Analytic Torsion

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Another milestone in the work of Singer is the invention of the analytic torsion [RS] which he together with Ray introduced in 1971 as an analytic counterpart of the Reidemeister torsion. The Reidemeister torsion (or the Reidemeister-Franz torsion) is a topological invariant of a compact manifold M and a representation of its fundamental group $\pi_1(M)$, which was introduced by Reidemeister in 1935 for 3-manifolds and generalized to higher dimensions by Franz and de Rham. In fact, the Reidemeister torsion is defined for every finite CW complex K and a unitary representation ρ of $\pi_1 := \pi_1(K)$ on a finite dimensional vector space V_ρ. Let $\tilde{K}$ be the simply connected covering space of K with π_1 acting on $\tilde{K}$ as group of deck transformations. Consider K as being embedded as a fundamental domain in $\tilde{K}$, so that $\tilde{K}$ is the set of translates of K under π_1. In this way, the real co-chain complex becomes an $\mathbb{R}(\pi_1)$-module. Let $C^*(K;\rho) := C^*(\tilde{K}) \otimes_{\mathbb{R}(\pi_1)} V_\rho$ be the twisted co-chain complex. In the real vector space $C^q(K;\rho)$ one can choose a preferred base $(x_i \otimes v_j)$, where x_i runs through the preferred base of the $\mathbb{R}(\pi_1)$-module $C^q(\tilde{K})$ given by the cells of K and v_j through an orthonormal base of V_ρ. A preferred base gives rise to an inner product in $C^*(K;\rho)$. Let $\delta\colon C^q(K;\rho) \to C^{q+1}(K;\rho)$ denote the co-boundary operator and δ^* the adjoint operator with respect to the inner product in $C^*(K;\rho)$. Define the combinatorial Laplacian $\Delta^{(c)}$ by $\Delta^{(c)} = \delta\delta^* + \delta^*\delta$. Assume that the cohomology $H^*(K;\rho)$ of the complex $C^*(K;\rho)$ vanishes. Then $\Delta^{(c)}$ is invertible and as shown by Ray and Singer, the Reidemeister torsion $\tau(K,\rho) \in \mathbb{R}^+$ of K and ρ is given by

$$\log \tau(K,\rho) = \frac{1}{2}\sum_{q=0}^{n}(-1)^{q+1} q \log\det(\Delta_q^{(c)}), \tag{1}$$

where n is the dimension of the top-dimensional cells of K. This is not quite the original definition of the Reidemeister torsion, but as shown by Ray and Singer, it is equivalent to the original one. If $H^*(K;\rho) \neq 0$, one has to choose a volume form $\mu \in \det H^*(K;\rho)$. For any choice of μ one can define the Reidemeister torsion $\tau_M(\rho;\mu) \in \mathbb{R}^+$, which depends on μ.

Let M be a closed Riemannian manifold and ρ an orthogonal representation of $\pi_1(M)$. Let $E_\rho \to M$ be the flat orthogonal vector bundle associated to ρ. Then by the Hodge de Rham theorem, $H^*(K;\rho)$ is identified with the space of harmonic forms with values in E_ρ. Using the global inner product on harmonic forms, we get a volume form μ. It turns out that $\tau_K(\rho;\mu)$ is invariant under subdivisions. Furthermore, any two smooth triangulations of M admit a common subdivision. Thus $\tau_K(\rho;\mu)$ is independent of the choice of K and we write $\tau_M(\rho) := \tau_K(\rho;\mu)$. This is the Reidemeister torsion of M with respect to ρ and $\mu \in \det H^*(M;E_\rho)$.

The original interest in Reidemeister torsion came from the fact that it is not a homotopy invariant, and so can distinguish spaces which are homotopy invariant but are not homeomorphic. Especially, Reidemeister used Reidemeister torsion to classify 3-dimensional lens spaces up to homeomorphism and this was generalized by Franz to higher dimensions. The classification includes examples of homotopy equivalent 3-dimensional manifolds which are not homeomorphic. Reidemeister torsion is closely related to Whitehead torsion, which is a more sophisticated invariant of chain complexes. It is related to the concept of simple homotopy equivalence.

