{A}_f)$ such that each $\phi \in A^2_{\mathrm{cusp}}(P, \chi, \sigma)$ is K_f-invariant. Then $E(\phi, \lambda, x)$ is also K_f-invariant for any $\phi \in A^2_{\mathrm{cusp}}(P, \chi, \sigma)$. As in (2.8), let $C \subset G(\mathbb{A}_f)$ be a set of representatives for the double cosets $G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\mathbb{R}) K_f$. For $c \in C$ let $\Gamma_c \subset G(\mathbb{R})$ be the arithmetic subgroup (2.7). Then

$$W/K_f \cong \bigsqcup_{c\in C} W_c$$

where W_c is a compact subset of $\Gamma_c Z_G(\mathbb{R})^0\backslash G(\mathbb{R})$. We may assume that ϕ is an eigenfunction of the Casimir element $\Omega_M \in Z(\mathfrak{m}_\mathbb{C})$ with eigenvalue μ which means that for any $x \in G(\mathbb{A})$, the function $\phi_x(m) = \phi(mx)$, $m \in M(\mathbb{R})$, satisfies

$$\Omega_M\phi_x = \mu\phi_x.$$

Let $c \in C$. Then as a function of $x \in G(\mathbb{R})$, we have

$$\Omega E(\phi,\lambda,cx) = r(\lambda)E(\phi,\lambda,cx) \tag{3.16}$$

where $\Omega \in Z(\mathfrak{g}_\mathbb{C})$ is the Casimir element and $r(\lambda)$ is a quadratic polynomial in λ which satisfies

$$|r(\lambda)| \le C(\mu^2 + \parallel \lambda \parallel^2), \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}, \tag{3.17}$$

with $C > 0$ independent of μ and λ. Let Δ be the Laplacian of $G(\mathbb{R})$ with respect to the left invariant metric. Then

$$\Delta = -\Omega_G + 2\Omega_{K_\infty},$$

where $\Omega_{K_\infty} \in Z((\mathfrak{k}_\infty)_\mathbb{C})$ is the Casimir element of K_∞. Let $\kappa = \sigma_\infty(\Omega_\infty)$. By (3.16) we get

$$\Delta E(\phi,\lambda,cx) = (-r(\lambda) + 2\kappa)E(\phi,\lambda,cx). \tag{3.18}$$

Let $\varphi \in C_0^\infty(\Gamma_c Z_G(\mathbb{R})^0\backslash G(\mathbb{R}))$ be such that $\varphi \equiv 1$ on W_c. We pick $T \in \mathfrak{a}_0$ so that (3.15) holds on $\operatorname{supp}\varphi \cdot K_f$. We note that

$$\Delta(\varphi\widehat{E}) = \varphi(\Delta\widehat{E}) + (\Delta\varphi)\widehat{E} - \langle\nabla\varphi,\nabla\widehat{E}\rangle \tag{3.19}$$

For $k \in \mathbb{N}$ let $\|\cdot\|_k$ be the k-Sobolev norm on $\Gamma_c Z_G(\mathbb{R})^0\backslash G(\mathbb{R})$ and $\|\cdot\|_{W_c,k}$ the Sobolev norm on W_c. By the fundamental elliptic estimate and (3.19) we get

$$\begin{aligned}\| \widehat{E} \|_{W_c,k} <\| \varphi\widehat{E} \|_k \leq \\ &\leq C(\| \varphi\widehat{E} \|_{k-2} + \| \Delta(\varphi\widehat{E}) \|_{k-2}) \\ &\leq C(\| \varphi\widehat{E} \|_{k-2} + \| \varphi\Delta\widehat{E} \|_{k-2}) \\ &+ C_1 \| \widehat{E} \|_{W_1,k-1},\end{aligned}$$

where $W_1 = \text{supp } \varphi$. Using (3.17) and (3.18), we obtain

$$\| \widehat{E}(\phi,\lambda) \|_{W_c,k} \leq C_2(1+ \| \lambda \|^2) \| \widehat{E}(\phi,\lambda) \|_{W_1,k-1} .$$

By iteration, we get

$$\| \widehat{E}(\phi,\lambda) \|_{W_c,n} \leq C_n(1+ \| \lambda \|^2)^n \| \widehat{E}(\phi,\lambda) \|_{W_n,0},$$

where $W_n \supset W_c$ is some open relatively compact subset of the space $\Gamma_c Z_G(\mathbb{R})^0 \backslash G(\mathbb{R})$. Suppose that our truncation parameter has been chosen so that (3.15) holds on $W_n \cdot K_f$. Then by the Sobolev embedding theorem we obtain

$$\begin{aligned}\sup_{x\in W_c} |\widehat{E}(\phi,\lambda,cx)| \leq C \| \widehat{E}(\phi,\lambda) \|_{W_c,n} \\ &\leq C_{n+1}(1+ \| \lambda \|^2) \| \Lambda^T\widehat{E}(\phi,\lambda) \| .\end{aligned}$$

Applying Proposition 3.2, the claimed estimate follows.

□

4. Residual Eisenstein series and intertwining operators

The main purpose of this section is to extend Theorem 2.4 to all intertwining operators acting on $A^2(P,\chi,\sigma)$. Our approach to this problem is based on Langland's description of the space $A^2(P,\chi,\sigma)$ in terms of iterated residues of cuspidal Eisenstein series [L1]. For the convenience of the reader we will summarize some of the basic features of this theory. For all details we refer the reader to [L1], [MoWa].

Let B be a parabolic subgroup. Given $\alpha \in \Phi_B$ and $c \in \mathbb{R}$, we set

$$H(\alpha,c) = \{\Lambda \in \mathfrak{a}_B^* \mid \Lambda(\alpha^\vee) = c\}. \tag{4.1}$$

An affine subspace $\mathfrak{t} \subset \mathfrak{a}_B^*$ is called admissible if $\mathfrak{t}$ is the intersection of hyperplanes of the form (4.1). Any such subspace $\mathfrak{t}$ can be written as

$$\mathfrak{t} = \Lambda + \tilde{\mathfrak{t}},$$

where $\tilde{\mathfrak{t}} \subset \mathfrak{a}_B^*$ is a linear subspace defined by root equations and Λ is vector in $\mathfrak{a}_B^*$ which is orthogonal to $\tilde{\mathfrak{t}}$. Let

$$\mathfrak{t}_{\mathbb{C}} = \Lambda + \tilde{\mathfrak{t}} \otimes \mathbb{C}$$

be the complexified affine subspace of $\mathfrak{a}_{B,\mathbb{C}}^*$. Then $\mathfrak{t}_{\mathbb{C}}$ is called an admissible subspace of $\mathfrak{a}_{B,\mathbb{C}}^*$.
Now suppose that $\mathfrak{t}_1 \supset \mathfrak{t}_2$ are two admissible subspaces of $\mathfrak{a}_B^*$ and that $\mathfrak{t}_2$ is of codimension one in $\mathfrak{t}_1$. Let $F(\Lambda)$ be a meromorphic function on $\mathfrak{t}_{1,\mathbb{C}}$ whose singularities lie along hyperplanes which are admissible as subspaces of $\mathfrak{a}_{B,\mathbb{C}}^*$. Choose a unit vector ν in $\tilde{\mathfrak{t}}_1$ which is normal to $\tilde{\mathfrak{t}}_2$. It is uniquely determined up to sign. Then we can define a meromorphic function $\mathrm{Res}_{(\nu,\mathfrak{t}_2)}F(\Lambda)$ on $\mathfrak{t}_{2,\mathbb{C}}$ by taking the residue with respect to $\mathfrak{t}_{2,\mathbb{C}}$. More precisely, let $\Lambda_0 \in \mathfrak{t}_{2,\mathbb{C}}$ be a point with the property that any singular hyperplane of F which contains Λ_0 also contains $\mathfrak{t}_{2,\mathbb{C}}$. Let Γ be a sufficiently small positively oriented circle about the origin in the complex plane. Then we put

$$\mathrm{Res}_{(\nu,\mathfrak{t}_2)}F(\Lambda_0) = \frac{1}{2\pi i}\int_\Gamma F(\Lambda_0 + z\nu)\, dz.$$

The singularities of $\mathrm{Res}_{(\nu,\mathfrak{t}_2)}F$ lie along the intersections of the singular hyperplanes of F ,which are different from $\mathfrak{t}_{2,\mathbb{C}}$, with $\mathfrak{t}_{2,\mathbb{C}}$.
Let $B \subset P$ be parabolic subgroups. Suppose that

$$\mathfrak{t}_0 \subset \mathfrak{t}_1 \subset \cdots \subset \mathfrak{t}_r = \mathfrak{a}_B^*$$

is a sequence of admissible affine subspaces such that for each i, $1 \leq i \leq r$, $\mathfrak{t}_{i-1}$ is of codimension 1 in $\mathfrak{t}_i$ and $\tilde{\mathfrak{t}}_0 = \mathfrak{a}_P^*$. Thus for each i there exists $\beta \in \Phi_B$ such that $\beta^\vee$ does not vanish on $\tilde{\mathfrak{t}}_i$ and

$$\tilde{\mathfrak{t}}_{i-1} = \{\Lambda \in \tilde{\mathfrak{t}}_i \mid \Lambda(\beta^\vee) = 0\}.$$

For each i we choose a unit vector $\nu_i \in \tilde{\mathfrak{t}}_i$ which is orthogonal to $\widetilde{\mathfrak{t}_{i-1}}$. Let S denote the sequence of affine subspaces together with the choices of unit normal vectors. Then we denote $\mathfrak{t}_0$ by

$$\mathfrak{t}_S = \Lambda_S + \mathfrak{a}_P^*.$$

Let F be a meromorphic function on $\mathfrak{a}_{B,\mathbb{C}}^*$ whose singularities lie along admissible hyperplanes. Then we can define inductively meromorphic functions F_i on $\mathfrak{t}_{i,\mathbb{C}}$ by

$$F_r = F \quad \text{and} \quad F_i = \mathrm{Res}_{(\nu_i,\mathfrak{t}_i)}F_{i+1}, \quad i = 0, \ldots, r-1.$$

Set

$$F_0 = \mathrm{Res}_S F.$$

This is the iterated residue of F with respect to S. It is a meromorphic function on

$$\mathfrak{t}_{S,\mathbb{C}} = \Lambda_S + \mathfrak{a}^*_{P,\mathbb{C}}.$$

Sometimes we will denote the iterated residue $\mathrm{Res}_S F(\Lambda_0)$ also by

$$\mathrm{Res}_{S,\Lambda\to\Lambda_0} F(\Lambda).$$

Recall that by Langlands' theory of Eisenstein series, $A^2(P,\chi,\sigma)$ is spanned by iterated residues of cuspidal Eisenstein series which are associated to $M_P(\mathbb{A})$. Furthermore $A^2(P,\chi,\sigma)$ can be identified with a space of automorphic forms on $M_P(\mathbb{A})$ [MoWa, p.37]. The Eisenstein series are attached to parabolic subgroups B^* of M_P. For any such B^*, there is a unique parabolic subgroup $B \subset P$ such that $B^* = B \cap M_P$ and $M_B = M_{B^*}$. Considered as automorphic forms on $G(\mathbb{A})$, the Eisenstein series are associated to $\phi \in A^2_{\mathrm{cusp}}(B,\chi,\sigma)$ and are defined by

$$E^P(B,\phi,\lambda,x) = \sum_{\gamma\in B(\mathbb{Q})\backslash P(\mathbb{Q})} e^{(\lambda+\rho_B^P)(H_B(\gamma x))}\phi(\gamma x), \tag{4.2}$$

where $\lambda \in \mathfrak{a}^*_{B,\mathbb{C}}$ and $\mathrm{Re}(\lambda) \in (\mathfrak{a}^*_B)^+ + \rho_B$. Sometimes we will denote $E^P(B,\phi,\lambda,x)$ simply by $E^P(\phi,\lambda,x)$. For $P = G$, we obtain the usual Eisenstein series (1.5). Because of the relation with the standard Eisenstein series on $M_P(\mathbb{A})$, it is clear that each $E^P(\phi,\lambda,x)$ has an analytic continuation to a meromorphic function of $\lambda \in \mathfrak{a}^*_{B,\mathbb{C}}$ satisfying the same properties as the standard Eisenstein series do.
Let B be any group in $\mathcal{P}_\chi$ which is contained in P. Let

$$F_B \colon \mathfrak{a}^*_{B,\mathbb{C}} \to A^2(B,\chi,\sigma)$$

be a meromorphic function whose singularities lie along hyperplanes. Then for fixed x, $E^P(F_B(\Lambda),\Lambda,x)$ is a meromorphic function on $\mathfrak{a}^*_{B,\mathbb{C}}$ whose singularities lie along hyperplanes. Let S be as above. Then

