

SPECTRAL ASYMPTOTICS FOR ARITHMETIC QUOTIENTS OF $SL(n, \mathbb{R})/SO(n)$

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ABSTRACT. In this paper we study the asymptotic distribution of the cuspidal spectrum of arithmetic quotients of the symmetric space $SL(n, \mathbb{R})/SO(n)$. In particular, we obtain Weyl's law with an estimation on the remainder term. This extends some of the main results of Duistermaat-Kolk-Varadarajan ([DKV79]) to this setting.

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1. INTRODUCTION

Let G be a reductive algebraic group over \mathbb{Q} , \mathbb{A} the ring of adèles of \mathbb{Q} , and ω a unitary character of $Z(\mathbb{Q})\backslash Z(\mathbb{A})$ where Z is the center of G . One of the fundamental problems in the theory of automorphic forms is to determine the spectral decomposition of the regular representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$. Langlands' theory essentially reduces the problem to the discrete part. There are deep conjectures of Arthur ([AG91]) which are aiming at the description of the discrete spectrum. However, these conjectures are out of reach at present.

One of the basic tools to study these problems is Arthur's trace formula. The main driving force behind Arthur's approach is the functoriality conjectures of Langlands. Consequently,

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the ability to compare the trace formula on two different groups is a key issue in Arthur's setup.

On the other hand, it is natural to ask what spectral information on the group itself can be inferred from the trace formula in higher rank. For example, it is well-known that the dimension of the space of automorphic forms with certain square-integrable Archimedean components can be computed using the trace formula ([Lan63]). More generally, there is an exact formula for traces of Hecke operators on automorphic forms whose Archimedean component lie in a discrete series L -packet with a non-singular Harish-Chandra parameter ([Art89]). (Cf. [GKM97] for a geometric counterpart.)

In the realm of spectral theory, a basic problem is to study the asymptotic distribution of the infinitesimal characters of the Archimedean components of cusp forms with a fixed K_∞ -type. In the simplest case one counts the Casimir eigenvalues of the Archimedean components of cusp forms. In fact, this was Selberg's motivation for developing the trace formula, with which he established the analogue of Weyl's law for the cuspidal spectrum of the quotient of the hyperbolic plane by a congruence subgroup [Sel56], [Sel89, p. 626–674]. This was extended to other rank one locally symmetric spaces by Reznikov [Rez93]. In higher rank, Duistermaat-Kolk-Varadarajan proved quite general results about the asymptotic distribution of the spherical spectrum for compact locally symmetric spaces [DKV79] and in particular gave an upper bound on the complementary spectrum. (Weyl's law itself, with a sharp remainder term, was proved earlier for the Laplace operator of any compact Riemannian manifold by Avakumović [Ava56] and this was generalized to any elliptic pseudo-differential operator by Hörmander [Hör68].) The first example of a non-uniform lattice in higher rank was treated by S. Miller [Mil01] who proved that for $\mathrm{SL}(3, \mathbb{Z})$, the cuspidal spectrum satisfies Weyl's law and the tempered cuspidal spectrum has density 1. Analogues of the Weyl's law for an arbitrary K_∞ -type were obtained by the second-named author for arithmetic quotients of $\mathrm{GL}(n)$, $n \geq 2$ in [Mül07]. For the spherical spectrum, Lindenstrauss and Venkatesh ([LV07]) showed that Weyl's law holds in great generality, proving a conjecture of Sarnak.

Our main goal in this paper is to extend the results of [DKV79] (with a slightly weaker error term) to the case of arithmetic quotients of the symmetric space X of positive-definite quadratic forms in $n \geq 2$ variables up to homothety. In particular, we derive Weyl's law with a remainder term strengthening the result of [Mül07] (for the spherical case).

Now we describe our results in more detail. Let $G = \mathrm{GL}(n)$. Let A_G be the group of scalar matrices with a positive real scalar, W the Weyl group and $K_\infty = \mathrm{O}(n)$. Let $\Pi_{\mathrm{cus}}(G(\mathbb{A}))$ (resp. $\Pi_{\mathrm{dis}}(G(\mathbb{A}))$) denote the set of irreducible unitary representations of $G(\mathbb{A})$ occurring in the cuspidal (resp. discrete) spectrum of $L^2(A_G G(\mathbb{Q}) \backslash G(\mathbb{A}))$ (necessarily with multiplicity one, cf. [JS81b], [JS81a], [MW89]). Let $\mathfrak{a}^* = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sum \lambda_i = 0\}$. Given $\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A}))$, denote by $\lambda_{\pi_\infty} \in \mathfrak{a}_\mathbb{C}^*/W$ the infinitesimal character of the Archimedean component π_∞ of π and for any subgroup K of $G(\mathbb{A})$ let \mathcal{H}_π^K be the space of K -invariant vectors in the representation space of π . We refer to §3 for normalization of various Haar

measures and the Plancherel measure $\beta(\lambda)$ on \mathfrak{ia}^* pertaining to $A_G \backslash G(\mathbb{R})$. The adelic version of our main result is the following Theorem.

Theorem 1.1. *Let $K = K_\infty K_f$ where K_f is an open compact subgroup of $G(\mathbb{A}_f)$ contained in some principal congruence subgroup $K_f(N)$ of level $N \geq 3$, and let $\Omega \subseteq \mathfrak{ia}^*$ be a W -invariant bounded domain with piecewise C^2 boundary. Then*

$$\sum_{\substack{\pi \in \Pi_{\text{cus}}(G(\mathbb{A})) \\ \lambda_{\pi_\infty} \in t\Omega}} \dim(\mathcal{H}_\pi^K) = \frac{\text{vol}(A_G G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f)}{|W|} \int_{t\Omega} \beta(\lambda) d\lambda + O(t^{d-1}(\log t)^{\max(n,3)})$$

as $t \rightarrow \infty$ where $t\Omega = \{t\lambda : \lambda \in \Omega\}$ and $d = \dim X$.

On the other hand for the ball $B_t(0)$ of radius t in $\mathfrak{a}_\mathbb{C}^*$ centered at the origin we have

$$\sum_{\substack{\pi \in \Pi_{\text{dis}}(G(\mathbb{A})) \\ \lambda_{\pi_\infty} \in B_t(0) \backslash \mathfrak{ia}^*}} \dim(\mathcal{H}_\pi^K) = O(t^{d-2})$$

as $t \rightarrow \infty$.

Thus, the complementary cuspidal spectrum, which, according to the Archimedean aspect of the Ramanujan conjecture for $\text{GL}(n)$, is not expected to exist at all, is at least of lower order of magnitude compared to the tempered spectrum. Theorem 1.1 (with an appropriate power of $\log t$, or perhaps even without it) is expected to hold for any reductive group G over \mathbb{Q} (where cuspidal spectrum is not necessarily tempered in general). Although the method of proof in principle carries over to any G , there are some issues which at present we do not know how to deal with in general.

One can rephrase Theorem 1.1 in more classical terms. Note that $X = A_G \backslash G(\mathbb{R}) / K_\infty = \text{SL}(n, \mathbb{R}) / \text{SO}(n)$. Let $\Gamma(N) \subseteq \text{SL}(n, \mathbb{Z})$ be the principal congruence subgroup of level N . Let $\Lambda_{\text{cus}}(\Gamma(N)) \subseteq \mathfrak{a}_\mathbb{C}^* / W$ be the cuspidal spectrum of the algebra of invariant differential operators of $\text{SL}(n, \mathbb{R})$, acting in $L^2(\Gamma(N) \backslash X)$. Given $\lambda \in \Lambda_{\text{cus}}(\Gamma(N))$, denote by $m(\lambda)$ the dimension of the corresponding eigenspace.

Corollary 1.2. *For $N \geq 3$ and Ω as before we have*

$$(1.1) \quad \sum_{\lambda \in \Lambda_{\text{cus}}(\Gamma(N)), \lambda \in t\Omega} m(\lambda) = \frac{\text{vol}(\Gamma(N) \backslash X)}{|W|} \int_{t\Omega} \beta(\lambda) d\lambda + O(t^{d-1}(\log t)^{\max(n,3)})$$

and

$$(1.2) \quad \sum_{\substack{\lambda \in \Lambda_{\text{cus}}(\Gamma(N)) \\ \lambda \in B_t(0) \backslash \mathfrak{ia}^*}} m(\lambda) = O(t^{d-2})$$

as $t \rightarrow \infty$.

The symmetric space X is endowed with a Riemannian structure defined using the Killing form. Taking the corresponding volume form and Laplacian operator Δ and applying Corollary 1.2 to the unit ball in \mathfrak{ia}^* we get

Corollary 1.3. *Let*

$$N_{\text{cus}}^{\Gamma(N)}(t) = \#\{j: \lambda_j \leq t^2\}$$

be the counting function for the cuspidal eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ (counted with multiplicity) of Δ on $\Gamma(N)\backslash X$, $N \geq 3$. Then

$$N_{\text{cus}}^{\Gamma(N)}(t) = \frac{\text{vol}(\Gamma(N)\backslash X)}{(4\pi)^{d/2}\Gamma(d/2+1)}t^d + O(t^{d-1}(\log t)^{\max(n,3)})$$

as $t \rightarrow \infty$.

The condition $N \geq 3$ is imposed for technical reasons. It guarantees that the principal congruence subgroup $\Gamma(N)$ is *neat* in the sense of Borel ([Bor69]), and in particular acts freely on X . This simplifies the analysis by eliminating the contribution of the non-unipotent conjugacy classes.

Our proof uses the method of Duistermaat-Kolk-Varadarajan ([DKV79]) who proved the same result (without the log factor in the error term) for any *compact* locally symmetric space. This simplifies and strengthens the argument of [Mül07] where the heat kernel was used instead.

In a nutshell, one has to show that for an appropriate family of test functions the main contribution in the trace formula comes in the geometric side from the identity conjugacy class, and in the spectral side from the cuspidal spectrum. Of course, in our case we have to consider the trace formula in Arthur's (non-invariant) form.

The main new technical difficulty is the analysis of the contribution of the unipotent conjugacy classes ([Art85]). These distributions are weighted orbital integrals and they can be analyzed by the method of stationary phase, just like the ordinary orbital integrals (cf. [DKV83] for the semi-simple case). A mitigating factor is that all unipotent orbits in the case of $\text{GL}(n)$ are of Richardson type, which somewhat simplifies the structure of the integral expression for these distributions.

On the spectral side, we have to show that the contribution of the continuous spectrum is of a lower order of magnitude. (This point is non-trivial in general, because it is rarely expected to hold for non-arithmetic non-uniform lattices ([PS85], [Sar86], [PS92]).) This is done as in [Mül07] by controlling this contribution in terms of Rankin-Selberg L -functions and using the analytic properties of the latter.

In contrast, the method of [LV07] completely avoids the contribution of the continuous spectrum by choosing appropriate test functions. This is an ingenious extension of the so-called simple trace formula (cf. [DKV84]) where in its ordinary form the continuous spectrum is removed at the price of a positive proportion of the cuspidal spectrum ([LM04]). It would be interesting to see whether the ideas of [LV07] can be pushed further to give tight upper bounds on the size of the continuous spectrum without appealing to the theory of L -functions (which is not always available).

The contents of the paper are as follows. We first give an outline of the proof of the Weyl law with remainder for the case of SL_2 in §2 by combining Hörmander's method and the

Selberg trace formula. The proof is not required for the higher rank case, but is given as a precursor for the general case (which is technically more difficult), in which Hörmander's method is replaced by the method of Duistermaat-Kolk-Varadarajan and the Selberg trace formula is substituted by Arthur's trace formula. In §3 we recall some basic facts about harmonic analysis on X and the spherical unitary dual and introduce some notation. The main section is §4 where we explain the higher rank setup and reduce the problem to an estimation of the contributions to the trace formula of the non-trivial conjugacy classes on the one hand (to the geometric side) and the non-discrete part of the spectrum on the other hand (to the spectral side). These estimations are carried out in sections 5 and 6 respectively. In the former we use the stationary phase method in its crudest form - that is, a simple application of the divergence theorem. In the latter we modify the analysis of [Mül07] to our setup.

It is natural to generalize our results to give the asymptotic behavior of traces of Hecke operators on the space of Maass forms with a uniform error term. This will have applications, among other things, to the distribution of low-lying zeros of automorphic L -functions of $GL(n)$ (cf. [ILS00] for the case $n = 2$). We hope to pursue this in a forthcoming paper.

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2. THE $SL(2)$ CASE

In this section we describe the main idea of the proof for the special case of $\Gamma \backslash \mathbb{H}$. This has also been discussed in [Mül]. For the convenience of the reader we recall the main steps. The case of a hyperbolic surface was first treated by Selberg in [Sel89, p. 668] (cf. also [Hej76]). Our method is a combination of Hörmander's method [Hör68] and the Selberg trace formula.

Let $\Gamma \subseteq SL(2, \mathbb{Z})$ be a congruence subgroup and let Δ be the Laplacian of the hyperbolic surface $\Gamma \backslash \mathbb{H}$. Let $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of Δ acting in $L^2(\Gamma \backslash \mathbb{H})$. We write $\lambda_j = \frac{1}{4} + r_j^2$ with $r_j \in \mathbb{R}_{\geq 0} \cup [0, \frac{1}{2}]i$. For $\lambda \geq 0$ let

$$N^\Gamma(\lambda) = \#\{j : |r_j| \leq \lambda\}$$

be the counting function for the eigenvalues, where each eigenvalue is counted with its multiplicity. Note that any eigenfunction with eigenvalue $\lambda_j \geq 1/4$ is cuspidal. Therefore $N^\Gamma(\lambda) = N_{\text{cus}}^\Gamma(\lambda) + C$ and it suffices to study the asymptotic behavior of $N^\Gamma(\lambda)$.

