LIMIT MULTIPLICITIES FOR PRINCIPAL CONGRUENCE SUBGROUPS OF GL(n)

TOBIAS FINIS, EREZ LAPID, AND WERNER MÜLLER

Abstract. We study the limiting behavior of the discrete spectra associated to the principal congruence subgroups of a reductive group over a number field. While this problem is well understood in the cocompact case (i.e. when the group is anisotropic modulo the center), we treat groups of unbounded rank. For the groups GL(n) we are able to show that the spectra converge to the Plancherel measure (the limit multiplicity property), and in general we obtain a substantial reduction of the problem. Our main tool is the recent refinement of the spectral side of Arthur's trace formula obtained in [27, 25], which allows us to show that for GL(n) the contribution of the continuous spectrum is negligible in the limit.

Contents

1. Introduction

Let G be an algebraically connected linear semisimple Lie group with a fixed choice of Haar measure. Since the group G is of type I, we can write unitary representations of G on separable Hilbert spaces as direct integrals (with multiplicities) over the unitary dual $\Pi(G)$, the set of isomorphism classes of irreducible unitary representations of G with the Fell topology (cf. [23]). An important case is the regular representation of $G \times G$ on $L^2(G)$, which can be decomposed as the direct integral of the tensor products $\pi \otimes \pi^*$ against the

Date: July 8, 2012.

2010 *Mathematics Subject Classification.* Primary 11F72; Secondary 22E40, 22E55, 11F70.

Key words and phrases. Limit multiplicities, lattices in Lie groups, trace formula.

Authors partially sponsored by grant # 964-107.6/2007 from the German-Israeli Foundation for Scientific Research and Development. First named author supported by DFG Heisenberg grant # FI 1795/1-1.

Plancherel measure μ_{pl} on $\Pi(G)$. The support of the Plancherel measure is called the *tempered dual* $\Pi(G)_{\mathrm{temp}} \subset \Pi(G)$.

Other basic objects of interest are the regular representations R_Γ of G on $L^2(\Gamma\backslash G)$ for lattices Γ in G. We will focus on the discrete part $L^2_{\mathrm{disc}}(\Gamma\backslash G)$ of $L^2(\Gamma\backslash G)$, namely the sum of all irreducible subrepresentations, and we denote by $R_{\Gamma,\mathrm{disc}}$ the corresponding restriction of R_Γ. For any $\pi \in \Pi(G)$ let $m_\Gamma(\pi)$ be the multiplicity of π in $L^2(\Gamma\backslash G)$. Thus,

$$m_\Gamma(\pi) = \dim \mathrm{Hom}_G(\pi, R_\Gamma) = \dim \mathrm{Hom}_G(\pi, R_{\Gamma,\mathrm{disc}}).$$

These multiplicities are known to be finite.[1] We define the discrete spectral measure on $\Pi(G)$ with respect to Γ by

$$\mu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi \in \Pi(G)} m_\Gamma(\pi)\delta_\pi,$$

where δ_π is the Dirac measure at π. While one cannot hope to describe the multiplicity functions m_Γ on $\Pi(G)$ explicity (apart from certain special cases, for example when π belongs to the discrete series), it is feasible and interesting to study asymptotic questions. The limit multiplicity problem concerns the asymptotic behavior of μ_Γ as $\mathrm{vol}(\Gamma\backslash G) \to \infty$.

More explicitly, let $\Gamma_1, \Gamma_2, \dots$ be a sequence of lattices in G. We say that the sequence (Γ_n) has the limit multiplicity property if the following two conditions are satisfied:

(1) For any Jordan measurable set $A \subset \Pi(G)_{\mathrm{temp}}$ we have
$$\mu_{\Gamma_n}(A) \to \mu_{\mathrm{pl}}(A) \quad \text{as } n \to \infty.$$
(2) For any bounded set $A \subset \Pi(G) \setminus \Pi(G)_{\mathrm{temp}}$ we have
$$\mu_{\Gamma_n}(A) \to 0 \quad \text{as } n \to \infty.$$

For the definition of a bounded subset of $\Pi(G)$ see §2 below. Recall that a Jordan measurable subset of $\Pi(G)_{\mathrm{temp}}$ is a bounded set such that $\mu_{\mathrm{pl}}(\partial A) = 0$, where $\partial A = \bar{A} - A^\circ$ is the boundary of A. The first condition can also be rephrased as

$$\lim_{n\to\infty} \mu_{\Gamma_n}(f) = \mu_{\mathrm{pl}}(f)$$

for any Riemann integrable function f on $\Pi(G)_{\mathrm{temp}}$, or equivalently, for any compactly supported continuous function.[2]

A great deal is known about the limit multiplicity problem for uniform lattices, where R_Γ decomposes discretely. The first results in this direction were proved by DeGeorge-Wallach ([17, 18]) for normal towers, i.e., descending sequences of finite index normal subgroups of a given uniform lattice with trivial intersection. Subsequently, Delorme ([22]) completely

[1] At least under a weak reduction-theoretic assumption on G and Γ ([44, p. 62]), which is satisfied if either G has no compact factors or if Γ is arithmetic (cf. [ibid., Theorem 3.3]).

[2] A Riemann integrable function on $\Pi(G)_{\mathrm{temp}}$ is a bounded compactly supported function which is continuous almost everywhere with respect to the Plancherel measure. Note here that the complement of a closed subset of Plancherel measure zero in the topological space $\Pi(G)_{\mathrm{temp}}$ is homeomorphic to a countable union of Euclidean spaces of bounded dimensions, and that under this homeomorphism the Plancherel density is given by a continuous function. The same is true in the case of p-adic reductive groups considered below.

resolved the limit multiplicity problem for this case in the affirmative. Recently, there has been big progress in proving limit multiplicity for much more general sequences of uniform lattices ([1]). In particular, families of non-commensurable lattices were considered for the first time.

In the case of non-compact quotients $\Gamma\backslash G$, where the spectrum also contains a continuous part, much less is known. Here, the limit multiplicity problem has been solved for normal towers of arithmetic lattices and discrete series L-packets $A \subset \Pi(G)$ (with regular parameters) by Rohlfs-Speh ([46]). Building on this work, the case of singleton sets A and normal towers of congruence subgroups has been solved by Savin ([49], cf. also [52]). Earlier results on the discrete series had been obtained by DeGeorge ([19]) and Barbasch-Moscovici ([10]) for groups of real rank one, and by Clozel ([15]) for general groups (but with a weaker statement). The limit multiplicity problem for the entire unitary dual has been solved for the standard congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ in [47] (cf. [28, p. 173], [21, §5]). In this case, a refined quantitative version of the limit multiplicity property for the non-tempered spectrum of the subgroups $\Gamma_0(N)$ has been proven by Iwaniec ([29]).[3] A partial result for certain normal towers of congruence arithmetic lattices defined by groups of $\mathbb{Q}$-rank one has been shown in [21]. Finally, generalizations to the distribution of Hecke eigenvalues have been obtained in [48] and [50].

In this paper we embark upon a general analysis of the case of non-compact quotients. We consider the entire unitary dual and groups of unbounded rank. The main problem is to show that the contribution of the continuous spectrum is negligible in the limit. This was known up to now only in the case of GL(2) (or implicitly in the very special situation considered in [46] and [49]). Our approach is based on a careful study of the spectral side of Arthur's trace formula in the recent form given in [27, 25]. As we shall see, this form is crucial for the analysis. Our results are unconditional only for the groups $\mathrm{GL}(n)$, but we obtain a substantial reduction of the problem in the general case.

Before stating our main result we shift to an adelic setting which allows one to incorporate Hecke operators into the picture (i.e., to consider the equidistribution of Hecke eigenvalues). Thus, let now G be a reductive group defined over a number field F and S a finite set of places of F containing the set S_∞ of all archimedean places. As usual, $G(F_S)^1$ denotes the kernel of the homomorphisms $|\chi| : G(F_S) \to \mathbb{R}^{>0}$ where χ ranges over the F-rational characters of G and $|\cdot|$ denotes the normalized absolute value on F_S^*. Similarly, we define a subgroup $G(\mathbb{A})^1$ of $G(\mathbb{A})$. Fix a Haar measure on $G(\mathbb{A})$. For any open compact subgroup K of $G(\mathbb{A}^S)$ let $\mu_K = \mu_K^{G,S}$ be the measure on $\Pi(G(F_S)^1)$ given by

$$\begin{aligned}\mu_K &= \frac{1}{\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1/K)} \sum_{\pi \in \Pi(G(F_S)^1)} \dim \mathrm{Hom}_{G(F_S)^1}(\pi, L^2(G(F)\backslash G(\mathbb{A})^1/K))\, \delta_\pi \\ &= \frac{\mathrm{vol}(K)}{\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)} \sum_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A}))} m_\pi \dim(\pi^S)^K\, \delta_{\pi_S}.\end{aligned}$$

[3]Recall that by Selberg's eigenvalue conjecture the non-tempered spectrum should consist only of the trivial representation in this case.

We say that a collection $\mathcal{K}$ of open compact subgroups of $G(\mathbb{A}^S)$ has the limit multiplicity property if $\mu_K \to \mu_{\mathrm{pl}}$ for $K \in \mathcal{K}$ in the sense that

(1) for any Jordan measurable subset $A \subset \Pi(G(F_S)^1)_{\mathrm{temp}}$ we have $\mu_K(A) \to \mu_{\mathrm{pl}}(A)$, $K \in \mathcal{K}$, and,
(2) for any bounded subset $A \subset \Pi(G(F_S)^1) \setminus \Pi(G(F_S)^1)_{\mathrm{temp}}$ we have $\mu_K(A) \to 0$, $K \in \mathcal{K}$.

Here, we write for example $\mu_K(A) \to \mu_{\mathrm{pl}}(A)$ to mean that for every $\epsilon > 0$ there are only finitely many subgroups $K \in \mathcal{K}$ such that $|\mu_K(A) - \mu_{\mathrm{pl}}(A)| \geq \epsilon$. We can again rephrase the first condition by saying that for any Riemann integrable function f on $\Pi(G(F_S)^1)_{\mathrm{temp}}$ we have

$$\mu_K(f) \to \mu_{\mathrm{pl}}(f), \quad K \in \mathcal{K}.$$

We remark that when G satisfies the strong approximation property with respect to S_∞ (which is equivalent to G being semisimple and simply connected and $G(F_\infty)$ having no compact factors), we have

$$G(F)\backslash G(\mathbb{A})/K \simeq \Gamma_K \backslash G(F_\infty)$$

for the lattice $\Gamma_K = G(F) \cap K$ in the connected semisimple Lie group $G(F_\infty)$. So, the previous setup for the collection of lattices Γ_K, K ranging over the open compact subgroups of $G(\mathbb{A}_{\mathrm{fin}})$, is contained in the current one. A similar connection can be made for general G, where however a single subgroup K will correspond to a finite set of lattices in $G(F_\infty)$.

An important step in the analysis of the limit multiplicity problem is to reduce it to a question about the trace formula. This is non-trivial not the least because of the complicated nature of the unitary dual. This step was carried out by Delorme in the case where S consists of the archimedean places ([22]). His argument was subsequently extended by Sauvageot to the general case ([48]), where he also axiomatized the essential property as a "density principle" (see §2 below). Using the result of Sauvageot, we can recast the limit multiplicity problem as follows. Let $\mathcal{H}(G(F_S)^1)$ be the algebra of smooth, compactly supported bi-K_S-finite functions on $G(F_S)^1$. For any $h \in \mathcal{H}(G(F_S)^1)$ let $\hat{h}$ be the function on $\Pi(G(F_S)^1)$ given by $\hat{h}(\pi) = \operatorname{tr} \pi(h)$. Note that we have

$$\mu_K(\hat{h}) = \frac{1}{\operatorname{vol}(G(F)\backslash G(\mathbb{A})^1)} \operatorname{tr} R_{\mathrm{disc}}(h \otimes \mathbf{1}_K)$$

and

$$\mu_{\mathrm{pl}}(\hat{h}) = h(1).$$

Then we have the following theorem.

Theorem 1 (Sauvageot)**.** *Suppose that the collection $\mathcal{K}$ has the property that for any function $h \in \mathcal{H}(G(F_S)^1)$ we have*

$$\mu_K(\hat{h}) \to h(1), \quad K \in \mathcal{K}. \tag{1}$$

Then limit multiplicity holds for $\mathcal{K}$.

We will recall how to obtain this result from Sauvageot's density principle in §2.

Given this reduction, it is natural to attack assertion (1) via the trace formula. In the cocompact case (i.e., when $G/Z(G)$ is anisotropic over F) one can use the Selberg trace formula. In the general case we use Arthur's (non-invariant) trace formula which expresses a certain distribution $h \mapsto J(h)$ on $C_c^\infty(G(\mathbb{A})^1)$ geometrically and spectrally ([9, 3, 5, 6, 7, 8]). The distribution J depends on the choice of a maximal F-split torus T_0 of G and a suitable maximal compact subgroup $\mathbf{K} = \mathbf{K}_S\mathbf{K}^S$ of $G(\mathbb{A})$ (cf. §3 below). The main terms on the geometric side are the elliptic orbital integrals, most notably the contribution $\operatorname{vol}(G(F)\backslash G(\mathbb{A})^1)h(1)$ of the identity element. The main term on the spectral side is $\operatorname{tr} R_{\mathrm{disc}}(h)$.

The relation (1) can now be broken down into the following two statements:

(2) $$\text{For any } h \in \mathcal{H}(G(F_S)^1) \text{ we have } J(h \otimes \mathbf{1}_K) - \operatorname{tr} R_{\mathrm{disc}}(h \otimes \mathbf{1}_K) \to 0,$$

and,

(3) $$\text{for any } h \in \mathcal{H}(G(F_S)^1) \text{ we have } J(h \otimes \mathbf{1}_K) \to \operatorname{vol}(G(F)\backslash G(\mathbb{A})^1)h(1).$$

We call these relations the *spectral* and *geometric limit properties*, respectively.

The spectral limit property is trivial in the cocompact case, since then $J(h) = \operatorname{tr} R_{\mathrm{disc}}(h)$. Also, for a tower $\mathcal{K}$ of normal subgroups K of $\mathbf{K}^S$ it is easy to see that for every $h \in \mathcal{H}(G(F_S)^1)$ we have in fact $J(h \otimes \mathbf{1}_K) = \operatorname{vol}(G(F)\backslash G(\mathbb{A})^1)h(1)$ for almost all $K \in \mathcal{K}$. This is Sauvageot's proof of the limit multiplicity property in this case.

In general both properties are nontrivial. We consider only the simplest collection of normal subgroups of $\mathbf{K}^S$, namely the *principal* congruence subgroups $\mathbf{K}^S(\mathfrak{n})$ of $\mathbf{K}^S$ for non-zero ideals $\mathfrak{n}$ of $\mathfrak{o}_F$ prime to S (see §4). In this case, the geometric limit property is a consequence of Arthur's analysis of the unipotent contribution to the trace formula in [7] (see §5, in particular Corollary 1). The main task is to prove the spectral limit property for this collection of subgroups. We are able to do this unconditionally for the groups $\mathrm{GL}(n)$, and consequently obtain the following main result.

Theorem 2. *Limit multiplicity holds for $G = \mathrm{GL}(n)$ over a number field F and the collection of all principal congruence subgroups $\mathbf{K}^S(\mathfrak{n})$ of $\mathbf{K}^S$.*

The key input for our approach to the spectral limit property is the refinement of the spectral expansion of Arthur's trace formula established in [27] (cf. Theorem 4 below). This result enables us to set up an inductive argument which relies on two conjectural properties, one global and one local, which we call (TWN) (tempered winding numbers) and (BD) (bounded degree), respectively. They are stated in §4 and are expected to hold for any reductive group G over a number field. Theorem 2 is proved for any group G satisfying these properties (Theorem 7). The global property (TWN) is a uniform estimate on the winding number of the normalizing scalars of the intertwining operators in the co-rank one case. For $\mathrm{GL}(n)$ this property follows from known results about the Rankin-Selberg L-functions (Proposition 1). In order to describe the local property (BD), recall that in the non-archimedean case the matrix coefficients of the local intertwining operators are rational functions of q^{-s}, where q is the cardinality of the residue field, and that the degrees of the denominators are bounded in terms of G only. Property (BD) gives an upper bound on the degree of the numerator in terms of the level. This property was studied in [26], where among other things it was proved for the groups $G = \mathrm{GL}(n)$. The import of property (BD) is that it yields a good bound for integrals of logarithmic derivatives of normalized intertwining operators (Proposition 2). The archimedean analogue of property (BD) (for general groups) had been established in [40, Appendix].

A technical feature of our proof is that the induction over the Levi subgroups of G does not work with the spectral limit property itself. Instead, what we prove by induction in §7 is that the collection of measures $\{\mu_{\mathbf{K}(\mathfrak{n})}^{G,S_\infty}\}$ is *polynomially bounded* in the sense of Definition 3, a property that already shows up in Delorme's work ([22]). This property is analyzed in §6, where we prove Proposition 6, a result on real reductive Lie groups which generalizes a part of Delorme's argument, and is (like Delorme's work) based on the Paley-Wiener theorem of Clozel-Delorme ([16]). Once we have that the collections $\{\mu_{\mathbf{K}_M(\mathfrak{n})}^{M,S_\infty}\}$ are polynomially bounded for all proper Levi subgroups M of G, we can deduce the spectral limit property for G (Corollary 3). The key technical estimate of §7, which is based on the refined spectral expansion and the bounds on intertwining operators of §4, is Lemma 10.

Remark 1. It follows from the classification of the discrete spectrum of $\mathrm{GL}(n)$ by Mœglin-Waldspurger ([38]) that the non-cuspidal discrete spectrum consists entirely of non-tempered representations. Therefore, at least for $\mathrm{GL}(n)$ the limit multiplicity property holds for the cuspidal spectrum as well. Once again, we expect that the same is true for other groups.

