ANALYTIC TORSION AND L^2-TORSION OF COMPACT LOCALLY SYMMETRIC MANIFOLDS

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Abstract. In this paper we study the analytic torsion and the L^2-torsion of compact locally symmetric manifolds. We consider the analytic torsion with respect to representations of the fundamental group which are obtained by restriction of irreducible representations of the group of isometries of the underlying symmetric space. The main purpose is to study the asymptotic behavior of the analytic torsion with respect to sequences of representations associated to rays of highest weights.

1. Introduction

Let G be a real, connected, semisimple Lie group without compact factors and with finite center. Let $K \subset G$ be a maximal compact subgroup. Then $\widetilde{X} = G/K$ is a Riemannian symmetric space of the noncompact type. Let $\Gamma \subset G$ be a discrete, torsion free, co-compact subgroup. Then $X = \Gamma\backslash\widetilde{X}$ is a compact oriented locally symmetric manifold. Let $d = \dim X$. Let τ be a finite-dimensional irreducible representation of G on a complex vector space V_τ. Denote by E_τ the flat vector bundle over X associated to the representation $\tau|_\Gamma$ of Γ. By [MtM, Lemma 3.1], E_τ can be equipped with a distinguished Hermitian fiber metric, called admissible. Let $\Delta_p(\tau)$ be the Laplace operator acting on E_τ-valued p-forms on X. Denote by $\zeta_p(s;\tau)$ the zeta function of $\Delta_p(\tau)$ (see [Sh]). Then the analytic torsion $T_X(\tau) \in \mathbb{R}^+$ is defined by

$$\log T_X(\tau) = \frac{1}{2}\sum_{p=0}^{d}(-1)^p p \frac{d}{ds}\zeta_p(s;\tau)\big|_{s=0} \tag{1.1}$$

(see [RS], [Mu2]). Since we have chosen distinguished metrics, we don't indicate the metric dependence of $T_X(\tau)$. We also consider the L^2-torsion $T_X^{(2)}(\tau)$ which is defined as in [Lo], using the Γ-trace of the heat operators on $\widetilde{X}$.

The main purpose of this paper is to study the asymptotic behavior of $T_X(\tau)$ and $T_X^{(2)}(\tau)$ for certain sequences of representations τ of G. This problem was first studied in [Mu3] in the context of hyperbolic 3-manifolds. The method used in this paper was based on the study of the twisted Ruelle zeta function. In [MP] we have developed a different and more simple method which we used to extend the results of [Mu3] to compact hyperbolic manifolds of any dimension. In the present paper, we generalize the results of the previous

Date: April 3, 2012.
1991 *Mathematics Subject Classification.* Primary: 58J52, Secondary: 53C53.
Key words and phrases. analytic torsion, locally symmetric manifolds.

papers to arbitrary compact locally symmetric spaces. Recently, Bismut, Ma, and Zhang [BMZ] studied the asymptotic behavior of the analytic torsion by a different method and in the more general context of analytic torsion forms on arbitrary compact manifolds. Furthermore, Bergeron and Venkatesh [BV] studied the asymptotic behavior of the analytic torsion if the flat bundle is kept fixed, but the discrete group varies in a tower $\{\Gamma_N\}_{N\in\mathbb{N}}$ of normal subgroups of finite index of Γ. They used this to study the growth of the torsion subgroup in the cohomology of arithmetic groups. In [MaM] the results of [Mu3] have been used to study the growth of the torsion in the cohomology of arithmetic hyperbolic 3-manifolds, if the lattice is kept fixed and the flat bundle varies. The results of the present paper will be used to study the growth of the torsion in the cohomology of arithmetic groups in higher rank cases.

Now we explain our results in more detail. Let $\delta(\widetilde{X}) = \operatorname{rank}_{\mathbb{C}}(G) - \operatorname{rank}_{\mathbb{C}}(K)$. Occasionally we will denote this number by $\delta(G)$. Let $\mathfrak{g}$ be the Lie algebra of G. Let $G_{\mathbb{C}}$ denote the simply connected complex Lie group corresponding to the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. We assume that G equals the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}$. Then the irreducible finite dimensional complex representations of G can be identified with the irreducible holomorphic representations of $G_{\mathbb{C}}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a fundamental Cartan subalgebra. Fix positive roots $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Let $\theta\colon \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution. For a highest weight $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ let τ_λ be the irreducible representation of G with highest weight λ. Then we denote by $\lambda_\theta \in \mathfrak{h}_{\mathbb{C}}^*$ the highest weight of $\tau_\lambda \circ \theta$, where we regard θ as an involution on G. Our main result is the following theorem.

Theorem 1.1. (i) *Let $\widetilde{X}$ be even dimensional or let $\delta(\widetilde{X}) \neq 1$. Then $T_X(\tau) = 1$ for all finite-dimensional representations τ of G.*
(ii)*Let $\widetilde{X}$ be odd-dimensional with $\delta(\widetilde{X}) = 1$. Let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ be a highest weight with $\lambda_\theta \neq \lambda$. For $m \in \mathbb{N}$ let $\tau_\lambda(m)$ be the irreducible representation of G with highest weight $m\lambda$. There exist constants $c > 0$ and $C_{\widetilde{X}} \neq 0$, which depends on $\widetilde{X}$, and a polynomial $P_\lambda(m)$, which depends on λ, such that*

$$\log T_X(\tau_\lambda(m)) = C_{\widetilde{X}} \operatorname{vol}(X) \cdot P_\lambda(m) + O\left(e^{-cm}\right)$$

as $m \to \infty$. Furthermore, there is a constant $C_\lambda > 0$ such that

$$P_\lambda(m) = C_\lambda \cdot m \dim(\tau_\lambda(m)) + R_\lambda(m),$$

where $R_\lambda(m)$ is a polynomial whose degree equals the degree of the polynomial $\dim(\tau_\lambda(m))$.

The coefficient of the highest order term of the polynomial $P_\lambda(m)$ can be determined using Weyl's dimension formula. Our main result can be also stated as follows. There exists a constant $C = C(\widetilde{X}, \lambda) \neq 0$, which depends on $\widetilde{X}$ and λ, such that

$$\log T_X(\tau_\lambda(m)) = C \operatorname{vol}(X) \cdot m \dim(\tau_\lambda(m)) + O\left(\dim(\tau_\lambda(m))\right) \tag{1.2}$$

as $m \to \infty$.

Part (i) of Theorem 1.1 extends a result of Moscovici and Stanton [MS1] who showed that $T_X(\rho) = 1$, if $\delta(\widetilde{X}) \geq 2$ and ρ is a unitary representation of Γ. Part (ii) is a consequence of the following two propositions. The first one shows that the asymptotic behavior of the analytic torsion with respect to the representations $\tau_\lambda(m)$ is determined by the asymptotic behavior of the L^2-torsion.

Proposition 1.2. *Let $\widetilde{X}$ be odd-dimensional with $\delta(\widetilde{X}) = 1$. Let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ be a highest weight. Assume that $\lambda_\theta \neq \lambda$. For $m \in \mathbb{N}$ let $\tau_\lambda(m)$ be the irreducible representation of G with highest weight $m\lambda$. Then there exists $c > 0$ such that*

$$\log T_X(\tau_\lambda(m)) = \log T_X^{(2)}(\tau_\lambda(m)) + O\left(e^{-cm}\right)$$

for all $m \in \mathbb{N}$.

The second result on which part (ii) of Theorem 1.1 relies is the computation of the L^2-torsion. The computation is based on the Plancherel formula. It gives

Proposition 1.3. *Let the assumptions be as in Proposition 1.2. There exists a constant $C_{\widetilde{X}}$, which depends on $\widetilde{X}$, and a polynomial $P_\lambda(m)$, which depends on λ, such that*

$$\log T_X^{(2)}(\tau_\lambda(m)) = C_{\widetilde{X}} \operatorname{vol}(X) \cdot P_\lambda(m), \quad m \in \mathbb{N}. \tag{1.3}$$

Moreover there is a constant $C_\lambda > 0$ such that

$$P_\lambda(m) = C_\lambda \cdot m \cdot \dim(\tau_\lambda(m)) + O\left(\dim(\tau_\lambda(m)\right) \tag{1.4}$$

as $m \to \infty$.

If we consider one of the odd-dimensional irreducible symmetric spaces $\widetilde{X}$ with $\delta(\widetilde{X}) = 1$ and choose λ to be a fundamental weight, the statements can be made more explicit.

Let $\widetilde{X} = \operatorname{Spin}(p,q)/(\operatorname{Spin}(p) \times \operatorname{Spin}(q))$, p,q odd, and $\widetilde{X} = G/K$. Let $n := (p+q-2)/2$. There are two fundamental weight $\omega_{f,n}^{\pm}$ which are not invariant under θ (see (6.45)). One has $\omega_{f,n}^- = (\omega_{f,n}^+)_\theta$. By equation (6.51), it suffices to consider the weight $\omega_{f,n}^+$. For $m \in \mathbb{N}$ let $\tau(m)$ be the representation with highest weight $m\omega_{f,n}^+$. By Weyl's dimension formula there exists a constant $C > 0$ such that

$$\dim(\tau(m)) = C m^{\frac{n(n+1)}{2}} + O\left(m^{\frac{n(n+1)}{2}-1}\right) \tag{1.5}$$

as $m \to \infty$. Let $\widetilde{X}_d$ be the compact dual of $\widetilde{X}$. Let

$$C_{p,q} = \frac{(-1)^{\frac{pq-1}{2}} 2\pi}{\operatorname{vol}(\widetilde{X}_d)} \binom{n}{\frac{p-1}{2}}. \tag{1.6}$$

Corollary 1.4. *Let $\widetilde{X} = \operatorname{Spin}(p,q)/(\operatorname{Spin}(p) \times \operatorname{Spin}(q))$, p,q odd, and $X = \Gamma\backslash\widetilde{X}$. With respect to the above notation we have*

$$\log T_X(\tau(m)) = C_{p,q} \operatorname{vol}(X) \cdot m \dim(\tau(m)) + O\left(m^{\frac{n(n+1)}{2}}\right)$$

as $m \to \infty$.

The case p arbitrary, $q = 1$ was treated in [MP] and the case $p = 3$, $q = 1$ in [Mu3]. In the latter case we have $\operatorname{Spin}(3,1) \cong \operatorname{SL}(2,\mathbb{C})$. The irreducible representation of $\operatorname{Spin}(3,1)$

with highest weight $\frac{1}{2}(m,m)$ corresponds to the m-th symmetric power of the standard representation $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ and we have

$$-\log T_X(\tau(m)) = \frac{1}{4\pi}\operatorname{vol}(X)m^2 + O(m).$$

The remaining case is $\widetilde{X} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. There are two fundamental weights ω_i, $i = 1,2$. Both are non-invariant under θ. Let $\tau_i(m)$, $i = 1,2$, be the irreducible representation with highest weight $m\omega_i$. By Weyl's dimensiona formula one has

$$\dim(\tau_i(m)) = \frac{1}{2}m^2 + O(m),$$

as $m \to \infty$. Let $\widetilde{X}_d$ be the compact dual of $\widetilde{X}$.

Corollary 1.5. *Let $\widetilde{X} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ and $X = \Gamma\backslash\widetilde{X}$. We have*

$$\log T_X(\tau_i(m)) = \frac{4\pi\operatorname{vol}(X)}{9\operatorname{vol}(\widetilde{X}_d)} m\dim(\tau_i(m)) + O(m^2)$$

as $m \to \infty$.

Using the equality of analytic and Reidemeister torsion [Mu2], we obtain corresponding statements for the Reidemeister torsion $\tau_X(\tau_\lambda(m))$. Especially we have

Corollary 1.6. *Let $X = \Gamma\backslash\widetilde{X}$ be a compact odd-dimensional locally symmetric manifold with $\delta(\widetilde{X}) = 1$. Let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ be a highest weight which satisfies $\lambda_\theta \neq \lambda$. Let $\tau_X(\tau_\lambda(m))$ be the Reidemeister torsion of X with respect to the representation $\tau_\lambda(m)$. Then $\operatorname{vol}(X)$ is determined by the set $\{\tau_X(\tau_\lambda(m))\colon m \in \mathbb{N}\}$.*

Finally we note that Bergeron and Venkatesh [BV] proved results of a similar nature, but in a different aspect. Let $\delta(\widetilde{X}) = 1$. Let $\Gamma \supset \Gamma_1 \supset \cdots \supset \Gamma_N \supset \cdots$ be a tower of subgroups of finite index with $\cap_N \Gamma_N = \{e\}$. A representation τ of G is called strongly acyclic, if the spectrum of the Laplacians $\Delta_p(\tau)$ on $\Gamma_N\backslash\widetilde{X}$ stays uniformly bounded away from zero. Then for a strongly acyclic representation τ they show that there is a constant $c_{G,\tau} > 0$ such that

$$\lim_{N\to\infty} \frac{\log T_{\Gamma_N\backslash\widetilde{X}}(\tau)}{[\Gamma\colon\Gamma_N]} = c_{G,\tau}\operatorname{vol}(\Gamma\backslash\widetilde{X}).$$

Next we explain our methods to prove Theorem 1.1. The first step is the proof of Proposition 1.2. We follow the proof used in [MP]. For an irreducible representation τ of G and $t > 0$ put

$$K(t,\tau) := \sum_{p=0}^{d} (-1)^p p \operatorname{Tr}\left(e^{-t\Delta_p(\tau)}\right).$$

Assume that $\tau|_\Gamma$ is acyclic, that is $H^*(X,E_\tau) = 0$. Then the analytic torsion is given by

$$\log T_X(\tau) := \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K(t,\tau)\,dt\right)\bigg|_{s=0}. \tag{1.7}$$

Now the key ingredient of the proof of Proposition 1.2 is the following lower bound for the spectrum of the Laplacians. For every highest weight λ which satisfies $\lambda_\theta \neq \lambda$, there exist $C_1, C_2 > 0$ such that

$$\Delta_p(\tau_\lambda(m)) \geq C_1 m^2 - C_2, \quad m \in \mathbb{N}, \tag{1.8}$$

(see Corollary 7.2). Since $\tau_\lambda(m)$ is acyclic and $\dim X$ is odd, $T_X(\tau_\lambda(m))$ is metric independent [Mu2]. Especially, it is invariant under rescaling of the metric. So we can replace $\Delta_p(\tau_\lambda(m))$ by $\frac{1}{m}\Delta_p(\tau_\lambda(m))$. Then

$$\begin{aligned}\log T_X(\tau(m)) =& \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1} K\left(\frac{t}{m}, \tau(m)\right) dt\right)\bigg|_{s=0} \\ &+ \frac{1}{2}\int_1^\infty t^{-1} K\left(\frac{t}{m}, \tau(m)\right) dt.\end{aligned} \tag{1.9}$$

It follows from (1.8) and standard estimations of the heat kernel that the second term on the right is $O(e^{-\frac{m}{8}})$ as $m \to \infty$. To deal with the first term, we use a preliminary form of the Selberg trace formula. It turns out that the contribution of the nontrivial conjugacy classes to the trace formula is also exponentially decreasing in m. Finally, the identity contribution equals $\log T_X^{(2)}(\tau_\lambda(m))$ up to a term, which is exponentially decreasing in m. This implies Proposition 1.2.

To deal with the L^2-torsion, we recall that for any τ, $\log T_X^{(2)}(\tau)$ it is defined in terms of the Γ-trace of the heat operators $e^{-t\widetilde{\Delta}_p(\tau)}$ on the universal covering [Lo]. In our case, $e^{-t\widetilde{\Delta}_p(\tau)}$ is a convolution operator and its Γ-trace equals the contribution of the identity to the spectral side of the Selberg trace formula applied to $e^{-t\Delta_p(\tau)}$. It follows that

$$\log T_X^{(2)}(\tau) = \operatorname{vol}(X) \cdot t_{\widetilde{X}}^{(2)}(\tau),$$

where $t_{\widetilde{X}}^{(2)}(\tau)$ depends only on $\widetilde{X}$ and τ. To compute $t_{\widetilde{X}}^{(2)}(\tau)$ we factorize $\widetilde{X}$ as $\widetilde{X} = \widetilde{X}_0 \times \widetilde{X}_1$, where $\delta(\widetilde{X}_0) = 0$ and $\widetilde{X}_1$ is irreducible with $\delta(\widetilde{X}_1) = 1$. Let $\tau = \tau_0 \otimes \tau_1$ be the corresponding decomposition of τ. Let $\widetilde{X}_{0,d}$ be the compact dual symmetric space of $\widetilde{X}_0$. Using a formula similar to [Lo, Proposition 11], we get

$$t_{\widetilde{X}}^{(2)}(\tau) = (-1)^{\dim(\widetilde{X}_0)/2} \frac{\chi(\widetilde{X}_{0,d})}{\operatorname{vol}(\widetilde{X}_{0,d})} \dim(\tau_0) \cdot t_{\widetilde{X}_1}^{(2)}(\tau_1).$$

This reduces the computation of $t_{\widetilde{X}}^{(2)}(\tau)$ to the case of an odd-dimensional irreducible symmetric space $\widetilde{X}$ with $\delta(\widetilde{X}) = 1$. From the classification of simple Lie groups it follows that the only possibilities for $\widetilde{X}$ are $\widetilde{X} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ or $\widetilde{X} = \mathrm{Spin}(p,q)/(\mathrm{Spin}(p) \times \mathrm{Spin}(q))$, p, q odd. Using the Plancherel formula, $t_{\widetilde{X}}^{(2)}(\tau)$ can be computed explicitly for these cases. Combined with Weyl's dimension formula, it follows that $t_{\widetilde{X}}^{(2)}(\tau_\lambda(m))$ is a polynomial in m. In this way we obtain our main result.

The paper is organized as follows. In section 2 we collect some facts about representations of reductive Lie groups. Section 3 is concerned with Bochner-Laplace operators on locally symmetric spaces. The main result are estimations of the heat kernel of a Bochner-Laplace operator. In section 4 we consider the analytic torsion in general. The main result of this section is Proposition 4.2, which establishes part (i) of Theorem 1.1. Section 5 is devoted to the study of the L^2-torsion. We reduce the study of the L^2-torsion to the case of an irreducible symmetric space $\widetilde{X}$ with $\delta(\widetilde{X}) = 1$. This case is then treated in section 6. Especially we establish Proposition 1.3 in this case. In section 7 we prove a lower bound for the spectrum of the twisted Laplace operators. This is the key result for the proof of Proposition 1.2. In the final section 8 we prove our main result, Theorem 1.1.

2. Preliminaries

In this section we summarize some facts about representations of reductive Lie groups.

2.1. Let G be a real reductive Lie group in the sense of [Kn2, p. 446]. Let $K \subset G$ be the associated maximal compact subgroup. Then G has only finitely many connected components. Denote by G^0 the component of the identity. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition.

We denote by $\hat{G}$ the unitary dual and by $\hat{G}_d$ the discrete series of G. By $\operatorname{Rep}(G)$ we denote the equivalence classes of irreducible finite-dimensional representations of G.

