RELATIVE ZETA FUNCTIONS, RELATIVE DETERMINANTS AND SCATTERING THEORY

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Abstract. We use the method of zeta function regularization to regularize the ratio $\det A/\det A_0$ of the determinants of two elliptic self-adjoint operators A, A_0 satisfying certain natural assumptions. This is of interest, especially, if the regularized determinants of the individual operators don't exist as, for example, in the case of elliptic operators on a noncompact manifold.

0. Introduction

In the present paper we use the method of zeta function regularization to regularize the ratio $\det A/\det A_0$ of the determinants of two elliptic self–adjoint operators A and A_0, acting in the space of smooth section of a vector bundle over a C^∞ manifold X. If X is closed, then the zeta function technique can be used to regularize the determinants of the individual operators A and A_0, and the above ratio is simply the ratio of these regularized determinants. However, if X is noncompact then, in general, an elliptic self–adjoint operator A on X has continuous spectrum and therefore, the zeta function of A cannot be defined. Since regularized determinants of elliptic operators play an important role in various fields of mathematics and physics (see [Mu6] for some references concerning applications), it is interesting to see how this technique can be extended to the case of elliptic operators on noncompact manifolds.

First, we recall some facts about the zeta function regularization of determinants of elliptic operators on compact manifolds. Let E be a complex vector bundle over a closed n–dimensional C^∞ manifold M and let $A : C^\infty(E) \to C^\infty(E)$ be an elliptic pseudo-differential operator of order $m > 0$. Suppose that A is symmetric and nonnegative (with respect to an inner product in $C^\infty(E)$ induced by the choice of a metric

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on M and a fibre metric in E). Then the zeta function of A is defined by

$$\zeta_A(s) = \sum_{\lambda_j > 0} \lambda_j^{-s}, \tag{0.1}$$

where the λ_j's run over the eigenvalues of A, and each eigenvalue is counted with its multiplicity. The series (0.1) is absolutely convergent in the half–plane $\mathrm{Re}(s) > n/m$ and admits a meromorphic extension to the entire complex plane [Se]. Moreover, $s = 0$ is not a pole of $\zeta_A(s)$. We note that the zeta function can also be expressed in terms of the heat operator by

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \big(\mathrm{Tr}(e^{-tA}) \;-\; \dim\ker A\big)\, dt, \quad \mathrm{Re}(s) > n/m, \tag{0.2}$$

and the well known heat expansion can be used to obtain the analytic continuation of $\zeta_A(s)$ (see e.g. [Gi]). Using the zeta function, the regularized determinant of A is defined by

$$\det A = \exp\left(-\frac{d}{ds} \zeta_A(s)\Big|_{s=0} \right). \tag{0.3}$$

The zeta function regularization was first introduced by Ray and Singer [RS1] to define a regularized determinant for the Laplacian on differential forms twisted by a flat bundle. Hawking [H] has used the same method to regularize quadratic path integrals on a curved background spacetime.

In some cases it is also possible to define a determinant for self–adjoint elliptic operators that are unbounded from below. For example, let $P : C^\infty(E) \to C^\infty(E)$ be an elliptic self–adjoint differential operator of order $m > 0$. For such operators, Atiyah, Patodi and Singer [APS1],[APS3] introduced the so called η-function which in the half–plane $\mathrm{Re}(s) > n/m$ is given by

$$\begin{aligned} \eta_P(s) &= \sum_{\lambda_j \neq 0} \frac{\mathrm{sign}\lambda_j}{|\lambda_j|^s} \\ &= \frac{1}{\Gamma\left((s+1)/2\right)} \int_0^\infty t^{(s-1)/2} \mathrm{Tr}\big(P e^{-tP^2}\big)\, dt. \end{aligned} \tag{0.4}$$

Here λ_j runs over the nonzero eigenvalues of P. The analytic function $\eta_P(s)$ has a meromorphic extension to $\mathbb{C}$ with isolated simple poles. The residues at all the poles are locally computable, and $s = 0$ is not a pole of $\eta_P(s)$ [APS3], [Gi]. The fact that $\eta_P(s)$ is regular at $s = 0$

is less obvious than in the case of the zeta function. Using $\eta_P(0)$, one can define a regularized determinant for P by

$$\det P = \det|P| \cdot \exp\biggl(\frac{\pi i}{2}\bigl(\eta_P(0) - \zeta_{|P|}(0)\bigr)\biggr). \tag{0.5}$$

[Si].

As explained above, for elliptic self–adjoint operators on a noncompact manifold X, in general, the zeta function (0.1) cannot be defined. To introduce an appropriate zeta function, we consider pairs (A, A_0) of elliptic self–adjoint operators such that A is a compactly supported perturbation of A_0. Then A_0 serves as a reference operator and our goal is to regularize $\det A/\det A_0$. In some special cases, this question has been studied in [Br], [GMS], [Lu], [JL] and [Mu2] (see §4). Assume, in addition, that A and A_0 are nonnegative. Then, under some natural assumptions, we can expect that $e^{-tA} - e^{-tA_0}$ is a trace class operator for each $t > 0$, and that $\operatorname{Tr}\left(e^{-tA} - e^{-tA_0}\right)$ has an asymptotic expansion as $t \to 0$. This holds, for example, for spinor Laplace operators on complete Riemannian manifolds [Bu]. In order to define the relative zeta function $\zeta(s; A, A_0)$ by a formula which is analogous to (0.2), we have to investigate the behaviour of $\operatorname{Tr}\left(e^{-tA} - e^{-tA_0}\right)$ as $t \to \infty$. For this purpose we use Krein's theory of the spectral shift function [Kr], [BY], [Y], which implies that the spectral shift function $\xi(\lambda) = \xi(\lambda; A, A_0)$ of (A, A_0) exists and the following trace formula holds

$$\operatorname{Tr}\left(e^{-tA} - e^{-tA_0}\right) = -t \int_0^\infty e^{-t\lambda}\xi(\lambda)\, d\lambda. \tag{0.6}$$

The spectral shift function is locally integrable and locally constant on the complement of $\sigma(A) \cup \sigma(A_0)$. Moreover, $e^{-t\lambda}\xi(\lambda)$ is absolutely integrable on $\mathbb{R}^+$. Thus, if the essential spectrum of A_0 has a positive lower bound, then the same holds for A and it follows from (0.6) that there exists $c > 0$ such that

$$\operatorname{Tr}\left(e^{-tA} - e^{-tA_0}\right) = \dim\ker A - \dim\ker A_0 + O(e^{-ct}) \tag{0.7}$$

as $t \to \infty$. This resembles the behaviour of the trace of the heat operator in the compact case and we can define a relative zeta function by

$$\zeta(s; A, A_0) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\left(\operatorname{Tr}\bigl(e^{-tA} - e^{-tA_0}\bigr) - b\right)\, dt, \quad \operatorname{Re}(s) \gg 0,$$

where $b = \dim\ker A - \dim\ker A_0$. However, if the continuous spectrum of A_0 extends down to zero, (0.7) does not hold. Then the large time behaviour of $\operatorname{Tr}\left(e^{-tA} - e^{-tA_0}\right)$ is more complicated and this is where scattering theory comes into play. First of all, by (0.6), it follows that the large time behaviour of $\operatorname{Tr}\left(e^{-tA} - e^{-tA_0}\right)$ is related to the behaviour of the spectral shift function $\xi(\lambda)$ near $\lambda = 0$. To study $\xi(\lambda)$ near $\lambda = 0$ we shall use scattering theory. Since $e^{-tA} - e^{-tA_0}$ is a trace class operator, the Birman-Krein invarinace principle for wave operators [K1], [K2] implies that the Wave operators $W_\pm(A, A_0)$ exist and are complete and therefore, the scattering matrix $S(\lambda) = S(\lambda; A, A_0)$, $\lambda \in \sigma_{ac}(A_0)$, exists. Furthermore, the determinant of the scattering matrix is related to the spectral shift function by

$$\det S(\lambda) = e^{-2\pi i \xi(\lambda)}, \quad \lambda \in \sigma_{ac}(A_0),$$

[BK], [BY], [Y]. Thus, the investigation of the large time behaviour of $\operatorname{Tr}\left(e^{-tA} - e^{-tA_0}\right)$ can be reduced to the study of $\log \det S(\lambda)$ near $\lambda = 0$. For typical examples (see §4), $\det S(\lambda)$ is differentiable near zero and (0.6) specializes to a more explicit trace formula involving the eigenvalues and the scattering phase shift $\frac{d}{d\lambda} \log \det S(\lambda)$. This trace formula can be used to determine the asymptotic behaviour of $\operatorname{Tr}\left(e^{-tA} - e^{-tA_0}\right)$ as $t \to \infty$. Then, by a modification of (0.2), we can define a relative zeta function $\zeta(s; A, A_0)$ which is a meromorphic function of $s \in \mathbb{C}$. This zeta function has some new features that do not occur in the compact case. For example, there are poles whose residues are related to the derivatives of $\det S(\lambda)$ at $\lambda = 0$ and therefore, these residues are not locally computable as in the compact case. If $\zeta(s; A, A_0)$ is holomorphic at $s = 0$, we can introduce the regularized ratio

$$\det(A, A_0) = \frac{\det A}{\det A_0}$$

by the analogous formula (0.3) with $\zeta_A(s)$ replaced by $\zeta(s; A, A_0)$.

We note that we may think of A_0 as being a reference operator and regard $\det(A, A_0)$ as a function of A. In particular, we can use $\det(A, A_0)$ in much the same way as $\det A$ is used in the case of elliptic operators on a compact manifold.

Next we briefly describe the content of this paper. In §1, we introduce relative zeta functions for pairs (H, H_0) of self–adjoint operators in a Hilbert space. In §2, we use Krein's theory of the spectral shift function to relate the large time behaviour of $\operatorname{Tr}\left(e^{-tA} - e^{-tA_0}\right)$ to properties of the scattering matrix near zero. Then relative determinants are introduced and studied in §3. Finally, in §4, we discuss several natural examples: the Schrödinger operator in $\mathbb{R}^n$, perturbations of the euclidean Laplacian, manifolds with cylindrical ends and surfaces with hyperbolic ends.

1. Relative zeta functions

In this section we study relative zeta functions in the abstract setting. Let $\mathcal{H}$ be a separable Hilbert space and let H, H_0 be two self–adjoint nonnegative linear operators in $\mathcal{H}$. We assume that the following conditions hold:

1) Let e^{-tH} and e^{-tH_0} be the heat semigroups associated to H and H_0, respectively. Then

$$e^{-tH} - e^{-tH_0} \text{ is a trace class operator for } t > 0. \tag{1.1}$$

2) As $t \to 0$, there exists an asymptotic expansion of the form

$$\operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} a_{jk} t^{\alpha_j} \log^k t \tag{1.2}$$

where $-\infty < \alpha_0 < \alpha_1 < \dots$ and $\alpha_k \to \infty$. Moreover, if $\alpha_j = 0$ we assume that $a_{jk} = 0$ for $k > 0$.

3) As $t \to \infty$, there exists an asymptotic expansion of the form

$$\operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) \sim \sum_{k=0}^{\infty} b_k t^{-\beta_k} \tag{1.3}$$

where $0 = \beta_0 < \beta_1 < \cdots$.

It follows from (1.2) that the integral

$$\int_0^1 t^{s-1} \operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) dt$$

is absolutely convergent in the half–plane $\operatorname{Re}(s) > -\alpha_0$ and has a meromorphic continuation to the whole complex plane. The possible poles occur at $s = -\alpha_j$, $j \in \mathbb{N}$, and at a given pole $s = -\alpha_j$, the Laurent expansion is given by

$$\sum_{k=0}^{k(j)} a_{jk} \frac{k!}{(s+\alpha_j)^{k+1}} + O(1). \tag{1.4}$$

Set

$$\zeta_1(s; H, H_0) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \operatorname{Tr}(e^{-tH} - e^{-tH_0})\, dt. \tag{1.5}$$

Then $\zeta_1(s; H, H_0)$ is a meromorphic function of $s \in \mathbb{C}$. Poles may occur at $s = -\alpha_j$, $j \in \mathbb{N}$, and the corresponding Laurent expansion is determined by (1.2). By our assumption about the asymptotic expansion (1.2), $\zeta_1(s; H; H_0)$ is holomorphic at $s = 0$. Next consider the integral

$$\int_1^\infty t^{s-1}\mathrm{Tr}\left(e^{-tH}-e^{-tH_0}\right)dt.$$

By (1.3), this integral is absolutely convergent in the half–plane $\mathrm{Re}(s) < \beta_0$ and admits a meromorphic continuation to $\mathbb{C}$. Set

$$\zeta_2(s;H,H_0)=\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\mathrm{Tr}\Big(e^{-tH}-e^{-tH_0}\Big)\,dt. \tag{1.6}$$

Then $\zeta_2(s;H,H_0)$ is a meromorphic function of $s\in\mathbb{C}$ with possible poles at $s=\beta_k,\ k\in\mathbb{N}^+$. All poles are simple and we have

$$\operatorname*{Res}_{s=\beta_k}\zeta_2(s;H,H_0)=\frac{b_k}{\Gamma(\beta_k)},\ k>0.$$

Furthermore, $\zeta_2\,(s;H,H_0)$ is holomorphic at $s=0$. Put

$$\zeta\,(s;H,H_0)=\zeta_1\,(s;H,H_0)+\zeta_2\,(s;H,H_0) \tag{1.7}$$

We call $\zeta\,(s;H,H_0)$ the *relative zeta function* of (H,H_0). Summarizing, we have proved

Proposition 1.1. *Assume that the pair (H,H_0) of nonnegative self–adjoint operators satisfies conditions (1.1) - (1.3). Then the relative zeta function $\zeta\,(s;H,H_0)$ is a meromorphic function of $s\in\mathbb{C}$. The set of poles of $\zeta\,(s;H,H_0)$ is contained in the set $\{-\alpha_j\mid j\in\mathbb{N}\}\cup\{\beta_k\mid k\in\mathbb{N}^+\}$ and $s=0$ is a regular point.*

Example 1.1. Let M be a closed Riemannian manifold of dimension n, let E be a Hermitian vector bundle over M and let $D_i\colon C^\infty(M,E)\longrightarrow C^\infty(M,E),\ i=0,1,$ be an elliptic self–adjoint differential operator of order $m_i>0$ which is nonnegative. Let $H_i,\ i=0,1,$ be the unique self–adjoint extension of D_i in $L^2(M,E)$. It is well–known that $\exp(-tH_i)$ is a trace class operator for $t>0$ and, as $t\to0$, there exists an asymptotic expansion

$$\mathrm{Tr}\ \left(e^{-tH_i}\right)\sim t^{-n/m_i}\sum_{j\geq0}a_{ij}t^j. \tag{1.8}$$

Let $h_i=\dim\mathrm{Ker}\ D_i,\ i=0,1.$ Then, as $t\to\infty$, we have

$$\mathrm{Tr}\,(e^{-tH_i})=h_i+O(e^{-ct}). \tag{1.9}$$

Thus, (1.2) and (1.3) are a consequence of (1.8) and (1.9). In the present case, the zeta function $\zeta_{H_i}(s)$ of H_i is defined by (0.1) and the relative zeta function of (H_1,H_0) can be expressed in terms of the absolute zeta functions by

$$\zeta(s;H_1,H_0)=\zeta_{H_1}(s)-\zeta_{H_0}(s), \tag{1.10}$$

which justifies the name *relative zeta function*.