We end this introduction with a few remarks on possible extensions of Theorems 2 and 7. For general sequences (Γ_n) of distinct irreducible lattices in a semisimple Lie group G, there is an obvious obstruction to the limit multiplicity property, namely the possibility that infinitely many lattices Γ_n contain a non-trivial subgroup Δ of the center of G, which forces the corresponding representations R_{Γ_n} to be Δ-invariant. By passing to the quotient G/Δ, we can assume that this is not the case. But even taking this trivial obstruction into account, the limit multiplicity property does not hold for arbitrary families. For instance, for $G = \mathrm{SL}_2(\mathbb{R})$ we can find a descending sequence of finite index normal subgroups Γ_n of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ such that for all n the multiplicity in $L^2(\Gamma_n\backslash G)$ of either one of the two lowest discrete series representations of G (or equivalently, the genus of the corresponding Riemann surface) is equal to one ([43]). Similarly, one can find a descending sequence of normal subgroups Γ_n of $\mathrm{SL}_2(\mathbb{Z})$ such that the limiting measure of the sequence (μ_{Γ_n}) has a strictly positive density on the entire complementary spectrum $\Pi(G) \setminus \Pi(G)_{\mathrm{temp}}$ ([45]). Note that in these examples the intersection of the finite index subgroups Γ_n is a non-central normal subgroup of Γ of infinite index, which accounts for the failure of the limit multiplicity property.[4] By Margulis's normal subgroup theorem such subgroups do not

[4]More precisely, in these examples the analog of the geometric limit property (3) fails. It follows from [46] (or alternatively by direct calculation) that the limit multiplicity property holds for the discrete series of $\mathrm{SL}_2(\mathbb{R})$ and arbitrary normal towers of subgroups of $\mathrm{SL}_2(\mathbb{Z})$, i.e., when the intersection of the normal subgroups is trivial.

exist for irreducible lattices Γ in semisimple Lie groups G of real rank at least two and without compact factors ([37, p. 4, Theorem 4'], cf. also [ibid., IX.6.14]). (The paper [1] is a major outgrowth of the Margulis normal subgroup theorem.) One expects that for irreducible arithmetic lattices the limit multiplicity property holds at least for any sequence of distinct *congruence* subgroups not containing non-trivial central elements. In the adelic setting, let G be a reductive group defined over a number field F such that the derived group G' of G is F-simple and simply connected. Then we expect the limit multiplicity property to be true for any collection $\mathcal{K}$ of open subgroups of a maximal compact subgroup $\mathbf{K}^S$ of $G(\mathbb{A}^S)$ for which no infinite subset has constant intersection with $G'(\mathbb{A}^S)$ or contains a non-trivial central element of $G(F)$. For this, a good understanding of the structure of these subgroups seems to be necessary to deal with both the geometric and the spectral sides. We hope to return to this problem in a future paper.

We thank the Centre Interfacultaire Bernoulli, Lausanne, and the Max Planck Institute for Mathematics, Bonn, where a part of this paper was worked out.

2. Sauvageot's density principle

In this section we recall the results of Sauvageot ([48]) and the proof of Theorem 1, providing a close link between the limit multiplicity problem and the trace formula. Recall that a subset $A \subset \Pi(G(F_S)^1)$ is *bounded*, if the Casimir eigenvalues λ_{π_∞} of the elements $\pi \in A$ are bounded and if in addition there exist a finite set $\mathcal{F} \subset \Pi(\mathbf{K}_\infty)$ and an open compact subgroup $K \subset G(F_{S-S_\infty})$ such that every $\pi \in A$ contains a $\mathbf{K}_\infty$-type in $\mathcal{F}$ and a non-trivial K-fixed vector.

The main result of [48] (Corollaire 6.2 and Théorème 7.3) is the following.[5]

Theorem 3 (Sauvageot)**.** *Let $\epsilon > 0$ be arbitrary.*

(1) *For any bounded set $A \subset \Pi(G(F_S)^1) \setminus \Pi_{\mathrm{temp}}(G(F_S)^1)$ there exists $h \in \mathcal{H}(G(F_S)^1)$ such that*
  (a) $\hat{h}(\pi) \geq 0$ *for all* $\pi \in \Pi(G(F_S)^1)$,
  (b) $\hat{h}(\pi) \geq 1$ *for all* $\pi \in A$,
  (c) $h(1) < \epsilon$.

(2) *For any Riemann-integrable function f on $\Pi_{\mathrm{temp}}(G(F_S)^1)$ there exist $h_1, h_2 \in \mathcal{H}(G(F_S)^1)$ such that*
  (a) $|f(\pi) - \hat{h}_1(\pi)| \leq \hat{h}_2(\pi)$ *for all* $\pi \in \Pi(G(F_S)^1)$,
  (b) $h_2(1) < \epsilon$.

As in [48], this result easily implies Theorem 1. We recall the argument. Let $A \subset \Pi(G(F_S)^1) \setminus \Pi_{\mathrm{temp}}(G(F_S)^1)$ be a bounded set. For any $\epsilon > 0$ let $h \in \mathcal{H}(G(F_S)^1)$ be as in the first part of Theorem 3. By assumption we have $|\mu_K(\hat{h}) - h(1)| < \epsilon$ for all but finitely many $K \in \mathcal{K}$. For all such K we have

$$\mu_K(A) \leq \mu_K(\hat{h}) \leq |\mu_K(\hat{h}) - h(1)| + h(1) < 2\epsilon.$$

[5] See the appendix of [50] for important corrections.

Similarly, let f be a Riemann-integrable function on $\Pi_{\mathrm{temp}}(G(F_S)^1)$. For any $\epsilon > 0$ let h_1 and h_2 be as in the second part of Theorem 3. By assumption, for all but finitely many $K \in \mathcal{K}$ we have $|\mu_K(\hat{h}_i) - h_i(1)| < \epsilon$, $i = 1, 2$. Then,

$$\begin{aligned} |\mu_K(f) - \mu_{\mathrm{pl}}(f)| &\le |\mu_K(f) - \mu_K(\hat{h}_1)| + |\mu_K(\hat{h}_1) - h_1(1)| + |h_1(1) - \mu_{\mathrm{pl}}(f)| \\ &\le |\mu_K(\hat{h}_1) - h_1(1)| + \mu_K(\hat{h}_2) + h_2(1) \\ &\le |\mu_K(\hat{h}_1) - h_1(1)| + |\mu_K(\hat{h}_2) - h_2(1)| + 2h_2(1) < 4\epsilon. \end{aligned}$$

Theorem 1 follows.

3. Review of the trace formula

In this section we recall Arthur's trace formula, and in particular the refinement of the spectral expansion obtained in [27].

3.1. **Notation.** We will mostly use the notation of [27]. As before, G is a reductive group defined over a number field F and $\mathbb{A}$ is the ring of adeles of F. For a finite place v of F let q_v be the cardinality of the residue field of v. As above, we fix a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v = \mathbf{K}_\infty \mathbf{K}_{\mathrm{fin}}$ of $G(\mathbb{A}) = G(F_\infty)G(\mathbb{A}_{\mathrm{fin}})$.

Let θ be the Cartan involution of $G(F_\infty)$ defining $\mathbf{K}_\infty$. It induces a Cartan decomposition $\mathfrak{g} = \mathrm{Lie}\, G(F_\infty) = \mathfrak{p} \oplus \mathfrak{k}$ with $\mathfrak{k} = \mathrm{Lie}\, \mathbf{K}_\infty$. We fix an invariant bilinear form B on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. This choice defines a Casimir operator Ω on $G(F_\infty)$, and we denote the Casimir eigenvalue of any $\pi \in \Pi(G(F_\infty))$ by λ_π. Similarly, we obtain a Casimir operator $\Omega_{\mathbf{K}_\infty}$ on $\mathbf{K}_\infty$ and write λ_τ for the Casimir eigenvalue of a representation $\tau \in \Pi(\mathbf{K}_\infty)$ (cf. [11, §2.3]). The form B induces a Euclidean scalar product $(X, Y) = -B(X, \theta(Y))$ on $\mathfrak{g}$ and all its subspaces. For $\tau \in \Pi(\mathbf{K}_\infty)$ we define $\|\tau\|$ as in [14, §2.2] (cf. also §6 below).

We fix a maximal F-split torus T_0 of G and let M_0 be its centralizer, which is a minimal Levi subgroup defined over F. We assume that the maximal compact subgroup $\mathbf{K} \subset G(\mathbb{A})$ is admissible with respect to M_0 ([4, §1]). Denote by A_0 the identity component of $T_0(\mathbb{R})$, which is viewed as a subgroup of $T_0(\mathbb{A})$ via the diagonal embedding of $\mathbb{R}$ into F_∞.

We write $\mathcal{L}$ for the (finite) set of Levi subgroups containing M_0, i.e., the set of centralizers of subtori of T_0. Let $W_0 = N_{G(F)}(T_0)/M_0$ be the Weyl group of (G, T_0), where $N_{G(F)}(H)$ is the normalizer of H in $G(F)$. For any $s \in W_0$ we choose a representative $w_s \in G(F)$. Note that W_0 acts on $\mathcal{L}$ by $sM = w_s M w_s^{-1}$.

Let now $M \in \mathcal{L}$. We write T_M for the split part of the identity component of the center of M. Set $A_M = A_0 \cap T_M(\mathbb{R})$ and $W(M) = N_{G(F)}(M)/M$, which can be identified with a subgroup of W_0. Denote by $\mathfrak{a}_M^*$ the $\mathbb{R}$-vector space spanned by the lattice $X^*(M)$ of F-rational characters of M and let $\mathfrak{a}_{M,\mathbb{C}}^* = \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. We write $\mathfrak{a}_M$ for the dual space of $\mathfrak{a}_M^*$, which is spanned by the co-characters of T_M. Let $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ be the homomorphism given by

$$e^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_{\mathbb{A}} = \prod_v |\chi(m_v)|_v$$

for any $\chi \in X^*(M)$ and denote by $M(\mathbb{A})^1 \subset M(\mathbb{A})$ the kernel of H_M. Let $\mathcal{L}(M)$ be the set of Levi subgroups containing M and $\mathcal{P}(M)$ the set of parabolic subgroups of G with Levi part M. We also write $\mathcal{F}(M) = \mathcal{F}^G(M) = \coprod_{L\in\mathcal{L}(M)} \mathcal{P}(L)$ for the (finite) set of parabolic subgroups of G containing M. Note that $W(M)$ acts on $\mathcal{P}(M)$ and $\mathcal{F}(M)$ by $sP = w_s P w_s^{-1}$. Denote by Σ_M the set of reduced roots of T_M on the Lie algebra of G. For any $\alpha \in \Sigma_M$ we denote by $\alpha^\vee \in \mathfrak{a}_M$ the corresponding co-root. Let $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ be the discrete part of $L^2(A_M M(F)\backslash M(\mathbb{A}))$, i.e., the closure of the sum of all irreducible subrepresentations of the regular representation of $M(\mathbb{A})$. We denote by $\Pi_{\mathrm{disc}}(M(\mathbb{A}))$ the countable set of equivalence classes of irreducible unitary representations of $M(\mathbb{A})$ which occur in the decomposition of $L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))$ into irreducibles.

For any $L \in \mathcal{L}(M)$ we identify $\mathfrak{a}_L^*$ with a subspace of $\mathfrak{a}_M^*$. We denote by $\mathfrak{a}_M^L$ the annihilator of $\mathfrak{a}_L^*$ in $\mathfrak{a}_M$. We set

$$\mathcal{L}_1(M) = \{L \in \mathcal{L}(M) : \dim \mathfrak{a}_M^L = 1\}$$

and

$$\mathcal{F}_1(M) = \bigcup_{L\in\mathcal{L}_1(M)} \mathcal{P}(L).$$

Note that the restriction of the scalar product $(\cdot,\cdot)$ on $\mathfrak{g}$ defined above gives $\mathfrak{a}_{M_0}$ the structure of a Euclidean space. In particular, this fixes Haar measures on the spaces $\mathfrak{a}_M^L$ and their duals $(\mathfrak{a}_M^L)^*$. We follow Arthur in the corresponding normalization of Haar measures on the groups $M(\mathbb{A})$ ([9, §1]).

3.2. **Intertwining operators.** Now let $P \in \mathcal{P}(M)$. We write $\mathfrak{a}_P = \mathfrak{a}_M$. Let U_P be the unipotent radical of P and M_P the unique $L \in \mathcal{L}(M)$ (in fact the unique $L \in \mathcal{L}(M_0)$) such that $P \in \mathcal{P}(L)$. Denote by $\Sigma_P \subset \mathfrak{a}_P^*$ the set of reduced roots of T_M on the Lie algebra $\mathfrak{u}_P$ of U_P. Let Δ_P be the subset of simple roots of P, which is a basis for $(\mathfrak{a}_P^G)^*$. Write $\mathfrak{a}_{P,+}^*$ for the closure of the Weyl chamber of P, i.e.

$$\mathfrak{a}_{P,+}^* = \{\lambda \in \mathfrak{a}_M^* : \langle\lambda, \alpha^\vee\rangle \geq 0 \text{ for all } \alpha \in \Sigma_P\} = \{\lambda \in \mathfrak{a}_M^* : \langle\lambda, \alpha^\vee\rangle \geq 0 \text{ for all } \alpha \in \Delta_P\}.$$

Denote by δ_P the modulus function of $P(\mathbb{A})$. Let $\bar{\mathcal{A}}_2(P)$ be the Hilbert space completion of

$$\{\phi \in C^\infty(M(F)U_P(\mathbb{A})\backslash G(\mathbb{A})) : \delta_P^{-\frac{1}{2}}\phi(\cdot x) \in L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))\ \forall x \in G(\mathbb{A})\}$$

with respect to the inner product

$$(\phi_1, \phi_2) = \int_{A_M M(F)U_P(\mathbb{A})\backslash G(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}\, dg.$$

Let $\alpha \in \Sigma_M$. We say that two parabolic subgroups $P, Q \in \mathcal{P}(M)$ are *adjacent* along α, and write $P|^\alpha Q$, if $\Sigma_P \cap -\Sigma_Q = \{\alpha\}$. Alternatively, P and Q are adjacent if the closure $\overline{PQ}$ of PQ belongs to $\mathcal{F}_1(M)$. Any $R \in \mathcal{F}_1(M)$ is of the form $\overline{PQ}$ for a unique unordered pair $\{P, Q\}$ of parabolic subgroups in $\mathcal{P}(M)$, namely P and Q are the maximal parabolic subgroups of R, and $P|^\alpha Q$ with $\alpha^\vee \in \Sigma_P^\vee \cap \mathfrak{a}_M^R$. Switching the order of P and Q changes α to $-\alpha$.

For any $P \in \mathcal{P}(M)$ let $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ be the extension of H_M to a left $U_P(\mathbb{A})$- and right $\mathbf{K}$-invariant map. Denote by $\mathcal{A}^2(P)$ the dense subspace of $\bar{\mathcal{A}}^2(P)$ consisting of its $\mathbf{K}$- and $\mathfrak{z}$-finite vectors, where $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}$. That is, $\mathcal{A}^2(P)$ is the space of automorphic forms ϕ on $U_P(\mathbb{A})M(F)\backslash G(\mathbb{A})$ such that $\delta_P^{-\frac{1}{2}}\phi(\cdot k)$ is a square-integrable automorphic form on $A_M M(F)\backslash M(\mathbb{A})$ for all $k \in \mathbf{K}$. Let $\rho(P,\lambda)$, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, be the induced representation of $G(\mathbb{A})$ on $\bar{\mathcal{A}}^2(P)$ given by

$$(\rho(P,\lambda,y)\phi)(x) = \phi(xy)e^{\langle \lambda, H_P(xy)-H_P(x)\rangle}.$$

It is isomorphic to $\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\left(L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A})) \otimes e^{\langle \lambda, H_M(\cdot)\rangle}\right)$.

For $P, Q \in \mathcal{P}(M)$ let

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

be the standard *intertwining operator* ([6, §1]), which is the meromorphic continuation in λ of the integral

$$[M_{Q|P}(\lambda)\phi](x) = \int_{U_Q(\mathbb{A})\cap U_P(\mathbb{A})\backslash U_Q(\mathbb{A})} \phi(nx)e^{\left\langle \lambda, H_P(nx)-H_Q(x)\right\rangle}\, dn, \quad \phi \in \mathcal{A}^2(P),\ x \in G(\mathbb{A}).$$

These operators satisfy the following properties.

(1) $M_{P|P}(\lambda) \equiv \mathrm{Id}$ for all $P \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$.
(2) For any $P, Q, R \in \mathcal{P}(M)$ we have $M_{R|P}(\lambda) = M_{R|Q}(\lambda) \circ M_{Q|P}(\lambda)$ for all $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. In particular, $M_{Q|P}(\lambda)^{-1} = M_{P|Q}(\lambda)$.
(3) $M_{Q|P}(\lambda)^* = M_{P|Q}(-\overline{\lambda})$ for any $P, Q \in \mathcal{P}(M)$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. In particular, $M_{Q|P}(\lambda)$ is unitary for $\lambda \in \mathrm{i}\mathfrak{a}^*_M$.
(4) If $P|^{\alpha}Q$ then $M_{Q|P}(\lambda)$ depends only on $\langle \lambda, \alpha^\vee \rangle$.

For any $P \in \mathcal{P}(M)$ we have a canonical isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_{\mathbb{C}}, K_\infty)$-modules

$$j_P : \operatorname{Hom}(\pi, L^2(A_M M(F)\backslash M(\mathbb{A}))) \otimes \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi) \to \mathcal{A}^2_\pi(P).$$

This gives rise to a Hilbert space structure on $\operatorname{Hom}(\pi, L^2(A_M M(F)\backslash M(\mathbb{A})))$ such that j_P becomes an isometry.

Suppose that $P|^{\alpha}Q$. The operator $M_{Q|P}(\pi, s)$ admits a normalization by a global factor $n_\alpha(\pi, s)$ which is a meromorphic function in s. We write

$$M_{Q|P}(\pi, s) \circ j_P = n_\alpha(\pi, s) \cdot j_Q \circ (\mathrm{Id} \otimes R_{Q|P}(\pi, s)) \tag{4}$$

where $R_{Q|P}(\pi, s) = \otimes_v R_{Q|P}(\pi_v, s)$ is the product of the locally defined normalized intertwining operators and $\pi = \otimes_v \pi_v$ ([6, §6], cf. [39, (2.17)]).

