# The spectral side of Arthur's trace formula

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**The trace formula is one of the most important tools in the theory of automorphic forms. It was invented in the 1950's by Selberg, who mostly studied the case of hyperbolic surfaces, and was later on developed extensively by Arthur in the generality of an adelic quotient of a reductive group over a number field. Here we provide an explicit expression for the spectral side, improving Arthur's fine spectral expansion. As a result, we obtain its absolute convergence for a wide class of test functions.**

Let $G$ be a reductive group defined over a number field $F$. A fundamental question in the theory of automorphic forms is the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$ as a representation of $G(\mathbb{A})$. Langlands essentially reduced this question to the discrete part (1). There are deep conjectures by Arthur about the structure of the discrete spectrum (2). However, by and large, these conjectures are wide open.

The trace formula was invented by Selberg in the 1950's, mostly in the context of hyperbolic surfaces of finite volume (3). Among other things, he used it to show that if $\Gamma$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ acting discontinuously on the hyperbolic plane $\mathbb{H}$, then the discrete spectrum of the Laplace operator $\Delta$ on $\Gamma\backslash\mathbb{H}$ obeys the Weyl law. More precisely, if $N_T(\Gamma)$ is the counting function for the number of linearly independent solutions of $(\Delta+\lambda)f=0$ with $\lambda < \frac{1}{4}+T^2$, then

$$N_T(\Gamma) = \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{4\pi}T^2 + O(T\log T).$$

It was realized by Eichler and, in a much broader scope, by Langlands, that comparing trace formulas on two different groups may yield important consequences to Langlands' functoriality principle, which roughly speaking relates the spectra of two locally symmetric spaces. A prerequisite for that is to develop the trace formula in a sufficiently explicit form in the context of adelic quotients $G(F)\backslash G(\mathbb{A})$. This task was carried out by Arthur in a long series of works.

The main source of difficulty in deriving the trace formula identity is the noncompactness of $G(F)\backslash G(\mathbb{A})$ when $G$ is not anisotropic. This results in boundary terms, both on the geometric side and the spectral side. Their analysis and combinatorics is a major theme in Arthur's work. Our purpose here is to give a refinement for the spectral expansion given by Arthur. In particular, we show that it is absolutely convergent in a strong sense for a wide class of test functions.

## Main Result

Let $G$ be a reductive group defined over a number field, which for simplicity of notation we assume to be $\mathbb{Q}$. Implicitly, all algebraic subgroups of $G$ we refer to are assumed to be defined over $\mathbb{Q}$. Fix a maximal $\mathbb{Q}$-split torus $T_0$ in $G$ and a suitable maximal compact subgroup $\mathbf{K}$ of $G(\mathbb{A})$. Let $\mathfrak{F}$ denote the finite set of parabolic subgroups of $G$ containing $T_0$ and let $\mathcal{L}$ denote the set of their Levi parts containing $T_0$. For $M \in \mathcal{L}$, we will use the following notation and conventions:

- $T_M$ – the split part of the center of $M$, and $A_M = T_M(\mathbb{R})^0$;
- $\mathfrak{a}_M^G$ – the $\mathbb{R}$-vector space spanned by the lattice of cocharacters of $T_M \cap G^{\mathrm{der}}$, $(\mathfrak{a}_M^G)^*$ – the dual space, both endowed with mutually dual Haar measures, $r = \dim \mathfrak{a}_M^G$;
- $W(M)$ – the finite group $N_G(M)/M$, and for any $s \in W(M)$, $M_s$ – the smallest subgroup in $\mathcal{L}$ containing $M$ and $s$, i.e., $\mathfrak{a}_{M_s}^G$ is the $+1$-eigenspace of $s$ acting on $\mathfrak{a}_M^G$;
- $\mathcal{P}(M)$ – the set of parabolic subgroups of $G$ with Levi $M$;
- for any $P \in \mathcal{P}(M)$, $\Delta_P^G$ (resp., $\Sigma_P^G$) – the set of simple (resp., reduced) coroots of $P$, and $\overline{P}$ – the opposite parabolic; we regard $\Delta_P^G$ as an $r$-tuple with a prescribed ordering;
- $P \xrightarrow{\alpha} Q$ (i.e., $P, Q \in \mathcal{P}(M)$ are adjacent along $\alpha^\vee \in \Delta_P^G$) iff $\Sigma_P^G \cap \Sigma_{\overline{Q}}^G = \{\alpha^\vee\}$, (in which case $P$ and $Q$ are maximal in $PQ$ – the parabolic subgroup generated by $P$ and $Q$);
- we fix $\mu_1,\dots,\mu_r \in (\mathfrak{a}_M^G)^*$ in general position; if $P \in \mathcal{P}(M)$ and $\alpha_1^\vee,\dots,\alpha_r^\vee \in \Sigma_P^G$ are linearly independent, let $\varpi_1,\dots,\varpi_r$ be their dual basis in $(\mathfrak{a}_M^G)^*$. This determines $P_i \xrightarrow{\alpha_i} Q_i$, $i=1,\dots,r$ by the condition that
$$\sum_{j\neq i}\left\langle \mu_j, \alpha_j^\vee\right\rangle \varpi_j \text{ lies in the Weyl chamber of } P_iQ_i. \qquad [1]$$
- For any $P \in \mathcal{P}(M)$, let
$$\mathcal{A}_P^2 = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} L^2_{\mathrm{disc}}(A_M M(\mathbb{Q})\backslash M(\mathbb{A})).$$

On this space, there is a family of induced representations $I_P(\lambda)$, $\lambda \in \mathrm{i}(\mathfrak{a}_M^G)^*$ and conjugation operators

$$\mathfrak{c}_s : \mathcal{A}_P^2 \to \mathcal{A}_{sP}^2 \quad s \in W(M).$$

The theory of Eisenstein series gives rise to intertwining maps from $I_P(\lambda)$ to the space of automorphic forms on $G(F)\backslash G(\mathbb{A})$, which furnish the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$. Additionally, it provides a family of unitary intertwining operators

$$M_{Q|P}(\lambda) : \mathcal{A}_P^2 \to \mathcal{A}_Q^2 \quad P, Q \in \mathcal{P}(M), \lambda \in \mathrm{i}(\mathfrak{a}_M^G)^*$$

satisfying $I_Q(\lambda) \circ M_{Q|P}(\lambda) = M_{Q|P}(\lambda) \circ I_P(\lambda)$ and the functional equations

$$M_{P_3|P_2}(\lambda) \circ M_{P_2|P_1}(\lambda) = M_{P_3|P_1}(\lambda) \quad \lambda \in \mathrm{i}(\mathfrak{a}_M^G)^*$$

for any triplet $P_1, P_2, P_3 \in \mathcal{P}(M)$. If $P \xrightarrow{\alpha} Q$ then $M_{Q|P}(\lambda) = M_{Q|P}(\langle\lambda, \alpha^\vee\rangle)$ is a function of a single variable.

Fix an open compact subgroup $K \subseteq G(\mathbb{A}_{\mathrm{fin}})$. The coset space $G(\mathbb{A})^1/K$ is a countable disjoint union of copies of $G(\mathbb{R})^1$, and in particular, a manifold. Denote by $\mathcal{F}(G(\mathbb{A})^1;K)$ the Fréchet space of smooth functions on $G(\mathbb{A})^1/K$ (i.e., smooth $K$-invariant functions on $G(\mathbb{A})^1$) for which the norms

$$\|f * X\|_{L^1(G(\mathbb{A})^1/K)} \quad X \in \mathcal{U}(\mathfrak{g}^1)$$

are finite.