We note that (1.8) is the general form of the short time asymptotic expansion of the trace of the heat operator of an elliptic operator on a compact manifold. In particular, there occur no logarithmic terms in this expansion. However, if we pass to the noncompact setting, then, in general, logarithmic terms naturally arise in the short time asymptotic expansion of the trace of the difference of two heat semigroups. Examples are hyperbolic manifolds or, more generally, manifolds with cusps. Furthermore, the large time behaviour (1.9) of the trace of the heat semigroup has also a very simple form. In the noncompact case, (1.3) is related to the behaviour of the continuous spectrum near zero. If the essential spectrum of H_0 has a positive lower bound, then the expansion (1.3) is similar to (1.9) (see Lemma 2.2).

Finally, we observe that a relative version of the η–function (0.4) can be defined in the same way. Suppose that D and D_0 are self–adjoint operators in $\mathcal{H}$ such that the following three conditions hold:

1) Let e^{-tD^2} and $e^{-tD_0^2}$ be the heat semigroups of the nonnegative selfadjoint operators D^2 and D_0^2, respectively. Then

$$De^{-tD^2} - D_0 e^{-tD_0^2} \text{ is a trace class operator for } t > 0. \tag{1.11}$$

2) As $t \to 0$, there exist asymptotic expansions of the form

$$\operatorname{Tr}\big(De^{-tD^2} - D_0 e^{-tD_0^2}\big) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} A_{jk} t^{\kappa_j} \log^k t, \tag{1.12}$$

where $-\infty < \kappa_0 < \kappa_1 < \cdots \to \infty$.

3) As $t \to \infty$, there exists an asymptotic expansion of the form

$$\operatorname{Tr}\big(De^{-tD^2} - D_0 e^{-tD_0^2}\big) \sim \sum_{k=0}^{\infty} B_k t^{-\theta_k} \tag{1.13}$$

where $0 < \theta_0 < \theta_1 < \cdots \to \infty$.

Then we can introduce partial η–functions by

$$\begin{aligned} &\eta_1(s; D, D_0) \\ &= \frac{1}{\Gamma\left((s+1)/2\right)} \int_0^1 t^{(s-1)/2} \operatorname{Tr}\big(De^{-tD^2} - D_0 e^{-tD_0^2}\big)\, dt, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Re}(s) > -\kappa_0; \end{aligned} \tag{1.14}$$

$$
\begin{aligned}
&\eta_2(s; D, D_0) \\
&= \frac{1}{\Gamma\left((s+1)/2\right)} \int_1^\infty t^{(s-1)/2} \operatorname{Tr}\left(De^{-tD^2} - D_0 e^{-tD_0^2}\right) dt, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Re}(s) < \theta_0.
\end{aligned} \tag{1.15}
$$

Using the asymptotic expansions (1.12) and (1.13), the partial η–functions can be continued analytically to the whole complex plane. Set

$$
\eta(s; D, D_0) = \eta_1(s; D, D_0) + \eta_2(s; D, D_0). \tag{1.16}
$$

This is the relative η–function associated to (D, D_0). If $\kappa_j \neq -1/2$ for all $j \in \mathbb{N}$, then $\eta(s; D, D_0)$ is regular at $s = 0$.

2. The spectral shift function and the large time asymptotic expansion

In this section, we study the spectral shift function of (H, H_0) and we investigate the connection between its behaviour at 0 (resp. ∞) and the asymptotic expansion (1.3) (resp. (1.2)).

First, we introduce some notation. Let A be a self–adjoint operator in a Hilbert space. Then we denote by $\sigma(A)$ the spectrum of A. Moreover, by $\sigma_{pp}(A), \sigma_{sc}(A), \sigma_{ac}(A)$ amd $\sigma_{ess}(A)$, respectively, we shall denote the pure point spectrum, the singular continuous, the absolutely continuous and the essential spectrum of A, respectively.

Next we recall some basic results of Krein [Kr] and Birman–Krein [BK] concerning the spectral shift function. See also [BY], [Y]. Let A and A_0 be bounded self–adjoint operators in $\mathcal{H}$ and suppose that $V = A - A_0$ is a trace class operator. Let $R_0(z) = (A_0 - z)^{-1}$. Then the *spectral shift function*

$$
\xi(\lambda) = \xi(\lambda; A, A_0) = \pi^{-1} \lim_{\epsilon \to 0} \arg\det(1 + VR_0(\lambda + i\epsilon)) \tag{2.1}
$$

exists for a.e. $\lambda \in \mathbb{R}$. Note that

$$
\arg\det(1 + VR_0(\lambda + i\epsilon))
$$

is well defined by the condition that it should tend to zero as $\epsilon \to \infty$. The spectral shift function $\xi(\lambda)$ is real valued, belongs to $L^1(\mathbb{R})$ and satifies

$$
\operatorname{Tr}(A - A_0) = \int_{\mathbb{R}} \xi(\lambda)\, d\lambda, \quad ||\xi||_{L^1} \leq ||A - A_0||_1, \tag{2.2}
$$

where $||\cdot||_1$ denotes the trace norm. Moreover, let $I(A, A_0)$ be the smallest interval containing $\sigma(A) \cup \sigma(A_0)$. Then

$$\xi(\lambda) = 0 \quad \text{for} \quad \lambda \notin I(A, A_0). \tag{2.3}$$

Let

$$\mathcal{G} = \left\{ f \colon \mathbb{R} \to \mathbb{R} \mid f \in L^1 \text{ and } \int_{\mathbb{R}} |\hat{f}(p)|(1 + |p|)\, dp < \infty \right\}.$$

Then for every $\varphi \in \mathcal{G}$, $\varphi(A) - \varphi(A_0)$ is a trace class operator and

$$\operatorname{Tr}(\varphi(A) - \varphi(A_0)) = \int_{\mathbb{R}} \varphi'(\lambda)\xi(\lambda)\, d\lambda. \tag{2.4}$$

Note that (2.4) determines the spectral shift function up to an additive constant and this constant is fixed by (2.3). Applied to our setting, we obtain

Proposition 2.1. *Let H, H_0 be two nonnegative self–adjoint operators in $\mathcal{H}$ and assume that $e^{-tH} - e^{-tH_0}$ is a trace class operator for $t > 0$. Then there exists a unique real valued locally integrable function $\xi(\lambda) = \xi(\lambda; H, H_0)$ on $\mathbb{R}$ such that for each $t > 0$, $e^{-t\lambda}\xi(\lambda) \in L^1(\mathbb{R})$ and the following conditions hold*

1) $\operatorname{Tr}(e^{-tH} - e^{-tH_0}) = -t\int_0^\infty e^{-t\lambda}\xi(\lambda)\, d\lambda.$

2) *For every $\varphi \in \mathcal{G}$, $\varphi(H) - \varphi(H_0)$ is a trace class operator and*

$$\operatorname{Tr}(\varphi(H) - \varphi(H_0)) = \int_{\mathbb{R}} \varphi'(\lambda)\xi(\lambda)\, d\lambda.$$

3) $\xi(\lambda) = 0$ *for* $\lambda < 0$.

Proof. First, we note that 2) and 3) determine $\xi(\lambda)$ uniquely. Indeed, if ξ_1 and ξ_2 are any locally integrable functions on $\mathbb{R}$ satisfying 2) and 3), then $\xi_1 - \xi_2$ is a constant which must be zero by 3). To prove existence, let $\xi_t(\lambda) = \xi(\lambda; e^{-tH}, e^{-tH_0})$ be the spectral shift function of (e^{-tH}, e^{-tH_0}). Then $\xi_t \in L^1(\mathbb{R})$ and by (2.3), $\xi_t(\lambda) = 0$ for $\lambda \notin [0, 1]$. Let $\varphi \in C_0^\infty(\mathbb{R})$ and set

$$f(\mu) = \begin{cases} \varphi(-\frac{1}{t}\ln\mu), & \text{if } 0 < \mu; \\ 0, & \text{if } \mu \leq 0. \end{cases}$$

Then $f \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} f \subset (0, \infty)$. Changing variables by $\mu = e^{-t\lambda}$ and using (2.4), we obtain

$$\operatorname{Tr}(\varphi(H) - \varphi(H_0)) = \int_{\mathbb{R}} \varphi'(\lambda)\xi_t(e^{-t\lambda})\, d\lambda.$$

which holds for every $\varphi \in C_0^\infty(\mathbb{R})$. Furthermore, we note that $\xi_t(e^{-t\lambda})$ is locally integrable and $\xi_t(e^{-t\lambda}) = 0$ for $\lambda < 0$. Hence, by uniqueness, $\xi_t(e^{-t\lambda})$ is independent of t. We denote this function by $\xi(\lambda) = \xi(\lambda; H, H_0)$. It satisfies 2) and 3). Moreover, using (2.2) we get 1) and $e^{-t\lambda}\xi(\lambda)$ is integrable on $\mathbb{R}$. The extension to functions in $\mathcal{G}$ is straight forward. □

The function $\xi(\lambda; H, H_0)$ is called the *spectral shift function* of H, H_0. From Proposition 2.1, it follows that, in order to determine the asymptotic behaviour of $\operatorname{Tr}(e^{-tH} - e^{-tH_0})$ as $t \to \infty$, we have to investigate the behaviour of the spectral shift function $\xi(\lambda)$ near $\lambda = 0$. As a first consequence of Proposition 2.1 we obtain

Lemma 2.2. *Suppose that $\sigma_{ess}(H_0) \subset [c, \infty]$ where $c > 0$. Then* $\ker H$ *and* $\ker H_0$ *are both finite dimensional and there exists $c_1 > 0$ such that*

$$\operatorname{Tr}(e^{-tH} - e^{-tH_0}) = \dim\ker H - \dim\ker H_0 + O(e^{-c_1 t}) \tag{2.5}$$

as $t \to \infty$.

Proof. First, we observe that by (1.1) and the invariance of the essential spectrum under compact perturbations [RS], it follows that $\sigma_{ess}(H) = \sigma_{ess}(H_0) \subset [c, \infty)$, $c > 0$. Therefore, H_0 and H have pure point spectrum in $[0, c)$. In particular, $\ker H_0$ and $\ker H$ are finite dimensional. Let λ_1 (resp. $\widetilde{\lambda}_1$) be the infimum of the nonzero spectrum of H (resp. H_0) and set $\mu = 1/2 \min(\lambda_1, \widetilde{\lambda}_1)$. By assumption, we have $0 < \mu \leq c$ and it follows from Proposition 2.1, 2), that

$$\xi(\lambda) = \dim\ker H_0 - \dim\ker H, \quad 0 \leq \lambda \leq \mu.$$

Put

$$\widetilde{\xi}(\lambda) = \xi(\lambda) + \dim\ker H - \dim\ker H_0, \quad \lambda \in \mathbb{R}.$$

Then we have $\widetilde{\xi}(\lambda) = 0$ for $\lambda < \mu$. Moreover, $\widetilde{\xi}(\lambda)e^{-t\lambda}$ is integrable as a function of $\lambda \in \mathbb{R}$. Hence, we obtain

$$-t\int_0^\infty e^{-t\lambda}\xi(\lambda) = \dim\ker H - \dim\ker H_0 - t\int_\mu^\infty e^{-t\lambda}\widetilde{\xi}(\lambda)\, d\lambda.$$

The integral on the right hand side can be estimated by

$$e^{-t\mu/2}\int_\mu^\infty |\widetilde{\xi}(\lambda)|e^{-t\lambda/2}\, d\lambda \leq Ce^{-t\mu/2}$$

for $t \geq 1$. Using Proposition 2.1, 1), we obtain (2.5). □

Thus, if the essential spectrum of H_0, and consequently also the essential spectrum of H, has a positive lower bound, then as $t \to \infty$, $\operatorname{Tr}(e^{-tH} - e^{-tH_0})$ behaves in the same way as the trace of the heat operator of an elliptic nonnegative self–adjoint differential operator on a compact manifold. Let

$$h = \dim \ker H - \dim \ker H_0.$$

If follows from (2.5) that under the assumption of Lemma 2.2, the relative zeta function can be defined by

$$\zeta(s; H, H_0) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\operatorname{Tr}(e^{-tH} - e^{-tH_0}) - h \right) \, dt$$

for $\operatorname{Re}(s) > -\alpha_0$. This is the analogue of (0.2). A special case is that $H, H_0 > 0$. Then (1.2) implies that for $\operatorname{Re}(s) > -\alpha_0$,

$$H^{-s} - H_0^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(e^{-tH} - e^{-tH_0} \right) \, dt,$$

and $H^{-s} - H_0^{-s}$ is a trace class operator for $\operatorname{Re}(s) > -\alpha_0$. Hence,

$$\zeta(s; H, H_0) = \operatorname{Tr}\left(H^{-s} - H_0^{-s} \right), \quad \operatorname{Re}(s) > -\alpha_0.$$

Now assume that the continuous spectrum of H_0 extends down to zero. Then the behaviour of the spectral shift function at $\lambda = 0$, in general, can be very complicated. We assume that near $\lambda = 0$, $\xi(\lambda)$ admits an asymptotic expansion in powers of λ, $\lambda \geq 0$. This implies (1.3). More precisely, we have

Lemma 2.3. *Suppose there exists $\epsilon > 0$ and a sequence*

$$0 \leq \gamma_0 < \gamma_1 < \gamma_2 < \cdots$$

with $\gamma_i \to \infty$ such that for every $N \in \mathbb{N}$,

$$\xi(\lambda) = \sum_{j=0}^{N} c_j \lambda^{\gamma_j} + O(\lambda^{\gamma_{N+1}}) \tag{2.6}$$

uniformly for $\lambda \in [0, \epsilon]$. Then there exists an asymptotic expansion of the form

$$\int_0^\infty \xi(\lambda) e^{-t\lambda} \, d\lambda \sim t^{-1} \left(\frac{b_0}{t^{\gamma_0}} + \frac{b_1}{t^{\gamma_1}} + \cdots \right)$$

as $t \to \infty$.

Proof. We split the integral as $\int_0^\epsilon + \int_\epsilon^\infty$. Since $\xi(\lambda) e^{-t\lambda}$ is integrable we obtain

$$\left| \int_\epsilon^\infty \xi(\lambda) e^{-t\lambda}\, d\lambda \right| \le C e^{-t\epsilon/2}, \quad t \ge 1.$$

Furthermore, note that for $\gamma \ge 0$ and $t \ge 1$, we have

$$\begin{aligned} \int_0^\epsilon \lambda^\gamma e^{-t\lambda}\, d\lambda &= \int_0^\infty \lambda^\gamma e^{-t\lambda}\, d\lambda - \int_\epsilon^\infty \lambda^\gamma e^{-t\lambda}\, d\lambda \\ &= \frac{C(\gamma)}{t^{\gamma+1}} + O(e^{-t\epsilon/2}). \end{aligned}$$

Replacing $\xi(\lambda)$ by (2.6), we get the desired result. □

To investigate the asymptotic behaviour of the spectral shift function near $\lambda = 0$, one may use the relation between the spectral shift function and the scattering phase [BK], [BY].