$$\mathrm{Res}_S E^P(F_B(\Lambda),\Lambda,x), \quad \Lambda \in \mathfrak{t}_{S,\mathbb{C}},$$

is a meromorphic function on $\mathfrak{t}_{S,\mathbb{C}}$. Let $(\mathfrak{a}^P_B)^*_{\mathbb{C}}$ be the orthogonal complement of $\mathfrak{a}^*_{P,\mathbb{C}}$ in $\mathfrak{a}^*_{B,\mathbb{C}}$, and let $S((\mathfrak{a}^P_B)^*_{\mathbb{C}})$ be the symmetric algebra on $(\mathfrak{a}^P_B)^*_{\mathbb{C}}$. If F_B is regular at a point Λ in $\mathfrak{t}_{S,\mathbb{C}}$, let

$$dF_B(\Lambda) = d_S F_B(\Lambda)$$

be the linear map from $S((\mathfrak{a}_B^P)^*_{\mathbb{C}})$ to $A^2(B,\chi,\sigma)$ which is obtained by expanding the analytic function

$$\lambda \to F_B(\lambda + \Lambda), \quad \lambda \in (\mathfrak{a}_B^P)^*_{\mathbb{C}},$$

as a Taylor series about $\lambda = 0$. Then $\mathrm{Res}_S E^P(F_B(\Lambda), \Lambda, x)$ depends only on $dF_B(\Lambda)$.
For each $B \in \mathcal{P}_\chi$ with $B \subset P$, Langlands constructs a finite set of admissible affine subspaces

$$\mathfrak{t} = \Lambda_{\mathfrak{t}} + \mathfrak{a}_P^*$$

of $\mathfrak{a}_B^*$, and for each $\mathfrak{t}$, a finite set $\mathcal{S}_B(P,\mathfrak{t})$ of sequences of admissible affine subspaces with distinguished unit normals as above, satisfying $\mathfrak{t}_S = \mathfrak{t}$ for each $S \in \mathcal{S}_B(P,\mathfrak{t})$ with the following properties: For any group $B \in \mathcal{P}_\chi$, $B \subset P$, let $\mathcal{S}_B(P)$ be the disjoint union over all $\mathfrak{t}$ of the sets $\mathcal{S}_B(P,\mathfrak{t})$. Suppose that for each such B, F_B is a meromorphic function from $\mathfrak{a}^*_{B,\mathbb{C}}$ to $A^2(B,\chi,\sigma)$ whose singularities lie along hyperplanes, and which is regular at each point Λ_S, $S \in \mathcal{S}_B(P)$. Then the function

$$\phi(x) = \sum_{\{B \in \mathcal{P}_\chi \mid B \subset P\}} \sum_{S \in \mathcal{S}_B(P)} \mathrm{Res}_{S,\Lambda \to \Lambda_S} E^P(F_B(\Lambda), \Lambda, x) \tag{4.3}$$

belongs to $A^2(P,\chi,\sigma)$ and $A^2(P,\chi,\sigma)$ is spanned by functions of this form. Recall that the condition $B \in \mathcal{P}_\chi$ implies that $A^2(B,\chi,\sigma)$ coincides with $A^2_{\mathrm{cusp}}(B,\chi,\sigma)$ so that F_B takes values in $A^2_{\mathrm{cusp}}(B,\chi,\sigma)$. Thus for regular values Λ, $E^P(F_B(\Lambda),\Lambda,x)$ can be identified with a cuspidal Eisenstein series for $M_P(\mathbb{A})$. Let

$$\Phi = \bigoplus_B \bigoplus_{S \in \mathcal{S}_B(P)} (d_S F_B)(\Lambda_S). \tag{4.4}$$

Then Φ belongs to the vector space

$$\bigoplus_B \bigoplus_{S \in \mathcal{S}_B(P)} \mathrm{Hom}(S((\mathfrak{a}_B^P)^*_{\mathbb{C}}), A^2(B,\chi,\sigma)) \tag{4.5}$$

and ϕ depends only on the vector Φ. Let $\mathcal{L}(P,\chi,\sigma)$ be the subspace of the vector space (4.5) consisting of all vectors of the form (4.4). Then $\Phi \to \phi$ defines a surjective map of $\mathcal{L}(P,\chi,\sigma)$ onto $A^2(P,\chi,\sigma)$.

Suppose that P_1 is a parabolic subgroup which is associated to P and let $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$. There exists a parabolic subgroup $B_1 \in \mathcal{P}_\chi$ with $B_1 \subset P_1$ and a unique element $t_B \in W(\mathfrak{a}_B, \mathfrak{a}_{B_1})$ such that the restriction of t_B to $\mathfrak{a}_P$ equals t and such that $t_B(\alpha) \in \Delta_{B_1}$ for every $\alpha \in \Delta_B$ which vanishes on $\mathfrak{a}_P$. Suppose that $S \in \mathcal{S}_B(P)$. Then we can transform S by t_B into a sequence tS of admissible affine subspaces of $\mathfrak{a}^*_{B_1}$ with distinguished unit normals associated to B_1 and P_1. It follows by symmetry that we may take for $\mathcal{S}_{B_1}(P_1)$ the collection

$$t\mathcal{S}_B(P) = \{tS \mid S \in \mathcal{S}_B(P)\}.$$

Now consider the intertwining operator

$$M_{P_1|P}(t,\lambda)\colon A^2(P,\chi,\sigma) \to A^2(P_1,\chi,\sigma),\ \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}.$$

Let $\phi \in A^2(P,\chi,\sigma)$. Then ϕ is the image under the map (4.3) of a vector Φ in $\mathcal{L}(P,\chi,\sigma)$ of the form (4.4) for a collection

$$\{F_B(\Lambda) \mid B \in \mathcal{P}_\chi,\ B \subset P\}$$

of functions as above. Let λ be a point in general position in $\mathfrak{a}^*_{P,\mathbb{C}}$. Then

$$\widetilde{F}_{B_1}(\Lambda_1) = M_{B_1|B}(t_B, t_B^{-1}\Lambda_1 + \lambda)F_B(t_B^{-1}\Lambda_1), \quad \Lambda_1 \in \mathfrak{a}^*_{B_1,\mathbb{C}}, \tag{4.6}$$

is a meromorphic function on $\mathfrak{a}^*_{B_1,\mathbb{C}}$ with values in $A^2(B_1,\chi,\sigma)$. By choosing λ properly, we can assume that $\widetilde{F}_{B_1}$ is regular at each point $\Lambda_{S_1},\ S_1 \in t\mathcal{S}_B(P)$. Then by [A6, Lemma 2.1], one has

$$\begin{aligned} M_{P_1|P}(t,\lambda)\phi(x) = \\ \sum_{\{B_1 \in \mathcal{P}_\chi \mid B_1 \subset P_1\}} \sum_{S_1 \in \mathcal{S}_{B_1}(P_1)} \operatorname{Res}_{S_1,\Lambda_1 \to \Lambda_{S_1}} E^{P_1}(\widetilde{F}_{B_1}(\Lambda_1), \Lambda_1, x). \end{aligned} \tag{4.7}$$

Thus in order to investigate $M_{P_1|P}(t,\lambda)$ as a meromorphic function of λ, we have to study the iterated residues on the right hand side of (4.7). We first establish the following lemma.

Lemma 4.1. *Let f_1 and f_2 be two entire functions on $\mathbb{C}^n$ of order $m \in \mathbb{N}$ whose zero divisors are hyperplanes. Let $f = f_1/f_2$ and let H be a singular hyperplane of f. Let $F = Res_H f$ be the residue of f with respect to H. Then there exist entire functions F_1 and F_2 on $H \cong \mathbb{C}^{n-1}$*

of order $\leq m$ such that $F = F_1/F_2$. The zero divisors of F_i, $i = 1, 2$, are hyperplanes in $\mathbb{C}^{n-1}$.

Proof. Suppose that H is given by the equation $\alpha(z) = c$ where $\alpha \in \operatorname{Hom}(\mathbb{C}^n, \mathbb{C})$ and $c \in \mathbb{C}$. Changing coordinates by $z \longmapsto Az + b$, $A \in \mathrm{U}(n, \mathbb{C})$, $b \in \mathbb{C}^n$, we may assume that H is defined by $z_1 = 0$. Then there exist $k_i \in \mathbb{N}$ and holomorphic functions g_i on $\mathbb{C}^n$, $i = 1, 2$, such that

$$f_i(z) = z_1^{k_i} g_i(z), \quad z \in \mathbb{C}^n, \quad \text{and} \quad g_i(0, z_2, \dots, z_n) \not\equiv 0. \tag{4.8}$$

Then

$$f(z) = z_1^{-l} \frac{g_1(z)}{g_2(z)},$$

where $l = k_2 - k_1$. Since $g_2 \restriction H \not\equiv 0$ and H is a singular hyperplane of f, we have $l \geq 1$. Let Γ be a circle in the z_1-plane of sufficiently small radius ε with center at the origin. Let $z' = (z_2, \dots, z_n) \in H \cong \mathbb{C}^{n-1}$ be such that $g_2(0, z') \neq 0$. Then by Cauchy's formula we get

$$\begin{aligned}
(\mathrm{Res}_H f)(z') &= \frac{1}{2\pi i} \int_\Gamma f(z_1, z')\, dz_1 \\
&= \frac{1}{2\pi i} \int_\Gamma z_1^{-l} \frac{g_1(z_1, z')}{g_2(z_1, z')}\, dz_1 \\
&= \frac{(-1)^l}{(l-1)!} \frac{\partial^{l-1}}{\partial z_1^{l-1}} \left(\frac{g_1(z_1, z')}{g_2(z_1, z')} \right) \Bigg|_{z_1 = 0}.
\end{aligned}$$

The right hand side can be written as $F_1(z')/F_2(z')$ where $F_2(z') = g_2(0, z')^l$ and $F_1(z')$ is a polynomial in $(\partial^p / \partial z_1^p)\,(g_j)(0, z')$, $j = 1, 2$, $0 \leq p \leq l - 1$. In particular, F_1 and F_2 are entire functions on $\mathbb{C}^{n-1}$ and

$$\mathrm{Res}_H f(z') = \frac{F_1(z')}{F_2(z')}.$$

It remains to estimate F_i. Let $p \in \mathbb{N}_0$. Then it follows from (4.8) that

$$\frac{\partial^p g_j}{\partial z_1^p}(0, z') = \frac{p!}{(k_j + p)!} \frac{\partial^{k_j + p} f_j}{\partial z_1^{k_j + p}}(0, z'). \tag{4.9}$$

By assumption, f_j is of order m. Hence, if we apply Cauchy's formula to the right hand side of (4.9), it follows that there exist $C_p, c_p > 0$ such that

$$\left|\frac{\partial^p g_j}{\partial z_1^p}(0,z')\right| \le C_p e^{c_p\|z'\|^m}, \quad z' \in \mathbb{C}^{n-1}.$$

This inequality implies that F_1 and F_2 are entire functions of order $\le m$.

□

Now we are ready to investigate the iterated residues in (4.7). Fix $B \in \mathcal{P}_\chi$ with $B \subset P$. Let $\phi_1, \dots, \phi_k$ be an orthonormal basis of $A^2(B,\chi,\sigma)$. Then

$$F_B(\Lambda) = \sum_{j=1}^k \psi_j(\Lambda)\phi_j,$$

where ψ_j, $1 \le j \le k$, are meromorphic functions on $\mathfrak{a}_{B,\mathbb{C}}^*$ which are holomorphic at Λ_S. Thus

$$E^P(F_B(\Lambda),\Lambda,x) = \sum_{i=1}^k \psi_j(\Lambda)E^P(\phi_j,\Lambda,x).$$

Let $\phi \in A^2(B,\chi,\sigma)$ and let ψ be a meromorphic function on $\mathfrak{a}_{B,\mathbb{C}}^*$ whose singularities lie along hyperplanes and which is holomorphic at Λ_S. Then

$$\operatorname{Res}_{S,\Lambda\to\Lambda_S}[\psi(\Lambda)E^P(\phi,\Lambda,x)]$$

depends only on the element $d_S\psi \in S((\mathfrak{a}_B^P)_{\mathbb{C}}^*)$ which is obtained from the Taylor expansion of the function $\lambda \to \psi(\lambda+\Lambda_S)$ of $\lambda \in (\mathfrak{a}_B^P)_{\mathbb{C}}^*$ at $\lambda = 0$.

In fact, finitely many components of $d_S\psi$ are sufficient. Hence, we can replace ψ by its Taylor polynomial of sufficiently high order. In particular, we can assume that $F_B(\Lambda)$ is a holomorphic function and that there exist $C, c > 0$ such that

$$\| F_B(\Lambda) \| \le Ce^{c\|\Lambda\|}, \quad \Lambda \in \mathfrak{a}_{B,\mathbb{C}}^*, \tag{4.10}$$

for every $B \in \mathcal{P}_\chi$ with $B \subset P$.