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**Theorem 1.** The spectral side of Arthur's trace formula is given by the sum-integral over

- a set of representatives $P \in \mathfrak{F}$ of associated classes of parabolic subgroups, (let $M \in \mathcal{L}$ be the Levi part);
- $s \in W(M)$, (set $k = \dim \mathfrak{a}_{M_s}^G$);
- $\lambda \in \mathrm{i}(\mathfrak{a}_{M_s}^G)^*$;
- $k$-tuples of coroots $\alpha_1^\vee, \dots, \alpha_k^\vee \in \Sigma_P^G$ whose projections to $\mathfrak{a}_{M_s}^G$ generate a lattice (denoted by $L$),

of

$$\mathrm{vol}(\mathfrak{a}_{M_s}^G/L)\frac{(-1)^k}{k!}|\det(s-1|\mathfrak{a}_M^{M_s})|^{-1}\times$$
$$\mathrm{tr}(M_{P|Q_k}(\lambda)M'_{Q_k|P_k}(\langle\lambda,\alpha_k^\vee\rangle)M_{P_k|Q_{k-1}}(\lambda)\dots M_{P_2|Q_1}(\lambda)$$
$$M'_{Q_1|P_1}(\langle\lambda,\alpha_1^\vee\rangle)M_{P_1|P}(\lambda)M_{P|sP}(\lambda)\mathfrak{c}_s I_P(f,\lambda))_{\mathcal{A}_P^2}$$

where the $P_i$ and $Q_i$s are determined by Eq. **1** and the choice of $\mu_1, \dots, \mu_r$, with $(\alpha_{k+1}^\vee, \dots, \alpha_r^\vee) = \Delta_{P\cap M_s}^{M_s}$. The sum-integral is absolutely convergent with respect to the trace norm for any $f \in \mathcal{F}(G(\mathbb{A})^1; K)$.

**Remarks**

1. The theorem explicates Arthur's fine spectral expansion (4) which was previously only known to be conditionally convergent, even for bi-$K$-finite compactly supported $f$. The key feature is that only first order derivatives of rank one intertwining operators appear.
2. The point of departure for Arthur's spectral expansion is the Maass–Selberg relations for the inner product of truncated Eisenstein series. The formula, proved by Langlands and Arthur (1, 5, 6), involves the operator limit
$$\lim_{\lambda\to 0}\sum_{Q\in\mathcal{P}(M)}\frac{M_{P|Q}(\mu)M_{Q|P}(\mu+\lambda)}{\prod_{\alpha^\vee\in\Delta_Q^G}\langle\lambda,\alpha^\vee\rangle}.$$
The main algebraic step in deriving Theorem 1 from Arthur's spectral expansion is to explicate this limit.
3. The most important part of the spectral side is the discrete part which is the sum of the invariant contributions
$$\frac{\mathrm{tr}(M_{P|sP}(0)\mathfrak{c}_s I_P(f,0))_{\mathcal{A}_P^2}}{|W(M)||\det(s-1|\mathfrak{a}_M^G)|}$$
of the pairs $(M, s)$ such that $M_s = G$. The term $M = G$, $s = 1$ is the trace of $f$ on the discrete spectrum of $G$. The fact that it is of trace class was proved in ref. 7 for the bi-$K$-finite case and in refs. 8 and 9 in general.
4. The role of noncompactly supported functions in the trace formula was highlighted in Langlands' idea of beyond endoscopy (10) [see also Venkatesh's thesis (11) on limiting forms of the trace formula].
5. On the geometric side, so far we know that the elliptic contribution converges absolutely for $f \in \mathcal{F}(G(\mathbb{A})^1; K)$.

**Example.** $G = \mathrm{GL}_3$, $M = T_0$ diagonal torus, $s = 1$. This is the most continuous (and noninvariant) part of the trace formula. Here $\mathcal{P}(T_0) = \{P_0, \dots, P_5\}$, $\Delta_{P_0}^G = (\alpha_1^\vee, \alpha_2^\vee)$, $\Sigma_{P_0}^G = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$ where $\alpha_3^\vee = \alpha_1^\vee + \alpha_2^\vee$, and

$$P_0 \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_3} P_2 \xrightarrow{\alpha_2} P_3 \xleftarrow{\alpha_1} P_4 \xleftarrow{\alpha_3} P_5 \xleftarrow{\alpha_2} P_0.$$