Let $\mathcal{H}_{ac}$ (resp. $\mathcal{H}_{0,ac}$) be the absolutely continuous subspace for H (resp. H_0) and let H_{ac} (resp. $H_{0,ac}$) be the restriction of H (resp. H_0) to $\mathcal{H}_{ac}$ (resp. $\mathcal{H}_{0,ac}$). Furthermore, let P_0 be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_{0,ac}$. It follows from (1.1) and the Birman–Krein invariance principle [BK], [K1] that the wave operators

$$W_\pm = W_\pm(H, H_0) = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} P_0 \tag{2.7}$$

exist and are complete. This means that the strong limit (2.7) exists and the operators $W_\pm$ define isometries of $\mathcal{H}_{0,ac}$ onto $\mathcal{H}_{ac}$, intertwining $H_{0,ac}$ and H_{ac}. In this context, the scattering operator S is defined by

$$S = W_-^* W_+. \tag{2.8}$$

This is a unitary operator on $\mathcal{H}_{0,ac}$ that commutes with $H_{0,ac.}$ Let $\sigma_0 = \sigma_{ac}(H_0)$ and let $\{E_0(\lambda)\}_{\lambda \in \sigma_0}$ be the spectral family of $H_{0,ac}$. Since S commutes with $H_{0,ac}$ we have

$$S = \int_{\sigma_0} S(\lambda)\, dE_0(\lambda)$$

where $S(\lambda) = S(\lambda; H, H_0)$ acts in the Hilbert space $\mathcal{H}(\lambda)$. The operator $S(\lambda)$ is called the *on–shell scattering matrix.* We observe that $S(\lambda)$ is a unitary operator in $\mathcal{H}(\lambda)$. Let I_λ be the identity of $\mathcal{H}(\lambda)$. Since $S(\lambda; H, H_0) = S(e^{-t\lambda}; e^{-tH}, e^{-tH_0})$ and $\exp(-tH) - \exp(-tH_0)$ is a trace class operator, it follows that

$$T(\lambda) = S(\lambda) - I_\lambda \tag{2.9}$$

is a trace class operator (cf. [BK], [BY]). The opeator $T(\lambda)$ is called the *scattering amplitude*. Thus, the Fredholm determinant $\det S(\lambda)$ of $S(\lambda)$ exists. Since $S(\lambda)$ is unitary, we have $|\det S(\lambda)| = 1,\ \lambda \in \sigma_{ac}(H_0)$. Hence, the scattering phase $s(\lambda) = \frac{1}{2\pi i}\log\det S(\lambda)$ is defined mod $\mathbb{Z}$. It was proved in [BK] that there is a distinguished choice for the scattering phase, namely the spectral shift function $\xi(\lambda)$. More precisely, the following identity holds

$$\det S(\lambda) = e^{-2\pi i \xi(\lambda)}, \quad \text{a.e. } \lambda \in \sigma_{ac}(H_0), \tag{2.10}$$

(see [BY], [Y]). This equality can be used to derive an expansion of $\xi(\lambda)$ near $\lambda = 0$. For $\epsilon > 0$, let $D_\epsilon = \{z \in \mathbb{C} \mid |z| < \epsilon\}$ and denote by $\widehat{D}_{\epsilon,m} \to D_\epsilon$ the ramified covering of order m, defined by $z \mapsto z^m$. Then, as a consequence of (2.10), we obtain

Proposition 2.4. *Let $\epsilon > 0$ and suppose that $(0,\epsilon) \subset \sigma_{ac}(H_0)$. Furthermore, assume that*

1) *$\xi(\lambda)$ is continuous on $(0,\epsilon)$;*
2) *There exists $m \in \mathbb{N}$ such that $\det S(\lambda)$, $\lambda \in (0,\epsilon)$, extends to a holomorphic function on $\widehat{D}_{\epsilon,m}$.*

Let $0 < \epsilon_1 < \epsilon$. Then there exists an expansion of the form

$$\xi(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^{k/m}, \quad \lambda \in [0,\epsilon_1), \tag{2.11}$$

for some $0 < \epsilon_1 < \epsilon^{1/m}$.

Proof. According to our assumption, $\det S(\lambda)$ is smooth on $(0,\epsilon)$. Since $|\det S(\lambda)| = 1,\ \lambda \in \sigma_{ac}(H_0)$, it follows that there exists $\delta \in C^\infty((0,\epsilon))$ such that $\det S(\lambda) = \exp(-2\pi i \delta(\lambda)),\ \lambda \in (0,\epsilon)$. On the other hand, ξ is continuous on $(0,\epsilon)$ and therefore, (2.10) implies that there exists $k \in \mathbb{Z}$ such that $\xi(\lambda) = \delta(\lambda) + k,\ \lambda \in (0,\epsilon)$. Hence, ξ is smooth on $(0,\epsilon)$. Differentiating (2.10) and using that $S(\lambda)$ is unitary, we get

$$\frac{d\xi(\lambda)}{d\lambda} = -\frac{1}{2\pi i}\operatorname{Tr}\left(S^*(\lambda)\frac{dS(\lambda)}{d\lambda}\right),\ \lambda \in (0,\epsilon). \tag{2.12}$$

Furthermore, since $\delta(\lambda)$ has a finite limit as $\lambda \to 0$, we get

$$\begin{aligned}\xi(\lambda) &= \xi(0+) - \frac{1}{2\pi i}\int_0^\lambda \operatorname{Tr}\left(S^*(u)\frac{dS(u)}{du}\right)\, du \\ &= \xi(0+) - \frac{1}{2\pi i}\int_0^{\lambda^{1/m}} \operatorname{Tr}\left(S^*(u^m)\frac{dS(u^m)}{du}\right)\, du \qquad (2.13)\end{aligned}$$

for $\lambda \in [0,\epsilon)$. By assumption, $\operatorname{Tr}(S^*(z^m)\, dS(z^m)/dz)$ is analytic on the disc $|z| < \epsilon_1 \le \epsilon^{1/m}$. Inserting the Taylor expansion on the right hand side of (2.13) we obtain the desired expansion (2.11). □

Let

$$\operatorname{Tr}\left(S^*(z^m)\frac{dS(z^m)}{dz}\right) = \sum_{k=0}^\infty d_k z^k,\ |z| < \epsilon_1,$$

be the Taylor expansion at $z = 0$. Then by (2.13), the coifficients of the expansion (2.11) are related to the d_k's by $c_k = d_{k-1}/k,\ k \ge 1$. Thus, under the assumptions of Proposition 2.4, the asymptotic expansion of $\operatorname{Tr}(e^{-tH} - e^{-tH_0})$ for $t \to \infty$ determines $\det S(\lambda)$ near $\lambda = 0$, up to an additive constant. This means that the coefficients b_k in (1.3) and therefore, the residues of the poles of $\zeta(s; H, H_0)$ at $s = k/m,\ k \in \mathbb{N}^+$, are nonlocal.

If we insert $\xi(\lambda)$ in the trace formula of Proposition 2.1 and integrate by parts, we obtain

$$\begin{aligned}&\operatorname{Tr}(e^{-tH} - e^{-tH_0}) \\ &\quad = -\xi(0+) + \frac{1}{2\pi i}\int_0^\epsilon e^{-t\lambda}\operatorname{Tr}\left(S^*(\lambda)\frac{dS(\lambda)}{d\lambda}\right)\, d\lambda + O(e^{-t\epsilon/2}) \\ &\quad = -\xi(0+) + \frac{1}{2\pi i}\int_0^{\epsilon_1} e^{-t\lambda^m}\operatorname{Tr}\left(S^*(\lambda^m)\frac{dS(\lambda^m)}{d\lambda}\right)\, d\lambda \qquad (2.14) \\ &\qquad + O(e^{-t\epsilon/2})\end{aligned}$$

as $t \to \infty$. This formula shows more explicitly how the scattering matrix determines the asymptotic behaviour of $\operatorname{Tr}(e^{-tH} - e^{-tH_0})$ as $t \to \infty$. Furthermore, in special cases, $\xi(0+)$ can be expressed in terms of spectral data.

It follows from (2.14) that

$$\operatorname{Tr}(e^{-tH} - e^{-tH_0}) = -\xi(0+) + O(t^{-1/m})$$

as $t \to \infty$. Inserting this equality in (1.6), we obtain the analytic extension of $\zeta_2(s; H, H_0)$ to the half–plane $\operatorname{Re}(s) < 1/m$.

Now, in order to define the relative determinant of H, H_0 by zeta function regularization, it is sufficient to know that $\zeta(s; H, H_0)$ is analytic in some half–plane $\mathrm{Re}(s) < \varepsilon$, $\varepsilon > 0$. Therefore, we may replace (1.3) by the following weaker assumption: There exist $b_0 \in \mathbb{C}$ and $\rho > 0$ such that

$$\mathrm{Tr}\left(e^{-tH} - e^{-tH_0}\right) = b_0 + O(t^{-\rho}) \tag{2.15}$$

as $t \to \infty$.

As above, this condition can be restated in terms of the behaviour of the spectral shift function near $\lambda = 0$. In fact, by Proposition 2.1, 1), (2.15) is equivalent to

$$\xi(\lambda) = -b_0 + O(\lambda^{\rho}) \tag{2.16}$$

as $\lambda \to 0+$. For this condition to hold, it is not necessary that the scattering matrix extends to an analytic function on some covering of a neighborhood of the origin. Much less stringent assumptions are sufficient. Now, suppose that (2.16) holds. Then $\zeta_2(s; H, H_0)$ has a meromorphic extension to the half–plane $\mathrm{Re}(s) < \rho$ which is given by

$$\begin{aligned} \zeta_2(s; H, H_0) = &- \frac{b_0}{\Gamma(s+1)} \\ &+ \frac{1}{\Gamma(s)} \int_1^{\infty} t^{s-1} \big(\mathrm{Tr}(e^{-tH} - e^{-tH_0}) - b_0\big)\, dt. \end{aligned} \tag{2.17}$$

This formula implies that $\zeta_2(s; H, H_0)$ is regular at $s = 0$.

Analogous results can be obtained for the small time asymptotic expansion (1.2). If we impose smoothness assumptions at $\lambda = \infty$ similar to those at $\lambda = 0$, then it follows that the asymptotic expansion (1.2) corresponds to an asymptotic expansion of the scattering phase shift $d/d\lambda(\log\det S(\lambda))$ as $\lambda \to \infty$.

Next, we briefly discuss relative η–functions. Details will appear elsewhere. Let D, D_0 be self–adjoint operators in $\mathcal{H}$, satisfying (1.11) and (1.12). Moreover, suppose that the spectra of D and D_0 have a common gap, i.e. there exists an interval $[a, b]$, $a < b$, such that

$$(\sigma(D) \cup \sigma(D_0)) \cap [a, b] = \emptyset. \tag{2.18}$$

Then, by a modification of the proof of Proposition 2.1, one can introduce a spectral shift function $\xi(\lambda) = \xi(\lambda; D, D_0)$. This function has the following properties:

1) $\xi \in L^1_{loc}(\mathbb{R})$ and $\xi(\lambda) = 0$ for $\lambda \in [a, b]$;

2) For all $\varphi \in C_0^{\infty}(\mathbb{R})$, $\varphi(D) - \varphi(D_0)$ is a trace class operator and

$$\mathrm{Tr}\left(\varphi(D) - \varphi(D_0)\right) = \int_{\mathbb{R}} \varphi'(\lambda)\xi(\lambda)\, d\lambda; \tag{2.19}$$

3)

$$\operatorname{Tr}\left(De^{-tD^2} - D_0 e^{-tD_0^2}\right) = \int_{\mathbb{R}} \frac{d}{d\lambda}(\lambda e^{-t\lambda^2})\xi(\lambda)\, d\lambda. \tag{2.20}$$

We note that 1) and 2) determine ξ uniquely. If, however, (2.18) is not satisfied, i.e. if $\sigma(D) = \sigma(D_0) = \mathbb{R}$, then the construction of the spectral shift function is slightly more complicated. In particular, ξ cannot be normalized by fixing its values outside the spectrum and therefore, ξ is determined by (2.19) only up to an additive constant.

By (2.20), the study of the behaviour of $\operatorname{Tr}\left(De^{-tD^2} - D_0 e^{-tD_0^2}\right)$ as $t \to \infty$ can be reduced to the investigation of the spectral shift function near $\lambda = 0$. To handle this problem, one can use, in the same way as above, the connection between the spectral shift function and the scattering phase. Indeed, since $De^{-tD^2} - D_0 e^{-tD_0^2}$ is a trace class operator for $t > 0$, it follows from the the Birman-Kato invarinace principle that the wave operators $W_\pm(D, D_0)$ exist and are complete [K2]. Hence, the scattering matrix $S(\lambda; D, D_0)$ also exists and it follows from [BK] that

$$\det S(\lambda; D, D_0) = e^{-2\pi i \xi(\lambda; D, D_0)}, \quad \lambda \in \sigma_{ac}(D_0).$$

As above, this equality can be used to get an expansion of $\xi(\lambda; D, D_0)$ near $\lambda = 0$, provided that the scattering matrix extends to a holomorphic function on some finite covering of a neighborhood of the origin.

But, in order to define the η–invariant, we only need that there exists $\epsilon > 0$ such that

$$\operatorname{Tr}\left(De^{-tD^2} - D_0 e^{-tD_0^2}\right) = O(t^{-(1/2+\epsilon)}) \tag{2.21}$$

as $t \to \infty$. Then (1.15) is absolutely convergent in the half–plane $\operatorname{Re}(s) < 2\epsilon$. Suppose, for example, that D and D_0 are invertible. Then there exists $\epsilon > 0$ such that (2.18) holds with respect to the interval $[-\epsilon, \epsilon]$. Thus, the spectral shift function vanishes on $[-\epsilon, \epsilon]$ and by (2.20), we obtain

$$\operatorname{Tr}\left(De^{-tD^2} - D_0 e^{-tD_0^2}\right) = O(t^{-\epsilon^2/2}). \tag{2.22}$$

Hence, in this case, the relative η–function can be defined by a single Mellin transform:

$$\begin{aligned} &\eta(s; D, D_0) \\ &= \frac{1}{\Gamma\left((s+1)/2\right)} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}\left(De^{-tD^2} - D_0 e^{-tD_0^2}\right) dt, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Re}(s) > -\kappa_0. \end{aligned} \tag{2.23}$$

3. Relative determinants

Suppose that (H, H_0) are nonnegative self–adjoint operators which satisfy (1.1), (1.2) and (2.15). Then the relative zeta function $\zeta(s; H, H_0)$ is well defined as a meromorphic function in the half–plane $\mathrm{Re}(s) < \rho$, $\rho > 0$, and $s = 0$ is not a pole of $\zeta(s; H, H_0)$. Therefore, generalizing (0.3), we may define the regularized *relative determinant* of (H, H_0) by

$$\det(H, H_0) = \exp\left(-\frac{d}{ds}\zeta(s; H, H_0)\Big|_{s=0} \right). \tag{3.1}$$

To justify the name "relative determinant", consider two self–adjoint operators H_0 and H_1 such that $\exp(-tH_i)$ is trace class for $t > 0$ and assume that (1.8) holds for both operators. Then the absolute determinants $\det H_i$ are defined and it follows from (1.10) that

$$\det(H_1, H_0) = \frac{\det H_1}{\det H_0}. \tag{3.2}$$

Using (2.17), the logarithm of the relative determinant can be written as

$$\begin{aligned} -\log\det(H, H_0) = &\frac{d}{ds}\zeta_1(s; H, H_0)\Big|_{s=0} - b_0\Gamma'(1) \\ &+ \int_1^\infty t^{-1}\big(\mathrm{Tr}\big(e^{-tH} - e^{-tH_0}\big) - b_0\big)\, dt. \end{aligned} \tag{3.3}$$

Furthermore, we may use the asymptotic expansion (1.2) to replace the derivative of $\zeta_1(s; H, H_0)$ at $s = 0$ by a more explicit formula. Namely, let $N \in \mathbb{N}$ be such that the exponent α_N in the expansion (1.2) satisfies $\alpha_N > 0$. Then it follows that

$$\begin{aligned} -\log\det(H, H_0) = &\sum_{j=0}^{N}\sum_{k=0}^{k(j)} a_{jk}\frac{k!}{\alpha_j^k} - a_{j_0}\Gamma'(1) \\ &+ \int_0^1 t^{-1}\left(\mathrm{Tr}(e^{-tH} - e^{-tH_0}) - \sum_{j=0}^{N}\sum_{k=0}^{k(j)} a_{jk} t^{\alpha_j}\log^k t \right) dt \\ &- b_0\Gamma'(1) + \int_1^\infty t^{-1}\left(\mathrm{Tr}(e^{-tH} - e^{-tH_0}) - b_0\right) dt. \end{aligned} \tag{3.4}$$

We shall now summmarize some of the elementary properties of the determinant. As an immediate consequence of the definition we get

Lemma 3.1. *Suppose that H_0, H_1 and H_2 are self–adjoint operators in $\mathcal{H}$ such that both (H_1, H_0) and (H_2, H_1) satisfy (1.1), (1.2) and (2.15). Then the following holds*

1) $\det(H_0, H_1) = \det(H_1, H_0)^{-1}$.