We will write $O(X)$ for any quantity which is bounded in absolute value by a constant multiple of X . Sometimes we write $O_p(X)$ to indicate dependence of the implied constant on additional parameters, but we often suppress it if it is understood from the context.

Next recall the Selberg trace formula [Sel56]. Let $E_k(z, s)$ be the Eisenstein series attached to the k -th cusp. The constant term of its Fourier expansion in the l -th cusp is of the form

$$\delta_{kl}y^s + C_{kl}(s)y^{1-s}.$$

Let $C(s) = (C_{kl}(s))$ be the corresponding scattering matrix and let $\phi(s) = \det C(s)$. Let $h \in C_c^\infty(\mathbb{R})$ and $\hat{h}(z) = \int_{\mathbb{R}} h(r)e^{irz} dr$. Then \hat{h} is entire and rapidly decreasing on horizontal strips. For $t \in \mathbb{R}$ put

$$\hat{h}_t(z) = \hat{h}(t-z) + \hat{h}(t+z).$$

Symmetrize the spectrum by $r_{-j} := -r_j$, $j \in \mathbb{N}$. For simplicity we assume that Γ contains no elements of finite order. Then the trace formula can be applied to \hat{h}_t and gives the following identity.

(2.1)

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \hat{h}(t-r_j) &= \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{2\pi} \int_{\mathbb{R}} \hat{h}(t-r)r \tanh(\pi r) dr + \sum_{\{\gamma\}_\Gamma} \frac{l(\gamma_0)}{\sinh\left(\frac{l(\gamma)}{2}\right)} h(l(\gamma)) \cos(tl(\gamma)) \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(t-r) \frac{\phi'}{\phi} \left(\frac{1}{2} + ir\right) dr - \frac{1}{2} \phi\left(\frac{1}{2}\right) \hat{h}(t) - \frac{m}{\pi} \int_{-\infty}^{\infty} \hat{h}(t-r) \frac{\Gamma'}{\Gamma}(1+ir) dr + \frac{m}{2} \hat{h}(t) - \\ &2m \ln 2 h(0). \end{aligned}$$

Here m is the number of cusps of $\Gamma \backslash \mathbb{H}$ and $\{\gamma\}_\Gamma$ runs over the hyperbolic conjugacy classes in Γ . Each such conjugacy class determines a closed geodesic τ_γ of $\Gamma \backslash \mathbb{H}$ and $l(\gamma)$ denotes the length of τ_γ . Each hyperbolic element γ is the power of a primitive hyperbolic element γ_0 . Since $h \in C_c^\infty(\mathbb{R})$, all series and integrals are absolutely convergent.

We use this formula to study the asymptotic behavior of the left hand side as $t \rightarrow \infty$. To this end we need to consider the asymptotic behavior of the terms on the right hand side. Using $|\tanh(x)| \leq 1$, $x \in \mathbb{R}$, it follows that

$$\int_{\mathbb{R}} \hat{h}(t-r)r \tanh(\pi r) dr = O(t), \quad |t| \rightarrow \infty.$$

For the second term we observe that the sum over the hyperbolic conjugacy classes is absolutely convergent. Hence the sum is uniformly bounded in t .

Next consider the integral involving the scattering matrix. For the principal congruence subgroup $\Gamma(N)$ the determinant of the scattering matrix $\phi(s) = \det C(s)$ has been computed by Huxley [Hej76]. It has the form

$$\phi(s) = (-1)^l A^{1-2s} \left(\frac{\Gamma(1-s)}{\Gamma(s)} \right)^k \prod_{\chi} \frac{L(2-2s, \bar{\chi})}{L(2s, \chi)},$$

where $k, l \in \mathbb{Z}$, $A > 0$, the product runs over Dirichlet characters χ to some modulus dividing N and $L(s, \chi)$ the Dirichlet L -function with character χ . Using Stirling's approximation formula to estimate the logarithmic derivative of the Gamma function and standard

estimations for the logarithmic derivative of Dirichlet L -functions on the line $\operatorname{Re}(s) = 1$, we get

$$(2.2) \quad \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right) = O(\log(|r|)), \quad |r| \rightarrow \infty.$$

This implies

$$\int_{\mathbb{R}} \hat{h}(t-r) \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right) dr = O(\log(|t|)), \quad |t| \rightarrow \infty.$$

In the same way we get

$$(2.3) \quad \int_{\mathbb{R}} \hat{h}(t-r) \frac{\Gamma'}{\Gamma}(1+ir) dr = O(\log(|t|)), \quad |t| \rightarrow \infty.$$

The remaining terms are bounded as $|t| \rightarrow \infty$. Summarizing, we obtain

$$(2.4) \quad \sum_{j=-\infty}^{\infty} \hat{h}(t-r_j) = O(|t|), \quad |t| \rightarrow \infty.$$

This result can be applied to estimate the number of eigenvalues in a neighborhood of a given point $\lambda \in \mathbb{R}$. As in the proof of [DG75, Lemma 2.3] choose $h \in C_c^\infty(\mathbb{R})$ such that $\hat{h} \geq 0$ and $\hat{h} > 0$ on $[-1, 1]$. Now note that there are only finitely many eigenvalues $\lambda_j = 1/4 + r_j^2$ with $r_j \notin \mathbb{R}$. Hence it suffices to consider the eigenvalues with $r_j \in \mathbb{R}$. For $\lambda \in \mathbb{R}$ we have

$$\#\{j : |r_j - \lambda| \leq 1, r_j \in \mathbb{R}\} \cdot \min\{\hat{h}(u) : |u| \leq 1\} \leq \sum_{r_j \in \mathbb{R}} \hat{h}(\lambda - r_j)$$

which is $O(1 + |\lambda|)$ by (2.4). It follows that

$$\#\{j : |r_j - \lambda| \leq 1\} = O(1 + |\lambda|)$$

for all $\lambda \in \mathbb{R}$. This local estimation is the basis of the following auxiliary result. Let h be even. Then

$$(2.5) \quad \sum_{|r_j| \leq \lambda} \left| \int_{\mathbb{R} - [-\lambda, \lambda]} \hat{h}(t-r_j) dt \right| + \sum_{|r_j| > \lambda} \left| \int_{-\lambda}^{\lambda} \hat{h}(t-r_j) dt \right| = O(\lambda)$$

for all $\lambda \geq 1$. The proof is elementary (see [Mül, Lemma 2.3]).

The next step is to integrate both sides of (2.1) over a finite interval $(-\lambda, \lambda)$ and study the asymptotic behavior as $\lambda \rightarrow \infty$ of the terms on the right hand side. To this end let $p(r)$ be a continuous even function on \mathbb{R} such that $p(r) = O(1 + |r|)$. Assume that $h(0) = 1$. Then it follows that

$$\int_{-\lambda}^{\lambda} \int_{\mathbb{R}} \hat{h}(t-r) p(r) dr dt = \int_{-\lambda}^{\lambda} p(r) dr + O(\lambda), \quad \lambda \rightarrow \infty.$$

[Mül, (2.10)]. Applying this to the functions $p(r) = r \tanh(\pi r)$, $\frac{\phi'}{\phi}(\frac{1}{2} + ir)$ and $\frac{\Gamma'}{\Gamma}(1 + ir)$ respectively and using (2.2) and Stirling's formula we obtain

$$\begin{aligned} \int_{-\lambda}^{\lambda} \int_{\mathbb{R}} \hat{h}(t-r) r \tanh(\pi r) dr dt &= \lambda^2 + O(\lambda), \\ \int_{-\lambda}^{\lambda} \int_{\mathbb{R}} \hat{h}(t-r) \frac{\phi'}{\phi}(\frac{1}{2} + ir) dr dt &= O(\lambda \log \lambda), \\ \int_{-\lambda}^{\lambda} \int_{\mathbb{R}} \hat{h}(t-r) \frac{\Gamma'}{\Gamma}(1 + ir) dr dt &= O(\lambda \log \lambda). \end{aligned}$$

The remaining terms on the right hand side of (2.1) stay bounded as $t \rightarrow \infty$. Hence their integral is of order $O(\lambda)$. Thus, it follows that for every even h such that $h(0) = 1$, we have

$$(2.6) \quad \int_{-\lambda}^{\lambda} \sum_{j=-\infty}^{\infty} \hat{h}(t-r_j) dt = \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{2\pi} \lambda^2 + O(\lambda \log \lambda)$$

as $\lambda \rightarrow \infty$.

We are now ready to prove Weyl's law. We choose h with $h(0) = 1$. Then the left-hand side of (2.6) is

$$\sum_{|r_j| \leq \lambda} \int_{\mathbb{R}} \hat{h}(t-r_j) dt - \sum_{|r_j| \leq \lambda} \int_{\mathbb{R} - [-\lambda, \lambda]} \hat{h}(t-r_j) dt + \sum_{|r_j| > \lambda} \int_{-\lambda}^{\lambda} \hat{h}(t-r_j) dt.$$

Using that $\int_{\mathbb{R}} \hat{h}(t-r) dt = h(0) = 1$, we get

$$2N^{\Gamma}(\lambda) = \int_{-\lambda}^{\lambda} \sum_j \hat{h}(t-r_j) dt + \sum_{|r_j| \leq \lambda} \int_{\mathbb{R} - [-\lambda, \lambda]} \hat{h}(t-r_j) dt - \sum_{|r_j| > \lambda} \int_{-\lambda}^{\lambda} \hat{h}(t-r_j) dt.$$

By (2.5) and (2.6) we obtain

$$N^{\Gamma}(\lambda) = \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{4\pi} \lambda^2 + O(\lambda \log \lambda).$$

3. PRELIMINARIES FOR THE HIGHER RANK CASE

3.1. Fix a positive integer n and let $G = \text{GL}(n)$, considered as an algebraic group over \mathbb{Q} . Let P_0 be the subgroup of upper triangular matrices of G , N_0 its unipotent radical and M_0 the group of diagonal matrices in G , which is a Levi subgroup of P_0 . Let A_G be the group of scalar matrices with a positive real scalar, $K_{\infty} = \text{O}(n)$ and $K_{\infty}^0 = \text{SO}(n)$. We recall some basic facts about harmonic analysis of $A_G \backslash G(\mathbb{R}) / K_{\infty}$. For more details cf. [Hel00]. Let A be the group of diagonal matrices with positive real diagonal entries and determinant 1. Let

$$A_G \backslash G(\mathbb{R}) = AN_0(\mathbb{R})K_{\infty}$$

be the Iwasawa decomposition. Let \mathfrak{a} be the Lie algebra of A and W the Weyl group. Then W acts on \mathfrak{a} and the Killing form gives a W -invariant inner product, as well as a Haar measure, on \mathfrak{a} . Recall the function $H : A_G \backslash G(\mathbb{R}) \rightarrow \mathfrak{a}$ defined by

$$H(\exp(X)nk) = X$$

for $X \in \mathfrak{a}$, $n \in N_0(\mathbb{R})$ and $k \in K_\infty$. Let $\rho \in \mathfrak{a}^*$ be such that $\delta_0(\exp X)^{\frac{1}{2}} = e^{\langle \rho, X \rangle}$ for $X \in \mathfrak{a}$ where δ_0 is the modulus function of $P_0(\mathbb{R})$.

We will fix Haar measures as in [DKV79, (3.29)]. Namely, on K_∞ and K_∞^0 we take the probability measures. The Haar measure on \mathfrak{a} defined above gives rise to a Haar measure on A (through the exponential map) and on \mathfrak{a}^* (the dual Haar measure). We transfer the latter to \mathfrak{ia}^* in the obvious way. On $N_0(\mathbb{R})$ we take the measure as in [ibid., p. 37] and on $A_G \backslash G(\mathbb{R})$ we take the measure which is compatible with the Iwasawa decomposition. On quotient spaces (such as X) we take the corresponding quotient measures.

Let ϕ_λ , $\lambda \in \mathfrak{a}_\mathbb{C}^*$ be the spherical function given by Harish-Chandra's formula

$$(3.1) \quad \phi_\lambda(g) = \int_{K_\infty} e^{\langle \lambda + \rho, H(kg) \rangle} dk = \int_{K_\infty^0} e^{\langle \lambda + \rho, H(kg) \rangle} dk.$$

Let $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)^W$ be the space of W -invariant Paley-Wiener functions on $\mathfrak{a}_\mathbb{C}^*$. We will denote by $C_c^\infty(G(\mathbb{R})//A_G K_\infty)$ the space of smooth bi- $A_G K_\infty$ -invariant functions which are compactly supported modulo A_G . The Harish-Chandra transform

$$\mathcal{H} : C_c^\infty(G(\mathbb{R})//A_G K_\infty) \rightarrow \mathcal{P}(\mathfrak{a}_\mathbb{C}^*)^W$$

is defined by

$$(\mathcal{H}f)(\lambda) = \int_A f(a)\phi_\lambda(a) da.$$

It is the composition of two maps: the Abel transform

$$\mathcal{A} : C_c^\infty(G(\mathbb{R})//A_G K_\infty) \rightarrow C_c^\infty(\mathfrak{a})^W$$

defined by

$$\mathcal{A}(f)(X) = \delta_0(\exp X)^{1/2} \int_{N_0(\mathbb{R})} f(\exp Xn) dn, \quad X \in \mathfrak{a}$$

and the Fourier-Laplace transform

$$\hat{h}(\lambda) = \int_{\mathfrak{a}} h(X)e^{\langle \lambda, X \rangle} dX, \quad h \in C_c^\infty(\mathfrak{a}), \quad \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

Thus $\mathcal{H}(f) = \widehat{\mathcal{A}(f)}$. The Abel transform is an isomorphism which respects support in the following sense. Given $R > 0$, put

$$V(R) = \{X \in \mathfrak{a} : \|X\| \leq R\}, \quad U(R) = K_\infty \exp V(R) K_\infty.$$

Then for all $h \in C_c^\infty(\mathfrak{a})^W$ with $\text{supp}(h) \subseteq V(R)$ one has $\text{supp}(\mathcal{A}^{-1}(h)) \subseteq U(R)$.

