THE ASYMPTOTICS OF THE RAY-SINGER ANALYTIC TORSION OF HYPERBOLIC 3-MANIFOLDS

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Abstract. In this paper we consider the analytic torsion of a closed hyperbolic 3-manifold associated with the m-th symmetric power of the standard representation of $\mathrm{SL}(2,\mathbb{C})$ and we study its asymptotic behavior if m tends to infinity. The leading coefficient of the asymptotic formula is given by the volume of the hyperbolic 3-manifold. It follows that the Reidemeister torsion associated with the symmetric powers determines the volume of a closed hyperbolic 3-manifold.

1. Introduction

Let X be a closed, oriented hyperbolic 3-manifold. Then there exists a discrete, torsion free, co-compact subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ such that $X = \Gamma\backslash\mathbb{H}^3$, where $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ is the 3-dimensional hyperbolic space. Let ρ be a finite-dimensional complex representation of Γ and let $E_\rho \to X$ be the associated flat vector bundle. Choose a Hermitian fiber metric h on E_ρ. Let $T_X(\rho; g, h)$ denote the Ray-Singer analytic torsion of the de Rham complex of E_ρ-valued differential forms [RS], where g denotes the hyperbolic metric. If ρ is acyclic, then $T_X(\rho; g, h)$ is metric independent [Mu1, Corollary 2.7]. In this case we denote it by $T_X(\rho)$.

For $m \in \mathbb{N}$ let $\tau_m = \mathrm{Sym}^m$ be the m-th symmetric power of the standard representation of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ and denote by E_{τ_m} the flat vector bundle associated to $\tau_m|_\Gamma$. It is well known that $H^*(X, E_{\tau_m}) = 0$. This follows, for example, from [BW, Chapt. VII, Theorem 6.7]. Hence the restriction of τ_m to Γ is an acyclic representation of Γ. Denote by $T_X(\tau_m)$ the analytic torsion with respect to $\tau_m|_\Gamma$. The purpose of this paper is to study the asymptotic behavior of $T_X(\tau_m)$ as $m \to \infty$. Our main result is the following theorem.

Theorem 1.1. *Let X be a closed, oriented hyperbolic 3-manifold. Then we have*

$$-\log T_X(\tau_m) = \frac{\mathrm{vol}(X)}{4\pi} m^2 + O(m) \tag{1.1}$$

as $m \to \infty$.

We note that there is an analogous result in the holomorphic setting. In [BV] Bismut and Vasserot studied the asymptotic behavior of the holomorphic Ray-Singer torsion for symmetric powers of a positive vector bundle.

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Let $\tau_X(\tau_m)$ denote the Reidemeister torsion of X with respect to $\tau_m|_\Gamma$ (see [Mu1]). Then by [Mu1, Theorem 1] we have $T_X(\tau_m) = \tau_X(\tau_m)$. Thus we obtain the following corollary.

Corollary 1.2. *Let X be a closed, oriented hyperbolic 3-manifold. Then we have*

$$-\log \tau_X(\tau_m) = \frac{\operatorname{vol}(X)}{4\pi} m^2 + O(m) \tag{1.2}$$

as $m \to \infty$.

This result has applications to the cohomology of arithmetic hyperbolic 3-manifolds. We will discuss this elsewhere. As an immediate corollary we get

Corollary 1.3. *Let X be a closed, oriented hyperbolic 3-manifold. Then* $\operatorname{vol}(X)$ *is determined by the set of Reidemeister torsions $\{\tau_X(\tau_m) \colon m \in \mathbb{N}\}$.*

Some remarks are in order. The Reidemeister torsion of a compact 3-manifold is known to be a topological invariant [Ch]. Therefore, it follows from Corollary 1.3 that the volume of a compact, oriented hyperbolic 3-manifold is a topological invariant. This is also a well known consequence of the Mostow-Prasad rigidity theorem [Mo, Pr].

There are only finitely many closed, oriented hyperbolic 3-manifolds with the same volume [T1, Theorem 3.6]. Therefore we get

Corollary 1.4. *A compact, oriented hyperbolic 3-manifold X is determined up to finitely many possibilities by the set $\{\tau_X(\tau_m) \colon m \in \mathbb{N}\}$ of Reidemeister torsion invariants.*

It is known [Zi] that the number of closed hyperbolic manifolds with a given volume can be arbitrarily large. Therefore the proof of the corollary does not give a uniform bound on the number of closed hyperbolic manifolds with the same set of Reidemeister torsion invariants.

Our approach to prove Theorem 1.1 is based on the expression of $T_X(\tau_m)$ in terms of the twisted Ruelle zeta function attached to τ_m. Recall that for a finite-dimensional complex representation ρ of Γ the twisted Ruelle zeta function $R_\rho(s)$ is defined for $\operatorname{Re}(s) \gg 0$ as the infinite product

$$R_\rho(s) = \prod_{\substack{[\gamma]\neq e \\ \text{prime}}} \det\left(\operatorname{Id} - \rho(\gamma) e^{-s\ell(\gamma)}\right), \tag{1.3}$$

where $[\gamma]$ runs over the nontrivial primitive conjugacy classes of Γ and $\ell(\gamma)$ denotes the length of the unique closed geodesic associated to $[\gamma]$. It follows from [Fr2] that $R_\rho(s)$ admits a meromorphic extension to the whole complex plane. If ρ is unitary and acyclic then $R_\rho(s)$ is regular at $s = 0$ and its value at zero satisfies

$$|R_\rho(0)| = T_X(\rho)^2$$

(see [Fr]). For an arbitrary unitary representation (which is not necessarily acyclic), the coefficient of the leading term of the Laurent expansion of $R_\rho(s)$ at $s = 0$ is given by the analytic torsion. The corresponding result holds for any compact, oriented hyperbolic manifold of dimension n [Fr]. In his thesis [Br] U. Bröcker has established a similar result for representations of the fundamental group that are restrictions of finite-dimensional irreducible representations of the isometry group $\mathrm{SO}_0(n,1)$ of the hyperbolic n-space. Unfortunately, his method is based on elaborate computations which are difficult to verify. This problem has been rectified by Wotzke in his thesis [Wo]. He gave a different proof which replaces Bröcker's explicite computations by the real version of Kostant's Bott-Borel-Weil theorem [Si].

To state the result for $n=3$ we need to introduce some notation. Let τ be a finite-dimensional, irreducible representation of $\mathrm{SL}(2,\mathbb{C})$, which we regard as real Lie group. Let θ be the Cartan involution of $\mathrm{SL}(2,\mathbb{C})$ with respect to $SU(2)$. Put $\tau_\theta=\tau\circ\theta$. Denote by $R_\tau(s)$ the twisted Ruelle zeta function for the restriction of τ to Γ. Let $E_\tau\to X$ be the flat vector bundle associated to $\tau|_\Gamma$. The flat bundle E_τ can be equipped with a canonical Hermitian fiber metric [MM]. Let $\Delta_p(\tau)$ be the corresponding Laplacian on E_τ-valued p-forms and denote by $T_X(\tau)$ the Ray-Singer analytic torsion associated to $\tau|_\Gamma$. Then the main result of [Wo] for $n=3$ is the following theorem.

Theorem 1.5. *Let τ be a finite-dimensional, irreducible representation of the real Lie group $\mathrm{SL}(2,\mathbb{C})$. Then we have*
1) *If $\tau_\theta\not\cong\tau$, then $R_\tau(s)$ is regular at $s=0$ and*

$$|R_\tau(0)|=T_X(\tau)^2.$$

2) *Let $\tau_\theta=\tau$. If $\tau\neq 1$, then the order $h(\tau)$ of $R_\tau(s)$ at $s=0$ is given by*

$$h(\tau)=-2\sum_{p=1}^{3}(-1)^p p\dim\ker\Delta_p(\tau). \tag{1.4}$$

and if $\tau=1$, the order equals $4-2\dim H^1(X,\mathbb{R})$. The leading term of the Laurent expansion of $R_\tau(s)$ at $s=0$ is given by

$$T_X(\tau)^2 s^{h(\tau)}.$$

The case of the trivial representation is covered by [Fr]. In this case the order of $R(s)$ at $s=0$ differs from (1.4).

It follows from Theorem 1.5 that in order to prove Theorem 1.1 it suffices to analyze the asymptotic behavior of $R_{\tau_m}(0)$ as $m\to\infty$. For this purpose we consider another type of twisted Ruelle zeta functions. Let A be the standard split torus of $\mathrm{SL}(2,\mathbb{C})$ and let M be the centralizer of A in $\mathrm{SU}(2)$ (see (2.1) for the explicit description). For $\sigma\in\hat{M}$ let $R(s,\sigma)$ be the Ruelle zeta function defined by (3.6). Using the decomposition of τ_m under the Cartan subgroup MA, it follows that $R_{\tau_m}(s)$ is the product of the twisted Ruelle zeta functions with shifted argument $R(s-(m/2-k),\sigma_{m-2k})$, $k=0,...,m$. This reduces the study of the asymptotic behavior of $T_X(\tau_{2m})$ (resp. $T_X(\tau_{2m+1})$) as $m\to\infty$ to the study of the behavior of $|R(k,\sigma_{2k})|$ and $|R(-k,\sigma_{2k})|$, (resp. $|R(k+1/2,\sigma_{2k+1})|$ and $|R(-k-1/2,\sigma_{-(2k+1)})|$), $k>2$, as $k\to\infty$. To analyze the behavior of $|R(k,\sigma_{2k})|$ (resp.

$|R(k+1/2, \sigma_{2k+1})|)$ as $k \to \infty$ we simply use the infinite product defining it. To deal with the remaining cases we use the functional equation which implies

$$|R(-s, \sigma_k)| = \exp\left(-4\pi^{-1} \operatorname{vol}(X) \operatorname{Re}(s)\right) |R(s, \sigma_{-k})|.$$

This is exactly how the volume of X appears in the asymptotic formula (1.1).

For the sake of completeness we include a proof of Theorem 1.5 which is based on results of [BO]. The starting point of the method is the observation that the flat bundle $E_\tau \to X$ is isomorphic to the locally homogeneous vector bundle defined by the restriction of τ to the maximal cocompact subgroup SU(2) of $\mathrm{SL}(2,\mathbb{C})$ (see [MM, Propostion 3.1]). Using this isomorphism the bundle E_τ can be equipped with a canonical Hermitian fiber metric induced from an invariant metric on the corresponding homogeneous vector bundle [MM, Lemma 3.1]. We define the Laplacian $\Delta_p(\tau)$ on E_τ-valued p-forms with respect to this metric on E_τ. Then, up to a constant, $\Delta_p(\tau)$ equals $-R(\Omega)$, where $R(\Omega)$ denotes the action of the Casimir operator on sections of the locally homogeneous bundle. This is the key fact which allows us to apply the Selberg trace formula to the heat kernel.

The paper is organized as follows. In section 2 we summerize some basic facts about hyperbolic 3-manifolds and analytic torsion. In section 3 we consider twisted Ruelle and Selberg zeta functions and establish some of their basic properties. In section 4 we introduce certain auxiliary elliptic operators which are needed to derive the determinant formula and to prove the functional equation for the Selberg zeta function. In the next section 5 we establish the functional equation for the Selberg and Ruelle zeta functions. In section 6 we use the determinant formula of [BO] to express the twisted Ruelle zeta function as a ratio of products of regularized determinants of the elliptic operators introduced in section 5. The determinant formula is one of main tools to study the leading term of the Laurent expansion of $R_\tau(s)$ at $s = 0$. In section 7 we give a proof of Theorem 1.5. In the final section 8 we proof Theorem 1.1.

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2. Preliminaries

2.1. Let $G = \mathrm{SL}(2,\mathbb{C})$ and $K = \mathrm{SU}(2)$. Then K is a maximal compact subgroup of G. We regard G as real Lie group and we recall that G is isomorphic to $\mathrm{Spin}_0(3,1)$ [Be]. Under this isomorphism, SU(2) is mapped to Spin(3). Thus G acts on the hyperbolic 3-space $\mathbb{H}^3$ and $\mathbb{H}^3 \cong G/K$. Let $\Gamma \subset G$ be a cocompact, torsion free discrete subgroup. Then $X = \Gamma\backslash\mathbb{H}^3$ is a compact, oriented hyperbolic 3-manifold and any such manifold is of this form.

Let $G = NAK$ be the standard Iwasawa decomposition of G and let M be the centralizer of A in K. Then

$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R}^+ \right\}, \quad M = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi] \right\}. \tag{2.1}$$

We use the natural normalization of the Haar measures for A, N, K and G as in [Kn, pp. 387-388]. In particular, we choose on K the Haar measure dk of total mass 1.

Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{n}$ denote the Lie algebras of G, K, A, M and N, respectively. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{2.2}$$

be the Cartan decomposition of $(\mathfrak{g}, \mathfrak{a})$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. Let α be the unique positive root of $(\mathfrak{g}, \mathfrak{a})$. Let $H \in \mathfrak{a}$ be such that $\alpha(H) = 1$. Let $\mathfrak{a}^+ \subset \mathfrak{a}$ be the positive Weyl chamber and let $A^+ = \exp(\mathfrak{a}^+)$. Let $W := W(\mathfrak{g}, \mathfrak{a})$ denote the restricted Weyl group.

Put $\mathfrak{h} = \mathfrak{m} \oplus \mathfrak{a}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. We identify $\mathfrak{h}$ with $\mathbb{R}^2$. Then the Weyl group W_G of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ acts on $\mathbb{C}^2$ by sign changes. So W_G has order 4. In a compatible ordering on $\mathfrak{h}_\mathbb{C}^*$ the only positive roots of the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ are α_1 and α_2 where $\alpha_1(H) = \alpha_2(H) = \alpha(H) = 1$ and $\alpha_1(iH) = -\alpha_2(iH) = i$. Let $\rho_G = \frac{1}{2}(\alpha_1 + \alpha_2)$.

Let B be the Killing form of $\mathfrak{g}$. Define a symmetric bilinear form on $\mathfrak{g}$ by

$$\langle Y_1, Y_2 \rangle = \frac{1}{4} B(Y_1, Y_2), \quad Y_1, Y_2 \in \mathfrak{g}. \tag{2.3}$$

Then $\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{p}$, negative definite on $\mathfrak{k}$ and we have $\langle \mathfrak{k}, \mathfrak{p} \rangle = 0$. The normalization is such that the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{p} \cong T_{eK}(G/K)$ induces the G-invariant Riemannian metric on $\mathbb{H}^3 = G/K$ which has constant curvature -1.

Let $\{Z_i\}$ be a basis of $\mathfrak{g}$ and let $\{Z^j\}$ be the basis of $\mathfrak{g}$ which is determined by $\langle Z_i, Z^j \rangle = \delta_{ij}$. Then the Casimir element $\Omega \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ is given by

$$\Omega = \sum_i Z_i Z^i. \tag{2.4}$$

Let $R(\Omega)$ be the differential operator induced by Ω on $C^\infty(\mathbb{H}^3)$. Then by Kuga's lemma we have

$$R(\Omega) = -\Delta,$$

where Δ is the hyperbolic Laplace operator on functions.

2.2. Let $\Gamma \subset G$ be a discrete, torsion free, cocompact subgroup. Then $X = \Gamma \backslash \mathbb{H}^3$ is a closed hyperbolic manifold. Given $\gamma \in \Gamma$, we denote by $[\gamma]$ the Γ-conjugacy class of γ. The set of all conjugacy classes of Γ will be denoted by $C(\Gamma)$. Let $\gamma \neq 1$. Then there exist $g \in G$, $m_\gamma \in M$, and $a_\gamma \in A^+$ such that

$$g \gamma g^{-1} = m_\gamma a_\gamma. \tag{2.5}$$

By [Wa1, Lemma 6.6], a_γ depends only on γ and m_γ is determined up to conjugacy in M. By definition there exists $\ell(\gamma) > 0$ such that

$$a_\gamma = \exp\left(\ell(\gamma) H\right). \tag{2.6}$$

Then $\ell(\gamma)$ is the length of the unique closed geodesic in X that corresponds to the conjugacy class $\{\gamma\}$. An element $\gamma \in \Gamma - \{e\}$ is called primitive, if it can not be written as $\gamma = \gamma_0^k$

for some $\gamma_0 \in \Gamma$ and $k > 1$. For every $\gamma \in \Gamma - \{e\}$ there exist a unique primitive element $\gamma_0 \in \Gamma$ and $n_\Gamma(\gamma) \in \mathbb{N}$ such that $\gamma = \gamma_0^{n_\Gamma(\gamma)}$.

2.3. Denote by $\hat{M}$ the set of unitary characters of M. Then $\hat{M} \cong \mathbb{Z}$ and the character σ_k that corresponds to $k \in \mathbb{Z}$ is given by

$$\sigma_k\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) = e^{ik\theta}. \tag{2.7}$$

The finite-dimensional irreducible representations of G, regarded as real Lie group, are parametrized by pairs of nonnegative integers [Kn, p. 32]. For $m \in \mathbb{N}_0$ let

$$\tau_m = \mathrm{Sym}^m \colon G \to \mathrm{GL}(S^m(\mathbb{C}^2))$$

be the m-th symmetric power of the standard representation of $G = \mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$. Denote by $\overline{\tau}_m$ the complex conjugate representation. Then

$$\tau_{m,n} = \tau_m \otimes \overline{\tau}_n \tag{2.8}$$

is the irreducible representation with highest weight (m,n). The restrictions of τ_m to MA decomposes as follows:

$$\tau_m\big|_{MA} = \bigoplus_{k=0}^{m} \sigma_{m-2k} \otimes e^{(\frac{m}{2}-k)\alpha}. \tag{2.9}$$

2.4. Let $P = MAN$ be the standard parabolic subgroup of G. We identify $\mathbb{C}$ with $\mathfrak{a}^*_\mathbb{C}$ by $z \to zH$. For $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ let $\pi_{n,\lambda}$ be the induced representation

$$\pi_{n,\lambda} = \mathrm{Ind}_P^G(\sigma_n \otimes e^{i\lambda} \otimes 1). \tag{2.10}$$

Note that this is the parametrization of the principal series used in [Kn, Chapt. XI, §2]. The representation $\pi_{n,\lambda}$ acts in the Hilbert space $\mathcal{H}_{n,\lambda}$ whose subspace of C^∞-vectors is given by

(2.11)

$$\mathcal{H}^\infty_{n,\lambda} = \left\{ f \in C^\infty(G, V_{\sigma_n}) \colon f(gman) = e^{-(i\lambda+1)(\log a)} \sigma_n(m)^{-1} f(g), \quad g \in G,\ man \in P \right\}.$$

If $\lambda \in \mathbb{R}$, then $\pi_{n,\lambda}$ is unitary. This family of representations is the unitary principal series. All $\pi_{n,\lambda}$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{R} \setminus \{0\}$, are irreducible. They have an explicit realization [Kn, Chapt. II, §4]. The Casimir eigenvalue $\pi_{n,\lambda}(\Omega)$ is given by

$$\pi_{n,\lambda}(\Omega) = -\lambda^2 + \frac{n^2}{4} - 1. \tag{2.12}$$

This follows from [Kn, Theorem 8.22]. It can be also verified using the explicit realization of $\pi_{n,\lambda}$. In the latter case one has to take into account that the identification of $\mathfrak{a}_\mathbb{C}$ with $\mathbb{C}$ is differnt from ours.

The nonunitarily induced representations

$$\pi^c_x = \mathrm{Ind}_P^G(1 \otimes e^x \otimes 1), \quad 0 < x < 1, \tag{2.13}$$

are unitarizable. This is the complementary series. The Casimir eigenvalue is given by

$$\pi_x^c(\Omega) = x^2 - 1. \tag{2.14}$$

This also follows from [Kn, Theorem 8.22] or can be verified using the explicit realization of π_x^c (see [Kn, Chapt. II, §4]). Denote by $\Theta_{n,\lambda} = \operatorname{tr} \pi_{n,\lambda}$ the character of $\pi_{n,\lambda}$.

2.5. Let $\tau \colon G \to \mathrm{GL}(V_\tau)$ be an irreducible finite-dimensional representation of G. Let E_τ be the flat vector bundle associated to the restriction $\tau|_\Gamma$ of τ to Γ. By [MM, Proposition 3.1] E_τ is canonically isomorphic to the locally homogeneous vector bundle associated to $\tau|_K$, i.e.,

$$E_\tau \cong (\Gamma\backslash G \times V_\tau)/K, \tag{2.15}$$

where K acts on $\Gamma\backslash G \times V_\tau$ by $(\Gamma g, v)\cdot k = (\Gamma g k, \tau(k)^{-1}v)$. So we may regard E_τ as locally homogeneous vector bundle equipped with a flat connection which, of course, is different from the canonical invariant connection on the homogeneous bundle.

The vector bundle E_τ can be equipped with a canonical fiber metric. By [MM, Lemma 3.1] there exists an inner product $\langle\cdot,\cdot\rangle$ on V_τ which satisfies

$$\begin{aligned} &(a) \qquad \langle \tau(Y)u, v\rangle = -\langle u, \tau(Y)v\rangle \text{ for all } Y \in \mathfrak{k},\ u, v \in V_\tau;\\ &(b) \qquad \langle \tau(Y)u, v\rangle = \langle u, \tau(Y)v\rangle \quad \text{for all } Y \in \mathfrak{p},\ u, v \in V_\tau. \end{aligned} \tag{2.16}$$

Such an inner product is called *admissible*. It is unique up to scaling. By (a) the inner product is invariant under $\tau(K)$ and therefore, it defines via (2.15) a Hermitian fiber metric in E_τ. Denote by $\Delta_p(\tau)$ the Laplacian on E_τ-valued p-forms with respect to an admissible metric on E_τ.

2.6. Let P be an elliptic differential operator acting on C^∞-sections of a smooth Hermitian vector bundle E over a compact Riemannian manifold X. The metrics g on X and h on E induce an inner product in $C^\infty(X,E)$. Suppose that with respect to this inner product the operator P is symmetric and nonnegative. Then the zeta function $\zeta(s;P)$, $s \in \mathbb{C}$, of P is defined as

$$\zeta(s;P) = \sum_{\lambda\in \operatorname{Spec}(P)\setminus\{0\}} m(\lambda)\lambda^{-s},$$

where $m(\lambda)$ denotes the multiplicity of the eigenvalue λ. The series converges absolutely and uniformly on compact subsets of $\operatorname{Re}(s) > \dim(X)/\operatorname{ord}(P)$. Moreover $\zeta(s;P)$ admits a meromorphic extension to $s \in \mathbb{C}$ which is holomorphic at $s = 0$ (see [Sh, Chapt. II]). Then the regularized determinant $\det P$ of P is defined as

$$\det P = \exp\left(-\frac{d}{ds}\zeta(s;P)\big|_{s=0}\right). \tag{2.17}$$

Assume that P is symmetric and bounded from below. Let $\lambda \in \mathbb{R}$ be such that $P + \lambda > 0$. Then $\det(P+\lambda)$ is defined by (2.17). Voros [Vo] has shown that the function $\lambda \mapsto \det(P+\lambda)$, defined for $\lambda \gg 0$, extends to an entire function $\det(P+s)$ of $s \in \mathbb{C}$ with zeros $-\lambda_j$ where $\lambda_j \in \operatorname{Spec}(P)$.

2.7. Finally we recall the definition of the Ray-Singer analytic torsion [RS]. Let χ be a finite-dimensional representation of $\pi_1(X)$ and let $E_\chi \to X$ be the associated flat vector bundle over X. Pick a Hermitian fiber metric h in E_χ and let $\Delta_p(\chi)\colon \Lambda^p(X,E_\chi) \to \Lambda^p(X,E_\chi)$ be the Laplacian on the space of E_χ-valued p-forms. Then $\Delta_p(\chi)$ is a nonnegative, second order elliptic differential operator. So it has a well defined regularized determinant, defined by (2.17). Then the analytic torsion is defined as the following weighted product of regularized determinants

$$T_X(\chi;g,h) = \prod_{p=1}^{3} \left(\det \Delta_p(\chi)\right)^{(-1)^{p+1}p/2}. \tag{2.18}$$

By definition $T_X(\chi;g,h)$ depends on g and h. However, if $\dim X$ is odd and χ is acyclic, i.e., $H^*(X,E_\chi)=0$, then $T_X(\chi;g,h)$ is independent of g and h [Mu1, Corollary 2.7]. In this case we denote it simply by $T_X(\chi)$.

In this paper we consider the special case where $X=\Gamma\backslash\mathbb{H}^3$ is a closed hyperbolic 3-manifold and χ is the restriction of a representation $\tau\colon G\to \mathrm{GL}(V_\tau)$ to Γ. Then, as explained above, the flat bundle E_τ carries an admissible metric. We denote the analytic torsion attached to $\tau|_\Gamma$ with respect to this metric by $T_X(\tau)$.

3. Twisted Ruelle and Selberg zeta functions

In this section we consider various kinds of twisted geometric zeta functions which are needed for the proof of our main result. We will use the notation introduced in section 2. First we recall the following estimation of the growth of the length spectrum. For $R>0$ we have

$$\#\big\{[\gamma]\in C(\Gamma)\colon \ell(\gamma)\le R\big\} \ll e^{2R} \tag{3.1}$$

[BO, (1.31)].