Let Q be a standard parabolic subgroup of G [Kn2, VII.7]. Then Q has a Langlands decomposition $Q = MAN$, where M is reductive and A is abelian. Q is called cuspidal if $\hat{M}_d \neq \emptyset$. Let $K_M = K \cap M$. Then K_M is a maximal compact subgroup of M.

Let $Q = MAN$ be cuspidal. For $(\xi, W_\xi) \in \hat{M}_d$ and $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, let

$$\pi_{\xi,\nu} = \operatorname{Ind}_Q^G(\xi \otimes e^{\nu} \otimes \operatorname{Id}) \tag{2.1}$$

be the induced representation acting by the left regular representation on the Hilbert space

$$\begin{aligned}\mathcal{H}_{\xi,\nu} = \big\{ f\colon G \to W_\xi\colon\, & f(gman) = e^{-(i\nu+\rho_Q)(\log a)}\xi(m)^{-1} f(g), \\ & \forall\, m \in M,\, a \in A,\, n \in N,\, g \in G,\, f|_K \in L^2(K, W_\xi)\big\}\end{aligned} \tag{2.2}$$

with norm given by

$$\|f\|^2 = \int_K |f(k)|^2_{W_\xi}\, dk.$$

If $\nu \in \mathfrak{a}^*$, then $\pi_{\xi,\nu}$ is unitarily induced. Denote by $\Theta_{\xi,\nu}$ the global character of $\pi_{\xi,\nu}$.

2.2. Next we recall some facts concerning the discrete series. Let G be a semisimple connected Lie group without compact factors and with finite center. Let $K \subset G$ be a maximal compact subgroup. Assume that $\delta(G) = 0$. Then G/K is even-dimensional. Let $n = \dim(G/K)/2$. Let $\mathfrak{t} \subset \mathfrak{k}$ be a compact Cartan subalgebra of $\mathfrak{g}$. Let $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the corresponding roots with Weyl-groups W_G, W_K. Then one can regard W_K as a subgroup of W_G. Let P be the weight lattice in $i\mathfrak{t}^*$. Let $\langle \cdot, \cdot \rangle$ be the inner product on $i\mathfrak{t}^*$ induced by the Killing form. Recall that $\Lambda \in P$ is called regular if $\langle \Lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then $\hat{G}_d$ is parametrized by the W_K-orbits of the regular elements of P, where W_K is the Weyl group of $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, [Kn1, Theorem 12.20, Theorem 9.20]. If Λ is a regular element of P, the corresponding discrete series will be denoted by ω_Λ. For $\pi \in \hat{G}$

we denote by χ_π the infinitesimal character of π. Let $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the center of the universal envelopping algebra of $\mathfrak{g}_C$. For a regular element $\Lambda \in \mathfrak{h}_\mathbb{C}^*$ let χ_Λ be the homomorphism of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$, defined by [Kn1, (8.32)]. By [Kn1, Theorem 9.20], the infinitesimal character of ω_Λ is given by χ_Λ. Fix positive roots $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ and let P^+ be the corresponding set of dominant weights. Let ρ_G be the half sum of the elements of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ Then we have the following proposition.

Proposition 2.1. *Let $\tau \in \mathrm{Rep}(G)$. Then for $\pi \in \hat{G}_d$ one has*

$$\dim\left(H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau)\right) = \begin{cases} 1, & \chi_\pi = \chi_{\check{\tau}},\ p = n; \\ 0, & \textit{else}. \end{cases}$$

Moreover, there are exactly $|W_G|/|W_K|$ distinct elements of $\hat{G}_d$ with infinitesimal character $\chi_{\check{\tau}}$, where $\check{\tau}$ is the contragredient representation of τ.

Proof. Let $\Lambda(\check{\tau}) \in P^+$ be the highest weight of τ. Clearly $\Lambda(\check{\tau}) + \rho_G$ is regular. Thus, since W_G acts freely on the regular elements, the proposition follows from [BW, Theorem I.5.3] and the above remarks on infinitesimal characters. $\square$

2.3. Let $Q = MAN$ be a standard parabolic subgroup. In general, M is neither semisimple nor connected. But M is reductive in the sense of [Kn2, p. 466]. Let $K_M = K \cap M$, let K_M^0 be the component of the identity, and let $\mathfrak{k}_\mathfrak{m} := \mathfrak{k} \cap \mathfrak{m}$ be its Lie algebra. Assume that $\mathrm{rank}(M) = \mathrm{rank}(K_M)$. Then M has a nonempty discrete series, which is defined as in [Kn1, XII,§8]. The explicit parametrization is given in [Kn1, Proposition 12.32], [Wa2, section 8.7.1].

3. Bochner Laplace operators

Let G be a semisimple connected Lie group without compact factors and with finite center. Let $K \subset G$ be a maximal compact subgroup. Let $\widetilde{X} = G/K$. Let Γ be a torsion free, cocompact discrete subgroup of G and let $X = \Gamma\backslash\widetilde{X}$.

Let ν be a finite-dimensional unitary representation of K on $(V_\nu, \langle\cdot,\cdot\rangle_\nu)$. Let

$$\widetilde{E}_\nu := G \times_\nu V_\nu$$

be the associated homogeneous vector bundle over $\widetilde{X}$. Denote by $R_g\colon \widetilde{E}_\nu \to \widetilde{E}_\nu$ the action of $g \in G$. The inner product $\langle\cdot,\cdot\rangle_\nu$ induces a G-invariant fiber metric $\widetilde{h}_\nu$ on $\widetilde{E}_\nu$. Let $\widetilde{\nabla}^\nu$ be the connection on $\widetilde{E}_\nu$ induced by the canonical connection on the principal K-fiber bundle $G \to G/K$. Then $\widetilde{\nabla}^\nu$ is G-invariant. Let

$$E_\nu := \Gamma\backslash\widetilde{E}_\nu$$

be the associated locally homogeneous bundle over X. Since $\tilde{h}_\nu$ and $\widetilde{\nabla}^\nu$ are G-invariant, they can be pushed down to a metric h_ν and a connection ∇^ν on E_ν. Let $C^\infty(\widetilde{X}, \widetilde{E}_\nu)$ resp. $C^\infty(X, E_\nu)$ denote the space of smooth sections of $\widetilde{E}_\nu$ resp. of E_ν. Let

$$C^\infty(G,\nu) := \{f : G \to V_\nu\colon f \in C^\infty,\ f(gk) = \nu(k^{-1})f(g),\ \forall g \in G, \forall k \in K\}. \tag{3.1}$$

Let $L^2(G,\nu)$ be the corresponding L^2-space. There is a canonical isomorphism

$$A : C^\infty(\widetilde{X},\widetilde{E}_\nu)\cong C^\infty(G,\nu) \tag{3.2}$$

which is defined by $Af(g)=R_g^{-1}(f(gK))$. It extends to an isometry

$$A\colon L^2(\widetilde{X},\widetilde{E}_\nu)\cong L^2(G,\nu). \tag{3.3}$$

Let

$$C^\infty(\Gamma\backslash G,\nu):=\{f\in C^\infty(G,\nu)\colon f(\gamma g)=f(g)\,\forall g\in G,\forall\gamma\in\Gamma\} \tag{3.4}$$

and let $L^2(\Gamma\backslash G,\nu)$ be the corresponding L^2-space. The isomorphisms (3.2) and (3.3) descend to isomorphisms

$$A : C^\infty(X,E_\nu)\cong C^\infty(\Gamma\backslash G,\nu),\quad L^2(X,E_\nu)\cong L^2(\Gamma\backslash G,\nu). \tag{3.5}$$

Let $\widetilde{\Delta}_\nu=\widetilde{\nabla^\nu}^*\widetilde{\nabla}^\nu$ be the Bochner-Laplace operator of $\widetilde{E}_\nu$. Since $\widetilde{X}$ is complete, $\widetilde{\Delta}_\nu$ with domain the space of smooth compactly supported sections is essentially self-adjoint [LM, p. 155]. Its self-adjoint extension will be denoted by $\widetilde{\Delta}_\nu$ too. With respect to the isomorphism (3.2) one has

$$\widetilde{\Delta}_\nu=-R(\Omega)+\nu(\Omega_K), \tag{3.6}$$

where R denotes the right regular representation of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ on $C^\infty(G,\nu)$ (see [Mi1, Proposition 1.1]). The heat operator

$$e^{-t\tilde{\Delta}_\nu}\colon L^2(G,\nu)\to L^2(G,\nu)$$

commutes with the action of G. Therefore, it is of the form

$$(e^{-t\tilde{\Delta}_\nu}\phi)(g)=\int_G H_t^\nu(g^{-1}g')(\phi(g'))\,dg' \tag{3.7}$$

where

$$H_t^\nu\colon G\to\operatorname{End}(V_\nu)$$

is in $C^\infty\cap L^2$ and satisfies the covariance property

$$H_t^\nu(k^{-1}gk')=\nu(k)^{-1}\circ H_t^\nu(g)\circ\nu(k'),\ \forall k,k'\in K,\forall g\in G. \tag{3.8}$$

It follows as in [BM, Proposition 2.4] that H_t^ν belongs to all Harish-Chandra Schwartz spaces $(\mathcal{C}^q(G)\otimes\operatorname{End}(V_\nu))$, $q>0$.

Now let $\|H_t^\nu(g)\|$ be the norm of $H_t^\nu(g)$ in $\operatorname{End}(V_\nu)$. Let $\widetilde{\Delta}_0$ be the Laplacian on functions on $\widetilde{X}$ and let H_t^0 be the associated heat kernel as above. We may use the principle of semigroup domination to bound $\|H_t^\nu(g)\|$ by the scalar heat kernel. Indeed we have

Proposition 3.1. *Let $\nu\in\hat{K}$. Then we have*

$$\|H_t^\nu(g)\|\le H_t^0(g)$$

for all $t\in\mathbb{R}^+$ and $g\in G$.

Proof. Let $K_\nu(t,x,y)$ be the kernel of $e^{-t\widetilde{\Delta}_\nu}$, acting in $L^2(\widetilde{X},\widetilde{E}_\nu)$. Denote by $|K_\nu(t,x,y)|$ the norm of the homomorphism

$$K_\nu(t,x,y) \in \operatorname{Hom}\left((\widetilde{E}_\nu)_y, (\widetilde{E}_\nu)_x\right).$$

It was proved in [Mu1, p. 325] that in the sense of distributions, one has

$$\left(\frac{\partial}{\partial t} + \widetilde{\Delta}_0\right) |K_\nu(t,x,y)| \leq 0,$$

where $\widetilde{\Delta}_0$ acts in the x-variable. Using (3.15) in [Mu1] one can proceed as in the proof of Theorem 4.3 of [DL] to show that

$$|K_\nu(t,x,y)| \leq K_0(t,x,y), \quad t \in \mathbb{R}^+,\, x,y \in \widetilde{X}, \tag{3.9}$$

where $K_0(t,x,y)$ is the kernel of $e^{-t\widetilde{\Delta}_0}$. See also [Gu, p. 7]. Now observe that

$$H_t^\nu(g^{-1}g') = R_g^{-1} \circ K_\nu(t,gK,g'K) \circ R_{g'} \quad \text{and} \quad H_t^0(g^{-1}g') = K_0(t,gK,g'K).$$

Since for each $x \in \widetilde{X}$, $R_g\colon (\widetilde{E}_\nu)_x \to (\widetilde{E}_\nu)_{g(x)}$ is an isometry, the proposition follows from (3.9). ☐

Now we pass to the quotient $X = \Gamma\backslash\widetilde{X}$. Let $\Delta_\nu = \nabla^{\nu *}\nabla^\nu$ be the Bochner-Laplace operator. It is essentially self-adjoint. Let R_Γ be the right regular representation of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ on $C^\infty(\Gamma\backslash G,\nu)$. By (3.6) it follows that with respect to the isomorphism (3.5) we have

$$\Delta_\nu = -R_\Gamma(\Omega) + \nu(\Omega_K). \tag{3.10}$$

Let $e^{-t\Delta_\nu}$ be the heat semigroup of Δ_ν, acting on $L^2(\Gamma\backslash G,\nu)$. Then $e^{-t\Delta_\nu}$ is represented by the smooth kernel

$$H_\nu(t,g,g') := \sum_{\gamma\in\Gamma} H_t^\nu(g^{-1}\gamma g'). \tag{3.11}$$

The convergence of the series in (3.11) can be established, for example, using Proposition 3.1 and the methods from the proof of Proposition 3.2 below. Put

$$h_t^\nu(g) := \operatorname{tr} H_t^\nu(g), \quad g \in G, \tag{3.12}$$

where $\operatorname{tr}\colon \operatorname{End}(V_\nu) \to \mathbb{C}$ is the matrix trace. Then the trace of the heat operator $e^{-t\Delta_\nu}$ is given by

$$\operatorname{Tr}(e^{-t\Delta_\nu}) = \int_{\Gamma\backslash G} \operatorname{tr} H_\nu(t,g,g)\, d\dot{g} = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} h_t^\nu(g^{-1}\gamma g) d\dot{g}. \tag{3.13}$$

Using results of Donnelly we now prove an estimate for the heat kernel H_t^0 of the Laplacian $\widetilde{\Delta}_0$ acting on $C^\infty(\widetilde{X})$.

Proposition 3.2. *There exist constants C_0 and c_0 such that for every $t \in (0,1]$ and every $g \in G$ one has*

$$\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq 1}} H_t^0(g^{-1}\gamma g) \leq C_0 e^{-c_0/t}.$$

Proof. For $x, y \in \widetilde{X}$ let $\rho(x,y)$ denote the geodesic distance of x, y. Since $K(t, gK, g'K) = H_t^0(g^{-1}g')$ is the kernel of $e^{-t\widetilde{\Delta}_0}$, it follows from [Do1, Theorem 3.3] that there exists a constant C_1 such that for every $g \in G$ and every $t \in (0,1]$ one has

$$H_t^0(g) \leq C_1 t^{-\frac{d}{2}} \exp\left(-\frac{\rho^2(gK, 1K)}{4t}\right). \tag{3.14}$$

Let $x \in \widetilde{X}$ and let $B_R(x)$ be the metric ball around x of radius R. Let $h > 0$ be the topological entropy of the geodesic flow of X (see [Ma]). There exists $C_2 > 0$ such that

$$\operatorname{vol} B_R(x) \leq C_2 e^{hR}, \quad R > 0 \tag{3.15}$$

[Ma]. Since Γ is cocompact and torsion-free, there exists an $\epsilon > 0$ such that $B_\epsilon(x) \cap \gamma B_\epsilon(x) = \emptyset$ for every $\gamma \in \Gamma - \{1\}$ and every $x \in \widetilde{X}$. Thus for every $x \in \widetilde{X}$ the union over all $\gamma B_\epsilon(x)$, where $\gamma \in \Gamma$ is such that $\rho(x, \gamma x) \leq R$ is disjoint and is contained in $B_{R+\epsilon}(x)$. Using (3.15) it follows that there exists a constant C_3 such that for every $x \in \widetilde{X}$ one has

$$\#\{\gamma \in \Gamma \colon \rho(x, \gamma x) \leq R\} \leq C_3 e^{hR}.$$

Hence there exists a constant $C_4 > 0$ such that for every $x \in \widetilde{X}$ one has

$$\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq 1}} e^{-\frac{\rho^2(\gamma x, x)}{8}} \leq C_4. \tag{3.16}$$

Now let

$$c_1 := \inf\{\rho(x, \gamma x) \colon \gamma \in \Gamma - \{1\},\ x \in \widetilde{X}\}.$$

We have $c_1 > 0$. Using (3.14) and (3.16), it follows that there are constants $c_0 > 0$ and $C_0 > 0$ such that for every $g \in G$ and $0 < t \leq 1$ we have

$$\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq 1}} H_t^0(g^{-1}\gamma g) \leq C_1 t^{-\frac{d}{2}} e^{-c_1^2/(8t)} \sum_{\substack{\gamma\in\Gamma\\ \gamma\neq 1}} e^{-\rho^2(\gamma gK, gK)/8} \leq C_0 e^{-c_0/t}.$$

□

4. The analytic torsion

Let τ be an irreducible finite-dimensional representation of G on V_τ. Let E_τ be the flat vector bundle over X associated to the restriction of τ to Γ. Let $\widetilde{E}^\tau$ be the homogeneous vector bundle associated to $\tau|_K$ and let $E^\tau := \Gamma\backslash\widetilde{E}^\tau$. There is a canonical isomorphism

$$E^\tau \cong E_\tau \tag{4.1}$$

[MtM, Proposition 3.1]. By [MtM, Lemma 3.1], there exists an inner product $\langle\cdot,\cdot\rangle$ on V_τ such that

(1) $\langle\tau(Y)u,v\rangle = -\langle u,\tau(Y)v\rangle$ for all $Y\in\mathfrak{k}$, $u,v\in V_\tau$
(2) $\langle\tau(Y)u,v\rangle = \langle u,\tau(Y)v\rangle$ for all $Y\in\mathfrak{p}$, $u,v\in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\tau|_K$ is unitary with respect to this inner product, it induces a metric on E^τ, and by (4.1) on E_τ, which we also call admissible. Let $\Lambda^p(E_\tau)=\Lambda^pT^*(X)\otimes E_\tau$. Let

$$\nu_p(\tau):=\Lambda^p\operatorname{Ad}^*\otimes\tau:\ K\to\operatorname{GL}(\Lambda^p\mathfrak{p}^*\otimes V_\tau). \tag{4.2}$$

Then there is a canonical isomorphism

$$\Lambda^p(E_\tau)\cong\Gamma\backslash(G\times_{\nu_p(\tau)}\Lambda^p\mathfrak{p}^*\otimes V_\tau). \tag{4.3}$$

of locally homogeneous vector bundles. Let $\Lambda^p(X,E_\tau)$ be the space of smooth E_τ-valued p-forms on X. The isomorphism (4.3) induces an isomorphism

$$\Lambda^p(X,E_\tau)\cong C^\infty(\Gamma\backslash G,\nu_p(\tau)), \tag{4.4}$$

where the latter space is defined as in (3.4). A corresponding isomorphism also holds for the spaces of L^2-sections. Let $\Delta_p(\tau)$ be the Hodge-Laplacian on $\Lambda^p(X,E_\tau)$ with respect to the admissible metric in E_τ. By [MtM, (6.9)] it follows that with respect to the isomorphism (4.4) one has

$$\Delta_p(\tau)f=-R_\Gamma(\Omega)f+\tau(\Omega)\operatorname{Id}f,\ \ f\in C^\infty(\Gamma\backslash G,\nu_p(\tau)). \tag{4.5}$$

Let

$$K(t,\tau):=\sum_{p=1}^d(-1)^pp\operatorname{Tr}(e^{-t\Delta_p(\tau)}). \tag{4.6}$$

and

$$h(\tau):=\sum_{p=1}^d(-1)^pp\dim H^p(X,E_\tau). \tag{4.7}$$

Then $K(t,\tau)-h(\tau)$ decays exponentially as $t\to\infty$ and it follows from (1.1) that

$$\log T_X(\tau)=\frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(K(t,\tau)-h(\tau))\,dt\right)\bigg|_{s=0}, \tag{4.8}$$

where the right hand side is defined near $s=0$ by analytic continuation of the Mellin transform. Let $\widetilde{E}_{\nu_p(\tau)}:=G\times_{\nu_p(\tau)}\Lambda^p\mathfrak{p}^*\otimes V_\tau$ and let $\widetilde{\Delta}_p(\tau)$ be the lift of $\Delta_p(\tau)$ to $C^\infty(\widetilde{X},\widetilde{E}_{\nu_p(\tau)})$. Then again it follows from [MtM, (6.9)] that on $C^\infty(G,\nu_p(\tau))$ one has

$$\widetilde{\Delta}_p(\tau)=-R(\Omega)+\tau(\Omega)\operatorname{Id}. \tag{4.9}$$