3.3. **The trace formula.** Arthur's trace formula gives two alternative expressions for a distribution J on $G(\mathbb{A})^1$. Note that this distribution depends on the choice of M_0 and $\mathbf{K}$. For $h \in C_c^\infty(G(\mathbb{A})^1)$, Arthur defines $J(h)$ as the value at the point $T = T_0$ specified in [4, Lemma 1.1] of a polynomial $J^T(h)$ on $\mathfrak{a}_{M_0}$ of degree at most $d_0 = \dim \mathfrak{a}^G_{M_0}$. Here, the polynomial $J^T(h)$ depends in addition on the choice of a parabolic subgroup $P_0 \in \mathcal{P}(M_0)$. Consider the equivalence relation on $G(F)$ defined by $\gamma \sim \gamma'$ whenever the semisimple parts of γ and γ' are $G(F)$-conjugate. Let $\mathcal{O}$ be the set of the resulting equivalence classes (which are in bijection with conjugacy classes of semisimple elements). The coarse geometric expansion ([9]) is

$$J^T(h) = \sum_{\mathfrak{o}\in\mathcal{O}} J^T_{\mathfrak{o}}(h), \tag{5}$$

where the summands $J^T_{\mathfrak{o}}(h)$ are again polynomials in T of degree at most d_0. Write $J_{\mathfrak{o}}(h) = J^{T_0}_{\mathfrak{o}}(h)$, which depends only on M_0 and $\mathbf{K}$. Then $J_{\mathfrak{o}}(h) = 0$ if the support of h is disjoint from all conjugacy classes of $G(\mathbb{A})$ intersecting $\mathfrak{o}$ (cf. [8, Theorem 8.1]). By [ibid., Lemma 9.1] (together with the descent formula of [4, §2]), for each compact set $\Omega \subset G(\mathbb{A})^1$ there exists a finite subset $\mathcal{O}(\Omega) \subset \mathcal{O}$ such that for h supported in Ω only the terms with $\mathfrak{o} \in \mathcal{O}(\Omega)$ contribute to (5). In particular, the sum is always finite. When $\mathfrak{o}$ consists of the unipotent elements of $G(F)$, we write $J^T_{\mathrm{unip}}(h)$ for $J^T_{\mathfrak{o}}(h)$.

We now turn to the spectral side. Let $L \supset M$ be Levi subgroups in $\mathcal{L}$, $P \in \mathcal{P}(M)$, and let $m = \dim \mathfrak{a}_L^G$ be the co-rank of L in G. Denote by $\mathfrak{B}_{P,L}$ the set of m-tuples $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee)$ of elements of $\Sigma_P^\vee$ whose projections to $\mathfrak{a}_L$ form a basis for $\mathfrak{a}_L^G$. For any $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee) \in \mathfrak{B}_{P,L}$ let $\mathrm{vol}(\underline{\beta})$ be the co-volume in $\mathfrak{a}_L^G$ of the lattice spanned by $\underline{\beta}$ and let

$$\begin{aligned}\Xi_L(\underline{\beta}) &= \{(Q_1, \dots, Q_m) \in \mathcal{F}_1(M)^m : \quad \beta_i^\vee \in \mathfrak{a}_M^{Q_i},\ i = 1, \dots, m\} \\ &= \{(\overline{P_1 P_1'}, \dots, \overline{P_m P_m'}) : \quad P_i|^{\beta_i} P_i',\ i = 1, \dots, m\}.\end{aligned}$$

For any smooth function f on $\mathfrak{a}_M^*$ and $\mu \in \mathfrak{a}_M^*$ denote by $D_\mu f$ the directional derivative of f along $\mu \in \mathfrak{a}_M^*$. For a pair $P_1|^\alpha P_2$ of adjacent parabolic subgroups in $\mathcal{P}(M)$ write

$$\delta_{P_1|P_2}(\lambda) = M_{P_2|P_1}(\lambda) D_\varpi M_{P_1|P_2}(\lambda) : \mathcal{A}^2(P_2) \to \mathcal{A}^2(P_2),$$

where $\varpi \in \mathfrak{a}_M^*$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$.[6] Equivalently, writing $M_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)$ for a meromorphic function Φ of a single complex variable, we have

$$\delta_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)^{-1} \Phi'(\langle \lambda, \alpha^\vee \rangle).$$

For any m-tuple $\mathcal{X} = (Q_1, \dots, Q_m) \in \Xi_L(\underline{\beta})$ with $Q_i = \overline{P_i P_i'}$, $P_i|^{\beta_i} P_i'$, denote by $\Delta_{\mathcal{X}}(P, \lambda)$ the expression

$$\frac{\mathrm{vol}(\underline{\beta})}{m!} M_{P_1'|P}(\lambda)^{-1} \delta_{P_1|P_1'}(\lambda) M_{P_1'|P_2'}(\lambda) \cdots \delta_{P_{m-1}|P_{m-1}'}(\lambda) M_{P_{m-1}'|P_m'}(\lambda) \delta_{P_m|P_m'}(\lambda) M_{P_m'|P}(\lambda).$$

In [27, pp. 179-180] we define a (purely combinatorial) map $\mathcal{X}_L : \mathfrak{B}_{P,L} \to \mathcal{F}_1(M)^m$ with the property that $\mathcal{X}_L(\underline{\beta}) \in \Xi_L(\underline{\beta})$ for all $\underline{\beta} \in \mathfrak{B}_{P,L}$.[7]

For any $s \in W(M)$ let L_s be the smallest Levi subgroup in $\mathcal{L}(M)$ containing w_s. We recall that $\mathfrak{a}_{L_s} = \{H \in \mathfrak{a}_M \mid sH = H\}$. Set

$$\iota_s = |\det(s-1)_{\mathfrak{a}_M^{L_s}}|^{-1}.$$

[6]Note that this definition differs slightly from the definition of $\delta_{P_1|P_2}$ in [27].

[7]The map $\mathcal{X}_L$ depends in fact on the additional choice of a vector $\underline{\mu} \in (\mathfrak{a}_M^*)^m$ which does not lie in an explicit finite set of hyperplanes. For our purposes, the precise definition of $\mathcal{X}_L$ is immaterial.

For $P \in \mathcal{F}(M_0)$ and $s \in W(M_P)$ let $M(P,s) : \mathcal{A}^2(P) \to \mathcal{A}^2(P)$ be as in [6, p. 1309]. $M(P,s)$ is a unitary operator which commutes with the operators $\rho(P,\lambda,h)$ for $\lambda \in \mathrm{i}\mathfrak{a}_{L_s}^*$. Finally, we can state the refined spectral expansion.

Theorem 4 ([27]). *For any $h \in C_c^\infty(G(\mathbb{A})^1)$ the spectral side of Arthur's trace formula is given by*

$$J(h) = \sum_{[M]} J_{\mathrm{spec},M}(h),$$

M ranging over the conjugacy classes of Levi subgroups of G (represented by members of $\mathcal{L}$), where

$$J_{\mathrm{spec},M}(h) = \frac{1}{|W(M)|} \sum_{s \in W(M)} \iota_s \sum_{\underline{\beta} \in \mathfrak{B}_{P,L_s}} \int_{\mathrm{i}(\mathfrak{a}_{L_s}^G)^*} \mathrm{tr}(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\lambda) M(P,s) \rho(P,\lambda,h)) \; d\lambda$$

with $P \in \mathcal{P}(M)$ arbitrary. The operators are of trace class and the integrals are absolutely convergent.

Note that here the term corresponding to $M = G$ is simply $J_{\mathrm{spec},G}(h) = \mathrm{tr}\, R_{\mathrm{disc}}(h)$.

4. Bounds on co-rank one intertwining operators

We now introduce the key global and local properties required for the proof of the spectral limit property, and verify that they are satisfied for the groups $\mathrm{GL}(n)$. These properties will be used to provide estimates for the contribution of the continuous spectrum to the spectral side of the trace formula.

We will use the notation $A \ll B$ to mean that there exists a constant c (independent of the parameters under consideration) such that $A \le cB$. If c depends on some parameters (say F) and not on others then we will write $A \ll_F B$.

Fix a faithful F-rational representation $\rho : G \to \mathrm{GL}(V)$ and an $\mathfrak{o}_F$-lattice Λ in the representation space V such that the stabilizer of $\hat{\Lambda} = \hat{\mathfrak{o}}_F \otimes \Lambda \subset \mathbb{A}_{\mathrm{fin}} \otimes V$ in $G(\mathbb{A}_{\mathrm{fin}})$ is the group $\mathbf{K}_{\mathrm{fin}}$. (Since the maximal compact subgroups of $\mathrm{GL}(\mathbb{A}_{\mathrm{fin}} \otimes V)$ are precisely the stabilizers of lattices, it is easy to see that such a lattice exists.) For any non-zero ideal $\mathfrak{n}$ of $\mathfrak{o}_F$ let

$$\mathbf{K}(\mathfrak{n}) = \{g \in G(\mathbb{A}_{\mathrm{fin}}) \,:\, \rho(g)v \equiv v \pmod{\mathfrak{n}\hat{\Lambda}}, \quad v \in \hat{\Lambda}\}$$

be the principal congruence subgroup of level $\mathfrak{n}$, an open normal subgroup of $\mathbf{K}_{\mathrm{fin}}$, and for $\mathfrak{n}$ prime to S let $\mathbf{K}^S(\mathfrak{n})$ be the corresponding open normal subgroup of $\mathbf{K}^S$. The groups $\mathbf{K}(\mathfrak{n})$ form a neighborhood base of the identity element in $G(\mathbb{A}_{\mathrm{fin}})$. For an open subgroup K of $\mathbf{K}_{\mathrm{fin}}$ let $\mathfrak{n}_K$ be the largest ideal of $\mathfrak{o}_F$ with $\mathbf{K}(\mathfrak{n}_K) \subset K$, and define the *level* of K as $\mathrm{level}(K) = \mathrm{N}(\mathfrak{n}_K)$, where $\mathrm{N}(\mathfrak{n}) = [\mathfrak{o}_F : \mathfrak{n}]$ denotes the ideal norm of $\mathfrak{n}$. Analogously, define $\mathrm{level}(K_v)$ for open subgroups $K_v \subset \mathbf{K}_v$.

4.1. **The global bound.** As in [39], for any $\pi \in \Pi(M(F_\infty))$ we define $\Lambda_\pi = \sqrt{\lambda_\pi^2 + \lambda_\tau^2}$, where τ is a lowest $\mathbf{K}_\infty$-type of $\mathrm{Ind}_P^G(\pi_\infty)$. (This is well-defined, because λ_τ is independent of τ.) Roughly speaking, Λ_π measures the size of π.

Definition 1. We say that the group G satisfies the property (TWN) (tempered winding number) if for any $M \in \mathcal{L}$, $M \neq G$, and any finite subset $\mathcal{F} \subset \Pi(\mathbf{K}_{M,\infty})$ there exists an integer $k > 1$ such that for any $\alpha \in \Sigma_M$ and any $\epsilon > 0$ we have

$$\int_{\mathrm{i}\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-k} \; ds \ll_{\mathcal{F},\epsilon} (1 + \Lambda_{\pi_\infty})^k \operatorname{level}(K_M)^\epsilon$$

for all open compact subgroups K_M of $\mathbf{K}_{M,\mathrm{fin}}$ and $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{\mathcal{F},K_M}$.

Since the normalizing factors $n_\alpha(\pi, s)$ arise from co-rank one situations, the property (TWN) is hereditary for Levi subgroups.

Remark 2. If we fix an open compact subgroup K_M, then the corresponding bound

$$\int_{\mathrm{i}\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-k} \; ds \ll_{K_M} (1 + \Lambda_{\pi_\infty})^k$$

is the content of [39, Theorem 5.3]. So, the point of (TWN) lies in the dependence of the bound on K_M.

Remark 3. In fact, we expect that

$$\int_T^{T+1} \left| \frac{n'_\alpha(\pi, \mathrm{i}t)}{n_\alpha(\pi, \mathrm{i}t)} \right| \; dt \ll 1 + \log(1 + T) + \log(1 + \Lambda_{\pi_\infty}) + \log \operatorname{level}(K_M) \tag{6}$$

for all $T \in \mathbb{R}$ and $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{K_M}$. This would give the following strengthening of (TWN):

$$\int_{\mathrm{i}\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-2} \; ds \ll 1 + \log(1 + \Lambda_{\pi_\infty}) + \log \operatorname{level}(K_M)$$

for any $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{K_M}$.

Remark 4. If G' is simply connected, then by [36, Lemma 1.6] (cf. also [26, Proposition 1]) we can replace $\operatorname{level}(K_M)$ by $\operatorname{vol}(K_M)^{-1}$ in the definition of (TWN) (as well as in (6)).

Proposition 1. *The estimate* (6) *holds for* $G = \mathrm{GL}(n)$ *with an implied constant depending only on* n *and* F. *In particular,* $\mathrm{GL}(n)$ *satisfies the property (TWN).*

Proof. The proposition follows from the fact that for $\mathrm{GL}(n)$ the global normalizing factors n_α can be expressed in terms of Rankin-Selberg L-functions and the known properties of these functions, which are collected and analyzed in [42, §§4,5]. Write $M \simeq \prod_{i=1}^r \mathrm{GL}(n_i)$, where the root α is trivial on $\prod_{i \geq 3} \mathrm{GL}(n_i)$, and let $\pi \simeq \otimes \pi_i$ with representations $\pi_i \in \Pi_{\mathrm{disc}}(\mathrm{GL}(n_i, \mathbb{A}))$. Let $L(s, \pi_1 \times \tilde{\pi}_2)$ be the completed Rankin-Selberg L-function associated to π_1 and π_2. It satisfies the functional equation

$$L(s, \pi_1 \times \tilde{\pi}_2) = \epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2) N(\pi_1 \times \tilde{\pi}_2)^{\frac{1}{2} - s} L(1 - s, \tilde{\pi}_1 \times \pi_2)$$

where $|\epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2)| = 1$ and $N(\pi_1 \times \tilde{\pi}_2) \in \mathbb{N}$ is the conductor (including the discriminant factor). We can then write

$$n_\alpha(\pi, s) = \frac{L(s, \pi_1 \times \tilde{\pi}_2)}{\epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2) N(\pi_1 \times \tilde{\pi}_2)^{\frac{1}{2} - s} L(s + 1, \pi_1 \times \tilde{\pi}_2)}.$$

By the proofs of [42, Proposition 4.5, 5.1], we have

$$\int_T^{T+1} \left| \frac{n'_\alpha(\pi, \mathrm{i}t)}{n_\alpha(\pi, \mathrm{i}t)} \right| \, dt \ll \log(T + \nu(\pi_1 \times \tilde{\pi}_2))$$

with

$$\nu(\pi_1 \times \tilde{\pi}_2) = N(\pi_1 \times \tilde{\pi}_2)(2 + c(\pi_1 \times \tilde{\pi}_2))$$

and $c(\pi_1 \times \tilde{\pi}_2)$ as in [ibid., (4.21)]. The discussion of [ibid.] shows that $\log(2+c(\pi_1 \times \tilde{\pi}_2)) \ll \log(1 + \Lambda_{\pi_\infty})$. Also, by [13] we have $\log N(\pi_1 \times \tilde{\pi}_2) \ll \log N(\pi_1) + \log N(\pi_2)$, where $N(\pi_i)$ is the usual conductor of the representation π_i. From the well known description of $N(\pi_i)$ ([30]) it follows that $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{K_M}$ implies $\log N(\pi_i) \ll \log \operatorname{level}(K_M)$. This completes the proof. $\square$

Remark 5. For general groups G the normalizing factors are given, at least up to local factors, by quotients of automorphic L-functions associated to the irreducible constituents of the adjoint action of the L-group ${}^L M$ of M on the unipotent radical of the corresponding parabolic subgroup of ${}^L G$ ([35]). To argue as above, we would need to know that these L-functions have finitely many poles and satisfy a functional equation with the associated conductor bounded by an arbitrary power of $\operatorname{level}(K_M)$ for automorphic representations $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{K_M}$. Unfortunately, finiteness of poles and the expected functional equation are not known in general. It is possible that for classical groups these properties are within reach. However, this may require some work.

4.2. **The contribution of the normalized local intertwining operators.**

Definition 2. We say that G satisfies the property (BD) (bounded degree) if there exists a constant c (depending only on G), such that for any $M \in \mathcal{L}$, $M \neq G$ and adjacent parabolic subgroups P, $Q \in \mathcal{P}(M)$, any finite place v of F, any open subgroup $K_v \subset \mathbf{K}_v$ and any smooth irreducible representation π_v of $M(F_v)$, the degrees of the numerators of the linear operators $R_{Q|P}(\pi_v, s)^{K_v}$ are bounded by $c \log_{q_v} \operatorname{level}(K_v)$ if $\mathbf{K}_v$ is hyperspecial, and by $c(\log_{q_v} \operatorname{level}(K_v) + 1)$ otherwise.

Remark 6. As in Remark 4, if G' is simply connected then we may replace $\operatorname{level}(K_v)$ by $\operatorname{vol}(K_v)^{-1}$.

Property (BD) is discussed in detail in [26] (see especially [ibid., Proposition 3]). It is hereditary for Levi subgroups. The main result of [26] (Theorem 1, taken together with Proposition 3) is the following.

Theorem 5. $\mathrm{GL}(n)$ *satisfies (BD).*

The relevance of (BD) to the trace formula is the following consequence, which we will prove in the remainder of this section.

Proposition 2. *Suppose that G satisfies (BD). Let $M \in \mathcal{L}$ and P, $Q \in \mathcal{P}(M)$ be adjacent parabolic subgroups. Then for all open subgroups $K \subset \mathbf{K}_{\mathrm{fin}}$ and all $\tau \in \Pi(\mathbf{K}_\infty)$ we have*

$$\int_{\mathrm{i}\mathbb{R}} \left\| R_{Q|P}(\pi, s)^{-1} R'_{Q|P}(\pi, s)\big|_{I_P^G(\pi)^{\tau,K}} \right\| (1 + |s|^2)^{-1} \, ds \ll 1 + \log(1 + \|\tau\|) + \log \operatorname{level}(K). \tag{7}$$

We remark that the dependence of the bound on τ is not essential for the limit multiplicity problem, but it is relevant for other asymptotic problems.

Let $z_1, \dots, z_m \in \mathbb{C}$ (not necessarily distinct) and let $b(z) = (z - z_1) \dots (z - z_m)$. Let V be any vector space over $\mathbb{C}$. By definition, a vector valued rational function on $\mathbb{C}$ of degree $\le m$ with denominator dividing $b(z)$ is a map $A : \mathbb{C} \setminus \{z_1, \dots, z_m\} \to V$ of the form

$$A(z) = \frac{1}{b(z)} \sum_{i=0}^{m} z^i v_i$$

for some vectors $v_0, \dots, v_m \in V$. We omit the reference to $z_1, \dots, z_m$ and b, if the precise choice of these parameters is unimportant.

For the next two lemmas, let V be a Banach space and V^* its dual space. The following lemma is a vector valued version of [27, Lemma 1], which we obtain as a consequence of a result of Borwein and Erdélyi in approximation theory ([12]).