2) *The operators (H_2, H_0) also satisfy (1.1), (1.2) and (2.15) and*

$$\det(H_2, H_0) = \det(H_2, H_1) \cdot \det(H_1, H_0). \tag{3.5}$$

Next, we consider some special cases. Let H be a nonnegative self–adjoint operator in $\mathcal{H}$ and suppose that H has eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. Let P_m denote the orthogonal projection of $\mathcal{H}$ onto the subspace of $\mathcal{H}$ spanned by the eigenvectors that correspond to $\lambda_1, \dots, \lambda_m$. Put $H_0 = (I - P_m)H$. Then $\exp(-tH)$ is a finite–rank perturbation of $\exp(-tH_0)$. In particular, (H, H_0) satisfies (1.1) – (1.3) and we have

$$\det(H, H_0) = \prod_{i=1}^{m} \lambda_i. \tag{3.6}$$

More generally, if the pair (H, H_0) satisfies (1.1) – (1.3), then the pair $((I - P_m)H, H_0)$ satisfies the same conditions and by (3.5), we get

$$\det(H, H_0) = \prod_{i=1}^{m} \lambda_i \cdot \det((I - P_m)H, H_0). \tag{3.7}$$

This means that we may split off any finite number of eigenvalues from the determinant.

Next let $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_c$ be the orthogonal decomposition into the subspaces that correspond to the point spectrum and the continuous spectrum of H, respectively. Let H_p and H_c denote the restriction of H to $\mathcal{H}_p$ and $\mathcal{H}_c$, respectively. Suppose that for each $t > 0$, $\exp(-tH_p)$ is a trace class operator and as $t \to 0, \operatorname{Tr}(\exp(-tH_p))$ admits an asymptotic expansion of the form (1.2). Then $\det H_p$ can be defined by zeta function regularization and using (3.5), one has

$$\det(H, H_0) = \det H_p \cdot \det(H_c, H_0). \tag{3.8}$$

Now let $a > 0$. Then $(H + a, H_0 + a)$ also satisfies (1.1) – (1.3) and the relative zeta function is given by

$$\zeta(s; H + a, H_0 + a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-ta} \operatorname{Tr}(e^{-tH} - e^{-tH_0})\, dt$$

for $\mathrm{Re}(s) > -\alpha_0$. The right hand side also makes sense if we replace a by any complex number z with $\mathrm{Re}(z) > 0$. We denote the corresponding analytic function by $\zeta(s, z; H, H_0)$. Using (1.2) it follows that for every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, $\zeta(s, z; H, H_0)$, regarded as function of s, admits a meromorphic continuation to $\mathbb{C}$ which is holomorphic at $s = 0$. Thus, generalizing (3.1) we set

$$\det(H + z, H_0 + z) = \exp\left(-\frac{\partial}{\partial s}\zeta(s, z; H, H_0)\Big|_{s=0} \right). \tag{3.9}$$

Again, if the operators H_0 and H_1 are as in (3.2), then

$$\det(H_1 + z, H_0 + z) = \frac{\det(H_1 + z)}{\det(H_0 + z)}. \tag{3.10}$$

We note that if H is a nonnegative self–adjoint operator such that $\exp(-tH)$ is trace class for $t > 0$ and (1.8) holds, then H has pure discrete spectrum consisting of a sequence of eigenvalues $0 \le \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty$ of finite multiplicities and $\det(H + z)$ is an entire function of z whose set of zeros equals $\{-\lambda_i\}_{i\in\mathbb{N}}$ and the multiplicities coincide. In particular, this shows that (3.10) is the quotient of two entire functions. We don't know yet, if this holds in general. Here we prove a weaker result.

Proposition 3.2. *Suppose that* (H, H_0) *satisfies (1.2). Then the relative determinant* $\det(H + z, H_0 + z)$ *is a holomorphic function of* $z \in \mathbb{C} - (-\infty, 0]$.

Proof. Let $\xi(\lambda)$ be the spectral shift function associated to (H, H_0). Let $c > 0$. By Proposition 2.1, 1), we have

$$\mathrm{Tr}(e^{-tH} - e^{-tH_0}) = -t\int_0^{2c} e^{-t\lambda}\xi(\lambda)\, d\lambda - t\int_{2c}^{\infty} e^{-t\lambda}\xi(\lambda)\, d\lambda.$$

Since $e^{-t\lambda}\xi(\lambda) \in L^1(\mathbb{R})$ for $t > 0$, the second integral is $O(e^{-tc})$ as $t \to \infty$. Furthermore, for every $N \in \mathbb{N}$, we have

$$\int_0^{2c} e^{-t\lambda}\xi(\lambda)\, d\lambda = \sum_{k=0}^{N} c_k t^k + O(t^{N+1})$$

as $t \to 0$. Together with (1.2) it follows that as $t \to 0$, $\int_{2c}^{\infty} e^{-t\lambda}\xi(\lambda)\, d\lambda$ has a complete asymptotic expansion similar to (1.2). This implies that the double integral

$$F(s,z)=\frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-tz}\int_{2c}^\infty e^{-t\lambda}\xi(\lambda)\,d\lambda\,dt$$

is absolutely convergent in the half–planes $\mathrm{Re}(s)>-\alpha_0$ and $\mathrm{Re}(z)>-c$. Moreover, as function of s, it admits a meromorphic continuation to $\mathbb{C}$ which is holomorphic at $s=0$.
Next consider the first integral. Note that ξ is absolutely integrable on $[0,2c]$. Hence for $\mathrm{Re}(z)>0$, we get

$$\begin{aligned}
-\frac{1}{\Gamma(s)}&\int_0^\infty t^s e^{-tz}\int_0^{2c} e^{-t\lambda}\xi(\lambda)\,d\lambda\,dt\\
&=-\frac{1}{\Gamma(s)}\int_0^{2c}\xi(\lambda)\int_0^\infty t^s e^{-t(z+\lambda)}\,dt\,d\lambda\\
&=-s\int_0^{2c}(z+\lambda)^{-(s+1)}\xi(\lambda)\,d\lambda.
\end{aligned}$$

Thus, for $\mathrm{Re}(z)>0$, we have

$$\begin{aligned}
&\det(H+z,H_0+z)\\
&=\exp\left\{\int_0^{2c}(z+\lambda)^{-1}\xi(\lambda)\,d\lambda+\frac{\partial F}{\partial s}(0,z)\right\}
\end{aligned}\tag{3.11}$$

The integral has an obvious extension to an analytic function of $z\in\mathbb{C}-(-\infty,0]$. Moreover, $\frac{\partial F}{\partial s}(0,z)$ is holomorphic in $\mathrm{Re}(z)>-c$. Since c is arbitrary, (3.11) gives the analytic continuation to $\mathbb{C}-(-\infty,0]$. $\square$

We now shall study the variation of the relative determinant. So, we consider a differentiable 1–parameter family H_u, $u\in(-\epsilon,\epsilon)$, of nonnegative self–adjoint operators in $\mathcal{H}$ and a nonnegative self–adjoint operator H in $\mathcal{H}$ such that for each $u\in(-\epsilon,\epsilon)$, the pair (H_u,H) satisfies (1.1), (1.2) and (2.15). By (3.5) we have

$$\det(H_u,H)=\det(H_u,H_0)\cdot\det(H_0,H),$$

so that we may take $H=H_0$. Let $\dot H_u=\frac{d}{du}H_u$. By Duhamel's principle, $\exp(-tH_u)$ is differentiable in u and

$$\frac{d}{du}\left(e^{-tH_u}\right)=\int_0^t e^{-vH_u}\dot H_u e^{-(t-v)H_u}\,dv.\tag{3.12}$$

Now assume that $\dot H_u e^{-tH_u}$ is a trace class operator for $t>0$ with trace norm uniformly bounded on compact subsets of $(0,\infty)$. Then it follows from (3.12) that $\frac{d}{du}\exp(-tH_u)$ is also a trace class operator for $t > 0$ and

$$\frac{d}{du}\operatorname{Tr}\left(e^{-tH_u} - e^{-tH_0}\right) = -t\operatorname{Tr}(\dot{H}_u e^{-tH_u}). \tag{3.13}$$

We shall now make the following two assumptions about the family H_u:

i) H_u is invertible for each $u \in (-\epsilon, \epsilon)$,
ii) As $t \to 0$, there exists an asymptotic expansion of the form

$$\begin{aligned}\operatorname{Tr}(\dot{H}_u H_u^{-1} e^{-tH_u}) \sim & \sum_{j=0}^{\infty}\sum_{k=0}^{k(j)} C_{jk}(u) t^{\gamma_j} \log^k t \\ & + C_1(u) + C_2(u)\log t\end{aligned} \tag{3.14}$$

and the exponents γ_j are such that $-\infty < \gamma_1 < \gamma_2 < \cdots$, $\gamma_j \neq 0$, $j \in \mathbb{N}$, and $\gamma_j \to \infty$.

Note that by i), there exists $c > 0$ such that

$$\operatorname{Tr}(\dot{H}_u H_u^{-1} e^{-tH_u}) = O(e^{-ct}) \tag{3.15}$$

as $t \to \infty$. Using i) and (3.13) – (3.15), it follows that for $\operatorname{Re}(s) \gg 0$,

$$\begin{aligned}\frac{d}{du}\zeta(s; H_u, H_0) &= \frac{1}{\Gamma(s)}\int_0^\infty t^s \frac{\partial}{\partial t}\operatorname{Tr}(\dot{H}_u H_u^{-1} e^{-tH_u})\, dt \\ &= -\frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{Tr}(\dot{H}_u H_u^{-1} e^{-tH_u})\, dt.\end{aligned} \tag{3.16}$$

By (3.14) and (3.15),

$$\int_0^\infty t^{s-1}\operatorname{Tr}(\dot{H}_u H_u^{-1} e^{-tH_u})\, dt$$

has an analytic continuation to a meromorphic function of $s \in \mathbb{C}$. The location of the poles is determined by the asymptotic expansion (3.14). In particular, the pole at $s = 0$ has order ≤ 2 and the Laurent expansion is given by

$$-\frac{C_2(u)}{s^2} + \frac{C_1(u)}{s} + \cdots$$

where $C_1(u)$ and $C_2(u)$ are the corresponding coefficients in (3.14). Using (3.16), we obtain

Proposition 3.3. *Suppose that the family H_u, $u \in (-\epsilon, \epsilon)$, satisfies assumptions i) and ii). Then* $\det(H_u, H_0)$ *is differentiable and*

$$\frac{d}{du} \log \det(H_u, H_0) = C_1(u) + \Gamma'(1) C_2(u) \tag{3.17}$$

where $C_1(u)$ denotes the constant term and $C_2(u)$ the coefficient of $\log t$ *in the asymptotic expansion (3.14).*

Corollary 3.4. *Let H_u, $u \in [0,1]$, be a differentiable family of self–adjoint operators in $\mathcal{H}$ such that for all $u \in [0,1]$, (1.1) and (1.2) holds for (H_u, H_0) and H_u satisfies conditions i) and ii). Then*

$$\det(H_1, H_0) = \exp\left\{ \int_0^1 (C_1(u) + \Gamma'(1) C_2(u))\, du \right\}. \tag{3.18}$$

Remark. A similar formula was proved in [GMS] for a certain class of elliptic invertible pseudodifferential operators on a compact manifold. For each of these operators the determinant can be defined via zeta function regularization and the relative determinant of any two such operators is the fraction of the determinants of the individual operators.

The assumption i) is, of course, rather restrictive and excludes many cases that are of interest for applications. However, using (3.4), differentiability can be established under more general assumptions. For example, we have

Proposition 3.5. *Let H_u, $u \in (-\varepsilon, \varepsilon)$, be a differentiable family of self–adjoint operators which satisfy (1.1), (1.2), (2.15) and in addition, the following conditions hold:*

a) *The constant b_0 and the exponent ρ in (2.15) are independent of u, and the coefficients of the expansion (1.2) are differentiable in u.*

b) *For $t > 0$, $\dot{H}_u e^{-tH_u}$ is a trace class operator and there exist $C > 0$ and $\delta > 0$ such that*

$$Tr\big(\dot{H}_u e^{-tH_u}\big) \le C t^{-(1+\delta)}$$

for $t \ge 1$ and $u \in (-\varepsilon, \varepsilon)$.

c) *As $t \to 0$, there exists an asymptotic expansion of the form*

$$Tr\big(\dot{H}_u e^{-tH_u}\big) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} C_{jk}(u) t^{\nu_j} \log^k t$$

where the exponents ν_j are such that $-\infty < \nu_1 < \nu_2 < \cdots$, $\nu_j \neq 1$, $j \in \mathbb{N}$, and $\nu_j \to \infty$.

Then the integral $\int_0^1 t^s \operatorname{Tr}\big(\dot H e^{-tH_u}\big)\,dt$ *is absolutely convergent in the half-plane* $\operatorname{Re}(s) > -\nu_1 - 1$ *and has a meromorphic extension to the half–plane* $\operatorname{Re}(s) < \delta$ *which is holomorphic at* $s = 0$. *Moreover,* $\det(H_u, H_0)$ *is a differentiable function of* u *and*

$$\frac{d}{du}\log\det(H_u,H_0) = \int_0^1 t^s\, Tr\big(\dot H e^{-tH_u}\big)\,dt\Big|_{s=0} + \int_1^\infty \operatorname{Tr}\left(\dot H_u e^{-tH_u}\right)\,dt.$$

Proof. We use formula (3.4). Since by b), $\operatorname{Tr}(\dot H_u e^{-tH_u})$ is absolutely integrable on $[1,\infty)$, it follows from a) and (3.13) that

$$\frac{d}{du}\int_1^\infty t^{-1}\left(\operatorname{Tr}(e^{-tH_u} - e^{-tH_0}) - b_0\right)\,dt = -\int_1^\infty \operatorname{Tr}\left(\dot H_u e^{-tH_u}\right)\,dt.$$

Furthermore, by c), $\int_0^1 t^s \operatorname{Tr}\left(\dot H_u e^{-tH_u}\right)\,dt$ has a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$, and

$$\frac{\partial}{\partial u}\zeta_1(s; D, D_0) = -\frac{1}{\Gamma(s)}\int_0^1 t^s \operatorname{Tr}\left(\dot H_u e^{-tH_u}\right)\,dt.$$

Taking derivatives at $s = 0$ gives the desired result. □

To conclude this section, we consider the case of two self–adjoint operators D, D_0 which are not necessarily bounded from below. A typical example are Dirac type operators on a complete manifold. Suppose that (1.11) and (1.12) hold for (D, D_0). For simplicity, we also assume that D and D_0 are invertible, i.e. 0 is not in the spectrum of D and D_0. Then the relative η–function $\eta(s; D, D_0)$ is defined by (2.23) and has a meromorphic extension to $\mathbb{C}$. If $-1/2$ does not occur among the exponents of the asymptotic expansion (1.12), then $\eta(s; D, D_0)$ is regular at $s = 0$ and therefore, we may define the relative η–invariant as $\eta(0; D, D_0)$. For example, if D and D_0 are Dirac type operators, then the asymptotic expansion (1.12) contains no negative exponents so that the relative η–invariant is given by

$$\eta(0; D, D_0) = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\operatorname{Tr}\left(De^{-tD^2} - D_0 e^{-tD_0^2}\right)\,dt.$$

Suppose also that D^2, D_0^2 satisfy (1.1) and (1.2). Since D and D_0 are invertible, Lemma 2.2 implies that (1.3) holds and the relative zeta function is given by

$$\begin{aligned}\zeta(s; D^2, D_0^2) &= \operatorname{Tr}\left((D^2)^{-s} - (D_0^2)^{-s}\right)\\ &= \operatorname{Tr}\left(|D|^{-2s} - |D_0|^{-2s}\right), \quad \operatorname{Re}(s) > -\kappa_0.\end{aligned}$$

Hence, the relative zeta function $\zeta(s;|D|,|D_0|)$ also exists as a meromorphic function on $\mathbb{C}$, and we have

$$\zeta(s;D^2,D_0^2)=\zeta(2s;|D|,|D_0|).$$

This formula implies that $\zeta(s;|D|,|D_0|)$ is regular at $s=0$ and therefore, $\det(|D|,|D_0|)$ can be defined by (3.1). Generalizing (0.5), we introduce the relative determinant for D, D_0 by

$$\begin{aligned}\det(D,D_0)=&\det(|D|,|D_0|)\\ &\cdot\exp\left(\frac{\pi i}{2}(\eta(0;D,D_0)-\zeta(0;|D|,|D_0|)\right).\end{aligned}\tag{3.19}$$

4. Examples

In this section we discuss a number of typical cases of operators (H, H_0) which satisfy the assumptions (1.1) - (1.3). Such operators arise very naturally both in geometry and physics. In particular, we will show how the general trace formula of Proposition 2.1 specializes in all these cases to a rather explicit trace formula involving the eigenvalues of the operator H and the scattering phase shift.