Writing $M_i = M_{P_{i+1}|P_i}$, $i = 0, 1, 2$ and $M_i = M_{P_i|P_{i+1}}$, $i = 3, 4, 5$ the $M_i$'s satisfy the commutativity constraint

$$M_{P_3|P_0}(\lambda) = M_2(\lambda_2)M_1(\lambda_3)M_0(\lambda_1) = M_3(\lambda_1)M_4(\lambda_3)M_5(\lambda_2)$$

for $\lambda \in \mathrm{i}(\mathfrak{a}_M^G)^*$, $\lambda_i = \langle\lambda, \alpha_i^\vee\rangle$, $i = 1, 2, 3$. The contribution is

$$\frac{1}{2}\int_{\lambda\in\mathrm{i}(\mathfrak{a}_M^G)^*}\mathrm{tr}[I_{P_0}(f,\lambda)M_{P_3|P_0}(\lambda)^{-1}$$
$$(M'_2(\lambda_2)M_1(\lambda_3)M'_0(\lambda_1) + M'_3(\lambda_1)M_4(\lambda_3)M'_5(\lambda_2)$$
$$+M_2(\lambda_2)M'_1(\lambda_3)M'_0(\lambda_1) + M'_3(\lambda_1)M'_4(\lambda_3)M_5(\lambda_2)$$
$$+M'_2(\lambda_2)M'_1(\lambda_3)M_0(\lambda_1) + M_3(\lambda_1)M'_4(\lambda_3)M'_5(\lambda_2))]\,d\lambda.$$

Here the choice of $\mu_1, \mu_2 \in (\mathfrak{a}_M^G)^*$ is immaterial. In general, for $G = \mathrm{GL}_n$, $M = T_0$, there are $n!n^{n-3}$ summands in the integrand.

## Combinatorial Setup

**Volumes of Polytopes.** Let $\mathfrak{P}$ be a polytope (i.e., the convex hull of finitely many points) in $V = \mathbb{R}^d$. Denote by $\mathfrak{P}^0$ (resp., $\mathfrak{P}^1$) the set of vertices (resp., edges) of $\mathfrak{P}$. We distinguish between an edge $e$ (often denoted $v_1 \xrightarrow{e} v_2$) and its vector $\vec{e} = v_2 - v_1$. (Several edges may have the same vector, i.e., they are translates of each other.)

There are several ways to compute the volume of $\mathfrak{P}$. We mention two of them. The first is a localization formula which was proven by Arthur for certain polytopes and was generalized by Brion and others (12, 13).

Assume that $\mathfrak{P}$ is simple, that is, every vertex has $d$ neighbors. Then the Fourier transform of the characteristic function of $\mathfrak{P}$ is given by

$$\int_{\mathfrak{P}} e^{\langle\lambda,u\rangle}\,du = (-1)^d\sum_{v\in\mathfrak{P}^0} v_{\vec{e}_1,\dots,\vec{e}_d}\frac{e^{\langle\lambda,v\rangle}}{\prod_{i=1}^d\langle\lambda,\vec{e}_i\rangle}\lambda \in \mathrm{i}V^* \qquad [2]$$

where $v \xrightarrow{e_i}$ are the edges emerging from $v$ and

$$v_{\vec{e}_1,\dots,\vec{e}_d} = \mathrm{vol}(V/\mathbb{Z}\vec{e}_1 + \dots + \mathbb{Z}\vec{e}_d)$$

is the volume of the parallelotope formed by $\vec{e}_1, \dots, \vec{e}_d$. In particular, $\mathrm{vol}(\mathfrak{P})$ is the limit of Eq. **2** as $\lambda \to 0$.