Using the definition (4.6) of $\widetilde{F}_{B_1}$ and Theorem 2.4, it follows that there exist entire functions $h(\Lambda)$ and $\widehat{M}_{B_1|B}(t_B,\Lambda)$ of $\Lambda \in \mathfrak{a}_{B,\mathbb{C}}^*$ of order $\le n+2$ such that

$$\widetilde{F}_{B_1}(\Lambda_1)=\frac{\widehat{M}_{B_1|B}(t_B,t_B^{-1}\Lambda_1+\lambda)F_B(t_B^{-1}\Lambda_1)}{h(t_B^{-1}\Lambda_1+\lambda)} \tag{4.11}$$

for $\Lambda_1\in\mathfrak{a}^*_{B_1,\mathbb{C}}$. Since $h(\Lambda)$ and $\widehat{M}_{B_1|B}(t_B,\Lambda)$ are of order $\le n+2$, we have

$$|h(t_B^{-1}\Lambda_1+\lambda)|\le Ce^{c\left(\|\Lambda_1\|^{n+2}+\|\lambda\|^{n+2}\right)} \tag{4.12}$$

and

$$\|\,\widehat{M}_{B_1|B}(t_B,t_B^{-1}\Lambda_1+\lambda)\,\|\le Ce^{c\left(\|\Lambda_1\|^{n+2}+\|\lambda\|^{n+2}\right)}. \tag{4.13}$$

As explained above, the Eisenstein series $E^{P_1}(\phi_1,\Lambda_1,x)$ may be regarded as standard Eisenstein series on $M_P(\mathbb{A})$. Therefore, the results of section 3 can be also applied to $E^{P_1}(\phi_1,\Lambda_1,x)$. Thus there exists an entire function $q(\Lambda_1)$ of $\Lambda_1\in\mathfrak{a}^*_{B_1,\mathbb{C}}$ of order $\le n+2$ such that for all $\phi_1\in A^2(B_1,\chi,\sigma)$, the function of $\Lambda_1\in\mathfrak{a}^*_{B_1,\mathbb{C}}$ defined by

$$\widehat{E}^{P_1}(\phi_1,\Lambda_1,x)=q(\Lambda_1)E^{P_1}(\phi_1,\Lambda_1,x)$$

is entire and Corollary 3.4 holds for $\widehat{E}^{P_1}(\phi_1,\Lambda_1,x)$. Hence, by 4.11 we get

$$\begin{aligned}&E^{P_1}(\tilde{F}_{B_1}(\Lambda_1),\Lambda_1,x)\\ &\qquad=\frac{\widehat{E}^{P_1}(\widehat{M}_{B_1|B}(t_B,t_B^{-1}\Lambda_1+\lambda)F_B(t_B^{-1}\Lambda_1),\Lambda_1,x)}{h(t_B^{-1}\Lambda_1+\lambda)q(\Lambda_1)}.\end{aligned} \tag{4.14}$$

By (4.12), the denominator is an entire function of $(\Lambda_1,\lambda)\in\mathfrak{a}^*_{B_1,\mathbb{C}}\times\mathfrak{a}^*_{P,\mathbb{C}}$ of order $\le n+2$. To estimate the numerator, we pick a compact subset $W\subset G(\mathbb{Q})Z_G(\mathbb{R})^0\backslash G(\mathbb{A})$. Then by Corollary 3.4 there exist $C,c>0$ such that for all $x\in W$, we have

$$\begin{aligned}&|\widehat{E}^{P_1}(\widehat{M}_{B_1|B}(t_B,t_B^{-1}\Lambda_1+\lambda)F_B(t_B^{-1}\Lambda_1),\Lambda_1,x)|\\ &\qquad\le C\,\|\,\widehat{M}_{B_1|B}(t_B,t_B^{-1}\Lambda_1+\lambda)\,\|\cdot\|\,F_B(t_B^{-1}\Lambda_1)\,\|\;\;e^{c\|\Lambda_1\|^{n+2}}.\end{aligned}$$

By (4.10) and (4.13), the right hand side is bounded from above by

$$C_1e^{c_1(\|\Lambda_1\|^{n+2}+\|\lambda\|^{n+2})}$$

for certain constants $C_1,c_1>0$. Hence for each $x\in G(\mathbb{A})$, the numerator on the right hand side of (4.14) is also an entire function of

$(\Lambda_1,\lambda)\in\mathfrak{a}^*_{B_1,\mathbb{C}}\times\mathfrak{a}^*_{P,\mathbb{C}}$ of order $\leq n+2$ and the estimates are uniform on compact subsets of $G(\mathbb{Q})Z_G(\mathbb{R})^0\backslash G(\mathbb{A})$. Now repeated application of Lemma 4.1 together with (4.7) implies that there exist an entire function $F(\lambda)$ of order $\leq n+2$ on $\mathfrak{a}^*_{P,\mathbb{C}}$ and an entire function $\widehat{M}_{P_1|P}(t,\lambda)$, $\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}$, with values in the bounded operators from $A^2(P,\chi,\sigma)$ to $A^2(P_1,\chi,\sigma)$ such that

$$M_{P_1|P}(t,\lambda)=\frac{\widehat{M}_{P_1|P}(t,\lambda)}{F(\lambda)},\quad \lambda\in\mathfrak{a}^*_{P,\mathbb{C}}. \tag{4.15}$$

Moreover, the above estimates imply that for every compact subset $W\subset G(\mathbb{Q})Z_G(\mathbb{R})^0\backslash G(\mathbb{A})$ and every $\phi\in A^2(P,\chi,\sigma)$, there exist $C,c>0$ such that

$$\sup_{x\in W}|\widehat{M}_{P_1|P}(t,\lambda)\phi(x)|\leq Ce^{c\|\lambda\|^{n+2}},\quad \lambda\in\mathfrak{a}^*_{P,\mathbb{C}}.$$

Since $A^2(P,\chi,\sigma)$ is finite dimensional, it follows that for each W as above there exist $C_1,c_1>0$ such that

$$\sup_{x\in W}|\widehat{M}_{P_1|P}(t,\lambda)\phi(x)|\leq C_1\parallel\phi\parallel e^{c_1\|\lambda\|^{n+2}} \tag{4.16}$$

for all $\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}$ and $\phi\in A^2(P,\chi,\sigma)$.
In order to proceed we need the following auxiliary result.

Lemma 4.2. *Let $\phi_1,\dots,\phi_m$ be linearly independent continuous functions on $N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A})$. Then there exist continuous functions $\psi_1,\dots,\psi_m\in C^0_c(N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}))$ such that*

$$\det\left(\langle\phi_i,\psi_j\rangle_{i,j=1}^m\right)\neq 0.$$

Proof. First we claim that there exist points $x_1,\dots,x_m$ in the space $N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A})$ such that

$$\det\left((\phi_i(x_j))_{i,j=1}^m\right)\neq 0. \tag{4.17}$$

It is easy to construct $x_1,\dots,x_m$ recursively. Since $\phi_1\neq 0$, there exists x_1 such that $\phi_1(x_1)\neq 0$. Suppose that $x_1,\dots,x_{k-1}$, $2\leq k\leq m$, have been constructed such that

$$\det\left((\phi_i(x_j))_{i,j=1}^{k-1}\right)\neq 0.$$

Let x_k be any point in $N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A})$. Then

$$\det\left((\phi_i(x_j))_{i,j=1}^k\right) = \sum_{j=1}^k (-1)^{j+k} A_j \phi_j(x_k) \tag{4.18}$$

where A_j is the determinant of the minor obtained by deleting the j-th line and the k-th row. In particular,

$$A_k = \det\left((\phi_i(x_j))_{i,j=1}^{k-1}\right)$$

which is nonzero by assumption. Since $\phi_1, \dots, \phi_m$ are linearly independent, we can pick x_k such that the right hand side of (4.18) is nonzero. Therefore, by finite induction, there exist $x_1, \dots, x_m$ such that (4.17) holds.
Now for each i, $1 \le i \le m$, we choose an approximation

$$\{\psi_{i,\varepsilon}\} \subset C_c^0(N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}))$$

of the delta function at x_i, that is, for every continuous function ϕ on $N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A})$ we have

$$\int_{N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A})} \psi_{i,\varepsilon}(x)\phi(x)\, dx \underset{\varepsilon\to 0}{\longrightarrow} \phi(x_i).$$

Then we obtain

$$\det\left(\langle\phi_i, \psi_{j,\varepsilon}\rangle_{i,j=1}^m\right) \underset{\varepsilon\to 0}{\longrightarrow} \det\left((\phi_i(x_j))_{i,j=1}^m\right) \neq 0.$$

Hence, for $\epsilon \ll 1$, $\psi_i = \psi_{i,\varepsilon}$, $1 \le i \le m$, is the desired set of compactly supported functions.

□

We can now establish Theorem 0.1.

Proof of Theorem 0.1. Formula (4.15) expresses $M_{P_1|P}(t,\lambda)$ as a quotient of entire functions and we know already that $F(\lambda)$ is of order $\le n+2$. Therefore, we are left with the estimation of the norm of $\widehat{M}_{P_1|P}(t,\lambda)$.
Let $\phi_1, \dots, \phi_m$ be an orthonormal basis of $A^2(P_1,\chi,\sigma)$ and let $\phi \in A^2(P,\chi,\sigma)$. Then it suffices to estimate $\langle \widehat{M}_{P_1|P}(t,\lambda)\phi, \phi_j\rangle$. Choose $\psi_1, \dots, \psi_m \in C_c^0(N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}))$ as in Lemma 4.2 such that $\det(\langle\phi_i,\psi_j\rangle_{i,j=1}^m) \neq 0$. Then

$$\langle \widehat{M}_{P_1|P}(t,\lambda)\phi, \psi_j\rangle = \sum_{j=1}^{m} \langle \widehat{M}_{P_1|P}(t,\lambda)\phi, \phi_i\rangle\langle \phi_i, \psi_j\rangle.$$

Let $A = (\langle \phi_i, \psi_j\rangle_{i,j=1}^{m})$. Then $\det A \neq 0$, and therefore, A is invertible. Let $A^{-1} = (b_{ij})$. Then

$$\langle \widehat{M}_{P_1|P}(t,\lambda)\phi, \phi_i\rangle = \sum_{j=1}^{m} b_{ij}\langle \widehat{M}_{P_1|P}(t,\lambda)\phi, \psi_j\rangle. \tag{4.19}$$

Since each ψ_j, $1 \leq j \leq m$, has compact support, it follows from (4.16) that there exist $C, c > 0$ such that

$$|\langle \widehat{M}_{P_1|P}(t,\lambda)\phi, \psi_j\rangle| \leq C \parallel \phi \parallel e^{c\|\lambda\|^{n+2}}$$

for $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ and $j = 1, \dots, m$. Therefore, by (4.19) we get

$$\parallel \widehat{M}_{P_1|P}(t,\lambda)\phi \parallel \leq C_1 \parallel \phi \parallel e^{c\|\lambda\|^{n+2}}, \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}.$$

This concludes the proof of Theorem 0.1.

□

Similar results hold for Eisenstein series. Suppose that $Q \supset P$ are parabolic subgroups. Let $\phi \in A^2(P,\chi,\sigma)$. Then by (4.3), ϕ equals a sum of iterated residues of cuspidal Eisenstein series which are determined by a collection $\{F_B \mid B \in \mathcal{P}_\chi, B \subset P\}$ of meromorphic functions F_B from $\mathfrak{a}^*_{B,\mathbb{C}}$ to $A^2(B,\chi,\sigma)$ satisfying the assumptions above. Let $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ be a point in general position. Then it follows from Theorem 7.1 of [L1] that

$$\begin{aligned} &E^Q(P,\phi,\lambda,x) \\ &= \sum_{\{B\in\mathcal{P}_\chi \mid B\subset P\}} \sum_{S\in\mathcal{S}_B(P)} \operatorname{Res}_{S,\Lambda\to\Lambda_S} E^Q(B, F_B(\Lambda), \Lambda+\lambda, x). \end{aligned} \tag{4.20}$$

We can now proceed in exactly the same way as above to prove Theorem 0.2.

Since the integral (1.6) is absolutely convergent for $\operatorname{Re}(\lambda) \in (\mathfrak{a}^*_P)^+ + \rho_P$, each intertwining operator $M_{P_1|P}(t,\lambda)$ is a holomorphic function on the tube domain $\{\lambda \in \mathfrak{a}^*_{P,\mathbb{C}} \mid \operatorname{Re}(\lambda) \in (\mathfrak{a}^*_P)^+ + \rho_P\}$. We shall now estimate $\parallel M_{P_1|P}(t,\lambda) \parallel$ on this domain. Such an estimate is known to exist for the restriction of the intertwining operator to the cuspidal subspace [HC, Lemma 38]. We shall extend it to the noncuspidal case. In addition to that, we need estimates that are uniform with respect to χ and σ_∞.