4.1. **The Schrödinger operator in $\mathbb{R}^n$.** A typical example is the Schrödinger operator $\Delta+V$ in $\mathbb{R}^n$ where $V\in C_0^\infty(\mathbb{R}^n)$ and Δ is the negative of the usual Laplacian of $\mathbb{R}^n$. I shall limit myself to this simple case, where V is both smooth and has compact support, although most of the results have generalizations to potentials with less regularity assumptions and the support property replaced by growth conditions. For details we refer the reader to [CV], [Gu1], [Gu2].

Regarded as operator in $L^2(\mathbb{R}^n)$ with domain $C_0^\infty(\mathbb{R}^n)$, $\Delta+V$ is essentially self–adjoint. Let H be the unique self–adjoint extension of $\Delta+V$ and let $H_0=\overline{\Delta}$. The spectrum of H consists of a finite number of eigenvalues

$$\lambda_1<\lambda_2\le\lambda_3\le\cdots\le\lambda_N<\lambda_{N+1}=\cdots=\lambda_{N'}=0,$$

where each eigenvalue is repeated according to its multiplicity, and an absolutely continuous spectrum which has multiplicity 2, if $n=1$, and infinite multiplicity, if $n>1$. Using Duhamel's formula

$$e^{-tH}-e^{-tH_0}=\int_0^t e^{-sH}Ve^{-(t-s)H_0}\,ds,$$

it is easy to see that $e^{-tH}-e^{-tH_0}$ is a trace class operator for $t>0$ (see also [CV], [Gu1], [Gu2]). Hence, the wave operators $W_\pm(H,H_0)$ exist and are complete, and the scattering operator S is defined by (2.8). Let

$$L^2(\mathbb{R}^n) \simeq L^2\left(\mathbb{R}^+, L^2(S^{n-1})\lambda^{n-1}\, d\lambda\right)$$

be the spectral resolution of H_0 defined by the Fourier transform

$$\widehat{f}(\lambda)(\omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\lambda<\omega,x>} f(x)\, dx.$$

Then the "on–shell scattering matrix" $S(\lambda) = S(\lambda; H, H_0)$ is given by

$$\widehat{Sf}(\lambda) = S(\lambda^2)\widehat{f}(\lambda),\ \lambda > 0.$$

Furthermore, $S(\lambda)$ is a unitary operator in $L^2(S^{n-1})$ which is of the form $S(\lambda) = \mathrm{Id} + T(\lambda)$ where $T(\lambda)$ is an integral operator with a smooth kernel $T(\lambda; \omega, \omega')$.

Since $V \in C_0^\infty(\mathbb{R}^n)$, it follows that for odd n, $S(\lambda^2)$ has an analytic continuation to a meromorphic operator valued function of $\lambda \in \mathbb{C}$. In the even–dimensional case a similar result is valid, except that $S(\lambda)$ extends to a meromorphic function on the logarithmic covering of $\mathbb{C}$, i.e. as a function of the variable $\log \lambda$. In particuar, this implies

Proposition 4.1. *The Fredholm determinant* $\det S(\lambda)$ *of* $S(\lambda)$ *is real analytic on* $(0, \infty)$.

Furthermore, since $e^{-tH} - e^{-tH_0}$, $t > 0$, is a trace class operator, the spectral shift function $\xi(\lambda; H, H_0)$ exists. In the present case, it can also be defined using the resolvents. Let $R_0(\lambda)$ and $R(\lambda)$ denote the resolvent of H_0 and H, respectively. Let $\mu \in \mathbb{R} - \sigma(H)$ and let $k > n/2$. Then $R(\mu)^k - R_0(\mu)^k$ is of the trace class. Hence, there exists the spectral shift function $\xi_\mu(\lambda) = \xi(\lambda; R(\mu)^k, R_0(\mu)^k)$ defined by (2.1) and, by the uniqueness of the spectral shift function, we get

$$\xi(\lambda; H, H_0) = \xi_\mu\left((\lambda - \mu)^{-k}; R(\mu)^k, R_0(\mu)^k\right).$$

Using Proposition 1 in [Gu1] (or Theorem III.2 in [Gu2]), it follows that $\xi(\lambda)$ is locally constant on the complement of $\sigma(H)$ and continuous on $(0, \infty)$. If $n \neq 2, 4$, then the discontinuity of $\xi(\lambda)$ at 0 is given by

$$\xi(0-) - \xi(0+) = m(0, H) + \epsilon, \tag{4.1}$$

where $m(0, H)$ is the multiplicity of the eigenvalue 0 of H and $\epsilon = 0$, if there is no resonance at 0 and $\epsilon = 1/2$ otherwise. If $n = 2, 4$, the description is more difficult. Let

$$\det S(0) = \lim_{\lambda \to 0+} \det S(\lambda).$$

We note that $\xi(0-) \in \mathbb{Z}$. Hence, using (2.10), we obtain

$$\det S(0) = (-1)^{2\epsilon}.$$

This implies that we may pick the branch of $\log \det S(\lambda)$ such that

$$\xi(0-) - \xi(0+) = m(0, H) + \frac{1}{2\pi i} \log \det S(0).$$

Then by Proposition 2.1 we get the following trace formula for $n \neq 2, 4$:

$$\begin{aligned} \operatorname{Tr}(e^{-tH} - e^{-tH_0}) = & \sum_{i=1}^{N'} e^{-t\lambda_i} + \frac{\log \det S(0)}{2\pi i} \\ & + \frac{1}{2\pi i} \int_0^\infty e^{-t\lambda} \frac{d}{d\lambda} \log \det S(\lambda)\, d\lambda, \end{aligned} \tag{4.2}$$

[Gu1], [Gu2]. If $n = 2, 4$, $(2\pi i)^{-1} \log \det S(0)$ has to be replaced by a different constant.

Let $K(t, x, y)$ be the kernel of $\exp(-tH)$. Using the heat equation method [Gi], it follows that as $t \to 0$, there exists an asymptotic expansion

$$\operatorname{Tr}(e^{-tH} - e^{-tH_0}) \sim (4\pi t)^{-n/2} \sum_{j=1}^{\infty} a_{j,n} t^j. \tag{4.3}$$

The first coefficients can be determined explicitly:

$$\begin{aligned} a_{1,n} &= -\int_{\mathbb{R}^n} V(x)\, dx,\ a_{2,n} = \frac{1}{2} \int_{\mathbb{R}^n} V^2(x)\, dx \\ a_{3,n} &= -\frac{1}{6} \int_{\mathbb{R}^n} \left(V^3(x) + \frac{1}{2} ||DV(x)||^2 \right) dx. \end{aligned}$$

The asymptotic expansion (4.3) corresponds to an asymptotic expansion of the scattering phase

$$s(\lambda) = \frac{1}{2\pi i} \log \det S(\lambda)$$

for high energy. Namely, for $\lambda \to \infty$ one has

$$\frac{ds}{d\lambda}(\lambda) \sim \lambda^{n/2-1} \sum_{j=1}^{\infty} \alpha_j(V) \lambda^{-j}, \tag{4.4}$$

and the coefficients $\alpha_j(V)$ are related to the coefficients in (4.3). We observe that the asymptotic expansion (4.4) holds for more general potentials (cf. [R]).

It remains to investigate the large time behaviour of $\operatorname{Tr}(e^{-tH}-e^{-tH_0})$. For this purpose we have to assume that n is odd. Then $\det S(\lambda^2)$ is real analytic on $(-c,\infty)$ for some $c>0$ and therefore it has a power series expansion at $\lambda=0$ which implies that there exists $\epsilon>0$ such that the series

$$\frac{ds}{d\lambda}(\lambda)=\frac{1}{2\pi i}\operatorname{Tr}\left(\frac{dS(\lambda)}{d\lambda}S^*(\lambda)\right)=\sum_{j=0}^{\infty}b_j\lambda^{j/2}. \tag{4.5}$$

is convergent for $0\le\lambda\le\epsilon$.

Now let P_1 be the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the orthogonal complement of the subspace spanned by the eigenfunctions of H that correspond to the negative eigenvalues. Let H_1 be the restriction of H to $\operatorname{Ran}P_1$. Then $H_1\ge 0$ and $H-H_1$ has finite rank. Let n be odd. Using (4.2), (4.3) and (4.5), it follows that (H_1,H_0) satisfies (1.1) – (1.3). Hence, the relative zeta function $\zeta(s;H_1,H_0)$ is well defined. Poles may occur at the points $s=k/2,\ k\in\mathbb{Z}$. All poles are simple and the residues can be determined from the expansions (4.3) and (4.5). The relative determinant $\det(H_1,H_0)$ is then given by (3.1). Using this determinant, we define the relative determinant $\det(\Delta+V,\Delta)$ by

$$\det(\Delta+V,\Delta)=\prod_{i=1}^{N}\lambda_i\cdot\det(H_1,H_0). \tag{4.6}$$

Remark. We recall from section 2, that in order to define the determinant, the full expansion (4.5) is not needed and may be replaced by weaker assumptions like (2.16). For example, there exist asymptotic expansions of the scattering matrix $S(\lambda)$ as $\lambda\to 0$ under more general assumptions about the potential (see [J1], [JK]).

If $W\in C_0^\infty(\mathbb{R}^n)$ is another potential, we may define the relative determinant $\det(\Delta+V+W,\Delta+V)$ in the same way, and by Lemma 3.1, we have

$$\det(\Delta+V+W,\Delta+V)=\det(\Delta+V+W,\Delta)\cdot\det(\Delta+V,\Delta)^{-1}. \tag{4.7}$$

If n is even, there is no expansion of the scattering phase near 0 in powers of λ. This problem can be eliminated if we shift the spectra of H and H_0 by a sufficiently large positive constant m. Let n be arbitrary.

Pick $m > 0$ such that $H + m > 0$. Then there exists $c > 0$ such that as $t \to \infty$, one has

$$\operatorname{Tr}\left(e^{-t(H+m)} - e^{-t(H_0+m)}\right) = (e^{-tc}).$$

If $t \to 0$, this trace has an asymtotic expansion similar to (4.3). Hence, the relative zeta function can be defined in the half–plane $\operatorname{Re}(s) > n/2$ in the usual way by

$$\begin{aligned} \zeta(s; H+m, H_0+m) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}\left(e^{-t(H+m)} - e^{-t(H_0+m}\right) \, dt \\ &= \operatorname{Tr}\left((H+m)^{-s} - (H_0+m)^{-s}\right). \end{aligned} \tag{4.8}$$

This zeta function was first introduced and studied by Guillopé [Gu1]. Using the trace formula (2.4), it follows that for $\operatorname{Re}(s) > n/2$

$$\begin{aligned} \zeta(s; H, H_0) = &\sum_{j=1}^{N'} (\lambda_j + m)^{-s} + \alpha m^{-s} \\ &+ \frac{1}{2\pi i} \int_0^\infty (\lambda + m)^{-s} \frac{d}{dz} \log \det S(\lambda) \, d\lambda, \end{aligned} \tag{4.9}$$

where α is determined by the singularity of $\xi(\lambda)$ at $\lambda = 0$. If $n \neq 2, 4$, then $\alpha = (2\pi i)^{-1} \log \det S(0)$. The analytic continuation of $\zeta(s; H + m, H_0 + m)$ is holomorphic at $s = 0$ and we can define the relative determinant by (3.1). Again, we denote this relative determinant by

$$\det(\Delta + V + m, \ \Delta + m).$$

We now consider the variation of $\det(\Delta + V + m, \Delta + m)$ with respect to V which can be computed using (3.17). First we note that

$$\begin{aligned} (H+m)^{-1} e^{-t(H+m)} &= \int_t^\infty e^{-s(H+m)} \, ds \\ &= (H+m)^{-1} e^{-(H+m)} + \int_t^1 e^{-s(H+m)} \, ds. \end{aligned}$$

Let $W \in C_0^\infty(\mathbb{R}^n)$. Using this formula together with the local heat expansion, it follows that as $t \to 0$, there is an asymptotic expansion of the form

$$\operatorname{Tr}\left(W(H+m)^{-1}e^{-t(H+m)}\right) \sim \sum_{\substack{j=1\\ j\neq n/2}}^{\infty} d_j(W)\, t^{-n/2+j} + c_1(W) + c_2(W)\log t$$

where

$$d_j(W) = \frac{1}{n/2-j}\int_{\mathbb{R}^n} W(x)a_{j-1,n}(x)\, dx$$

$$\begin{aligned} c_1(W) &= \operatorname{Tr}\left(W(H+m)^{-1}e^{-(H+m)}\right) \\ &+ \sum_{\substack{j=0\\ j\neq n/2-1}}^{p} (j+1-n/2)^{-1}\int_{\mathbb{R}^n} W(x)a_{j,n}(x)\, dx \\ &+ \int_{\mathbb{R}^n} W(x)\int_0^1 \left(K(s,x,x) - \sum_{j=0}^{p} a_{j,n}(x)s^{-n/2+j}\right) ds\, dx, \end{aligned}$$

$$c_2(W) = -\int_{\mathbb{R}^n} W(x)a_{n/2-1,n}(x)\, dx;$$

Thus, by (3.17) we have

$$\frac{\delta}{\delta V}\log\det(\Delta+V+m,\Delta+m) = c_1(\delta V) + \Gamma'(1)c_2(\delta V).$$

Now assume that n is odd and consider $\det(\Delta+V,\Delta)$. Since the continuous spectrum of $\Delta+V$ contains 0, $\det(\Delta+V,\Delta)$ may not be differentiable in all directions. But we can use Proposition 3.5 to determine under what assumptions about the potential and the variation, $\det(\Delta_u,\Delta_0)$ will be differentiable. Suppose, for example, that $V\geq 0$, $V\neq 0$, and let $W\in C_0^\infty(\mathbb{R}^n)$ be such that $V+uW\geq 0$ for $|u|\leq\epsilon$. Let $\Delta_u = \Delta+V+uW$ and let $K_u(x,y,t)$ be the kernel of $e^{-t\Delta_u}$. Then Δ_u has no L^2-eigenfunctions for $|u|\leq\epsilon$ and by Duhamel's principle, it follows that

$$\begin{aligned} &K_u(x,y,t) - (4\pi t)^{-n/2}e^{-\|x-y\|^2/4t} \\ &\quad = -(4\pi)^{-n/2}\int_0^t s^{-n/2}\int_{\mathbb{R}^n} K_u(x,z,t-s) \\ &\qquad \times (V+uW)(z)e^{-\|z-y\|^2/4s}\, dz\, ds. \end{aligned} \tag{4.10}$$

Hence, if $|u|\leq\epsilon$, we get

$$K_u(x,y,t) \leq (4\pi t)^{-n/2}e^{-\|x-y\|^2/4t}. \tag{4.11}$$

Inserting this estimate into (4.10), we obtain

$$\operatorname{Tr}\left(e^{-t\Delta_u} - e^{-t\Delta}\right) = O(t^{-n/2+1}) \tag{4.12}$$

as $t \to \infty$ and $|u| \le \epsilon$. Using (4.11) and (4.12), it is easy to verify that the assumptions of Proposition 3.5 are satisfied. Thus the relative determinant $\det(\Delta + V + uW, \Delta)$ is differentiable for $|u| \le \epsilon$.