Let $\beta(\lambda) = |c(\lambda)c(\rho)^{-1}|^{-2}$ be the Plancherel measure on \mathfrak{ia}^* . The c -function is given by the Gindikin-Karpelevic formula cf. [DKV79, p. 46]. In our case, identifying \mathfrak{a}^* with $\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sum \lambda_i = 0\}$ we have

$$c(\lambda)^{-1} = \prod_{1 \leq i < j \leq n} \phi(\lambda_i - \lambda_j)$$

where we set

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \mathbf{\Gamma}(s/2), \quad \phi(s) = \frac{\Gamma_{\mathbb{R}}(s+1)}{\Gamma_{\mathbb{R}}(s)}.$$

For any $h \in C_c^\infty(\mathfrak{a})$ define

$$(3.2) \quad \mathcal{B}h(x) = \frac{1}{|W|} \int_{\mathfrak{ia}^*} \hat{h}(\lambda) \phi_{-\lambda}(x) \beta(\lambda) d\lambda.$$

By the Plancherel theorem $\mathcal{B}h \in C_c^\infty(G(\mathbb{R})//A_G K_\infty)$ and $\mathcal{A}(\mathcal{B}h) = h^W$ where

$$h^W(a) = \frac{1}{|W|} \sum_{w \in W} h(wa).$$

By Stirling's formula we have

$$\phi(z) = O\left(1 + |z|^{\frac{1}{2}}\right), \quad \frac{\phi'(z)}{\phi(z)} = O\left(\frac{1}{1 + |z|}\right)$$

for $\operatorname{Re} z = 0$. We infer that

$$(3.3) \quad \beta(\lambda) = O(\tilde{\beta}(\lambda)), \quad \lambda \in \mathfrak{ia}^*,$$

where

$$(3.4) \quad \tilde{\beta}(\lambda) = \prod_{i < j} (1 + |\lambda_i - \lambda_j|)$$

Moreover, for any $\xi \in \mathfrak{a}^*$, denoting by D_ξ the directional derivative along ξ , we have

$$(3.5) \quad D_\xi \beta(\lambda) = O_\xi \left((1 + \|\lambda\|)^{d-r-1} \right), \quad \lambda \in \mathfrak{ia}^*$$

where $r = \dim \mathfrak{a}$ and $d = \dim X$. In fact, the analogues of these bounds hold for any G – cf. [DKV79, §3]. Of course, in our case $r = n - 1$ and $d = n(n + 1)/2 - 1$. It will be convenient to set

$$\tilde{\beta}(t, \lambda) = \prod_{i < j} (t + |\lambda_i - \lambda_j|)$$

so that $\tilde{\beta}(\lambda) = \tilde{\beta}(1, \lambda)$. Note that

$$(3.6) \quad \tilde{\beta}(t, \lambda) = O\left((t + \|\lambda\|)^{d-r}\right), \quad \lambda \in \mathfrak{ia}^*$$

and that

$$(3.7) \quad \tilde{\beta}(t, \lambda_1 + \lambda_2) = O\left(\tilde{\beta}(t + \|\lambda_1\|, \lambda_2)\right).$$

3.2. If P is a parabolic subgroup of G containing M_0 then it is defined over \mathbb{Q} (since G is split) and it admits a unique Levi component M_P (also defined over \mathbb{Q}) containing M_0 . We call M_P a semi-standard Levi subgroup of G and denote by \mathcal{L} the set of semi-standard Levi subgroups of G . Any $M \in \mathcal{L}$ is isomorphic to $\mathrm{GL}(m_1) \times \cdots \times \mathrm{GL}(m_r)$ with $m_1 + \cdots + m_r = n$. As a lattice \mathcal{L} is isomorphic to the partition lattice whose elements are the partitions of $\{1, \dots, n\}$, ordered by refinement. We will use superscript M to denote notation pertaining to M . For example, \mathfrak{a}^M is the product of the \mathfrak{a} 's corresponding to the $\mathrm{GL}(m_i)$'s and β^M is the Plancherel measure with respect to M , etc. We also set \mathfrak{a}_M to be the orthogonal complement of \mathfrak{a}^M in \mathfrak{a} . Similarly, we write $\mathfrak{a}^* = \mathfrak{a}_M^* \oplus (\mathfrak{a}^M)^*$. For any $\mu \in \mathfrak{a}^*$ we write $\mu = \mu_M + \mu^M$ corresponding to this decomposition.

Lemma 3.1. *Suppose that $M \neq G$. Then*

$$\tilde{\beta}^M(\lambda^M)(1 + \|\lambda\|) = O(\tilde{\beta}(\lambda)), \quad \lambda \in \mathfrak{ia}^*.$$

Proof. Passing to a larger Levi subgroup M only strengthens the inequality. Therefore, we can assume without loss of generality that M is maximal. By conjugating M and λ by a Weyl element, we can assume that M is a standard Levi isomorphic to $\mathrm{GL}(k) \times \mathrm{GL}(n-k)$, $k \geq 1$. In this case,

$$\tilde{\beta}(\lambda) = \tilde{\beta}(\lambda^M) \prod_{(i,j): 1 \leq i \leq k < j \leq n} (1 + |\lambda_i - \lambda_j|).$$

It remains to observe that

$$1 + \|\lambda\| = O\left(\prod_{(i,j): 1 \leq i \leq k < j \leq n} (1 + |\lambda_i - \lambda_j|)\right).$$

In fact,

$$\|\lambda\| = O\left(\sum_{(i,j): 1 \leq i \leq k < j \leq n} |\lambda_i - \lambda_j|\right),$$

simply because the linear map $\lambda \mapsto (\lambda_i - \lambda_j)_{1 \leq i \leq k < j \leq n}$ from \mathfrak{a}^* to $\mathbb{R}^{k(n-k)}$ is injective. \square

3.3. For each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let $I(\lambda)$ be the unique spherical irreducible subquotient of the induced representation $\mathrm{Ind}_{P_0(\mathbb{R})}^{G(\mathbb{R})}(e^{\langle \lambda, H(\cdot) \rangle})$. Set

$$\mathfrak{a}_{\mathrm{un}}^* = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : I(\lambda) \text{ is unitarizable}\}.$$

It is known that $\|\mathrm{Re} \lambda\| \leq \|\rho\|$ for any $\lambda \in \mathfrak{a}_{\mathrm{un}}^*$.

Note that for any $M \in \mathcal{L}$ we have

$$(\mathfrak{a}_{\mathrm{un}}^M)^* + \mathfrak{ia}_{M}^* \subseteq \mathfrak{a}_{\mathrm{un}}^*.$$

For any $w \in W$ let $\mathfrak{a}_{w, \pm 1}^*$ be the \pm -eigenspaces of w on \mathfrak{a}^* . It is well-known that $\mathfrak{a}_{w, +1}^* = \mathfrak{a}_{M_w}^*$ where M_w is the smallest $M \in \mathcal{L}$ such that w belongs to the Weyl group W_M of M (cf. [Art82b, p. 1299], [OT92, Theorem 6.27]). Define

$$\mathfrak{a}_w^* = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : w\lambda = -\bar{\lambda}\} = \mathfrak{a}_{w, -1}^* + \mathfrak{ia}_{w, +1}^* = \mathfrak{a}_{w, -1}^* + \mathfrak{ia}_{M_w}^*.$$

Since every unitarizable representation is isomorphic to its hermitian dual, and since $I(\lambda) \simeq I(\lambda')$ if and only if λ' is in the Weyl orbit of λ , we infer that

$$\mathfrak{a}_{\text{un}}^* \subseteq \bigcup_{w \in W} \mathfrak{a}_w^*.$$

For any $M \in \mathcal{L}$ define

$$\begin{aligned} \mathfrak{a}_{\text{hm}, \subseteq M}^* &= \bigcup_{w \in W_M} \mathfrak{a}_w^*, \\ \mathfrak{a}_{\text{un}, \not\subseteq M}^* &= \mathfrak{a}_{\text{un}}^* \setminus \mathfrak{a}_{\text{hm}, \subseteq M}^*. \end{aligned}$$

3.4. Consider now the principal congruence subgroups. Let

$$N = \prod_p p^{r_p}, \quad r_p \geq 0.$$

Set

$$K_p(N) = \{k \in G(\mathbb{Z}_p) : k \equiv 1 \pmod{p^{r_p} \mathbb{Z}_p}\}$$

and

$$K_f(N) = \prod_{p < \infty} K_p(N).$$

Then $K_f(N)$ is an open compact subgroup of $G(\mathbb{A}_f)$. The determinant defines a map which fibers $A_G G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f(N)$ over

$$\mathbb{R}^+ \mathbb{Q}^* \backslash \mathbb{A}^* / \prod_p \{r \in \mathbb{Z}_p^* : a \equiv 1 \pmod{p^{r_p} \mathbb{Z}_p}\} \cong (\mathbb{Z}/N\mathbb{Z})^*.$$

The fiber of any point is $\text{SL}(n, \mathbb{R})$ -invariant, and as an $\text{SL}(n, \mathbb{R})$ space isomorphic to

$$\text{SL}(n, \mathbb{Q}) \backslash \text{SL}(n, \mathbb{A}) / (\text{SL}(n, \mathbb{A}) \cap K_f(N)).$$

By strong approximation for $\text{SL}(n)$ we have $\text{SL}(n, \mathbb{A}) \cap (\text{SL}(n, \mathbb{R}) \cdot K_f(N)) = \Gamma(N)$, where $\Gamma(N) \subseteq \text{SL}(n, \mathbb{Z})$ is the principal congruence subgroup of level N . Thus each fiber is isomorphic to $\Gamma(N) \backslash \text{SL}(n, \mathbb{R})$, and therefore

$$(3.8) \quad A_G G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f(N) \cong \bigsqcup_{(\mathbb{Z}/N\mathbb{Z})^*} (\Gamma(N) \backslash \text{SL}(n, \mathbb{R})).$$

We fix a Haar measure on $G(\mathbb{A}_f)$. This defines a Haar measure on $A_G \backslash G(\mathbb{A})$ through $A_G \backslash G(\mathbb{A}) = A_G \backslash G(\mathbb{R}) G(\mathbb{A}_f)$ and our previous choice of Haar measure on $A_G \backslash G(\mathbb{R})$. On any open subgroup of $G(\mathbb{A}_f)$ we take the restricted measure. Note that the expression $\text{vol}(A_G G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f)$ does not depend on the choice of Haar measure on $G(\mathbb{A}_f)$.

4. THE METHOD OF DUISTERMAAT-KOLK-VARADARAJAN

Our strategy to prove Theorem 1.1 is to use the method of Duistermaat-Kolk-Varadarajan [DKV79], in conjunction with Arthur's (non-invariant) trace formula [Art78] which replaces the Selberg trace formula in the non-compact case. Arthur's trace formula is an identity of distributions

$$(4.1) \quad J_{\text{geo}}(f) = J_{\text{spec}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1).$$

The distributions J_{geo} and J_{spec} will be recalled in §5 and §6 respectively. We apply the trace formula identity to a certain class of test functions described below. In order to apply the argument of [DKV79], the main technical difficulty is to show that, in a suitable sense, the main terms in the trace formula, in both the geometric and the spectral side, are exactly those which occur in the compact case.

To make this more precise, fix a compact open subgroup K_f of $G(\mathbb{A}_f)$. We assume that $K_f \subseteq K_f(N)$ for some $N \geq 3$. For $h \in C_c^\infty(\mathfrak{a})$ we define $\mathcal{F}(h)$ to be the restriction to $G(\mathbb{A})^1$ of the function $\mathcal{B}(h) \otimes \mathbf{1}_{K_f}$ on $G(\mathbb{A})$, where $\mathcal{B}(h)$ is defined by (3.2) and $\mathbf{1}_{K_f}$ is the characteristic function of K_f in $G(\mathbb{A}_f)$ normalized by $\text{vol}(K_f)^{-1}$. For $t \geq 1$ let $h_t \in C_c^\infty(\mathfrak{a})$ be defined by $h_t(X) = t^r h(tX)$ for $X \in \mathfrak{a}$. We have $\widehat{h}_t = \widehat{h}(t^{-1}\cdot)$. Also, for $\mu \in \mathfrak{ia}^*$ we set $h_\mu = h e^{-\langle \mu, \cdot \rangle}$. (Hopefully, this does not create any confusion with the previous notation h_t .) Finally let $h_{t,\mu} = (h_t)_\mu$ so that $\widehat{h_{t,\mu}} = \widehat{h}(t^{-1}(\cdot - \mu))$.