Recall that $\mathfrak{X}$ is the set of W_0 conjugacy classes of pairs (M_B, r_B), where B is a standard parabolic subgroup and r_B is an irreducible cuspidal automorphic representation of $M_B(\mathbb{A})^1$. Given any irreducible unitary representation π of $M_B(\mathbb{A})^1$, let μ_π be the eigenvalue of the Casimir operator of $M_B(\mathbb{R})$, acting in the Gårding space $\mathcal{H}^\infty_{\pi_\infty}$ of the Archimedean constituent π_∞ of π. Given $\chi \in \mathfrak{X}$ and $(M_B, r_B) \in \chi$, the Casimir eigenvalue μ_{r_B} depends only on the class χ and we denote it by μ_χ. Let τ be a unitary irreducible representation of K_∞. Then we denote by $\lambda_\tau = \tau(\Omega_{K_\infty})$ the eigenvalue of the Casimir operator Ω_{K_∞} of K_∞. We shall now consider a slightly different space of automorphic forms. Let K_f be a compact open subgroup of $G(\mathbb{A}_f)$ and let $\tau \in \Pi(K_\infty)$. Let $A^2(P,\chi,\tau)^{K_f}$ be the subspace of $A^2(P,\chi)$ of all K_f- invariant automorphic forms which transform under K_∞ according to τ. The uniform estimate we are aiming at is based on the following lemma.

Lemma 4.3. *Let P be a standard parabolic subgroup and let $\mathfrak{S}_P$ be a Siegel domain with respect to P. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. Then there exist $C, c > 0$ and $N \in \mathbb{N}$, such that for all $\chi \in \mathfrak{X}$, $\tau \in \hat{K}_\infty$, and $\phi \in A^2(P,\chi,\tau)^{K_f}$,*

$$|\phi(x)| \le C(1 + \mu_\chi^2 + \lambda_\tau^2)^N \parallel \phi \parallel \exp(c\rho_{P_0}(H_{P_0}(x))), \quad x \in \mathfrak{S}_P.$$

Proof. We observe that $A^2(P,\chi,\tau)^{K_f}$ is also spanned by sums of iterated residues of the form (4.3). Consider one of the iterated residues in (4.3), say

$$\psi(x) = \mathrm{Res}_{S,\Lambda\to\Lambda_S} E^P(F_B(\Lambda), \Lambda, x),$$

where $B \in \mathcal{P}_\chi$ and $S \in \mathcal{S}_B(P)$. Then for each $x \in G(\mathbb{A})$, the function $\psi_x(m) = \psi(mx)$ of $m \in M(\mathbb{R})$ is an eigenfunction of the Casimir operator Ω_M of $M(\mathbb{R})$ with eigenvalue $\parallel \Lambda_S \parallel^2 - \parallel \rho_B \parallel^2 + \mu_\chi$. Furthermore, it follows from Langlands' construction that $\parallel \Lambda_S \parallel \le \|\rho_B\|$. Thus we can assume that $\phi \in A^2(P,\chi,\tau)^{K_f}$ is such that for any $x \in G(\mathbb{A})$,

$$\Omega_M \phi_x = \lambda \phi_x \quad \text{and} \quad |\lambda - \mu_\chi| \le \parallel \rho_B \parallel^2.$$

Set

$$K_f^M = M(\mathbb{A}) \cap K_f.$$

Given $k \in K_\infty$, let ϕ_k be the function defined by $\phi_k(x) = \phi(xk)$, $x \in G(\mathbb{A})$. Then ϕ_k can be identified with a square integrable automorphic form on $M(\mathbb{Q})A(\mathbb{R})^0\backslash M(\mathbb{A})$ which is K_f^M-invariant. Recall from section 2 that there exists a finite subset $C \subset M(\mathbb{A}_f)$ such that

$$M(\mathbb{Q})\backslash M(\mathbb{A})/K_f^M \cong \bigsqcup_{c\in C}(\Gamma_c\backslash M(\mathbb{R}))$$

where Γ_c is an arithmetic subgroup of $M(\mathbb{R})$. Therefore, it suffices to consider $\phi \in L^2(\Gamma_c A(\mathbb{R})^0\backslash M(\mathbb{R}), \tau)$ which satisfies

$$\Omega_M\phi = \lambda\phi, \quad |\lambda - \mu_\chi| \leq \| \rho_B \|^2 . \tag{4.21}$$

Let Δ be the Laplacian of $M(\mathbb{R})$ with respect to the left-invariant metric on $M(\mathbb{R})$. Let Ω_K be the Casimir element of K_∞^M. Then

$$\Delta = -\Omega_M + 2\Omega_K.$$

Let $\lambda_\tau = \tau(\Omega_K)$. Then by (4.21), we get

$$\Delta\phi = (-\lambda + 2\lambda_\tau)\phi. \tag{4.22}$$

Now observe that the sectional curvature $k_{M(\mathbb{R})}$ of $\Gamma_c A(\mathbb{R})^0\backslash M(\mathbb{R})$ has a uniform bound $|k_{M(\mathbb{R})}| \leq C_1$. Given $x \in \Gamma_c A(\mathbb{R})^0\backslash M(\mathbb{R})$, let $i(x)$ denote the injectivity radius at x. Set

$$\tilde{\imath}(x) = \min\{C_1^{-1/2}, i(x)\}.$$

Let $p = \dim(A(\mathbb{R})^0\backslash M(\mathbb{R}))$ and put $N = [p/4] + 1$. Then by [CGT, (1.26)], we get for $x \in \Gamma_c A(\mathbb{R})^0\backslash M(\mathbb{R})$,

$$\begin{aligned}|\phi(x)| &\leq C(\tilde{\imath}(x)^{-p/2} \| \phi \|_{B_{\tilde{\imath}(x)}(x)} + \cdots + \tilde{\imath}(x)^{2N-p/2} \| \Delta^N\phi \|_{B_{\tilde{\imath}(x)}(x)}) \\ &\leq C(\tilde{\imath}(x)^{-p/2} \| \phi \| + \cdots + \tilde{\imath}(x)^{2N-p/2} \| \Delta^N\phi \|).\end{aligned}$$

Applying (4.22), it follows that

$$|\phi(x)| \leq C\tilde{\imath}(x)^{-p/2}(1 + \mu_\chi^2 + \lambda_\tau^2)^{p/2} \| \phi \| . \tag{4.23}$$

Using the description of the metric on $\mathfrak{S}_P$ [Bo2], it is easy to see that there exist $C > 0$ such that

$$Ce^{-2\rho_{P_0}(H_{P_0}(x))} \leq \tilde{\imath}(x), \quad x \in \mathfrak{S}_P.$$

Combined with (4.23), the lemma follows.

□

We can now estimate the intertwining operator. Let $P \in \mathcal{P}(M)$, $P_1 \in \mathcal{P}(M_1)$ and let $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$. Suppose that $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ is such that $\mathrm{Re}(\lambda) \in (\mathfrak{a}^*_P)^+ + \rho_P$. Then for any $\phi \in A^2(P, \chi, \sigma)$, $M_{P_1|P}(t, \lambda)\phi(x)$ is given by the absolutely convergent integral (1.6). Then we obtain

$$|M_{P_1|P}(t,\lambda)\phi(x)| \le e^{-(\mathrm{Re}(t\lambda)+\rho_{P_1})(H_{P_1}(x))} \int_{N_1(\mathbb{A})\cap w_t N(\mathbb{A}) w_t^{-1}\backslash N_1(\mathbb{A})} |\phi(w_t^{-1}nx)| e^{(\mathrm{Re}(\lambda)+\rho_P)(H_P(w_t^{-1}nx))}\, dn.$$

Since

$$\mathrm{Vol}((N_1(\mathbb{Q}) \cap w_t N(\mathbb{Q}) w_t^{-1})\backslash(N_1(\mathbb{A}) \cap w_t N(\mathbb{A}) w_t^{-1})) = 1,$$

we can also integrate over

$$(N_1(\mathbb{Q}) \cap w_t N(\mathbb{Q}) w_t^{-1})\backslash N_1(\mathbb{A}).$$

Now observe that

$$N_1(\mathbb{Q}) \cap w_t P(\mathbb{Q}) w_t^{-1} = N_1(\mathbb{Q}) \cap w_t N(\mathbb{Q}) w_t^{-1}$$

(see[MoWa, p.94]). Given $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, put

$$\phi_\lambda(x) = \phi(x) e^{(\lambda+\rho_P)(H_P(x))}.$$

Then, as in [MoWa, p.89], we get

$$\int_{N_1(\mathbb{Q})\cap w_t P(\mathbb{Q}) w_t^{-1}\backslash N_1(\mathbb{A})} |\phi_\lambda(w_t^{-1}nx)|\, dn \le \\ \le \int_{N_1(\mathbb{Q})\backslash N_1(\mathbb{A})} \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} |\phi_\lambda(\gamma n x)|\, dn.$$

Suppose that $\phi \in A^2(P, \chi, \tau)^{K_f}$. Let $\mathfrak{S}_P$ be a Siegel domain with respect to P. By Lemma 4.3 there exist $C, c > 0$ and $N \in \mathbb{N}$ such that for all $m \in \mathfrak{S}_P$

$$|\phi(mk)| \le C(1 + \mu_\chi^2 + \lambda_\tau^2)^N \parallel \phi \parallel \exp(c\rho_{P_0}(H_{P_0}(m))). \tag{4.24}$$

Let $P_0 \in \mathcal{P}(M_0)$ and suppose that $P_0 \subset P$. We may assume that $c \ge 2$. Let $y \colon G(\mathbb{A}) \to \mathbb{R}^*_+$ be a function which satisfies

$$y(nmx) = e^{c\rho_{P_0}(H_{P_0}(m))} y(x), \tag{4.25}$$

for all $n \in N_0(\mathbb{A})$, $m \in M_0(\mathbb{A})$, $x \in G(\mathbb{A})$, and

$$1 < y(k) \text{ for all } k \in K.$$

For $x \in G(\mathbb{A})$, put

$$E^M(x) = \sum_{\gamma \in M(\mathbb{Q}) \cap P_0(\mathbb{Q}) \backslash M(\mathbb{Q})} y(\gamma x).$$

This is an Eisenstein series for $M(\mathbb{A})$. By our choice of y, it is convergent and defines an automorphic form on $N(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A})$. Using (4.24) and (4.25), it follows as in [MoWa, p.88] that

$$|\phi_\lambda(x)| \leq C(1 + \mu_\chi^2 + \lambda_\tau^2)^N \parallel \phi \parallel e^{\mathrm{Re}(\lambda)(H_P(x))} E^M(x)$$

for all $x \in G(\mathbb{A})$. Now observe that

$$\begin{aligned}
&\sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\mathrm{Re}(\lambda)(H_P(\gamma x))} E^M(\gamma x) \\
&\quad = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\mathrm{Re}(\lambda)(H_P(\gamma x))} \sum_{\gamma' \in P_0(\mathbb{Q}) \cap M(\mathbb{Q}) \backslash M(\mathbb{Q})} e^{c\rho_{P_0}(H_{P_0}(\gamma' \gamma x))} \\
&\quad = \sum_{\gamma \in P_0(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{(\mathrm{Re}(\lambda) + c\rho_{P_0})(H_{P_0}(\gamma x))}.
\end{aligned}$$

Let $\omega \subset N_1(\mathbb{A})$ be a compact subset such that $N_1(\mathbb{Q})\omega = N_1(\mathbb{A})$. Given $m \in M(\mathbb{A})$, let $W \subset G(\mathbb{A})$ be any compact subset such that $\omega m K \subset W$. It follows as in [HC, p.45] that there exist $C_1 > 0$ and $H_1 \in \mathfrak{a}_0$ such that

$$\sum_{\gamma \in P_0(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{(\lambda + \rho_{P_0})(H_{P_0}(\gamma x))} \leq C_1 \frac{e^{(\mathrm{Re}(\lambda) + \rho_{P_0})(H_1')}}{\prod_{\alpha \in \Delta_0} |\langle \mathrm{Re}(\lambda) + \rho_{P_0}, \alpha \rangle|}$$

for $\lambda \in (\mathfrak{a}_0^*)^+ + \rho_{P_0}$, $x \in W$. Since $\prod_{\alpha \in \Delta_0} |\langle \mathrm{Re}(\lambda) + \rho_{P_0}, \alpha \rangle| \geq C_2$ for $\lambda \in (\mathfrak{a}_0^*)^+ + \rho_{P_0}$, we get

$$\begin{aligned}
&\int_{N_1(\mathbb{Q}) \backslash N_1(\mathbb{A})} \sum_{\gamma \in P_0(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{(\mathrm{Re}(\lambda) + c\rho_0)(H_{P_0}(\gamma n m k))} \, dn \\
&\qquad \leq \sup_{x \in W} \sum_{\gamma \in P_0(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{(\mathrm{Re}(\lambda) + c\rho_{P_0})(H_{P_0}(\gamma x))} \\
&\qquad \leq C e^{(\mathrm{Re}(\lambda) + \rho_{P_0})(H_1)}.
\end{aligned}$$

Summarizing, we have proved the following lemma.