If T is an endomorphism of a finite-dimensional vector space, denote by S^kT the k-th symmetric power of T. Let $\overline{\mathfrak{n}}=\theta\mathfrak{n}$ be the negative root space. Then for $\sigma\in\hat{M}$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ the twisted Selberg zeta function is defined by

$$Z(s,\sigma) = \prod_{\substack{[\gamma]\neq e\\ \text{prime}}} \prod_{k=0}^{\infty} \det\left(1-\left(\sigma(m_\gamma)\otimes S^k\left(\mathrm{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}\right)\right)e^{-(s+1)\ell(\gamma)}\right), \tag{3.2}$$

where $[\gamma]$ runs over the non-trivial primitive conjugacy classes in Γ. By [BO, (3.6)] we have

$$\log Z(s,\sigma) = -\sum_{[\gamma]\neq e} \frac{\sigma(m_\gamma)e^{-\ell(\gamma)}}{\det(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})n_\Gamma(\gamma)}\, e^{-s\ell(\gamma)}. \tag{3.3}$$

It follows from (3.1) that the series converges absolutely and uniformly in the half-plane $\mathrm{Re}(s)>1$. Therefore the infinite product converges absolutely and uniformly in the half-plane $\mathrm{Re}(s)>1$. Furthermore by [BO, Theorem 3.15] it has a meromorphic extension to the entire complex plane and satisfies a functional equation [BO, Theorem 3.18]. To state the functional equation we need some notation. Let $w \in W_A$ be the non-trivial element. Then w acts on $\hat{M}$ by

$$(w\sigma)(m) = \sigma(m_w^{-1} m m_w), \quad m \in M,\, \sigma \in \hat{M},$$

where m_w is a representative of w in the normalizer of $\mathfrak{a}$ in K. Thus $w\sigma_k = \sigma_{-k}$, $k \in \mathbb{Z}$. For each $\sigma \in \hat{M}$ there is an associated Dirac operator $D_X(\sigma)$ acting in a Clifford bundle $E_\sigma \to X$ [BO, p.29]. Let $\eta(D_X(\sigma))$ denote the eta invariant of $D_X(\sigma)$. Let P_σ be the Plancherel polynomial with respect to σ. If the Haar measures are normalized as in [Kn, pp. 387-388] and $\mathfrak{a}_\mathbb{C}$ is identified with $\mathbb{C}$ by $z \in \mathbb{C} \mapsto zH \in \mathfrak{a}_\mathbb{C}$, then by [Kn, Theorem 11.2] (up to a minor correction) it is given by

$$P_{\sigma_k}(z) = \frac{1}{4\pi^2}\left(\frac{k^2}{4} - z^2\right), \quad k \in \mathbb{Z}, \tag{3.4}$$

where $\sigma_k \in \hat{M}$ is the character defined by (2.7). We note that our definition of $P_\sigma(z)$ differs from the definition of $P_\sigma(z)$ in [BO, p. 56].

Then the functional equation satisfied by $Z(s,\sigma)$ is the following equality

$$Z(s,\sigma) = e^{i\pi\eta(D_X(\sigma))} \exp\left\{-4\pi \operatorname{vol}(X) \int_0^s P_\sigma(r)\, dr\right\} Z(-s, w\sigma) \tag{3.5}$$

[BO, Theorem 3.18]. Our formula differs from the formula in [BO, Theorem 3.18]. This is due to the the different definition of $P_\sigma(z)$. Since the functional equation plays an important role in this paper, we will give a separate proof for the functional equation of the symmetrized Selberg zeta function in section 5.

A related dynamical zeta function is the twisted Ruelle zeta function $R(s,\sigma)$ which is defined by

$$R(s,\sigma) = \prod_{\substack{[\gamma]\neq e \\ \text{prime}}} \left(1 - \sigma(m_\gamma) e^{-s\ell(\gamma)}\right), \tag{3.6}$$

where, as above, $[\gamma]$ runs over the non-trivial primitive conjugacy classes in Γ. Note that $R(s,\sigma_0)$ equals the usual Ruelle zeta function

$$R(s) = \prod_{\substack{[\gamma]\neq e \\ \text{prime}}} \left(1 - e^{-s\ell(\gamma)}\right). \tag{3.7}$$

Ruelle zeta functions of this type have been studied by Fried, and Bunke and Olbrich [BO]. The two zeta functions are closely related. Namely the Ruelle zeta function can be expressed in terms Selberg zeta functions as follows.

Lemma 3.1. *For every $\sigma \in \hat{M}$ we have*

$$R(s,\sigma) = \frac{Z(s+1,\sigma) Z(s-1,\sigma)}{Z(s, \sigma\otimes\sigma_2) Z(s,\sigma\otimes\sigma_{-2})}. \tag{3.8}$$

Proof. By [BO, (3.4)] we have

$$\log R(s,\sigma) = -\sum_{[\gamma]\neq e} \frac{\sigma(m_\gamma)}{n_\Gamma(\gamma)} e^{-s\ell(\gamma)}. \tag{3.9}$$

Using (3.3) we get

$$\begin{aligned}\log & Z(s+1,\sigma) + \log Z(s-1,\sigma) - \log Z(s,\sigma\otimes\sigma_2) - \log Z(s,\sigma\otimes\sigma_{-2}) \\ &= -\sum_{[\gamma]\neq e} \frac{\sigma(m_\gamma)(1-\sigma_2(m_\gamma)e^{-\ell(\gamma)} - \sigma_{-2}(m_\gamma)e^{-\ell(\gamma)} + e^{-2\ell(\gamma)})}{\det(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}})n_\Gamma(\gamma)} e^{-s\ell(\gamma)} \\ &= -\sum_{[\gamma]\neq e} \frac{\sigma(m_\gamma)}{n_\Gamma(\gamma)} e^{-s\ell(\gamma)}.\end{aligned} \tag{3.10}$$

Together with (3.9) the lemma follows. □

Put

$$\theta_X(\sigma) := 2\eta(D_X(\sigma)) - \eta(D_X(\sigma\otimes\sigma_2)) - \eta(D_X(\sigma\otimes\sigma_{-2})), \quad \sigma\in\hat{M}.$$

We summarize the main properties of $R(s,\sigma)$ by the following proposition.

Proposition 3.2. *For each $\sigma\in\hat{M}$ we have*

1) *The infinite product (3.6) is absolutely convergent in the half-plane* $\operatorname{Re}(s) > 2$.
2) *$R(s,\sigma)$ admits a meromorphic extension to whole complex plane.*
3) *$R(s,\sigma)$ satisfies the following functional equation.*

$$R(s,\sigma) = e^{i\pi\theta_X(\sigma)} \exp\left(4\pi^{-1}\operatorname{vol}(\Gamma\backslash\mathbb{H}^3)s\right) R(-s,w\sigma). \tag{3.11}$$

Proof. 1) follows from the estimation (3.1). The meromorphic extension is established in [BO, Chap. 4] and the functional equation is proved in [BO, Theorem 4.5]. It follows from Lemma 3.1 and the functional equation of the Selberg zeta function. Namely using (3.8) and (3.5) we get

$$\begin{aligned}\frac{R(s,\sigma)}{R(-s,w\sigma)} = e^{i\pi\theta_X(\sigma)} \exp\Bigg(-4\pi\operatorname{vol}(X)\bigg\{\int_0^{s+1} P_{\sigma_k}(r)\,dr + \int_0^{s-1} P_{\sigma_k}(r) \\ - \int_0^s P_{\sigma_{k+2}}(r)\,dr - \int_0^s P_{\sigma_{k-2}}(r)\,dr\bigg\}\Bigg).\end{aligned}$$

It follows from (3.4) by a simple computation that

$$\int_0^{s+1} P_{\sigma_k}(r)\,dr + \int_0^{s-1} P_{\sigma_k}(r) q - \int_0^s P_{\sigma_{k+2}}(r)\,dr - \int_0^s P_{\sigma_{k-2}}(r)\,dr = -\frac{s}{\pi^2}$$

which implies 3). □

Now let $\tau\colon G \to \mathrm{GL}(V)$ be a representation in a finite-dimensional complex vector space V. We fix a norm $\|\cdot\|$ in V. The restriction $\tau|_{MA}$ of τ to MA decomposes into characters:

$$\tau|_{MA} = \bigoplus_{k\in I} \sigma_k \otimes e^{\nu_k\alpha}, \tag{3.12}$$

where $I \subset \mathbb{Z}$ is finite and $\nu_k \in \frac{1}{2}\mathbb{Z}$. Let $c = \max\{|\nu_k|\colon k \in I\}$. Given $g \in G$, we denote by $a(g) \in A^+$ the A^+-component of g with respect to the Cartan decomposition $G = KA^+K$. It follows from (3.12) that there exists $C_1 > 0$ such that

$$\| \tau(g) \| \le C_1 e^{c\alpha(\log a(g))}, \quad g \in G.$$

This implies that there exists $c_2 > 0$ such that

$$\| \tau(\gamma) \| \le C e^{c_2\ell(\gamma)}, \quad \gamma \in \Gamma \setminus \{1\}.$$

Therefore, the infinite product

$$R_\tau(s) = \prod_{\substack{[\gamma]\neq e\\ \text{prime}}} \det\left(\mathrm{I} - \tau(\gamma)e^{-s\ell(\gamma)}\right) \tag{3.13}$$

is absolutely convergent in the half-plane $\mathrm{Re}(s) > c_2 + 2$. By (3.12) we have

$$\det\left(\mathrm{I} - \tau(\gamma)e^{-s\ell(\gamma)}\right) = \det\left(\mathrm{I} - \tau(m_\gamma a_\gamma)e^{-s\ell(\gamma)}\right) = \prod_{k\in I} \det\left(1 - \sigma_k(m_\gamma)e^{-(s-\nu_k)\ell(\gamma)}\right).$$

Taking the product of both sides over all non-trivial primitive conjugacy classes, we get

$$R_\tau(s) = \prod_{k\in I} R(s-\nu_k, \sigma_k), \quad \mathrm{Re}(s) > c_2 + 2. \tag{3.14}$$

The right hand side is a meromorphic function on $\mathbb{C}$. This implies that $R_\tau(s)$ admits a meromorphic continuation to $\mathbb{C}$.

Using (3.8), it follows that $R_\tau(s)$ can also be expressed in terms of twisted Selberg zeta functions. This formula can be simplified using Kostant's Bott-Borel-Weil theorem [Ko] which we recall next. Let

$$\mu_p\colon MA \to \mathrm{GL}(\Lambda^p \mathfrak{n}_{\mathbb{C}}), \quad p = 0, 1, 2, \tag{3.15}$$

be the p-th exterior power of the adjoint representation of MA on $\mathfrak{n}_{\mathbb{C}}$. It decomposes into characters as follows

$$\mu_0 = \sigma_0, \quad \mu_1 = (\sigma_2 \otimes e^\alpha) \oplus (\sigma_{-2} \otimes e^\alpha), \quad \mu_2 = \sigma_0 \otimes e^{2\alpha}. \tag{3.16}$$

Denote by $\tilde{\mu}_p$ the contragredient representation of the representation (3.15). Given $(m, n) \in \mathbb{Z} \times Z$, we define a character $\chi_{(m,n)}\colon MA \to \mathbb{C}^\times$ by

$$\chi_{(m,n)} = \sigma_{m-n} \otimes e^{\frac{m+n}{2}\alpha}. \tag{3.17}$$

Lemma 3.3. *Let τ be an irreducible representation of G with highest weight $\Lambda_\tau \in \mathbb{N}_0 \times \mathbb{N}_0$. We have the following identity of characters of MA.*

$$\sum_{p=0}^{2}(-1)^p \operatorname{tr}\tilde{\mu}_p \cdot \operatorname{tr}\tau = \sum_{w\in W_G}(-1)^{\ell(w)}\chi_{w(\Lambda_\tau+\rho_G)-\rho_G}, \tag{3.18}$$

where $\ell(w)$ denotes the length of w.