Throughout the rest of the paper let $d = \dim X$ and $r = \dim \mathfrak{a}$. For $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ and $t > 0$ let

$$B_t(\mu) = \{v \in \mathfrak{a}_{\mathbb{C}}^* : \|v - \mu\| \leq t\}$$

be the ball of radius t around μ . If $t = 1$ we will often suppress it from the notation.

Also, for $h \in C_c^\infty(\mathfrak{a})$ we set

$$\mathcal{M}(\widehat{h})(\lambda) = \max_{\nu \in B_{1+\|\rho\|}(\lambda)} |\widehat{h}(\nu)| \quad \text{and} \quad \mathcal{N}(h) = \int_{\mathfrak{ia}^*} \tilde{\beta}(\lambda) \mathcal{M}(\widehat{h})(\lambda) d\lambda,$$

where $\tilde{\beta}(\lambda)$ is defined by (3.4). We need the following auxiliary result.

Lemma 4.1. *For $h \in C_c^\infty(\mathfrak{a})$, $t \geq 1$, and $\mu \in \mathfrak{ia}^*$ we have*

$$(4.2) \quad \mathcal{N}(h_{t,\mu}) = O_h(t^r \tilde{\beta}(t, \mu)).$$

Proof. For $t \geq 1$ and $\mu \in \mathfrak{ia}^*$ we have $\mathcal{M}(\widehat{h_{t,\mu}})(\lambda) \leq \mathcal{M}(\widehat{h})(t^{-1}(\lambda - \mu))$. Hence

$$\mathcal{N}(h) \leq \int_{\mathfrak{ia}^*} \tilde{\beta}(\lambda) \mathcal{M}(\widehat{h})(t^{-1}(\lambda - \mu)) d\lambda = t^r \int_{\mathfrak{ia}^*} \tilde{\beta}(t\lambda + \mu) \mathcal{M}(\widehat{h})(\lambda) d\lambda.$$

By (3.7) we get

$$\tilde{\beta}(t\lambda + \mu) = O(\tilde{\beta}(t\|\lambda\|, \mu)) = O((1 + \|\lambda\|)^{d-r} \tilde{\beta}(t, \mu)).$$

The lemma follows since for any $N \in \mathbb{N}$ we have $\mathcal{M}(\widehat{h})(\lambda) = O_{h,N}((1 + \|\lambda\|)^{-N})$ for $\lambda \in \mathfrak{ia}^*$. \square

Let $\Omega \subseteq \mathfrak{ia}^*$ be a bounded open subset such that $\partial\Omega = \overline{\Omega} \setminus \Omega$ is piecewise C^2 . For $t > 0$ let $t\Omega = \{t\mu : \mu \in \Omega\}$.

The following Propositions will be proved in §5 and §6 respectively.

Proposition 4.2. *We have*

- (1) *There exists a neighborhood ω_0 of 0 in \mathfrak{a} such that*

$$J_{\text{geo}}(\mathcal{F}(h)) = O(\mathcal{N}(h)).$$

for all $h \in C_c^\infty(\mathfrak{a})$ supported in ω_0 .

- (2) *For every such h we have*

$$\int_{t\Omega} J_{\text{geo}}(\mathcal{F}(h_\mu)) d\mu = \text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1) \int_{t\Omega} \mathcal{F}(h_\mu)(1) d\mu + O_{h,\Omega}(t^{d-1}(\log t)^{\max(n,3)})$$

as $t \rightarrow \infty$.

Analogously,

Proposition 4.3. *We have*

- (1) $J_{\text{spec}}(\mathcal{F}(h)) = J_{\text{dis}}(\mathcal{F}(h)) + O(\mathcal{N}(h))$ *for every $h \in C_c^\infty(\mathfrak{a})$ where*

$$(4.3) \quad J_{\text{dis}}(\mathcal{F}(h)) = \sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A}))} \dim(\mathcal{H}_\pi^K) \hat{h}^W(\lambda_{\pi_\infty}).$$

- (2) $\int_{t\Omega} J_{\text{spec}}(\mathcal{F}(h_\mu)) d\mu = \int_{t\Omega} J_{\text{dis}}(\mathcal{F}(h_\mu)) d\mu + O_{h,\Omega}(t^{d-1} \log t)$ *as $t \rightarrow \infty$.*

Using (4.1) and (4.2) we infer

Corollary 4.4. *For every $h \in C_c^\infty(\mathfrak{a})$ supported in ω_0 we have*

- (1) $J_{\text{dis}}(\mathcal{F}(h)) = O(\mathcal{N}(h))$. *In particular, for $\mu \in \mathfrak{ia}^*$ and $t \geq 1$, we have*

$$J_{\text{dis}}(\mathcal{F}(h_{t,\mu})) = O_h(t^x \tilde{\beta}(t, \mu)).$$

- (2) $\int_{t\Omega} J_{\text{dis}}(\mathcal{F}(h_\mu)) d\mu = \text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1) \int_{t\Omega} \mathcal{F}(h_\mu)(1) d\mu + O_{h,\Omega}(t^{d-1}(\log t)^{\max(n,3)})$ *as $t \rightarrow \infty$.*

In the rest of this section we will prove the main Theorem using Corollary 4.4. We first rewrite the expression for J_{dis} as follows. There exist congruence subgroups $\Gamma_i \subseteq G(\mathbb{R})$, $i = 1, \dots, l$, such that

$$A_G G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f \cong \bigsqcup_{i=1}^l (\Gamma_i \backslash G(\mathbb{R})^1).$$

Let $\Lambda_{\text{dis}}(\Gamma_i) \subset \mathfrak{a}_\mathbb{C}^*$ be the (symmetrized) spectrum of the algebra of invariant differential operators of $\text{SL}(n, \mathbb{R})$ acting on the discrete subspace $L_{\text{dis}}^2(\Gamma_i \backslash X)$ of the Laplace operator on $\Gamma_i \backslash X$. For $\lambda \in \Lambda_{\text{dis}}(\Gamma_i)$ let $m_i(\lambda)$ denote the dimension of the corresponding eigenspace

normalized by the size of the orbit of λ under W (cf. [DKV79, (3.25)]). If $\lambda \notin \Lambda_{\text{dis}}(\Gamma_i)$ set $m_i(\lambda) = 0$. Finally, set

$$\Lambda_{\text{dis}} = \Lambda_{\text{dis}}(K_f) := \cup_{i=1}^l \Lambda_{\text{dis}}(\Gamma_i), \quad m(\lambda) = \sum_{i=1}^l m_i(\lambda).$$

By the well-known relation between automorphic forms and automorphic representations we get

$$m(\lambda) = \sum_{\pi: \lambda_{\pi_\infty} = \lambda} \dim \mathcal{H}_\pi^K$$

for any $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and therefore

$$(4.4) \quad J_{\text{dis}}(\mathcal{F}(h)) = \sum_{\lambda \in \Lambda_{\text{dis}}(K_f)} m(\lambda) \hat{h}^W(\lambda).$$

For any bounded subset $\Omega \subset \mathfrak{a}_\mathbb{C}^*$ we set

$$m(\Omega) = \sum_{\lambda \in \Omega} m(\lambda).$$

Then

$$(4.5) \quad \sum_{\substack{\pi \in \Pi_{\text{dis}}(G(\mathbb{A})) \\ \lambda_{\pi_\infty} \in t\Omega}} \dim(\mathcal{H}_\pi^K) = m(t\Omega).$$

Following [DKV79] we first establish a bound on the spectrum near a given point $\mu \in \mathfrak{a}^*$. For this we will only use the first part of Corollary 4.4.

Proposition 4.5. *We have*

$$m(B_t(\mu)) = O\left(t^r \tilde{\beta}(t, \mu)\right)$$

for all $\mu \in \mathfrak{ia}^*$ and $t \geq 1$. More precisely, for $\mu \in \mathfrak{a}^*$ we have

$$m(\{\nu \in \mathfrak{a}_\mathbb{C}^*: \|\text{Im } \nu - \mu\| \leq t\}) = O\left(t^r \tilde{\beta}(t, \mu)\right).$$

Proof. First note that the second part follows from the first one since $\text{Re } \lambda$ is bounded if $m(\lambda) > 0$. We will prove the first part for $\mu \in \mathfrak{ia}_M^*$ by induction on the co-rank of M . (The case $M = M_0$ is the statement of the Proposition.) For the case $M = G$, we have $\mu = 0$ and we argue as in [DKV79, Proposition 6.4]. Fix $h \in C_c^\infty(A)^W$ of small support such that $\hat{h} \geq 0$ on $\mathfrak{a}_{\text{un}}^*$, and $|\hat{h}| \geq 1$ on $B(0)$ ([ibid., Lemma 6.2]). Then $m(B_t(0)) \leq J_{\text{dis}}(\mathcal{F}(h_t))$, and $J_{\text{dis}}(\mathcal{F}(h_t)) = O(t^d)$ by Corollary 4.4. We deduce the case $M = G$.

For the induction step we proceed as in the proof of [ibid., Proposition 7.1].¹ Let h be as before. Then $\widehat{h_{t,\mu}} \geq 0$ on $\mathfrak{a}_{\text{hm},\subseteq M}^*$ and $|\widehat{h_{t,\mu}}| \geq 1$ on $B_t(\mu)$. Thus,

$$\begin{aligned} |J_{\text{dis}}(\mathcal{F}(h_{t,\mu}))| &= \left| \sum_{\lambda \in \Lambda_{\text{dis}}} m(\lambda) \widehat{h_{t,\mu}}(\lambda - \mu) \right| \geq m(B_t(\mu) \cap \mathfrak{a}_{\text{hm},\subseteq M}^*) \\ &\quad - \sum_{\lambda \in \mathfrak{a}_{\text{un},\not\subseteq M}^*} m(\lambda) |\widehat{h_{t,\mu}}(\lambda - \mu)|. \end{aligned}$$

Hence,

$$\begin{aligned} m(B_t(\mu)) &= m(B_t(\mu) \cap \mathfrak{a}_{\text{hm},\subseteq M}^*) + m(B_t(\mu) \cap \mathfrak{a}_{\text{un},\not\subseteq M}^*) \\ &\leq |J_{\text{dis}}(\mathcal{F}(h_{t,\mu}))| + \sum_{\lambda \in \mathfrak{a}_{\text{un},\not\subseteq M}^*} m(\lambda) |\widehat{h_{t,\mu}}(\lambda - \mu)| + m(B_t(\mu) \cap \mathfrak{a}_{\text{un},\not\subseteq M}^*). \end{aligned}$$

We will bound each of the terms on the right-hand side separately. For the first term we use Corollary 4.4. To bound the other two terms consider \mathfrak{a}_w^* with $w \notin W_M$, or equivalently, $M_w \not\subseteq M$. We have

$$\mathfrak{a}_w^* \cap \mathfrak{ia}_M^* = \mathfrak{i}(\mathfrak{a}_{M_w}^* \cap \mathfrak{a}_M^*) = \mathfrak{ia}_{M'}^*$$

where $M' \in \mathcal{L}$ is generated by M and M_w . Therefore the kernels of the maps $(\nu, \mu) \mapsto \nu - \mu, \nu - \mu_{M'}$ from $\mathfrak{a}_w^* \times \mathfrak{ia}_M^*$ to $\mathfrak{a}_{\mathbb{C}}^*$ coincide. It follows that $\|\nu - \mu_{M'}\| + \|\mu^{M'}\| = O(\|\nu - \mu\|)$ for $\nu \in \mathfrak{a}_w^*$ and $\mu \in \mathfrak{ia}_M^*$. In particular, there exists c such that $B_t(\mu) \cap \mathfrak{a}_w^* = \emptyset$ unless $\|\mu^{M'}\| \leq ct$, and in this case

$$B_t(\mu) \cap \mathfrak{a}_w^* \subseteq B_{ct}(\mu_{M'}) \cap \mathfrak{a}_w^*.$$

Therefore, either $m(B_t(\mu) \cap \mathfrak{a}_w^*) = 0$, or

$$m(B_t(\mu) \cap \mathfrak{a}_w^*) \leq m(B_{ct}(\mu_{M'})) = O(t^r \tilde{\beta}(t, \mu_{M'})) = O(t^r \tilde{\beta}(t, \mu))$$

by the induction hypothesis and (3.7). Since $\mathfrak{a}_{\text{un},\not\subseteq M}^* \subseteq \cup_{w \notin W_M} \mathfrak{a}_w^*$ we obtain

$$m(B_t(\mu) \cap \mathfrak{a}_{\text{un},\not\subseteq M}^*) = O(t^r \tilde{\beta}(t, \mu)).$$

Therefore for any $N > 0$, $\sum_{\lambda \in \mathfrak{a}_{\text{un},\not\subseteq M}^*} m(\lambda) |\widehat{h}(t^{-1}(\lambda - \mu))|$ is bounded by a constant multiple of

$$\sum_{k=1}^{\infty} m(B_{tk}(\mu) \cap \mathfrak{a}_{\text{un},\not\subseteq M}^*) k^{-N} = O\left(\sum_{k=1}^{\infty} k^{-N+r} t^r \tilde{\beta}(kt, \mu)\right).$$

For N sufficiently large, this is $O(t^r \tilde{\beta}(t, \mu))$. This completes the induction step and the proof of the Proposition. \square

Corollary 4.6. $m(B_t(0) \setminus \mathfrak{ia}^*) = O(t^{d-2})$.