Let $e^{-t\widetilde{\Delta}_p(\tau)}$ be the corresponding heat semigroup on $L^2(G,\nu_p(\tau))$. It is a smoothing operator which commutes with the action of G. Therefore, it is of the form

$$\left(e^{-t\widetilde{\Delta}_p(\tau)}\phi\right)(g)=\int_G H_t^{\tau,p}(g^{-1}g')\phi(g')\,dg',\quad \phi\in L^2(G,\nu_p(\tau),\ \ g\in G,$$

where the kernel

$$H_t^{\tau,p}\colon G\to\operatorname{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau) \tag{4.10}$$

belongs to $C^\infty\cap L^2$ and satisfies the covariance property

$$H_t^{\tau,p}(k^{-1}gk')=\nu_p(\tau)(k)^{-1}H_t^{\tau,p}(g)\nu_p(\tau)(k') \tag{4.11}$$

with respect to the representation (4.2). Moreover, for all $q>0$ we have

$$H_t^{\tau,p}\in(\mathcal{C}^q(G)\otimes\operatorname{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau))^{K\times K}, \tag{4.12}$$

where $\mathcal{C}^q(G)$ denotes Harish-Chandra's L^q-Schwartz space. The proof is similar to the proof of Proposition 2.4 in [BM]. Now we come to the heat kernel of $\Delta_p(\tau)$. First the integral kernel of $e^{-t\Delta_p(\tau)}$, regarded as an operator in $L^2(\Gamma\backslash G,\nu_p(\tau))$, is given by

$$H^{\tau,p}(t;g,g'):=\sum_{\gamma\in\Gamma}H_t^{\tau,p}(g^{-1}\gamma g'), \tag{4.13}$$

As in section 3 this series converges absolutely and locally uniformly. Therefore the trace of the heat operator $e^{-t\Delta_p(\tau)}$ is given by

$$\operatorname{Tr}\left(e^{-t\Delta_p(\tau)}\right)=\int_{\Gamma\backslash G}\operatorname{tr}H^{\tau,p}(t;g,g)\,d\dot{g},$$

where tr denotes the trace $\operatorname{tr}\colon\operatorname{End}(V_\nu)\to\mathbb{C}$. Let

$$h_t^{\tau,p}(g):=\operatorname{tr}H_t^{\tau,p}(g). \tag{4.14}$$

Using (4.13) we obtain

$$\operatorname{Tr}\left(e^{-t\Delta_p(\tau)}\right)=\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}h_t^{\tau,p}(g^{-1}\gamma g)\,d\dot{g}. \tag{4.15}$$

Put

$$k_t^\tau=\sum_{p=1}^d(-1)^p p\,h_t^{\tau,p}. \tag{4.16}$$

Then it follows that

$$K(t,\tau)=\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}k_t^\tau(g^{-1}\gamma g)\,d\dot{g}. \tag{4.17}$$

Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$. Then (4.17) can be written as

$$K(t,\tau)=\operatorname{Tr}R_\Gamma(k_t^\tau). \tag{4.18}$$

We shall now compute the Fourier transform of k_t^τ. To begin with let π be an admissible unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. Set

$$\tilde{\pi}(H_t^{\tau,p}) = \int_G \pi(g) \otimes H_t^{\tau,p}(g)\, dg.$$

This defines a bounded operator on $\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau$. As in [BM, pp. 160-161] it follows from (4.11) that relative to the splitting

$$\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau = (\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K \oplus \left[(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K\right]^\perp,$$

$\tilde{\pi}(H_t^{\tau,p})$ has the form

$$\tilde{\pi}(H_t^{\tau,p}) = \begin{pmatrix} \pi(H_t^{\tau,p}) & 0 \\ 0 & 0 \end{pmatrix}$$

with $\pi(H_t^{\tau,p})$ acting on $(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K$. Using (4.9) it follows as in [BM, Corollary 2.2] that

$$\pi(H_t^{\tau,p}) = e^{t(\pi(\Omega) - \tau(\Omega))}\, \mathrm{Id} \tag{4.19}$$

on $(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K$. Let $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{e_j\}_{j=1}^m$ be orthonormal bases of $\mathcal{H}_\pi$ and $\Lambda^p \mathfrak{p}^* \otimes V_\tau$, respectively. Then we have

$$\begin{aligned} \operatorname{Tr} \tilde{\pi}(H_t^{\tau,p}) &= \sum_{n=1}^\infty \sum_{j=1}^m \langle \tilde{\pi}(H_t^{\tau,p})(\xi_n \otimes e_j), (\xi_n \otimes e_j) \rangle \\ &= \sum_{n=1}^\infty \sum_{j=1}^m \int_G \langle \pi(g)\xi_n, \xi_n \rangle \langle H_t^{\tau,p}(g) e_j, e_j \rangle\, dg \\ &= \sum_{n=1}^\infty \int_G h_t^{\tau,p}(g) \langle \pi(g)\xi_n, \xi_n \rangle\, dg \\ &= \operatorname{Tr} \pi(h_t^{\tau,p}). \end{aligned} \tag{4.20}$$

Let $\pi \in \hat{G}$ and let Θ_π denote its character. Then it follows from (4.16), (4.19) and (4.20) that

$$\Theta_\pi(k_t^\tau) = e^{t(\pi(\Omega) - \tau(\Omega))} \sum_{p=1}^d (-1)^p p \cdot \dim(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K. \tag{4.21}$$

Now we consider the case of a principle series representation. Let Q be a standard cuspidal parabolic subgroup. Let $Q = MAN$ be the Langlands decomposition of Q. Denote by $\mathfrak{a}$ the Lie algebra of A. Let $K_M = K \cap M$. Let (ξ, W_ξ) be a discrete series representation of M and let $\nu \in \mathfrak{a}_\mathbb{C}^*$. Let $\pi_{\xi,\nu}$ be the induced representation and let $\Theta_{\xi,\nu}$ be the global character of $\pi_{\xi,\nu}$ (see section 2).

Proposition 4.1. *Let $Y \in \mathfrak{a}$ be a unit vector and let $\mathfrak{p}_Y$ be the orthogonal complement of Y in $\mathfrak{p}$. Then*

(i) $\Theta_{\xi,\nu}(k_t^\tau) = e^{t(\pi_{\xi,\nu}(\Omega) - \tau(\Omega))} \dim \left(W_\xi \otimes (\Lambda^{\mathrm{odd}} \mathfrak{p}_Y^* - \Lambda^{\mathrm{ev}} \mathfrak{p}_Y^*) \otimes V_\tau\right)^{K_M},$

(ii) $\Theta_{\xi,\nu}(k_t^\tau)=0$ *if* $\dim\mathfrak{a}_\mathfrak{q}\geq 2$.

Proof. By Frobenius reciprocity [Kn1, p. 208] and (4.21) we get

$$\Theta_{\xi,\nu}(k_t^\tau)=e^{t(\pi_{\xi,\nu}(\Omega)-\tau(\Omega))}\sum_{p=1}^{d}(-1)^p p\dim\left(W_\xi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^{K_M}.$$

Now

$$\mathfrak{p}^*=\mathbb{R}Y^*\oplus\mathfrak{p}_Y^*$$

as K_M-module. Therefore, in the Grothendieck ring of K_M we have

$$\begin{aligned}\sum_{p=1}^{d}(-1)^p p\Lambda^p\mathfrak{p}^*&=\sum_{p=1}^{d}(-1)^p p[\Lambda^p\mathfrak{p}_Y^*\oplus\Lambda^{p-1}\mathfrak{p}_Y^*]^K\\&=\sum_{p=1}^{d}(-1)^p p\Lambda^p\mathfrak{p}_Y^*+\sum_{p=0}^{d-1}(-1)^{p+1}(p+1)\Lambda^p\mathfrak{p}_Y^*\\&=\sum_{p=0}^{d}(-1)^{p+1}\Lambda^p\mathfrak{p}_Y^*.\end{aligned}\tag{4.22}$$

Tensoring with W_ξ and V_τ and taking K_M-invariants, we obtain (i).

To prove (ii), suppose that there is a nonzero $H\in\mathfrak{a}\cap\mathfrak{p}_Y$. Since M centralizes H, $\varepsilon(H)+i(H)$ is a K_M intertwining operator between $\Lambda^{\mathrm{ev}}\mathfrak{p}_Y^*$ and $\Lambda^{\mathrm{odd}}\mathfrak{p}_Y^*$, and non-trivial since $H\neq 0$. Hence $\Lambda^{\mathrm{ev}}\mathfrak{p}_Y^*$ and $\Lambda^{\mathrm{odd}}\mathfrak{p}_Y^*$ are equivalent as K_M-modules and (ii) follows. ☐

Proposition 4.2. *Assume that* $\delta(\tilde{X})\geq 2$ *or that* $\tilde{X}$ *is even-dimensional. Then* $T_X(\tau)=1$ *for all finite-dimensional irreducible representations* τ *of* G.

Proof. Let

$$R_\Gamma=\bigoplus_{\pi\in\hat{G}}m_\Gamma(\pi)\pi$$

be the decomposition of the right regular representation R_Γ of G on $L^2(\Gamma\backslash G)$, see [Wa1, section 1]. Then by (4.18) we have

$$K(t,\tau)=\sum_{\pi\in\hat{G}}m_\Gamma(\pi)\Theta_\pi(k_t^\tau).\tag{4.23}$$

The series on the right hand side is absolutely convergent. First assume that $\delta(X)\geq 2$. By [De, section 2.2] the Grothendieck group of all admissible representations of G is generated by the representations $\pi_{\xi,\lambda}$, where $\pi_{\xi,\lambda}$ is associated to some standard cuspidal parabolic subgroup Q of G as in (2.1). Since $\delta(X)\geq 2$ one has $\Theta_{\xi,\lambda}(k_t^\tau)=0$ for every such representation by Proposition 4.1. Thus one has $\Theta_\pi(k_t^\tau)=0$ for every irreducible unitary representation of G. By (4.23) it follows that $K(t,\tau)=0$. Let $h(\tau)$ be as in (4.7). Since $K(t,\tau)-h(\tau)$ decays exponentially as $t\to\infty$, it follows that $K(t,\tau)-h(\tau)=0$ and using (4.8), the first statement follows.

Now assume that $d = \dim \widetilde{X}$ is even. Note that as K-modules we have

$$\Lambda^p \mathfrak{p}^* \cong \Lambda^{d-p} \mathfrak{p}^*, \quad p = 0, \dots, d.$$

Since d is even, it follows that in the representation ring $R(K)$ we have the following equality

$$\sum_{p=0}^{d} (-1)^p p \Lambda^p \mathfrak{p}^* = \frac{d}{2} \sum_{p=0}^{d} (-1)^p \Lambda^p \mathfrak{p}^*.$$

Let $(\pi, \mathcal{H}_\pi) \in \hat{G}$. Then it follows from (4.21) that

$$\Theta_\pi(k_t^\tau) = \frac{d}{2} e^{t(\pi(\Omega)-\tau(\Omega))} \sum_{p=0}^{d} (-1)^p \dim(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K.$$

Let $\mathcal{H}_{\pi,K}$ be the subspace of $\mathcal{H}_\pi$ consisting of all smooth K-finite vectors. Then

$$(\mathcal{H}_{\pi,K} \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K = (\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K.$$

Thus the $(\mathfrak{g}, K)$-cohomology $H^*(\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau)$ is computed from the Lie algebra cohomology complex $([\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau]^K, d)$ (see [BW]). Using the Poincaré principle we get

$$\Theta_\pi(k_t^\tau) = \frac{d}{2} e^{t(\pi(\Omega)-\tau(\Omega))} \sum_{p=0}^{d} (-1)^p \dim H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau). \tag{4.24}$$

Now by [BW, II.3.1, I.5.3] we have

$$H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau) = \begin{cases} [\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau]^K, & \pi(\Omega) = \tau(\Omega); \\ 0, & \pi(\Omega) \neq \tau(\Omega). \end{cases} \tag{4.25}$$

Hence for every $\pi \in \hat{G}$ one has $\Theta_\pi(k_t^\tau) \in \mathbb{Z}$ and $\Theta_\pi(k_t^\tau)$ is independent of $t > 0$. Thus by (4.23), $K(t, \tau)$ is independent of $t > 0$. Let $h(\tau)$ be defined by (4.7). Then $K(t, \tau) - h(\tau) = O(e^{-ct})$ as $t \to \infty$. Hence $K(t, \tau) = h(\tau)$. By (4.8) it follows that $T_X(\tau) = 1$. $\square$

5. L^2-torsion

In this section we study the L^2-torsion $T_X^{(2)}(\tau)$. For its definition we refer to [Lo]. Actually, in [Lo] only the case of the trivial representation τ_0 has been discussed. However the extension to a nontrivial τ is straight forward. The definition is based on the Γ-trace of the heat operator $e^{-t\widetilde{\Delta}_p(\tau)}$ on the universal covering $\widetilde{X}$ (see [Lo]). For our purposes, it suffices to introduce the L^2-torsion for representations τ on $\widetilde{X}$ which satisfy $\tau_\theta \not\cong \tau$.
Let $h_t^{\tau,p}$ be the function defined by (4.14). By homogeneity it follows that in our case the Γ-trace is given by

$$\operatorname{Tr}_\Gamma \left(e^{-t\widetilde{\Delta}_p(\tau)} \right) = \operatorname{vol}(X) h_t^{\tau,p}(1). \tag{5.1}$$

In order to define the L^2-torsion we need to know the asymptotic behavior of $h_t^{\tau,p}(1)$ as $t\to 0$ and $t\to\infty$. First we consider the behavior as $t\to 0$. Using (4.15) we have

$$\operatorname{vol}(X)h_t^{\tau,p}(1)=\operatorname{Tr}\left(e^{-t\Delta_p(\tau)}\right)-\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma-\{1\}}h_t^{\tau,p}(g^{-1}\gamma g)\,d\dot{g}. \tag{5.2}$$

To deal with the second term on the right, we consider the representation $\nu_p(\tau)$ of K which is defined by (4.2), and for $p=0,\dots,n$ we put

$$E_p(\tau):=\tau(\Omega)\operatorname{Id}-\nu_p(\tau)(\Omega_K), \tag{5.3}$$

which we regard as endomorphism of $\Lambda^p\mathfrak{p}^*\otimes V_\tau$. It defines endomorphisms of $\Lambda^pT^*(\widetilde{X})\otimes\widetilde{E}_\tau$ and of $\Lambda^pT^*(X)\otimes E_\tau$. By (3.6) and (4.9) for the Bochner-Laplace operator $\widetilde{\Delta}_{\nu_p(\tau)}$ and the Hodge-Laplace operator $\widetilde{\Delta}_p(\tau)$ on the bundle $\widetilde{E}_{\nu_p(\tau)}$ we have

$$\widetilde{\Delta}_p(\tau)=\widetilde{\Delta}_{\nu_p(\tau)}+E_p(\tau). \tag{5.4}$$

Similarly, by (3.10) and (4.5) for the corresponding operators on $E_{\nu_p(\tau)}$ we have

$$\Delta_p(\tau)=\Delta_{\nu_p(\tau)}+E_p(\tau). \tag{5.5}$$

Let $\nu_p(\tau)=\oplus_{\sigma\in\hat{K}}m(\sigma)\sigma$ be the decomposition of $\nu_p(\tau)$ into irreducible representations. This induces a corresponding decomposition of the homogeneous vector bundle

$$\widetilde{E}_{\nu_p(\tau)}=\bigoplus_{\sigma\in\hat{K}}m(\sigma)\widetilde{E}_\sigma.$$

With respect to this decomposition we have

$$E_p(\tau)=\bigoplus_{\sigma\in\hat{K}}m(\sigma)\left(\tau(\Omega)-\sigma(\Omega_K)\right)\operatorname{Id}_{V_\sigma}, \tag{5.6}$$

where $\sigma(\Omega_K)$ is the Casimir eigenvalue of σ and V_σ is the representation space of σ, and

$$\widetilde{\Delta}_{\nu_p(\tau)}=\bigoplus_{\sigma\in\hat{K}}m(\sigma)\widetilde{\Delta}_\sigma. \tag{5.7}$$

This shows that $\widetilde{\Delta}_{\nu_p(\tau)}(\tau)$ commutes with $E_p(\tau)$. Let $H_t^{\nu_p(\tau)}$ be the kernel of $e^{-t\widetilde{\Delta}_{\nu_p(\tau)}}$ and let $H_t^{\tau,p}$ be the kernel of $e^{-t\widetilde{\Delta}_p(\tau)}$. Using (5.4) we get

$$H_t^{\tau,p}(g)=e^{-tE_p(\tau)}\circ H_t^{\nu_p(\tau)}(g),\quad g\in G. \tag{5.8}$$

Let $c\in\mathbb{R}$ be such that $E_p(\tau)\geq c$. By Proposition 3.1 it follows that

$$\|H_t^{\tau,p}(g)\|\leq e^{-ct}H_t^0(g),\quad t\in\mathbb{R}^+,\, g\in G. \tag{5.9}$$

Taking the trace in $\operatorname{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau)$ we get

$$\sum_{\gamma\in\Gamma-\{1\}}|h_t^{\tau,p}(g^{-1}\gamma g)|\leq\binom{d}{p}\dim(\tau)e^{-ct}\sum_{\gamma\in\Gamma-\{1\}}H_t^0(g^{-1}\gamma g),\quad t\in\mathbb{R}^+,\, g\in G. \tag{5.10}$$

Thus by Proposition 3.2 there exist $C_1, c_1 > 0$ such that

(5.11) $$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma-\{1\}} |h_t^{\tau,p}(g^{-1}\gamma g)|\, d\dot{g} \le C_1 e^{-c_1/t}$$

for $0 < t \le 1$. Thus by (5.2)

$$h_t^{\tau,p}(1) = \frac{1}{\operatorname{vol}(X)} \operatorname{Tr}\left(e^{-t\Delta_p(\tau)}\right) + O(e^{-c_1/t})$$

for $0 < t \le 1$. Using the asymptotic expansion of $\operatorname{Tr}\left(e^{-t\Delta_p(\tau)}\right)$ (see [Gi]), it follows that there is an asymptotic expansion

(5.12) $$h_t^{\tau,p}(1) \sim \sum_{j=0}^{\infty} a_j t^{-d/2+j}$$

as $t \to 0$. To study the behavior of $h_t^{\tau,p}(1)$ as $t \to \infty$, we use the Plancherel theorem, which can be applied since $h_t^{\tau,p}$ is a K-finite Schwarz function. Let π be an admissible unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. It follows from (4.19) and (4.20) that

$$\operatorname{Tr}\pi(h_t^{\tau,p}) = e^{t(\pi(\Omega)-\tau(\Omega))} \dim\left(\mathcal{H}_\pi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau\right)^K.$$

Let $Q = MAN$ be a standard parabolic subgroup of G. Let (ξ, W_ξ) be a discrete series representation of M. Let $\langle\cdot,\rangle$ denote the inner product on the real vector space $\mathfrak{a}^*$ induced by the Killing form. Fix positive restricted roots of $\mathfrak{a}$ and let $\rho_{\mathfrak{a}}$ denote the corresponding half-sum of these roots. Define a constant $c(\xi)$ by

(5.13) $$c(\xi) := -\langle\rho_{\mathfrak{a}}, \rho_{\mathfrak{a}}\rangle + \xi(\Omega_M).$$

Recall that for $\nu \in \mathfrak{a}^*$ one has

(5.14) $$\pi_{\xi,\nu}(\Omega) = -\langle\nu,\nu\rangle + c(\xi).$$

Then by the Plancherel theorem, [HC, Theorem 3] and (5.14) we have

$$h_t^{\tau,p}(1) = \sum_Q \sum_{\xi\in\hat{M}_d} e^{-t(\tau(\Omega)-c(\xi))} \int_{\mathfrak{a}^*} e^{-t\|\nu\|^2} \dim\left(\mathcal{H}_{\xi,\nu} \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau\right)^K p_\xi(i\nu)\, d\nu.$$

Here the outer sum is over all association classes of standard cuspidal parabolic subgroups of G and $p_\xi(i\nu)$, the Plancherel-density associated to $\pi_{\xi,\nu}$, is of polynomial growth in ν. Let $K_M = K \cap M$. By Frobenius reciprocity we have

(5.15) $$\dim\left(\mathcal{H}_{\xi,\nu} \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau\right)^K = \dim\left(W_\xi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau\right)^{K_M}.$$

Thus we get

(5.16) $$h_t^{\tau,p}(1) = \sum_Q \sum_{\xi\in\hat{M}_d} \dim\left(W_\xi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau\right)^{K_M} e^{-t(\tau(\Omega)-c(\xi))} \int_{\mathfrak{a}^*} e^{-t\|\nu\|^2} p_\xi(i\nu)\, d\nu.$$

The exponents of the exponential factors in front of the integrals are controlled by the following lemma.