Following a suggestion of Arnold Shapiro, Ray and Singer were looking for an analytic description of the Reidemeister torsion of a closed Riemannian manifold. The inspiration for the definition came from formula (1). Let $(\Lambda^*(M;E_\rho),d)$ be the de Rham complex of E_ρ-valued differential forms on M and let $\Delta := dd^* + d^*d$ be the Laplace operator acting in $\Lambda^*(M;E_\rho)$, where d^* is the formal adjoint of d with respect to the inner product induced by the Riemmanian metric g and the fibre metric in E_ρ. Then the idea is to replace $C^*(K;\rho)$ by $\Lambda^*(M;E_\rho)$ and the combinatorial Laplacian $\Delta_q^{(c)}$ by the Hodge Laplacian Δ_q. The problem is that Δ_q is acting in an infinite dimensional space and the determinant is not well defined. To overcome this problem, one uses the zeta function regularization. Regarded as unbounded operator in the Hilbert space of L^2-forms of degree q with values in E_ρ, Δ_q is essentially self-adjoint and non-negative. Since M is compact, it follows that Δ_q has a pure point spectrum consisting of eigenvalues $\lambda_j \geq 0$, $j \in \mathbb{N}$, of finite multiplicity $m(\lambda_j)$. Let $\zeta_q(s;\rho) := \sum_{\lambda_j>0} m(\lambda_j)\lambda_j^{-s}$, $s \in \mathbb{C}$, be the spectral zeta function. As shown by Seeley, the series converges absolutely and uniformly on compact subsets of the half plane $\mathrm{Re}(s) > n/2$, admits a meromorphic extension to $\mathbb{C}$ and is holomorphic at $s = 0$. Then the regularized determinant of Δ_q is defined by $\det(\Delta_q) := \exp(-\frac{d}{ds}\zeta_q(s;\rho)\big|_{s=0})$. Replacing formally $\det(\Delta_q^{(c)})$ in (1) by the regularized determinant of Δ_q has led Ray and Singer to the following definition of the analytic torsion $T_M(\rho) \in \mathbb{R}^+$

$$\log T_M(\rho) := \frac{1}{2}\sum_{q=0}^{n}(-1)^q q \frac{d}{ds}\zeta_q(s;\rho)\big|_{s=0}. \tag{2}$$

By its definition, $T_M(\rho)$ depends on the whole spectrum of the Laplace operators Δ_q, $q = 0,\dots,n$, and is therefore a more sophisticated spectral invariant. Ray and Singer proved that the analytic torsion satisfies the same formal properties as the Reidemeister torsion, which supported their conjecture that

$$T_M(\rho) = \tau_M(\rho) \tag{3}$$

for any orthogonal representation ρ. The Ray-Singer conjecture was eventually proved independently by Cheeger [Ch] and Müller [Mu] (and is now often referred to as the Cheeger-Müller theorem). The proofs of Cheeger and Müller are different, but similar in spirit. The strategy of both proofs is to show that $T_M(\rho) - \tau_M(\rho)$ remains invariant under surgery which reduces the problem to the case of the sphere for which the equality can be verified explicitly. The proof of Cheeger is based on analytic surgery methods. Müller uses the Whitney approximation of the de Rham complex by the co-chain complex and the finite element approximation of eigenvalues, which goes back to the work of Dodziuk and Patodi.

The equality of analytic torsion and Reidemeister torsion has been extended in various ways. Müller has shown that (3) holds for unimodular representations ρ of $\pi_1(M)$

($|\det\rho(\gamma)| = 1$ for all $\gamma \in \pi_1(M)$). Recently, this result has found some interesting applications to the study of the cohomology of arithmetic groups (see below).