¹The formulation of Proposition 4.5 is slightly stronger than [DKV79, Theorem 7.3]. This makes the induction step a bit smoother.

Indeed suppose that $\lambda \in \mathfrak{a}_{\text{un}}^* \cap \mathfrak{a}_w^*$ with $w \neq 1$. Then $\text{Re } \lambda$ is bounded while $\text{Im } \lambda \in \mathfrak{a}_{M_w}^*$ and $M_w \neq M_0$. Thus,

$$m(B_t(0) \setminus \mathfrak{ia}^*) \leq \sum_{M_0 \neq M \in \mathcal{L}} m(B_t(0) \cap \mathfrak{ia}_M^* + C)$$

for some compact set C of $\mathfrak{a}_{\mathbb{C}}^*$. Each of the sets $B_t(0) \cap \mathfrak{ia}_M^* + C$, $M \neq M_0$, is covered by $O(t^{r-1})$ balls of bounded radius whose centers are in $B_t(0) \cap \mathfrak{ia}_M^*$. The corollary follows from Proposition 4.5 since $\tilde{\beta}(R, \mu) = O_R(\|\mu\|^{d-r-1})$ for $\mu \in \mathfrak{ia}_M^*$.

We are now ready to prove Theorem 1.1. Let $\Omega \subseteq \mathfrak{ia}^*$ be a bounded domain with piecewise C^2 boundary. Fix $h \in C_c^\infty(\mathfrak{a})^W$ with $h(0) = 1$. Using (4.4) and the second part of Corollary 4.4 we get

$$\sum_{\lambda \in \Lambda_{\text{dis}}} m(\lambda) \int_{t\Omega} \hat{h}(\lambda - \mu) d\mu = \text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1) \int_{t\Omega} \mathcal{F}(h_\mu)(1) d\mu + O(t^{d-1}(\log t)^{\max(n,3)}).$$

as $t \rightarrow \infty$. By Plancherel inversion

$$\int_{t\Omega} \mathcal{F}(h_\mu)(1) d\mu = \frac{1}{|W| \text{vol}(K_f)} \int_{t\Omega} \int_{\mathfrak{ia}^*} \hat{h}(\lambda - \mu) \beta(\lambda) d\lambda d\mu.$$

Note that $\text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1) \text{vol}(K_f)^{-1} = \text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1 / K_f)$. As in [DKV79, §8] we will show that

$$(4.6) \quad \int_{t\Omega} \int_{\mathfrak{ia}^*} \hat{h}(\lambda - \mu) \beta(\lambda) d\lambda d\mu - \int_{t\Omega} \beta(\lambda) d\lambda = O(t^{d-1}),$$

$$(4.7) \quad \sum_{\lambda \in \Lambda_{\text{dis}}} m(\lambda) \int_{t\Omega} \hat{h}(\lambda - \mu) d\mu - m(t\Omega) = O(t^{d-1}).$$

From the description of the residual spectrum ([MW89]) it follows that $\Lambda_{\text{dis}} \setminus \Lambda_{\text{cus}} \subseteq \mathfrak{a}_{\mathbb{C}}^* \setminus \mathfrak{ia}^*$. Therefore, in view of Corollary 4.6 we can replace Λ_{dis} by Λ_{cus} in (4.7). Altogether, this gives Theorem 1.1 by (4.5).

To prove (4.6) we can write its left-hand side as

$$(4.8) \quad \int_{t\Omega'} \int_{t\Omega} \hat{h}(\lambda - \mu) \beta(\lambda) d\mu d\lambda - \int_{t\Omega} \int_{t\Omega'} \hat{h}(\lambda - \mu) \beta(\lambda) d\mu d\lambda$$

where Ω' is the complement of Ω in \mathfrak{ia}^* (cf. [DKV79, p. 84]). For $\kappa \geq 0$ let

$$\partial_\kappa(t\Omega) = \{\nu \in \mathfrak{ia}^* : \inf_{\mu \in t\partial\Omega} \|\nu - \mu\| \leq \kappa\}.$$

By separating the integrals in (4.8) into shells $k-1 \leq \|\lambda - \mu\| < k$, $k \in \mathbb{N}$, we can bound the sum by

$$\sum_{k=1}^{\infty} \int_{\partial_k(t\Omega)} \int_{(B_k(\lambda) \setminus B_{k-1}(\lambda)) \cap \mathfrak{ia}^*} |\hat{h}(\lambda - \mu)| \beta(\lambda) d\mu d\lambda.$$

Using (3.3) and the rapid decay of \hat{h} , for any $N > 0$ this is bounded by a constant multiple of

$$\sum_{k=1}^{\infty} k^{-N} k^r \operatorname{vol}(\partial_k(t\Omega))(k+t)^{d-r}.$$

Since Ω has a piecewise C^2 boundary we can cover $\partial_k(t\Omega)$ by $O(t^{r-1})$ balls of radius k and (4.6) follows. Similarly, we write the left-hand side of (4.7) as

$$(4.9) \quad - \sum_{\lambda \in \Lambda_{\text{dis}}, \lambda \in t\Omega} m(\lambda) \int_{t\Omega'} \hat{h}(\lambda - \mu) d\mu + \sum_{\lambda \in \Lambda_{\text{dis}}, \lambda \in t\Omega'} m(\lambda) \int_{t\Omega} \hat{h}(\lambda - \mu) d\mu \\ + \sum_{\lambda \in \Lambda_{\text{dis}}, \lambda \notin \mathfrak{ia}^*} m(\lambda) \int_{t\Omega} \hat{h}(\lambda - \mu) d\mu.$$

We proceed as above and divide the domains $t\Omega \times t\Omega'$, $t\Omega' \times t\Omega$ and $(\mathfrak{a}_{\mathbb{C}}^* \setminus \mathfrak{ia}^*) \times t\Omega$ into shells $k-1 < \|\lambda - \mu\| < k$, $k \in \mathbb{N}$, and estimate the corresponding sum-integral on each shell. In this way we bound (4.9) by

$$\sum_{k=1}^{\infty} k^{-N} (m(\partial_k(t\Omega)) + m(B_{R(k+t)}(0) \setminus \mathfrak{ia}^*)) k^r$$

where R is such that $\Omega \subset B_R(0)$. We can now use the condition on $\partial\Omega$, Proposition 4.5 and Corollary 4.6 to infer (4.7).

Next we show that Theorem 1.1 implies Corollary 1.2. Let $\Omega \subset \mathfrak{ia}^*$ be as in Theorem 1.1. If K_f equals the principal congruence subgroup $K_f(N)$ then it follows from (3.8) that

$$\sum_{\substack{\pi \in \Pi_{\text{cus}}(G(\mathbb{A})) \\ \lambda_{\pi_\infty} \in t\Omega}} \dim(\mathcal{H}_\pi^K) = \varphi(N) \sum_{\lambda \in \Lambda_{\text{cus}}(\Gamma(N)), \lambda \in t\Omega} m(\lambda).$$

and

$$\operatorname{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1 / K_f(N)) = \varphi(N) \operatorname{vol}(\Gamma(N) \backslash X)$$

where $\varphi(N) = \#[(\mathbb{Z}/N\mathbb{Z})^*]$. Thus, Theorem 1.1 in this case amounts to Corollary 1.2.

Corollary 1.3 is derived from Corollary 1.2 exactly as in [DKV79, p. 87-88]. Note that the main term in Corollary 1.2 differs from that of [DKV79, Theorem 8.8] by a certain factor $\sigma(G)$ ([ibid., (8.12)]). This is because of the different normalization of measures on X and \mathfrak{ia}^* . This has no effect on Corollary 1.3 where (as in [ibid., (8.33)]) it is assumed that both the volume and Laplacian are with respect to the same Riemannian structure.

Remark 1. All the Propositions in this section carry over, in the obvious way, to groups which are products of $\operatorname{GL}(m)$'s. The proof of Proposition 4.5 depends on the validity of Propositions 4.2 and 4.3. Proposition 4.2 will be proved in the next section. We will prove Proposition 4.3 in §6 below by induction on n . The induction hypothesis, together with the validity of Proposition 4.2 guarantees the validity of Proposition 4.5 for any proper Levi subgroup of G .

5. THE GEOMETRIC SIDE

In this section we study the geometric side J_{geo} of the trace formula and prove Proposition 4.2. By the coarse geometric expansion we can write J_{geo} as a sum of distributions

$$J_{\text{geo}}(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1),$$

parameterized by semisimple conjugacy classes of $G(\mathbb{Q})$. The distribution $J_{\mathfrak{o}}(f)$ is the value at $T = 0$ of the polynomials $J_{\mathfrak{o}}^T(f)$ defined in [Art78]. In particular, following Arthur, we write $J_{\text{unip}}(f)$ for the contribution corresponding to the class of $\{1\}$.

The first step is to show that it suffices to deal with $J_{\text{unip}}(f)$.

Lemma 5.1. *Let $K_f(N)$ be a principal congruence subgroup of level $N \geq 3$. There exists a bi- K_∞ -invariant compact neighborhood ω of $K_f(N)$ in $G(\mathbb{A})^1$ which does not contain any $x \in G(\mathbb{A})$ whose semisimple part is conjugate to a non-unipotent element of $G(\mathbb{Q})$.*

Proof. Taking the coefficients of the characteristic polynomial gives rise to a conjugation invariant algebraic map from G to the affine n -space, and therefore to a continuous map $q : G(\mathbb{A}) \rightarrow \mathbb{A}^n$. Let $T = q(K_\infty \cdot K_f)$. The discreteness of \mathbb{Q}^n implies that there exists a neighborhood ω of $K_\infty \cdot K_f$ such that

$$q(\omega) \cap \mathbb{Q}^n \subseteq T \cap \mathbb{Q}^n.$$

By passing to a smaller neighborhood we can assume that ω is bi- K_∞ -invariant and compact.

To prove the Lemma it suffices to show that

$$(5.1) \quad T \cap \mathbb{Q}^n = \{q(1)\}.$$

Indeed, if the semi-simple part of $x \in \omega$ is conjugate to $\gamma \in G(\mathbb{Q})$ then $q(x) = q(\gamma) \in q(\omega) \cap \mathbb{Q}^n \subseteq T \cap \mathbb{Q}^n$ and therefore by (5.1) $q(\gamma) = q(1)$ so that γ is unipotent.

To show (5.1), let f be a monic polynomial over \mathbb{Q} with coefficients in T . Since $T_p \subseteq \mathbb{Z}_p^n$ for all p , we infer that f has integral coefficients, so that the roots of f are algebraic integers. Moreover, by the condition at ∞ , the roots of f have absolute value 1. It follows from Dirichlet's unit theorem, that the roots of f are roots of unity. Let $g \in K_p(N)$. Then $g - 1 \in NM_n(\mathbb{Z}_p)$ and therefore the roots of the characteristic polynomial of g are congruent to 1 modulo N in the ring of integers of the algebraic closure of \mathbb{Q}_p . Thus, the roots of f are roots of unity which are congruent to 1 modulo N in the ring of integers of the corresponding cyclotomic field. By Serre's lemma ([Mum70, p. 207]) they are all 1, and (5.1) follows. \square

Corollary 5.2. *Let ω be as in Lemma 5.1 and suppose that $f \in C_c^\infty(G(\mathbb{A})^1)$ is supported in ω . Then*

$$J_{\text{geo}}(f) = J_{\text{unip}}(f).$$

Proof. Let $\mathfrak{o} \in \mathcal{O}$. By [Art78, Theorem 8.1] the distribution $J_{\mathfrak{o}}^T(f)$ is given by

$$J_{\mathfrak{o}}^T(f) = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})^1} j_{\mathfrak{o}}^T(x, f) dx,$$

where $j_{\mathfrak{o}}^T(x, f)$ is defined in [ibid., p. 947]. It follows from Lemma 5.1 and the definition of $j_{\mathfrak{o}}^T(x, f)$ that $J_{\mathfrak{o}}^T(f) = 0$ unless \mathfrak{o} corresponds to $\{1\}$. \square

To analyze $J_{\text{unip}}(f)$ we use Arthur's fundamental result ([Art85, Corollaries 8.3 and 8.5]) to express $J_{\text{unip}}(f)$ in terms of weighted orbital integrals. To state the result we recall some facts about weighted orbital integrals. Let S be a finite set of places of \mathbb{Q} containing ∞ . Set

$$\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v, \quad \text{and} \quad G(\mathbb{Q}_S) = \prod_{v \in S} G(\mathbb{Q}_v).$$

Let $M \in \mathcal{L}$ and $\gamma \in M(\mathbb{Q}_S)$. The general weighted orbital integrals $J_M(\gamma, f)$ defined in ([Art88]) are distributions on $G(\mathbb{Q}_S)$. Let

$$G(\mathbb{Q}_S)^1 = G(\mathbb{Q}_S) \cap G(\mathbb{A})^1$$

and write $C_c^\infty(G(\mathbb{Q}_S)^1)$ for the space of functions on $G(\mathbb{Q}_S)^1$ obtained by restriction of functions in $C_c^\infty(G(\mathbb{Q}_S))$. If γ belongs to the intersection of $M(\mathbb{Q}_S)$ with $G(\mathbb{Q}_S)^1$, one can obviously define the corresponding weighted orbital integral as linear form on $C_c^\infty(G(\mathbb{Q}_S)^1)$. Since for $\text{GL}(n)$ all conjugacy classes are stable, the expression of $J_{\text{unip}}(f)$ in terms of weighted orbital integrals simplifies. For $M \in \mathcal{L}$ let $(\mathcal{U}_M(\mathbb{Q}))$ be the (finite) set of unipotent conjugacy classes of $M(\mathbb{Q})$. Then by ([Art85, Corollaries 8.3 and 8.5]) we get

$$(5.2) \quad J_{\text{unip}}(f) = \text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1) f(1) + \sum_{\substack{M \in \mathcal{L} \\ M \neq G}} \sum_{u \in (\mathcal{U}_M(\mathbb{Q}))} a^M(u) J_M(u, f)$$

for $f \in C_c^\infty(G(\mathbb{Q}_S)^1)$, where $a^M(u)$ are certain constants which we fortunately do not need to worry about for the purpose of this paper.