Proof. Let $L = MA$. For a finite-dimensional L-module W denote by $\operatorname{ch}_L(W)$ the element in the character ring $R(L)$. Let V_τ be an irreducible G-module with highest weight Λ_τ. By the analog of a result of Kostant [Ko, Theorem 5.14], for real Lie algebras [BW, Theorem III.3.1], [Si], we have

$$\sum_{p=0}^{2}(-1)^p \operatorname{ch}_L(H^p(\mathfrak{n}, V_\tau)) = \sum_{w\in W_G}(-1)^{\ell(w)}\chi_{w(\Lambda_\tau+\rho_G)-\rho_G}, \tag{3.19}$$

where $H^p(\mathfrak{n}, V_\tau)$ denotes the Lie algebra cohomology. By the Poincaré principle [Ko, (7.2.3)] we have

$$\sum_{p=0}^{2}(-1)^p \operatorname{ch}_L(\Lambda^p\mathfrak{n}^* \otimes V_\tau) = \sum_{p=0}^{2}(-1)^p \operatorname{ch}_L(H^p(\mathfrak{n}, V_\tau)). \tag{3.20}$$

Here L acts on $\mathfrak{n}^*$ via the contragredient representation of the adjoint representation. Combining (3.19) and (3.20), the lemma follows. In fact, in the present case the lemma could also be proved by an elementary computation, using the parametrization (2.8). $\square$

We are now ready to prove the formula which expresses $R_\tau(s)$ as a fraction of twisted Selberg zeta functions. For $w \in W_G$ write

$$\chi_{w(\Lambda_\tau+\rho_G)-\rho_G} = \sigma_{\tau,w} \otimes e^{(\lambda_{\tau,w}-1)\alpha}, \tag{3.21}$$

where $\sigma_{\tau,w} \in \hat{M}$ and $\lambda_{\tau,w} \in \mathbb{R}$.

Proposition 3.4. *Let τ be an irreducible finite-dimensional representation of G. Then we have*

$$R_\tau(s) = \prod_{w\in W_G} Z(s-\lambda_{\tau,w}, \sigma_{\tau,w})^{(-1)^{\ell(w)}}. \tag{3.22}$$

Proof. Recall that for an endomorphism W of a finite-dimensional vector space we have

$$\det(\operatorname{Id} - W) = \sum_{k=0}^{\infty}(-1)^k \operatorname{tr}(\Lambda^k W). \tag{3.23}$$

Let $m \in M$ and $a \in A$. Note that $\tilde{\mu}_p(ma) = \Lambda^p \operatorname{Ad}(ma)_{\bar{\mathfrak{n}}}$. Hence if we apply (3.23) to $\tilde{\mu}_p$ we get

$$\sum_{p=0}^{2}(-1)^p \operatorname{tr}\tilde{\mu}_p(ma) = \det\left(\operatorname{Id} - \operatorname{Ad}(ma)_{\bar{\mathfrak{n}}}\right).$$

Using (3.18) and (3.21), we get

$$\operatorname{tr}\tau(ma)=\sum_{w\in W_G}(-1)^{\ell(w)}\frac{\sigma_{\tau,w}(m)}{\det\left(\operatorname{Id}-\operatorname{Ad}(ma)_{\overline{\mathfrak{n}}}\right)}e^{(\lambda_{\tau,w}-1)\alpha(\log a)}. \tag{3.24}$$

Next we have

$$\begin{aligned}\log R_\tau(s)&=\sum_{\substack{[\gamma]\neq e\\ \text{prime}}}\operatorname{tr}\log\left(\operatorname{Id}-\tau(\gamma)e^{-s\ell(\gamma)}\right)\\ &=-\sum_{\substack{[\gamma]\neq e\\ \text{prime}}}\sum_{k=1}^\infty\frac{\operatorname{tr}\left(\tau(\gamma)e^{-s\ell(\gamma)}\right)^k}{k}\\ &=-\sum_{[\gamma]\neq e}\frac{\operatorname{tr}\tau(\gamma)}{n_\Gamma(\gamma)}e^{-s\ell(\gamma)}.\end{aligned} \tag{3.25}$$

Now let $\gamma\in\Gamma\setminus\{e\}$, $\gamma\sim m_\gamma a_\gamma$. Then $\log a_\gamma=\ell(\gamma)H$. Inserting (3.24) on the right hand side of (3.25), we get

$$\log R_\tau(s)=\sum_{w\in W_G}(-1)^{\ell(w)+1}\sum_{[\gamma]\neq e}\frac{\sigma_{\tau,w}(m_\gamma)}{\det\left(\operatorname{Id}-\operatorname{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}\right)n_\Gamma(\gamma)}e^{-(s-\lambda_{\tau,w}+1)\ell(\gamma)}$$

By [BO, (3.6)], the right hand side equals

$$\sum_{w\in W_G}(-1)^{\ell(w)}\log Z(s-\lambda_{\tau,w},\sigma_{\tau,w}),$$

which proves the proposition. $\square$

We also need to consider symmetrized Ruelle and Selberg zeta functions. Recall that the nontrivial element $w_A\in W_A$ acts on $\hat{M}$ by $w_A\sigma_k=\sigma_{-k}$. Let $\sigma\in\hat{M}\setminus\{\sigma_0\}$. Put

$$S(s,\sigma):=Z(s,\sigma)Z(s,w_A\sigma). \tag{3.26}$$

This is the symmetrized Selberg zeta function. Let $\theta\colon G\to G$ be the Cartan involution. Put

$$\tau_\theta=\tau\circ\theta. \tag{3.27}$$

Note that $\tau_p=\operatorname{Sym}^p$ satisfies $\tau_p\circ\theta=\overline{\tau}_p,\quad$ and $\quad\overline{\tau}_p\circ\theta=\tau_p$. Thus θ acts on the highest weights by

$$\theta(m,n)=(n,m),\quad(m,n)\in\mathbb{N}_0\times\mathbb{N}_0. \tag{3.28}$$

By (2.8) it follows that an irreducible finite-dimensional representation τ of G satisfies $\tau_\theta=\tau$, if and only if $\tau=\tau_{m,m}$ for some $m\in\mathbb{N}_0$.

Proposition 3.5. *Let τ be an irreducible finite-dimensional representation of G. Then we have*

$$R_\tau(s)R_{\tau_\theta}(s)=\prod_{w\in W_G} S(s-\lambda_{\tau,w},\sigma_{\tau,w})^{(-1)^{\ell(w)}},\quad \tau_\theta\ncong\tau, \tag{3.29}$$

and

$$R_{\tau_{m,m}}(s)=Z(s-(m+1),\sigma_0)Z(s+m+1,\sigma_0)S(s,\sigma_{2m+2})^{-1},\quad m\in\mathbb{N}_0. \tag{3.30}$$

Proof. Put

$$\Xi(\tau)=\{w(\Lambda_\tau+\rho_G)-\rho_G\colon w\in W_G\}.$$

Let $\tau=\tau_{m,n}$. Then we have

$$\Xi(\tau)=\{(m,n),(-(m+2),n),(m,-(n+2)),(-(m+2),-(n+2))\}.$$

By (3.17) and (3.21), it follows that

$$\begin{aligned}\{(\sigma_{\tau,w},\lambda_{\tau,w})\colon w\in W_G\}=\Big\{&\left(\sigma_{m-n},(m+n)/2+1\right),\left(\sigma_{-(m+n+2)},(n-m)/2\right),\\ &\left(\sigma_{m+n+2},(m-n)/2\right),\left(\sigma_{n-m},-(m+n)/2-1\right)\Big\}.\end{aligned} \tag{3.31}$$

Assume that $m\neq n$. Using that $(\tau_{m,n})_\theta=\tau_{n,m}$ and (3.31), it follows that

$$\begin{aligned}\{(\sigma_{\tau,w},\lambda_{\tau,w})\colon w\in W_G\}&\cup\{(\sigma_{\tau_\theta,w},\lambda_{\tau_\theta,w})\colon w\in W_G\}\\ &=\{(\sigma_{\tau,w},\lambda_{\tau,w}),(w_A\sigma_{\tau,w},\lambda_{\tau,w})\colon w\in W_G\}.\end{aligned}$$

By (3.22) the first equality follows. Now assume that $\tau_\theta=\tau$. By (3.28) there exists $m\in\mathbb{N}_0$ such that $\tau=\tau_{m,m}$. In this case we get

$$\{(\sigma_{\tau,w},\lambda_{\tau,w})\colon w\in W_G\}=\left\{(\sigma_0,m+1),(\sigma_{-2(m+1)},0),(\sigma_{2(m+1)},0),(\sigma_0,-(m+1)\right\}. \tag{3.32}$$

Using again (3.22) and (3.26), we get (3.30). □

4. Bochner-Laplace operators

In this section we study certain auxiliary elliptic operators which are needed to derive the determinant formula and the functional equation for the Selberg zeta function. These operators were first introduced by Bunke and Olbrich [BO].

Let $w_A\in W_A$ be the nontrivial element. It acts on $\sigma_k\in\hat{M}$ by $w_A\sigma_k=\sigma_{-k}$. Thus, if $k\neq 0$, then σ_k is not W_A-invariant. For $l\in\mathbb{N}_0$ let $\nu_l\in\hat{K}$ denote the irreducible representation of $K=\mathrm{SU}(2)$ of highest weight l. Then we have

$$\nu_l|_M=\bigoplus_{k=0}^{l}\sigma_{l-2k}. \tag{4.1}$$

Let $R(K)$ and $R(M)$ denote the representation rings of K and M, respectively. The inclusion $i\colon M\to K$ induces the restriction map $i^*\colon R(K)\to R(M)$. From (4.1) we get

$$\begin{aligned}&i^*(\nu_l-\nu_{l-2})=\sigma_l+\sigma_{-l},\quad l\in\mathbb{N},\ l\geq 2;\\ &i^*(\nu_1)=\sigma_1+\sigma_{-1},\quad i^*(\nu_0)=\sigma_0.\end{aligned} \tag{4.2}$$

It follows from (4.2) that for every $\sigma \in \hat{M}$ there exists a unique $\xi_\sigma \in R(K)$ such that

$$i^*(\xi_\sigma) = \sigma + w_A\sigma. \tag{4.3}$$

Then we have

$$\xi_\sigma = \sum_{\nu \in \hat{K}} m_\nu(\sigma)\nu. \tag{4.4}$$

with $m_\nu(\sigma) \in \{0, \pm 1\}$ for $\sigma \neq \sigma_0$ and $m_{\nu_l}(\sigma_0) = 0$, if $l \neq 0$, and $m_{\nu_0}(\sigma_0) = 2$.

Given $\nu \in \hat{K}$, let $\tilde{E}_\nu$ denote the associated homogeneous vector bundle over G/K and $E_\nu = \Gamma\backslash\tilde{E}_\nu$ the corresponding locally homogeneous bundle over X. For $\sigma \in \hat{M}$ and $\nu \in \hat{K}$ let $m_\nu(\sigma)$ be defined by (4.4). Put

$$E(\sigma) = \bigoplus_{\substack{\nu \\ m_\nu(\sigma) \neq 0}} E_\nu. \tag{4.5}$$

This bundle has a canonical grading

$$E(\sigma) = E^+(\sigma) \oplus E^-(\sigma) \tag{4.6}$$

defined by the sign of $m_\nu(\sigma)$.

Let $\tilde{A}_\nu$ be the elliptic G-invariant differential operator on $C^\infty(G/K, \tilde{E}_\nu) \cong (C^\infty(G) \otimes V_\nu)^K$ which is induced by $-\Omega$, where $\Omega \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ is the Casimir element. Let

$$\tilde{\Delta}_\nu = (\nabla^\nu)^* \nabla^\nu$$

be the connection Laplacian associated to the canonical invariant connection ∇^ν of $\tilde{E}_\nu$. By [Mia, Proposition 1.1] we have

$$\tilde{A}_\nu = \tilde{\Delta}_\nu - \nu(\Omega_K), \tag{4.7}$$

where $\Omega_K \in \mathcal{Z}(\mathfrak{k}_\mathbb{C})$ is the Casimir element of K. Being G-invariant, $\tilde{A}_\nu$ descends to an elliptic operator

$$A_\nu \colon C^\infty(X, E_\nu) \to C^\infty(X, E_\nu). \tag{4.8}$$

It follows from (4.7) that A_ν is symmetric and bounded from below. For $l \in \mathbb{Z}$ put

$$c(\sigma_l) = \frac{l^2}{4} - 1. \tag{4.9}$$

Define the operator $A(\sigma)$ acting on $C^\infty(X, E(\sigma))$ by

$$A(\sigma) := \bigoplus_{\substack{\nu \\ m_\nu(\sigma) \neq 0}} A_\nu + c(\sigma). \tag{4.10}$$

Obviously, $A(\sigma)$ preserves the grading of $E(\sigma)$.

By (4.7) the elliptic operator A_ν is symmetric and bounded from below. Therefore the heat operator e^{-tA_ν} is well defined and is a trace class operator. Given $\sigma \in \hat{M}$, put

$$K(t;\sigma) = \sum_{\nu\in\hat{K}} m_\nu(\sigma)\,\mathrm{Tr}(e^{-tA_\nu}), \tag{4.11}$$

where $m_\nu(\sigma)$ is defined by (4.4). Our next goal is to use the Selberg trace formula to express $K(t,\sigma)$ in terms of the length of the closed geodesics.