Lemma 1. *Let $\mathbf{S^1}$ be the unit circle in $\mathbb{C}$ with the standard Lebesgue measure $|dz|$. Suppose that $A : \mathbb{C} \setminus \{z_1, \dots, z_m\} \to V$, $z_1, \dots, z_m \notin \mathbf{S^1}$, is a rational map of degree $\le m$ such that $\|A(z)\| \le 1$ for all $z \in \mathbf{S^1}$. Then*

$$\int_{\mathbf{S^1}} \|A'(z)\| \; |dz| \le 2\pi m.$$

Proof. Suppose that $b(z) = (z - z_1) \dots (z - z_m)$ is such that $b(z)A(z)$ is polynomial of degree $\le m$. For any $w \in \mathbb{C} \setminus \mathbf{S^1}$ let ϕ_w be the Möbius transformation $\phi_w(z) = \frac{1-\bar{w}z}{z-w}$ so that $|\phi_w(z)| = 1$ on $\mathbf{S^1}$ and

$$\int_{\mathbf{S^1}} |\phi_w'(z)| \; |dz| = 2\pi. \tag{8}$$

Let $\phi_>$, $\phi_<$ be the two Blaschke products

$$\phi_{\gtrless}(z) = \prod_{j : |z_j| \gtrless 1} \phi_{z_j}(z).$$

By assumption, for any unit vector $w \in V^*$ the function $f(z) = (A(z), w)$ satisfies $|f(z)| \le 1$ on $\mathbf{S^1}$ and $b(z)f(z)$ is a polynomial of degree $\le m$. Therefore, by [12, Theorem 1] we deduce that

$$|f'(z)| \le \max(|\phi_>'(z)|, |\phi_<'(z)|), \quad z \in \mathbf{S^1}.$$

Thus,

$$\|A'(z)\| \le \max(|\phi_>'(z)|, |\phi_<'(z)|) \le |\phi_>'(z)| + |\phi_<'(z)|, \quad z \in \mathbf{S^1}.$$

Integrating this inequality over $\mathbf{S^1}$ and using (8) we obtain the lemma. □

Analogously, we have

Lemma 2. *Suppose that $A : \mathbb{C} \setminus \{z_1, \dots, z_m\} \to V$ is a rational map with denominator dividing $b(z) = (z - z_1) \dots (z - z_m)$ and satisfying $\|A(z)\| \le 1$ for all $z \in \mathrm{i}\mathbb{R}$. Write*

$z_j = u_j + \mathrm{i}v_j$, $j = 1, \dots, m$. *Then*

$$\int_{\mathrm{i}\mathbb{R}} \|A'(z)\| \frac{|dz|}{1+|z|^2} \le 2\pi \sum_{j=1}^m \frac{|u_j|+1}{(|u_j|+1)^2+v_j^2} \le 2\pi m.$$

Proof. The proof is similar. For any $w \in \mathbb{C}$ let $\phi_w(z) = \frac{z+\bar{w}}{z-w}$. Applying [12, Theorem 4][8] we conclude as before that

$$\|A'(z)\| \le \max(|\phi'_>(z)|, |\phi'_<(z)|) \le |\phi'_>(z)| + |\phi'_<(z)|, \quad z \in \mathrm{i}\mathbb{R},$$

where now

$$\phi_{\gtrless}(z) = \prod_{j:\operatorname{Re} z_j \gtrless 0} \phi_{z_j}(z).$$

It remains to observe that for any $w = u + \mathrm{i}v \in \mathbb{C} \setminus \mathrm{i}\mathbb{R}$ we have

$$\int_{\mathrm{i}\mathbb{R}} |\phi'_w(z)| \frac{|dz|}{1+|z|^2} = 2\pi \frac{|u|+1}{(|u|+1)^2+v^2}.$$

Indeed, we have $|\phi'_w(z)| = \frac{2|u|}{|z-w|^2} = \frac{2|u|}{u^2+(t-v)^2}$ for $z = \mathrm{i}t$, $t \in \mathbb{R}$, so that

$$\int_{\mathrm{i}\mathbb{R}} |\phi'_w(z)| \frac{|dz|}{1+|z|^2} = \int_{\mathbb{R}} \frac{2|u|}{(u^2+(z-v)^2)(1+z^2)} \, dz.$$

By the residue theorem this is equal to

$$2\pi \left(\frac{|u|}{u^2+(\mathrm{i}-v)^2} + \frac{1}{1+(v+\mathrm{i}|u|)^2} \right) = \frac{2\pi}{v+\mathrm{i}(|u|-1)} \left(\frac{|u|}{v-\mathrm{i}(|u|+1)} + \frac{1}{v+\mathrm{i}(|u|+1)} \right)$$
$$= \frac{2\pi(|u|+1)}{v^2+(|u|+1)^2}$$

as claimed. □

Proof of Proposition 2. For each finite place v of F let $\mathfrak{p}_v$ be the maximal ideal of the ring of integers of F_v and let $f_v \ge 0$ be the minimum integer with $\mathbf{K}_v(\mathfrak{p}_v^{f_v}) \subset K$. Then $\operatorname{level}(K) = \prod_v N_v$ with $N_v = q_v^{f_v}$. Without loss of generality we may assume that $K = \prod_v \mathbf{K}_v(\mathfrak{p}_v^{f_v}) = K_v K^v$. Write

$$\begin{aligned} R(\pi,s)^{-1}R'(\pi,s)\big|_{I(\pi)^{\tau,K}} &= R_\infty(\pi_\infty,s)^{-1}R'_\infty(\pi_\infty,s)\big|_{I(\pi_\infty)^\tau} \otimes \mathrm{Id}_{I(\pi^\infty)^K} \\ &\quad + \sum_{v \text{ finite}} R_v(\pi_v,s)^{-1}R'_v(\pi_v,s)\big|_{I(\pi_v)^{K_v}} \otimes \mathrm{Id}_{I(\pi^v)^{\tau,K^v}} . \end{aligned}$$

Recall that the operators $R_v(\pi_v, s)$ are unitary for $\operatorname{Re} s = 0$.

Consider first the case where v is finite. By property (BD) we have $R_v(\pi_v,s)\big|_{I(\pi_v)^{K_v}} = A_v(q_v^{-s})$, where A_v satisfies the conditions of Lemma 1 (with respect to the operator norm)

[8]This result is misstated on [27, p. 190].

with $m = m_v \ll \log_{q_v} N_v$ if $\mathbf{K}_v$ is hyperspecial, and $m_v \ll \log_{q_v} N_v + 1$ otherwise. Thus,

$$\begin{aligned}\int_{\mathrm{i}\mathbb{R}} \|R_v(\pi_v,s)^{-1}R'_v(\pi_v,s)\big|_{I(\pi_v)^{K_v}}\| \frac{ds}{1+|s|^2} &= \int_{\mathrm{i}\mathbb{R}} \|R'_v(\pi_v,s)\big|_{I(\pi_v)^{K_v}}\| \frac{ds}{1+|s|^2} \\ &\le 2\sum_{n=0}^{\infty}(1+\frac{4\pi^2n^2}{(\log q_v)^2})^{-1}\int_0^{\frac{2\pi\mathrm{i}}{\log q_v}} \|R'_v(\pi_v,s)\big|_{I(\pi_v)^{K_v}}\|\ ds \\ &\ll (\log q_v)\int_0^{\frac{2\pi\mathrm{i}}{\log q_v}} \|R'_v(\pi_v,s)\big|_{I(\pi_v)^{K_v}}\|\ ds = (\log q_v)\int_{\mathbf{S}^1}\|A'_v(z)\|\ |dz|.\end{aligned}$$

By Lemma 1, the last integral is $\ll (\log q_v)(\log_{q_v} N_v) = \log N_v$, if $\mathbf{K}_v$ is hyperspecial, and $\ll 1 + \log N_v$ otherwise. Note here that the last case occurs only for v in a finite set that depends only on $\mathbf{K}_{\mathrm{fin}}$.

Regarding the archimedean contribution, it follows from [40, Proposition A.2] that $R_\infty(\pi_\infty,s)\big|_{I(\pi_\infty)^\tau}$ satisfies the conditions of Lemma 2 with $b(s) = \prod_{j=1}^r\prod_{k=1}^m(s-\rho_j+ck)$, where

- $c > 0$ depends only on M,
- r and $u_j = \mathrm{Re}(\rho_j)$, $j = 1,\dots,r$, are bounded in terms of G only,
- $m \ll 1 + \|\tau\|$.

Write $\rho_j = u_j + \mathrm{i}v_j$. By Lemma 2 we infer that

$$\begin{aligned}\int_{\mathrm{i}\mathbb{R}} \|R_\infty(\pi_\infty,s)^{-1}R'_\infty(\pi_\infty,s)\big|_{I(\pi_\infty)^\tau}\|(1+|s|^2)^{-1}\ ds &= \int_{\mathrm{i}\mathbb{R}} \|R'_\infty(\pi_\infty,s)\big|_{I(\pi_\infty)^\tau}\|(1+|s|^2)^{-1}\ ds \\ &\ll \sum_{j=1}^r\sum_{k=1}^m \frac{|u_j-ck|+1}{(|u_j-ck|+1)^2+v_j^2} \ll 1+\log(1+\|\tau\|).\end{aligned}$$

Altogether,

$$\begin{aligned}\int_{\mathrm{i}\mathbb{R}} \|R(\pi,s)^{-1}R'(\pi,s)\big|_{I(\pi)^{\tau,K}}\|(1+|s|)^{-2}\ ds &\ll 1+\log(1+\|\tau\|)+\sum_{v\text{ finite}}\log N_v \\ &= 1+\log(1+\|\tau\|)+\log\mathrm{level}(K),\end{aligned}$$

as required. □

5. The geometric limit property

We now study the geometric side of the trace formula and prove the geometric limit property for the principal congruence subgroups $\mathbf{K}^S(\mathfrak{n})$, where S is a finite set of places of F containing S_∞. In addition to $\mathbf{K}$ and M_0 we fix in this section a parabolic subgroup $P_0 \in \mathcal{P}(M_0)$. Recall Arthur's distribution J^T_{unip} on $G(\mathbb{A})^1$, the contribution of the unipotent elements of $G(F)$ to the trace formula (5), which is a polynomial in $T \in \mathfrak{a}_{M_0}$ of degree at most $d_0 = \dim \mathfrak{a}^G_{M_0}$ ([7]). It can be split into the contributions of the finitely many $G(\bar F)$-conjugacy classes of unipotent elements of $G(F)$. It is well known ([ibid., Corollary 4.4]) that the contribution of the unit element is simply the constant polynomial $\mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)h(1)$. Write

$$J^T_{\mathrm{unip}-\{1\}}(h) = J^T_{\mathrm{unip}}(h) - \mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)h(1), \quad h \in C_c^\infty(G(\mathbb{A})^1).$$

The distribution J_{unip} is defined as $J^{T_0}_{\mathrm{unip}}$ for a certain vector $T_0 \in \mathfrak{a}_{M_0}$ depending only on G, and analogously for $J_{\mathrm{unip}-\{1\}}$. We want to estimate the latter distribution for the functions $h = h_S \otimes \mathbf{1}_{K^S(\mathfrak{n})}$.

For any compact subset $\Omega \subset G(F_S)^1$ we write $C_\Omega^\infty(G(F_S)^1)$ for the Fréchet space of all smooth functions on $G(F_S)^1$ supported in Ω equipped with the seminorms $\sup_{x\in\Omega}|(Xh)(x)|$, where X ranges over the left-invariant differential operators on $G(F_\infty)$.

Proposition 3. *For any compact subset $\Omega \subset G(F_S)^1$ there exists a seminorm $||\cdot||$ on $C_\Omega^\infty(G(F_S)^1)$ such that*

$$|J_{\mathrm{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})| \le \frac{(1+\log \mathrm{N}(\mathfrak{n}))^{d_0}}{\mathrm{N}(\mathfrak{n})}||h_S||$$

for all $h_S \in C_\Omega^\infty(G(F_S)^1)$ and all integral ideals $\mathfrak{n}$ of $\mathfrak{o}_F$ prime to S.

Remark 7. Let $G = \mathrm{GL}(2)$, $\mathbf{K}(\mathfrak{n})$ the standard principal congruence subgroups, and assume for simplicity $S = S_\infty$. Then we have the explicit formula

$$\begin{aligned} J_{\mathrm{unip}-\{1\}}(h_\infty \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})}) \;&=\; \frac{\mathrm{vol}(T(\mathbb{A})^1/T(F))}{\mathrm{N}(\mathfrak{n})}\Bigg(\int_{F\otimes\mathbb{R}}\int_{\mathbf{K}_\infty} h_\infty(k^{-1}\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)k)\log|x|_\infty\,dk\,dx \\ &+(\gamma_F - \log \mathrm{N}(\mathfrak{n}))\int_{F\otimes\mathbb{R}}\int_{\mathbf{K}_\infty} h_\infty(k^{-1}\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)k)\;dk\,dx\Bigg), \end{aligned}$$

where γ_F denotes the constant term of the Laurent expansion of ζ_F at $s = 1$. This shows that (regarding the dependency on $\mathrm{N}(\mathfrak{n})$) the estimate of Proposition 3 is best possible in this case.

Proposition 3 will be proved below. It has the following consequence.

Corollary 1 (Geometric limit property)**.** *For any $h_S \in C_c^\infty(G(F_S)^1)$ we have*

$$\lim_{\mathfrak{n}} J(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) = \mathrm{vol}(G(F)\backslash G(\mathbb{A})^1)h_S(1).$$

Proof. Fix $h_S \in C_c^\infty(G(F_S)^1)$ and let $\Omega_S \subset G(F_S)^1$ be the support of h_S. Then the support of the test function $h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}$ is $\Omega_S\mathbf{K}^S(\mathfrak{n})$, which for any $\mathfrak{n}$ is a subset of the compact set $\Omega_S\mathbf{K}^S$, and therefore there are only finitely many classes $\mathfrak{o} \in \mathcal{O}$ that contribute to the geometric side of the trace formula (5) for the functions $h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}$. Moreover, for a fixed class $\mathfrak{o} \in \mathcal{O}$ different from the unipotent class, the set of all $G(\mathbb{A})$-conjugacy classes of elements of $\mathfrak{o}$ meets $\Omega_S\mathbf{K}^S(\mathfrak{n})$ only for at most finitely many $\mathfrak{n}$. Therefore, the geometric side reduces to $J_{\mathrm{unip}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})$ for all but finitely many $\mathfrak{n}$, and the assertion follows from Proposition 3. ∎

The proof of Proposition 3 consists of a slight extension of Arthur's arguments in [7]. The case where $F = \mathbb{Q}$ and $\mathfrak{n}$ is a power of a fixed prime is in fact already covered by

Arthur's arguments. However, we will give a detailed account, since our setting allows also for considerable simplifications compared to Arthur's paper. We also remark that when we restrict the prime divisors of $\mathfrak{n}$ to a fixed finite set, we can appeal directly to Arthur's fine geometric expansion ([8]) to obtain the geometric limit property (cf. [21, Proposition 1.7]).

The first ingredient of the proof is an asymptotic formula for $J^T_{\mathrm{unip}-\{1\}}$ obtained by Arthur. Let $\mathcal{U} \subset G$ be the unipotent variety of G, i.e., the Zariski closure of the set of unipotent elements of $G(F)$. Recall that we fixed a Euclidean norm $||\cdot||$ on $\mathfrak{a}_{M_0}$ and let $d(T) = \min_{\alpha\in\Delta_0} \langle \alpha, T\rangle$ for $T \in \mathfrak{a}_{M_0}$. For a parabolic subgroup $P \supset P_0$ write $A_P = A_{M_P}$ and set $A_P(T_1) = \{a \in A_{M_P} : \langle \alpha, H_P(a) - T_1\rangle > 0, \quad \alpha \in \Delta_P\}$ for $T_1 \in \mathfrak{a}_{M_0}$. As in [9, p. 941], we fix a suitable vector T_1, which depends only on G, P_0 and $\mathbf{K}$, such that $G(\mathbb{A}) = U_0(\mathbb{A})M_0(\mathbb{A})^1 A_{P_0}(T_1)\mathbf{K}$. Finally, recall the truncation function $F(\cdot, T) = F^G(\cdot, T)$ for $T \in \mathfrak{a}_{M_0}$, which is the characteristic function of a compact subset of $G(F)\backslash G(\mathbb{A})^1$ ([9, p. 941], [7, p. 1242]). By [7, Theorem 4.2] and the discussion in [ibid., §3], we have the following estimate.

Proposition 4 (Arthur)**.** *There exist an integer $k \geq 1$, left-invariant differential operators $X_1, \dots, X_k$ on $G(F_\infty)$, and positive numbers m, ϵ and ϵ_0 with the following property. For any compact set $\Omega \subset G(\mathbb{A})^1$ there exist a positive constant d_Ω and a seminorm $||\cdot||_\Omega$ on $C^\infty(\Omega)$ of the form*

$$||h||_\Omega = c(\Omega) \sum_{i=1}^{k} \sup_{x\in G(\mathbb{A})^1} |(X_i h)(x)|$$

with $c(\Omega) > 0$, such that for all non-zero ideals $\mathfrak{n}$ of $\mathfrak{o}_F$ with $\mathbf{K}(\mathfrak{n})\Omega\mathbf{K}(\mathfrak{n}) = \Omega$ and any bi-$\mathbf{K}(\mathfrak{n})$-invariant function $h \in C^\infty(\Omega)$ we have

$$\left| J^T_{\mathrm{unip}-\{1\}}(h) - \int_{G(F)\backslash G(\mathbb{A})^1} F(x, T) \sum_{\gamma\in\mathcal{U}(F),\, \gamma\neq 1} h(x^{-1}\gamma x)\, dx \right| \leq ||h||_\Omega \, \mathrm{N}(\mathfrak{n})^m e^{-\epsilon d(T)} \tag{9}$$

for all $T \in \mathfrak{a}_{M_0}$ with $||T|| \geq d_\Omega$ and $d(T) \geq \epsilon_0 ||T||$.

Here, the constant d_Ω is such that the integral formula on [7, p. 1240] for the polynomial $J^T_{\mathrm{unip}}(h)$ is valid for vectors T with $d(T) \geq \epsilon_0 d_\Omega$ for all h with support contained in Ω (cf. [9, Theorem 7.1] and its proof for the existence of such a constant).