Finally, Bismut and Zhang treated the general case of an arbitrary finite dimensional representation ρ. The framework is slightly different. They work with the metrics on $\det H^*(M, E_\rho)$ induced by the analytic torsion and the Thom-Smale complex associated to a Morse function on M. In general, an equality does not hold anymore. There appears a defect term, which, however, can be described explicitly. The defect is a kind of obstruction for ρ being unimodular. Bismut and Zhang gave a completely new proof, which uses the Witten deformation of the de Rham complex associated to the Morse function.

The equivariant case was first studied by Lott and Rothenberg and then by Lück. Also Bismut and Zhang extended their result to the equivariant case, using again the Witten deformation of the de Rahm complex.

It is natural to try to generalize (3) to other classes of manifolds. The first obvious case are compact manifolds with a non-empty boundary. A corresponding result was announced by Cheeger in his paper proving the Ray-Singer conjecture. Then Lück has derived a formula using the double of a compact Riemannian manifold M with boundary which reduces the problem to the equivariant case. This approach requires that the metric of M is a product near the boundary. On the analytic side one has to impose absolute or relative boundary conditions for the Laplacians. The resulting formula comparing analytic and Reidemeister torsion involves a correction term which is given by the Euler characteristic $\chi(\partial M)$ of the boundary. The general case, i.e., without any assumption on the behavior of the metric near the boundary, was treated by Ma and Brüning. The boundary contribution, which is called anomaly, is in general more complicated.

There were various attempts to generalize (3) to singular spaces. The first case are manifolds with conical singularities. The study of analytic and topological torsion on singular spaces with conical singularities started with work of A. Dar. She proved that on singular spaces with isolated conical singularities, the analytic torsion is well-defined. On the combinatorial side she used the middle intersection complex to define the intersection Reidemeister torsion, which one expected to be equal to the analytic torsion. This turned out not to be true. There are recent results by Albin-Rochon-Sher, Hartman-Spreafico, and Ludwig who establish a formula relating analytic torsion and intersection torsion. This is not an equality, but the defect term can be described explicitly.

Guided by the definition of the real analytic torsion, Ray and Singer introduced an analog of the analytic torsion for complex manifolds. The role of the flat vector bundle is played by a holomorphic vector bundle $E \to X$ over a compact complex manifold and the de Rham complex is replaced by the $\bar{\partial}$-complex of $(0, q)$-forms with values in E. The complex analytic torsion has found important applications in arithmetic-algebraic geometry and theoretical physics. This will be discussed in a separate section.

The equality of analytic torsion and Reidemeister torsion has recently found interesting application in the study of the growth of torsion in the cohomology of arithmetic groups. This idea goes back to Bergeron and Venkatesh. The origin of this kind of applications is the following observation. Let X be a compact Riemannian manifold and let ρ be a representation of $\pi_1(X)$ on a finite dimensional real vector space V. Suppose that there exists a lattice $M \subset V$ which is invariant under $\pi_1(X)$. Let $\mathcal{M}$ be the associated local system of finite rank free $\mathbb{Z}$-modules over X. Note that $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{R} = E_\rho$. The cohomology $H^*(X;\mathcal{M})$ of X with coefficients in $\mathcal{M}$ is a finitely generated abelian group. Suppose that ρ is acyclic, i.e., $H^*(X, E_\rho) = 0$. Then $H^*(X;\mathcal{M})$ is a finite abelian group. Denote by $|H^q(X;\mathcal{M})|$ the order of $H^q(X;\mathcal{M})$. Then as observed by Cheeger, the Reidemeister torsion satisfies