Let $\tilde{A}_\nu$ be the lift of A_ν to the universal covering $\tilde{X} = G/K$. It acts in the space of smooth sections of the homogeneous vector bundle $\tilde{E}_\nu$ associated to ν. With respect to the isomorphism $C^\infty(G/K,\tilde{E}_\nu) \cong (C^\infty(G)\otimes V_\nu)^K$ we have

$$\tilde{A}_\nu = -R(\Omega)\otimes \mathrm{Id}_{V_\nu}\,.$$

Let $e^{-t\tilde{A}_\nu}$, $t>0$, the heat semigroup generated by $\tilde{A}_\nu$. This is a smoothing operator on $L^2(G/K,\tilde{E}_\nu) \cong (L^2(G)\otimes V_\nu)^K$ which commutes with the action of G. Therefore it is of the form

$$\left(e^{-t\tilde{A}_\nu}\phi\right)(g) = \int_G H^\nu_t(g^{-1}g')\phi(g')\,dg',\quad \phi\in (L^2(G)\otimes V_\nu)^K,\quad g\in G,$$

where the kernel $H^\nu_t\colon G\to \mathrm{End}(V_\nu)$ is C^∞, L^2, and satisfies the covariance property

$$H^\nu_t(k^{-1}gk') = \nu(k)^{-1}\circ H^\nu_t(g)\circ\nu(k'),\quad k,k'\in K,\ g\in G. \tag{4.12}$$

Actually, a much stronger result holds. For $q>0$ let $\mathcal{C}^q(G)$ be Harish-Cahndra's L^q-Schwartz space. Then we have

$$H^\nu_t \in (\mathcal{C}^q(G)\otimes \mathrm{End}(V_\tau))^{K\times K} \tag{4.13}$$

for all $q>0$. The proof is similar to the proof of Proposition 2.4 in [BM]. By standard arguments it follows that the kernel of the heat operator e^{-tA_ν} is given by

$$H^\nu(t;x,x') = \sum_{\gamma\in\Gamma} H^\nu_t(g^{-1}\gamma g'), \tag{4.14}$$

where $x,x'\in X$ and $x=\Gamma gK$ and $x'=\Gamma g'K$. Therefore the trace of the heat operator e^{-tA_ν} is given by

$$\mathrm{Tr}\left(e^{-tA_\nu}\right) = \int_X \mathrm{tr}\, H^\nu(t;x,x)\,dx,$$

where tr denotes the trace $\mathrm{tr}\colon \mathrm{End}(E_{\nu,x})\to\mathbb{C}$ for $x\in X$. Let

$$h^\nu_t(g) = \mathrm{tr}\, H^\nu_t(g).$$

Using (4.12) and (4.14), it follows that

$$\mathrm{Tr}\left(e^{-tA_\nu}\right) = \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma} h^\nu_t(g^{-1}\gamma g)\,dg. \tag{4.15}$$

Let R_Γ denote the right regular representation of G on $L^2(\Gamma\backslash G)$. Then (4.15) can be written as

$$\operatorname{Tr}\left(e^{-tA_\nu}\right) = \operatorname{Tr} R_\Gamma(h_t^\nu). \tag{4.16}$$

Let

$$h_t^\sigma = \sum_{\nu\in\hat{K}} m_\nu(\sigma) h_t^\nu. \tag{4.17}$$

Then by (4.11) and (4.16) we get

$$K(t;\sigma) = \operatorname{Tr} R_\Gamma\left(h_t^\sigma\right), \quad t > 0.$$

We can now apply the Selberg trace formula [Wa1] . We use the notation introduced in section 2. Let $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$, where θ is the Cartan involution. For $\gamma \in \Gamma \setminus \{e\}$ put

$$D(\gamma) = e^{\ell(\gamma)} \det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)_{\overline{\mathfrak{n}}}).$$

Let $\Theta_{n,\lambda}$ denote the character of the principal series representation $\pi_{n,\lambda}$, where $n \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$. Then the Selberg trace formula gives

$$\begin{aligned} K(t;\sigma) &= \operatorname{Vol}(X) h_t^\sigma(e) \\ &+ \frac{1}{2\pi} \sum_{[\gamma]\neq e} \frac{\ell(\gamma)}{n_\Gamma(\gamma) D(\gamma)} \sum_{n\in\mathbb{Z}} \overline{\sigma_n(m_\gamma)} \int_{\mathbb{R}} \Theta_{n,\lambda}(h_t^\sigma) e^{-i\ell(\gamma)\lambda}\, d\lambda. \end{aligned} \tag{4.18}$$

Note that $h_t^\sigma(e)$ can also be expressed in terms of characters. By (4.13), each h_t^ν belongs to $\mathcal{C}^q(G)$ for all $q > 0$. Therefore h_t^σ is in $\mathcal{C}^q(G)$. Hence we can apply the Plancherel formula for G (see [Kn, Theorem 11.2]). With respect to the normalizations of Haar measures used in [Kn] and the definition of the Plancherel polynom (3.4), we have

$$h_t^\sigma(e) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \Theta_{n,\lambda}(h_t^\sigma) P_{\sigma_n}(i\lambda)\, d\lambda. \tag{4.19}$$

To continue we need to compute the characters $\Theta_{n,\lambda}(h_t^\sigma)$. First by (4.17) we have

$$\Theta_{n,\lambda}(h_t^\sigma) = \sum_{\nu\in\hat{K}} m_\nu(\sigma) \Theta_{n,\lambda}(h_t^\nu), \tag{4.20}$$

which reduces the problem to the computation of $\Theta_{n,\lambda}(h_t^\nu)$. For any unitary representation π of G on a Hilbert space $\mathcal{H}_\pi$ set

$$\tilde{\pi}(H_t^\nu) = \int_G \pi(g) \otimes H_t^\nu(g)\, dg.$$

This defines a bounded operator on $\mathcal{H}_\pi \otimes V_\nu$. As in [BM, pp. 160-161] it follows from (4.12) that relative to the splitting

$$\mathcal{H}_\pi \otimes V_\nu = \left(\mathcal{H}_\pi \otimes V_\nu\right)^K \oplus \left[\left(\mathcal{H}_\pi \otimes V_\nu\right)^K\right]^\perp,$$

$\tilde{\pi}(H_t^\nu)$ has the form

$$\tilde{\pi}(H_t^\nu) = \begin{pmatrix} \pi(H_t^\nu) & 0 \\ 0 & 0 \end{pmatrix}$$

with $\pi(H_t^\nu)$ acting on $(\mathcal{H}_\pi \otimes V_\tau)^K$. Then it follows as in [BM, Corollary 2.2] that

$$\pi(H_t^\nu) = e^{t\pi(\Omega)}\, \mathrm{Id}, \tag{4.21}$$

where Id is the identity on $(\mathcal{H}_\pi \otimes V_\nu)^K$. Let $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{e_j\}_{j=1}^m$ be orthonormal bases of $\mathcal{H}_\pi$ and V_ν, respectively. Then we have

$$\begin{aligned} \operatorname{Tr} \pi(H_t^\nu) &= \sum_{n=1}^\infty \sum_{j=1}^m \langle \pi(H_t^\nu)(\xi_n \otimes e_j), (\xi_n \otimes e_j)\rangle \\ &= \sum_{n=1}^\infty \sum_{j=1}^m \int_G \langle \pi(g)\xi_n, \xi_n\rangle \langle H_t^\nu(g) e_j, e_j\rangle \, dg \\ &= \sum_{n=1}^\infty \int_G h_t^\nu(g) \langle \pi(g)\xi_n, \xi_n\rangle \, dg \\ &= \operatorname{Tr} \pi(h_t^\nu). \end{aligned} \tag{4.22}$$

Together with (4.21) we get

$$\operatorname{Tr} \pi(h_t^\nu) = e^{t\pi(\Omega)} \dim \left(\mathcal{H}_\pi \otimes V_\nu\right)^K. \tag{4.23}$$

Now we consider a unitary principal series representation $\pi_{n,\lambda}$ Let $[\nu|_M : \sigma_n]$ denote the multiplicity of $\sigma_n \in \hat{M}$ in $\nu|_M$. It equals 0 or 1. For any representation π of G denote by $\pi^\vee$ the contragredient representation of π. By Frobenius reciprocity [Kn, p. 208], we have

$$\dim \left(\mathcal{H}_{n,\lambda} \otimes V_\nu\right)^K = [\pi_{n,\lambda}^\vee|_K \colon \nu] = [\pi_{-n,-\lambda}|_K \colon \nu] = [\nu|_M \colon \sigma_{-n}] = [\nu|_M \colon \sigma_n].$$

Combined with (4.23), we obtain

$$\Theta_{n,\lambda}(h_t^\nu) = e^{t\pi_{n,\lambda}(\Omega)} [\nu|_M : \sigma_n].$$

Using (4.20), (4.4) and (4.3), we get

$$\Theta_{n,\lambda}(h_t^\sigma) = e^{t\pi_{n,\lambda}(\Omega)} \sum_{\nu\in\hat{K}} m_\nu(\sigma)[\nu|_M : \sigma_n] = e^{t\pi_{n,\lambda}(\Omega)}[\sigma + w_A\sigma : \sigma_n]. \tag{4.24}$$

The Casimir eigenvalue $\pi_{n,\lambda}(\Omega)$ is given by (2.12). Using the definition of $c(\sigma)$ by (4.9) it can be written as

$$\pi_{n,\lambda}(\Omega) = -\lambda^2 + c(\sigma_n). \tag{4.25}$$

Now we can put our computations together. Let $k \in \mathbb{N}_0$. If we insert (4.24) in (4.18) and (4.19) and use (4.25), we get

$$\begin{aligned} K(t;\sigma_k) = e^{tc(\sigma_k)} \Bigg(& 2\operatorname{vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_{\sigma_k}(i\lambda)\, d\lambda \\ & + \sum_{[\gamma]\neq e} \frac{\ell(\gamma)}{n_\Gamma(\gamma)} L_{\mathrm{sym}}(\gamma;\sigma_k) \frac{e^{-\ell(\gamma)^2/(4t)}}{(4\pi t)^{1/2}} \Bigg), \end{aligned} \tag{4.26}$$

where

$$L_{\mathrm{sym}}(\gamma,\sigma) = \frac{(\sigma(m_\gamma) + (w_A\sigma)(m_\gamma))e^{-\ell(\gamma)}}{\det\left(\operatorname{Id} - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}}\right)}. \tag{4.27}$$

Using the definition of $A(\sigma)$ by (4.10) together with (4.11), we finally get

Proposition 4.1. *For every $\sigma \in \hat{M}$ we have*

$$\operatorname{Tr}_s\left(e^{-tA(\sigma)}\right) = 2\operatorname{vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda)\, d\lambda + \sum_{[\gamma]\neq e} \frac{\ell(\gamma)}{n_\Gamma(\gamma)} L_{\mathrm{sym}}(\gamma;\sigma) \frac{e^{-\ell(\gamma)^2/(4t)}}{(4\pi t)^{1/2}}. \tag{4.28}$$

5. The functional equation of the Selberg zeta function

One of the main ingredients of the proof of Theorem 1.1 is the functional equation (3.5) satisfied by the Selberg zeta function $Z(s,\sigma)$. In particular, it is important to determine the sign in the exponential factor. We include a proof of the functional equation for the symmetrized Selberg zeta function which suffices for our purpose.