Thus, by (3.3), in order to prove Proposition 4.2 it suffices to show that for every M , $u \in \mathcal{U}_M(\mathbb{Q})$ and $h \in C_c^\infty(\mathfrak{a})$ with $\text{supp } h \subset \omega_0$,

$$(5.3) \quad J_M(u, \mathcal{F}(h)) = O \left(\int_{i\mathfrak{a}^*} |\hat{h}(\lambda)| |\beta(\lambda)| d\lambda \right),$$

and if Ω is bounded with a piecewise C^2 boundary then for all $M \neq G$, $u \in \mathcal{U}_M(\mathbb{Q})$,

$$(5.4) \quad \int_{t\Omega} J_M(u, \mathcal{F}(h_\mu)) d\mu = O(t^{d-1} (\log t)^{\max(r+1, 3)})$$

as $t \rightarrow \infty$.

Before studying the distributions $J_M(u, f)$ we first examine a special case of unipotent orbital integrals. Recall that any parabolic subgroup P of (any reductive group) G has an open orbit on its unipotent radical N_P (acting by conjugation). The corresponding conjugacy class $\mathcal{O}_P = \mathcal{O}_P^G$ in G is a unipotent orbit which is called a Richardson orbit. We have $\dim \mathcal{O}_P = 2 \dim N_P$ ([CM93, Theorem 7.1.1]). For $G = \text{GL}(n)$, all orbits are

Richardson ([ibid., Theorem 7.2.3]) and $G(F)$ (resp. $P(F)$) acts transitively on $\mathcal{O}_P(F)$ (resp. $N_P(F) \cap \mathcal{O}_P(F)$) over any field F .

The following Lemma is probably well-known but we were unable to locate a convenient reference in the literature.

Lemma 5.3. *Let $P = MN$ be a parabolic subgroup of $G = \mathrm{GL}(n)$. Then the orbital integral of $f \in C_c^\infty(G(\mathbb{Q}_S))$ along $\mathcal{O}_P(\mathbb{Q}_S)$ is given by*

$$\int_{K_S} \int_{N(\mathbb{Q}_S)} f(k^{-1}uk) \, du \, dk$$

where K_S is the maximal compact subgroup of $G(\mathbb{Q}_S)$.

Proof. Fix a representative $u \in \mathcal{O}_P(\mathbb{Q}_S) \cap N(\mathbb{Q}_S)$. Then, denoting by C_g^H the centralizer of g in H , we have

$$\dim C_u^G = \dim G - \dim \mathcal{O}_P = \dim G - 2 \dim N = \dim P - \dim N = \dim C_u^P.$$

We can therefore write the orbital integral as

$$\int_{K_S} \int_{C_u^P(\mathbb{Q}_S) \backslash P(\mathbb{Q}_S)} f(k^{-1}p^{-1}upk) \delta_P(p)^{-1} \, dp \, dk.$$

The map $p \mapsto p^{-1}up$ is a submersion from $P(\mathbb{Q}_S)$ to $N(\mathbb{Q}_S)$. Therefore the functional

$$g \in C_c^\infty(N(\mathbb{Q}_S)) \mapsto \int_{C_u^P(\mathbb{Q}_S) \backslash P(\mathbb{Q}_S)} g(p^{-1}up) \delta(p^{-1}) \, dp$$

is absolutely continuous with respect to the Haar measure of $N(\mathbb{Q}_S)$. Since the orbit of u under $P(\mathbb{Q}_S)$ is dense in $N(\mathbb{Q}_S)$, the only absolutely continuous measure μ on $N(\mathbb{Q}_S)$ satisfying $\mu(\mathrm{Ad}(p)f) = \delta(p)\mu(f)$ is the Haar measure. The Lemma follows. \square

We fix an open compact subgroup $K_f \subseteq G(\mathbb{A}_f)$ with $K_f \subset K(N)$, $N \geq 3$ and put

$$K = K_\infty \cdot K_f.$$

Let ω be as in Lemma 5.1. Note that there exists a neighborhood ω_0 of 0 in \mathfrak{a} such that if $h \in C_c^\infty(\mathfrak{a})^W$ is supported in ω_0 then $f = \mathcal{F}(h)$ is supported in ω . Let S be a sufficiently large set of places containing ∞ and the primes dividing N .

The distributions $J_M(\cdot, f)$ are based on the usual orbital integrals studied by Harish-Chandra and Ranga-Rao. Unlike in the case $n = 2$, their Fourier transforms are not explicit in general, so we cannot argue as in §2 (cf. (2.3)). Instead, following Arthur, we write $J_M(\cdot, f)$ in terms of weighted orbital integrals, namely as the expression on top of p. 256 in [Art88]². In the case where γ is unipotent, that is, $\sigma = 1$ in the notation of [ibid.], the formula simplifies since the outer integration disappears and only $Q = G$ contributes. (Note that the function v_Q^Q vanishes on K unless $Q = G$.) The derivation of [Art88, p. 256] is based on an expression of the invariant orbital integral in $M(\mathbb{Q}_S)$ and in particular on the formula on [ibid., p. 246]. In our case all unipotent orbits of M are

²Note the following typo: v_M^Q should be replaced by w_M^Q .

stable and are of Richardson type. Therefore, any unipotent u of $M(\mathbb{Q})$ belongs to $\mathcal{O}_{Q \cap M}^M$ for some parabolic Q of G and we can use Lemma 5.3 instead of [loc. cit.] to simplify the expression for $J_M(u, f)$ to

$$J_M(u, f) = \int_K \int_U f(k^{-1}xk)w_M(x) dx$$

where U is an appropriate compact subset of $N_Q(\mathbb{Q}_S)$ (depending on the support of f) and the weight function $w_M(x)$ is described in [ibid., Lemma 5.4]: it is a finite linear combination of functions of the form $\prod_{i=1}^r \log \|p_i(x_{v_i})\|_{v_i}$ where p_i are polynomials on N into an affine space, $v_i \in S$, $i = 1, \dots, r$ (not necessarily distinct) and

$$\|(y_1, \dots, y_m)\|_v = \begin{cases} \max(|y_1|_v, \dots, |y_m|_v), & v < \infty, \\ |y_1|_v^2 + \dots + |y_m|_v^2, & v = \infty. \end{cases}$$

(The fact that the product is over r terms is implicit in [loc. cit.] but follows from the proof.) Using the Plancherel theorem, and viewing ϕ_λ as a function on $N_Q(\mathbb{Q}_S)$ depending only on the Archimedean component, we have

$$J_M(u, \mathcal{F}(h)) = \int_{\mathfrak{ia}^*} \hat{h}(\lambda)\beta(\lambda) \left(\int_U w_M(x)\psi(x)\phi_{-\lambda}(x) dx \right) d\lambda$$

where $\psi(x) = \int_{K_S} \mathbf{1}_{K_f}(k^{-1}xk) dk$. Since $|\phi_\lambda| \leq 1$, the inner integral is majorized by

$$\int_U |w_M(x)| dx,$$

which converges by [Art88, Lemma 6.1], and (5.3) follows.

Now, by a change of variable we can write

$$J_M(u, \mathcal{F}(h_\mu)) = \int_{\mathfrak{ia}^*} \hat{h}(\lambda)\beta(\lambda + \mu) \left(\int_U w_M(x)\psi(x)\phi_{-\lambda-\mu}(x) dx \right) d\lambda.$$

It follows from [ibid., Lemma 7.1] that for some $a > 0$ we have

$$\int_{\{n \in U : |w_M(x)| > aR\}} |w_M(x)| dx = O\left(e^{-R^{1/r}}\right).$$

Taking $R = (\log t)^r$ we get

$$\int_U w_M(x)\psi(x)\phi_{-\lambda-\mu}(x) dx = \int_{U_t} w_M(x)\psi(x)\phi_{-\lambda-\mu}(x) dx + O(1/t)$$

where we set $U_t = \{x \in U : |w_M(x)| < a(\log t)^r\}$. Therefore, using (3.3) and (3.6) we obtain

$$J_M(u, \mathcal{F}(h_\mu)) = \int_{\mathfrak{ia}^*} \hat{h}(\lambda)\beta(\lambda + \mu) \left(\int_{U_t} w_M(x)\psi(x)\phi_{-\lambda-\mu}(x) dx \right) d\lambda + O\left((1 + \|\mu\|^{d-r})/t\right).$$

In order to prove (5.4), we use Harish-Chandra's formula for the spherical function (3.1) to write $J_M(u, \mathcal{F}(h_\mu))$, up to an error term of $O((1 + \|\mu\|^{d-r})/t)$ as

$$\int_{\mathfrak{ia}^*} \hat{h}(\lambda) \int_{K_\infty^0} \int_{U_t} e^{\langle -\lambda + \rho, H(kx) \rangle} w_M(x) \psi(x) \beta(\lambda + \mu) e^{-\langle \mu, H(kx) \rangle} dx dk d\lambda.$$

Integrating over $t\Omega$ and interchanging the order of integration, we obtain that up to an error of order $O(t^{d-r-1})$, the left-hand side of (5.4) is equal to

$$t^r \int_{\mathfrak{ia}^*} \hat{h}(\lambda) \int_{K_\infty^0} \int_{U_t} e^{\langle -\lambda + \rho, H(kx) \rangle} w_M(x) \psi(x) \int_\Omega \beta(\lambda + t\mu) e^{-t\langle \mu, H(kx) \rangle} d\mu dx dk d\lambda.$$

We first estimate the integral over Ω . The trivial estimate using (3.3) and (3.6) gives $O((\|\lambda\| + t)^{d-r})$. To go further, fix a regular $\xi \in \mathfrak{a}^*$ and let ν be the outward pointing normal vector field at the boundary $\partial\Omega$. Let $k \in K_\infty^0$ and $x \in N(\mathbb{R})$ be such that $\langle \xi, H(kx) \rangle \neq 0$. Then by the divergence theorem, we get

$$\begin{aligned} \int_\Omega \beta(\lambda + t\mu) e^{-t\langle \mu, H(kx) \rangle} d\mu &= \frac{-1}{t \langle \xi, H(kx) \rangle} \int_\Omega \beta(\lambda + t\mu) D_\xi e^{-t\langle \mu, H(kx) \rangle} d\mu = \\ &= \frac{1}{\langle \xi, H(kx) \rangle} \int_\Omega D_\xi \beta(\lambda + t\mu) e^{-t\langle \mu, H(kx) \rangle} d\mu - \frac{1}{t \langle \xi, H(kx) \rangle} \int_{\partial\Omega} \beta(\lambda + t\sigma) e^{-t\langle \sigma, H(kx) \rangle} \langle \nu(\sigma), \xi \rangle d\sigma. \end{aligned}$$

We estimate the integrands on the right-hand side by (3.5) and (3.3) respectively. We obtain

$$\begin{aligned} \int_\Omega (D_\xi \beta)(\lambda + t\mu) e^{-t\langle \mu, H(kx) \rangle} d\mu &= O(1 + \|\lambda\|^{d-r-1} + t^{d-r-1}), \\ \int_{\partial\Omega} \beta(\lambda + t\sigma) e^{-t\langle \sigma, H(kx) \rangle} \langle \nu(\sigma), \xi \rangle d\sigma &= O(1 + \|\lambda\|^{d-r} + t^{d-r}). \end{aligned}$$

Thus,

$$\int_\Omega \beta(\lambda + t\mu) e^{-t\langle \mu, H(kx) \rangle} d\mu = O\left(\frac{1 + \|\lambda\|^{d-r} + t^{d-r-1}}{|\langle \xi, H(kx) \rangle|}\right).$$

Set $F(k, x) = \langle \xi, H(kx) \rangle$ on $K_\infty^0 \times U$ and for any $\epsilon > 0$ let $V_{<\epsilon} = \{(k, x) \in K_\infty^0 \times U : |F(k, x)| < \epsilon\}$. Similarly for $V_{\geq\epsilon}$.

It follows that the left-hand side of (5.4) is majorized by

$$(5.5) \quad O(t^d (\log t)^r) \int_{\mathfrak{a}^*} |\hat{h}(\lambda)| (1 + \|\lambda\|)^{d-r} d\lambda \left(\text{vol}(V_{<\epsilon}) + \int_{V_{\geq\epsilon}} \frac{dy}{t|F(y)|} \right)$$

uniformly for $\epsilon > 0$.

Next, we study the critical points of F on $K_\infty^0 \times N_Q(\mathbb{R})$. Let $K_L = K_\infty^0 \cap Q(\mathbb{R})$. Note that K_L acts on $K_\infty^0 \times N_Q(\mathbb{R})$ by the free action $m(k, x) = (km^{-1}, mxm^{-1})$, and that F is invariant under this action. The following Lemma (and its proof) hold for any real reductive group.