The second formula, following an argument of McMullen-Schneider (14), expresses the mixed volume of $d$ polytopes $\mathfrak{P}_1, \dots, \mathfrak{P}_d$ in $V$ as the sum of volumes of parallelotopes in a noncanonical way. (see Fig. 1.) Let $\underline{\mu} = (\mu_1, \dots, \mu_d) \in (V^*)^d$ be in general position, and let $\mathcal{X}_{\underline{\mu}}(\mathfrak{P}_1, \dots, \mathfrak{P}_d) \subseteq \mathfrak{P}_1^1 \times \dots \times \mathfrak{P}_d^1$ consist of the $d$-tuples $(e_1, \dots, e_d)$, for which there exists $\lambda \in V^*$ such that the maximum of $\lambda - \mu_i$ on $\mathfrak{P}_i$ is attained on the edge $e_i$. This $\lambda$, if it exists, is the solution of the system of linear equations

$$\langle\lambda - \mu_i, \vec{e}_i\rangle = 0, \quad i = 1, \dots, d.$$

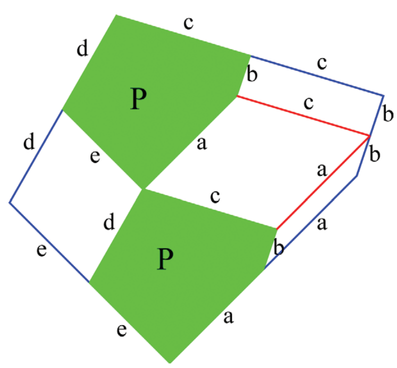


**Fig. 1.** Tessellating $2P$ by two copies of $P$ and four parallelograms formed by a (nonunique) set of pairs of sides of $P$. The area of $P$ is one half of the sum of the parallelograms areas.

The mixed volume is the sum over $(e_1,\ldots,e_d)\in\mathcal{X}_{\underline{\mu}}$ of $v_{\vec{e_1},\ldots,\vec{e_d}}$. In particular, for any polytope $\mathfrak{P}$ we obtain

$$\mathrm{vol}(\mathfrak{P})=\frac{1}{d!}\sum_{(e_1,\ldots,e_d)\in\mathcal{X}^{\mathfrak{P}}_{\underline{\mu}}} v_{\vec{e_1},\ldots,\vec{e_d}} \qquad [3]$$

where we write $\mathcal{X}^{\mathfrak{P}}_{\underline{\mu}}$ for $\mathcal{X}_{\underline{\mu}}(\mathfrak{P},\ldots,\mathfrak{P})$.

The expression in Eq. **3** is particularly simple in the case where $\mathcal{Z}$ is a zonotope – that is, a translate of the Minkowski sum of line segments $\sum_{i=1}^{n}[0,1]v_i$. In that case, the formula reduces to

$$\mathrm{vol}(\mathcal{Z})=\sum_{1\le i_1<\cdots<i_d\le n} v_{v_{i_1},\ldots,v_{i_d}}. \qquad [4]$$

In fact, one can tessellate $\mathcal{Z}$ by $\binom{n}{d}$ parallelotopes with sides $v_{i_1},\ldots,v_{i_d}$ (15). We write $\mathfrak{T}(\mathcal{Z})$ for the set of linearly independent $d$-tuples of $v_i$s.

More generally, one considers a deformed zonotope $\mathfrak{P}$ (subordinate to $\mathcal{Z}$), characterized by the property that its normal fan coincides with that of $\mathcal{Z}$ (and hence, is defined by the hyperplane arrangement in $V^*$ dual to the lines spanned by the $v_i$s). In particular, the face lattice of $\mathfrak{P}$ coincides with that of $\mathcal{Z}$. The choice of $\underline{\mu}$ gives rise to a bijection $\ell_{\underline{\mu}}:\mathfrak{T}(\mathcal{Z})\to\mathcal{X}^{\mathfrak{P}}_{\underline{\mu}}$ with the property that $\ell_{\underline{\mu}}(v_{i_1},\ldots,v_{i_d})=(e_1,\ldots,e_d)$ with $\vec{e_j}$ proportional to $v_{i_j}$, $j=1,\ldots,d$. We have (see Fig. 2)

$$\mathrm{vol}(\mathfrak{P})=\frac{1}{d!}\sum_{\tau\in\mathfrak{T}(\mathcal{Z})} v_{\ell_{\underline{\mu}}(\tau)}.$$