Let $\sigma \in \hat{M}$. Note that $A(\sigma)$ is a second order elliptic differential operator on a compact manifold. Therefore it is essentially self-adjoint and the unique self-adjoint extension of $A(\sigma)$ has pure point spectrum consisting of a sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ of finite multiplicities. It follows from Weyl's law that

$$\sum_{\lambda_i > 0} \lambda_i^{-2} < \infty. \tag{5.1}$$

Therefore the resolvent $(A(\sigma)+s^2)^{-1}$, $\operatorname{Re}(s^2) \gg 0$, is a Hilbert-Schmidt operator. Let $\operatorname{Re}(s_i^2) \gg 0$,$i = 1,2$. By the resolvent equation we have

$$(A(\sigma)+s_1^2)^{-1} - (A(\sigma)+s_2^2)^{-1} = (s_2^2 - s_1^2)(A(\sigma)+s_1^2)^{-1} \circ (A(\sigma)+s_2^2)^{-1}.$$

Thus the right hand side is a product of Hilbert-Schmidt operators and therefore, it is a trace class operator. Hence $(A(\sigma)+s_1^2)^{-1} - (A(\sigma)+s_2^2)^{-1}$ is a trace class operator. Now observe that

$$(A(\sigma)+s^2)^{-1} = \int_0^\infty e^{-ts^2} e^{-tA(\sigma)}\, dt.$$

Furthermore we have the heat expansion

$$\operatorname{Tr}\left(e^{-tA(\sigma)}\right) \sim \sum_{j\geq 0} a_j t^{-3/2+j} \tag{5.2}$$

as $t \to +0$. Let $\operatorname{Re}(s^2), \operatorname{Re}(s_0^2) \gg 0$. Then it follows from (5.2) that

$$\operatorname{Tr}_s\left((A(\sigma)+s^2)^{-1} - (A(\sigma)+s_0^2)^{-1}\right) = \int_0^\infty (e^{-ts^2} - e^{-ts_0^2}) \operatorname{Tr}_s\left(e^{-tA(\sigma)}\right)\, dt. \tag{5.3}$$

Now we replace $\operatorname{Tr}_s\left(e^{-tA(\sigma)}\right)$ by the right hand side of (4.28). First note that for $\operatorname{Re}(s) > 0$ we have

$$\int_0^\infty e^{-ts^2} \frac{e^{-\ell(\gamma)^2/(4t)}}{\sqrt{4\pi t}} = \frac{1}{2s} e^{-s\ell(\gamma)} \tag{5.4}$$

Furthermore by Cauchy's theorem we have

$$\begin{split}\int_0^\infty (e^{-ts^2} - e^{-ts_0^2}) \left(\int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda)\, d\lambda\right) dt &= \int_{\mathbb{R}} \frac{s_0^2 - s^2}{(\lambda^2+s^2)(\lambda^2+s_0^2)} P(i\lambda)\, d\lambda \\ &= \frac{\pi}{s} P_\sigma(s) - \frac{\pi}{s_0} P_\sigma(s_0).\end{split} \tag{5.5}$$

For the last equality we used that $P_\sigma(s)$ is an even polynomial. By (5.4) and (5.5), we get

$$\begin{split}\operatorname{Tr}_s\left((A(\sigma)+s^2)^{-1} - (A(\sigma)+s_0^2)^{-1}\right) &= 2\pi \operatorname{vol}(X) \left(\frac{P_\sigma(s)}{s} - \frac{P_\sigma(s_0)}{s_0}\right) \\ &+ \frac{1}{2s} \sum_{[\gamma]\neq e} \frac{\ell(\gamma)}{n_\Gamma(\gamma)} L_{\mathrm{sym}}(\gamma;\sigma) e^{-s\ell(\gamma)} - \frac{1}{2s_0} \sum_{[\gamma]\neq e} \frac{\ell(\gamma)}{n_\Gamma(\gamma)} L_{\mathrm{sym}}(\gamma;\sigma) e^{-s_0\ell(\gamma)}.\end{split} \tag{5.6}$$

By (3.3) we have

$$\sum_{[\gamma]\neq e} \frac{\ell(\gamma)}{n_\Gamma(\gamma)} L_{\mathrm{sym}}(\gamma;\sigma) e^{-s\ell(\gamma)} = \frac{Z'(s,\sigma)}{Z(s,\sigma)} + \frac{Z'(s,w_A\sigma)}{Z(s,w_A\sigma)},$$

which is the logarithmic derivative of the symmetrized Selberg zeta function $S(s,\sigma)$ defined by (3.26). Thus we get

$$\begin{split}\operatorname{Tr}_s\left((A(\sigma)+s^2)^{-1} - (A(\sigma)+s_0^2)^{-1}\right) =& 2\pi \operatorname{vol}(X) \left(\frac{P_\sigma(s)}{s} - \frac{P_\sigma(s_0)}{s_0}\right) \\ &+ \frac{1}{2s} \frac{S'(s,\sigma)}{S(s,\sigma)} - \frac{1}{2s_0} \frac{S'(s_0,\sigma)}{S(s_0,\sigma)}.\end{split} \tag{5.7}$$

Put

$$\Xi(s,\sigma) = \exp\left(4\pi \operatorname{vol}(X) \int_0^s P_\sigma(r)\, dr\right) S(s,\sigma).$$

Then (5.7) can be rewritten as

$$\mathrm{Tr}_s\left((A(\sigma)+s^2)^{-1}-(A(\sigma)+s_0^2)^{-1}\right)=\frac{1}{2s}\frac{\Xi'(s,\sigma)}{\Xi(s,\sigma)}-\frac{1}{2s_0}\frac{\Xi'(s_0,\sigma)}{\Xi(s_0,\sigma)}. \tag{5.8}$$

From this equality one can deduce the existence of the meromorphic extension of $S(s,\sigma)$ and determine the location of the singularities, i.e., zeros and poles of $S(s,\sigma)$. Let $\lambda_1<\lambda_2<\cdots$ be the eigenvalues of $A(\sigma)$. For each λ_j let $\mathcal{E}(\lambda_j)$ be the eigenspace of $A(\sigma)$ with eigenvalue λ_j. Put

$$m_s(\lambda_j,\sigma)=\dim_{\mathrm{gr}}\mathcal{E}(\lambda_j).$$

If $\lambda_j<0$, we choose the square root $\sqrt{\lambda_j}$ which has positive imaginary part. Put

$$s_j^{\pm}=\pm i\sqrt{\lambda_j},\quad j\in\mathbb{N}.$$

Proposition 5.1. *The Selberg zeta function $S(s,\sigma)$, defined by (3.26), has a meromorphic extension to $s\in\mathbb{C}$. The set of singularities of $S(s,\sigma)$ equals $\{s_j^{\pm}\colon j\in\mathbb{N}\}$. If $\lambda_j\neq 0$, then the order of $S(s,\sigma)$ at both s_j^+ and s_j^- is equal to $m_s(\lambda_j,\sigma)$. The order of the singularity at $s=0$ is $2m_s(0,\sigma)$.*

Proof. The left hand side of (5.8) equals

$$\sum_{j=1}^{\infty}m_s(\lambda_j,\sigma)\left\{\frac{1}{s^2+\lambda_j}-\frac{1}{s_0^2+\lambda_j}\right\}.$$

By (5.1) the series converges absolutely and uniformly on compact subsets which shows that it is a meromorphic function of $s\in\mathbb{C}$ and the only poles are simple and occur exactly at the points $\{s_j^{\pm}\colon j\in\mathbb{N}\}$. Hence the logarithmic derivative of $\Xi(s,\sigma)$ is a meromorphic function with the same poles. Let $\lambda_j\neq 0$. Then

$$\frac{2s}{s^2+\lambda_j}=\frac{1}{s-s_j^+}+\frac{1}{s-s_j^-}.$$

It follows that $s_j^{\pm}$ are simple poles of $\Xi'(s,\sigma)\cdot\Xi(s,\sigma)^{-1}$ with residue $m_s(\lambda_j,\sigma)$. Hence the order of $\Xi(s,\sigma)$ at $s_j^{\pm}$ equals $m_s(\lambda_j,\sigma)$. In the same way it follows that the order of $\Xi(s,\sigma)$ at $s=0$ equals $2m_s(0,\sigma)$. □

Now subtract from (5.8) the same equation for $-s$ and multiply by $2s$. Then we get

$$\frac{\Xi'(s,\sigma)}{\Xi(s,\sigma)}+\frac{\Xi'(-s,\sigma)}{\Xi(-s,\sigma)}=0,$$

which shows that the logarithmic derivative of $\Xi(s)\cdot\Xi(-s)^{-1}$ equals zero. Therefore $\Xi(s)\cdot\Xi(-s)^{-1}$ is constant. By Proposition 5.1 the order of $S(s,\sigma)$ at zero is even. Hence

$$\lim_{s\to 0}\frac{\Xi(s)}{\Xi(-s)}=1.$$

This implies $\Xi(s) = \Xi(-s)$. Since $P_\sigma(z)$ is even, we obtain the following functional equation for $S(s,\sigma)$:

$$S(s,\sigma) = \exp\left(-8\pi \operatorname{vol}(X) \int_0^s P_\sigma(r)\, dr\right) S(-s,\sigma). \tag{5.9}$$

Note that $\overline{Z(s,\sigma_m)} = Z(\overline{s},\sigma_{-m})$. Hence for $s \in \mathbb{R}$ we have $S(s,\sigma) = |Z(s,\sigma)|^2$. Then (5.9) is reduced to

$$|Z(s,\sigma)| = \exp\left(-4\pi \operatorname{vol}(X) \int_0^s P_\sigma(r)\, dr\right) |Z(-s,\sigma)|, \quad s \in \mathbb{R}. \tag{5.10}$$

6. The determinant formula

By [BO, Theorem 3.19] the twisted Selberg zeta function can be expressed as a graded regularized determinant of $A(\sigma)$. We include a simple proof of this formula for our case.

First we recall the notion of the graded regularized determinant of an elliptic self-adjoint operator. Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Hermitian vector bundle over a compact Riemannian manifold. Let $P\colon C^\infty(Y,E) \to C^\infty(Y,E)$ be an elliptic differential operator which is symmetric and bounded from below. Assume that P preserves the grading, i.e., assume that with respect to the decomposition

$$C^\infty(Y,E) = C^+(Y,E^+) \oplus C^\infty(Y,E^-)$$

P takes the form

$$P = \begin{pmatrix} P^+ & 0 \\ 0 & P^- \end{pmatrix}.$$

Then we define the graded determinant $\det_{\mathrm{gr}}(P)$ of P by

$$\det_{\mathrm{gr}}(P) = \frac{\det(P^+)}{\det(P^-)}. \tag{6.1}$$

Given $\sigma \in \hat{M}$, let $A(\sigma)$ be the elliptic operator defined by (4.10). It acts in a graded vector bundle. Hence the graded determinant $\det_{\mathrm{gr}}(s^2 + A(\sigma))$ is defined. Let $P_\sigma(r)$ be the Plancherel polynomial (3.4). By [BO, Theorem 3.19] the twisted symmetrized Selberg zeta function $S(s,\sigma)$ can be expressed by the graded determinant as follows.

Proposition 6.1. *We have*

$$S(s;\sigma) = \det_{\mathrm{gr}}\left(s^2 + A(\sigma)\right) \exp\left(-4\pi \operatorname{vol}(X) \int_0^s P_\sigma(r)\, dr\right), \quad \sigma \neq \sigma_0, \tag{6.2}$$

and

$$Z(s;\sigma_0) = \det\left(s^2 - 1 + \Delta\right) \exp\left((6\pi)^{-1} \operatorname{vol}(X) s^3\right), \tag{6.3}$$

where Δ is the Laplace operator on $C^\infty(X)$ and det *is the usual regularized determinant.*

Proof. We give a simple proof of this formula. Let $\sigma \in \hat{M}$. For $\mathrm{Re}(s^2) \gg 0$ let

$$\zeta(z,s) = \int_0^\infty e^{-ts^2} \operatorname{Tr}_s\left(e^{-tA(\sigma)}\right) t^{z-1}\, dt. \tag{6.4}$$

The integral converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(z) > 3/2$. It admits an extension to a meromorphic function of $z \in \mathbb{C}$ which is differentiable in s. It is regular at $z = 0$ and we have

$$\zeta(z,s) = -\log\det(A(\sigma) + s^2) + O(z). \tag{6.5}$$

Furthermore for $\mathrm{Re}(z) > 3/2$ we have

$$-\frac{1}{2s}\frac{d}{ds}\zeta(z,s) = \int_0^\infty e^{-ts^2} \operatorname{Tr}_s\left(e^{-tA(\sigma)}\right) t^{z}\, dt. \tag{6.6}$$

Thus by (6.5) we get

$$\begin{aligned} \frac{1}{2s}\frac{d}{ds}\log\det\nolimits_{\mathrm{gr}}\left(A(\sigma)+s^2\right) &- \frac{1}{2s_0}\frac{d}{ds}\log\det\nolimits_{\mathrm{gr}}\left(A(\sigma)+s^2\right)\big|_{s=s_0} \\ &= \lim_{z\to 0}\left(-\frac{1}{2s}\frac{d}{ds}\zeta(z,s) + \frac{1}{2s_0}\frac{d}{ds}\zeta(z,s)\big|_{s=s_0}\right) \\ &= \int_0^\infty (e^{-ts^2} - e^{-ts_0^2}) \operatorname{Tr}_s\left(e^{-tA(\sigma)}\right)\, dt. \end{aligned} \tag{6.7}$$