We now need to bound the truncated integral

$$\int_{G(F)\backslash G(\mathbb{A})^1} F(x, T) \sum_{\gamma\in\mathcal{U}(F),\, \gamma\neq 1} (h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})(x^{-1}\gamma x)\, dx$$

in terms of $\mathrm{N}(\mathfrak{n})$. For any fixed value of T the integral approaches zero as $\mathrm{N}(\mathfrak{n}) \to \infty$ by the dominated convergence theorem. We make this quantitative as follows.

Lemma 3. *Let $\Omega_S \subset G(F_S)^1$ be a compact set. Then there exists a constant $C(\Omega_S)$, depending only on G and Ω_S, such that*

$$(10)\quad \int_{G(F)\backslash G(\mathbb{A})^1} F(x,T) \sum_{\gamma \in G(F),\, \gamma \neq 1} |(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})(x^{-1}\gamma x)|\, dx \leq C(\Omega_S) \frac{\sup|h_S|}{\mathrm{N}(\mathfrak{n})} (1 + ||T||)^{d_0}.$$

for all bounded measurable functions h_S on $G(F_S)^1$ with support contained in Ω_S.

The proof of this estimate is based on an elementary estimate for a lattice-point counting problem that we will prove first. As a preparation we need the following result from algebraic number theory.

Lemma 4. *Let F be a number field, Λ a fractional ideal of F and $D \subset F \otimes \mathbb{R}$ a compact set. Then there exists a positive constant $C(D,\Lambda)$ such that for all positive real numbers a and non-zero integral ideals $\mathfrak{n}$ we have*

$$|aD \cap (\mathfrak{n}\Lambda - \{0\})| \leq \frac{C(D,\Lambda)}{\mathrm{N}(\mathfrak{n})} a^{[F:\mathbb{Q}]}.$$

Proof. This lemma is an immediate consequence of [34, p. 102, Theorem 0], which provides the upper bound $\max(1, \frac{C(D,\Lambda)}{\mathrm{N}(\mathfrak{n})} a^{[F:\mathbb{Q}]})$ for the cardinality of the intersection $aD \cap \mathfrak{n}\Lambda$. $\square$

Lemma 5. *Let $\mathfrak{u}_P$ be the Lie algebra of the unipotent radical U_P of a standard parabolic subgroup $P = M_P U_P$ of G, $\Lambda \subset \mathfrak{u}_P(F)$ an $\mathfrak{o}_F$-lattice and $D \subset \mathfrak{u}_P(F \otimes \mathbb{R})$ a compact set. Then there exists a positive constant $C = C(P,D,\Lambda)$ such that for all $a \in A_P(T_1)$ and all non-zero integral ideals $\mathfrak{n}$ we have*

$$|\mathrm{Ad}(a)D \cap (\mathfrak{n}\Lambda - \{0\})| \leq \frac{C}{\mathrm{N}(\mathfrak{n})} \delta_P(a).$$

Proof. Let $e_1, \dots, e_n$ be a basis of $\mathfrak{u}_P$ consisting of eigenvectors with respect to T_M and let $\alpha_1, \dots, \alpha_n \in \Sigma_P$ be the associated eigencharacters. Without loss of generality we can assume that $\Lambda = \sum_i \Lambda_i e_i$ with fractional ideals $\Lambda_1, \dots, \Lambda_n \subset F$ and $D = \sum_i D_i e_i$ with compact sets $D_1, \dots, D_n \subset F \otimes \mathbb{R}$. Since a non-zero vector is a vector with at least one non-zero coordinate, we can estimate

$$|\mathrm{Ad}(a)D \cap (\mathfrak{n}\Lambda - \{0\})| \leq \sum_{i=1}^n |\alpha_i(a)D_i \cap (\mathfrak{n}\Lambda_i - \{0\})| \prod_{j \neq i} |\alpha_j(a)D_j \cap \mathfrak{n}\Lambda_j| \,.$$

We now use the estimate of Lemma 4 for $|\alpha_i(a)D_i \cap (\mathfrak{n}\Lambda_i - \{0\})|$, while for the other coordinates we use the trivial estimate $|\alpha_j(a)D_j \cap \mathfrak{n}\Lambda_j| \leq C(D_j,\Lambda_j)\alpha_j(a)^{[F:\mathbb{Q}]} + 1$. This gives the desired result, since the values $\alpha(a) = e^{\langle \alpha, H_P(a) \rangle}$, $\alpha \in \Sigma_P$, are bounded from below. $\square$

Proof of Lemma 3. By Arthur's discussion in [7, §5], we can bound the left-hand side of (10) by

$$(1 + ||T||)^{d_0} \sup_{a \in A_{P_0}(T_1)} \delta_{P_0}(a)^{-1} \sum_{\gamma \in \mathcal{U}(F),\, \gamma \neq 1} \phi(a^{-1}\gamma a),$$

where

$$\phi(x) = \int_\Gamma |(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})(y^{-1}xy)|\,dy$$

for a compact set $\Gamma \subset G(\mathbb{A})^1$ and a Radon measure dy on Γ depending only on G, P_0 and $\mathbf{K}$. Of course, we can assume that $\Gamma = \prod_v \Gamma_v$ with $\Gamma_v = \mathbf{K}_v$ for all $v \notin T$, where $T \supset S_\infty$ is a finite set of places of F. In a second step, Arthur reduces to the estimation of

$$(1 + ||T||)^{d_0} \sum_{P \supset P_0} \sum_{\mu \in M_P(F)} \sup_{a_1 \in A_P(T_1)} \delta_P(a_1)^{-1} \sum_{\nu \in U_P(F)\colon \mu\nu \neq 1} \phi_\mu(a_1^{-1}\nu a_1),$$

where

$$\phi_\mu(u) = \sup_{b \in B} \delta_{P_0}(b)^{-1}\phi(b^{-1}\mu ub), \quad u \in U_P(\mathbb{A}),$$

for a fixed compact set $B \subset A_0$.

Here, for a given P we need to sum only over all μ belonging to the intersection of $M_P(F)$ with a compact set that depends only on Ω_S, or equivalently over a finite subset of $M_P(F)$ that depends only on Ω_S. Considering each possibility for μ separately, we see that for all but at most finitely many $\mathfrak{n}$ (depending on Ω_S) only $\mu = 1$ will contribute. Furthermore, from the definition of ϕ_1 we can estimate

$$\phi_1(u) \le C_1(\Omega_S) \sup|h_S| \mathbf{1}_{\Omega'_S \operatorname{Ad}(\Gamma^S)(\mathbf{K}^S(\mathfrak{n})) \cap U_P(\mathbb{A})}(u), \quad u \in U_P(\mathbb{A}),$$

with a constant $C_1(\Omega_S)$ and a compact set $\Omega'_S \subset G(F_S)^1$ that depend only on Ω_S. There exist exponents $e_v \ge 0$ for $v \notin S$, with $e_v = 0$ for $v \notin T$, such that

$$\operatorname{Ad}(\Gamma_v)(\mathbf{K}_v(\mathfrak{p}_v^f)) \subset \mathbf{K}_v(\mathfrak{p}_v^{f-e_v})$$

for $f \ge e_v$. Write $\mathfrak{n} = \prod_{v \notin S} \mathfrak{p}_v^{f_v}$. We conclude that $\operatorname{Ad}(\Gamma^S)(\mathbf{K}^S(\mathfrak{n})) \subset \prod_{v \notin S} L_{v,f_v}$, where for $v \notin S$ and $f \ge 0$ we set $L_{v,f} = \operatorname{Ad}(\Gamma_v)(\mathbf{K}_v)$ in case $f < e_v$ (which implies $v \in T$) and $L_{v,f} = \mathbf{K}_v(\mathfrak{p}_v^{f-e_v})$, otherwise. Identify the unipotent radical U_P with its Lie algebra $\mathfrak{u}_P$ via the exponential map. Then everything reduces to an application of Lemma 5 (with $\mathfrak{n}$ replaced by $\mathfrak{n}' = \prod_{v\colon f_v \ge e_v} \mathfrak{p}_v^{f_v - e_v}$). □

To finish the argument we follow Arthur's interpolation argument in [7, pp. 1252-1254]. We formulate the precise technical statement in the following lemma, which is a slight variant of [5, Lemma 5.2]. The proof is omitted.

Lemma 6. *Let $\mathfrak{a}_0$ be a Euclidean vector space with norm $||\cdot||$, Δ_0 a set of linearly independent elements of $\mathfrak{a}_0^*$, $d_0 \ge 0$ an integer and $\epsilon_0 > 0$. Then there exists a constant a with the following property. For polynomials q on $\mathfrak{a}_0$ of degree $\le d_0$ and real numbers $A > 0$, $B > 1$ with*

$$|q(T)| \le A(1 + ||T||)^{d_0}$$

for all vectors $T \in \mathfrak{a}_0$ with $\langle \alpha, T\rangle \ge \max(\epsilon_0||T||, B)$, $\alpha \in \Delta_0$, we have

$$|q(T)| \le aAB^{d_0}(1 + ||T||)^{d_0}$$

for any $T \in \mathfrak{a}_0$.

Proof of Proposition 3. Write $N = \mathrm{N}(\mathfrak{n})$. Given $h_S \in C_c^\infty(G(F_S)^1)$ with support contained in a compact set $\Omega_S \subset G(F_S)^1$, we combine (9) and Lemma 3 to obtain

$$|J^T_{\mathrm{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})| \le ||h_S||_{\Omega_S} \left(N^m e^{-\epsilon d(T)} + N^{-1}(1+||T||)^{d_0}\right)$$

for all $T \in \mathfrak{a}_0$ with $||T|| \ge d_{\Omega_S}$ and $d(T) \ge \epsilon_0 ||T||$, where $||\cdot||_{\Omega_S}$ is a suitable seminorm on $C^\infty(\Omega_S)$. This implies

$$|J^T_{\mathrm{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})| \le 2||h_S||_{\Omega_S} N^{-1}(1+||T||)^{d_0}$$

for all $T \in \mathfrak{a}_0$ with $||T|| \ge \max(d_{\Omega_S}, \frac{m+1}{\epsilon\epsilon_0}\log N)$ and $d(T) \ge \epsilon_0||T||$. Applying Lemma 6 to the polynomial $J^T_{\mathrm{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})$ and the point $T_0 \in \mathfrak{a}_0$, we obtain the assertion. □

Remark 8. An alternative proof of Corollary 1 might be given by replacing $\mathcal{U}(F)$ by $G(F)$ in the arguments above and using [2, p. 267, Theorem 1].

For use in a planned future paper, we note that the arguments above actually yield the following extension of Proposition 3.

Proposition 5. *There exists a seminorm $||\cdot||$ on $C_c^\infty(G(F_S)^1)$ such that*

$$|J_{\mathrm{unip}-\{1\}}(h_S \otimes h^S)| \le \frac{(1+\log \mathrm{N}(\mathfrak{n}))^{d_0}}{\mathrm{N}(\mathfrak{n}')}||h_S||$$

for all $h_S \in C_c^\infty(G(F_S)^1)$, all pairs of integral ideals $\mathfrak{n}$ and $\mathfrak{n}'$ of $\mathfrak{o}_F$ prime to S with $\mathfrak{n}'$ a divisor of $\mathfrak{n}$, and all bi-$\mathbf{K}^S(\mathfrak{n})$-invariant functions h^S on $\mathbf{K}^S(\mathfrak{n}')$ with $|h^S| \le 1$.

6. Polynomially bounded collections of measures

As a preparation for our proof of the spectral limit property, we prove in this section a proposition on real reductive Lie groups, which extends an argument of Delorme in [22]. Let temporarily G_∞ be the group of real points of a connected reductive group defined over $\mathbb{R}$, or, slightly more generally, the quotient of such a group by a connected subgroup of its center. In the end, we will apply the results to $G_\infty = G(F_\infty)^1$, of course. Let K_∞ be a maximal compact subgroup of G_∞ and θ the associated Cartan involution. We now consider Levi subgroups M and parabolic subgroups P defined over $\mathbb{R}$. All Levi subgroups are supposed to be θ-stable. Factor each Levi subgroup M as a direct product $M = A_M M^1$, where A_M is the largest central subgroup of M isomorphic to a power of $\mathbb{R}^{>0}$, and let $\mathfrak{a}_M = \mathrm{Lie}\, A_M$. We identify representations of M^1 with representations of M on which A_M acts trivially. Fix a minimal θ-stable Levi subgroup M_0. As in §3.1, we fix an invariant bilinear form B on $\mathrm{Lie}\, G_\infty$, which induces Euclidean norms on all its subspaces and therefore Hermitian norms on the spaces $\mathfrak{a}^*_{M,\mathbb{C}}$. Then for each $r > 0$ we define $\mathcal{H}(G_\infty)_r$ as the subspace of $\mathcal{H}(G_\infty)$ of functions with support contained in the compact set $K_\infty \exp(\{x \in \mathfrak{a}_{M_0} : \|x\| \le r\})K_\infty$.

For any $k \ge 0$ let

$$\|f\|_k = \sum_{X_i} \|f \star X_i\|_{L^1(G_\infty)} = \sum_{X_i} \|X_i \star f\|_{L^1(G_\infty)}$$

where X_i ranges over a basis of $\mathcal{U}(\mathrm{Lie}\, G_\infty \otimes \mathbb{C})_{\le k}$, with the usual filtration.

Let $\mathrm{Irr}(G_\infty)$ be the set of all irreducible admissible representations of G_∞ up to infinitesimal equivalence. The unitary dual $\Pi(G_\infty)$ can be viewed as a subset of $\mathrm{Irr}(G_\infty)$ in a natural way. For $\pi \in \mathrm{Irr}(G_\infty)$ denote its infinitesimal character by χ_π and its Casimir eigenvalue (which depends only on χ_π) by λ_π. For any $\mu \in \Pi(K_\infty)$ let $\mathrm{Irr}(G_\infty)_\mu$ be the set of irreducible representations containing μ as a K_∞-type. More generally, for any subset $\mathcal{F}$ of $\Pi(K_\infty)$ we write $\mathrm{Irr}(G_\infty)_{\mathcal{F}} = \cup_{\tau\in\mathcal{F}} \mathrm{Irr}(G_\infty)_\tau$. We call a subset $\mathcal{F} \subset \Pi(K_\infty)$ *saturated*, if for each $\mu \in \mathcal{F}$ all $\mu' \in \Pi(K_\infty)$ with $\|\mu'\| \le \|\mu\|$ are also contained in $\mathcal{F}$. Recall that here $\|\cdot\|$ is defined as in [14, §2.2]. More precisely, let K_∞^0 be the connected component of the identity of K_∞. Then $\|\mu\| = \|\chi_\mu + 2\rho\|^2$, where χ_μ denotes the highest weight of any irreducible constituent of $\mu|_{K_\infty^0}$ with respect to a maximal torus of K_∞^0 (and the choice of a system of positive roots), and ρ is as usual one half times the corresponding sum of all positive roots.

We write $\mathcal{D}$ for the set of all conjugacy classes of pairs (M, δ) consisting of a Levi subgroup M of G_∞ and a discrete series representation δ of M^1. For any $\underline{\delta} \in \mathcal{D}$ let $\mathrm{Irr}(G_\infty)_{\underline{\delta}}$ be the set of all irreducible representations which arise by the Langlands quotient construction from the irreducible constituents of $I_M^L(\delta)$ for Levi subgroups $L \supset M$. Here, I_M^L denotes (unitary) induction from an arbitrary parabolic subgroup of L with Levi subgroup M to L. We then have a disjoint decomposition

$$\mathrm{Irr}(G_\infty) = \coprod_{\underline{\delta}\in\mathcal{D}} \mathrm{Irr}(G_\infty)_{\underline{\delta}}$$

and consequently

$$\Pi(G_\infty) = \coprod_{\underline{\delta}\in\mathcal{D}} \Pi(G_\infty)_{\underline{\delta}}.$$

For $\pi \in \mathrm{Irr}(G_\infty)$ we write $\underline{\delta}(\pi)$ for the unique element $\underline{\delta} \in \mathcal{D}$ with $\pi \in \mathrm{Irr}(G_\infty)_{\underline{\delta}}$. We introduce a partial order on $\mathcal{D}$ as in [14, §2.3], using the lowest K_∞-types of $I_M^G(\delta)$: $\underline{\delta} \prec \underline{\delta}'$ if and only if $\|\mu\| < \|\mu'\|$ for lowest K_∞-types μ and μ' of $I_M^G(\delta)$ and $I_{M'}^G(\delta')$, respectively.

When $\mathcal{F} \subset \Pi(K_\infty)$ is saturated, we have $\mathrm{Irr}(G_\infty)_{\mathcal{F}} = \cup_{\underline{\delta}\in\mathcal{D}_{\mathcal{F}}} \mathrm{Irr}(G_\infty)_{\underline{\delta}}$, where $\mathcal{D}_{\mathcal{F}}$ is the set of all $\underline{\delta} \in \mathcal{D}$ with $\mathrm{Irr}(G_\infty)_{\underline{\delta}} \cap \mathrm{Irr}(G_\infty)_{\mathcal{F}} \ne \emptyset$. For $\underline{\delta} \in \mathcal{D}$ we let $\mathcal{F}(\underline{\delta})$ be the finite saturated set of all $\mu' \in \Pi(K_\infty)$ with $\|\mu'\| \le \|\mu\|$ for a lowest K_∞-type μ of $I_M^G(\delta)$.

Proposition 6. *Let $\mathfrak{M}$ be a set of Borel measures on $\Pi(G_\infty)$. Then the following conditions on $\mathfrak{M}$ are equivalent:*

(1) *For all $\underline{\delta} \in \mathcal{D}$ there exist positive constants $N_{\underline{\delta}}$ and $C_{\underline{\delta}}$ such that*

$$\mu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \le R\}) \le C_{\underline{\delta}}(1 + R)^{N_{\underline{\delta}}}$$

for all $\mu \in \mathfrak{M}$ and $R > 0$.

(2) *There exists $r > 0$ such that for each finite set $\mathcal{F} \subset \Pi(K_\infty)$ the supremum $\sup_{\mu\in\mathfrak{M}}|\mu(\hat{f})|$ is a continuous seminorm on $\mathcal{H}(G_\infty)_{r,\mathcal{F}}$.*

(3) *For each $r > 0$ and each finite set $\mathcal{F} \subset \Pi(K_\infty)$ the supremum $\sup_{\mu\in\mathfrak{M}}|\mu(\hat{f})|$ is a continuous seminorm on $\mathcal{H}(G_\infty)_{r,\mathcal{F}}$.*

(4) *For each finite set $\mathcal{F} \subset \Pi(K_\infty)$ there exists an integer $k = k(\mathcal{F})$ with $\sup_{\mu \in \mathfrak{M}} \mu(g_{k,\mathcal{F}}) < \infty$, where $g_{k,\mathcal{F}}$ is the non-negative function on $\Pi(G_\infty)$ defined by*

$$g_{k,\mathcal{F}}(\pi) = (1 + |\lambda_\pi|)^{-k}$$

for $\pi \in \Pi(G_\infty)_{\mathcal{F}}$, and $g_{k,\mathcal{F}}(\pi) = 0$, otherwise.