Lemma 5.4. *F is a Morse function on $K_L \backslash (K_\infty^0 \times N_Q(\mathbb{R}))$. Its critical points are the cosets $K_L(w, 1)$, $w \in W$.*

Proof. Let \mathfrak{g}_1 be the Lie Algebra of $G_1(\mathbb{R}) = \mathrm{SL}(n, \mathbb{R})$, and let $\mathfrak{g}_1 = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. We will write $\langle \cdot, \cdot \rangle$ for the Killing form of \mathfrak{g}_1 and use it to identify \mathfrak{a}^* with \mathfrak{a} . Suppose that (k, x) is a critical point. In particular, k is a critical point for the function $F(\cdot, x)$. By [DKV83, Lemma 5.3] this means that $kx \in AK_\infty^0$. Let \mathfrak{n}_Q denote the Lie algebra of $N_Q(\mathbb{R})$ and we use the notation F_X , $X \in \mathfrak{k} \oplus \mathfrak{n}_Q$ for the X -directional derivative. Then for any $Y \in \mathfrak{n}_Q$

$$0 = F_Y(k, x) = \frac{d}{dt} \langle \xi, H(k \exp(tY)x) \rangle \Big|_{t=0} = \langle \xi, H(\cdot) \rangle_{\mathrm{Ad}(k)Y} (kx),$$

where the latter denotes the directional derivative along $\mathrm{Ad}(k)Y$ of the function $\langle \xi, H(\cdot) \rangle$. By [DKV83, Corollary 5.2] and the fact that $kx \in AK_\infty^0$ we get

$$0 = \langle \xi, \mathrm{Ad}(k)Y \rangle.$$

Thus, $X := \mathrm{Ad}(k^{-1})\xi \in \mathfrak{n}_Q^\perp \cap \mathfrak{p} = \mathfrak{m}_Q \cap \mathfrak{p}$. Therefore, there exists $k' \in K_L$ such that $\xi' = \mathrm{Ad}(k')X \in \mathfrak{a}$. In particular, $\xi' \in \mathrm{proj}_{\mathfrak{a}}(\mathrm{Ad}(K_\infty^0)^{-1}\xi)$ and $\|\xi'\| = \|\xi\|$. On the other hand, by Kostant's convexity theorem [Hel00, p. 473] it follows that $\mathrm{proj}_{\mathfrak{a}}(\mathrm{Ad}(K_\infty^0)^{-1}\xi)$ equals to the convex hull of $w\xi$, $w \in W$. We infer that $\xi' = w^{-1}\xi$ for some w . Therefore, $k \in C_K(\xi)wk'$. Since ξ is regular, $C_K(\xi) = C_K(A) \subseteq K_L$ and thus $k \in WK_L$. Write $kx = ak_1$ and $k = wk_2$ with $a \in A$ and $k_1 \in K_\infty^0$ and $k_2 \in K_L$. The equality $ak_1 = wk_2x$ gives $w^{-1}k_1 = a'x'k_2$ where $a' = w^{-1}a^{-1}w \in A$ and $x' = k_2xk_2^{-1} \in N_Q(\mathbb{R})$. By the uniqueness of the Iwasawa decomposition we obtain $x' = 1$ and therefore $n = 1$.

Consider the Hessian as a symmetric bilinear form on the tangent space $\mathfrak{k}_L \backslash \mathfrak{k} \oplus \mathfrak{n}_Q$. Note that the Killing form defines a perfect pairing on $\mathfrak{n}_Q \times \mathfrak{k}_L \backslash \mathfrak{k}$. Since $F(wk, 1) = 1$ for all k , $\mathfrak{k}_L \backslash \mathfrak{k}$ is totally isotropic for the Hessian. For $Y \in \mathfrak{n}_Q$ we get as above

$$F_Y(wk, 1) = \frac{d}{dt} \langle \xi, H(wke^{tY}) \rangle \Big|_{t=0} = \frac{d}{dt} \langle \xi, H(e^{t\mathrm{Ad}(wk)Y}) \rangle \Big|_{t=0} = \langle \xi, \mathrm{Ad}(wk)Y \rangle.$$

Thus, for $X \in \mathfrak{k}_L \backslash \mathfrak{k}$ we get

$$F_{XY}(w, 1) = \langle \xi, \mathrm{Ad}(w) \mathrm{ad}(X)Y \rangle = \langle \mathrm{Ad}(w)^{-1}\xi, \mathrm{ad}(X)Y \rangle = \langle \mathrm{ad}(\mathrm{Ad}(w)^{-1}\xi)Y, X \rangle.$$

Hence, the Hessian is not singular since $\mathrm{ad}(\mathrm{Ad}(w)^{-1}\xi)$ is non-singular on \mathfrak{n}_Q . \square

The estimation (5.4) will follow from (5.5) by taking $\epsilon = 1/t$ and using the following Lemma applied to F . (Note that $\dim \mathfrak{n}_Q > 1$ unless $n = 2$.)

Lemma 5.5. *Let X be an orientable manifold of dimension $m \geq 2$ with a nowhere vanishing differential m -form, and let dx be the corresponding measure. Let f be a Morse function on X . Then for any compact $Y \subseteq X$*

$$\begin{aligned} \mathrm{vol}\{x \in Y : |f(x)| < \delta\} &= O(\delta(-\log \delta)^\eta) \\ \int_{\{x \in Y : |f(x)| \geq \delta\}} \frac{1}{|f(x)|} dx &= O((-\log \delta)^{1+\eta}) \end{aligned}$$

for $\delta < \frac{1}{2}$, where $\eta = 1$ if $m = 2$ and $\eta = 0$ otherwise.

Proof. The question is local. Therefore, by the Morse Lemma it suffices to consider the following 3 cases.

- (1) $f \neq 0$ on Y .
- (2) $X = \mathbb{R}^m$ and $f(\underline{x}) = x_1$.
- (3) $X = \mathbb{R}^m$, $m = p + q$ and

$$f(\underline{x}, \underline{y}) = |\underline{x}|^2 - |\underline{y}|^2 \quad \underline{x} \in \mathbb{R}^p, \underline{y} \in \mathbb{R}^q.$$

The first case is trivial. In the second case the volume is bounded by δ while the integral is bounded by $-\log \delta$. Consider the last case. If $pq = 0$ the volume is the volume of a ball of radius $\sqrt{\delta}$ which is $O(\delta^{m/2})$. The integral reduces to

$$\int_{\underline{x} \in \mathbb{R}^m: \delta \leq |\underline{x}|^2 \leq 1} \frac{1}{|\underline{x}|^2} d\underline{x} = O\left(\int_{\delta^2}^1 r^{m-1} r^{-2} dr\right) = O(-\log \delta).$$

For $pq > 0$ the volume of the intersection of the unit disc with

$$\{(\underline{x}, \underline{y}) : \underline{x} \in \mathbb{R}^p, \underline{y} \in \mathbb{R}^q, ||\underline{x}|^2 - |\underline{y}|^2| < \delta\}$$

is given by a constant multiple of

$$\int_{0 \leq r, s \leq 1: |r^2 - s^2| < \delta} r^{p-1} s^{q-1} dr ds.$$

If $p = q = 1$ then this is majorized by

$$\int_{0 \leq x, y \leq 1: xy < \delta} dx dy = \delta - \delta \log \delta.$$

Otherwise, it is majorized by

$$\int_{0 \leq x, y \leq 1: xy < \delta} x + y dx dy \leq \delta$$

Similarly, the integral reduces to

$$\int_{\substack{|\underline{x}|, |\underline{y}| \leq 1 \\ ||\underline{x}|^2 - |\underline{y}|^2| \geq \delta}} \frac{1}{||\underline{x}|^2 - |\underline{y}|^2|} d\underline{x} d\underline{y}.$$

Using polar coordinates for $\underline{x}, \underline{y}$, we obtain

$$\int_{\substack{0 \leq r, s \leq 1 \\ |r^2 - s^2| \geq \delta}} \frac{r^{p-1} s^{q-1}}{|r^2 - s^2|} dr ds.$$

For $p = q = 1$ this is majorized by

$$\int_{\substack{0 \leq x, y \leq 1 \\ xy \geq \delta}} \frac{1}{xy} dx dy = \int_{\delta}^1 \left(\int_{\delta/x}^1 \frac{dy}{y} \right) \frac{dx}{x} = \int_{\delta}^1 (\log x - \log \delta) \frac{dx}{x} = \frac{1}{2} (\log \delta)^2.$$

Otherwise, it is majorized by

$$\int_{\substack{0 \leq x, y \leq 1 \\ xy \geq \delta}} \frac{x+y}{xy} dx dy = 2 \int_{\delta}^1 \left(\int_{\delta/x}^1 \frac{dy}{y} \right) dx = 2 \int_{\delta}^1 (\log x - \log \delta) dx \leq -2 \log \delta$$

as required. \square

This concludes the proof of Proposition 4.2.

6. THE SPECTRAL SIDE

In this section we analyze the spectral side of the trace formula for $G = \mathrm{GL}(n)$, following [Mül07]. By [MS04, Theorem 0.1], the spectral side J_{spec} is given as a finite linear combination

$$J_{\mathrm{spec}}(f) = J_{\mathrm{dis}}(f) + \sum_{M \in \mathcal{L}: M \neq G} \sum_{s \in W(\mathfrak{a}_M)} a_{M,s} J_{M,s}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1),$$

of distributions J_{dis} and $J_{M,s}$. To describe these distributions, we need to introduce some more notation. Fix $M \in \mathcal{L}$ and let $\mathcal{P}(M)$ be the set of parabolic subgroups of G (necessarily defined over \mathbb{Q}) for which M is a Levi component. For $P \in \mathcal{P}(M)$, we write $\mathfrak{a}_P = \mathfrak{a}_M$ and let $\mathcal{A}^2(P)$ be the space of automorphic forms on $N_P(\mathbb{A})M_P(\mathbb{Q}) \backslash G(\mathbb{A})$ which are square integrable on $M_P(\mathbb{Q}) \backslash M_P(\mathbb{A})^1 \times K$ [Art82a, p. 1249]. Given $Q \in \mathcal{P}(M)$ let, $W(\mathfrak{a}_P, \mathfrak{a}_Q)$ be the set of all linear isomorphisms from \mathfrak{a}_P to \mathfrak{a}_Q which are restrictions of elements of W . For $s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$ let

$$M_{Q|P}(s, \lambda) : \mathcal{A}^2(P) \rightarrow \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}_{P,\mathbb{C}}^*,$$

be the intertwining operator [Art82b, §1], which is a meromorphic function of $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$. Set

$$M_{Q|P}(\lambda) := M_{Q|P}(1, \lambda).$$

Fix $P \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{ia}_M^*$. For $Q \in \mathcal{P}(M)$ and $\Lambda \in \mathfrak{ia}_M^*$ define

$$\mathfrak{M}_Q(P, \lambda, \Lambda) = M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda).$$

Then

$$(6.1) \quad \{\mathfrak{M}_Q(P, \lambda, \Lambda) : \Lambda \in \mathfrak{ia}_M^*, Q \in \mathcal{P}(M)\}$$

is a (G, M) family with values in the space of operators on $\mathcal{A}^2(P)$ ([ibid., p. 1310]). Let $L \in \mathcal{L}$ with $L \supset M$. Then the (G, M) family (6.1) has an associated (G, L) family

$$\{\mathfrak{M}_{Q_1}(P, \lambda, \Lambda) : \Lambda \in \mathfrak{ia}_L^*, Q_1 \in \mathcal{P}(L)\}$$

[ibid., p. 1297]. For a parabolic group P let

$$\theta_P(\lambda) = \mathrm{vol}(\mathfrak{a}_P / \mathbb{Z}(\Delta_P^\vee))^{-1} \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee), \quad \lambda \in \mathfrak{ia}_P^*,$$

where $\mathbb{Z}(\Delta_P^\vee)$ is the lattice in \mathfrak{a}_P generated by the co-roots α^\vee , $\alpha \in \Delta_P$. Then

$$\mathfrak{M}_L(P, \lambda, \Lambda) = \sum_{Q_1 \in \mathcal{P}(L)} \mathfrak{M}_{Q_1}(P, \lambda, \Lambda) \theta_{Q_1}(\Lambda)^{-1}$$

extends to a smooth function on \mathfrak{ia}_L^* . In particular, set

$$\mathfrak{M}_L(P, \lambda) := \mathfrak{M}_L(P, \lambda, 0) = \lim_{\substack{\Lambda \in \mathfrak{ia}_L^* \\ \Lambda \rightarrow 0}} \left(\sum_{Q_1 \in \mathcal{P}(L)} \text{vol}(\mathfrak{a}_{Q_1}/\mathbb{Z}(\Delta_{Q_1}^\vee)) M_{Q_1|P}(\lambda)^{-1} \frac{M_{Q_1|P}(\lambda + \Lambda)}{\prod_{\alpha \in \Delta_{Q_1}} \Lambda(\alpha^\vee)} \right),$$

where for each $Q_1 \in \mathcal{P}(L)$, Q is a group in $\mathcal{P}(M_P)$ which is contained in Q_1 . Let $\rho(P, \lambda)$ be the induced representation of $G(\mathbb{A})$ on $\overline{\mathcal{A}}^2(P)$. Then the distribution $J_{M,s}$ is given by

$$(6.2) \quad J_{M,s}(f) = \int_{\mathfrak{ia}_L^*} \text{tr}(\mathfrak{M}_L(P, \lambda) M_{P|P}(s, 0) \rho(P, \lambda, f)) d\lambda$$

where L is the minimal Levi containing M and s and P is any element of $\mathcal{P}(M)$. By [MS04, Theorem 0.1] this integral is absolutely convergent with respect to the trace norm. (See [FLM] for a more general result, where the independence of the above expression on P is also explained.) Implicit here is that $\mathfrak{M}_L(P, \lambda) M_{P|P}(s, 0) \rho(P, \lambda, f)$ extends to a trace-class operator on $\overline{\mathcal{A}}^2(P)$ for almost all λ . Finally, $J_{\text{dis}}(f)$ is defined to be $J_{G,1}(f)$.