Lemma 5.1. *Let $(\tau, V_\tau) \in \mathrm{Rep}(G)$. Assume that $\tau \ncong \tau_\theta$. Let $Q = MAN$ be a cuspidal parabolic subgroup of G. Let $\xi \in \hat{M}_d$ and assume that* $\dim\left(W_\xi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau\right)^{K_M} \neq 0$. *Then one has*

$$\tau(\Omega) - c(\xi) > 0.$$

Proof. Assume that $\tau(\Omega) - c(\xi) \leq 0$. Then by (5.14) there exists a $\nu_0 \in \mathfrak{a}^*$ such that

$$\pi_{\xi,\nu_0}(\Omega) = \tau(\Omega).$$

Together with (5.15), our assumption and [BW, Proposition II.3.1] it follows that

$$\dim\left(H^p(\mathfrak{g}, K; \mathcal{H}_{\xi,\nu_0,K} \otimes V_\tau)\right) \neq 0,$$

where $\mathcal{H}_{\xi,\nu_0,K}$ are the K-finite vectors in $\mathcal{H}_{\xi,\nu_0}$ Since $\tau \ncong \tau_\theta$, this is a contradiction to the first statement of [BW, Proposition II. 6.12]. ☐

Let $\tau \in \mathrm{Rep}(G)$ and assume that τ satisfies $\tau \ncong \tau_\theta$. It follows from (5.16) and Lemma 5.1 that there exists $c > 0$ such that

$$h_t^{\tau,p}(1) = O\left(e^{-ct}\right) \tag{5.17}$$

as $t \to \infty$. Using (5.12) and (5.17) it follows from standard methods, see for example [Gi], that the Mellin transform

$$\int_0^\infty h_t^{\tau,p}(1) t^{s-1}\, dt$$

converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$ if $d = \dim(\tilde{X})$ is odd and has at most a simple pole at $s = 0$ for $d = \dim(\tilde{X})$ even. Thus we can define the L^2-torsion $T_X^{(2)}(\tau) \in \mathbb{R}^+$ by

$$\log T_X^{(2)}(\tau) := \frac{1}{2} \sum_{p=1}^{d} (-1)^p p \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}_\Gamma \left(e^{-t\widetilde{\Delta}_p(\tau)} \right) t^{s-1}\, dt \right) \bigg|_{s=0}, \tag{5.18}$$

where the right hand side is defined near $s = 0$ by analytic continuation. For $t > 0$ let

$$K^{(2)}(t, \tau) := \sum_{p=1}^{d} (-1)^p p h_t^{\tau,p}(1). \tag{5.19}$$

Put

$$t_{\widetilde{X}}^{(2)}(\tau) := \frac{1}{2} \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty K^{(2)}(t,\tau) t^{s-1}\, dt \right) \bigg|_{s=0}. \tag{5.20}$$

Then $t_{\widetilde{X}}^{(2)}(\tau)$ depends only on the symmetric space $\widetilde{X}$ and τ, and we have

$$\log T_X^{(2)}(\tau) = \mathrm{vol}(X) \cdot t_{\widetilde{X}}^{(2)}(\tau). \tag{5.21}$$

Next we establish an auxiliary result concerning the twisted Euler characteristic. We let $\tau \in \mathrm{Rep}(G)$ be arbitrary. Let $\mathcal{H}^p(X, E_\tau) := \ker \Delta_p(\tau)$ be the space of E_τ-valued harmonic p-forms. Let

$$\chi(X, E_\tau) := \sum_{p=0}^{d} (-1)^p \dim \mathcal{H}^p(X, E_\tau)$$

be the twisted Euler characteristic. Furthermore, let $\tilde{X}_d$ denote the compact dual of $\tilde{X}$.

Proposition 5.2. *If $\delta(\tilde{X}) \neq 0$, we have $\chi(X, E_\tau) = 0$. If $\delta(\tilde{X}) = 0$, one has*

$$\chi(X, E_\tau) = (-1)^n \operatorname{vol}(X) \frac{\chi(\tilde{X}_d)}{\operatorname{vol}(\tilde{X}_d)} \dim(\tau), \tag{5.22}$$

where $n = \dim(\widetilde{X})/2$.

Proof. Let $\pi \in \hat{G}$. It follows from (4.19) and (4.21) that

$$\sum_{p=0}^{d} (-1)^p \Theta_\pi(h_t^{p,\tau}) = e^{t(\pi(\Omega) - \tau(\Omega))} \sum_{p=0}^{d} (-1)^p \dim(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K.$$

Using [BW, II.3.1] and the Poincaré principle as in the proof of Proposition 4.2, we get

$$\sum_{p=0}^{d} (-1)^p \Theta_\pi(h_t^{p,\tau}) = \sum_{p=0}^{d} (-1)^p \dim H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau). \tag{5.23}$$

Now by [BW, Theorem I.5.3] it follows that if $H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau) \neq 0$, then $\chi_\pi = \chi_{\check{\tau}}$, where $\check{\tau}$ is the contragredient representation of τ. By [Kn1, Corollary 10.37, Corollary 9.2] there are only finitely many representations $\pi \in \hat{G}$ with a given infinitesimal character. Thus if $Q = MAN$ is a fundamental parabolic subgroup with $Q \neq G$ and if $\xi \in \hat{M}_d$, it follows that there are only finitely many $\lambda \in \mathfrak{a}^*$ such that

$$\sum_{p=0}^{d} (-1)^p \Theta_{\xi,\lambda}(h_t^{p,\tau}) \neq 0. \tag{5.24}$$

Hence by the Plancherel-Theorem, [HC, Theorem 3] and (5.23) we get

$$\sum_{p=0}^{d} (-1)^p h_t^{p,\tau}(1) = \sum_{p=0}^{d} (-1)^p \sum_{\pi \in \hat{G}_d} d(\pi) \dim H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau), \tag{5.25}$$

where $\hat{G}_d$ denotes the discrete series of G and $d(\pi)$ denotes the formal degree of π. The sum is finite. Let

$$b_p^{(2)}(X, E_\tau) := \lim_{t \to \infty} \operatorname{Tr}_\Gamma \left(e^{-t\tilde{\Delta}_p(\tau)} \right)$$

be the L^2-Betti number. Using that (5.25) is independent of t and (5.1), we get

$$\operatorname{vol}(X)\sum_{p=0}^{d}(-1)^p h_t^{p,\tau}(1)=\sum_{p=0}^{d}(-1)^p b_p^{(2)}(X,E_\tau)=\chi^{(2)}(X,E_\tau). \tag{5.26}$$

By the Γ-index theorem of Atiyah [At] we have $\chi^{(2)}(X,E_\tau)=\chi(X,E_\tau)$. Hence by (5.25) and (5.26) we get

$$\chi(X,E_\tau)=\operatorname{vol}(X)\cdot\sum_{p=0}^{d}(-1)^p\sum_{\pi\in\hat{G}_d}d(\pi)\dim H^p(\mathfrak{g},K;\mathcal{H}_{\pi,K}\otimes V_\tau). \tag{5.27}$$

If $\delta(\tilde{X})\neq 0$ then $\hat{G}_d$ is empty and hence, this sum equals zero, which proves the first statement. Now assume that $\delta(\tilde{X})=0$. Then $\tilde{X}$ is even-dimensional. Let $\dim(\tilde{X})=2n$. We keep the notation from section 2.2. By [Ol, Corollary 5.2] for $\Lambda'=w(\Lambda(\check{\tau})+\rho_G)$, $w\in W_G/W_K$ one has

$$d(\omega_{\Lambda'})=\frac{\dim(\tau)}{\operatorname{vol}(\tilde{X}_d)}$$

and so together with Proposition 2.1 we get

$$\sum_{p=0}^{d}(-1)^p\sum_{\pi\in\hat{G}_d}d(\pi)\dim H^p(\mathfrak{g},K;\mathcal{H}_{\pi,K}\otimes V_\tau)=(-1)^n\frac{1}{\operatorname{vol}(\tilde{X}_d)}\#(W_G/W_K)\dim(\tau). \tag{5.28}$$

Finally, by [Bo, page 175] one has

$$\#(W_G/W_K)=\chi(\tilde{X}_d).$$

Applying equation (5.28), the proof of the Proposition follows. □

Remark 1. We remark that if X is Hermitian and τ is the trivial representation, then equation (5.22) reduces to Hirzebruch's Proportionality principle.

Now we assume that $\delta(\widetilde{X})=1$ and that $\widetilde{X}$ is odd-dimensional. By the classification of simple Lie groups we have $\widetilde{X}=\widetilde{X}_0\times\widetilde{X}_1$, where $\delta(\widetilde{X}_0)=0$ and $\widetilde{X}_1=\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ or $\widetilde{X}_1=\mathrm{Spin}(p,q)/(\mathrm{Spin}(p)\times\mathrm{Spin}(q))$, p,q odd. Let $\widetilde{X}_0=G_0/K_0$ and let $G_1=\mathrm{SL}(3,\mathbb{R})$, $K_1=\mathrm{SO}(3)$ or $G_1=\mathrm{Spin}(p,q)$, $K_1=\mathrm{Spin}(p)\times\mathrm{Spin}(q)$, p,q odd. Let $G=G_0\times G_1$. Let τ be a finite-dimensional irreducible representation of G and assume that $\tau\ncong\tau_\theta$. Then $\tau=\tau_0\otimes\tau_1$, where τ_i is an irreducible representation of G_i, $i=0,1$, and $\tau_1\ncong\tau_{1,\theta}$.

Proposition 5.3. *Let $\delta(\widetilde{X})=1$ and assume that $\widetilde{X}$ is odd-dimensional. Let $\widetilde{X}=\widetilde{X}_0\times\widetilde{X}_1$, where $\widetilde{X}_1$ is an odd-dimensional irreducible symmetric space with $\delta(\widetilde{X}_1)=1$. Let τ be a finite-dimensional irreducible representation of G with $\tau\ncong\tau_\theta$. Then*

$$t^{(2)}_{\widetilde{X}}(\tau)=(-1)^{\dim\widetilde{X}_0/2}\frac{\chi(\widetilde{X}_{0,d})}{\operatorname{vol}(\widetilde{X}_{0,d})}\dim\tau_0\cdot t^{(2)}_{\widetilde{X}_1}(\tau_1).$$

Proof. Let $\widetilde{E} \to \widetilde{X}$ be the homogeneous vector bundle associated to $\tau|_K$. Similarly, let $\widetilde{E}_i \to \widetilde{X}_i$ be the homogeneous vector bundle associated to $\tau_i|_{K_i}$, $i = 0, 1$. Then $\widetilde{E} \cong \widetilde{E}_1 \boxtimes \widetilde{E}_2$ and

$$\Lambda^k(\widetilde{X}, \widetilde{E}) \cong \bigoplus_{p+q=k} \left(\Lambda^p(\widetilde{X}_0, \widetilde{E}_0) \otimes \Lambda^q(\widetilde{X}_1, \widetilde{E}_1) \right).$$

With respect to this decomposition we have

$$\widetilde{\Delta}_k(\tau) = \bigoplus_{p+q=k} \left(\widetilde{\Delta}_p(\tau_0) \otimes \mathrm{Id} + \mathrm{Id} \otimes \widetilde{\Delta}_q(\tau_1) \right).$$

Let $H_t^{\tau,k}$ and $H_t^{\tau_i,p}$, $i = 0, 1$, be the corresponding heat kernels. Then it follows that $H_t^{\tau,k} = \oplus_{p+q=k} H_t^{\tau_0,p} \otimes H_t^{\tau_1,q}$. Hence for $h_t^{\tau,k} = \operatorname{tr} H_t^{\tau,k}$ and $h_t^{\tau_i,p} = \operatorname{tr} H_t^{\tau_i,p}$, $i = 0, 1$, we have

$$h_t^{\tau,k} = \sum_{p+q=k} h_t^{\tau_0,p} \cdot h_t^{\tau_1,q}.$$

Using this equality, we get

$$\begin{aligned} \sum_{k=0}^{d} (-1)^k k\, h_t^{\tau,k}(1) &= \sum_{p=0}^{d_1} \sum_{q=0}^{d_2} (-1)^{p+q} (p+q)\, h_t^{\tau_1,p}(1) \cdot h_t^{\tau_2,q}(1) \\ &= \sum_{p=0}^{d_1} (-1)^p\, h_t^{\tau_1,p}(1) \cdot \sum_{q=0}^{d_2} (-1)^q q\, h_t^{\tau_2,q}(1) \\ &\quad + \sum_{q=0}^{d_2} (-1)^q\, h_t^{\tau_2,q}(1) \cdot \sum_{p=0}^{d_1} (-1)^p p\, h_t^{\tau_1,p}(1). \end{aligned} \tag{5.29}$$

Let $\Gamma_i \subset G_i$, $i = 0, 1$, any cocompact, torsion free discrete subgroup. The existence of the Γ_i follows from our assumptions on the G_i stated in the introduction and from results of Borel [Bor]. Put $X_i = \Gamma_i \backslash \widetilde{X}_i$ and $E_i = \Gamma \backslash \widetilde{E}_i$. By (5.26) and the remark following it we have

$$\sum_{p=0}^{d} (-1)^p h_t^{\tau_i,p}(1) = \frac{\chi(X_i)}{\operatorname{vol}(X_i)}, \quad i = 0, 1. \tag{5.30}$$

Taking the Mellin transform of (5.29) and using (5.30) and Proposition 5.2, the proposition follows. ∎

6. The asymptotics of the L^2-torsion for $\delta(\widetilde{X}) = 1$

In this section we study the asymptotic behaviour of the L^2-torsion of an odd-dimensional irreducible symmetric space $\widetilde{X}$ with $\delta(\widetilde{X}) = 1$. Then we can assume that either $G = \mathrm{Spin}(p, q)$, p, q odd, and $K = \mathrm{Spin}(p) \times \mathrm{Spin}(q)$, or $G = \mathrm{SL}_3(\mathbb{R})$ and $K = \mathrm{SO}(3)$. To compute the L^2 torsion in these cases, we need some preparation. Let $Q = MAN$ be a fundamental parabolic subgroup of G, i.e. we have $\dim(A) = 1$. Let M^0 be the identity component of M and let $\mathfrak{m}$ be its Lie algebra. Then in our case $\mathfrak{m}$ is always semisimple.

Let $K_M := K \cap M$, let K_M^0 be the identity component of K_M and let $\mathfrak{k}_{\mathfrak{m}} := \mathfrak{k} \cap \mathfrak{m}$ be its Lie algebra. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}_{\mathfrak{m}}$. Then $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{m}$ and of $\mathfrak{k}$. Moreover $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

Let $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, $\Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, $\Delta((\mathfrak{k}_{\mathfrak{m}})_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the corresponding roots. Then there is a canonical inclusion $\Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \hookrightarrow \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Fix a positive restricted root $e_1 \in \mathfrak{a}^*$ and fix positive roots $\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. In this way we obtain positive roots $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Let ρ_G resp. ρ_M be the half sums of the elements of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and $\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, respectively. By our choices we have $\rho_G|_{\mathfrak{m}} = \rho_M$.

Let

$$T := \{m \in K_M \colon \operatorname{Ad}(m)|_{\mathfrak{t}} = \operatorname{Id}\}.$$

Then we have

$$T = \{k \in K \colon \operatorname{Ad}(k)|_{\mathfrak{t}} = \operatorname{Id}\}.$$

Thus T is connected. Let N_{K_M} and $N_{K_M^0}$ be the normalizers of $\mathfrak{t}$ in K_M and K_M^0, respectively. Let $W_{K_M} := N_{K_M}/T$ and let $W_{\mathfrak{k}_{\mathfrak{m}}} = N_{K_M^0}/T$ be the Weyl group of $\Delta((\mathfrak{k}_{\mathfrak{m}})_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Moreover we let $W_{\mathfrak{m}}$ be the Weyl group of $\Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Finally we let

$$W(A) := \{k \in K \colon \operatorname{Ad}(k)\mathfrak{a} = \mathfrak{a}\}/K_M.$$

The following lemma is certainly well-known and has already been used by Olbrich, [Ol, page 15]. However, for the sake of completeness, we include a proof here.