4.2. **Perturbations of the Euclidean Laplacian.** Let g be a complete Riemannian metric on $\mathbb{R}^n$ and let $g_{ij}(x)$, $1 \le i,j \le n$, be the components of $g(x)$ with respect to the standard basis dx_i of $T_x^*\mathbb{R}^n$. Let $G = \det(g)$ and let $g^{kl}(x)$ denote the components of $(g_{ij}(x))^{-1}$. Furthermore, let W be a complex vector space of dimension m with Hermitian inner product $\langle \cdot, \cdot \rangle$. Let $A = (A_1, \dots, A_n)$ be a smooth Yang–Mills potential, i.e. the A_j are C^∞ functions on $\mathbb{R}^n$ with values in the Hermitian operators on W. Then we consider the following differential operator

$$\begin{aligned}\Delta &= \Delta_{g,A} \\ &= -G^{-1/4} \sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_j} + iA_j\right) \sqrt{G} g^{jk} \left(\frac{\partial}{\partial x_k} + iA_k\right) G^{-1/4},\end{aligned} \tag{4.13}$$

acting in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n) \otimes W$. Note that this class of operators contains, for example, all spinor Laplacians on $\mathbb{R}^n$, i.e. the squares of twisted Dirac operators. For $p > 0$, we make the following assumption:

For every $\alpha \in \mathbb{N}^n$, there exists $C_\alpha > 0$ such that

$$\sum_{ij} |\partial^\alpha (g_{ij}(x) - \delta_{ij})| + \sum_i |\partial^\alpha A_i(x)| \le C_\alpha (1 + ||x||^2)^{-p-|\alpha|} \tag{4.14}$$

In [CKS] it was proved that $\Delta_{g,A}$, regarded as operator in $\mathcal{H}$ with domain $C_0^\infty(\mathbb{R}^n) \otimes W$, is essentially self–adjoint. Let $H = H_{g,A}$ be the corresponding self–adjoint extension. Furthermore let

$$\Delta_0 = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \otimes \mathrm{Id}_W$$

and let H_0 be the self–adjoint extension of Δ_0. H and H_0 are nonnegative self–adjoint operators in $\mathcal{H}$. The following results, which we summarize as a proposition, were established in [CKS].

Proposition 4.2. *Suppose that $p > 1$. Then*

a) *The wave operators $W_\pm(H, H_0)$ exist and are complete.*

b) *H has no point spectrum.*
c) *The singular continuous spectrum of H is empty.*

If $p > n$, it follows from [Ro], [Br] that for $t > 0$, $e^{-tH} - e^{-tH_0}$ is a trace class operator and as $t \to 0$, there is an asymptotic expansion of the form

$$\operatorname{Tr}(e^{-tH} - e^{-tH_0}) \sim \sum_{j \geq 0} a_j t^{-n/2+j}. \tag{4.15}$$

Thus for $m > 0$, the pair of operators $(H+m,\ H_0+m)$ satisfies (1.1) – (1.3) and the relative zeta function $\zeta(s; H+m, H_0+m)$ can be defined in $\operatorname{Re}(s) > n/2$ by the usual formula

$$\zeta(s; H+m, H_0+m) = \operatorname{Tr}\left((H+m)^{-s} - (H_0+m)^{-s}\right). \tag{4.16}$$

Remark. In a more general context, such zeta functions have been studied by Bruneau [Br] in his thesis. Bruneau considers a certain class of elliptic operators in $\mathbb{R}^n$ which, for example, includes the Laplacians for asymptotically equal metrics and also the operators of the type (4.13). He then studies relative zeta functions like (4.16).

In order to define the relative zeta function for (H, H_0) itself, we need some information about the behaviour of the scattering matrix at $\lambda = 0$. This requires more stringent assumptions. For example, we may assume that

$$g_{ij}(x) = \delta_{ij}\ ,\ i,j = 1,\dots,n,\ A_j(x) = 0,\ j = 1,\dots,n \text{ for } \parallel x \parallel \geq R.$$

Then $\Delta_{g,A} = \Delta_0$ outside a compact set. If n is odd, it follows from Theorem 1.1 of [SZ] that $(H - \lambda^2)^{-1}$, $\operatorname{Im}(\lambda) > 0$ has a continuation to a meromorphic function of $\lambda \in \mathbb{C}$ with values in the operators from $L^2_{comp}(\mathbb{R}^n) \otimes W$ to $L^2_{loc}(\mathbb{R}^n) \otimes W$. Let $S(\lambda) = S(\lambda; H, H_0)$ be the scattering matrix. Then, using the analytic continuation of the resolvent, it follows that $S(\lambda^2)$ extends to a meromorphic function of $\lambda \in \mathbb{C}$. We now can procced in essentially the same way as in the case of the Schrödinger operator in odd dimensions. In particular, the relative determinant $\det(\Delta_{g,A}, \Delta_0)$ exists and we can study it as a function of the metric g and the gauge field A.

In the same way, one can study Dirac operators on $\mathbb{R}^n$ coupled to a gauge field. For example, consider the Dirac operator on $\mathbb{R}^3$:

$$D_A = c\sum_{j=1}^{3}\gamma_j\left(\frac{\partial}{\partial x_j}+iA_j(x)\right)+mc^2\gamma_4+V(x),\quad x\in\mathbb{R}^3,$$

where $c>0$ is the speed of light, $\gamma_1,...\gamma_4$ are the Dirac matrices, V is the electric potential, m is the mass and $A=(A_1,A_2,A_3)$ is the magnetic field. We assume that the potential V and the magnetic field A are C^∞ and satisfy the following assumption:

For each $\alpha\in\mathbb{N}^3$, there exists $C_\alpha>0$ such that

$$|\partial^\alpha A(x)|+|\partial^\alpha V(x)|\le C_\alpha(1+|x|^2)^{-p-|\alpha|},\quad x\in\mathbb{R}^3,$$

where $p>0$ is independent of α.

Then D_A is essentially self–adjoint in $(L^2(\mathbb{R}^3))^4$. Let D_0 be the free Dirac operator, i.e. the Dirac operator with magnetic field $A=0$. If $p>1$, then the wave operators $W_\pm(D_A,D_0)$ exist and are complete. The singularly continuous spectrum of D_A is empty and the essential spectrum of D_A equals $(-\infty,-mc^2]\cup[mc^2,\infty)$ (cf. [Th]). In particular, if $m>0$, then the continuous spectrum of D_A has a gap at zero. It is proved in [Br], [Br1] that the operators (D_A,D_0) satisfy (1.11) and (1.12). Due to the gap of the continuous spectrum, $\operatorname{Tr}(D_A\exp(-tD_A^2)-D_0\exp(-tD_0^2))$ is exponentially decreasing for $t\to\infty$. Thus we can define the relative eta invariant $\eta(D_A,D_0)$. Moreover, $\det(|D_A|,|D_0|)$ also exists and therefore, $\det(D_A,D_0)$ exists too and is given by (3.19).

4.3. **Manifolds with cylindrical ends.** Let M be a compact n–dimensional C^∞ manifold with smooth boundary Y. Let X be the non-compact manifold obtained from M by gluing the bottom of the half–cylinder $\mathbb{R}^+\times Y$ to the boundary of M, i.e.

$$X=M\cup_Y(\mathbb{R}^+\times Y)$$

We equip X with a Riemannian metric g which is a product on $\mathbb{R}^+\times Y$, i.e. on $\mathbb{R}^+\times Y$, g takes the form

$$g=du^2+h,$$

where h is the pull–back of a metric on Y. Then X equipped with g is called a "manifold with a cylindrical end". In this way, we obtain a natural class of complete Riemannian metrics on X. We also may consider larger classes of metrices which are obtained by perturbations of cylindrical end metrics. For example, we may require that the perturbation together with all its derivatives is exponentially decreasing if $u \to \infty$, where u is the radial variable on the cylinder. If we change coordinates by $v = e^{-u}$, then we get a metric on the compact manifold M. Let x be a defining function of ∂M. Then near the boundary, the metric g has the form

$$\frac{dx^2}{x^2} + h,$$

where h is a semi–positive metric on M which restricts to a non–degenerate metric on ∂M. These are the exact b–metrics studied by Melrose [Me1], [Me2]. Another possible condition would be to require that the perturbation and a finite number of its derivatives decay like $(1+u^2)^{-p}$, for some $p > 1$.

Here we shall restrict attention to metrics which are exact products on the cylinder. But most of the results extend to larger classes of metrics; for example, to the class of exact b–metrics. Let X be equipped with a cylindrical end metric and let Δ be the corresponding Laplacian. Then Δ is essentially self–adjoint and we continue to denote its unique self–adjoint extension by Δ. Now, we shall briefly describe some facts concerning the spectral decomposition of Δ. For details see [Me1], [Mu3]. The spectrum of Δ is the union of a pure point spectrum $\sigma_{pp}(\Delta)$ and the absolutely continuous spectrum $\sigma_{ac}(\Delta)$. The point spectrum $\sigma_{pp}(\Delta)$ consists of eigenvalues of finite multiplicity with no finite point of accumulation [Do]. Let

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty$$

be the sequence of eigenvalues where each eigenvalue is repeated according to its mulitplicity. Let

$$N_{pp}(\lambda) = \# \{ j \mid \lambda_j \le \lambda \}.$$

It was proved by Christiansen and Zworski [CZ] that there exists $C > 0$ such that

$$N_{pp}(\lambda) \le C(1+\lambda)^{n/2}, \ \lambda \ge 0, \tag{4.17}$$

$n = \dim X$. The first result of this type is due to Donnelly [Do], who proved a similar bound with n replaced by $2n-1$. We note that (4.17) is the optimal bound [CZ]. The estimate (4.17) implies that for $t > 0$,

$$\sum_i e^{-t\lambda_i} < \infty. \tag{4.18}$$

Let $L^2_p(X) \subset L^2(X)$ be the subspace spanned by the L^2–eigenfunctions of Δ. Then it follows from (4.18) that $e^{-t\Delta}|L^2_p(X)$ is a trace class operator and

$$\operatorname{Tr}\left(e^{-t\Delta}|L^2_p(X)\right)=\sum_j e^{-t\lambda_j}. \tag{4.19}$$

Now note that on $\mathbb{R}^+\times Y$, one has

$$\Delta=-\frac{\partial^2}{\partial u^2}+\Delta_Y. \tag{4.20}$$

Therefore, standard perturbation theory implies that the continuous spectrum of Δ is governed by the Laplacian Δ_Y of Y. We regard the right hand side of (4.20) as operator in $L^2(\mathbb{R}^+\times Y)$ with domain $C_0^\infty(\mathbb{R}^+\times Y)$ and impose Dirichlet boundary conditions at the bottom of the half–cylinder. Let Δ_0 be the corresponding self–adjoint extension. Then $e^{-t\Delta}-e^{-t\Delta_0}$ is a trace class operator for $t>0$. Hence, by the Birman-Kato invariance principle, the wave operators $W_\pm(\Delta,\Delta_0)$ exist and are complete. Therefore, Δ_{ac} is unitarily equivalent to Δ_0. Let

$$0=\mu_0<\mu_1<\mu_2<\cdots\to\infty$$

be the sequence of eigenvalues of Δ_Y. Then it follows that

$$\sigma_{ac}(\Delta)=\bigcup_{j=0}^\infty[\mu_j,\infty),$$

i.e. the μ_j's are the thresholds of the continuous spectrum. At each threshold there starts a new branch of the continuous spectrum with multiplicity equal to the rank of the corresponding eigenspace.

A more explicit description of the continuous spectrum can be given in terms of generalized eigenfunctions. Let $\mathcal{E}(\mu_k)$ be the eigenspace for the eigenvalue μ_k of Δ_Y. Let $\phi\in\mathcal{E}(\mu_k)$ and let $\lambda>\mu_k$ be different from all thresholds. Then there exists a unique eigenfunction $E(\phi,\lambda)\in C^\infty(X)$ of Δ with eigenvalue λ which on $\mathbb{R}^+\times Y$ has the following expansion

$$\begin{aligned}E(\phi,\lambda,(u,y))=&e^{iu\sqrt{\lambda-\mu_k}}\phi(y)\\&+\sum_{\mu_l<\lambda}e^{-iu\sqrt{\lambda-\mu_l}}\left(T_{kl}(\lambda)\phi\right)(y)+\Psi,\end{aligned} \tag{4.21}$$

where $\Psi\in L^2(\mathbb{R}^+\times Y)$ and $T_{kl}(\lambda):\mathcal{E}(\mu_k)\to\mathcal{E}(\mu_l)$ is a linear operator. Furthermore, each generalized eigenfunction $E(\phi,\lambda)$ extends to a meromorphic function of $\lambda\in\Sigma$ where Σ is the minimal Riemann surface to which all functions $\sqrt{\lambda-\mu_k}$, $k\in\mathbb{N}$, extend to be holomorphic. This is a consequence of the extension of the resolvent $(\Delta-\lambda)^{-1}$, as operator in certain weighted L^2 spaces, to a meromorphic function of $\lambda\in\Sigma$ (see [Mu4], [Me1]). More precisely, let $\rho\in C^\infty(X)$ be such that $\rho(u,y)=u$

for $(u,y) \in [1,\infty) \times Y$. For $\delta \in \mathbb{R}$, let $L^2_\delta(X)$ be the Hilbert space of all measurable functions on X which are square integrable with respect to the measure $e^{2\rho(x)}dx$. Then the following holds:

Theorem 4.3. *Let $\delta > 0$. Then the resolvent $(\Delta - \lambda)^{-1/2}$, as operator from $L^2_\delta(X)$ to $L^2_{-\delta}(X)$, extends to a holomorphic function of $\lambda \in \Sigma$.*

For the proof see [Me1]. One also can adapt the method described in [Mu4].

Let

$$S_{kl}(\lambda) = \left(\frac{\lambda - \mu_k}{\lambda - \mu_l}\right)^{1/4} T_{kl}(\lambda) \tag{4.22}$$

and put

$$S(\lambda) = \bigoplus_{\mu_k,\mu_l<\lambda} S_{kl}(\lambda).$$

Then

$$S(\lambda)\colon \bigoplus_{\mu_k<\lambda} \mathcal{E}(\mu_k) \to \bigoplus_{\mu_k<\lambda} \mathcal{E}(\mu_k)$$

is equal to the scattering matrix $S(\lambda;\Delta,\Delta_0)$. Since the generalized eigenfunctions are meromorphic functions of $\lambda \in \Sigma$, the same holds for $T_{kl}(\lambda)$ and therefore, for $S_{kl}(\lambda)$ too. This leads to the following result about the structure of the scattering matrix.