Lemma 4.4. *Let $N = 2\dim(M(\mathbb{R})/A(\mathbb{R})^0)$. For every compact subset $\omega \subset M(\mathbb{A})$, there exist $C > 0$ and $H_1 \in \mathfrak{a}_0$ such that for all $\chi \in \mathfrak{X}$, $\tau \in \hat{K}_\infty$, and $\phi \in A^2(P,\chi,\tau)^{K_f}$, one has*

$$\begin{aligned} &|M_{P_1|P}(t,\lambda)\phi(mk)| \\ &\qquad \le C \parallel \phi \parallel (1+\mu_\chi^2+\lambda_\tau^2)^N e^{\mathrm{Re}(\lambda)(H_1)}, \end{aligned}$$

for all $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ with $\mathrm{Re}(\lambda) \in (\mathfrak{a}_P^*)^+ + \rho_P$, $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$, $m \in \omega$, and $k \in K$.

Let $\phi_1, \dots, \phi_l$ be an orthonormal basis of $A^2(P,\chi,\tau)^{K_f}$. Choose functions $\psi_1, \dots, \psi_l \in C_c^0(M(\mathbb{Q})\backslash M(\mathbb{A})^1 K)$ as in Lemma 4.2 with respect to $\phi_1, \dots, \phi_l$ such that

$$\det(\langle \phi_i, \psi_j \rangle_{i,j=1}^l) \neq 0.$$

Let (b_{rs}) be the inverse matrix to $(\langle \phi_i, \psi_j \rangle_{i,j=1}^l)$. Then similarly to (4.19), we get

$$\langle M_{P_1|P}(t,\lambda)\phi, \phi_i \rangle = \sum_{j=1}^l b_{ij} \langle M_{P_1|P}(t,\lambda)\phi, \psi_j \rangle.$$

Applying Lemma 4.4, we obtain the following corollary.

Corollary 4.5. *Let the notation be as above. There exist $C > 0$ and $H_1 \in \mathfrak{a}_0$, such that for all $\chi \in \mathfrak{X}$, $\tau \in \hat{K}_\infty$, $\phi \in A^2(P,\chi,\tau)^{K_f}$ and $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$,*

$$\parallel M_{P_1|P}(t,\lambda) \parallel \le C(1+\mu_\chi^2+\lambda_\tau^2)^N e^{\mathrm{Re}(\lambda)(H_1)}$$

for all $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ with $\mathrm{Re}(\lambda) \in (\mathfrak{a}_P^*)^+ + \rho_P$.

5. The rank one intertwining operators

We shall now investigate the rank one intertwining operators. Let $P \in \mathcal{P}(M)$ be of relative rank one and let $P^- = N^- M$ be the opposite parabolic group to P. Let $P_1, \dots, P_r$, $P_1 = P$, be a set of representatives for $\mathcal{P}(M)/W(\mathfrak{a}_P)$. Then $r = 1$ or $r = 2$. Let $\mathcal{P} = \mathcal{P}(M)$. Put

$$A^2(\mathcal{P},\chi,\sigma) = \bigoplus_{j=1}^r A^2(P_j,\chi,\sigma). \tag{5.1}$$

We define the intertwining operator

$$M(z)\colon A^2(\mathcal{P},\chi,\sigma) \to A^2(\mathcal{P},\chi,\sigma) \tag{5.2}$$

as a block matrix in the same way as in section 2, putting

$$M(z) = M_{P|P}(-1, z\lambda), \ z \in \mathbb{C}, \quad \text{if} \quad r = 1, \tag{5.3}$$

and

$$M(z) = \begin{pmatrix} 0 & M_{P_1|P_2}(1, -z\lambda) \\ M_{P_2|P_1}(1, z\lambda) & 0 \end{pmatrix}, \quad z \in \mathbb{C}, \tag{5.4}$$

if $r = 2$. Then $M(z)$ is a meromorphic function of $z \in \mathbb{C}$ which, as in section 2, satisfies

$$M(z)M(-z) = \mathrm{Id}, \quad M(z)^* = M(\overline{z}), \quad z \in \mathbb{C}. \tag{5.5}$$

Our main purpose is to study the meromorphic function $\det M(z)$. As an immediate consequence of Theorem 0.1, we get

Theorem 5.1. *There exist entire functions $h_1(z)$ and $h_2(z)$ of order $\leq n+2$ such that*

$$\det M(z) = \frac{h_1(z)}{h_2(z)}, \quad z \in \mathbb{C}.$$

The following result about the poles of $M(z)$ in the half-plane $\mathrm{Re}(z) \geq 0$ has recently been proved by J. Franke [Fr].

Theorem 5.2. *The poles of $M(z)$ in the half-plane $\mathrm{Re}(z) \geq 0$ are all real.*

Since $M(z)$ is holomorphic for $\mathrm{Re}(z) > \|\rho_P\|$, it follows that all poles of $M(z)$ in the half-plane $\mathrm{Re}(z) \geq 0$ are contained in the interval $(0, \|\rho_P\|]$. In particular, the number of poles of $M(z)$ in $\mathrm{Re}(z) \geq 0$ is finite.

Remark. Recall that for the cuspidal intertwining operator $M_{\mathrm{cusp}}(z)$ a stronger result holds. The poles of $M_{\mathrm{cusp}}(z)$ in $(0, \|\rho_P\|]$ are all simple. The same holds for any cuspidal Eisenstein series $E(\phi, z)$, $\phi \in A^2_{\mathrm{cusp}}(\mathcal{P},\chi,\sigma)$: The poles of $E(\phi, z)$ in $\mathrm{Re}(z) \geq 0$ are all simple and contained in $(0, \|\rho_P\|]$. Furthermore, any residue of a cuspidal Eisenstein series $E(\phi, z)$ at a pole $z_0 \in (0, \|\rho_P\|]$ is square integrable. Each of these statements is false in the noncuspidal case. A counterexample is discussed by Franke in [Fr].

Our next goal is to express $\det M(z)$ as a Blaschke product. For the cuspidal rank one intertwining operators $M_{\text{cusp}}(z)$, such a formula was derived in [Mu1, (6.8)]. To generalize this formula, we need the following auxiliary result.

Proposition 5.3. *Let h_1 and h_2 be two entire functions of order $p \geq 0$. Put $h = h_1/h_2$ and assume that h satisfies the following two conditions:*
1) *$h(z)$ is holomorphic in $\operatorname{Im}(z) > 0$.*
2) *$|h(u)| = 1$ for $u \in \mathbb{R}$.*

Then there exist $C, c > 0$ such that $|h(z)| \leq Ce^{c|z|^p}$ for all z with $\operatorname{Im}(z) \geq 0$.

Proof. To prove the proposition, we shall use some results from Nevanlinna theory. For details we refer to [Ne1, Chap. VI].
Put
$$\ln^+ x = \max(\ln x, 0), \quad x \geq 0.$$
Let f be a meromorphic function on $\mathbb{C}$. For any $r \geq 0$, put
$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\theta})| \, d\theta. \tag{5.6}$$
Furthermore, let $n(r, \infty)$ denote the number of poles of f in the closed disc $D_r = \{z \in \mathbb{C} \mid |z| \leq r\}$ where each pole is counted with its multiplicity. Let
$$N(r, f) = \int_0^r \{n(x, \infty) - n(0, \infty)\} \frac{dx}{x} + n(0, \infty) \ln r. \tag{5.7}$$
Since the function $x \longmapsto n(x, \infty)$ is monoton increasing, it follows that
$$N(r, f) \geq 0 \quad \text{for} \quad r \geq 1. \tag{5.8}$$
Furthermore, one has
$$N(e^u, f) = \int_{-\infty}^u \{n(e^v, \infty) - n(0, \infty)\} \, dv + n(0, \infty) u$$
which shows that $N(e^u, f)$ is a convex function of u. The characteristic function of f is then defined by
$$T(r, f) = m(r, f) + N(r, f), \quad r \geq 0. \tag{5.9}$$
Given $a \in \mathbb{C}$, put

$$m(r,a) = m\left(r, \frac{1}{f-a}\right) \text{ and } N(r,a) = N\left(r, \frac{1}{f-a}\right).$$

Then the first main theorem of Nevanlinna theory states that for all $a \in \mathbb{C}$,

$$m(r,a) + N(r,a) = T(r,f) + \varphi(r,a), \tag{5.10}$$

where $\varphi(r,a)$ is bounded as $r \to \infty$ [Ne1, p.166]. More precisely, let c be the first nonvanishing coefficient of the Laurent expansion of $f(z)-a$ at $z=0$. Then

$$|\varphi(r,a)| \leq |\ln|c|| + \ln^+|a| + \ln 2.$$

In particular, if we put $a=0$ in (5.10), then we get

$$T(r,f^{-1}) = T(r,f) + \varphi(r), \tag{5.11}$$

and $\varphi(r)$ is bounded as $r \to \infty$. Furthermore, if f_1 and f_2 are two meromorphic functions, then one has

$$T(r,f_1 f_2) \leq T(r,f_1) + T(r,f_2) \tag{5.12}$$

[Ne1, p.170].
Now let h_1 and h_2 be two entire functions of order p, which means that there exist $C, c > 0$ such that

$$|h_i(z)| \leq Ce^{c|z|^p} \quad \text{for all} \quad z \in \mathbb{C}. \tag{5.13}$$

Put

$$h(z) = \frac{h_1(z)}{h_2(z)}, \quad z \in \mathbb{C}.$$

If we apply (5.11) and (5.12) to h, we get

$$T(r,h) \leq T(r,h_1) + T(r,h_2^{-1}) = T(r,h_1) + T(r,h_2) + \varphi(r), \tag{5.14}$$

where $\varphi(r)$ stays bounded as $r \to \infty$. Since h_1 and h_2 are entire, it follows that

$$T(r,h_j) = m(r,h_j) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+|h_j(re^{i\theta})|\, d\theta, \quad j=1,2.$$

Now by (5.13) we have

$$\ln^+ |h_j(z)| \le C(1+|z|^p).$$

Using (5.8) and (5.14), we obtain

$$m(r,h) \le T(r,h) \le Cr^p, \quad r \ge 1. \tag{5.15}$$

Now assume that $h(z)$ is holomorphic in

$$\omega_R = \{z \in \mathbb{C} \mid |z| \le R,\ \operatorname{Im}(z) \ge 0\}.$$

Let $z = re^{i\varphi}$, $r \ge 0$. Then the following formula of Nevanlinna[Ne2, p. 29], [Di, p. 18] holds for any z in the interior of ω_R.

$$\begin{aligned} \ln|h(z)| = & \frac{1}{\pi}\int_{-R}^{R} \ln|h(t)| \left\{ \frac{r\sin\varphi}{|z-t|^2} - \frac{R^2 r\sin\varphi}{|R^2 - zt|^2} \right\} dt \\ & + \frac{1}{2\pi}\int_0^{\pi} \ln|h(Re^{i\theta})| \left\{ \frac{R^2-r^2}{|Re^{i\theta}-z|^2} - \frac{R^2-r^2}{|Re^{-i\theta}-z|^2} \right\} d\theta \\ & - \sum_{\substack{|a_m|<R \\ \operatorname{Im}(a_m)>0}} \ln\left| \frac{R^2 - \overline{a}_m z}{R(z-a_m)} \frac{R(z-\overline{a}_m)}{R^2 - a_m z} \right|, \end{aligned} \tag{5.16}$$

where a_m runs over the zeros of h in ω_R. If $|h(t)| = 1$ for $t \in \mathbb{R}$, the first integral on the right hand side vanishes. Suppose that

$$|z| \le \frac{R}{2}. \tag{5.17}$$

Then the second integral on the right hand side of (5.16) can be estimated by

$$\begin{aligned} \frac{1}{2\pi}\int_0^{\pi} & \left|\ln|h(Re^{i\theta})|\right| \, d\theta \cdot 2\frac{R^2}{(R-r)^2} \\ & \le \frac{4}{\pi}\int_0^{\pi} \left|\ln|h(Re^{i\theta})|\right| \, d\theta \\ & = \frac{4}{\pi}\int_0^{\pi} \ln^+\left|h(Re^{i\theta})\right| \, d\theta + \frac{4}{\pi}\int_0^{\pi} \ln^+\left|h(Re^{i\theta})\right|^{-1} \, d\theta. \end{aligned}$$