Lemma 6.1. *One has*

$$\frac{|W_{K_M}|}{|W_{\mathfrak{k}_{\mathfrak{m}}}|} \cdot |W(A)| = 2.$$

Proof. By [Kn2, Proposition 7.19 (b)] one has $\#\,(M/M^0) = \#\,(K_M/K_M^0)$. Let $k \in K_M$. Then $\operatorname{Ad}(k)\mathfrak{t}$ is a maximal torus in $\mathfrak{k}_{\mathfrak{m}}$ and thus there exists a $k^0 \in K_M^0$ such that $\operatorname{Ad}(k)\mathfrak{t} = \operatorname{Ad}(k^0)\mathfrak{t}$. Hence every element of K_M/K_M^0 has a representative in N_{K_M} and thus there are canonical isomorphisms $K_M/K_M^0 \cong N_{K_M}/N_{K_M^0} \cong W_{K_M}/W_{\mathfrak{k}_{\mathfrak{m}}}$. In other words $|W_{K_M}|/|W_{\mathfrak{k}_{\mathfrak{m}}}|$ equals the number of components of M. Let $\mathfrak{a}_{\mathfrak{p}}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$, let $\Delta_{\mathfrak{a}_{\mathfrak{p}}}$ be the corresponding restricted roots and let $W(\Delta_{\mathfrak{a}_{\mathfrak{p}}})$ be the corresponding Weyl-group. One has $W(\Delta_{\mathfrak{a}_{\mathfrak{p}}}) = N_K(\mathfrak{a}_{\mathfrak{p}})/Z_K(\mathfrak{a}_{\mathfrak{p}})$, where $N_K(\mathfrak{a}_{\mathfrak{p}})$ resp. $Z_K(\mathfrak{a}_{\mathfrak{p}})$ are the normalizer resp. centralizer of $\mathfrak{a}_{\mathfrak{p}}$ in K. Moreover by [Kn2, Proposition 8.85] each element of $W(A)$ has a representative in $N_K(\mathfrak{a}_{\mathfrak{p}})$, i.e can be extended to an element of $W(\Delta_{\mathfrak{a}_{\mathfrak{p}}})$ which fixes $\mathfrak{a}$. Now a case-by-case study easily implies that $W(\Delta_{\mathfrak{a}_{\mathfrak{p}}})$ contains such an element which is non-trivial if and only if $G = \operatorname{Spin}(p, 1)$. In this case M is connected. In all other cases, M has exactly two components. This proves the Lemma. $\square$

Let $H_1 \in \mathfrak{a}$ with $e_1(H_1) = 1$. Then we normalize the Killing form B by $1/B(H_1, H_1)$. We let $\|\cdot\|$ be the corresponding norm on the real vector-space $i\mathfrak{t}^* \oplus \mathfrak{a}^*$. Let Ω be the Casimir element with respect to the normalized Killing form. Then for $\tau \in \operatorname{Rep}(G)$ with highest weight $\Lambda(\tau)$ we have

$$\tau(\Omega) = \|\Lambda(\tau) + \rho_G\|^2 - \|\rho_G\|^2 \,. \tag{6.31}$$

The restriction of the normalized Killing form to $\mathfrak{m}$ is non-degenerate and Ad-invariant. Let Ω_M be the corresponding Casimir element. For $\sigma \in \mathrm{Rep}(M^0)$ with highest weight $\Lambda(\sigma) \in i\mathfrak{t}^*$ we define

$$c(\sigma) := \|\Lambda(\sigma) + \rho_M\|^2 - \|\rho_G\|^2 . \tag{6.32}$$

Then one has $c(\sigma) = \chi_\sigma(\Omega_M) - \|\rho_G|_{\mathfrak{a}}\|^2$ and thus one has

$$c(\sigma) = c(\check{\sigma}) \tag{6.33}$$

for every $\sigma \in \mathrm{Rep}(M^0)$. Let $W_{\mathfrak{g}} := W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be the Weyl group of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and for $w \in W_{\mathfrak{g}}$ let $\ell(w)$ be its length with respect to the simple roots defined by $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Finally let

$$W^1 := \{w \in W_{\mathfrak{g}} \colon w^{-1}\alpha > 0 \quad \forall \alpha \in \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\}.$$

The subspace $\mathfrak{n}$ is even-dimensional and we write $\dim(\mathfrak{n}) = 2n$. For $k = 0, \dots, 2n$ let $H^k(\mathfrak{n}; V_\tau)$ be the Lie-algebra cohomology of $\mathfrak{n}$ with coefficients in V_τ. Then the $H^k(\mathfrak{n}; V_\tau)$ are M^0A-modules and their decomposition into irreducible M^0A-components can be described by the following theorem of Kostant.

Proposition 6.2. *In the sense of M^0A-modules one has*

$$H^k(\mathfrak{n}; V_\tau) \cong \sum_{\substack{w \in W^1 \\ \ell(w) = k}} V_{\tau(w)},$$

where $V_{\tau(w)}$ is the M^0A module with highest weight $w(\Lambda(\tau) + \rho_G) - \rho_G$.

Proof. See for example [Wr, Theorem 2.5.1.3]. ☐

Corollary 6.3. *As M^0A-modules we have*

$$\bigoplus_{k=0}^{2n} (-1)^k \Lambda^k \mathfrak{n}^* \otimes V_\tau = \bigoplus_{w \in W^1} (-1)^{l(w)} V_{\tau(w)}.$$

Proof. This follows from Proposition 6.2 and the Poincaré principle [Ko, (7.2.3)]. ☐

For $w \in W^1$ let $\sigma_{\tau,w} \in \mathrm{Rep}(M^0)$ be the finite-dimensional irreducible representation of M^0 with highest weight

$$\Lambda(\sigma_{\tau,w}) := w(\Lambda(\tau) + \rho_G)|_{\mathfrak{t}} - \rho_M, \tag{6.34}$$

and let $\lambda_{\tau,w} \in \mathbb{R}$ be such that

$$w(\Lambda(\tau) + \rho_G)|_{\mathfrak{a}} = \lambda_{\tau,w} e_1. \tag{6.35}$$

Then we have the following corollary about the Casimir eigenvalue.

Proposition 6.4. *For every $w \in W^1$ one has*

$$\tau(\Omega) = \lambda_{\tau,w}^2 + c(\sigma_{\tau,w}).$$

Proof. By (6.31) we have

$$\begin{aligned}\tau(\Omega) &= \|\Lambda(\tau)+\rho_G\|^2 - \|\rho_G\|^2 = \|w(\Lambda(\tau)+\rho_G)\|^2 - \|\rho_G\|^2 \\ &= \|\lambda_{\tau,w}e_1\|^2 + \|\Lambda(\sigma_{\tau,w})+\rho_M\|^2 - \|\rho_G\|^2 = \lambda_{\tau,w}^2 + c(\sigma_{\tau,w}).\end{aligned}$$

□

Let k_t^τ be defined by (4.16). Our next goal is to compute the Fourier transform of k_t^τ. Note that, since T is connected, it follows from [Wa2, section 6.9, section 8.7.1] that for every discrete series representation ξ of M over W_ξ there exists a discrete series representation ξ^0 of M^0 over W_{ξ^0} such that ξ is induced from ξ^0. Moreover, since M^0 is semisimple, the discrete series of M^0 is parametrized as in section 2.2. By [Wa2, section 8.7.1], two discrete series representations ξ_1^0 and ξ_2^0 of M^0 with corresponding parameters $\Lambda_{\xi_1^0}, \Lambda_{\xi_2^0}$ as in section 2.2 induce the same discrete series representation of M if and only if $\Lambda_{\xi_1^0}$ and $\Lambda_{\xi_2^0}$ are W_{K_M}-conjugate. For $\xi \in \hat{M}_d$ and $\lambda \in \mathbb{C}$ we let $\pi_{\xi,\lambda} := \pi_{\xi,\lambda e_1}$, $\Theta_{\xi,\lambda} := \Theta_{\xi,\lambda e_1}$.

Proposition 6.5. *Let $\xi \in \hat{M}_d$ with infinitesimal character $\chi(\xi)$. Let $\mathfrak{p}_{\mathfrak{m}} := \mathfrak{p} \cap \mathfrak{m}$ and let $v := \frac{1}{2}\dim \mathfrak{p}_{\mathfrak{m}}$. Then for $\lambda \in \mathbb{C}$ one has*

$$\Theta_{\xi,\lambda}(k_t^\tau) = (-1)^v \sum_{\substack{w\in W^1 \\ \chi(\xi)=\chi(\check{\sigma}_{\tau,w})}} (-1)^{\ell(w)+1} e^{-t(\lambda^2+\lambda_{\tau,w}^2)}.$$

Proof. One has

$$\pi_{\xi,\lambda}(\Omega) = -\lambda^2 + \|\Lambda_\xi\|^2 - \|\rho_G\|^2.$$

Thus if $\sigma \in \operatorname{Rep}(M^0)$ is such that $\chi_\sigma = \chi_\xi$ one has

$$\pi_{\xi,\lambda}(\Omega) = -\lambda^2 + c(\sigma). \tag{6.36}$$

Let ξ^0, Λ_{ξ^0} be as above. Then $\xi|_{K_M}$ is induced from $\xi^0|_{K_M^0}$ and by Frobenius reciprocity one has

$$[\Lambda^p p^* \otimes \mathcal{H}_\xi \otimes V_\tau]^K = [\Lambda^p \mathfrak{p}^* \otimes W_\xi \otimes V_\tau]^{K_M} = [\Lambda^p \mathfrak{p}^* \otimes W_{\xi_0} \otimes V_\tau]^{K_M^0}.$$

Thus by (4.19) one has

$$\Theta_{\xi,\lambda}(k_t^\tau) = e^{t(\pi_{\xi,\lambda}(\Omega)-\tau(\Omega))} \sum_{p=0}^d (-1)^p p \, [\Lambda^p \mathfrak{p}^* \otimes W_{\xi^0} \otimes V_\tau]^{K_M^0}.$$

Let $\mathfrak{p}_Y$ be as in Proposition 4.1. Since $\dim \mathfrak{a} = 1$, it follows that as K_M^0 modules $\mathfrak{p}_Y \cong \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{n}$. Using (4.22), it follows that as K_M^0 modules we have

$$\sum_{p=0}^d (-1)^p p \Lambda^p \mathfrak{p}^* = \sum_{p=0}^d (-1)^{p+1} \Lambda^p(\mathfrak{p}_{\mathfrak{m}}^* \oplus \mathfrak{n}^*) = \sum_{k=0}^{2n} (-1)^{k+1} \left(\Lambda^{\mathrm{ev}} \mathfrak{p}_{\mathfrak{m}}^* - \Lambda^{\mathrm{odd}} \mathfrak{p}_{\mathfrak{m}}^*\right) \otimes \Lambda^k \mathfrak{n}^*.$$

Thus together with Corollary 6.3 and the Poincaré principle one gets

$$\sum_{p=0}^{d}(-1)^p p\left[\Lambda^p\mathfrak{p}^*\otimes W_{\xi^0}\otimes V_\tau\right]^{K_M^0}. = \sum_{w\in W^1}(-1)^{\ell(w)+1}\left[\left(\Lambda^{\mathrm{ev}}\mathfrak{p}_{\mathfrak{m}}^*-\Lambda^{\mathrm{odd}}\mathfrak{p}_{\mathfrak{m}}^*\right)\otimes W_{\xi^0}\otimes V_{\tau(w)}\right]^{K_{M^0}}$$
$$= \sum_{w\in W^1}(-1)^{\ell(w)+1}\chi(\mathfrak{m},K_{M^0};W_{\xi^0}\otimes V_{\tau(w)}),$$

where $\chi(\mathfrak{m},K_M^0;W_{\xi^0}\otimes V_{\tau(w)})$ denotes the Euler-characteristic of the $(\mathfrak{m},K_M^0)$-cohomology with coefficients in the M^0-module $V_{\tau(w)}\otimes W_{\xi^0}$. Thus the proposition follows from Proposition 2.1, Proposition 6.4, equation (6.36) and equation (6.33). $\square$

Next we come to the Plancherel measures. For $\xi\in\hat{M}_d$ we let $\xi^0\in\hat{M}_d^0$ be as above. Fix a regular $\Lambda_{\xi^0}\in i\mathfrak{t}^*$ corresponding to ξ^0 as in section 2.2 and let $\Lambda_\xi:=\Lambda_{\xi^0}$. Choose positive roots $\Delta^+(\mathfrak{m}_\mathbb{C},\mathfrak{t}_\mathbb{C};\Lambda_\xi)$ such that Λ_ξ is dominant with respect to these roots. Let $\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C};\Lambda_\xi)$ be positive roots defined via $\Delta^+(\mathfrak{m}_\mathbb{C},\mathfrak{t}_\mathbb{C};\Lambda_\xi)$ and e_1 and let ρ_{G,Λ_ξ} be the half-sum of the elements of $\Delta^+(\mathfrak{m}_\mathbb{C},\mathfrak{t}_\mathbb{C};\Lambda_\xi)$. For $\lambda\in\mathbb{R}$ we let $\mu_\xi(\lambda)$ be the Plancherel measure of $\pi_{\xi,\lambda}$. Then there exists a polynomial $P_\xi(z)$ such that one has

$$\mu_\xi(\lambda)=P_\xi(i\lambda). \tag{6.37}$$

The polynomial $P_\xi(z)$ is given as follows. There exists a constant $c_{\tilde{X}}$ which depends only on $\tilde{X}$ such that one has

$$P_\xi(z)=(-1)^n c_{\tilde{X}}\prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C};\Lambda_\xi)}\frac{\langle\alpha,\Lambda_\xi+ze_1\rangle}{\langle\alpha,\rho_{G,\Lambda_\xi}\rangle}, \tag{6.38}$$

[Kn1, Theorem 13.11], [Wa3, Theorem 13.5.1]. By [Ol, Lemma 5.1] and our normalizations one has

$$c_{\tilde{X}}=\frac{1}{|W(A)|\operatorname{vol}(\tilde{X}_d)}. \tag{6.39}$$

Note that $P_\xi(z)$ is an even polynomial in z. Now let $w\in W_\mathfrak{m}$. We regard $W_\mathfrak{m}$ as a subgroup of $W_\mathfrak{g}$. Then if we replace Λ_ξ by $w\Lambda_\xi$, we have to replace $\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{t}_\mathbb{C};\Lambda_\xi)$ by $w\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{t}_\mathbb{C};\Lambda_\xi)$. This implies that $P_\xi(z)$ depends only on the $W_\mathfrak{m}$-orbit of Λ_ξ or equivalently on the infinitesimal character $\chi(\xi)$ of ξ. Thus if for $\sigma\in\operatorname{Rep}(M^0)$ with highest weight $\Lambda(\sigma)$ we let

$$P_\sigma(z):=(-1)^n c_{\tilde{X}}\prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})}\frac{\langle\alpha,\Lambda(\sigma)+\rho_M+ze_1\rangle}{\langle\alpha,\rho_G\rangle}, \tag{6.40}$$

where $c_{\tilde{X}}$ is as in (6.38), it follows that $P_\xi(\lambda)=P_\sigma(\lambda)$ if $\chi(\sigma)=\chi(\xi)$. Putting everything together, we obtain the following corollary.

Proposition 6.6. *Let $\tau\in\operatorname{Rep}(G)$ and assume that $\tau\ncong\tau_\theta$. Then one has*

$$\log T_X^{(2)}(\tau)=(-1)^v\pi\operatorname{vol}(X)\frac{|W_\mathfrak{m}|}{|W_{K_M}|}\sum_{w\in W^1}(-1)^{\ell(w)}\int_0^{|\lambda_{\tau,w}|}P_{\check{\sigma}_{\tau,w}}(t)dt.$$

Proof. For a given regular and integral $\Lambda \in i\mathfrak{t}^*$ there are exactly $|W_{\mathfrak{m}}|/|W_{K_M}|$ distinct elements of $\hat{M}_d$ with infinitesimal character χ_Λ. Thus if one combines the Plancherel-Theorem with Proposition 4.1, Proposition 6.5, equation (6.37) and the previous remarks one obtains

$$k_t^\tau(1) = (-1)^v \frac{|W_{\mathfrak{m}}|}{|W_{K_M}|} \sum_{w \in W^1} (-1)^{\ell(w)+1} e^{-t\lambda_{\tau,w}^2} \int_{\mathbb{R}} e^{-t\lambda^2} P_{\check{\sigma}_{\tau,w}}(i\lambda) d\lambda.$$

We let

$$I(t,\tau) := \operatorname{vol}(X) k_t^\tau(1).$$

By the computations below one has $|\lambda_{\tau,w}| > 0$ for every $w \in W^1$. Thus, since is $P_\sigma(\lambda)$ is an even polynomial of degree $2n$ for each $\sigma \in \hat{M}^0$, for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 2n+1$ the integral

$$\mathcal{M}I(s,\tau) := \int_0^\infty t^{s-1} I(t,\tau) dt$$

exists. Moreover, by [Fr], Lemma 2 and Lemma 3, $\mathcal{M}I(s,\tau)$ has a meromorphic continuation to $\mathbb{C}$ which is regular at 0 and if $\mathcal{M}I(\tau)$ denotes its value at 0 one has

$$\mathcal{M}I(\tau) = 2\pi \operatorname{vol}(X)(-1)^v \frac{|W_{\mathfrak{m}}|}{|W_{K_M}|} \sum_{w \in W^1} (-1)^{\ell(w)} \int_0^{|\lambda_{\tau,w}|} P_{\check{\sigma}_{\tau,w}}(\lambda)\, d\lambda.$$

By definition one has

$$\log T_X^{(2)}(\tau) = \frac{1}{2} \mathcal{M}I(\tau)$$

and the proposition follows. $\square$

Now let $G = \operatorname{Spin}(p,q)$, p,q odd, $p = 2p_1 + 1$, $q = 2q_1 + 1$. Let $n := p_1 + q_1$. Let $K = \operatorname{Spin}(p) \times \operatorname{Spin}(q)$ and $\widetilde{X} = G/K$. Then $\dim(\widetilde{X}) = pq$. The normalized Killing form is given by

$$\langle X, Y \rangle := \frac{1}{2n-2} B(X,Y).$$

We equip $\tilde{X}$ with the Riemannian metric defined by the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{p}$. We have $\mathfrak{m} \cong \mathfrak{so}(p-1, q-1)$. We realize the fundamental Cartan subalgebra as follows. Let

$$H_1 := E_{p,p+1} + E_{p+1,p}. \tag{6.41}$$

Then we put

$$\mathfrak{a} = \mathbb{R} H_1.$$

Moreover we let

$$H_i := \begin{cases} \sqrt{-1}(E_{2i-3,2i-2} - E_{2i-2,2i-3}), & 2 \le i \le p_1 + 1 \\ \sqrt{-1}(E_{2i-1,2i} - E_{2i,2i-1}) & p_1 + 1 < i \le n+1. \end{cases} \tag{6.42}$$

Then

$$\mathfrak{t} := \bigoplus_{i=2}^{n+1} \sqrt{-1} H_i$$

is a Cartan subalgebra of $\mathfrak{m}$ and

$$\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}$$

is a Cartan subalgebra of $\mathfrak{g}$. Define $e_i \in \mathfrak{h}_{\mathbb{C}}^*$, $i = 1, \dots, n+1$, by

$$e_i(H_j) = \delta_{i,j},\ 1 \leq i, j \leq n+1.$$

Then the sets of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ are given by

$$\begin{aligned} \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) &= \{\pm e_i \pm e_j,\ 1 \leq i < j \leq n+1\} \\ \Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) &= \{\pm e_i \pm e_j,\ 2 \leq i < j \leq n+1\}. \end{aligned}$$