Assume that $\sigma \neq \sigma_0$. Together with (5.3) and (5.7) we get

$$\frac{d}{ds}\log\det\nolimits_{\mathrm{gr}}\left(A(\sigma)+s^2\right) = \frac{d}{ds}\log S(s,\sigma) + 4\pi\operatorname{vol}(X) P_\sigma(s) + bs \tag{6.8}$$

for some $b \in \mathbb{C}$. Integrating this equality gives

$$\log S(s,\sigma) = -4\pi\operatorname{vol}(X)\int_0^s P_\sigma(r)\, dr + \log\det\nolimits_{\mathrm{gr}}\left(A(\sigma)+s^2\right) + \frac{b}{2}s^2 + c \tag{6.9}$$

for some $c \in \mathbb{C}$. In order to determine the constants b and c we take $s \in \mathbb{R}$ and consider the asymptotic behavior of both sides of (6.9) as $s \to \infty$. By (3.3) it follows that $\log S(s,\sigma) \to 0$ as $s \to \infty$. Next consider the behavior of $\log\det_{\mathrm{gr}}(A(\sigma)+s^2)$ for $s \in \mathbb{R}$ and $s \to \infty$. By (6.5) we have

$$\log\det\nolimits_{\mathrm{gr}}\left(A(\sigma)+s^2\right) = -\frac{d}{dz}\left(z\zeta(z,s)\right)\big|_{z=0}. \tag{6.10}$$

Denote the first term on the right hand side of (4.28) by $I(t,\sigma)$ and the second by $H(t,\sigma)$. Let $\mathrm{Re}(z) > 3/2$. Then by (4.28) and (6.4) we get

$$\zeta(z,s) = \int_0^\infty e^{-ts^2} I(t,\sigma) t^{z-1}\, dt + \int_0^\infty e^{-ts^2} H(t,\sigma) t^{z-1}\, dt. \tag{6.11}$$

It follows from the definition of $H(t,\sigma)$ that the integral $\int_0^\infty e^{-ts^2} H(t,\sigma) t^{z-1}\, dt$ is an entire function of $z \in \mathbb{C}$ and for every compact subset $\omega \subset \mathbb{C}$ there exist $C, c > 0$ such that

$$\left|\frac{d}{dz}\int_0^\infty e^{-ts^2} H(t,\sigma) t^{z-1}\, dt\right| \le C\, e^{-cs^2} \quad z \in \omega,\ s \ge 0. \tag{6.12}$$

To deal with the first integral on the right hand side of (6.11), we note that

$$\int_0^\infty e^{-ts^2}\left(\int_{\mathbb{R}} e^{-t\lambda^2}\lambda^{2j}\,d\lambda\right)dt=\Gamma(j+1/2)\Gamma(-j-1/2+z)s^{2j-2z+1}. \tag{6.13}$$

By (3.4) the Plancherel polynomial $P_\sigma(z)$ is of the form $P_\sigma(z)=a_1+a_2z^2$. Using the definition of $I(t,\sigma)$ and (6.13), we get

$$\begin{aligned}\frac{d}{dz}\left(z\int_0^\infty e^{-ts^2}I(t,\sigma)t^{z-1}\,dt\right)\bigg|_{z=0}&=-4\pi\operatorname{vol}(X)\left(a_1s-\frac{a_2}{3}s^3\right)\\&=-4\pi\operatorname{vol}(X)\int_0^s P_\sigma(r)\,dr.\end{aligned} \tag{6.14}$$

Together with (6.10), (6.11), and (6.12) we obtain

$$\log\det{}_{\mathrm{gr}}\left(A(\sigma)+s^2\right)=4\pi\operatorname{vol}(X)\int_0^\infty P_\sigma(r)\,dr+O(e^{-cs^2})$$

for $s\in\mathbb{R}$, $s\to\infty$. This implies that the constants b and c in (6.9) are zero. Exponentiating (6.9), we get (6.2). The proof of (6.3) is similar. □

Remark. From the statement of Theorem 3.19 in [BO] it is not apparent that the determinant is the graded determinant. However, it is the general understanding in [BO] that the trace of a trace class operator on a graded bundle is the super trace corresponding to the grading (see [BO, p. 29]). Consequently regularized determinants of elliptic operators on graded bundles are always understood in [BO] as graded determinants.

Now let τ be an irreducible, finite-dimensional representation of G with highest weight $\Lambda_\tau=(m,n)$. For $w\in W_G$ let $\sigma_{\tau,w}\in\hat{M}$ and $\lambda_{\tau,w}$ be defined by (3.21). Let

$$\Delta(w)=\bigoplus_{\substack{\nu\\ m_\nu(\sigma_{\tau,w})\neq 0}} A_\nu+\tau(\Omega). \tag{6.15}$$

This is an elliptic operator acting on $C^\infty(X,E(\sigma_{\tau,w}))$. Using (3.31), an explicite computation shows that for all $w\in W_G$ we have

$$\lambda_{\tau,w}^2+c(\sigma_{\tau,w})=\frac{1}{2}\left(m(m+2)+n(n+2)\right)=\tau(\Omega). \tag{6.16}$$

Using (6.16), and (6.15), it follows that

$$A(\sigma_{\tau,w})+\lambda_{\tau,w}^2=\Delta(w) \tag{6.17}$$

as operators on $C^\infty(X,E(\sigma_{\tau,w}))$. Then it follows from (6.2) that

$$S(s-\lambda_{\tau,w};\sigma_{\tau,w})=\det{}_{\mathrm{gr}}(s^2-2\lambda_{\tau,w}s+\Delta(w))\exp\left(-4\pi\operatorname{vol}(X)\int_0^{s-\lambda_{\tau,w}}P_{\sigma_{\tau,w}}(r)\,dr\right),$$

if $\sigma_{\tau,w}\neq\sigma_0$. If $\sigma_{\tau,w}=\sigma_0$, we use (6.3), which leads to a similar formula

Proposition 6.2. *Let $\tau_\theta \ncong \tau$. There is a constant $c = c(\tau)$ such that*

$$R_\tau(s)R_{\tau_\theta}(s) = e^{c\operatorname{vol}(X)s} \prod_{w\in W_G} \det\nolimits_{\mathrm{gr}}(s^2 - 2\lambda_{\tau,w}s + \Delta(w))^{(-1)^{\ell(w)}}. \tag{6.18}$$

Proof. By assumption we have $\tau = \tau_{m,n}$ with $m \neq n$. It follows from (3.31) that $\sigma_{\tau,w} \ncong \sigma_0$ for all $w \in W_G$. Put

$$F(s) = \sum_{w\in W_G} (-1)^{\ell(w)+1} \int_0^{s-\lambda_{\tau,w}} P_{\sigma_{\tau,w}}(r)\, dr. \tag{6.19}$$

Then it follows from (3.29) and (6.2) that

$$R_\tau(s)R_{\tau_\theta}(s) = e^{-4\pi\operatorname{vol}(X)F(s)} \prod_{w\in W_G} \det\nolimits_{\mathrm{gr}}\left(s^2 - 2s\lambda_{\tau,w} + \Delta(w)\right)^{(-1)^{\ell(w)}}. \tag{6.20}$$

Using (3.4) and (3.31), an explicite computation gives

$$F(s) = -\pi^{-2}(m+1)(n+1)s.$$

□

Now we consider the case $\tau_\theta = \tau$. Then $\tau = \tau_{m,m}$ for some $m \in \mathbb{N}_0$.

Proposition 6.3. *Let $m \in \mathbb{N}_0$. There exists a constant $c = c(m)$ such that*

$$R_{\tau_{m,m}}(s) = e^{c\operatorname{vol}(X)s} \cdot \frac{\det\left((s+m+1)^2 - 1 + \Delta\right)\det\left((s-m-1)^2 - 1 + \Delta\right)}{\det_{\mathrm{gr}}\left(s^2 + A(\sigma_{2m+2})\right)}. \tag{6.21}$$

Proof. Put

$$F_m(s) = \frac{1}{6\pi}\left((s+m+1)^3 + (s-m-1)^3\right) + 4\pi \int_0^s P_{\sigma_{2m+2}}(r)\, dr.$$

Using (3.30), (6.2) and (6.3), it follows that

$$R_{\tau_{m,m}}(s) = e^{\operatorname{vol}(X)F_m(s)} \cdot \frac{\det\left((s+m+1)^2 - 1 + \Delta\right)\det\left((s-m-1)^2 - 1 + \Delta\right)}{\det_{\mathrm{gr}}\left(s^2 + A(\sigma_{2m+2})\right)}.$$

Using (3.4), it follows that $F_m(s) = 2\pi^{-1}(m+1)^2 s$. □

7. Proof of Theorem 1.5

Since [Wo] has not been published yet, we include a proof of Theorem 1.5. Let $\tau\colon G \to \mathrm{GL}(V_\tau)$ be an irreducible finite-dimensional representation with associated flat bundle E_τ equipped with an admissible metric. Let $\Delta_p(\tau)$ be the Laplacian on E_τ-valued p-forms. Let

$$K(t,\tau) := \sum_{p=1}^{3} (-1)^p p \operatorname{Tr}\left(e^{-t\Delta_p(\tau)}\right) \tag{7.1}$$

and

$$q(\tau)=\sum_{p=1}^{3}(-1)^p p\dim\ker H^p(X,E_\tau).$$

Then by definition of the analytic torsion we have

$$\log T_X(\tau)=\frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty\left(K(t,\tau)-q(\tau)\right)t^{s-1}\,dt\right)\Bigg|_{s=0},\tag{7.2}$$

where the right hand side is defined near $s=0$ by analytic continuation of the Mellin transform. The first step of the proof is to apply the trace formula to express $K(t,\tau)$ in terms of the length of closed geodesics. This is the basis for the relation between analytic torsion and the twisted Ruelle zeta function.

Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Let $x_0=eK$. Recall that there is a canonical isomorphism $T_{x_0}(G/K)\cong\mathfrak{p}$. Let R_Γ denote the right regular representation of G on $L^2(\Gamma\backslash G)$ (resp. $C^\infty(\Gamma\backslash G)$). Using (2.15), we get a canonical isomorphism

$$\Lambda^p(X,E_\tau)\cong\left(C^\infty(\Gamma\backslash G)\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^K,\tag{7.3}$$

where K acts by $k\in K\mapsto R_\Gamma(k)\otimes\Lambda^p\operatorname{Ad}^*_{\mathfrak{p}}(k)\otimes\tau(k)$. There is a similar isomorphism for the space $L^2\Lambda^p(X,E_\tau)$ of L^2-sections of $\Lambda^pT^*X\otimes E_\tau$. With respect to the isomorphism (7.3), we have the following generalization of Kuga's lemma

$$\Delta_p(\tau)=-R_\Gamma(\Omega)\otimes\operatorname{Id}+\tau(\Omega)\operatorname{Id},\tag{7.4}$$

(see [MM, (6.9)]), where Ω is the Casimir element and $\tau(\Omega)$ is the Casimir eigenvalue of τ.

Let $\tilde\Delta_p(\tau)$ be the lift of $\Delta_p(\tau)$ to the universal covering $\tilde X=G/K$. Let $e^{-t\tilde\Delta_p(\tau)}$, $t>0$, be the corresponding heat semigroup. This is a smoothing operator on

$$L^2\Lambda^p(\tilde X;\tilde E_\tau)\cong(L^2(G)\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau)^K,$$

which commutes with the action of G. Therefore, it is of the form

$$\left(e^{-t\tilde\Delta_p(\tau)}\phi\right)(g)=\int_G H_t^{\tau,p}(g^{-1}g')\phi(g')\,dg',\quad \phi\in(L^2(G)\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau)^K,\;\; g\in G,$$

where the kernel $H_t^{\tau,p}\colon G\to\operatorname{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau)$ belongs to $C^\infty\cap L^2$ and satisfies the covariance property

$$H_t^{\tau,p}(k^{-1}gk')=\nu_p(\tau)(k)^{-1}H_t^{\tau,p}(g)\nu_p(\tau)(k'),\tag{7.5}$$

with respect to the representation

$$\nu_p(\tau):=\Lambda^p\operatorname{Ad}^*_K\otimes\tau\colon K\to\operatorname{GL}(\Lambda^p\mathfrak{p}^*\otimes V_\tau).\tag{7.6}$$

Moreover, for all $q>0$ we have

$$H_t^{\tau,p}\in\left(\mathcal{C}^q(G)\otimes\operatorname{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau)\right)^{K\times K},\tag{7.7}$$

where $\mathcal{C}^q(G)$ denotes Harish-Cahndra's L^p-Schwartz space. The proof is similar to the proof of Proposition 2.4 in [BM]. Let

$$h_t^{\tau,p}(g) = \operatorname{tr} H_t^{\tau,p}(g).$$

Repeating the arguments which we used to prove (4.16), we get

$$\operatorname{Tr}\left(e^{-t\Delta_p(\tau)}\right) = \operatorname{Tr} R_\Gamma(h_t^{\tau,p}). \tag{7.8}$$