Proposition 4.4. *The scattering matrix $S(\lambda)$ is a unitary operator of finite rank which depends smoothly on $\lambda \in (\mu_{j-1},\mu_j)$ for all $j \in \mathbb{N}$, and its rank changes when λ crosses a threshold. Moreover, the coefficients of $S(\lambda)$ extend to meromorphic functions on Σ.*

We note that if $0 < \lambda < \mu_1$, then $S(\lambda)$ is a scalar and $S(\lambda^2)$ extends to a meromorphic function on $\{z \in \mathbb{C} \mid |z| < \mu_1\}$ which is regular at $\lambda = 0$. In particular, $S(\lambda)$ is a holomorphic function in a small neighborhood of the origin. Thus, condition 2) of Proposition 2.4 is satisfied and it remains to show that the spectral shift function $\xi(\lambda) = \xi(\lambda;\Delta,\Delta_0)$ is continuous in some interval $(0,\epsilon)$, $\epsilon > 0$. To study $\xi(\lambda)$ one may use the definition (2.1) of the spectral shift function and proceed along lines similar to [JK2]. However, for our present purpose, we also can use the trace formula proved by Christiansen [Ch] for exact b-metrics (see [Me2, Chapter 7]). The trace that is used by Christiansen is a regularization of the trace for operators with continuous kernels, called b-trace, which was introduced by Melrose [Me1]. The trace formula computes the b-trace of $\cos\big(t\sqrt{\Delta}\big)$, as tempered distribution, in terms of spectral data. But the b-trace of $\cos\big(t\sqrt{\Delta}\big)$ is closely related to the trace of $\cos\big(t\sqrt{\Delta}\big) - \cos\big(t\sqrt{\Delta_0}\big)$, taken in the distributional sense.

Therefore, the b-trace formula gives rise to a relative trace formula. For example, the b-trace of $e^{-t\Delta}$ is given by

$$\text{b-}\operatorname{Tr}\left(e^{-t\Delta}\right) = \lim_{R\to\infty}\left\{\int_{X_R} e^{-t\Delta}(x,x)\,dx - \frac{R}{\sqrt{4\pi t}}\operatorname{Tr}\left(e^{-t\Delta_Y}\right)\right\}$$

where

$$X_R = M\cup_Y([0,R]\times Y).$$

Now note that the kernel of $e^{-t\Delta_0}$ equals

$$\frac{1}{\sqrt{4\pi t}}\left\{\exp\left(-\frac{(u-u')^2}{4t}\right) - \exp\left(-\frac{(u+u')^2}{4t}\right)\right\} e^{-t\Delta_Y}(y,y').$$

Thus, we get

$$\begin{aligned}
&\operatorname{Tr}\left(e^{-t\Delta} - e^{-t\Delta_0}\right)\\
&\qquad = \lim_{R\to\infty}\left\{\int_{X_R} e^{-t\Delta}(x,x)\,dx - \int_{[0,R]\times Y} e^{-t\Delta_0}(x,x)\,dx\right\}\\
&\qquad = \lim_{R\to\infty}\left\{\int_{X_R} e^{-t\Delta}(x,x)\,dx - \frac{R}{\sqrt{4\pi t}}\operatorname{Tr}\left(e^{-t\Delta_Y}\right)\right\}\\
&\qquad\quad + \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-u^2/t}\,du\operatorname{Tr}\left(e^{-t\Delta_Y}\right)\\
&\qquad = \text{b-}\operatorname{Tr}\left(e^{-t\Delta}\right) + \frac{1}{4}\operatorname{Tr}\left(e^{-t\Delta_Y}\right).
\end{aligned}$$

Since $e^{-t\Delta}$ can be expressed in terms of $\cos\left(t\sqrt{\Delta}\right)$ by

$$e^{-t\Delta} = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-u^2/4t}\cos\left(u\sqrt{\Delta}\right)du,$$

the application of the b-trace formula leads to the following relative trace formula

$$\begin{aligned}
\operatorname{Tr}\left(e^{-t\Delta} - e^{-t\Delta_0}\right) = &\sum_j e^{-t\lambda_j} + \frac{1}{2}\sum_{k=1}^\infty \dim\mathcal{E}(\mu_k)e^{-t\mu_k}\\
&+ \frac{1}{4}\operatorname{Tr}(S(0))\\
&+ \frac{1}{2\pi i}\int_0^\infty e^{-t\lambda}\frac{d}{d\lambda}\log\det S(\lambda)\,d\lambda.
\end{aligned}\tag{4.23}$$

Remark. An analogous trace formula holds for $\varphi(\Delta)-\varphi(\Delta_0)$ where φ is any function in $C_0^\infty(\mathbb{R})$. It follows from Proposition 2.1 that in order to establish the trace formula, it is sufficient to determine the spectral shift function up to an additive constant. On the other hand, one can compute the spectral shift function explicitly using the eigenfunction expansion of the corresponding kernels. This approach was used in [Mu5] to compute the spectral shift function for spinor Laplacians on finite volume locally symmetric spaces of $\mathbb{Q}$–rank one. The computations for manifolds with cylindrical ends are similar, in fact, they are easier.

Using the trace formula (4.23) together with the fact that $S(\lambda^2)$ is holomorphic in a neighborhood of $\lambda = 0$, it follows that as $t \to \infty$, we have an asymptotic expansion of the form

$$\operatorname{Tr}\left(e^{-t\Delta} - e^{-t\Delta_0}\right) \sim \dim\ker\Delta + \frac{1}{4}\operatorname{Tr}\left(S(0)\right) + \sum_{j=1}^{\infty} c_j t^{-j/2}.$$

To determine the asymptotic behaviour as $t \to 0$, we may use the standard parametrix construction which implies that there is an asymptotic expansion of the form

$$\operatorname{Tr}\left(e^{-t\Delta} - e^{-t\Delta_0}\right) \sim \sum_{j=0}^{\infty} a_j t^{-n/2+j}$$

as $t \to 0$. Hence, conditions (1.1) – (1.3) hold and the relative zeta function $\zeta(s;\Delta,\Delta_0)$ is a meromorphic function of $s \in \mathbb{C}$. Using the trace formula (4.23), the zeta function can be expressed in terms of the eigenvalues and the scattering phase shift. First we note that by the main result of [CZ], the scattering phase satisfies

$$\int_0^{\Lambda} \frac{d}{dz} \log\det S(\lambda)\, d\lambda = O(\Lambda^n)$$

as $\Lambda \to \infty$. Let $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$. Then, using (4.23), it follows that for $\operatorname{Re}(s) > n$, we have

$$\begin{aligned}\zeta(s;\Delta+z,\Delta_0+z) &= \sum_j (\lambda_j+z)^{-s} + \sum_k (\mu_k+z)^{-s}\\ &\quad + \frac{1}{2\pi i}\int_0^{\infty} (\lambda+z)^{-s} \frac{d}{d\lambda}\log\det S(\lambda)\, d\lambda. \qquad (4.24)\end{aligned}$$

The series and the integral are absolutely convergent for s in the given half–plane. There is a similar formula for $z = 0$, except that the integral has to be defined by analytic continuation using the expansion

$$\log\det S(\lambda) = \sum_{j=0}^{\infty} d_j \lambda^{j/2}, \quad |\lambda| < \varepsilon.$$

This expansion is a consequence of the analyticity of $S(\lambda^2)$ in a neighborhood of $\lambda = 0$.

Everything that has been said for the Laplacian Δ on functions extends without any difficulties to generalized Dirac operators. Suppose, for example, that X is a spin manifold and let $F \to X$ be a Hermitian vector bundle over X with a Hermitian connection. We assume that all structures are adapted to the product structure on $\mathbb{R}^+ \times Y$. Let S be the spinor bundle of X and let

$$D : C^\infty(S \otimes F) \to C^\infty(S \otimes F)$$

be the associated twisted Dirac operator. Then, on $\mathbb{R}^+ \times Y$, D has the following form

$$D = \gamma \left(\frac{\partial}{\partial u} + D_Y \right) \tag{4.25}$$

where γ denotes the Clifford multiplication by the exterior normal vector field to Y and D_Y is a twisted Dirac operator on Y. We note that D is essentially self-adjoint in $L^2(S \otimes F)$ and we shall denote its unique self–adjoint extension also by D. Let $H = D^2$ and let H_0 be the closure of

$$-\frac{\partial^2}{\partial u^2} + D_Y^2 \colon C_0^\infty(\mathbb{R}^+ \times Y, S \otimes F) \to L^2(\mathbb{R}^+ \times Y, S \otimes F)$$

obtained by imposing Dirichlet boundary conditions. Then, as above, H, H_0 satisfy the conditions (1.1) - (1.3) and therefore, $\zeta(s; H, H_0)$ and $\det(H, H_0)$ are well defined. Since, by (4.25), D determines uniquely the operator H_0, $\det(H, H_0)$ depends only on D and we shall denote it simply by $\det(D^2)$.

Now assume that $\dim X$ is even. Then the spinor bundle splits as

$$S = S^+ \oplus S^-$$

and, with respect to this splitting, D takes the form

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

Then D_-D_+ and D_+D_- have unique self–adjoint extensions H^+ and H^-, respectively. Note that $H = H^+ \oplus H^-$. Similarly, we get $H_0^\pm$ such that $H_0 = H_0^+ \oplus H_0^-$. Then the relative determinants $\det(H^\pm, H_0^\pm)$ can be defined as above and we shall denote them by $\det(D_-D_+)$ and $\det(D_+D_-)$, respectively.

Next we consider families of Dirac operators on manifolds with cylindrical ends. Let M and B be connected Riemannian manifolds and assume that B is compact. Let $\pi \colon M \to B$ be a Riemannian submersion whose fibres $Z_b = \pi^{-1}(b)$ are $2k$–dimensional manifolds with cylindrical ends. We assume that Z_b is an oriented spin manifold. Such families can be constructed from fibrations of manifolds with boundary along the lines of [BC]. Let $F \to M$ be a Hermitian vector bundle which is adapted to the product structure and let D_b be the twisted Dirac operator on the fibre Z_b. Let $\mathbb{R}^+ \times Y_b$ be the cylindrical end of Z_b. Then on $\mathbb{R}^+ \times Y_b$, D_b has the form

$$D_b = \gamma_b \left(\frac{\partial}{\partial u} + D_{Y_b} \right).$$

Let $\mathcal{D}$ denote the corresponding family of Dirac operators. Assume that $0 \notin \operatorname{Spec}(D_{Y_b})$ for all $b \in B$. Then for each $b \in B$, D_b has discrete spectrum near zero, and we may construct the determinant line bundle

$$\mathcal{L} = \det(\ker \mathcal{D}_+)^* \otimes \det(\ker \mathcal{D}_-)$$

in the same way as in [BF1]. Using the determinant $\det(D_{-,b}D_{+,b})$ defined as above, we can equip $\mathcal{L}$ with the Quillen metric.

More generally, we may assume that $\dim \ker D_{Y_b}$ is constant. This means that the orthogonal projections P_b of $L^2(S \otimes F|Y_b)$ onto $\ker D_{Y_b}$ are a smooth family, i.e. $\ker D_{Y_b}$, $b \in B$, gives rise to a smooth vector bundle $\ker D_Y$ over B. Then we modify the Dirac operator as follows. Let $f \in C^\infty(\mathbb{R})$ be such that $f(u) = 0$ for $u \leq 1$ and $f(u) = 1$ for $u \geq 2$. Put

$$\widetilde{D}_b = D_b - \gamma_b f(u) P_b.$$

By our assumption, we get an operator with smooth coefficients

$$\widetilde{\mathcal{D}} \colon C^\infty(M, S \otimes F) \to C^\infty(M, S \otimes F).$$

Although $\widetilde{\mathcal{D}}$ is not a differential operator, it is easy to see that $\widetilde{\mathcal{D}}$ has similar spectral properties as $\mathcal{D}$. Furthermore, note that on $[2, \infty) \times Y_b$, we have

$$\widetilde{D}_b = \gamma_b \left(\frac{\partial}{\partial u} + D_{Y_b} - P_b \right)$$

and by definition, $\widetilde{D}_{Y_b} = D_{Y_b} - P_b$ is invertible for all $b \in B$. This means that the continuous spectrum of $\widetilde{D}_b$ has a gap at zero. Thus, we may proceed as above and construct the determinant line bundle $\widetilde{\mathcal{L}}$ for the family $\widetilde{\mathcal{D}}$ and equip $\widetilde{\mathcal{L}}$ with the Quillen metric. The following lemma implies that the two line bundles are in fact the same.

Lemma 4.5. *Let $\ker \widetilde{D}_b$ and $\ker D_b$ denote the spaces of L^2–solutions of $\widetilde{D}_b$ and D_b, respectively. Then we have*

$$\ker \widetilde{D}_b = \ker D_b, \quad b \in B.$$

Proof. To simplify notation, we skip the subscript b. Let ϕ_j, $j \in \mathbb{Z}$, be an orthonoarmal basis of eigenfunctions of D_Y and let λ_j be the corresponding eigenvalues. If $\varphi \in \ker D$, then on $\mathbb{R}^+ \times Y$, φ has the following expansion:

$$\varphi(u,y) = \sum_{\lambda_j > 0} a_j e^{-\lambda_j u} \phi_j(y).$$

Thus $P\varphi = 0$ and therefore, $\varphi \in \ker \widetilde{D}$. Now let $\psi \in \ker \widetilde{D}$. The restriction of ψ to $\mathbb{R}^+ \times Y$ is given by

$$\psi(u,y) = \sum_{\lambda_j > 0} b_j e^{-\lambda_j u} \phi_j(y) \; + \sum_{\lambda_k = 0} c_k \exp\left(\int_0^u f(u)\,du\right) \phi_k(y).$$

Since $f(u) = 1$ for $u > 2$ and ψ is square integrable, it follows that $c_k = 0$ for all k. Thus $P\psi = 0$ and hence, $D\psi = 0$. $\square$

Thus, the determinant line bundle $\widetilde{\mathcal{L}}$ associated to $\widetilde{\mathcal{D}}$ coincides with $\mathcal{L}$. Since for each $b \in B$, the spectrum of $\widetilde{D}_b$ has a gap at zero, we can construct the Quillen metric as above.

4.4. **Surfaces with hyperbolic ends.** Let (X, g) be a complete surface of finite area such that the Gaussian curvature K of (X, g) satisfies $K \equiv -1$ in the complement of a compact set. We call (X, g) a surface with hyperbolic ends. Any such surface X admits a decomposition of the form

$$X = X_0 \cup Y_1 \cup \cdots \cup Y_m,$$

where X_0 is a compact surface with smooth boundary and

$$Y_i \cong [1, \infty) \times S^1, \; i = 1, \dots, m,$$

and the metric on Y_i equals

$$ds^2 = \frac{dy^2 + dx^2}{y^2}$$

where $(y, x) \in [1, \infty) \times S^1$. Each end Y_i is called a cusp of X. Special cases are hyperbolic surfaces of finite area, i.e. surfaces with $K \equiv -1$ everywhere and $\mathrm{Area}(X) < \infty$. Then there exists a discrete torsion free subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ of finite co-volume such that $X = \Gamma \backslash H$ where H denotes the upper half-plane equipped with the Poincaré metric. Examples are the well known surfaces $\Gamma(N)\backslash H$ where $\Gamma(N) \subset SL(2, \mathbb{Z})$ is the principal convergence subgroup of level N.

Let Δ be the Laplacian of X. Then Δ is essentially self–adjoint in $L^2(X)$. The structure of the spectrum $\sigma(\Delta)$ of Δ is well known (see e.g. [Mu2]). Namely, $\sigma(\Delta)$ is the union of the absolutely continuous spectrum $\sigma_{ac}(\Delta)$, and the pure point spectrum $\sigma_{pp}(H)$. The point spectrum consists of a sequence of eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$

of finite multiplicity. The only possibly point of accumulation of the eigenvalue sequence is ∞ and, as proved by Colin de Verdiere [CV2], for a generic metric there exist only finitely many eigenvalues which are all contained in [0,1/4).