We fix positive systems of roots by

$$\begin{aligned} \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) &:= \{e_i + e_j,\ i \neq j\} \sqcup \{e_i - e_j,\ i < j\} \\ \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) &:= \{e_i + e_j,\ i \neq j,\ i, j \geq 2\} \sqcup \{e_i - e_j,\ 2 \leq i < j\}. \end{aligned}$$

The finite-dimensional irreducible representations τ of G are parametrized by their highest weights

$$\begin{aligned} \Lambda(\tau) =& k_1(\tau)e_1 + \dots + k_{n+1}(\tau)e_{n+1}, \quad (k_1(\tau), \dots k_{n+1}(\tau)) \in \mathbb{Z}\left[\frac{1}{2}\right]^{n+1} \\ & k_1(\tau) \geq k_2(\tau) \geq \dots \geq k_n(\tau) \geq |k_{n+1}(\tau)|\,. \end{aligned} \tag{6.43}$$

Let Λ be a highest weight and let τ_Λ be the associated irreducible representation of G. Recall that we denote by Λ_θ the highest weight of the representation $\tau_\Lambda \circ \theta$. If Λ is a highest weight as in (6.43), then

$$\Lambda_\theta = k_1(\tau)e_1 + \dots + k_n(\tau)e_n - k_{n+1}(\tau)e_{n+1}. \tag{6.44}$$

Thus the fundamental weights which are not invariant under θ are the weights

$$\omega_{f,n}^+ := \sum_{j=1}^{n+1} \frac{1}{2} e_j; \quad \omega_{f,n}^- := (\omega_{f,n}^+)_\theta = \sum_{j=1}^{n} \frac{1}{2} e_j - \frac{1}{2} e_{n+1}. \tag{6.45}$$

The finite-dimensional irreducible representations σ of M^0 are parametrized by their highest weights

$$\begin{aligned} \Lambda(\sigma) =& k_2(\sigma)e_2 + \dots + k_{n+1}(\sigma)e_{n+1}, \quad (k_2(\sigma), \dots, k_{n+1}(\sigma)) \in \mathbb{Z}\left[\frac{1}{2}\right]^{n}, \\ & k_2(\sigma) \geq k_3(\sigma) \geq \dots \geq k_n(\sigma) \geq |k_{n+1}(\sigma)|\,. \end{aligned} \tag{6.46}$$

For $\sigma \in \mathrm{Rep}(M^0)$ with highest weight $\Lambda(\sigma)$ as in (6.46) we let $w_0\sigma \in \mathrm{Rep}(M^0)$ be the representation with highest weight

$$\Lambda(w_0\sigma) := k_2(\sigma)e_2 + \dots + k_n(\sigma)e_n - k_{n+1}(\sigma)e_{n+1}.$$

Then for every $\sigma \in \mathrm{Rep}(M^0)$ one has $\check{\sigma} = \sigma$ if n is even and $\check{\sigma} = w_0\sigma$ if n is odd. Applying equation (6.40) this implies that

$$P_\sigma(\lambda) = P_{w_0\sigma}(\lambda) = P_{\check{\sigma}}(\lambda) \tag{6.47}$$

for every $\sigma \in \mathrm{Rep}(M^0)$.

Let $\tau \in \mathrm{Rep}(G)$ with highest weight $\tau_1 e_1 + \cdots + \tau_{n+1}e_{n+1}$. For $k = 0, \ldots n$ let

$$\lambda_{\tau,k} = \tau_{k+1} + n - k \tag{6.48}$$

and let $\sigma_{\tau,k}$ be the irreducible representation of M with highest weight

$$\Lambda_{\sigma_{\tau,k}} := (\tau_1 + 1)e_2 + \cdots + (\tau_k + 1)e_{k+1} + \tau_{k+2}e_{k+2} + \cdots + \tau_{n+1}e_{n+1}. \tag{6.49}$$

Then as in [MP, section 2.7] one has

$$\begin{aligned} \{(\lambda_{\tau,w}, \sigma_{\tau,w}, l(w))\colon w \in W^1\} = \{&(\lambda_{\tau,k}, \sigma_{\tau,k}, k)\colon k = 0, \ldots, n\} \\ &\sqcup \{(-\lambda_{\tau,k}, w_0\sigma_{\tau,k}, 2n-k)\colon k = 0, \ldots, n\}. \end{aligned} \tag{6.50}$$

Combining (6.44), (6.47) and (6.50) and Proposition 6.6 it follows that

$$T_X^{(2)}(\tau) = T_X^{(2)}(\tau_\theta) \tag{6.51}$$

for each $\tau \in \mathrm{Rep}(G)$. Now for $p, q \in \mathbb{N}$ we let

$$C_{p,q} = \frac{(-1)^{\frac{pq-1}{2}} 2\pi}{\mathrm{vol}(\tilde{X}_d)} \binom{\frac{p+q-2}{2}}{\frac{p-1}{2}}. \tag{6.52}$$

Then we have

Proposition 6.7. *Let $\widetilde{X} = \mathrm{Spin}(p,q)/(\mathrm{Spin}(p) \times \mathrm{Spin}(q))$, p, q odd, and $X = \Gamma\backslash\widetilde{X}$. Let $\Lambda \in \mathfrak{h}_{\mathbb{C}}^*$ be a highest weight with $\Lambda_\theta \neq \Lambda$. For $m \in \mathbb{N}$ let $\tau_\Lambda(m)$ be the irreducible representation of $\mathrm{Spin}(p,q)$ with highest weight $m\Lambda$. There exists a polynomial $P_\Lambda(m)$ whose coefficients depend only on Λ, such that for all $m \in \mathbb{N}$ we have*

$$\log T_X^{(2)}(\tau_\Lambda(m)) = C_{p.q}\,\mathrm{vol}(X) P_\Lambda(m).$$

Moreover there is a constant $C_\Lambda > 0$, which depends on Λ, such that

$$P_\Lambda(m) = C_\Lambda \cdot m \dim(\tau_\Lambda(m)) + O\left(\dim(\tau_\Lambda(m))\right) \tag{6.53}$$

as $m \to \infty$. If $\Lambda = \omega_{f,n}^{\pm}$ is one of the fundamental weights that are not invariant under θ, then $C_\Lambda = 1$.

Proof. Let $\Lambda = \tau_1 e_1 + \cdots + \tau_{n+1}e_{n+1}$. By (6.44) and (6.51) we may assume that $\tau_{n+1} > 0$. Put $\tau(m) := \tau_\Lambda(m)$. Then

$$\lambda_{\tau(m),k} = m\tau_{k+1} + n - k, \quad k = 0, \ldots, n, \tag{6.54}$$

and by Proposition 6.6, (6.50) and (6.47) we have

$$\log T_X^{(2)}(\tau(m)) = 2\pi\,\mathrm{vol}(X)(-1)^v \frac{|W_{\mathfrak{m}}|}{|W_{K_M}|} \sum_{k=0}^{n} (-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)\,dt.$$

In the hyperbolic case the term $(-1)^v|W_{\mathfrak{m}}|/|W_{K_M}|$ equals 1. Therefore this equation agrees with [MP, (5.18)]. Note that $2n = \dim \mathfrak{n}$. Let $c_{\tilde{X}}$ be defined by (6.39) and put

$$P_\Lambda(m) := \frac{(-1)^n}{c_{\tilde{X}}} \sum_{k=0}^{n} (-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)\, dt. \tag{6.55}$$

Then it follows from (6.40) and (6.50) that P_Λ is a polynomial in m whose coefficients depend only on Λ. By definition one has

$$\log T_X^{(2)}(\tau(m)) = 2\pi \operatorname{vol}(X)(-1)^{v+n} \frac{|W_{\mathfrak{m}}|}{|W_{K_M}|} c_{\tilde{X}} P_\Lambda(m).$$

So it remains to compute the constant. By (6.39) and Lemma 6.1 one has

$$\frac{|W_{\mathfrak{m}}|}{|W_{K_M}|} c_{\tilde{X}} = \frac{|W_{\mathfrak{m}}|}{|W_{\mathfrak{k}_{\mathfrak{m}}}|} \frac{1}{2 \operatorname{vol}(\tilde{X}_d)}.$$

Recall that $\mathfrak{m}_{\mathbb{C}} \cong \mathfrak{so}(2n, \mathbb{C})$, $(\mathfrak{k}_{\mathfrak{m}})_{\mathbb{C}} \cong \mathfrak{so}(2p_1, \mathbb{C}) \oplus \mathfrak{so}(2q_1, \mathbb{C})$ and so by [Kn2, page 685] one has $|W_{\mathfrak{m}}| = n!2^{n-1}$, $|W_{\mathfrak{k}_{\mathfrak{m}}}| = p_1!q_1!2^{n-2}$. Hence, as in [Ol, Proposition 1.3], one has

$$\frac{|W_{\mathfrak{m}}|}{|W_{\mathfrak{k}_{\mathfrak{m}}}|} = 2 \binom{\frac{p+q-2}{2}}{\frac{p-1}{2}}.$$

Furthermore one has $v = \frac{\dim \mathfrak{p}_{\mathfrak{m}}}{2} = \frac{(p-1)(q-1)}{2}$ and thus we get $v + n = \frac{pq-1}{2}$. This proves the first part of the proposition.

To determine the highest order term of the polynomial $P_\Lambda(m)$, we proceed as in [MP, Lemma 5.4] to show that

$$P_{\sigma_{\tau(m),k}}(t) = (-1)^{n+k} c_{\tilde{X}} \dim(\tau(m)) \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{t^2 - \lambda_{\tau(m),j}^2}{\lambda_{\tau(m),k}^2 - \lambda_{\tau(m),j}^2}.$$

Denote the product on the right by $\Pi_k(t;m)$. Then it follows from (6.55) that

$$P_\Lambda(m) = \dim(\tau(m)) \cdot \sum_{k=0}^{n} \int_0^{\lambda_{\tau(m),k}} \Pi_k(t;m)\, dt. \tag{6.56}$$

To deal with the sum, we follow [BV, 5.9.1]. Put $\lambda_{\tau(m),n+1} = 0$. Then the finite sequence $\lambda_{\tau(m),k}$, $k = 0, \ldots, n+1$ is strictly decreasing. For $k = 0, \ldots, n$ set

$$Q_k(t;m) := \sum_{j=0}^{k} \Pi_j(t;m).$$

Then $Q_k(t;m)$ is the unique even polynomial of degree $\leq 2n$ which satisfies

$$Q_k(\pm\lambda_{\tau(m),j}) = \begin{cases} 1, & \text{if } j \leq k, \\ 0, & \text{if } n \geq j > k. \end{cases} \tag{6.57}$$

Moreover we have

$$\sum_{k=0}^{n}\int_{0}^{\lambda_{\tau(m),k}}\Pi_k(t;m)\,dt=\sum_{k=0}^{n}\int_{\lambda_{\tau(m),k+1}}^{\lambda_{\tau(m),k}}Q_k(t;m)\,dt. \tag{6.58}$$

As proved in [BV, Sect. 5.9.1], each integral on the right is positive. This can be seen as follows. By (6.57), the polynomial Q_k' has a root in each interval $[\lambda_{\sigma_{\tau(m),j+1}},\lambda_{\sigma_{\tau(m),j}}]$, $[-\lambda_{\sigma_{\tau(m),j}},-\lambda_{\sigma_{\tau(m),j+1}}]$ for $0\leq j<n$, $j\neq k$ and a root in $[-\lambda_{\sigma_{\tau(m),n}},\lambda_{\sigma_{\tau(m),n}}]$. Since Q_k' is of degree $\leq 2n-1$, it follows that Q_k is either constant or strictly increasing on $[\lambda_{\sigma_{\tau(m),k+1}},\lambda_{\sigma_{\tau(m),k}}]$. Furthermore, $Q_n(t;m)$ is a polynomial of degree $2n$, which is equal to 1 at $2n+2$ pairwise distinct points. Hence $Q_n\equiv 1$. Thus by (6.54) and (6.58) we get

$$\begin{aligned}(n+1)(m\tau_1+n)=(n+1)\lambda_{\tau(m),0}&\geq\sum_{k=0}^{n}(\lambda_{\tau(m),k}-\lambda_{\tau(m),k+1})\\ &\geq\sum_{k=0}^{n}\int_{0}^{\lambda_{\tau(m),k}}\Pi_k(t;m)\,dt\geq\tau_{n+1}m.\end{aligned} \tag{6.59}$$

Since $P_\Lambda(m)$ is a polynomial in m, it follows that there exists $C_\Lambda\geq\tau_{n+1}>0$ such that (6.53) holds . If Λ is one of the fundamental weights $\omega_{f,n}^{\pm}$, defined by (6.45), then it follows as in [MP, Section 5] that $C_\Lambda=1$. This proves the second part of the proposition. $\square$

Finally we turn to the case $G=\mathrm{SL}_3(\mathbb{R})$, $K=\mathrm{SO}(3)$. We define our fundamental Cartan subalgebra as follows. Let

$$H_1:=\operatorname{diag}(1,1,-2);\quad \mathfrak{a}:=\mathbb{R}H_1.$$

Then we have $\mathfrak{m}=\mathfrak{sl}_2(\mathbb{R})$, if $\mathfrak{sl}_2(\mathbb{R})$ is embedded into $\mathfrak{g}$ as an upper left block. Let

$$H_2:=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \mathfrak{t}:=\mathbb{R}T_1$$

embedded into $\mathfrak{g}$ as an upper left block. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{m}$ and

$$\mathfrak{h}:=\mathfrak{a}\oplus\mathfrak{t} \tag{6.60}$$

is a θ-stable fundamental Cartan subalgebra of $\mathfrak{g}$. Note that $\mathfrak{h}$ is different from the usual Cartan subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$ which consist of all diagonal matrices of trace 0. Define $f_1\in\mathfrak{a}^*$ and $f_2\in i\mathfrak{t}^*$ by

$$f_1(H_1)=3;\quad f_2(H_2)=i.$$

We fix f_1 as a positive restricted root of $\mathfrak{a}$. Then we can define positive roots by

$$\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C}):=\{f_1-f_2,\ f_1+f_2,\ 2f_2\};\quad \Delta^+(\mathfrak{m}_\mathbb{C},\mathfrak{t}_\mathbb{C})=\{2f_2\}.$$

Under our normalization one has

$$\langle f_1,f_1\rangle=1;\quad\langle f_2,f_2\rangle=\frac{1}{3};\quad\langle f_1,f_2\rangle=0. \tag{6.61}$$

One easily sees that $\dim \mathfrak{n} = 2$, hence $n = 1$. Moreover by [Kn2, page 485] one has $|W(A)| = 1$. For $k \in \mathbb{N}$ let $\sigma_k \in \mathrm{Rep}(M^0)$ be of highest weight kf_2. Then it follows from (6.40) and (6.39) that

$$P_{\sigma_k}(z) = -\frac{9}{8\,\mathrm{vol}(\tilde{X}_d)}\,(k+1)\left(z^2 - \left(\frac{k+1}{3}\right)^2\right). \tag{6.62}$$

Define $e_i \in \tilde{\mathfrak{h}}_{\mathbb{C}}^*$ by $e_i(\mathrm{diag}(t_1,t_2,t_3)) = \sum_j \delta_{i,j} t_j$. Then one can choose positive roots

$$\Delta^+(\mathfrak{g}_{\mathbb{C}}, \tilde{\mathfrak{h}}_{\mathbb{C}}) := \{e_1 - e_2,\ e_1 - e_3,\ e_2 - e_3\} \tag{6.63}$$

and there is a standard inner-automorphism Φ of $\mathfrak{g}_{\mathbb{C}}$ which sends $\mathfrak{h}_{\mathbb{C}}$ to $\tilde{\mathfrak{h}}_{\mathbb{C}}$ and which satisfies

$$\Phi^*(e_1 - e_2) = 2f_2; \quad \Phi^*(e_1 - e_3) = f_1 + f_2; \quad \Phi^*(e_2 - e_3) = f_1 - f_2. \tag{6.64}$$

The fundamental weights $\tilde{\omega}_1, \tilde{\omega}_2 \in \tilde{\mathfrak{h}}_{\mathbb{C}}^*$ are given by

$$\tilde{\omega}_1 = \frac{2}{3}(e_1 - e_2) + \frac{1}{3}(e_2 - e_3)$$

and

$$\tilde{\omega}_2 = \frac{1}{3}(e_1 - e_2) + \frac{2}{3}(e_2 - e_3).$$

Thus the fundamental weights $\omega_1, \omega_2 \in \mathfrak{h}_{\mathbb{C}}^*$ are given by

$$\omega_1 := \Phi^*(\tilde{\omega}_1) = \frac{1}{3}f_1 + f_2; \quad \omega_2 := \Phi^*(\tilde{\omega}_2) = \frac{2}{3}f_1. \tag{6.65}$$

If Λ is a weight, $\Lambda = \tau_1\omega_1 + \tau_2\omega_2$, $\tau_1, \tau_2 \in \mathbb{N}^0$, then a standard computation shows that

$$\Lambda_\theta = \tau_2\omega_1 + \tau_1\omega_2. \tag{6.66}$$

Now we fix $\tau_1, \tau_2 \in \mathbb{N}_0$, $\tau_1 + \tau_2 > 0$ and for $m \in \mathbb{N}$ we let $\tau(m)$ be the representation of G with highest weight

$$\Lambda(\tau(m)) := m\tau_1\omega_1 + m\tau_2\omega_2. \tag{6.67}$$

We let $\tilde{W}_{\mathfrak{g}}$ be the Weyl-group of $\Delta(\mathfrak{g}_{\mathbb{C}}, \tilde{\mathfrak{h}}_{\mathbb{C}})$. Then $\tilde{W}_{\mathfrak{g}}$ consists of all permutations of e_1, e_2, e_3. Let

$$\tilde{W}^1 := (\Phi^*)^{-1}W^1 = \{w \in \tilde{W}_{\mathfrak{g}} \colon w^{-1}(e_1 - e_2) > 0\}.$$

Then one has

$$\{(w, \ell(w)); w \in \tilde{W}^1\} = \left\{(\mathrm{Id}, 0);\ \left(\begin{pmatrix} e_1 & e_2 & e_3 \\ e_1 & e_3 & e_2 \end{pmatrix}, 1\right);\ \left(\begin{pmatrix} e_1 & e_2 & e_3 \\ e_3 & e_1 & e_2 \end{pmatrix}, 2\right)\right\}.$$

By a direct computation we get

$$
\begin{aligned}
&\{w(\Lambda(\tau(m))+\tilde{\rho}_G),\ell(w); w\in\tilde{W}^1\}\\
&\quad=\left\{\left(\frac{2m\tau_1+m\tau_2+3}{3}(e_1-e_2)+\frac{m\tau_1+2m\tau_2+3}{3}(e_2-e_3);0\right),\right.\\
&\qquad\left(\frac{2m\tau_1+m\tau_2+3}{3}(e_1-e_2)+\frac{m\tau_1-m\tau_2}{3}(e_2-e_3);1\right),\\
&\qquad\left.\left(\frac{-m\tau_1+m\tau_2}{3}(e_1-e_2)+\frac{-2m\tau_1-m\tau_2-3}{3}(e_2-e_3);2\right)\right\}.
\end{aligned}
\tag{6.68}
$$

As in [BV, 5.9.2] we introduce the following constants

$$
A_1(\tau(m)):=\frac{m\tau_1+1}{2};\quad A_2(\tau(m)):=\frac{m\tau_1+m\tau_2+2}{2};\quad A_3(\tau(m)):=\frac{m\tau_2+1}{2}
\tag{6.69}
$$

and

$$
C_1(\tau(m)):=\frac{m\tau_1+2m\tau_2+3}{3};\ C_2(\tau):=\frac{m\tau_1-m\tau_2}{3};\ C_3(\tau):=\frac{2m\tau_1+m\tau_2+3}{3}.
\tag{6.70}
$$

Note that on $\tilde{\mathfrak{h}}_{\mathbb{C}}^*$ one has $\tilde{\omega}_1=e_1;\quad \tilde{\omega}_2=e_1+e_2$, since the matrices in $\tilde{\mathfrak{h}}_{\mathbb{C}}^*$ have trace 0. Then, combining (6.64) and (6.68), we get

$$
\begin{aligned}
\{\left(\Lambda(\sigma_{\tau(m),w}),\lambda_{\tau(m),w},\ell(w)\right); w\in W^1\}&=\{((2A_1(\tau(m))-1)f_2,C_1(\tau(m)),0),\\
&((2A_2(\tau(m))-1)f_2,C_2(\tau(m)),1),((2A_3(\tau(m))-1)f_2,-C_3(\tau(m)),2)\}.
\end{aligned}
$$

Thus if we apply (6.62) we obtain

$$
\begin{aligned}
&\sum_{w\in W^1}(-1)^{\ell(w)}\int_0^{|\lambda_{\tau(m),w}|}P_{\sigma_{\tau(m),w}}(t)dt\\
&=-\frac{1}{\operatorname{vol}(\tilde{X}_d)}C_{\mathrm{SL}_3(\mathbb{R})}\sum_{k=1}^{3}(-1)^{k+1}A_k(\tau(m))\int_0^{|C_k(\tau(m))|}\left(\frac{9}{4}t^2-A_k(\tau(m))^2\right)dt\\
&=-\frac{1}{\operatorname{vol}(\tilde{X}_d)}\sum_{k=1}^{3}(-1)^{k+1}\frac{A_k(\tau(m))|C_k(\tau(m))|}{4}\left(3C_k(\tau(m))^2-4A_k(\tau(m))^2\right).
\end{aligned}
\tag{6.71}
$$

We can now prove our main result about the L^2-torsion for the case $G=SL_3(\mathbb{R})$.