By (5.15), the first integral can be estimated by CR^p for $R \ge 1$. By symmetry, (5.15) holds also for h^{-1}. Hence, the second integral can also be estimated by $C'R^p$, $R \ge 1$, for some constant $C' > 0$. It remains to estimate the sum. Since $\operatorname{Im}(a_m) \ge 0$, we have

$$\left|\frac{z-\overline{a}_m}{z-a_m}\right| \geq 1 \quad \text{for} \quad \mathrm{Im}(z) \geq 0.$$

This implies that

$$-\sum_{\substack{|a_m|<R\\ \mathrm{Im}(a_m)>0}} \ln\left|\frac{R(z-\overline{a}_m)}{R(z-a_m)}\right| \leq 0.$$

Furthermore, by (5.17) we have $|a_m z| \leq R^2/2$. Hence we get

$$\frac{1}{3} = \frac{R^2-R^2/2}{R^2+R^2/2} \leq \left|\frac{R^2-\overline{a}_m z}{R^2-a_m z}\right| \leq \frac{R^2+R^2/2}{R^2-R^2/2} = 3.$$

Thus

$$-\sum_{\substack{|a_m|<R\\ \mathrm{Im}(a_m)>0}} \ln\left|\frac{R^2-\overline{a}_m z}{R^2-a_m z}\right| \leq \ln 3 \sum_{\substack{|a_m|<R\\ \mathrm{Im}(a_m)>0}} 1 = \ln 3\; n(R),$$

where $n(R)$ denotes the number of zeros, counted with their multiplicity, of $h(z)$ in $|z| \leq R$. Since h is the quotient of the entire functions h_1 and h_2, it is clear that $n(R)$ is bounded from above by the number $n_1(R)$ of zeros of $h_1(z)$ in $|z| \leq R$, where again each zero is counted with its multiplicity. To estimate $n_1(R)$, we use Jensen's formula [Ne1, p.164]. Assume that $h_1(0) \neq 0$. Then

$$\int_0^R n_1(x)\frac{dx}{x} + \ln|h_1(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|h_1(Re^{i\varphi})|\, d\varphi.$$

Let

$$M(R) = \max\{|h_1(z)| \mid |z| = R\}.$$

By (5.13) we have

$$\ln M(R) \leq C_1 R^p \quad \text{for} \quad R \geq 1.$$

Choose R so large that $M(2R) > 1$. Then we get

$$\int_0^{2R} n_1(x)\frac{dx}{x} \leq \ln M(2R) - \ln|h_1(0)|.$$

Using that $n_1(x)$ in monoton increasing, it follows that

$$\begin{aligned} n_1(R) &\leq \frac{1}{\ln 2} \int_R^{2R} n_1(x) \frac{dx}{x} \\ &\leq \frac{1}{\ln 2} \left(\ln M(2R) - \ln |h_1(0)| \right) \\ &\leq C_2 R^p \end{aligned}$$

for $R \geq 1$. Now we put $R = 2r$ in (5.16), where $R = |z|$. Combining our estimates, it follows that there exists $c > 0$ such that

$$\ln |h(z)| \leq c|z|^p, \quad |z| \geq 2, \mathrm{Im}(z) \geq 0.$$

Hence there exists $C > 0$ such that

$$|h(z)| \leq C e^{c|z|^p}, \quad \mathrm{Im}(z) \geq 0.$$

$\square$

Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. Let $\Pi(K, K_f)$ denote the subset of $\Pi(K)$ consisting of all $\sigma \in \Pi(K)$ such that $\sigma_f | K_f = \mathrm{Id}$. Given $\sigma \in \Pi(K, K_f)$, we denote by λ_σ the Casimir eigenvalue of σ_∞.

Lemma 5.4. *Given $\sigma \in \Pi(K)$ and $\chi \in \mathfrak{X}$, let $M(z; \chi, \sigma)$ be the intertwining operator acting in $A^2(\mathcal{P}, \chi, \sigma)$ and let $d_{\chi,\sigma} = \dim A^2(\mathcal{P}, \chi, \sigma)$. Set $a(\chi, \sigma) = (1 + \log(1 + \lambda_\sigma^2 + \mu_\chi^2)) d_{\chi,\sigma}$. There exists $c > 0$ such that*

$$e^{-ca(\chi,\sigma)\,\mathrm{Re}(z)} |\det M(z; \chi, \sigma)| \leq 1 \tag{5.18}$$

for all $\sigma \in \Pi(K, K_f)$, $\chi \in \mathfrak{X}$ and z with $\mathrm{Re}(z) \geq \|\rho_P\|$.

Proof. It follows from Corollary 4.5 that there exist $C, c > 0$ and $N \in \mathbb{N}$, such that for all $\sigma \in \Pi(K, K_f)$, $\chi \in \mathfrak{X}$ and $\phi, \psi \in A^2(\mathcal{P}, \chi, \sigma)$, one has

$$|\langle M(z; \chi, \sigma)\phi, \psi \rangle| \leq C \parallel \phi \parallel\parallel \psi \parallel (1 + \lambda_\sigma^2 + \mu_\chi^2)^N e^{c\,\mathrm{Re}(z)}$$

for $\mathrm{Re}(z) \geq \|\rho_P\|$. Hence we get

$$\begin{aligned} |\det M(z; \chi, \sigma)| &\leq e^{c_1 (1 + d_{\chi,\sigma}) \mathrm{Re}(z)} (1 + \lambda_\sigma^2 + \mu_\chi^2)^{N d_{\chi,\sigma}} \\ &\leq e^{c_2 ((1 + \log(1 + \lambda_\sigma^2 + \mu_\chi^2)) d_{\chi,\sigma}) \mathrm{Re}(z)} \end{aligned}$$

for $\mathrm{Re}(z) \geq \|\rho_P\|$ and a certain constant $c_2 > 0$ which is independent of χ.

$\square$

By Theorem 5.2, $\det M(z)$ has only finitely many poles in $\mathrm{Re}(z) \geq 0$, which we denote by $u_1, \dots, u_l$. These poles are all real and contained in $(0, \|\rho_P\|]$. Thus $u_1, \dots, u_l \in (0, \|\rho_P\|]$. For each i, $1 \leq i \leq l$, let m_i be the order of the pole u_i. Let $q_1 = \exp(c(1 + \log(1 + \lambda_\sigma^2 + \mu_\chi^2))d_{\chi,\sigma})$ where c is the constant in (5.18). Put

$$\xi(z) = q_1^{-z} \prod_{i=1}^{l} \left(\frac{z - u_i}{z + u_i} \right)^{m_i} \det M(z). \tag{5.19}$$

Lemma 5.5. *The analytic function $\xi(z)$ has the following properties:*

1) $\xi(z)\xi(-z) = 1$ *for all* $z \in \mathbb{C}$.
2) $\overline{\xi(z)} = \xi(\overline{z})$ *for all* $z \in \mathbb{C}$.
3) $|\xi(z)| = 1$ *for* $\mathrm{Re}(z) = 0$.
4) $\xi(z)$ *is holomorphic in the half-plane* $\mathrm{Re}(z) > 0$ *and satisfies* $|\xi(z)| \leq 1$ *for* $\mathrm{Re}(z) \geq 0$.

Proof. By (5.5) we have

$$\det M(z) \det M(-z) = 1 \quad \text{and} \quad \overline{\det M(z)} = \det M(\overline{z}), \quad z \in \mathbb{C}. \tag{5.20}$$

Using this fact and the definition of $\xi(z)$, we obtain 1), 2) and 3). The first part of 4) is clear from the definition of $\xi(z)$. To prove the second part, consider the function $\tilde{\xi}(z) = \xi(iz)$. By Theorem 5.1, there exist entire functions f_1 and f_2 of order $\leq n+2$ such that $\tilde{\xi} = f_1/f_2$. By the above results, assumptions 1) and 2) of Proposition 5.3 are satisfied for $\tilde{\xi}$. Hence there exist $C, c > 0$ such that

$$|\xi(z)| \leq C e^{c|z|^{n+2}}, \quad \mathrm{Re}(z) \geq 0. \tag{5.21}$$

By Lemma 5.4 we have

$$|\xi(z)| \leq 1 \quad \text{for} \quad \mathrm{Re}(z) \geq \|\rho_P\|. \tag{5.22}$$

In particular, $|\xi(z)| \leq 1$ for $\mathrm{Re}(z) = \|\rho_P\|$. By 3), we have $|\xi(z)| = 1$ on $\mathrm{Re}(z) = 0$. Since (5.21) holds for $0 \leq \mathrm{Re}(z) \leq \|\rho_P\|$, application of a Phragmen-Lindelöf-type theorem implies that $|\xi(z)| \leq 1$ for $0 \leq \mathrm{Re}(z) \leq |\rho_P|$. Together with (5.22), we obtain 4).

□

By Lemma 5.5, $\xi(z)$ is an analytic function in the half-plane $\mathrm{Re}(z) > 0$ which is continuous and bounded on $\mathrm{Re}(z) \geq 0$. Hence we can apply Carleman′s theorem [T, Section 3.71] which implies the following lemma.

Lemma 5.6. *Let η run over all zeros of $\xi(z)$ in the half-plane* $\operatorname{Re}(z) > 0$*, where each zero is counted with its multiplicity. Then the series*

$$\sum_{\eta} \frac{\operatorname{Re}(\eta)}{|\eta|^2}$$

converges.

From the definition (5.19) of $\xi(z)$ it is clear, that the same statement holds for the zeros of $\det M(z)$. By (5.20) we may as well take the sum over all poles of $\det M(z)$ and get a convergent series.
By Theorem 5.1, there exist entire functions $h_1(z)$ and $h_2(z)$ of order $\leq n+2$ such that

$$\xi(z) = \frac{h_1(z)}{h_2(z)}, \quad z \in \mathbb{C}.$$

Let η be a zero of $\xi(z)$. Then it follows from Lemma 5.5, that $\overline{\eta}$ is also a zero and $-\eta, -\overline{\eta}$ are poles of $\xi(z)$. Note that $\xi(z)$ can have only finitely many zeros in $(0, \infty)$. We denote these zeros by $\rho_1, ..., \rho_m$. For $p \in \mathbb{N}$ let

$$E(u,p) = (1-u)\exp(u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}), \quad u \in \mathbb{C}.$$

Then by Hadamard's factorization theorem we get

$$\xi(z) = e^{P(z)} \prod_{i=1}^{m} \frac{z-\rho_i}{z+\rho_i} \cdot \frac{\prod_{\eta} E\left(\frac{z}{\eta}, n+2\right) E\left(\frac{z}{\overline{\eta}}, n+2\right)}{\prod_{\eta} E\left(\frac{z}{-\eta}, n+2\right) E\left(\frac{z}{-\overline{\eta}}, n+2\right)},$$

where η runs over the zeros of $\xi(z)$ in the quadrant $\operatorname{Re}(z) > 0$, $\operatorname{Im}(z) > 0$, and each zero is counted with its multiplicity. $P(z)$ is a polynomial of order $n+2$. Using Lemma 5.6, we can argue in exactly the same way as in [Mu1, pp. 517-518] to show that

$$\xi(z) = e^{Q(z)} \prod_{i=1}^{m} \frac{z-\rho_i}{z+\rho_i} \cdot \prod_{\eta} \frac{(z-\eta)(z-\overline{\eta})}{(z+\eta)(z+\overline{\eta})}, \tag{5.23}$$

where

$$Q(z) = az + \pi i l, \quad z \in \mathbb{C},$$

for some $a < 0$ and $l \in \mathbb{Z}$. Moreover, the infinite product equals

$$\prod_{\eta} \left(1 - 4z \frac{\operatorname{Re}(\eta)}{(z+\eta)(z+\overline{\eta})} \right),$$

which is absolutely convergent by Lemma 5.6. Thus we have proved the following theorem.

Theorem 5.7. *Let $q = q_1 e^a$ with q_1 and $a < 0$ defined above. Let $u_1, \dots, u_l \in (0, \|\rho_P\|]$ denote the poles of $\det M(z)$ in the half-plane $\operatorname{Re}(z) \geq 0$, and let η run over all poles, counted with their orders, of $\det M(z)$ in the half-plane $\operatorname{Re}(z) < 0$. Then*

$$\det M(z) = \det M(0) q^z \prod_{j=1}^{l} \left(\frac{z+u_j}{z-u_j} \right)^{m_j} \prod_{\eta} \frac{z+\overline{\eta}}{z-\eta}, \quad z \in \mathbb{C}, \tag{5.24}$$

and the infinite product is absolutely convergent.