**Root Zonotope.** A particularly interesting zonotope is associated to a root system $\Phi$. It is given by

$$\mathcal{Z}_\Phi=\frac{1}{2}\sum_{\alpha\in\Phi_+}[-\alpha^\vee,\alpha^\vee]$$

and is called the root zonotope. Alternatively, $\mathcal{Z}_\Phi$ is the convex hull of the orbit under the Weyl group of $\rho^\vee=\frac{1}{2}\sum_{\alpha>0}\alpha^\vee$. If $\Phi$ is defined by a reductive group $G$ over $\mathbb{Q}$, then we write $\mathcal{Z}^G=\mathcal{Z}_\Phi$. The faces of $\mathcal{Z}^G$ correspond to $\mathfrak{F}$, and the dimension of a face is the semisimple rank of the corresponding parabolic subgroup. Moreover, the Levi parts of two parabolic subgroups coincide if and only if the corresponding faces are parallel. In particular, the vertices of $\mathcal{Z}^G$ correspond to the minimal parabolic subgroups containing $T_0$ and the edges to pairs of adjacent parabolic subgroups.

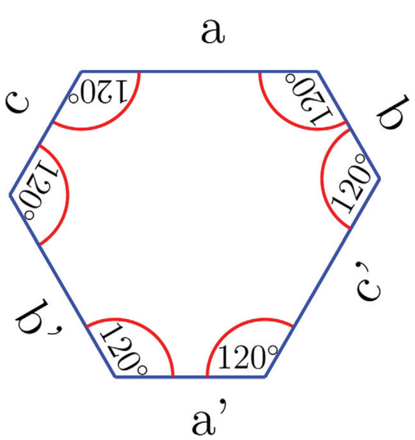


**Fig. 2.** A deformed regular hexagon whose sides lengths are constrained by the equations $a+b=a'+b'$, $a+c=a'+c'$. Its area is $ab+ac+bc+a'b'+a'c'+b'c'=ab+ac'+bc'+a'b'+a'c+b'c=ab'+ac+b'c+a'b+a'c'+bc'$ where an equilateral triangle with unit side is normalized to have unit area.

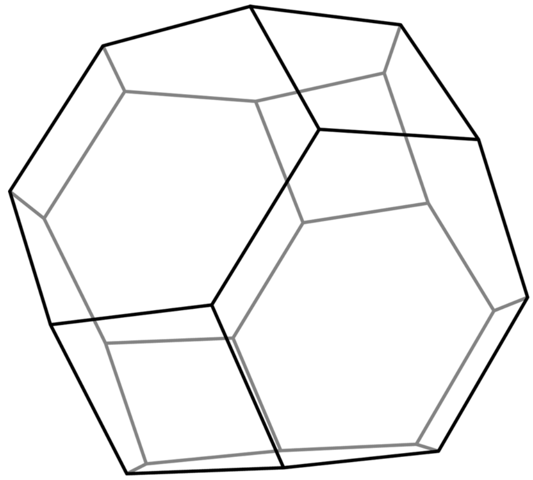

**Fig. 3.** The permutohedron for $n=4$ (courtesy of Bill Casselman). The faces correspond to semistandard parabolic subgroups of GL(4) (the maximal ones correspond to the eight hexagons and six squares and the rank one parabolic subgroups to the 36 edges).

An element of $\mathfrak{T}(\mathcal{Z}^G)$ is simply a (linear algebra theoretic) basis of coroots. Denoting by $r$ the $\mathbb{Q}$-semisimple rank of $G$ we get by Eq. **4**

$$\mathrm{vol}(\mathcal{Z}^G)=\frac{1}{r!}\sum_{(\alpha_1^\vee,\ldots,\alpha_r^\vee)\in\mathfrak{T}(\mathcal{Z}^G)} v_{\alpha_1^\vee,\ldots,\alpha_r^\vee}.$$