Proposition 6.8. *Let $\widetilde{X}=\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ and $X=\Gamma\backslash\widetilde{X}$. Let $\Lambda\in\mathfrak{h}_{\mathbb{C}}^*$ be a highest weight with $\Lambda_\theta\neq\Lambda$. For $m\in\mathbb{N}$ let $\tau_\Lambda(m)$ be the irreducible representation of $\mathrm{SL}(3,\mathbb{R})$ with highest weight $m\Lambda$. There exists a polynomial P_Λ whose coefficients depend only on Λ such that*

$$
\log T_X^{(2)}(\tau_\Lambda(m))=\frac{\pi\operatorname{vol}(X)}{\operatorname{vol}(\widetilde{X}_d)}P_\Lambda(m).
$$

Moreover, there exists a constant $C(\Lambda)>0$ depending only on Λ such that

$$
P_\Lambda(m)=C(\Lambda)m\dim(\tau_\Lambda(m))+O(\dim(\tau_\Lambda(m))),
$$

as $m \to \infty$. If Λ equals one of the fundamental weights $\omega_{f,i}$ then $C(\Lambda) = 4/9$.

Proof. There exist $\tau_1, \tau_2 \in \mathbb{N}_0$, $\tau_1 \neq \tau_2$, such that $\Lambda = \tau_1\omega_1 + \tau_2\omega_2$. Put $\tau(m) := \tau_\Lambda(m)$. Then by Proposition 6.6, equation (6.69), (6.70) and (6.71), the first statement is proved and it remains to consider the asymptotic behavior of the polynomial P_Λ. We differ two cases. First we assume that $\tau_1\tau_2 \neq 0$. Then if we put

$$\alpha_4(\tau) := \begin{cases} -\frac{\tau_2^4}{18} + \frac{2\tau_1^3\tau_2}{9} + \frac{\tau_1^2\tau_2^2}{3}; & \tau_1 \geq \tau_2 \\ -\frac{\tau_1^4}{18} + \frac{2\tau_2^3\tau_1}{9} + \frac{\tau_1^2\tau_2^2}{3}; & \tau_2 \geq \tau_1, \end{cases}$$

an explicit computation using equation (6.69), (6.70) and (6.71) shows that

$$\sum_{w \in W^1} (-1)^{\ell(w)} \int_0^{|\lambda_{\tau(m),w}|} P_{\sigma_{\tau(m),w}}(t)dt = -\frac{\alpha_4(\tau)}{\mathrm{vol}(\tilde{X}_d)} m^4 + O(m^3),$$

as $m \to \infty$. Note that $\alpha_4(\tau) > 0$ by our assumption on τ_1 and τ_2. Now we assume that $\tau_1\tau_2 = 0$. Then if we define

$$\alpha_3(\tau) := \frac{2(\tau_1^3 + \tau_2^3)}{9},$$

an explicit computation using equation (6.69), (6.70) and (6.71) gives

$$\sum_{w \in W^1} (-1)^{\ell(w)} \int_0^{\lambda_{\tau(m),w}} P_{\sigma_{\tau(m),w}}(t)dt = -\frac{\alpha_3(\tau)}{\mathrm{vol}(\tilde{X}_d)} m^3 + O(m^2),$$

as $m \to \infty$. For $\mathrm{SL}_3(\mathbb{R})$ one has $v = 1$ and using Lemma 6.1 one gets $\frac{|W_{\mathfrak{m}}|}{|W_{K_M}|} = 1$. Moreover, every element of $\mathrm{Rep}(M^0)$ is self-dual. Thus using Proposition 6.6 we obtain

$$\log T_X^{(2)}(\tau(m)) = \mathrm{vol}(X) \frac{\pi\alpha_4(\tau)}{\mathrm{vol}(\tilde{X}_d)} m^4 + O(m^3)$$

as $m \to \infty$, if $\tau_1\tau_2 \neq 0$, and

$$\log T_X^{(2)}(\tau(m) = \mathrm{vol}(X) \frac{\pi\alpha_3(\tau)}{\mathrm{vol}(\tilde{X}_d)} m^3 + O(m^2),$$

as $m \to \infty$, if $\tau_1\tau_2 = 0$. Now we define constants

$$d_3(\tau) := \frac{\tau_1^2\tau_2 + \tau_2^2\tau_1}{2}; \quad d_2(\tau) := \left(\frac{4\tau_1\tau_2 + \tau_1^2 + \tau_2^2}{2} \right).$$

Then by Weyl's dimension formula one has

$$\dim \tau(m) = d_3(\tau)m^3 + d_2(\tau)m^2 + O(m),$$

as $m \to \infty$. Note that $d_3(\tau) > 0$ for $\tau_1\tau_2 \neq 0$ and that $d_3(\tau) = 0$, $d_2(\tau) > 0$ for $\tau_1\tau_2 = 0$. This completes the proof of the proposition. $\square$

7. Lower bounds of the spectrum

In this section we assume that $\delta(\widetilde{X}) = 1$ and that $\widetilde{X}$ is odd-dimensional. Our goal is to establish the lower bound (1.8) for the spectrum of the Laplace operators $\Delta_p(\tau_\lambda(m))$. To this end we use (5.5), which reduces the problem to the estimation from below of the endomorphism $E_p(\tau_\lambda(m))$.

First we introduce some notation. Let $\widetilde{X} = G/K$. Recall that we assume that $G \subset G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is the simply connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. By the classification of simple Lie groups there is a decomposition $\widetilde{X} = \widetilde{X}_0 \times \widetilde{X}_1$, where $\delta(\widetilde{X}_0) = 0$ and where $\widetilde{X}_1$ is an irreducible symmetric space with $\delta(\widetilde{X}_0) = 1$. Since $\widetilde{X}_0$ is even-dimensional, the dimension of $\widetilde{X}_1$ is odd. Let $G = G_0 \times G_1$ be the corresponding decomposition of G. Then $\delta(G_0) = 0$ and $G_1 = \operatorname{Spin}(p,q)$, p, q odd, or $G_1 = \mathrm{SL}(3,\mathbb{R})$. Let $\mathfrak{g}_i$, $i = 0,1$ be the Lie algebra of G_i. Let $\mathfrak{t}_0 \subset \mathfrak{g}_0$ be a compact Cartan subalgebra and let $\mathfrak{h}_1 \subset \mathfrak{g}_1$ be a fundamental Cartan subalgebra. Then $\mathfrak{h}_1$ is of split rank one. Put

$$\mathfrak{h} := \mathfrak{t}_0 \oplus \mathfrak{h}_1.$$

Then $\mathfrak{h}$ is a Cartan subalgebra of split rank one. Let $(\tau, V_\tau) \in \operatorname{Rep}(G)$ with highest weight $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$. Then $\lambda = \lambda_0 + \lambda_1$, where $\lambda_0 \in \mathfrak{t}_{0,\mathbb{C}}^*$ and $\lambda_1 \in \mathfrak{h}_{1,\mathbb{C}}^*$ are highest weights. Let $\theta\colon \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution. Assume that $\lambda_\theta \neq \lambda$. Then λ_1 satisfies $(\lambda_1)_\theta \neq \lambda_1$. Let $(\tau_i, V_{\tau_i}) \in \operatorname{Rep}(G_i)$, $i = 0,1$, be the representations with highest weight λ_i. Then $\tau \cong \tau_0 \otimes \tau_1$. Let

$$\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$$

be the Cartan decomposition of $\mathfrak{g}_i$, $i = 0,1$. We may choose $\mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$. Then we have

$$\Lambda^p \mathfrak{p}^* \otimes V_\tau \cong \bigoplus_{r+s=p} (\Lambda^r \mathfrak{p}_0^* \otimes V_{\tau_0}) \otimes (\Lambda^s \mathfrak{p}_1^* \otimes V_{\tau_1})$$

Let $\Omega_i \in \mathcal{Z}(\mathfrak{g}_{i,\mathbb{C}})$, $i = 1,2$, be the Casimir operator of $\mathfrak{g}_i$. Then $\Omega = \Omega_0 \otimes \operatorname{Id} + \operatorname{Id} \otimes \Omega_1$. Similarly, we have $\Omega_K = \Omega_{0,K} \otimes \operatorname{Id} + \operatorname{Id} \otimes \Omega_{1,K}$. Set

$$\nu_{i,p}(\tau_i) := \Lambda^p \operatorname{Ad}_{\mathfrak{p}_i}^* \otimes \tau_i \colon K_i \to \operatorname{GL}(\Lambda^p \mathfrak{p}_i^* \otimes V_{\tau_i}), \quad i = 0,1.$$

Let

$$E_{i,p}(\tau_i) := \tau_i(\Omega_i) \operatorname{Id}_i - \nu_p(\tau_i)(\Omega_{i,K}), \quad i = 0,1, \tag{7.1}$$

be the corresponding endomorphisms acting in $\Lambda^p \mathfrak{p}_i^* \otimes V_{\tau_i}$. Then it follows that

$$E_p(\tau) = \bigoplus_{r+s=p} \left(E_{0,r}(\tau_0) \otimes \operatorname{Id} + \operatorname{Id} \otimes E_{1,s}(\tau_1)\right). \tag{7.2}$$

Therefore it suffices to estimate $E_{i,p}(\tau_i)$, $i = 0,1$.

Let us first recall the general formula for the Casimir eigenvalues. We let $\mathfrak{g}$ be a semisimple real Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{b}$, $\mathfrak{b} \subset \mathfrak{p}$, be a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Let the associated groups G and K be as in the introduction. Let $\|\cdot\|$ denote the norm induced by the (suitably normalized) Killing form on the real vector space $i\mathfrak{t}^* \oplus \mathfrak{b}^*$. Fix positive roots $\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$, $\Delta^+(\mathfrak{k}_\mathbb{C},\mathfrak{t}_\mathbb{C})$ and let ρ_G resp. ρ_K be the half sums of the positive roots. Let τ be an irreducible finite-dimensional complex representation of G with highest weight $\Lambda(\tau)\in i\mathfrak{t}^*\oplus\mathfrak{b}^*$ and let ν be an irreducible unitary representation of K with highest weight $\Lambda(\nu)\in i\mathfrak{t}^*$. Then we have

$$\tau(\Omega)=\|\Lambda(\tau)+\rho_G\|^2-\|\rho_G\|^2;\quad \nu(\Omega_K)=\|\Lambda(\nu)+\rho_K\|^2-\|\rho_K\|^2. \tag{7.3}$$

We have the following general bound, which we use to deal with $E_{0,p}(\tau_0)$.

Lemma 7.1. *Let $\lambda\in\mathfrak{h}_\mathbb{C}^*$ be a highest weight. Given $m\in\mathbb{N}$, let $\tau_\lambda(m)$ be the irreducible representation with highest weight $m\lambda$. There exists $C>0$ such that*

$$E_p(\tau_\lambda(m))\geq -Cm$$

for all $p=0,\dots,d$ and $m\in\mathbb{N}$.

Proof. Let $\tau\in\operatorname{Rep}(G)$ with highest weight $\Lambda(\tau)$. Let $\nu'\in\hat{K}$ with highest weight $\Lambda(\nu')\in i\mathfrak{t}^*$. Assume that $[\tau|_K\colon\nu']\neq 0$. We claim that there is a weight λ of τ such that $\Lambda(\nu')=\lambda|_\mathfrak{t}$. To see this, let V_τ be the space of the representation τ and let $V_\tau(\Lambda(\nu'))$ be the eigenspace of $\mathfrak{t}$ with eigenvalue $\Lambda(\nu')$. Then $V_\tau(\Lambda(\nu'))$ is invariant under $\mathfrak{h}$. So it decomposes into joint eigenspaces of $\mathfrak{h}$. Let λ be the weight of one of these eigenspaces. Then $\lambda|_\mathfrak{t}=\Lambda(\nu')$. Now we note that as a weight of τ, λ belongs to the convex hull of the Weyl group orbit of $\Lambda(\tau)$ (see [Ha, Theorem 7.41]). Thus we get

$$\|\Lambda(\tau)\|\geq\|\lambda\|\geq\|\lambda|_\mathfrak{t}\|=\|\Lambda(\nu')\|. \tag{7.4}$$

Now let $\nu\in\widehat{K}$ with $[\nu_p(\tau)\colon\nu]\neq 0$. Then by [Kn2, Proposition 9.72] there exists $\nu'\in\widehat{K}$ with $[\tau|_K\colon\nu']\neq 0$ of highest weight $\Lambda(\nu')\in i\mathfrak{t}^*$ and $\mu\in i\mathfrak{t}^*$ which is a weight of ν_p such that the highest weight $\Lambda(\nu)$ of ν is given by $\mu+\Lambda(\nu')$. Since $\Lambda(\tau)$ is dominant we have

$$\|\Lambda(\tau)+\rho_G\|^2\geq\|\Lambda(\tau)\|^2.$$

Thus by (7.4) we get

$$\begin{aligned}\|\Lambda(\tau)+\rho_G\|^2-\|\Lambda(\nu)+\rho_K\|^2&\geq\|\Lambda(\tau)\|^2-\|\Lambda(\nu')\|^2-2\|\mu+\rho_K\|\cdot\|\Lambda(\nu')\|-\|\mu+\rho_K\|^2\\&\geq -2\|\mu+\rho_K\|\cdot\|\Lambda(\tau)\|-\|\mu+\rho_K\|^2.\end{aligned}$$

There is $C>0$ such that $\|\mu+\rho_K\|\leq C$ for all weights μ of ν_p. Hence there is $C_1>0$ such that for all $\tau\in\operatorname{Rep}(G)$ one has

$$\|\Lambda(\tau)+\rho_G\|^2-\|\Lambda(\nu)+\rho_K\|^2\geq -C_1(\|\Lambda(\tau)\|+1) \tag{7.5}$$

for all $\nu\in\widehat{K}$ with $[\nu_p(\tau)\colon\nu]\neq 0$. Now we apply this to $\tau_\lambda(m)$. By definition of $\tau_\lambda(m)$ we have $\Lambda(\tau_\lambda(m))=m\lambda$. Using (7.5) and (7.3), the lemma follows. ∎

Now we turn to the estimation of $E_{1,p}(\tau_1)$. In this case we have either $G_1=\operatorname{Spin}(p,q)$, p,q odd, or $G=\operatorname{SL}(3,\mathbb{R})$. We deal with these cases separately.

7.1. **The case $G = \mathrm{Spin}(p,q)$.** Let $p = 2p_1 + 1$, $q = 2q_1 + 1$. Let $n := p_1 + q_1$. Let $K = \mathrm{Spin}(p) \times \mathrm{Spin}(q)$ and $\widetilde{X} = G/K$. Then $\dim(\widetilde{X}) = pq$. We let $\mathfrak{t}$ and $\mathfrak{h}$ be as in section 6. Also the Killing form will be normalized as in this section. Then we have the following lemma.

Lemma 7.2. *Let $\Lambda \in \mathfrak{h}_{\mathbb{C}}^*$ be given as $\Lambda = k_1 e_1 + \dots + k_{n+1} e_{n+1}$, $k_1 \geq k_2 \geq \dots \geq k_{n+1} \geq 0$. Let $\Lambda' \in \mathfrak{h}_{\mathbb{C}}^*$ belong to the convex hull of the set $\{w\Lambda,\ w \in W_G\}$ and let $\lambda \in i\mathfrak{t}^*$ be given by $\lambda := \Lambda'|_{\mathfrak{t}}$. Then one has*

$$\|\lambda\|^2 \leq \sum_{i=1}^{n} k_i^2.$$

Proof. Recall that the Weyl group W_G consist of permutations and even sign changes of the $e_1, \dots, e_{n+1}$. Thus there exist $\alpha_1, \dots, \alpha_m \in (0,1)$, $\sum_{j=1}^m \alpha_j = 1$, and for each $j = 1, \dots, m$ a $\sigma_j \in S^{n+1}$, the symmetric group, and a sequence $\epsilon_{j,1}, \dots, \epsilon_{j,n+1} \in \{\pm 1\}$ such that

$$\Lambda' = \sum_{j=1}^{m} \alpha_j \left(\sum_{i=1}^{n+1} \epsilon_{j,i} k_i e_{\sigma_j(i)} \right).$$

Thus one has

$$\lambda = \sum_{j=1}^{m} \alpha_j \left(\sum_{\substack{i=1 \\ \sigma_j(i) \neq 1}}^{n+1} \epsilon_{j,i} k_i e_{\sigma_j(i)} \right)$$

and so one gets

$$\|\lambda\| \leq \sum_{j=1}^{m} \alpha_j \left\| \sum_{\substack{i=1 \\ \sigma_j(i) \neq 1}}^{n+1} \epsilon_{j,i} k_i e_{\sigma_j(i)} \right\| = \sum_{j=1}^{m} \alpha_j \sqrt{\sum_{\substack{i=1 \\ \sigma_j(i) \neq 1}}^{n+1} k_i^2} \leq \sum_{j=1}^{m} \alpha_j \sqrt{\sum_{i=1}^{n} k_i^2} = \sqrt{\sum_{i=1}^{n} k_i^2}.$$

For the last inequality we used that the k_i's satisfy $k_1 \geq k_2 \geq \dots \geq k_{n+1}$. $\square$

Now we let $\Lambda(\tau) \in \mathfrak{h}_{\mathbb{C}}^*$ be given by

$$\Lambda(\tau) := \tau_1 e_1 + \dots + \tau_{n+1} e_{n+1}, \quad \tau_1 \geq \tau_2 \geq \dots \geq \tau_{n+1} > 0.$$

For $m \in \mathbb{N}$ we let $\tau(m)$ be the representation of G with highest weight

$$\Lambda(\tau(m)) := m\Lambda(\tau).$$

Then we have the following proposition.