One can use (5.24) to compute the logarithmic derivative of $\det M(z)$ which gives

$$\begin{aligned} \frac{d}{dz} \log \det M(z) = \log q - \sum_{j=1}^{l} \frac{2 m_j u_j}{z^2 - u_j^2} \\ - \sum_{\substack{\eta \\ \operatorname{Re}(\eta)<0}} \frac{2 \operatorname{Re}(\eta)}{(z-\eta)(z+\overline{\eta})}. \end{aligned} \tag{5.25}$$

Putting $z = i\lambda$, $\lambda \in \mathbb{R}$, we get

$$\begin{aligned} \frac{d}{dz} \log \det M(i\lambda) = \log q + \sum_{j=1}^{l} \frac{2 m_j u_j}{\lambda^2 + u_j^2} \\ + \sum_{\substack{\eta \\ \operatorname{Re}(\eta)<0}} \frac{2 \operatorname{Re}(\eta)}{\operatorname{Re}(\eta)^2 + (\lambda - \operatorname{Im}(\eta))^2}. \end{aligned} \tag{5.26}$$

6. Poles of intertwining operators

In this section we shall investigate the poles of the rank one intertwining operators. Our main purpose is to obtain estimates, which are uniform in χ and σ_∞, of the number of poles of any matrix coefficient of $M(z)$ in a circle of radius R. In [Mu1] and [Mu2], we derived such estimates for cuspidal rank one intertwining operators. The proof can not be extended directly, but we will use the results of the previous sections to reduce the general case to the cuspidal one.

Suppose that $\dim(\mathfrak{a}_M/\mathfrak{a}_G) = 1$. Let $P \in \mathcal{P}(M)$. Let $\chi \in \mathfrak{X}$ and $\sigma \in \Pi(K)$. As in section 4, we denote by μ_χ the Casimir eigenvalue of χ and by λ_σ the Casimir eigenvalue of σ_∞. Throughout this section, we shall fix an open compact subgroup K_f of $G(\mathbb{A}_f)$ and, as above, we denote by $\Pi(K, K_f)$ the subset of $\Pi(K)$ consisting of all $\sigma \in \Pi(K)$ such that $\sigma_f|K_f = \mathrm{Id}$. To begin with, we prove the following lemma.

Lemma 6.1. *Let $m = \dim G$. There exists $C > 0$ such that for all $\chi \in \mathfrak{X}$ and $\sigma \in \Pi(K, K_f)$, we have*

$$\dim A^2(\mathcal{P}, \chi, \sigma) \leq C(1 + |\mu_\chi| + |\lambda_\sigma|)^{5m}.$$

Proof. Let $K_f^M = M(\mathbb{A}_f) \cap K_f$. Then K_f^M is an open compact subgroup of $M(\mathbb{A})$. As explained in section 2, there exist arithmetic subgroups $\Gamma_c \subset M(\mathbb{R})$, parametrized by a finite subset $C \subset M(\mathbb{A}_f)$, such that

$$M(\mathbb{Q})\backslash M(\mathbb{A})/K_f^M \cong \bigsqcup_{c \in C} (\Gamma_c \backslash M(\mathbb{R}))$$

(see (2.8)). Let $K_\infty^M = M(\mathbb{R}) \cap K_\infty$ and put

$$X_M = Z_M(\mathbb{R})^0 \backslash M(\mathbb{R})/K_\infty^M.$$

Let σ_∞^M be the restriction of σ to K_∞^M. Then σ_∞^M defines a homogeneous vector bundle $\tilde{E}_\sigma$ over X_M which can be pushed down to a locally homogeneous vector bundle E_σ over each $\Gamma_c \backslash X_M$. Let $\Delta_\sigma = (\nabla^\sigma)^* \nabla^\sigma$ be the Bochner-Laplace operator of the canonical connection ∇^σ on E_σ. This is an elliptic operator acting in $C^\infty(\Gamma_c \backslash X_M, E_\sigma)$, and with respect to the inner product induced by the metrics on $\Gamma_c \backslash X_M$ and E_σ, Δ_σ is formally self-adjoint. The point spectrum of Δ_σ consists of a sequence of eigenvalues $0 \leq \mu_1 < \mu_2 < \cdots$ with finite-dimensional eigenspaces $\mathcal{E}(\mu_i)$. Given $\mu \in \mathbb{R}^+$, let

$$L^2_{disc,\mu}(\Gamma_c \backslash X_M, E_\sigma) = \bigoplus_{\mu_i \leq \mu} \mathcal{E}(\mu_i).$$

Now observe that there is a canonical isomorphism

$$C^\infty(\tilde{E}_\sigma) \cong (C^\infty(Z_M(\mathbb{R})^0\backslash M(\mathbb{R})) \otimes V_{\sigma_\infty})^{K_\infty^M}.$$

With respect to this isomorphism, one has

$$\tilde{\Delta}_\sigma = -R(\Omega_M) \otimes \mathrm{Id} + \lambda_\sigma \mathrm{Id}, \tag{6.1}$$

where R is the right regular representation (see [Mu2, Section3]). Let

$$\nu = |\mu_\chi| + |\lambda_\sigma| + \|\rho_P\|. \tag{6.2}$$

Using the identifications described in section 2, together with (6.1) and (4.21), it follows that for each $\sigma \in \Pi(K, K_f)$, $A^2(\mathcal{P}, \chi, \sigma)$ can be identified with a subspace of

$$\bigoplus_{c\in C} \dim(\sigma_\infty) L^2_{disc,\nu}(\Gamma_c\backslash X_M, E_\sigma).$$

By Theorem 0.2 of [Mu2], we have

$$\dim L^2_{disc,\nu}(\Gamma_c\backslash X_M, E_\sigma) \leq C(1 + \lambda_\sigma^{4p} + \nu^{4p}), \tag{6.3}$$

where $p = \dim M(\mathbb{R})$. Furthermore, by Weyl's dimension formula, there exists $C_1 > 0$ and $l \leq 1/2 \dim K_\infty$ such that

$$\dim(\tau) \leq C_1(1 + \lambda_\tau^l) \tag{6.4}$$

for all $\tau \in \widehat{K}_\infty$. Combining (6.2) - (6.4), we obtain the desired estimation of $\dim A^2(\mathcal{P}, \chi, \sigma)$.

$\square$

First we deal with the cuspidal intertwining operators. Let $M_{\mathrm{cusp}}(z; \chi, \sigma)$ be the restriction of $M(z; \chi, \sigma)$ to the space $A^2_{\mathrm{cusp}}(\mathcal{P}, \chi, \sigma)$. We recall that for $P \in \mathcal{P}_\chi$, $A^2_{\mathrm{cusp}}(P, \chi, \sigma) = A^2(P, \chi, \sigma)$ and $A^2_{\mathrm{cusp}}(P, \chi, \sigma) = \{0\}$ if $P \notin \mathcal{P}_\chi$. Therefore,

$$M_{\mathrm{cusp}}(z; \chi, \sigma) = \begin{cases} M(z; \chi, \sigma) & ; \quad \text{if } P \in \mathcal{P}_\chi, \\ 0 & ; \quad \text{if } P \notin \mathcal{P}_\chi. \end{cases}$$

Let $\eta \in \mathbb{C}$. If η is a pole of $\det M_{\mathrm{cusp}}(z; \chi, \sigma)$, let $\nu(\eta; \chi, \sigma)$ denote the order of the pole η and if η is not a pole of $\det M_{\mathrm{cusp}}(z; \chi, \sigma)$, we put $\nu(\eta; \chi, \sigma) = 0$. Given $R \geq 0$, let

$$N_{\mathrm{cusp}}(R; \chi, \sigma) = \sum_{\substack{\eta\in\mathbb{C} \\ |\eta|\leq R}} \nu(\eta; \chi, \sigma).$$

Theorem 6.2. *Let $m = \dim G$. There exists $C > 0$ such that for all $\chi \in \mathfrak{X}$, $\sigma \in \Pi(K, K_f)$ and $R \geq 0$, one has*

$$N_{\text{cusp}}(R; \chi, \sigma) \leq C(1 + R^2 + \mu_\chi^2 + \lambda_\sigma^2)^{3m}.$$

Proof. By Theorem 5.2, $\det M_{\text{cusp}}(z; \chi, \sigma)$ has only finitely many poles in the half-plane $\operatorname{Re}(z) \geq 0$ which are all real. Let $u_1, ..., u_l \in (0, \|\rho_P\|]$ denote these poles. Put

$$N_{\text{cusp}}^+(\chi, \sigma) = \sum_{i=1}^{l} \nu(u_i; \chi, \sigma).$$

By the adèlic version of Theorem 7.3 of [Mu2] there exists $C > 0$ such that

$$N_{\text{cusp}}^+(\chi, \sigma) \leq C(1 + |\mu_\chi|^{3m} + |\lambda_\sigma|^{3m}) \tag{6.5}$$

for all $\chi \in \mathfrak{X}$ and $\sigma \in \Pi(K, K_f)$. Let

$$N_{\text{cusp}}^-(R; \chi, \sigma) = N_{\text{cusp}}(R; \chi, \sigma) - N_{\text{cusp}}^+(\chi, \sigma).$$

In order to estimate $N_{\text{cusp}}^-(R; \chi, \sigma)$, we first observe that by the adèlic version of Theorem 7.1 of [Mu2] there exists $C > 0$ such that for all $\chi \in \mathfrak{X}$, $\sigma \in \Pi(K, K_f)$ and $R \geq 0$, we have

$$\begin{aligned} &\left| \int_{-R}^{R} \frac{d}{dz} \log \det M_{\text{cusp}}(iu; \chi, \sigma)\, du \right| \\ &\qquad\qquad \leq C(1 + R^{2m} + \mu_\chi^{2m} + \lambda_\sigma^{2m}). \end{aligned} \tag{6.6}$$

Next observe that for any $R \geq 0$, we have

$$\sum_{j=1}^{l} 2\nu(u_j; \chi, \sigma) \int_{-R}^{R} \frac{u_j}{u_j^2 + \lambda^2}\, d\lambda \leq 2\pi N_{\text{cusp}}^+(\chi, \sigma). \tag{6.7}$$

Using (5.26), (6.5), (6.6) and (6.7), it follows that for all $R \geq 0$, $\chi \in \mathfrak{X}$ and $\sigma \in \Pi(K, K_f)$,

$$\begin{aligned} &\left| 2R \log q + \int_{-R}^{R} \sum_{\substack{\eta \\ \operatorname{Re}(\eta) < 0}} \frac{2 \operatorname{Re}(\eta)}{\operatorname{Re}(\eta)^2 + (\lambda - \operatorname{Im}(\eta))^2}\, d\lambda \right| \\ &\qquad \leq 2\pi N_{\text{cusp}}^+(\chi, \sigma) + C(1 + R^2 + \mu_\chi^2 + \lambda_\sigma^2)^{2m} \\ &\qquad \leq C_1(1 + R^2 + \mu_\chi^2 + \lambda_\sigma^2)^{2m}, \end{aligned} \tag{6.8}$$

where $C_1 > 0$ is independent of χ and σ. We distinguish two cases:

a) $q \geq 1$. Then it follows from the definition of q that

$$0 \leq \log q \leq c(1+\log(1+\lambda_\sigma^2+\mu_\chi^2))\, d_{\chi,\sigma},$$

where $d_{\chi,\sigma} = \dim A^2(\mathcal{P},\chi,\sigma)$ and $c>0$ is independent of χ and $\sigma \in \Pi(K,K_f)$. By Lemma 6.1, it follows that

$$|\log q| \leq C(1+\mu_\chi^2+\lambda_\sigma^2)^{3m}.$$

Hence, by (6.8) we get

$$\begin{aligned}\int_{-R}^{R} \sum_{\substack{\eta\\ \operatorname{Re}(\eta)<0}} \frac{|\operatorname{Re}(\eta)|}{\operatorname{Re}(\eta)^2+(\lambda-\operatorname{Im}(\eta))^2}\, d\lambda \\ \leq C_2(1+R^2+\mu_\chi^2+\lambda_\sigma^2)^{3m}.\end{aligned} \tag{6.9}$$

b) $q<1$. Then $\log q<0$. Since all terms in the series on the left-hand side of (6.8) are negative, (6.9) holds in this case too.