Definition 3. We call a collection $\mathfrak{M}$ of measures satisfying the equivalent conditions of Proposition 6 *polynomially bounded.*

For the proof, we first need to recall the classification of tempered and admissible representations of G_∞, as well as the Paley-Wiener theorem. We first recall Vogan's classification of irreducible admissible representations. For (M, δ) as above, and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, consider the induced representation $\pi_{\delta,\lambda}$ (with respect to any parabolic subgroup containing M as a Levi subgroup). Its semi-simplification depends only on the K_∞-conjugacy class of the triple (M, δ, λ). Vogan defines the R-group R_δ of δ, a finite group of exponent two, as well as its subgroup $R_{\delta,\lambda}$. The dual group $\hat{R}_\delta$ acts simply transitively on the set $A(\delta)$ of lowest K_∞-types of $\pi_{\delta,\lambda}$. We then have a decomposition of the representation $\pi_{\delta,\lambda}$ as a direct sum of $|R_{\delta,\lambda}|$ many representations $\pi_{\delta,\lambda}(\mu)$, where μ is an orbit of $R^\perp_{\delta,\lambda}$ in $A(\delta)$ ([51, 6.5.10, 6.5.11]):

$$\pi_{\delta,\lambda} = \bigoplus_{\mu \in A(\delta)/R^\perp_{\delta,\lambda}} \pi_{\delta,\lambda}(\mu).$$

We call the $\pi_{\delta,\lambda}(\mu)$ the basic representations. Each basic representation $\pi_{\delta,\lambda}(\mu)$ has a unique irreducible subquotient $\bar{\pi}_{\delta,\lambda}(\mu)$ containing a K_∞-type in the orbit μ. Alternatively, this subquotient can also be constructed as a Langlands quotient ([51, 6.6.14, 6.6.15]). This construction sets up a bijection $\bar{\pi}_{\delta,\lambda}(\mu) = \pi \mapsto \sigma_\pi = \pi_{\delta,\lambda}(\mu)$ between infinitesimal equivalence classes of irreducible admissible representations π and basic representations σ_π, where the latter are interpreted as elements of the Grothendieck group of admissible representations ([51, 6.5.13]). By definition, the parametrization is compatible with the disjoint decomposition of $\mathrm{Irr}(G_\infty)$ according to the elements of $\mathcal{D}$.

The distributions $\operatorname{tr} \sigma_\pi$ for $\pi \in \mathrm{Irr}(G_\infty)$ form a basis of the Grothendieck group of admissible representations. More precisely, we have the following relations expressing the characters of irreducible representations $\pi \in \mathrm{Irr}(G_\infty)$ in terms of the characters of basic representations:

$$\operatorname{tr} \pi(\phi) = \operatorname{tr} \sigma_\pi(\phi) + \sum_{\pi' : \underline{\delta}(\pi) \prec \underline{\delta}(\pi'),\, \chi_\pi = \chi_{\pi'}} n(\pi, \pi') \operatorname{tr} \sigma_{\pi'}(\phi) \tag{11}$$

with certain integers $n(\pi, \pi')$ ([51, 6.6.7]). Note that here the sum on the right-hand side is finite. For our purposes, all we need to know about the integers $n(\pi, \pi')$ is the following uniform boundedness property ([22, Proposition 2.2]).

Lemma 7 (Vogan)**.** *For each group G_∞ there exists a constant n_{G_∞} such that*

$$\sum_{\pi' : \underline{\delta}(\pi) \prec \underline{\delta}(\pi'),\, \chi_\pi = \chi_{\pi'}} |n(\pi, \pi')| \le n_{G_\infty} \tag{12}$$

for all $\pi \in \mathrm{Irr}(G_\infty)$.

For the Paley-Wiener theorem, we need to group the basic representations into series of induced representations, which gives a slightly different parametrization. We use the concept of a non-degenerate limit of discrete series introduced in [32, 33]. Let $\underline{\delta} \in \mathcal{D}$ with representative (M, δ). Whenever L is a Levi subgroup containing M and $I_M^L(\delta)$ decomposes as a sum of non-degenerate limit of discrete series representations δ' of L^1, we call the resulting pairs (L, δ') *affiliated* with the class $\underline{\delta}$ ([16, Définition 2]). These representations are precisely those irreducible constituents of the representations $I_M^L(\delta)$ for $L \supset M$, which are not itself irreducibly induced from any smaller Levi subgroup. The Levi subgroups $L \supset M$ appearing here are those for which $\mathfrak{a}_L^*$ is the fixed space of $\mathfrak{a}_M^*$ under one of the subgroups $R_{\delta,\lambda} \subset R_\delta$.

We can then rewrite any representation $\pi_{\delta,\lambda}(\mu)$ in the form $\pi_{\delta',\lambda}$ for $M \subset L$, $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^* \subset \mathfrak{a}_{M,\mathbb{C}}^*$, and (L, δ') affiliated with $\mathcal{D}$, such that the intermediate induction to the largest Levi subgroup $L_{\mathrm{Re}\,\lambda}$ with $\mathrm{Re}\,\lambda \in \mathfrak{a}^*_{L_{\mathrm{Re}\,\lambda}}$ is irreducible (and tempered). (To see this, combine [51, 6.6.14, 6.6.15] with [16, (2.1), (2.2)].) Note that the tempered dual of G_∞ can either be parametrized as the set of all basic representations $\pi_{\delta,\lambda}(\mu)$ with $\mathrm{Re}\,\lambda = 0$ (which are always irreducible), or as the set of all *irreducible* induced representations $\pi_{\delta',\lambda}$, $\mathrm{Re}\,\lambda = 0$, where δ' is a non-degenerate limit of discrete series.

Recall the definition of the Paley-Wiener space $\mathcal{PW}(\mathfrak{a})_r$, $r > 0$, of a Euclidean vector space $\mathfrak{a}$. It is the space of all entire functions F on the complexified dual $\mathfrak{a}_{\mathbb{C}}^*$ such that the Paley-Wiener norms

$$\|F\|_{r,n} = \sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} (1 + \|\lambda\|)^n e^{-r\|\mathrm{Re}\,\lambda\|} |F(\lambda)|, \quad n \geq 0,$$

are finite. The Paley-Wiener norms give $\mathcal{PW}(\mathfrak{a})_r$ the structure of a Fréchet space and the Fourier transform is a topological isomorphism between $C^\infty(\{x \in \mathfrak{a} : \|x\| \leq r\})$ and $\mathcal{PW}(\mathfrak{a})_r$.

Now let $\underline{\delta} \in \mathcal{D}$. Consider the finite set $\mathcal{D}'(\underline{\delta})$ of all pairs (M, δ), where M is a standard Levi subgroup of G_∞, $\delta \in \Pi(M^1)$ a non-degenerate limit of discrete series, and (M, δ) is affiliated with $\underline{\delta}$. The Paley-Wiener space $\mathcal{PW}_{r,\underline{\delta}}$ is then defined as the space of all elements $F = (F_{(M,\delta)}) \in \prod_{(M,\delta)\in\mathcal{D}'(\underline{\delta})} \mathcal{PW}(\mathfrak{a}_M)_r$ fulfilling the following conditions:

(1) Whenever the triples (M, δ, λ) and (M', δ', λ') are conjugate by an element of K_∞, we have $F_{(M',\delta')}(\lambda') = F_{(M,\delta)}(\lambda)$.
(2) Whenever for $M \subset M'$ we have a decomposition

$$I_M^{M'}(\delta_M) = \bigoplus_{i=1}^m \delta_{M'}^{(i)}$$

with (M, δ_M), $(M', \delta_{M'}^{(i)}) \in \mathcal{D}'(\underline{\delta})$, the corresponding identity

$$F_{(M,\delta_M)}(\lambda) = \sum_{i=1}^m F_{(M',\delta_{M'}^{(i)})}(\lambda), \quad \lambda \in \mathfrak{a}_{M',\mathbb{C}}^* \subset \mathfrak{a}_{M,\mathbb{C}}^*,$$

holds.

For any finite set $\mathcal{F} \subset \Pi(K_\infty)$ the space $\mathcal{PW}_{r,\mathcal{F}}$ is defined as $\prod_{\underline{\delta} \in \mathcal{D}_{\mathcal{F}}} \mathcal{PW}_{r,\underline{\delta}}$. These Paley-Wiener spaces have in a natural way the structure of Fréchet spaces, and we define for each $n \geq 0$ the Paley-Wiener norm $\|F\|_{r,n}$ of $F \in \mathcal{PW}_{r,\mathcal{F}}$ to be the maximum of the norms $\|F_{(M,\delta)}\|_{r,n}$, where $(M,\delta) \in \mathcal{D}'(\underline{\delta})$, $\underline{\delta} \in \mathcal{D}_{\mathcal{F}}$. (Cf. [16, Appendice C] for a concrete combinatorial description of these spaces.)

We can now quote the Paley-Wiener theorem of Clozel-Delorme ([16, Théorème 1, Théorème 1']).

Theorem 6 (Clozel-Delorme)**.** *For any finite saturated set $\mathcal{F} \subset \Pi(K_\infty)$ and any $r > 0$ the natural continuous map of Fréchet spaces $T_{r,\mathcal{F}} : \mathcal{H}(G_\infty)_{r,\mathcal{F}} \to \mathcal{PW}_{r,\mathcal{F}}$ given by $f \mapsto (\operatorname{tr} \pi_{\delta,\lambda}(f))$ is surjective.*

Remark 9. By the open mapping theorem, a continuous surjection of Fréchet spaces is automatically open. This applies to the surjections $T_{r,\mathcal{F}}$ of Theorem 6. In concrete terms, this means that not only for every $F \in \mathcal{PW}_{r,\mathcal{F}}$ there exists $\phi \in \mathcal{H}(G_\infty)_{r,\mathcal{F}}$ with $T_{r,\mathcal{F}}(\phi) = F$, but that the following stronger statement is true. Given an integer $k \geq 0$, for every $F \in \mathcal{PW}_{r,\mathcal{F}}$ there exists a preimage $\phi \in \mathcal{H}(G_\infty)_{r,\mathcal{F}}$ of F under $T_{r,\mathcal{F}}$, which satisfies $\|\phi\|_k \leq c(r,\mathcal{F},k)\|F\|_{r,n(r,\mathcal{F},k)}$, where $c(r,\mathcal{F},k)$ and $n(r,\mathcal{F},k)$ depend only on r,$\mathcal{F}$ and k.

We now turn to the proof of Proposition 6, which is an extension of an argument of Delorme (cf. the proof of [22, Proposition 3.3]). As in [ibid.], the proof is based on the existence of certain test functions on G_∞, however, in comparison to Delorme's argument we also need to bound the seminorms of these functions. We therefore recall the construction in some detail. The first elementary lemma ([24, Lemma 6.3]) asserts the existence of functions with certain properties of the Fourier transform.

Lemma 8 (Duistermaat-Kolk-Varadarajan)**.** *Let $\mathfrak{a}$ be a real vector space, W a finite Coxeter group acting on $\mathfrak{a}$ and $r > 0$. Then for any $t \geq 1$ there exists a function $h(t,\cdot) \in C^\infty(\{x \in \mathfrak{a} : \|x\| \leq r\})^W$ such that its Fourier transform $\hat{h}(t,\cdot) \in \mathcal{PW}(\mathfrak{a})_r^W$ has the following properties.*

1. *$\hat{h}(t,\lambda)$ is non-negative real for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for which there exists an element $w \in W$ with $w(\lambda) = -\bar{\lambda}$.*
2. *$|\hat{h}(t,\lambda)| \geq 1$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\|\lambda\| \leq t$.*
3. *For all $m \geq 0$ there exists a positive constant $c(m)$ such that the Paley-Wiener norm $\|\hat{h}(t,\cdot)\|_{r,m}$ is bounded by $c(m)t^m$.*
4. *For all $a > 0$ and $m \geq 0$ there exists a positive constant $c(m,a)$ with*
$$|\hat{h}(t,\lambda)| \leq c(m,a)\frac{t^m}{(1 + \|\operatorname{Im} \lambda\|)^m}$$
for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\|\operatorname{Re} \lambda\| \leq a$.

The following lemma is a strengthening of [22, Proposition 3.2] (we have added the third assertion).

Lemma 9. *Let $\underline{\delta} \in \mathcal{D}$ with representative (M,δ), $k \geq 0$ an integer and $t \geq 1$. Then there exist a function $\phi_{\underline{\delta}}^{t,k} \in \mathcal{H}_{r,\mathcal{F}(\underline{\delta})}(G_\infty)$ and an integer $r_k \geq 0$ with the properties that*

(1) $\operatorname{tr}\sigma_\pi(\phi^{t,k}_{\underline{\delta}}) = 0$ *for all* $\pi \in \operatorname{Irr}(G_\infty)$ *with* $\underline{\delta}(\pi) \neq \underline{\delta}$.
(2) $\operatorname{tr}\sigma(\phi^{t,k}_{\underline{\delta}}) = [R_\delta : R_{\delta,\lambda}]\hat{h}(t,\lambda)$ *for all basic representations* $\sigma = \pi_{\delta,\lambda}(\mu)$, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, $\mu \in A(\delta)$.
(3) $\|\phi^{t,k}_{\underline{\delta}}\|_k \leq t^{r_k}$.

Proof. We apply the Paley-Wiener theorem to the following element $F^t = F^t_{(M',\delta')}$ of $\mathcal{PW}_{r,\mathcal{F}}$, where $\mathcal{F} = \mathcal{F}(\underline{\delta})$. Note that since M is a cuspidal Levi subgroup, the Weyl group $W(A_M) = N_K(A_M)/C_K(A_M)$ is a finite Coxeter group ([31]). Apply Lemma 8 to the vector space $\mathfrak{a}_M$ and the group $W(A)$ to obtain functions $\hat{h}(t,\cdot) \in \mathcal{PW}(\mathfrak{a}_M)^{W(A)}_r$. Set $F^t_{(M',\delta')}(\lambda) = 0$ whenever (M',δ') is not affiliated with $\underline{\delta}$. On the other hand, if (M',δ') is affiliated with $\underline{\delta}$, then we have a decomposition $I^{M'}_M(\delta) = \oplus_{i=1}^S \delta'_i$, where $\delta'_1 = \delta'$ and S is a divisor of $|R_\delta|$. We then set $F^t_{(M',\delta')}(\lambda) = \frac{|R_\delta|}{S}\hat{h}(t,\lambda)$. By the $W(A)$-invariance of $\hat{h}(t,\cdot)$ and the transitivity of induction, this indeed defines an element F^t of $\mathcal{PW}_{r,\mathcal{F}}$. Moreover, by the third assertion of Lemma 8, every Paley-Wiener norm of F^t is bounded by a power of t. The Paley-Wiener theorem (Theorem 6) and Remark 9 provide for each $k \geq 0$ a preimage $\phi^{t,k}_{\underline{\delta}} \in \mathcal{H}_{r,\mathcal{F}}(G_\infty)$ which by construction satisfies the first and third properties. Writing $\pi_{\delta,\lambda}(\mu) \simeq \pi_{\delta',\lambda}$ as above, we see that it also satisfies the second property. □

Corollary 2. *The test functions* $\phi^{t,k}_{\underline{\delta}} \in \mathcal{H}_{r,\mathcal{F}}(G_\infty)$ *have the following additional properties:*

(1) $\operatorname{tr}\sigma_\pi(\phi^{t,k}_{\underline{\delta}}) \geq 1$ *for all* $\pi \in \Pi(G_\infty)_{\underline{\delta}}$ *with* $|\lambda_\pi| \leq t^2 - c_{\underline{\delta}}$, *where* $c_{\underline{\delta}}$ *is a constant depending only on* $\underline{\delta}$.
(2) *For all* $m \geq 0$ *there exists a constant* $C_{\underline{\delta},m}$, *depending only on* $\underline{\delta}$ *and* m, *such that*

$$0 \leq \operatorname{tr}\sigma_\pi(\phi^{t,k}_{\underline{\delta}}) \leq C_{\underline{\delta},m}\frac{t^{2m}}{(1+|\lambda_\pi|)^m}$$

for all $\pi \in \Pi(G_\infty)_{\underline{\delta}}$.

Proof. Let $\bar{\pi}_{\delta,\lambda} \simeq \pi \in \Pi(G_\infty)$ and $\sigma_\pi = \pi_{\delta,\lambda}(\mu)$. Since π is unitary, we need to have $w(\lambda) = -\bar{\lambda}$ for an element $w \in W(A)_\delta$ ([22, (2.1)]). By Lemma 9, the trace $\operatorname{tr}\sigma_\pi(\phi^{t,k}_{\underline{\delta}})$ is an integer multiple of $\hat{h}(t,\lambda)$. By the first property of Lemma 8, it is therefore nonnegative real.

Furthermore, the Casimir eigenvalue of π can be computed as $\lambda_\pi = -\|\operatorname{Im}\lambda\|^2 + \|\operatorname{Re}\lambda\|^2 - \|\chi_\delta\|^2 - c_M$ for a constant c_M (cf. [11, §3.2, (2)]). Again by unitarity, we have $\|\operatorname{Re}\lambda\| \leq \|\rho_P\|$, where ρ_P is half the sum of the positive roots of a parabolic P with Levi subgroup M ([22, (2.2)]). Therefore, we obtain $|\lambda_\pi| \geq \|\lambda\|^2 - c_{\underline{\delta}}$ for a constant $c_{\underline{\delta}}$.

To show the first assertion, $|\lambda_\pi| \leq t^2 - c_{\underline{\delta}}$ implies that $\|\lambda\| \leq t$, and by the second property of Lemma 8 we obtain $\hat{h}(t,\lambda) \geq 1$. For the second assertion, we use the fourth property of Lemma 8 and the boundedness of $\|\operatorname{Re}\lambda\|$ together with the fact that $[R_\delta : R_{\delta,\lambda}]$ is obviously bounded by $|R_\delta|$. □

Proof of Proposition 6. Let $\mathfrak{M}$ be a collection of Borel measures on $\Pi(G_\infty)$. It is easy to see that the fourth condition of the proposition implies the third and second conditions, since we can estimate $|\hat{f}(\pi)| \leq c_k \|f\|_{2k} g_{k,\mathcal{F}}(\pi)$ for $f \in \mathcal{H}(G_\infty)_{\mathcal{F}}$, $\pi \in \Pi(G_\infty)$ and $k \geq 0$, with a constant c_k depending only on k (and G_∞).