More generally, given a Levi subgroup $M$, we denote by $\mathcal{Z}^G_M\subseteq\mathfrak{a}^G_M$ the projected root zonotope

$$\mathcal{Z}^G_M=\frac{1}{2}\sum_{\alpha^\vee\in\Sigma^G_P}[-\alpha^\vee,\alpha^\vee]$$

[for any $P\in\mathcal{P}(M)$]. It is obtained from $\mathcal{Z}^G$ by projection onto $\mathfrak{a}^G_M$. These zonotopes are simple. We identify the set of vertices of $\mathcal{Z}^G_M$ with $\mathcal{P}(M)$. The combinatorics of $\mathcal{Z}^G_M$ play a ubiquitous role in the trace formula as well as in other contexts. For instance, the volume of deformed zonotopes subordinate to $\mathcal{Z}^G_M$ define the weight factors in the weighted orbital integrals occurring in the geometric side of the trace formula.

**Example.** Let $G=\mathrm{GL}_n$ and $T_0$ the diagonal torus. The zonotope $\mathcal{Z}^G$ is the $n-1$ dimensional permutohedron $\Pi_n$. It is given by the convex hull in $\mathbb{R}^n$ of $(\sigma(1),\ldots,\sigma(n))$ where $\sigma$ ranges over the permutations of $\{1,\ldots,n\}$. (see Fig. 3.) The faces of $\Pi_n$ correspond to ordered partitions of $\{1,\ldots,n\}$, two faces being parallel if and only if the underlying unordered partitions are the same. The corresponding parabolic and Levi subgroups are given by generalized block upper triangular and block diagonal matrices respectively. The set $\mathfrak{T}(\mathcal{Z}^G)$ correspond to trees on $\{1,\ldots,n\}$ with an ordering of the edges. They all span the coroot lattice. Thus, by Cayley's formula

$$\mathrm{vol}(\mathcal{Z}^G)=\#\text{ of trees}=n^{n-2}.$$

**Intertwining Families.** Let $\mathbb{C}[[V]]$ be the algebra of formal power series in $V$, i.e., the completion of the symmetric algebra $\mathrm{Sym}(V)$ of $V$.

By definition, an intertwining family with respect to a polytope $\mathfrak{P}$ consists of the following data:

1. $\forall v\in\mathfrak{P}^0$ a finite-dimensional vector space $W_v$; and
2. $\forall v_1,v_2\in\mathfrak{P}^0, A_{v_2|v_1}\in\mathbb{C}[[V]]\otimes\mathrm{Hom}(W_{v_1},W_{v_2})$,

satisfying the following conditions:

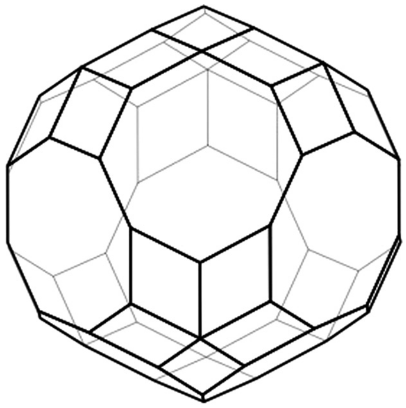

**Fig. 4.** A variant of the monotone path polytope for the root zonotope of type $A_4$. It has 62 vertices, 100 edges (10 directions, 10 in each direction) and 40 faces (10 octagons and 30 squares). It plays a role in the proof of Theorem 2. Reprinted from Discrete Math. 241, **S. Felnser** and **G. Ziegler**, *Zonotopes associated with higher Bruhat orders*, 301–312, ©2001, with permission from Elsevier.

1. $\forall v \in \mathfrak{P}^0, A_{v|v} = 1 \otimes \mathrm{Id}_{W_v}$;
2. $\forall v_1, v_2, v_3 \in \mathfrak{P}^0, A_{v_3|v_2} \circ A_{v_2|v_1} = A_{v_3|v_1}$; and
3. $\forall v_1 \xrightarrow{e} v_2, A_{v_2|v_1} \in \mathbb{C}[[\vec{e}]] \otimes \mathrm{Hom}(W_{v_1}, W_{v_2})$.