$$\tau_X(\rho) = \prod_{q=0}^{n} |H^q(X;\mathcal{M})|^{(-1)^{q+1}}. \tag{4}$$

Using the equality of $T_X(\rho) = \tau_X(\rho)$, (4) provides an analytic tool to study the torsion subgroups in the cohomology. This has been used by Bergeron and Venkatesh in the following way. Let $\mathbf{G}$ be a connected semisimple algebraic group over $\mathbb{Q}$ and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic subgroup. Let $G = \mathbf{G}(\mathbb{R})$. Then Γ is a lattice in G. An example is $G = \mathrm{SL}(n,\mathbb{R})$ and $\Gamma = \mathrm{SL}(n,\mathbb{Z})$. We assume that Γ is co-compact and torsion free. Let $K \subset G$ be a maximal compact subgroup and $\widetilde{X} := G/K$ the associated Riemannian symmetric space of non-positive curvature. Then Γ acts freely on $\widetilde{X}$ and $X := \Gamma\backslash\widetilde{X}$ is a compact locally symmetric manifold. Consider a decreasing sequence of congruence subgroups $\cdots \subset \Gamma_{j+1} \subset \Gamma_j \subset \cdots \subset \Gamma$ with $\cap_j \Gamma_j = \{1\}$. Put $X_j := \Gamma\backslash\widetilde{X}$, $j \in \mathbb{N}$. Then $X_j \to X$ is a finite covering. Let ϱ be a rational irreducible representation of $\mathbf{G}$ on a finite dimensional $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$. Then there is a lattice $M \subset V_{\mathbb{Q}}$, which is invariant under $\varrho(\Gamma)$. Let $V = V_{\mathbb{Q}} \otimes \mathbb{R}$ and $\rho := \varrho|_\Gamma$. Then $M \subset V$ is a Γ-invariant lattice. Let $T_{X_j}(\rho)$ be the analytic torsion of the covering X_j of X with respect to $\rho|_{\Gamma_j}$. By an appropriate assumption on the highest weight of ϱ, there is a uniform spectral gap at the origin for all Laplacians on X_j, uniformly in $j \in \mathbb{N}$. In this case, it follows that the limit $\log T_{X_j}(\rho)/\operatorname{vol}(X_j)$ as $j \to \infty$ exists and equals a constant $t^{(2)}_{\widetilde{X}}(\rho)$ which depends only on $\widetilde{X}$ and ρ. In fact, $\operatorname{vol}(X) \cdot t^{(2)}_{\widetilde{X}}(\rho)$ is the L^2-torsion of X. Let $H^q(\Gamma_j; M)$ be the cohomology of Γ_j with coefficients in the Γ_j-module M. Note that $H^q(\Gamma_j; M) \cong H^q(X_j;\mathcal{M})$, where $\mathcal{M}$ is the local system associated to M. Then as shown by Bergeron and Venkatesh, it follows from (4) combined with (3)

$$\liminf_{j\to\infty} \sum_q \frac{\log |H^q(\Gamma_j; M)|}{[\Gamma\colon \Gamma_j]} \geq C_{G,M}, \tag{5}$$

where the sum runs over the integers q such that $q + \frac{\dim(X)-1}{2}$ is odd and $C_{G,L} \geq 0$. Moreover, if the fundamental rank $\operatorname{rank}_{\mathbb{C}}(G) - \operatorname{rank}_{\mathbb{C}}(K)$ is 1, then $C_{G,M} > 0$. The latter condition is satisfied, for example, for hyperbolic manifolds and $X = \Gamma\backslash\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ with $n = 3, 4$. For these cases it follows from (5) that the order of the torsion of the cohomology grows exponentially. There is a conjecture with a more precise statement saying that the exponential growth happens exactly in the middle degree. Müller and Rochon extended this result to the case of finite volume hyperbolic manifolds. This includes Bianchi subgroups $\Gamma_D = \mathrm{SL}(2,\mathcal{O}_D)$ of $\mathrm{SL}(2,\mathbb{C})$, where $\mathcal{O}_D$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$, $D > 0$, square free. The complementary case is if the lattice is fixed and the rank of the module M increases. This case has been studied by Müller, Pfaff, and Rochon with analogous results on the growth of torsion.

There are other interesting developments related to real analytic torsion which, due to the limited space, could not be discussed here.

References

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