For $k \geq 0$ and $\underline{\delta} \in \mathcal{D}$ let $g_{k,\underline{\delta}} = (1 + |\lambda_\pi|)^{-k}$ for $\pi \in \Pi(G_\infty)_{\underline{\delta}}$, and extend this function by zero to all of $\Pi(G_\infty)$.

For a given $\underline{\delta} \in \mathcal{D}$, consider the following two statements:

(13) There exist nonnegative constants $C_{\underline{\delta}}$ and $N_{\underline{\delta}}$ such that

$$\mu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \leq R\}) \leq C_{\underline{\delta}}(1 + R)^{N_{\underline{\delta}}} \text{ for all } \mu \in \mathfrak{M} \text{ and } R \geq 0.$$

$$\text{There exists an integer } k_{\underline{\delta}} > 0 \text{ such that } \sup_{\mu \in \mathfrak{M}} \mu(g_{k,\underline{\delta}}) < \infty. \tag{14}$$

It is easy to see that these statements are equivalent: if (13) is satisfied, then we can bound

$$\mu(g_{k,\underline{\delta}}) \leq \sum_{N \geq 0} \mu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : N \leq |\lambda_\pi| \leq N + 1\})(N + 1)^{-k} \leq C_{\underline{\delta}} \sum_{N \geq 0} \frac{(N + 2)^{N_{\underline{\delta}}}}{(N + 1)^k},$$

which is bounded independently of $\mu \in \mathfrak{M}$ for $k \geq N_{\underline{\delta}} + 2$. On the other hand, we clearly have

$$(1 + R)^{-k} \mu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \leq R\}) \leq \mu(g_{k,\underline{\delta}}),$$

which gives the other implication.

Observe now that the first condition of the proposition is just (13) for arbitrary $\underline{\delta}$. Moreover, the fourth condition is clearly equivalent to (14) for arbitrary $\underline{\delta}$. Therefore, the first and fourth conditions of the proposition are equivalent.

Therefore, it remains only to prove (13) or (14) for arbitrary $\underline{\delta}$ assuming the second condition of the proposition. We will prove them by induction on $\underline{\delta}$, i.e., for a given $\underline{\delta}$ we assume that (14) is satisfied for all $\underline{\delta}' \prec \underline{\delta}$ and are going to prove (13) for $\underline{\delta}$. For this, consider the test functions $\phi^{t,k}_{\underline{\delta}}$ constructed above for $t = (R + c_\delta)^{1/2}$, where $R \geq 0$. By assumption, for a suitable $r > 0$ for each finite set $\mathcal{F}$ the supremum $\sup_{\mu \in \mathfrak{M}} |\mu(\hat{f})|$ is a continuous seminorm on $\mathcal{H}(G_\infty)_{r,\mathcal{F}}$. Taking $f = \hat{\phi}^{t,k}_{\underline{\delta}}$ and using the third assertion of Lemma 9, we obtain that for a suitable value of k all absolute values $|\mu(\hat{\phi}^{t,k}_{\underline{\delta}})|$, $\mu \in \mathfrak{M}$, can be bounded by a polynomial in t, or equivalently by $D_{\underline{\delta}} R^{m_{\underline{\delta}}}$ for some constants $D_{\underline{\delta}}$ and $m_{\underline{\delta}}$. Write

$$\mu(\hat{\phi}^{t,k}_{\underline{\delta}}) = \int \operatorname{tr} \pi(\phi^{t,k}_{\underline{\delta}}) d\mu(\pi).$$

Inserting (11) into this equation, we obtain

$$\mu(\hat{\phi}^{t,k}_{\underline{\delta}}) = \int (\operatorname{tr} \sigma_\pi)(\hat{\phi}^{t,k}_{\underline{\delta}}) d\mu(\pi) + \int \left[\sum_{\pi' : \underline{\delta}(\pi) \prec \underline{\delta}(\pi'),\, \chi_\pi = \chi_{\pi'}} n(\pi, \pi') \operatorname{tr} \sigma_{\pi'}(\hat{\phi}^{t,k}_{\underline{\delta}}) \right] d\mu(\pi).$$

By the first assertion of Corollary 2, the first integral provides an upper bound for the measure of the set $\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \le R\}$:

$$\mu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \le R\}) \le \int (\operatorname{tr} \sigma_\pi)(\hat{\phi}^{t,k}_{\underline{\delta}}) d\mu(\pi).$$

Regarding the second integral, only π' with $\underline{\delta}(\pi') = \underline{\delta}$ can contribute, and we can estimate their contribution using the second assertion of Corollary 2:

$$0 \le \operatorname{tr} \sigma_{\pi'}(\hat{\phi}^{t,k}_{\underline{\delta}}) \le C_{\underline{\delta},m} \frac{t^{2m}}{(1+|\lambda_{\pi'}|)^m} = C_{\underline{\delta},m} \frac{(R+c_{\underline{\delta}})^m}{(1+|\lambda_\pi|)^m},$$

since π and π' have the same infinitesimal character. Using (12) to bound the absolute values of the integers $n(\pi, \pi')$, we obtain

$$\begin{aligned} \mu(\{\pi \in \Pi(G_\infty)_{\underline{\delta}} : |\lambda_\pi| \le R\}) &\le \mu(\hat{\phi}^{t,k}_{\underline{\delta}}) - \int \left[\sum_{\pi' : \underline{\delta}(\pi) \prec \underline{\delta}(\pi'),\, \chi_\pi = \chi_{\pi'}} n(\pi, \pi') \operatorname{tr} \sigma_{\pi'}(\hat{\phi}^{t,k}_{\underline{\delta}}) \right] d\mu(\pi) \\ &\le \mu(\hat{\phi}^{t,k}_{\underline{\delta}}) + n_{G_\infty} C_{\underline{\delta},m} (R + c_{\underline{\delta}})^m \sum_{\underline{\delta}' \prec \underline{\delta}} \mu(g_{m,\underline{\delta}'}). \end{aligned}$$

By the above, the first summand is here bounded by $D_{\underline{\delta}} R^{m_{\underline{\delta}}}$ independently of $\mu \in \mathfrak{M}$, while for suitable m the sum $\sum_{\underline{\delta}' \prec \underline{\delta}} \mu(g_{m,\underline{\delta}'})$ is bounded independently of μ by the induction hypothesis. We conclude that (13) holds for $\underline{\delta}$. $\square$

We remark that the proof simplifies for the groups $\mathrm{GL}(n)$, since in this case the tempered basic representations $\pi_{\delta,\lambda}$, $\operatorname{Re} \lambda = 0$, are always irreducible, the R-groups are trivial and the sets $\mathcal{D}'(\underline{\delta})$ are therefore singletons. The Paley-Wiener space $\mathcal{PW}_{r,\underline{\delta}}$ is then just the space of W_δ-invariant functions in $\mathcal{PW}(\mathfrak{a}_M)_r$, where (M, δ) is a representative of $\underline{\delta}$ and W_δ denotes the stabilizer of δ inside the Weyl group $W(A_M)$.

7. The spectral limit property

We now come back to the global situation and consider the question whether the collection of measures $\{\mu_K^{G,S_\infty}\}_{K \in \mathcal{K}}$ on $G(F_\infty)^1$ associated to a set $\mathcal{K}$ of open subgroups K of $\mathbf{K}_{\mathrm{fin}}$ is polynomially bounded. We conjecture that this is true for the set of all open subgroups of $\mathbf{K}_{\mathrm{fin}}$. Note that each finite set $\mathcal{K}$ is known to have this property ([41]). So, as in the case of property (TWN) considered above, the issue is to control the dependence on K.

Remark 10. Deitmar and Hoffmann ([20]) have shown unconditionally that for any G the collection of measures $\{\mu^{G,S_\infty}_{\mathbf{K}(\mathfrak{n}),\mathrm{cusp}}\}$, where $\mu^{G,S_\infty}_{\mathbf{K}(\mathfrak{n}),\mathrm{cusp}}$ is the analog of $\mu^{G,S_\infty}_{\mathbf{K}(\mathfrak{n})}$ for the *cuspidal* spectrum, is polynomially bounded. (In fact, they obtain a more precise statement.) However, for our argument we need to know the corresponding statement for the full discrete spectrum.

Our results in this direction are Lemmas 11 and 12 below, which we will use to prove the spectral limit property for principal congruence subgroups in Corollary 3, thereby finishing our argument. Recall the spectral expansion of Theorem 4, which expresses Arthur's distribution $J(h)$ as a sum of contributions $J_{\mathrm{spec},M}(h)$ associated to the conjugacy classes of Levi subgroups M of G. Also recall properties (TWN) and (BD) from §4. The technical heart of our argument is contained in the following lemma. We use freely the notation introduced in §3.

Lemma 10. *Suppose that G satisfies properties (TWN) and (BD). Furthermore, let $M \in \mathcal{L}$, $M \neq G$, and assume that the set of measures $\{\mu^{M,S_\infty}_{\mathbf{K}_M(\mathfrak{n})}\}$ is polynomially bounded. Let $S \supset S_\infty$ be a finite set of places of F. Then for any finite set $\mathcal{F} \subset \Pi(\mathbf{K}_\infty)$ there exists an integer $k \geq 0$ such that for any open subgroup $K_S \subset \mathbf{K}_{S-S_\infty}$, for all $0 < \delta < \dim U_P$ and any $s \in N_G(M)/M$, $\underline{\beta} \in \mathfrak{B}_{P,L_s}$ and $\mathcal{X} \in \Xi_{L_s}(\underline{\beta})$ we have*

$$\int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} \|\Delta_{\mathcal{X}}(P,\lambda)M(P,s)\rho(P,\lambda,h\otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})\|_{1,\mathcal{A}^2(P)}\ d\lambda \ll_{K_S,\mathcal{F},\delta} \|h\|_k \,\mathrm{N}(\mathfrak{n})^{-\delta} \tag{15}$$

for all $h \in \mathcal{H}(G(F_S)^1)_{\mathcal{F},K_S}$ and all integral ideals $\mathfrak{n}$ of $\mathfrak{o}_F$ prime to S, where $\|\cdot\|_1$ denotes the trace norm.

Proof. We argue as in [27, §5] (cf. also [39]). First note that we may omit $M(P,s)$ in (15), since it is a unitary operator and hence does not affect the trace norm. Let Δ be the operator $\mathrm{Id}-\Omega+2\Omega_{\mathbf{K}_\infty}$, where Ω (resp. $\Omega_{\mathbf{K}_\infty}$) is the Casimir operator of $G(F_\infty)$ (resp. $\mathbf{K}_\infty$). Since the groups $\mathbf{K}(\mathfrak{n})$ form a neighborhood base of the identity in $\mathbf{K}_{\mathrm{fin}}$, we can find an ideal $\mathfrak{n}_0$ with $K_S \supset \mathbf{K}_S(\mathfrak{n}_0)$. For any $k > 0$ we bound the left-hand side of (15) by

$$\int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} \|\Delta_{\mathcal{X}}(P,\lambda)\rho(P,\lambda,\Delta)^{-2k}\|_{1,\mathcal{A}^2(P)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\mathcal{F}}} \|\rho(P,\lambda,\Delta^{2k}\star h\otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})})\|\ d\lambda$$
$$\leq \mathrm{vol}(\mathbf{K}^S(\mathfrak{n}))\|\Delta^{2k}\star h\|_{L^1(G(F_S)^1)} \int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} \|\Delta_{\mathcal{X}}(P,\lambda)\rho(P,\lambda,\Delta)^{-2k}\|_{1,\mathcal{A}^2(P)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\mathcal{F}}}\ d\lambda.$$

Consider the integral on the right-hand side.[9] For any $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and $\tau \in \Pi(\mathbf{K}_\infty)$, the operator $\rho(P,\lambda,\Delta)$ acts by the scalar $\mu(\pi,\lambda,\tau) = 1+\|\lambda\|^2-\lambda_\pi+2\lambda_\tau-e_P$ on $\mathcal{A}^2_\pi(P)^\tau$, where e_P is a constant depending only on P (cf. [11, §3.2, (2)]). Since it is easy to see that $e_P \leq 0$, we have

$$\mu(\pi,\lambda,\tau)^2 \geq \frac{1}{4}(1+\|\lambda\|^2+\lambda_\pi^2+\lambda_\tau^2) \geq \frac{1}{4}(1+\|\lambda\|^2+\Lambda_\pi^2).$$

Therefore,

$$\int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} \|\Delta_{\mathcal{X}}(P,\lambda)\rho(P,\lambda,\Delta)^{-2k}\|_{1,\mathcal{A}^2(P)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\mathcal{F}}}\ d\lambda \leq$$
$$\sum_{\tau\in\mathcal{F}}\sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A}))} \int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} \|\Delta_{\mathcal{X}}(P,\lambda)\|_{1,\mathcal{A}^2_\pi(P)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau}}\mu(\pi,\lambda,\tau)^{-2k}\ d\lambda.$$

[9] In the corresponding formula [27, (5.1)] the restriction to the K_0-fixed part was mistakenly omitted.

Estimating $\|A\|_1 \le \dim V \|A\|$ for any linear operator A on a finite-dimensional Hilbert space V, we bound the previous expression by

$$\sum_{\tau\in\mathcal{F}} \sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A}))} \dim \mathcal{A}^2_\pi(P)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau} \int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} \|\Delta_{\mathcal{X}}(P,\lambda)\|_{\mathcal{A}^2_\pi(P)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau}} \mu(\pi,\lambda,\tau)^{-2k}\, d\lambda$$

which is in turn bounded by
(16)

$$\sum_{\substack{\tau\in\mathcal{F},\\ \pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A}))}} (1+\Lambda_\pi)^{-k} \dim \mathcal{A}^2_\pi(P)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau} \int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} (1+\|\lambda\|)^{-k} \prod_{i=1}^m \|\delta_{P_i|P_i'}(\lambda)\big|_{\mathcal{A}^2_\pi(P_i')^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau}}\|\, d\lambda.$$

We first need to estimate the integral over $\mathrm{i}(\mathfrak{a}^G_{L_s})^*$. Let $\mathcal{F}_M \subset \Pi(\mathbf{K}_{M,\infty})$ be the finite set of all irreducible components of restrictions of elements of $\mathcal{F}$ to $\mathbf{K}_{M,\infty}$. Then by Frobenius reciprocity only those $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ with $\pi_\infty \in \Pi(M(F_\infty))^{\mathcal{F}_M}$ can contribute to (16).

Let $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee)$ and introduce the new coordinates $s_i = \langle \lambda, \beta_i^\vee \rangle$, $i = 1, \dots, m$, on $(\mathfrak{a}^G_{L_s,\mathbb{C}})^*$. By (4) we can write

$$\delta_{P_i|P_i'}(\lambda) = \frac{n'_{\beta_i}(\pi,s_i)}{n_{\beta_i}(\pi,s_i)} \operatorname{Id} + j_{P_i'} \circ (\operatorname{Id} \otimes R(\pi,s_i)^{-1} R'(\pi,s_i)) \circ j_{P_i'}^{-1}.$$

Property (TWN) and Proposition 2 (which is based on property (BD)) together yield the estimate

$$\int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} (1+\|\lambda\|)^{-k} \prod_{i=1}^m \|\delta_{P_i|P_i'}(\lambda)\big|_{\mathcal{A}^2_\pi(P_i')^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau}}\|\, d\lambda \ll (1+\Lambda_{\pi_\infty})^N \,\mathrm{N}(\mathfrak{n}_0\mathfrak{n})^\epsilon \tag{17}$$

for all $\epsilon > 0$ and all sufficiently large N and k (possibly depending on τ).

Consider now the dimensions of the spaces of automorphic forms appearing in (16). We have

$$\begin{aligned} \dim \mathcal{A}^2_\pi(P)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau} &= m_\pi \dim \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi)^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau} \\ &= m_\pi \dim \operatorname{Ind}_{P(F_\infty)}^{G(F_\infty)}(\pi_\infty)^\tau \dim \operatorname{Ind}_{P(\mathbb{A}_{\mathrm{fin}})}^{G(\mathbb{A}_{\mathrm{fin}})}(\pi_{\mathrm{fin}})^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n})}, \end{aligned}$$

where

$$m_\pi = \dim \operatorname{Hom}(\pi, L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A}))).$$

Note that here the factor $\dim \operatorname{Ind}_{P(F_\infty)}^{G(F_\infty)}(\pi_\infty)^\tau$ is bounded by $(\dim\tau)^2$. Since $\mathbf{K}(\mathfrak{n}_0\mathfrak{n})$ is a normal subgroup of $\mathbf{K}_{\mathrm{fin}}$, we have

$$\dim \operatorname{Ind}_{P(\mathbb{A}_{\mathrm{fin}})}^{G(\mathbb{A}_{\mathrm{fin}})}(\pi_{\mathrm{fin}})^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n})} = [\mathbf{K}_{\mathrm{fin}} : (\mathbf{K}_{\mathrm{fin}} \cap P(\mathbb{A}_{\mathrm{fin}}))\mathbf{K}(\mathfrak{n}_0\mathfrak{n})] \dim \pi_{\mathrm{fin}}^{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}.$$

Using the factorization $\mathbf{K}_{\mathrm{fin}} \cap P(\mathbb{A}_{\mathrm{fin}}) = (\mathbf{K}_{\mathrm{fin}} \cap M(\mathbb{A}_{\mathrm{fin}}))(\mathbf{K}_{\mathrm{fin}} \cap U(\mathbb{A}_{\mathrm{fin}}))$, we can rewrite this as

$$\begin{aligned} \dim \operatorname{Ind}_{P(\mathbb{A}_{\mathrm{fin}})}^{G(\mathbb{A}_{\mathrm{fin}})}(\pi_{\mathrm{fin}})^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n})} &= \operatorname{vol}(\mathbf{K}(\mathfrak{n}_0\mathfrak{n}))^{-1} [\mathbf{K}_{\mathrm{fin}} \cap U(\mathbb{A}_{\mathrm{fin}}) : \mathbf{K}(\mathfrak{n}_0\mathfrak{n}) \cap U(\mathbb{A}_{\mathrm{fin}})]^{-1} \\ &\qquad \operatorname{vol}(\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})) \dim \pi_{\mathrm{fin}}^{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}. \end{aligned}$$

Identifying U with its Lie algebra $\mathfrak{u}$ via the exponential map, it is easy to see that

$$[\mathbf{K}_{\mathrm{fin}} \cap U(\mathbb{A}_{\mathrm{fin}}) : \mathbf{K}(\mathfrak{n}_0\mathfrak{n}) \cap U(\mathbb{A}_{\mathrm{fin}})]^{-1} \le C \, \mathrm{N}(\mathfrak{n}_0\mathfrak{n})^{-\dim U}$$

for a constant C depending only on G.