A basic example for an intertwining family is given by

$$W_v = \mathbb{C}\ \forall v \in \mathfrak{P}^0 \text{ and } A_{v_2|v_1} = e^{v_2 - v_1} \forall v_1, v_2 \in \mathfrak{P}^0. \qquad \textbf{[5]}$$

In the case where $\mathfrak{P} = \mathcal{Z}_M^G$, another example of intertwining families, which motivates our definition, is the Taylor expansion of the intertwining operators (restricted to a fixed $\mathbf{K}$ and $\mathfrak{z}$-type) around any point in $\mathrm{i}(\mathfrak{a}_M^G)^*$.

Suppose from now on that $\mathfrak{P}$ is simple. For any $v \in \mathfrak{P}^0$ set

$$\theta_v = v_{\vec{e}_1,\ldots,\vec{e}_d}^{-1} \prod_{i=1}^{d} \vec{e}_i \in \mathrm{Sym}(V),$$

where $v \xrightarrow{e_i}$ are the edges emerging from $v$. Fix a base point $v_0 \in \mathfrak{P}^0$. Then for an intertwining family we have

$$(-1)^d \sum_{v \in \mathfrak{P}^0} \frac{A_{v_0|v}(0) A_{v|v_0}}{\theta_v} \in \mathbb{C}[[V]] \otimes \mathrm{End}(W_{v_0}).$$

The point is that the singularities cancel in pairs. We denote the value at 0 by $\mathcal{M}^{v_0}(A) \in \mathrm{End}(W_{v_0})$.

Clearly, $\mathcal{M}^{v_0'}(A) = A_{v_0'|v_0}(0) \mathcal{M}^{v_0}(A) A_{v_0|v_0'}(0)$ for any $v_0' \in \mathfrak{P}^0$.

For any edge $v_1 \xrightarrow{e} v_2$ we introduce the logarithmic derivative with respect to $v_1 \xrightarrow{e} v_2$

$$\Delta_e A := \frac{A_{v_1|v_2}(0) A_{v_2|v_1} - \mathrm{Id}_{W_{v_1}}}{\vec{e}} \in \mathbb{C}[[\vec{e}]] \otimes \mathrm{End}(W_{v_1})$$
$$\mathcal{D}_e(A) = \Delta_e A(0).$$

For a $d$-tuple $\tau$ of edges $v_i \xrightarrow{e_i} v_i'$, define

$$\mathcal{D}_\tau^{v_0}(A) = A_{v_0|v_d}(0) \mathcal{D}_{e_d}(A) A_{v_d|v_{d-1}}(0) \mathcal{D}_{e_{d-1}}(A)$$
$$\ldots \mathcal{D}_{e_2}(A) A_{v_2|v_1}(0) \mathcal{D}_{e_1}(A) A_{v_1|v_0}(0) \in \mathrm{End}(W_{v_0}).$$

Clearly $\mathcal{D}_\tau^{v_0'}(A) = A_{v_0'|v_0}(0) \mathcal{D}_\tau^{v_0}(A) A_{v_0|v_0'}(0)$ for any $v_0' \in \mathfrak{P}^0$.

**Conjecture 1.** For any intertwining family with respect to a simple polytope $\mathfrak{P}$ and $\underline{\mu} \in (V^*)^d$ in general position we have

$$\mathcal{M}^{v_0}(A) = \frac{1}{d!} \sum_{\tau \in \mathcal{X}_{\underline{\mu}}^{\mathfrak{P}}} \nu_\tau \mathcal{D}_\tau^{v_0}(A).$$

For the basic example in Eq. **5**, the conjecture reduces to the formula in Eq. **3** in light of Eq. **2**. In general, the conjecture can be viewed as a noncommutative generalization of Eq. **3**.†

The main input for the proof of Theorem 1, besides Arthur's spectral expansion, is

**Theorem 2.** Conjecture 1 holds for projected root zonotopes.

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