Using (6.9) we obtain

$$\begin{aligned} N^-_{\text{cusp}}(R;\chi,\sigma) &\int_{-1}^{1} \frac{1}{1+\lambda^2}\, d\lambda \\ &\leq \sum_{\substack{|\eta|\leq R\\ \operatorname{Re}(\eta)<0}} \int_{-2R}^{2R} \frac{|\operatorname{Re}(\eta)|}{\operatorname{Re}(\eta)^2+(\lambda-\operatorname{Im}(\eta))^2}\, d\lambda \\ &\leq \int_{-2R}^{2R} \sum_{\substack{\eta\\ \operatorname{Re}(\eta)<0}} \frac{|\operatorname{Re}(\eta)|}{\operatorname{Re}(\eta)^2+(\lambda-\operatorname{Im}(\eta))^2}\, d\lambda \\ &\leq C(1+R^2+\mu_\chi^2+\lambda_\sigma^2)^{3m}. \end{aligned}$$

□

Let $\Phi_1,\dots,\Phi_d$ be an orthonormal basis in $A^2(\mathcal{P},\chi,\sigma)$. If η is a pole of $M_{\text{cusp}}(z;\chi,\sigma)$, let $\kappa_{ij}(\eta,\chi,\sigma)$ be the order of $\langle M_{\text{cusp}}(z;\chi,\sigma)\Phi_i,\Phi_j\rangle$ at η. Set $\kappa(\eta;\chi,\sigma)=\max_{i,j}\kappa_{ij}(\eta;\chi,\sigma)$. If η is not a pole of $M_{\text{cusp}}(z;\chi,\sigma)$, set $\kappa(\eta;\chi,\sigma)=0$.

Corollary 6.3. *Let $m=\dim G$. There exists $C>0$ such that for all $\chi\in\mathfrak{X}$, $\sigma\in\Pi(K,K_f)$ and $R\geq 0$, one has*

$$\sum_{\substack{\eta\in\mathbb{C}\\ |\eta|\leq R}} \kappa(\eta;\chi,\sigma) \leq C(1+R^2+\mu_\chi^2+\lambda_\sigma^2)^{4m}.$$

proof. We split the sum as follows

$$\sum_{|\eta|\leq R} \kappa(\eta;\chi,\sigma) = \sum_{\substack{|\eta|\leq R\\ \mathrm{Re}(\eta)<0}} \kappa(\eta;\chi,\sigma) + \sum_{\substack{|\eta|\leq R\\ \mathrm{Re}(\eta)>0}} \kappa(\eta;\chi,\sigma).$$

Let $\eta \in \mathbb{C}$, $\mathrm{Re}(\eta) > 0$. If η is a pole of $M_{\mathrm{cusp}}(z;\chi,\sigma)$ then η is real and $\nu(\eta;\chi,\sigma) = 1$. Thus the second sum equals the number of poles of $M_{\mathrm{cusp}}(z;\chi,\sigma)$ in $\mathrm{Re}(z) \geq 0$. Using the adèlic version of Corollary 6.2 of [Mu2], this number can be estimated by $C(1+\mu_\chi^2+\lambda_\sigma^2)^{2m}$, where $m = \dim G$ and $C > 0$ is a constant which is independent of χ and σ. It remains to estimate the first sum. Let $\eta \in \mathbb{C}$, $\mathrm{Re}(\eta) < 0$, be a pole of $M_{\mathrm{cusp}}(z;\chi,\sigma)$. We distinguish two cases.
a) $M_{\mathrm{cusp}}(z;\chi,\sigma)$ is holomorphic in $-\eta$. Then it follows as in the proof of Corollary 7.6 of [Mu1] that

$$\kappa(\eta;\chi,\sigma) \leq \nu(\eta;\chi,\sigma)$$

b) $-\eta$ is a pole of $M_{\mathrm{cusp}}(z;\chi,\sigma)$. Arguing again as in the proof of Corollary 7.6 of [Mu1], we get

$$\kappa(\eta;\chi,\sigma) \leq \nu(\eta;\chi,\sigma) + 2\dim A^2(P,\chi,\sigma).$$

Thus

$$\sum_{\substack{|\eta|\leq R\\ \mathrm{Re}(\eta)<0}} \kappa(\eta;\chi,\sigma) \leq N_{\mathrm{cusp}}(R,\chi,\sigma) + 2\dim A^2(P,\chi,\sigma) N^+_{\mathrm{cusp}}(\chi,\sigma).$$

Using Lemma 6.1, (6.5) and Theorem 6.2, the desired estimate follows. □

Let $P \in \mathcal{P}(M)$, $P_1 \in \mathcal{P}(M_1)$ and let $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$. Using (2.18) and (2.19) this result extends to all matrix coefficients of

$$M_{P_1|P}(t,\lambda) : A^2_{\mathrm{cusp}}(P,\chi,\sigma) \to A^2_{\mathrm{cusp}}(P_1,\chi,\sigma).$$

i.e. we have

Corollary 6.4. *Let $m = \dim G$. There exists $C > 0$ such that for all $\chi \in \mathfrak{X}$, $\sigma \in \Pi(K, K_f)$, $R \geq 0$, $\phi \in A^2_{\mathrm{cusp}}(P,\chi,\sigma)$ and $\psi \in A^2_{\mathrm{cusp}}(P_1,\chi,\sigma)$, the number of poles, counted with their order, of the meromorphic function $\lambda \in \{\Lambda \in \mathfrak{a}^*_{M,\mathbb{C}} \mid\mid\mid \Lambda \| \leq R\} \mapsto \langle M_{P_1|P}(t,\lambda)\phi, \psi\rangle$ in is bounded by $C(1+R^2+\mu_\chi^2+\lambda_\sigma^2)^{4m}$.*

We can now proceed to estimate the number of poles of the matrix coefficients of all rank one intertwining operators.

As above, let $P \in \mathcal{P}(M)$, $P_1 \in \mathcal{P}(M_1)$ and $t \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$. We recall that $A^2(P, \chi, \sigma)$ is spanned by sums of iterated residues of the form (4.3), and for any such ϕ, $M_{P_1|P}(t, \lambda)\phi$ is given by formula (4.7). Now consider any of the cuspidal intertwining operators

$$M_{B_1|B}(t_B, \Lambda) : A^2_{\mathrm{cusp}}(B, \chi, \sigma) \to A^2_{\mathrm{cusp}}(B_1, \chi, \sigma) \tag{6.10}$$

occurring in (4.7). Given $S \in \mathcal{S}_B(P)$, let $\Sigma(B, S)$ be the set of all points $\lambda_0 \in \mathfrak{a}^*_{P,\mathbb{C}}$ such that there exist $\Phi \in A^2_{\mathrm{cusp}}(B, \chi, \sigma)$, $\Psi \in A^2_{\mathrm{cusp}}(B_1, \chi, \sigma)$ and a singular hyperplane of $\langle M_{B_1|B}(t_B, \Lambda)\Phi, \Psi\rangle$ which intersects $\mathfrak{t}_{S,\mathbb{C}} = \Lambda_S + \mathfrak{a}^*_{P,\mathbb{C}}$ transversally in the point $\Lambda_S + \lambda_0$. Then the following Lemma is an immediate consequence of (4.6) and (4.7).

Lemma 6.5. *For each $\phi \in A^2(P, \chi, \sigma)$ and $\psi \in A^2(P_1, \chi, \sigma)$, the set of poles of $\langle M_{P_1|P}(t, \lambda)\phi, \psi\rangle$ is contained in the set*

$$\bigcup_{\{B \in \mathcal{P}_\chi | B \subset P\}} \bigcup_{S \in \mathcal{S}_B(P)} \Sigma(B, S).$$

As explained in section 4, the points Λ_S, $S \in \mathcal{S}_B(P)$, are all contained in the compact set $\{\Lambda \in (\mathfrak{a}^P_B)^* \mid \| \Lambda \| \leq \|\rho_P\|\}$. Using this fact and Lemma 6.5, the estimation of the number of poles, counted with their order, of $\langle M_{P_1|P}(t, \lambda)\phi, \psi\rangle$ in the disk $D_R = \{\lambda \in \mathfrak{a}^*_{P,\mathbb{C}} \mid \| \lambda \| \leq R\}$ can be reduced to the following problem: Consider any matrix coefficient

$$a_{\Phi,\Psi}(\Lambda) = \langle M_{B_1|B}(t_B, \Lambda)\Phi, \Psi\rangle$$

of the cuspidal intertwining operator (6.10), where $\Phi \in A^2_{\mathrm{cusp}}(B, \chi, \sigma)$ and $\Psi \in A^2_{\mathrm{cusp}}(B_1, \chi, \sigma)$. Let $\mathfrak{h} = \{\mathcal{H}_1, \dots, \mathcal{H}_r\}$ be a set of singular hyperplanes of $a_{\Phi,\Psi}(\Lambda)$ satisfying

$$\bigcap_k \mathcal{H}_k = \{\Lambda_{\mathfrak{h}}\} \text{ and } \| \Lambda_{\mathfrak{h}} \| \leq \|\rho_P\| + R. \tag{6.11}$$

Let $m(\mathcal{H}_k)$ denote the order of the singularity of $a_{\Phi,\Psi}(\Lambda)$ along $\mathcal{H}_k$ and put

$$m(\mathfrak{h}) = \sum_{k=1}^{r} m(\mathcal{H}_k).$$

Put

$$N_B(R;\chi,\sigma)=\sum_{\mathfrak{h}} m(\mathfrak{h}),$$

where the sum runs over all systems $\mathfrak{h}$ of singular hyperplanes of of $a_{\Phi,\Psi}(\Lambda)$ such that (6.11) holds. Then by Lemma 6.5, our problem of estimating the number of poles of $\langle M_{P_1|P}(t,\lambda)\phi,\psi\rangle$ in D_R is reduced to the estimation of each of the numbers $N_B(R;\chi,\sigma)$. By (2.19) we can assume that $B_1\in\mathcal{P}(M_B)$. Furthermore, by (2.20) and the discussion following this equation, it suffices to consider $M_{B_1|B}(1,\Lambda)$. Finally, as in (2.22), there is a sequence of adjacent chambers $C_{i_1},\dots,C_{i_N}$ in $\mathfrak{a}_B$ with associated parabolic subgroups $B_{i_k}\in\mathcal{P}(M_B)$ such that $B_{i_1}=B$, $B_{i_N}=B_1$, and

$$\begin{aligned} M_{B_1|B}(1,\Lambda) = M_{B_{i_N}|B_{i_{N-1}}}(1,\Lambda)\cdots \\ \cdots M_{B_{i_3}|B_{i_2}}(1,\Lambda)M_{B_{i_2}|B_{i_1}}(1,\Lambda),\quad \Lambda\in\mathfrak{a}^*_{B,\mathbb{C}}. \end{aligned} \tag{6.12}$$

By (2.25), each $M_{B_{i_{k+1}}|B_{i_k}}(1,\Lambda)$ is induced from a rank one cuspidal intertwining operator

$$M_{B^*_{i_{k+1}}|B^*_{i_k}}(1,\Lambda^*): A^2(B^*_{i_k},\chi,\sigma')\to A^2(B^*_{i_{k+1}},\chi,\sigma'),$$

where Λ^* is the projection of Λ to $\mathfrak{a}^*_{B^*_{i_k},\mathbb{C}}$. For each k, $k=1,...,N-1$, we choose an orthonormal basis $\phi_{1,k},...,\phi_{d_k,k}$ in $A^2(B^*_{i_k},\chi,\sigma')$. Given a pole η^* of $M_{B^*_{i_{k+1}}|B^*_{i_k}}(1,\Lambda^*)$, we denote by $\kappa_{ij,k}(\eta^*;\chi,\sigma')$ the order of $\langle M_{B^*_{i_{k+1}}|B^*_{i_k}}(1,\Lambda^*)\phi_{i,k},\phi_{j,k+1}\rangle$ in the possible pole η^*. Put

$$\kappa_k(\eta^*;\chi,\sigma')=\max_{i,j}\kappa_{ij,k}(\eta^*;\chi,\sigma').$$

Then it follows from (6.12) that there exists $c>0$, independent of χ and σ, such that

$$N_B(R;\chi,\sigma)\le\prod_{k=1}^{N-1}\left(\sum_{\|\eta^*\|\le c+R}\kappa_k(\eta^*;\chi,\sigma')\right).$$

The right hand side can be estimated by Corollary 6.4. Since N is bounded by the number of chambers in $\mathfrak{a}_B$, we get

Proposition 6.6. *Let $m=\dim G$ and let r be the number of chambers in $\mathfrak{a}_B$ where B is any minimal parabolic subgroup of G. Then there exists $C>0$ such that for all $\chi\in\mathfrak{X}$, $\sigma\in\Pi(K,K_f)$, $R\ge 0$, $\phi\in A^2(P,\chi,\sigma)$ and $\psi\in A^2(P_1,\chi,\sigma)$, the number of poles, counted with their order, of $\langle M_{P_1|P}(t,\lambda)\phi,\psi\rangle$ in the disc of radius R is bounded by $C(1+R^2+\mu_\chi^2+\lambda_\sigma^2)^{4rm}$.*

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