The absolutely continuous spectrum equals $[1/4,\infty)$ with multiplicity equal to the number of cusps m. Let Δ_0 be the self–adjoint extension of the operator

$$-\sum_{i=1}^{m} y_i^2 \frac{d^2}{dy_i^2} \colon \bigoplus_{i=1}^{m} C_0^\infty([1,\infty)) \to \bigoplus_{i=1}^{m} L^2\left([1,\infty), y_i^{-2} dy_i\right)$$

with respect to Dirichlet boundary conditions. If we regard a function on $[1,\infty)$ as a function on $[1,\infty)\times S^1 \cong Y_i$ which is independent of the second variable, then we get a canonical inclusion

$$\bigoplus_{i=1}^{m} L^2\left([1,\infty), y_i^{-2} dy_i\right) \subset L^2(X). \tag{4.26}$$

Hence, we may consider $e^{-t\Delta_0}$ as an operator in $L^2(X)$ which is equal to zero in the orthogonal complement of the image of the inclusion (4.26). With this identification, it was proved in [Mu1] that $e^{-t\Delta} - e^{-t\Delta_0}$ is a trace class operator for every $t>0$. Thus Δ_{ac} and Δ_0 are unitarily equivalent. Moreover, there exists a complete set of generalized eigenfunctions $E_i(z,s)$, $i=1,...,m$, which are meromorphic functions of $s\in\mathbb{C}$ satisfying

$$\Delta E_i(z,s) = s(1-s)E_i(z,s), \quad s\in\mathbb{C}.$$

Furthermore, each $E_i(z,s)$ is regular on the line $\operatorname{Re}(s)=1/2$. The restriction of $E_i(z,s)$ to the cusp Y_j can be expanded in a Fourier series and the zeroth Fourier coefficient has the form

$$\delta_{ij} y_j^s + C_{ij}(s) y_j^{1-s}.$$

Put $C(s) = \big(C_{ij}(s)\big)$. Then $C(s)$ is a meromorphic matrix valued function which satisfies the following functional equation

$$C(s)C(1-s) = \mathrm{Id}.$$

Moreover, the scattering matrix $S(\lambda) = S(\lambda;\Delta,\Delta_0)$ is related to $C(s)$ by

$$S\left(\frac{1}{4}+\lambda^2\right) = C\left(\frac{1}{2}+i\lambda\right), \quad \lambda\in\mathbb{R}. \tag{4.27}$$

Hence $S(\lambda)$ has an extension to a meromorphic function on the double covering of $\mathbb{C}$ defined by $\lambda = s(1-s)$ and we have

$$S(s(1-s)) = C(s).$$

Using the generalized eigenfunctions, the spectral shift function $\xi(\lambda) = \xi(\lambda; \Delta, \Delta_0)$ can be computed explicitly. Together with Proposition 2.1, this leads to the following trace formula

$$\begin{aligned}
\operatorname{Tr}\left(e^{-t\Delta} - e^{-t\Delta_0}\right) &= \sum_j e^{-t\lambda_j} + \frac{1}{4}\operatorname{Tr}\left(S(1/4)\right) e^{-t/4} \\
&\quad - \frac{1}{2\pi i}\int_{1/4}^{\infty} e^{-t\lambda}\frac{d}{d\lambda}\log\det S(\lambda)\, d\lambda \\
&= \sum_j e^{-t\lambda_j} + \frac{1}{4}\operatorname{Tr}\left(C(1/2)\right) e^{-t/4} \qquad (4.28)\\
&\quad - \frac{1}{4\pi i}\int_{-\infty}^{\infty} e^{-(1/4+\lambda^2)t}\frac{d}{d\lambda}\log\det C(1/2+i\lambda)\, d\lambda
\end{aligned}$$

where we have used (4.27).

Remark. There is a similar trace formula for $\cos\left(t\sqrt{\Delta}\right) - \cos\left(t\sqrt{\Delta_0}\right)$. If X is hyperbolic, i.e. $X = \Gamma\backslash H$, then the left hand side of this trace formula can be rewritten as a sum of orbital integrals where the sum runs over the conjugacy classes in Γ. This is then the Selberg trace formula.

It follows from (4.28) that as $t \to \infty$, we have

$$\operatorname{Tr}\left(e^{-t\Delta} - e^{-t\Delta_0}\right) = 1 + O(e^{-ct})$$

for some $c > 0$. Furthermore, using Theorem 8.20 of [Mu1], we obtain the following asymptotic expansion for $t \to 0$:

$$\begin{aligned}
\operatorname{Tr}\left(e^{-t\Delta} - e^{-t\Delta_0}\right) &= \frac{\text{Area}}{4\pi t} + \frac{m}{2}\frac{\log t}{\sqrt{4\pi t}} \\
&\quad + \frac{3\gamma m}{2}\frac{1}{\sqrt{4\pi t}} + \frac{\chi(X)}{6} + O(\sqrt{t}) \qquad (4.29)
\end{aligned}$$

Here γ is Euler's constant and $\chi(X)$ is the Euler characteristic of X. The higher order terms are similar. This is an example where logarithmic terms arise naturally. Summarizing, it follows that Δ, Δ_0 satisfy (1.1) - (1.3) and therefore, the relative zeta function $\zeta(s; \Delta, \Delta_0)$ exists.

Using (4.28), it follows that for $\operatorname{Re}(s) > 1$, the relative zeta function is given by

$$\begin{aligned}
\zeta(s; \Delta, \Delta_0) &= \sum_{\lambda_j > 0} \lambda_j^{-s} + 4^{s-1}\operatorname{Tr}(C(1/2)) \\
&\quad - \frac{1}{2\pi i}\int_{1/4}^{\infty} \lambda^{-s}\frac{d}{d\lambda}\log\det S(\lambda)\, d\lambda.
\end{aligned}$$

By (4.29), $\zeta(s;\Delta,\Delta_0)$ has a meromorphic continuation to $\mathbb{C}$ which is holomorphic at $s=0$. In particular, we can define the relative determinant $\det(\Delta,\Delta_0)$ by (3.1). Since Δ_0 is uniquely determined by Δ, we shall denote the determinant by $\det\Delta$.

Remark. A different regularization of a relative determiant for surfaces with hyperbolic ends was introduced by Lundelius [Lu]. We recall his definition. Let X_i, $i=1,2$, be two surfaces with hyperbolic ends and assume that the number of ends is the same for both surfaces. Let Δ_i, $i=1,2$, be the Laplacian of X_i. By assumption there exist open relatively compact subsets $K_i\subset X_i$ such that $X_1-K_1=X_2-K_2$ and $\Delta_1|(X_1-K_1)=\Delta_2|(X_2-K_2)$. Let

$$\mathcal{H}=L^2(X_1)\oplus L^2(K_2)\cong L^2(X_2)\oplus L^2(K_1).$$

Then we may regard Δ_i as unbounded operator in $\mathcal{H}$ which is zero in $L^2(K_j)$, $i\neq j$, $i,j=1,2$. Moreover, $e^{-t\Delta_1}-e^{-t\Delta_2}$ is a trace class operator for $t>0$ and (1.2), (1.3) hold for (Δ_1,Δ_2). Thus, one can define $\det(\Delta_1,\Delta_2)$. This is the relative determinant used by Lundelius. It is closely related to our definition. In fact, it follows from the definitions that

$$\det(\Delta_1,\Delta_2)=\frac{\det\Delta_1}{\det\Delta_2}.$$

Next consider the differentiability of the determinant. It follows from Proposition 3.5 that $\det\Delta$ is a differentiable function on the space of all metrics with hyperbolic ends. For example, let g_0 be any metric with hyperbolic ends and let $f\in C_0^\infty(X)$. Put

$$g_u=e^{uf}g_0,\quad u\in\mathbb{R}.$$

Then g_u is also a metric with hyperbolic ends. Let Δ_{g_u} be the Laplacian with respect to g_u. Then

$$\Delta_{g_u}=e^{-uf}\Delta_{g_0}$$

and it follows as in [OPS] that

$$-\frac{d}{du}\log\det\Delta_{g_u}\big|_{u=0}=\int_X f(x)\left(\frac{K_0(x)}{12\pi}-\frac{1}{A_0}\right)d\mu_0(x)$$

where $K_0(x)$, A_0 and $d\mu_0$ are the Gaussian curvature, the area and the volume element of (X,g_0), respectively. Thus one can study critical points of $\det\Delta$ for finite area surfaces with hyperbolic ends in the same way as in [OPS].

In the present case, $\det\Delta$ can be expressed in a different way using the eigenvalues of Δ and the *resonances*. Here resonances are defined in terms of an analytic continuation of the resolvent of Δ to a meromorphic function with values in the linear operators from $L^2_{comp}(X)$ to

$L^2_{loc}(X)$ (see [Mu4]). A resonance is then, by definition, a pole of the analytic continuation of the resolvent which does not correspond to an eigenvalue in the sense that the pole is either not an eigenvalue or its multiplicity (defined in a proper sense) is bigger than the dimension of the eigenspace. Let $R(\Delta)$ be the set of all poles of the analytic continuation of the resolvent. For each $\eta \in R(\Delta)$ one can define an algebraic multiplicity $m(\eta)$. In fact, there is a nonself–adjoint operator B – the generator of the Lax–Phillips semigroup – such that $R(\Delta)$ equals the set of generalized eigenvalues of B and $m(\eta)$ is then the algebraic multiplicity of this eigenvalue [Mu2].

We note that resonances can also be defined as poles of the scattering matrix which we regard as a function on the double covering of $\mathbb{C}$ defined by $\lambda = s(1-s)$.

Now the resonance set $R(\Delta)$ can be used as a discrete set of spectral parameters in the same way as the eigenvalues are used in the case of a compact Riemannian manifold. For example, there is an analogue of Weyl's formula [Mu2], [Pa], and also a trace formula [Mu2]. In particular, one can introduce the *resonance zeta function* which is defined by

$$\zeta_{Res}(s) = \sum_{\substack{\eta \in R(\Delta) \\ \eta \neq 1}} \frac{m(\eta)}{(1-\eta)^s}. \tag{4.30}$$

This series is absolutely convergent for $\operatorname{Re}(s) > 2$ and admits a meromorphic continuation to $\mathbb{C}$ which is holomorphic at $s=0$. Using this zeta function, we define a second regularized determinant by

$$\det{}_{Res}\Delta = \exp\left(-\frac{d}{ds}\zeta_{Res}(s)\Big|_{s=0}\right).$$

Then formally, one has

$$\det{}_{Res}\Delta = \prod_{\substack{\eta \in R(\Delta) \\ \eta \neq 1}} |1-\eta|^{2m(\eta)}.$$

Furthermore, the two determinants are closely related. Namely, we have

$$\det \Delta = \exp\left(\frac{\operatorname{Area}(X)}{8\pi} - \frac{3\pi\gamma}{2} m\right) \det{}_{Res}\Delta$$

where γ is Euler's constant [Mu2]. More generally, for $\mathrm{Re}(z) > 1$, we can define

$$\zeta_{Res}(s; z) = \sum_{\substack{\eta \in R(\Delta) \\ \eta \neq 1}} \frac{m(\eta)}{(z-\eta)^s}, \ \mathrm{Re}(s) > 2. \tag{4.31}$$

Then, as a function of s, $\zeta_{Res}(s; z)$ has a meromorphic continuation to $\mathbb{C}$ and $s = 0$ is not a pole. Put

$$\det{}_{Res}(\Delta + z - 1) = \exp\left(-\frac{\partial}{\partial s}\zeta_{Res}(s; z)\Big|_{s=0}\right).$$

For a hyperbolic surface $\Gamma \backslash H$ of finite area, $\det_{Res}(\Delta + z - 1)$ can be expressed in terms of the Selberg zeta function $Z_\Gamma(s)$ and the determinant of the scattering matrix $S(\lambda)$. First recall that

$$Z_\Gamma(s) = \prod_\gamma \prod_{k=0}^\infty \left(1 - e^{-(s+k)\ell(\gamma)}\right), \ \mathrm{Re}(s) > 1,$$

where γ runs through the primitive closed geodesics and $\ell(\gamma)$ is the length of γ. Furthermore, we write the spectral parameter λ as usually as $\lambda = z(1-z)$, $z \in \mathbb{C}$, and regard the scattering matrix as a function of z. Then

$$\begin{aligned} &\det{}^2_{Res}(\Delta + z - 1) \\ &\qquad = \det S(z) Z_\Gamma(z)^2 Z_\infty(z)^2 \Gamma(z + 1/2)^{-2m} e^{c(2z-1)+d}, \end{aligned} \tag{4.32}$$

where c and d are certain constants that depend on $\mathrm{Area}(\Gamma \backslash H)$ and m, and

$$Z_\infty(s) = \left((2\pi)^s \Gamma_2(s)^2/\Gamma(s)\right)^{\mathrm{Area}(\Gamma\backslash H)/2\pi}$$

with $\Gamma_2(s)$ being the double Gamma function (see [Mu2]). It follows from (4.32) that $\det_{Res}(\Delta + z - 1)$ extends to an analytic function on $\mathbb{C}$ with the set of zeros equal to $R(\Delta)$. We claim that this holds in general.

If $\Gamma = SL(2, \mathbb{Z})$, then the scattering matrix is a function $S(z)$ which is given in terms of the Riemannian zeta function $\zeta(s)$ by

$$S(z) = \sqrt{\pi}\frac{\Gamma(z - 1/2)}{\Gamma(z)} \frac{\zeta(2z-1)}{\zeta(2z)}.$$

[He]. Thus in this case, the resonances are precisely the numbers $\rho/2$ where ρ runs over the nontrivial zeros of the Riemann zeta function. Furthermore, $\det_{Res}(\Delta + z - 1)$ can be factorized in the product of two determinants where the first one is defined in terms of the eigenvalues similar to (0.3) and the second one is defined in terms of the nontrivial zeros ρ of $\zeta(s)$. We describe the second determinant. For $\mathrm{Re}(z) > 1$ consider the Dirichlet series

$$\zeta(s,z) = (2\pi)^s \sum_\rho (z-\rho)^{-s} \tag{4.33}$$

where $\arg(z-\rho) \in (-\pi/2, \pi/2)$. It was proved in [De] that (4.33) converges absolutely for $\mathrm{Re}(s) > 1$ and for fixed z, it has an analytic continuation to a holomorphic function of $s \in \mathbb{C}\backslash\{1\}$. Moreover, one has

$$\begin{aligned} 2^{-1/2}(2\pi)^{-2}\pi^{-z/2}\Gamma(z/2)\zeta(z)z(z-1) \\ = \exp\left(-\frac{\partial}{\partial s}\xi(s,z)\Big|_{s=0}\right). \end{aligned} \tag{4.34}$$

The right hand side is then the determinant associated with the resonances. Note that (4.33) resembles (4.31). This analogy becomes even closer if we recall Colin de Verdiere's result that for a generic metric, the number of eigenvalues is finite [CV2].

Remark. In conclusion, one can say that for a surface with hyperbolic ends, the resonances can be used as a substitute for the eigenvalues of the Laplacian of a compact surface. It would be very interesting to see if in other cases, the resonances play a similar role.

Everything that we described here for surfaces can be extended to the case of manifolds with ends of hyperbolic type as studied in [Mu1]. Furthermore, the Laplacian on forms can be treated in the same way. In particular, we can introduce the L^2 analytic torsion for hyperbolic manifolds of finite volume.

Another interesting problem is the investigation of the finer structure of the distribution of resonances in the case of surfaces with hyperbolic ends. This should be seen in connection with quantum chaos [Sa].

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