Putting things together, we obtain for (15) the bound

$$C(\mathfrak{n}_0, \mathcal{F}, \epsilon) \|h\|_{4k} \, \mathrm{N}(\mathfrak{n})^{\epsilon - \dim U} \operatorname{vol}(\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})) \sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{\mathcal{F}_M}} (1 + |\lambda_{\pi_\infty}|)^{-k} m_\pi \dim \pi_{\mathrm{fin}}^{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}$$

for sufficiently large k. By assumption, the set of measures $\{\mu^{M,S_\infty}_{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}\}$ is polynomially bounded. Therefore the fourth condition of Proposition 6 yields the existence of an integer k, depending only on $\mathcal{F}_M$, such that

$$\mu^{M,S_\infty}_{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}(g_{k,\mathcal{F}_M}) = \frac{\operatorname{vol}(\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n}))}{\operatorname{vol}(M(F)\backslash M(\mathbb{A})^1)} \sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))^{\mathcal{F}_M}} (1 + |\lambda_{\pi_\infty}|)^{-k} m_\pi \dim \pi_{\mathrm{fin}}^{\mathbf{K}_M(\mathfrak{n}_0\mathfrak{n})}$$

is bounded independently of $\mathfrak{n}$. This proves the assertion. □

Remark 11. Note that (6) yields the following improvement of (17):

$$\int_{\mathrm{i}(\mathfrak{a}^G_{L_s})^*} \prod_{i=1}^m (1+\|\lambda\|)^{-2m} \|\delta_{P_i|P_i'}(\lambda)\big|_{\mathcal{A}^2_\pi(P_i')^{\mathbf{K}(\mathfrak{n}_0\mathfrak{n}),\tau}}\| \, d\lambda \ll (1+\log(1+\Lambda_{\pi_\infty})+\log \mathrm{N}(\mathfrak{n}_0\mathfrak{n}))^m. \tag{18}$$

This implies a slightly improved version of Lemma 10, in which the expression $\mathrm{N}(\mathfrak{n})^{-\delta}$ is replaced by $(1 + \log \mathrm{N}(\mathfrak{n}))^m \, \mathrm{N}(\mathfrak{n})^{-\dim U_P}$.

We are now in a position to prove that for $M \in \mathcal{L}$ the collection of measures $\{\mu^{M,S_\infty}_{\mathbf{K}_M(\mathfrak{n})}\}$ on $\Pi(M(F_\infty)^1)$ is polynomially bounded.[10]

Lemma 11. *Let G be anisotropic modulo the center. Then the collection of measures $\{\mu^{G,S_\infty}_K\}$, where K ranges over the open subgroups of $\mathbf{K}_{\mathrm{fin}}$, is polynomially bounded.*

Proof. In this case, the trace formula for a test function $h \in C_c^\infty(G(F_\infty)^1)$ can be written as

$$\operatorname{vol}(G(F)\backslash G(\mathbb{A})^1) \mu^{G,S_\infty}_K(\hat{h}) = \sum_{[\gamma]} \operatorname{vol}(G(F)_\gamma \backslash G(\mathbb{A})^1_\gamma) \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} (h \otimes \mathbf{1}_K)(g^{-1}\gamma g) dg,$$

where summation is over the conjugacy classes of $G(F)$. Clearly, for all h with support contained in a fixed set $\Omega_\infty \subset G(F_\infty)^1$, we can bound the absolute value of the right hand side by

$$\left[\sum_{[\gamma]} \operatorname{vol}(G(F)_\gamma \backslash G(\mathbb{A})^1_\gamma) \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} (\mathbf{1}_{\Omega_\infty \mathbf{K}_{\mathrm{fin}}})(g^{-1}\gamma g) dg\right] \sup|h|$$

independently of K. Therefore, $\sup|\mu^{G,S_\infty}_K(\hat{h})|$ is a continuous seminorm on every space $\mathcal{H}(G(F_\infty)^1)_{r,\mathcal{F}}$, and by Proposition 6 we obtain the assertion. □

[10]Variants of Lemma 11 have been previously established in [22, Proposition 3.3] and [20].

Lemma 12. *Suppose that G satisfies (TWN) and (BD). Then for each $M \in \mathcal{L}$ the collection of measures $\{\mu^{M,S_\infty}_{\mathbf{K}_M(\mathfrak{n})}\}$, $\mathfrak{n}$ ranging over the integral ideals of $\mathfrak{o}_F$, is polynomially bounded.*

Proof. We do induction over the semisimple rank of M, using Lemma 11 as the base. For the induction step, we can assume the assertion for all groups in $\mathcal{L}\backslash\{G\}$ and have the task to establish it for G itself. Fix $r > 0$ and apply the trace formula to $h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})}$, where $h \in \mathcal{H}(G(F_\infty)^1)_{r,\mathcal{F}}$. By Theorem 4, we have

$$\operatorname{vol}(G(F)\backslash G(\mathbb{A})^1)\mu^{G,S_\infty}_{\mathbf{K}(\mathfrak{n})}(\hat{h}) = J(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})}) - \sum_{[M],\, M\neq G} J_{\mathrm{spec},M}(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})}).$$

Now, for each single choice of $\mathfrak{n}$ the absolute value $|J(h\otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})})|$ is a continuous seminorm by Arthur ([9]). Moreover, as in the proof of Corollary 1, for all $\mathfrak{n}$ outside of a finite set depending only on r we have $J(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})}) = J_{\mathrm{unip}}(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})})$. Therefore it follows from our analysis of the geometric side in Proposition 3 that $\sup_{\mathfrak{n}}|J(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})})|$ is a continuous seminorm on $\mathcal{H}(G(F_\infty)^1)_{r,\mathcal{F}}$. By Theorem 4, for each $M \neq G$ the absolute value $|J_{\mathrm{spec},M}(h\otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})})|$ is (up to a constant depending only on G) bounded by a finite sum of integrals of the form (15). Applying Lemma 10 (with $S = S_\infty$), we see that the spectral terms $\sup_{\mathfrak{n}}|J_{\mathrm{spec},M}(h \otimes \mathbf{1}_{\mathbf{K}(\mathfrak{n})})|$ are also continuous seminorms on $\mathcal{H}(G(F_\infty)^1)_{r,\mathcal{F}}$. By Proposition 6, we conclude that the collection $\{\mu^{G,S_\infty}_{\mathbf{K}(\mathfrak{n})}\}$ is polynomially bounded. □

As before, let S be a finite set of places containing S_∞.

Corollary 3 (Spectral limit property)**.** *Suppose that G satisfies (TWN) and (BD). Then we have the spectral limit property for the set of subgroups $\mathbf{K}^S(\mathfrak{n})$, where $\mathfrak{n}$ ranges over the integral ideals of $\mathfrak{o}_F$ prime to S.*

Proof. From Lemma 12 we get that for each $M \in \mathcal{L}$ the collection of measures $\{\mu^{M,S_\infty}_{\mathbf{K}_M(\mathfrak{n})}\}$ is polynomially bounded. Therefore we can apply Theorem 4 and Lemma 10 again to conclude that for each $h \in \mathcal{H}(G(F_S)^1)$ we have

$$J_{\mathrm{spec},M}(h \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) \to 0$$

for all $M \neq G$ and therefore

$$J(h \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) - \operatorname{tr} R_{\mathrm{disc}}(h \otimes \mathbf{1}_{\mathbf{K}^S(\mathfrak{n})}) \to 0,$$

which is the spectral limit property. □

Theorem 7. *Suppose that G satisfies (TWN) and (BD). Then limit multiplicity holds for the set of subgroups $\mathbf{K}^S(\mathfrak{n})$, where $\mathfrak{n}$ ranges over the integral ideals of $\mathfrak{o}_F$ prime to S.*

Proof. The geometric limit property has been established in Corollary 1, and the spectral limit property in Corollary 3. By Theorem 1, we obtain the result. □

This finishes also the proof of Theorem 2, since (TWN) and (BD) have been verified for $G = \mathrm{GL}(n)$ in Proposition 1 and Theorem 5, respectively.

Remark 12. In the situation of Theorem 7 we actually obtain the following quantitative statement: for any finite set $\mathcal{F} \subset \Pi(\mathbf{K}_\infty)$ there exists an integer $k \geq 0$ such that for any open subgroup $K_S \subset \mathbf{K}_{S-S_\infty}$ we have

$$|\mu^{G,S}_{\mathbf{K}^S(\mathfrak{n})}(\hat{h}) - h(1)| \ll_{K_S,\mathcal{F}} \frac{(1+\log \mathrm{N}(\mathfrak{n}))^{d_0}}{\mathrm{N}(\mathfrak{n})} \|h\|_k \tag{19}$$

for all $h \in \mathcal{H}(G(F_S)^1)_{\mathcal{F},K_S}$ and all integral ideals $\mathfrak{n}$ of $\mathfrak{o}_F$ prime to S. (Here we use that $\dim U_P = 1$ can occur only if the derived group G' is isogenous to $\mathrm{SL}(2)$, and that (6) is known in this case.) We expect that one should be able to replace $\mathrm{N}(\mathfrak{n})$ in the denominator by $\mathrm{N}(\mathfrak{n})^{d_{\min}/2}$, where $d_{\min}$ is the dimension of the minimal unipotent orbit of G. In view of Lemma 10 this would follow from a corresponding improvement of Proposition 3. To deduce from (19) an estimate for $|\mu^{G,S}_{\mathbf{K}^S(\mathfrak{n})}(A) - \mu_{\mathrm{pl}}(A)|$ for subsets $A \subset \Pi(G)$, a quantitative version of the density principle (Theorem 3) would be necessary.

References

[1] Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet. On the growth of Betti numbers of locally symmetric spaces. *C. R. Math. Acad. Sci. Paris*, 349(15-16):831–835, 2011.

[2] James Arthur. Eisenstein series and the trace formula. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 253–274. Amer. Math. Soc., Providence, R.I., 1979.

[3] James Arthur. A trace formula for reductive groups. II. Applications of a truncation operator. *Compositio Math.*, 40(1):87–121, 1980.

[4] James Arthur. The trace formula in invariant form. *Ann. of Math. (2)*, 114(1):1–74, 1981.

[5] James Arthur. On a family of distributions obtained from Eisenstein series. I. Application of the Paley-Wiener theorem. *Amer. J. Math.*, 104(6):1243–1288, 1982.

[6] James Arthur. On a family of distributions obtained from Eisenstein series. II. Explicit formulas. *Amer. J. Math.*, 104(6):1289–1336, 1982.

[7] James Arthur. A measure on the unipotent variety. *Canad. J. Math.*, 37(6):1237–1274, 1985.

[8] James Arthur. On a family of distributions obtained from orbits. *Canad. J. Math.*, 38(1):179–214, 1986.

[9] James G. Arthur. A trace formula for reductive groups. I. Terms associated to classes in $G(\mathbf{Q})$. *Duke Math. J.*, 45(4):911–952, 1978.

[10] Dan Barbasch and Henri Moscovici. L^2-index and the Selberg trace formula. *J. Funct. Anal.*, 53(2):151–201, 1983.

[11] A. Borel and H. Garland. Laplacian and the discrete spectrum of an arithmetic group. *Amer. J. Math.*, 105(2):309–335, 1983.

[12] Peter Borwein and Tamás Erdélyi. Sharp extensions of Bernstein's inequality to rational spaces. *Mathematika*, 43(2):413–423 (1997), 1996.

[13] C. J. Bushnell and G. Henniart. An upper bound on conductors for pairs. *J. Number Theory*, 65(2):183–196, 1997.

[14] L. Clozel and P. Delorme. Le théorème de Paley-Wiener invariant pour les groupes de Lie réductifs. *Invent. Math.*, 77(3):427–453, 1984.

[15] Laurent Clozel. On limit multiplicities of discrete series representations in spaces of automorphic forms. *Invent. Math.*, 83(2):265–284, 1986.

[16] Laurent Clozel and Patrick Delorme. Le théorème de Paley-Wiener invariant pour les groupes de Lie réductifs. II. *Ann. Sci. École Norm. Sup. (4)*, 23(2):193–228, 1990.

[17] David L. de George and Nolan R. Wallach. Limit formulas for multiplicities in $L^2(\Gamma\backslash G)$. *Ann. of Math. (2)*, 107(1):133–150, 1978.

[18] David L. DeGeorge and Nolan R. Wallach. Limit formulas for multiplicities in $L^2(\Gamma\backslash G)$. II. The tempered spectrum. *Ann. of Math. (2)*, 109(3):477–495, 1979.

[19] David Lee DeGeorge. On a theorem of Osborne and Warner. Multiplicities in the cuspidal spectrum. *J. Funct. Anal.*, 48(1):81–94, 1982.

[20] Anton Deitmar and Werner Hoffman. Spectral estimates for towers of noncompact quotients. *Canad. J. Math.*, 51(2):266–293, 1999.

[21] Anton Deitmar and Werner Hoffmann. On limit multiplicities for spaces of automorphic forms. *Canad. J. Math.*, 51(5):952–976, 1999.

[22] Patrick Delorme. Formules limites et formules asymptotiques pour les multiplicités dans $L^2(G/\Gamma)$. *Duke Math. J.*, 53(3):691–731, 1986.

[23] Jacques Dixmier. *Les C^*-algèbres et leurs représentations*. Deuxième édition. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars Éditeur, Paris, 1969.

[24] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan. Spectra of compact locally symmetric manifolds of negative curvature. *Invent. Math.*, 52(1):27–93, 1979.

[25] Tobias Finis and Erez Lapid. On the spectral side of Arthur's trace formula—combinatorial setup. *Ann. of Math. (2)*, 174(1):197–223, 2011.

[26] Tobias Finis, Erez Lapid, and Werner Müller. On the degrees of matrix coefficients of intertwining operators. available at `http://www.ma.huji.ac.il/~erezla/publications.html`.

[27] Tobias Finis, Erez Lapid, and Werner Müller. On the spectral side of Arthur's trace formula—absolute convergence. *Ann. of Math. (2)*, 174(1):173–195, 2011.

[28] Henryk Iwaniec. Nonholomorphic modular forms and their applications. In *Modular forms (Durham, 1983)*, Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., pages 157–196. Horwood, Chichester, 1984.

[29] Henryk Iwaniec. Small eigenvalues of Laplacian for $\Gamma_0(N)$. *Acta Arith.*, 56(1):65–82, 1990.

[30] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika. Conducteur des représentations du groupe linéaire. *Math. Ann.*, 256(2):199–214, 1981.

[31] A. W. Knapp. Weyl group of a cuspidal parabolic. *Ann. Sci. École Norm. Sup. (4)*, 8(2):275–294, 1975.

[32] A. W. Knapp and Gregg J. Zuckerman. Classification of irreducible tempered representations of semisimple groups. *Ann. of Math. (2)*, 116(2):389–455, 1982.

[33] A. W. Knapp and Gregg J. Zuckerman. Classification of irreducible tempered representations of semisimple groups. II. *Ann. of Math. (2)*, 116(3):457–501, 1982.

[34] Serge Lang. *Algebraic number theory*, volume 110 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1994.

[35] Robert P. Langlands. *Euler products*. Yale University Press, New Haven, Conn., 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1967, Yale Mathematical Monographs, 1.

[36] Alexander Lubotzky. Subgroup growth and congruence subgroups. *Invent. Math.*, 119(2):267–295, 1995.

[37] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[38] C. Mœglin and J.-L. Waldspurger. Le spectre résiduel de $\mathrm{GL}(n)$. *Ann. Sci. École Norm. Sup. (4)*, 22(4):605–674, 1989.

[39] W. Müller. On the spectral side of the Arthur trace formula. *Geom. Funct. Anal.*, 12(4):669–722, 2002.

[40] W. Müller and B. Speh. Absolute convergence of the spectral side of the Arthur trace formula for GL_n. *Geom. Funct. Anal.*, 14(1):58–93, 2004. With an appendix by E. M. Lapid.
[41] Werner Müller. The trace class conjecture in the theory of automorphic forms. *Ann. of Math. (2)*, 130(3):473–529, 1989.
[42] Werner Müller. Weyl's law for the cuspidal spectrum of SL_n. *Ann. of Math. (2)*, 165(1):275–333, 2007.
[43] Morris Newman. A complete description of the normal subgroups of genus one of the modular group. *Amer. J. Math.*, 86:17–24, 1964.
[44] M. Scott Osborne and Garth Warner. *The theory of Eisenstein systems*, volume 99 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.
[45] R. Phillips and P. Sarnak. The spectrum of Fermat curves. *Geom. Funct. Anal.*, 1(1):80–146, 1991.
[46] Jürgen Rohlfs and Birgit Speh. On limit multiplicities of representations with cohomology in the cuspidal spectrum. *Duke Math. J.*, 55(1):199–211, 1987.
[47] Peter Sarnak. A note on the spectrum of cusp forms for congruence subgroups. *preprint*, 1983.
[48] François Sauvageot. Principe de densité pour les groupes réductifs. *Compositio Math.*, 108(2):151–184, 1997.
[49] Gordan Savin. Limit multiplicities of cusp forms. *Invent. Math.*, 95(1):149–159, 1989.
[50] Sug Woo Shin. Plancherel density theorem for automorphic representations. *Israel J. Math.*, to appear.
[51] David A. Vogan, Jr. *Representations of real reductive Lie groups*, volume 15 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1981.
[52] Nolan R. Wallach. Limit multiplicities in $L^2(\Gamma\backslash G)$. In *Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989)*, volume 1447 of *Lecture Notes in Math.*, pages 31–56. Springer, Berlin, 1990.

Freie Universität Berlin, Institut für Mathematik, Arnimallee 3, D-14195 Berlin, Germany
E-mail address: finis@math.fu-berlin.de

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel
E-mail address: erezla@math.huji.ac.il

Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, D-53115 Bonn, Germany
E-mail address: mueller@math.uni-bonn.de