We will now show Proposition 4.3, arguing as in [Mül07, §5]. Fix K_f and let $K = K_\infty K_f$. We can assume without loss of generality that $h \in C_c^\infty(\mathfrak{a})^W$. Let $f = \mathcal{F}(h)$. We will expand (6.2) according to $\pi \in \Pi_{\text{dis}}(M(\mathbb{A})^1)$. For each such $\pi = \pi_\infty \otimes \pi_f$ let $\mathcal{A}_\pi^2(P)$ be the subspace of $\mathcal{A}^2(P)$ of functions ϕ such that for each $x \in G(\mathbb{A})$, the function $\phi_x(m) := \phi(mx)$, $m \in M_P(\mathbb{A})$, transforms under $M_P(\mathbb{A})$ according to the representation π . Let $M_{Q_1|P}(s, \pi, \lambda)$ and $\mathfrak{M}_L(P, \pi, \lambda)$ denote the restriction of $M_{Q_1|P}(s, \lambda)$ and $\mathfrak{M}_L(P, \lambda)$, respectively, to $\mathcal{A}_\pi^2(P)$. Let $\mathcal{A}_P^2(\pi)^K$ the subspace of K -invariant functions. This is a finite-dimensional space. Let Π_K be the orthogonal projection of $\overline{\mathcal{A}_\pi^2(P)}$ onto $\mathcal{A}_P^2(\pi)^K$. Note that $\mathcal{A}_P^2(\pi)^K = 0$, unless the induced representation $I_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_\infty)$ has a non-zero K_∞ -fixed vector. Let $\lambda_{\pi_\infty} \in (\mathfrak{a}^M)_\mathbb{C}^*/W_M$ be the infinitesimal character of π_∞ . Then

$$\rho(P, \lambda, f) = \hat{h}(\lambda_{\pi_\infty} + \lambda) \Pi_K$$

on $\mathcal{A}_P^2(\pi)$ and by (6.2) we get

$$(6.3) \quad J_{M,s}(f) = \sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A})^1)} \int_{\mathfrak{ia}_L^*} \hat{h}(\lambda_{\pi_\infty} + \lambda) \text{tr}(\mathfrak{M}_L(P, \pi, \lambda) M_{P|P}(s, \pi, 0) \Pi_K) d\lambda.$$

In particular, for $M = G$ we obtain (4.3) since $L_{\text{dis}}^2(G(\mathbb{Q}) \backslash G(\mathbb{A})^1)$ is multiplicity free,

To deal with the terms $M \neq G$ we first recall the normalized intertwining operators. The normalizing factors $r_{Q_1|P}(\pi, \lambda)$, described in [Mül07, §5], are given by

$$r_{Q_1|P}(\pi, \lambda) = \prod_{\alpha \in \Sigma_P \cap \Sigma_{\overline{Q}}} r_\alpha(\pi, \langle \lambda, \alpha^\vee \rangle),$$

where each $r_\alpha(\pi, \cdot)$ is a meromorphic function on \mathbb{C} which is given in terms of Rankin-Selberg L -functions. More precisely, if M has the form

$$M = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_r)$$

with $n = n_1 + \cdots + n_r$ then we can write $\pi = \pi_1 \otimes \cdots \otimes \pi_r$, where $\pi_i \in \Pi_{\mathrm{dis}}(\mathrm{GL}(n_i, \mathbb{A}))$, and a root α corresponds to an ordered pair (i, j) of distinct integers between 1 and r . The normalizing factor $r_\alpha(\pi, s)$ is then given by

$$r_\alpha(\pi, s) = \frac{L(s, \pi_i \times \tilde{\pi}_j)}{L(1+s, \pi_i \times \tilde{\pi}_j)\epsilon(s, \pi_i \times \tilde{\pi}_j)},$$

where $L(s, \pi_i \times \tilde{\pi}_j)$ is the global Rankin-Selberg L -function and $\epsilon(s, \pi_i \times \tilde{\pi}_j)$ is the corresponding ϵ -factor.

The normalized intertwining operator is defined by

$$N_{Q|P}(\pi, \lambda) := r_{Q|P}(\pi, \lambda)^{-1} M_{Q|P}(\pi, \lambda), \quad \lambda \in \mathfrak{a}_{M, \mathbb{C}}^*.$$

For a Levi subgroup L let $\mathcal{F}(L)$ be the set of all parabolic subgroups containing L . Using the basic properties of (G, M) families ([Art82b, p. 1329]), one gets

$$\mathfrak{M}_L(P, \pi, \lambda) = \sum_{S \in \mathcal{F}(L)} \mathfrak{N}'_S(P, \pi, \lambda) \nu_L^S(P, \pi, \lambda),$$

where the operator $\mathfrak{N}'_S(P, \pi, \lambda)$ is obtained from the (G, L) family attached to $N_{Q|P}(\pi, \lambda)$ (cf. [Mül07, (6.10)]) and $\nu_L^S(P, \pi, \lambda)$ is defined in terms of the normalizing factors (see below).

Therefore, by (6.3) $J_{M,s}(f)$ equals

$$\sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} \sum_{S \in \mathcal{F}(L)} \int_{\mathfrak{ia}_L^*} \hat{h}(\lambda_{\pi_\infty} + \lambda) \nu_L^S(P, \pi, \lambda) \mathrm{tr}(M_{P|P}(s, \pi, 0) \mathfrak{N}'_S(P, \pi, \lambda)_K) d\lambda.$$

where $\mathfrak{N}'_S(P, \pi, \lambda)_K$ denotes the restriction of $\mathfrak{N}'_S(P, \pi, \lambda)$ to $\mathcal{A}_\pi^2(P)^K$. By [Mül07, (6.13)]

$$\|\mathfrak{N}'_S(P, \pi, \lambda)_K\| = O_K(1)$$

for all $\lambda \in \mathfrak{ia}_M^*$ and $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)$. Since $M_{P|P}(s, \pi, 0)$ is unitary, $J_{M,s}(f)$ is bounded by a constant multiple of

$$(6.4) \quad \sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A})^1)} \sum_{S \in \mathcal{F}(L)} \dim(\mathcal{A}_\pi^2(P)^K) \int_{\mathfrak{ia}_L^*} |\hat{h}(\lambda_{\pi_\infty} + \lambda)| \cdot |\nu_L^S(P, \pi, \lambda)| d\lambda.$$

The function $\nu_L^S(P, \pi, \lambda)$ can be described as follows. If F is a subset of $\Sigma(G, A_M)$, let $F_L^\vee = \{\alpha_L^\vee : \alpha \in F\}$ (as a multiset). Suppose that $S \in \mathcal{P}(L_1)$ with $L_1 \supset L$. Then by [Art82b, Proposition 7.5] we have

$$\nu_L^S(P, \pi, \lambda) = \sum_F \mathrm{vol}(\mathfrak{a}_L^{L_1} / \mathbb{Z}(F_L^\vee)) \cdot \prod_{\alpha \in F} \frac{r'_\alpha(\pi, \langle \lambda, \alpha^\vee \rangle)}{r_\alpha(\pi, \langle \lambda, \alpha^\vee \rangle)}$$

where F runs over all subsets of $\Sigma(L_1, A_M)$ such that F_L^V is a basis of $\mathfrak{a}_L^{L^1}$. The argument of [Mül07, Proposition 5.1] in conjunction with [ibid., Lemmas 5.3 and 5.4] gives

$$\int_T^{T+1} \left| \frac{r'(\pi_1 \otimes \pi_2, it)}{r(\pi_1 \otimes \pi_2, it)} \right| dt = O(\log(|T| + \|\lambda_{\pi_1, \infty}\| + \|\lambda_{\pi_2, \infty}\| + 2))$$

for all $\pi_i \in \Pi_{\text{dis}}(\text{GL}(m_i, \mathbb{A}))$, $i = 1, 2$ and any $T \in \mathbb{R}$. Therefore,

$$(6.5) \quad \int_{B(\mu) \cap \mathfrak{ia}_L^*} |\nu_L^S(P, \pi, \lambda)| d\lambda = O(\log^l(\|\lambda_{\pi_\infty} + \mu\| + 2))$$

for any $\pi \in \Pi_{\text{dis}}(M(\mathbb{A})^1)$ and $\mu \in \mathfrak{ia}_L^*$ where $l = \dim \mathfrak{a}_L^*$. On the other hand, upon replacing K_f by an open subgroup which is normal in $K_f(1) = \prod_{p < \infty} \text{GL}(n, \mathbb{Z}_p)$ we can use ([Mül07, (3.5) and (3.7)]) to obtain

$$(6.6) \quad \sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A})^1): \lambda_{\pi_\infty} = \lambda} \dim(\mathcal{A}_\pi^2(P)^K) \leq [K_f(1) : K_f] m_M^{K_f \cap M(\mathbb{A}_f)}(\lambda)$$

for any $\lambda \in (\mathfrak{a}_\mathbb{C}^M)^*$ where $m_M^{K_f \cap M(\mathbb{A}_f)}$ measures the multiplicity with respect to the locally symmetric space $M(F) \backslash M(\mathbb{A})^1 / (K_f \cap M(\mathbb{A}_f))$.

We are now ready to prove Proposition 4.3 by induction on n . For $\mu \in (\mathfrak{a}_\mathbb{C}^M)^*$ let $B^M(\mu)$ be the ball in $(\mathfrak{a}_\mathbb{C}^M)^*$ of radius 1 with center μ . Using (6.6) and taking into account Remark 1 (end of section 4) we can use the induction hypothesis to apply Proposition 4.5 to M to infer that

$$(6.7) \quad \sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A})^1): \text{Im } \lambda_{\pi_\infty} \in B^M(\mu)} \dim(\mathcal{A}_\pi^2(P)^K) = O(\tilde{\beta}^M(\mu))$$

for any $\mu \in (\mathfrak{a}^M)^*$.

Let $\mathfrak{a}_M^L = \mathfrak{a}_M \cap \mathfrak{a}^L$ and denote by $(\mathfrak{a}_M^L)^\perp = (\mathfrak{a}^M)^* \oplus \mathfrak{a}_L^*$ its annihilator in \mathfrak{a}^* . Let \mathcal{Z} be a lattice in $\mathfrak{i}(\mathfrak{a}_M^L)^\perp$ such that the balls $B(\mu) \cap \mathfrak{i}(\mathfrak{a}_M^L)^\perp$, $\mu \in \mathcal{Z}$ cover $\mathfrak{i}(\mathfrak{a}_M^L)^\perp$. We estimate (6.4) by splitting the sum and the integral to the sets $\text{Im } \lambda_{\pi_\infty} + \lambda \in B(\mu) \cap \mathfrak{i}(\mathfrak{a}_M^L)^\perp$, $\mu \in \mathcal{Z}$, and recalling that $\text{Re } \lambda_{\pi_\infty} \leq \|\rho^M\|$. We obtain that (6.4) is bounded by

$$\sum_{\mu \in \mathcal{Z}} \max_{B_{1+\|\rho^M\|}(\mu)} |\hat{h}| \sum_{S \in \mathcal{F}(L)} \sum_{\substack{\pi \in \Pi_{\text{dis}}(M(\mathbb{A})^1) \\ \text{Im } \lambda_{\pi_\infty} \in B^M(\mu^M)}} \dim(\mathcal{A}_\pi^2(P)^K) \int_{B(\mu_L) \cap \mathfrak{ia}_L^*} |\nu_L^S(P, \pi, \lambda)| d\lambda.$$

By (6.5) and (6.7) this is majorized by

$$(6.8) \quad \sum_{\mu \in \mathcal{Z}} \max_{B_{1+\|\rho^M\|}(\mu)} |\hat{h}| \tilde{\beta}^M(\mu^M) \log^l(\|\mu\| + 2).$$

Using Lemma 3.1, the series can be estimated by $O(\mathcal{N}(h))$. Thus, $J_{M,s}(\mathcal{F}(h)) = O(\mathcal{N}(h))$. This gives the first part of Proposition 4.3. To show the second part note that for $M \neq G$ we have

$$\tilde{\beta}^M(\mu^M) = \begin{cases} 1, & \text{if } n = 2; \\ O((\|\mu\| + 1)^{d-r-2}), & \text{otherwise.} \end{cases}$$

Thus, by (6.8) we have

$$\int_{t\Omega} J_{M,s}(\mathcal{F}(h_\mu)) d\mu = \begin{cases} O_{h,\Omega}(t^{d-2} \log^r(t+2)), & \text{if } n > 2, \\ O_{h,\Omega}(t^{d-1} \log(t+2)), & \text{if } n = 2. \end{cases}$$

This concludes the proof of Proposition 4.3.

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