Proposition 7.3. *There exists a constant C such that*

$$E_p(\tau(m)) \geq m^2 \tau_{n+1} - Cm$$

for all m.

Proof. Recall that $\nu_p(\tau(m)) = \tau(m)|_K \otimes \nu_p$. Let $\nu \in \hat{K}$ be such that $[\nu_p(\tau(m)) : \nu] \neq 0$. By [Kn2, Proposition 9.72], there exists a $\nu' \in \hat{K}$ with $[\tau(m) : \nu'] \neq 0$ of highest weight $\lambda(\nu') \in \mathfrak{b}_{\mathbb{C}}^*$ and a $\mu \in \mathfrak{b}_{\mathbb{C}}^*$ which is a weight of ν_p such that the highest weight $\lambda(\nu)$ of ν is given by $\mu + \lambda(\nu')$. As shown in the proof of Lemma 7.1, there is a weight $\widetilde{\Lambda} \in \mathfrak{h}_{\mathbb{C}}^*$ of $\tau(m)$ such that $\lambda(\nu') = \widetilde{\Lambda}|_{\mathfrak{t}}$. By [Ha, Theorem 7.41], $\widetilde{\Lambda}$ belongs to the convex hull of the Weyl group orbit of $\Lambda(\tau(m))$. Thus, applying (7.3) and Lemma 7.2, we obtain constants $C_1, C_2 > 0$, which are independent of m, such that

$$\nu(\Omega_K) = \|\lambda(\nu) + \rho_K\|^2 - \|\rho_K\|^2 \leq \|\lambda(\nu')\|^2 + C_1(1 + \|\lambda(\nu')\|) \leq m^2 \left(\sum_{j=1}^{n} \tau_j^2 \right) + C_2 m.$$

One the other hand, by (7.3) we have

$$\begin{aligned} \tau(m)(\Omega) = \|\Lambda(\tau(m)) + \rho_G\|^2 - \|\rho_G\|^2 &= \sum_{j=1}^{n+1} (m\tau_j + n + 1 - j)^2 - \sum_{j=1}^{n+1} (n + 1 - j)^2 \\ &\geq m^2 \sum_{j=1}^{n+1} \tau_j^2. \end{aligned}$$

This implies the proposition. $\square$

7.2. **The case $G = \mathrm{SL}(3, \mathbb{R})$.** We use the notation of section 6. We choose the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, which is defined by (6.60). The fundamental weights $\omega_i \in \mathfrak{h}_{\mathbb{C}}^*$, $i = 1, 2$, are given by (6.65). Let $\Lambda \in \mathfrak{h}_{\mathbb{C}}^*$ be a highest weight. For $m \in \mathbb{N}$ let $\tau_\Lambda(m)$ be the irreducible representation with highest weight $m\Lambda$.

Proposition 7.4. *Assume that Λ satisfies $\Lambda_\theta \neq \Lambda$. Then there exists $C_\Lambda > 0$ such that*

$$E_p(\tau_\Lambda(m)) \geq \frac{1}{9} m^2 - C_\Lambda m$$

for all $m \in \mathbb{N}$ and $p = 0, \ldots, 5$.

Proof. There exist $\tau_1, \tau_2 \in \mathbb{N}_0$ such that $\Lambda = \tau_1 \omega_1 + \tau_2 \omega_2$. Note that by (6.63) and (6.64) one has $\rho_G = f_1 + f_2$. Then by (6.65) and (6.61) we get

$$\tau_\Lambda(m)(\Omega) = \|m\Lambda + \rho_G\|^2 - \|\rho_G\|^2 = \frac{4(\tau_1^2 + \tau_1\tau_2 + \tau_2^2)}{9} m^2 + \frac{4(\tau_1 + \tau_2)}{3} m.$$

Next recall that there is a natural isomorphism $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ (see [Ha, Sect. 4.9]). Furthermore if we embed $\mathfrak{sl}(2, \mathbb{C})$ into $\mathfrak{g}_{\mathbb{C}}$ as an upper left block then $\mathfrak{t}_{\mathbb{C}}$ is isomorphic to a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. For $j \in \mathbb{N}$ let ν_j denote the representation of $\mathfrak{k}_{\mathbb{C}}$ with highest weight jf_2. Then we deduce from the branching law from $\mathrm{GL}_3(\mathbb{C})$ to $\mathrm{GL}_2(\mathbb{C})$, [GW, Theorem 8.1.1] that

$$\tau_\Lambda(m)|_{\mathfrak{k}_{\mathbb{C}}} = \bigoplus_{j=0}^{m\tau_1} \bigoplus_{k=0}^{m\tau_2} \nu_{j+k}.$$

If we use

$$\nu_j(\Omega_K) = \frac{j^2}{3} + \frac{2}{3}j.$$

and argue as in the proof of Proposition 7.3, we obtain a constant C which is independent of τ_1, τ_2 and m such that for every $\nu \in \hat{K}$ with $[\nu_p(\tau(m)) : \nu] \neq 0$ for some p one has

$$\nu(\Omega_K) \leq \frac{(m(\tau_1 + \tau_2) + C)^2}{3} + \frac{2\,(m(\tau_1 + \tau_2) + C)}{3}.$$

Thus we obtain a constant C_Λ such that for every m and every p one has

$$E_p(\tau_\Lambda(m)) \geq \frac{(\tau_1 - \tau_2)^2}{9} m^2 - C_\Lambda m.$$

By (6.66) the condition $\Lambda_\theta \neq \Lambda$ is equivalent to $\tau_1 \neq \tau_2$. This proves the Proposition. ☐

Now we can summarize our results.

Proposition 7.5. *Let $\delta(\widetilde{X}) = 1$ and assume that $\dim(\widetilde{X})$ is odd. Let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ be a highest weight with $\lambda_\theta \neq \lambda$. For $m \in \mathbb{N}$ let $\tau_\lambda(m)$ be the irreducible representation of G with highest weight $m\lambda$. There exist $C_1, C_2 > 0$ such that*

$$E_p(\tau_\lambda(m)) \geq C_1 m^2 - C_2$$

for all $p = 0, \ldots, d$ and $m \in \mathbb{N}$.

Proof. Let $\lambda = \lambda_0 + \lambda_1$ with $\lambda_0 \in \mathfrak{t}_{0,\mathbb{C}}^*$ and $\lambda_1 \in \mathfrak{h}_{1,\mathbb{C}}^*$ highest weights, and assume that $(\lambda_1)_\theta \neq \lambda_1$. Let $\tau_i(m)$, $i = 0, 1$, be the irreducible representations of G_i with highest weight $m\lambda_i$. Then $\tau(m) = \tau_0(m) \otimes \tau_1(m)$. Let $E_{0,p}(\tau_0(m))$ and $E_{1,p}(\tau_1(m))$ be defined by (7.1). By Lemma 7.1 there exists $C > 0$ such that

$$E_{0,p}(\tau_0(m)) \geq -Cm$$

for all $p = 0, \ldots, d$ and $m \in \mathbb{N}$. Furthermore, by Proposition 7.3 and Proposition 7.4 there exist $C_3, C_4 > 0$ such that

$$E_{1,p}(\tau_1(m)) \geq C_3 m^2 - C_4$$

for all $p = 0, \ldots, d$ and $m \in \mathbb{N}$. Combined with (7.2) the proof follows. ☐

Corollary 7.6. *Let the assumptions be as in Proposition 7.5. There exist constants $C_1, C_2 > 0$ such that*

$$\Delta_p(\tau_\lambda(m)) \geq C_1 m^2 - C_2$$

for all $p = 0, \ldots, d$ and $m \in \mathbb{N}$.

Proof. Recall that the Bochner-Laplace operator satisfies $\Delta_{\nu_p(\tau(m))} \geq 0$. Hence the corollary follows from (5.5) and Proposition 7.5. ☐

8. Proof of the main results

First assume that $\delta(\widetilde{X}) \neq 1$. If $\delta(\widetilde{X}) = 0$, then $\dim \widetilde{X}$ is even. Hence, it follows from Proposition 4.2 that $T_X(\tau) = 1$ for all finite-dimensional irreducible representations of G, which proves part (i) of Theorem 1.1.

Now assume that $\delta(\widetilde{X}) = 1$ and that $d = \dim(X)$ is odd. Let $\mathfrak{h} \subset \mathfrak{g}$ be a fundamental Cartan subalgebra. Let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ be a highest weight with $\lambda_\theta \neq \lambda$. For $m \in \mathbb{N}$ let $\tau(m)$ be the irreducible representation of G with highest weight $m\lambda$. Then $\tau(m) \ncong \tau(m)_\theta$ for all $m \in \mathbb{N}$. Hence by [BW, Theorem 6.7] we have $H^p(X, E_{\tau(m)}) = 0$ for all $p = 0, \ldots, d$. Then by (4.8) we have

$$\log T_X(\tau(m)) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} K(t, \tau(m))\, dt\right)\Bigg|_{s=0}. \tag{8.1}$$

Since $\tau(m)$ is acyclic and $\dim X$ is odd, $T_X(\tau(m))$ is metric independent [Mu2, Corollary 2.7]. Especially we can rescale the metric by $\sqrt{m}$ without changing $T_X(\tau(m))$. Equivalently we can replace $\Delta_p(\tau(m))$ by $\frac{1}{m}\Delta_p(\tau(m))$. Using (8.1) we get

$$\log T_X(\tau(m)) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} K\left(\frac{t}{m}, \tau(m)\right) dt\right)\Bigg|_{s=0}.$$

To continue, we split the t-integral into the integral over $[0,1]$ and the integral over $[1,\infty)$. This leads to

$$\begin{aligned}\log T_X(\tau(m)) =& \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1} K\left(\frac{t}{m}, \tau(m)\right) dt\right)\Bigg|_{s=0} \\ &+ \frac{1}{2}\int_1^\infty t^{-1} K\left(\frac{t}{m}, \tau(m)\right) dt.\end{aligned} \tag{8.2}$$

We first consider the second term on the right hand side. To this end we need the following lemma.

Lemma 8.1. *Let $h_t^{\tau(m),\, p}$ be defined by (4.14) and let H_t^0 be the heat kernel of the Laplacian $\widetilde{\Delta}_0$ on $C^\infty(\widetilde{X})$. There exist $m_0 \in \mathbb{N}$ and $C > 0$ such that for all $m \geq m_0$, $g \in G$, $t \in (0,\infty)$ and $p \in \{0, \ldots, d\}$ one has*

$$\left| h_t^{\tau(m),\, p}(g) \right| \leq C \dim(\tau(m)) e^{-t\frac{m^2}{2}} H_t^0(g).$$

Proof. Let $p \in \{0, \ldots, n\}$. Let $H_t^{\nu_p(\tau(m))}$ be the kernel of $e^{-t\widetilde{\Delta}_{\nu_p(\tau(m))}}$ and let $H_t^{\tau(m),\, p}$ be the kernel of $e^{-t\widetilde{\Delta}_p(\tau(m))}$. By (5.8) we have

$$H_t^{\tau(m),\, p}(g) = e^{-tE_p(\tau(m))} \circ H_t^{\nu_p(\tau(m))}(g).$$

Thus by Proposition 3.1 and Proposition 7.5 there exists an m_0 such that for $m \geq m_0$ one has

$$\left\| H_t^{\tau(m),\, p}(g) \right\| \leq e^{-t\frac{m^2}{2}} H_t^0(g). \tag{8.3}$$

Taking the trace in $\operatorname{End}(\Lambda^p \mathfrak{p}^* \otimes V_{\tau(m)})$ for every $p \in \{0, \ldots, d\}$, the lemma follows. $\square$

Using (4.17), (4.16) and Lemma 8.1, we obtain

$$\begin{aligned}\left|K\left(\frac{t}{m},\tau(m)\right)\right| &\leq Ce^{-\frac{m}{2}t}\dim(\tau(m))\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}H^0_{t/m}(g^{-1}\gamma g)\,d\dot{g}\\ &= Ce^{-\frac{m}{2}t}\dim(\tau(m))\operatorname{Tr}(e^{-\frac{t}{m}\Delta_0}).\end{aligned}$$

Furthermore, by the heat asymptotic [Gi] we have

$$\operatorname{Tr}(e^{-\frac{1}{m}\Delta_0}) = C_d\operatorname{vol}(X)m^{d/2} + O\left(m^{(d-1)/2}\right)$$

as $m\to\infty$. Hence there exists $C_1>0$ such that

$$\left|K\left(\frac{t}{m},\tau(m)\right)\right| \leq C_1 m^{d/2}\dim(\tau(m))e^{-\frac{m}{2}t},\quad t\geq 1.$$

Thus we obtain

$$\left|\int_1^\infty t^{-1}K\left(\frac{t}{m},\tau(m)\right)\,dt\right| \leq C_2 m^{d/2}\dim(\tau(m))e^{-m/4}.$$

Using Weyl's dimension formula, it follows that

$$\int_1^\infty t^{-1}K\left(\frac{t}{m},\tau(m)\right)\,dt = O\left(e^{-m/8}\right). \tag{8.4}$$

Now we turn to the first term on the right hand side of (8.2). We need to estimate $K(t,\tau(m))$ for $0<t\leq 1$. To this end we use (4.17) to decompose $K(t,\tau(m))$ into the sum of two terms: The contribution of the identity

$$I(t,\tau(m)) := \operatorname{vol}(X)k_t^{\tau(m)}(1), \tag{8.5}$$

where $k_t^{\tau(m)}$ is defined by (4.16), and the remaining term

$$H(t,\tau(m)) := \int_{\Gamma\backslash G}\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq 1}}k_t^{\tau(m)}(g^{-1}\gamma g)\,d\dot{g}$$

First we consider $H(t,\tau(m))$. Using Proposition 8.1 and Proposition 3.2, it follows that for every $m\geq m_0$ and every $t\in(0,1]$ we have

$$\begin{aligned}\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq 1}}\left|k_t^{\tau(m)}(g^{-1}\gamma g)\right| &\leq Ce^{-t\frac{m^2}{2}}\dim(\tau(m))\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq 1}}H_t^0(g^{-1}\gamma g)\\ &\leq C_1\dim(\tau(m))e^{-t\frac{m^2}{2}}e^{-c_0/t}.\end{aligned}$$

Hence using Weyl's dimension formula we get $c_1>0$ such that

$$\left|H\left(\frac{t}{m},\tau(m)\right)\right| \leq C_2 e^{-c_1 m}e^{-c_1/t},\quad 0<t\leq 1.$$

This implies that

$$(8.6)\quad \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}H\left(\frac{t}{m},\tau(m)\right)\,dt\right)\bigg|_{s=0}=\int_0^1 t^{-1}H\left(\frac{t}{m},\tau(m)\right)\,dt=O\left(e^{-c_1 m}\right)$$

as $m\to\infty$.

It remains to consider the contribution of the identity $I(t,\tau(m))$. By Lemma 8.1 there exists $C>0$ such that for all $m\geq m_0$ and $p=0,\dots,d$ we have

$$|h_t^{\tau(m),p}(1)|\leq C\dim(\tau(m))e^{-t\frac{m^2}{2}}H_t^0(1).$$

Next we estimate $H_t^0(1)$ using the Plancherel-Theorem. Since the function $H_t^0(1)$ is K-biinvariant, the Plancherel-Theorem for $H_t^0(1)$ reduces to the spherical Plancherel theorem [He, Theorem 7.5]. Thus if $Q=MAN$ is a fixed minimal standard parabolic subgroup, it follows from (5.14) that

$$H_t^0(1)=e^{-t\|\rho_{\mathfrak{a}}\|^2}\int_{\mathfrak{a}^*}e^{-t\|\nu\|^2}\beta(\nu)d\nu,$$

where $\beta(\nu)$ is the spherical Plancherel-density. Thus there exists $C_1>0$ such that $|H_t^0(1)|\leq C_1$ for $t\geq 1$. Hence, by (4.16) we get

$$|k_t^{\tau(m)}(1)|\leq C_2\dim(\tau(m))e^{-t\frac{m^2}{2}}$$

for $t\geq 1$ and $m\geq m_0$. By (8.5) and Weyl's dimension formula it follows that there exist $C,c>0$ such that

$$(8.7)\qquad \left|I\left(\frac{t}{m},\tau(m)\right)\right|\leq Ce^{-cm}e^{-ct}$$

for $t\geq 1$ and $m\geq m_0$. Hence we get

(8.8)

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}I\left(\frac{t}{m},\tau(m)\right)\,dt\right)\bigg|_{s=0}&=\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}I\left(\frac{t}{m},\tau(m)\right)\,dt\right)\bigg|_{s=0}\\&\quad+O\left(e^{-cm}\right)\end{aligned}$$

for $m\geq m_0$. To deal with the first term on the right, we note that by (5.12) and the definition of $k_t^{\tau(m)}$ by (4.16), $k_t^{\tau(m)}(1)$ has an asymptotic expansion of the form

$$k_t^{\tau(m)}(1)\sim\sum_{j=0}^\infty c_j t^{-d/2+j}$$

as $t\to 0$. Since we are assuming that $d=\dim(X)$ is odd, the expansion has no constant term. This implies that

$$\int_0^\infty t^{s-1}I(t,\tau(m))\,dt$$

is holomorphic at $s=0$. Therefore we get

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} I\left(\frac{t}{m},\tau(m)\right)\, dt\right)\Bigg|_{s=0} = \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} I\left(t,\tau(m)\right)\, dt\right)\Bigg|_{s=0}.$$

By definition, the right hand side equals $\log T_X^{(2)}(\tau(m))$, where $T_X^{(2)}(\tau(m))$ is the L^2-torsion. Combined with (8.2), (8.4) and (8.6) we obtain

$$\log T_X(\tau(m)) = \log T_X^{(2)}(\tau(m)) + O\left(e^{-cm}\right) \tag{8.9}$$

as $m \to \infty$. This proves Proposition 1.2. □

Combining Proposition 5.3 with Proposition 6.7 and Proposition 6.8, we obtain Proposition 1.3. Together with Proposition 1.2 we obtain part (ii) of Theorem 1.1.

Corollary 1.4 follows from Proposition 6.7 and Corollary 1.5 follows from Proposition 6.8.

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