Put

$$k_t^\tau = \sum_{p=1}^{3}(-1)^p p\, h_t^{\tau,p}. \tag{7.9}$$

By (7.1) we have

$$K(t,\tau) = \operatorname{Tr} R_\Gamma(k_t^\tau).$$

We can now apply the Selberg trace formula [Wa1]. Let the notation be as in (4.18). Then we get

$$\begin{aligned} K(t,\tau) &= \operatorname{Vol}(X) k_t^\tau(e) \\ &\quad + \frac{1}{2\pi}\sum_{[\gamma]\neq e}\frac{\ell(\gamma)}{n_\Gamma(\gamma)D(\gamma)}\sum_{n\in\mathbb{Z}}\overline{\sigma_n(m_\gamma)}\int_{\mathbb{R}}\Theta_{n,\lambda}(k_t^\tau)e^{-i\ell(\gamma)\lambda}\,d\lambda, \end{aligned} \tag{7.10}$$

where the notation is the same as in (4.18). The characters $\Theta_{n,\lambda}(k_t^\tau)$ can be computed in the same was as in section 4. Let π be a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. Set

$$\tilde\pi(H_t^{\tau,p}) = \int_G \pi(g)\otimes H_t^{\tau,p}(g)\,dg.$$

This defines a bounded operator on $\mathcal{H}_\pi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau$. As in [BM, pp. 160-161] it follows from (7.5) that relative to the splitting

$$\mathcal{H}_\pi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau = \left(\mathcal{H}_\pi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^K \oplus \left[\left(\mathcal{H}_\pi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^K\right]^\perp,$$

$\tilde\pi(H_t^{\tau,p})$ has the form

$$\tilde\pi(H_t^{\tau,p}) = \begin{pmatrix}\pi(H_t^{\tau,p}) & 0\\ 0 & 0\end{pmatrix}$$

with $\pi(H_t^{\tau,p})$ acting on $\left(\mathcal{H}_\pi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^K$. Using (7.4) it follows as in [BM, Corollary 2.2] that

$$\pi(H_t^{\tau,p}) = e^{t(\pi(\Omega)-\tau(\Omega))}\operatorname{Id}$$

on $\left(\mathcal{H}_\pi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^K$. As in (4.22) we get

$$\operatorname{tr}\pi(H_t^{\tau,p}) = \operatorname{tr}\pi(h_t^{\tau,p}). \tag{7.11}$$

Now let π be a unitary principal series representation $\pi_{n,\lambda}$ acting in the Hilbert space $\mathcal{H}_{n,\lambda}$. Using (7.11) and (2.12) we get

$$\Theta_{n,\lambda}(h_t^{\tau,p}) = e^{-t(\lambda^2+1-n^2/4+\tau(\Omega))}\dim\left(\mathcal{H}_{n,\lambda}\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^K. \tag{7.12}$$

Denote by $\mathbb{C}_n$ the M-module defined by σ_n. By Frobenius reciprocity [Kn, p. 208] we have

$$\dim\left(\mathcal{H}_{n,\lambda}\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^K=\dim\left(\mathbb{C}_n\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^M.$$

and by (7.9) we get

$$\Theta_{n,\lambda}(k_t^\tau)=e^{-t(\lambda^2+1-n^2/4+\tau(\Omega))}\sum_{p=1}^{3}(-1)^p p\dim\left(\mathbb{C}_n\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau\right)^M.$$

Choose an orthonormal basis of $\mathfrak{p}$ as in [Mil, p.9]. Using this basis it follows that as M-modules, $\mathfrak{p}$ and $\mathfrak{a}\oplus\mathfrak{n}$ are equivalent. Thus we get

$$\sum_{p=1}^{3}(-1)^p p\,\Lambda^p\mathfrak{p}^*=\sum_{p=1}^{3}(-1)^p p\left(\Lambda^p\mathfrak{n}^*+\Lambda^{p-1}\mathfrak{n}^*\right)=\sum_{p=0}^{2}(-1)^{p+1}\Lambda^p\mathfrak{n}^*. \tag{7.13}$$

Therefore, the Fourier transformation of k_t^τ is given by

$$\Theta_{n,\lambda}(k_t^\tau)=e^{-t(\lambda^2+1-n^2/4+\tau(\Omega))}\sum_{p=0}^{2}(-1)^{p+1}\dim(\mathbb{C}_n\otimes\Lambda^p\mathfrak{n}^*\otimes V_\tau)^M. \tag{7.14}$$

This formula can be simplified using the real version of Konstant's Bott-Borel-Weil theorem [Si]. We apply Lemma 3.3 to determine the $n\in\mathbb{Z}$ for which $\dim(\mathbb{C}_n\otimes\Lambda^p\mathfrak{n}^*\otimes V_\tau)^M\neq 0$. We decompose the characters on the right hand side of (3.18) according to (3.21). Let $\sigma_{\tau,w}\in\hat{M}$ and $\lambda_{\tau,w}\in\frac{1}{2}\mathbb{Z}$ be defined by (3.21). Using (3.18) and (6.16), we get

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\overline{\sigma_n(m_\gamma)}\int_{\mathbb{R}}\Theta_{n,\lambda}(k_t^\tau)e^{-i\ell(\gamma)\lambda}\,d\lambda\\&\qquad=\sum_{w\in W_G}(-1)^{\ell(w)+1}\sigma_{\tau,w}(m_\gamma)e^{-t\lambda_{\tau,w}^2}\frac{e^{-\ell(\gamma)^2/(4t)}}{(4\pi t)^{1/2}}.\end{aligned} \tag{7.15}$$

Next we consider the contribution of the identity to (7.10). By (7.7), k_t^τ is in $\mathcal{C}^q(G)$ for all $q>0$. Therefore we can apply the Plancherel formula for G (see [Kn, Theorem 11.2]). With respect to the normalizations of Haar measures used in [Kn], we have

$$k_t^\tau(e)=\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\Theta_{n,\lambda}(k_t^\tau)P_{\sigma_n}(i\lambda)\,d\lambda,$$

where $P_{\sigma_n}(z)$ is the Plancherel polynom (3.4). Repeating the arguments that led to (7.15), we get

$$k_t^\tau(e)=\sum_{w\in W_G}(-1)^{\ell(w)+1}e^{-t\lambda_{\tau,w}^2}\int_{\mathbb{R}}e^{-t\lambda^2}P_{\sigma_{\tau,w}}(i\lambda)\,d\lambda. \tag{7.16}$$

Combined with (7.10) and (7.15), we obtain

$$K(t,\tau)=\sum_{w\in W_G}(-1)^{\ell(w)+1}e^{-t\lambda^2_{\tau,w}}\Bigg(\operatorname{vol}(X)\int_{\mathbb{R}}e^{-t\lambda^2}P_{\sigma_{\tau,w}}(i\lambda)\,d\lambda \\ +\sum_{\{\gamma\}\neq\{e\}}\frac{\ell(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma_{\tau,w})\frac{e^{-\ell(\gamma)^2/(4t)}}{(4\pi t)^{1/2}}\Bigg), \tag{7.17}$$

where $L(\gamma,\sigma)$ is defined by

$$L(\gamma,\sigma)=\frac{\sigma(m_\gamma)e^{-\ell(\gamma)}}{\det\left(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}}\right)}. \tag{7.18}$$

Unfortunately, the constants $\lambda_{\tau,w}$ appearing in the exponential factors prevent us from applying the Mellin transform to this formula directly. This problem occurred already in [Fr]. To overcome this problem we use the auxiliary operators introduced in section 4.

Using (4.26) and (6.16), it follows that for $w\in W_G$ we have

$$e^{\tau(\Omega)t}K(t,\sigma_{\tau,w})=e^{-t\lambda^2_{\tau,w}}\Bigg(2\operatorname{vol}(X)\int_{\mathbb{R}}e^{-t\lambda^2}P_{\sigma_{\tau,w}}(i\lambda)\,d\lambda \\ +\sum_{[\gamma]\neq 1}\frac{\ell(\gamma)}{n_\Gamma(\gamma)}\left(L(\gamma;\sigma_{\tau,w})+L(\gamma;w_A(\sigma_{\tau,w}))\right)\frac{e^{-\ell(\gamma)^2/(4t)}}{(4\pi t)^{1/2}}\Bigg). \tag{7.19}$$

Next observe that by (3.31) there exists a decomposition

$$W_G=W_0\sqcup W_1,$$

with $|W_i|=2$, $i=1,2$, and a bijection $j\colon W_0\to W_1$ such that for $w\in W_0$ we have

$$\sigma_{\tau,j(w)}=w_A(\sigma_{\tau,w}),\quad \lambda_{\tau,j(w)}=-\lambda_{\tau,w}.$$

Hence by (7.17) and (7.19) we get

$$K(t,\tau)=\frac{1}{2}\sum_{w\in W_G}(-1)^{\ell(w)+1}e^{\tau(\Omega)t}K(t;\sigma_{\tau,w}). \tag{7.20}$$

This equality can be expressed in a slightly different way as follows. Denote by Tr_s the supertrace with respect to the grading of $E(\sigma_{\tau,w})$. Using the definition of $K(t,\sigma_{\tau,w})$ by (4.11) and the definition of $\Delta(w)$ by (6.15), we get

$$K(t,\tau)=\frac{1}{2}\sum_{w\in W_G}(-1)^{\ell(w)+1}\,\mathrm{Tr}_s\left(e^{-t\Delta(w)}\right). \tag{7.21}$$

To continue we need to determine the location of the spectrum of the operators $A(\sigma)$.

Lemma 7.1. *For $\sigma\in\hat{M}$ we have $A(\sigma)\geq -1$. Moreover, if $k\notin\{0,\pm 2\}$, then $A(\sigma_k)>-1$.*

Proof. Let $\hat{G}$ denote the unitary dual of G. Let

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus}_{\pi\in\hat{G}} m_\Gamma(\pi)\mathcal{H}_\pi \tag{7.22}$$

be the spectral decomposition of the right regular representation of G on $L^2(\Gamma\backslash G)$. Let (ν, V_ν) be an irreducible unitary representation of K. Then $L^2(X, E_\nu) \cong (L^2(\Gamma\backslash G) \otimes V_\nu)^K$. Using (7.22), we get

$$\left(L^2(\Gamma\backslash G) \otimes V_\nu\right)^K = \widehat{\bigoplus}_{\pi\in\hat{G}} m_\Gamma(\pi)\left(\mathcal{H}_\pi \otimes V_\nu\right)^K. \tag{7.23}$$

This decomposition corresponds to the spectral resolution of A_ν as follows. Assume that $m_\Gamma(\pi)\dim(\mathcal{H}_\pi \otimes V_\nu)^K \neq 0$. Then $m_\Gamma(\pi)(\mathcal{H}_\pi \otimes V_\nu)^K$ is an eigenspace of A_ν with eigenvalue $-\pi(\Omega)$. Note that $\hat{G}$ is the union of the trivial representation, the unitary principal series $\pi_{k,\lambda}$ with $k \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$, and the complementary series π_x^c with $0 < x < 1$ [KS, Proposition 49], [Kn, Theorem 16.2] (where for the latter reference the different parametrization of the induced representations has to be taken into account).

First consider the principal series $\pi_{n,\lambda}$. By Frobenius reciprocity [Kn, p. 208] we have for $l \in \mathbb{N}_0$

$$\dim(\mathcal{H}_{\pi_{k,\lambda}} \otimes V_{\nu_l})^K = [\pi_{k,\lambda}^\vee|_K : \nu_l] = [\nu_l|_M : \sigma_{-k}] = [\nu_l|_M : \sigma_k]. \tag{7.24}$$

By (4.1) it follows that $(\mathcal{H}_{\pi_{k,\lambda}} \otimes V_{\nu_l})^K \neq 0$ implies $l \geq k$. Moreover, by (4.2) it follows that $m_{\nu_l}(\sigma_m) \neq 0$ implies $m \geq l$. Thus if $m_\nu(\sigma_m) \neq 0$ and $(\mathcal{H}_{\pi_{k,\lambda}} \otimes V_\nu)^K \neq 0$, then we have $m \geq k$. Hence if $\nu \in \hat{K}$ and $\sigma \in \hat{M}$ are such that $m_\nu(\sigma) \neq 0$ and $(\mathcal{H}_{\pi_{k,\lambda}} \otimes V_\nu)^K \neq 0$, then it follows from (2.12) and (4.9) that

$$-\pi_{k,\lambda}(\Omega) + c(\sigma) \geq 0. \tag{7.25}$$

Next consider the complementary series. By (4.9) we have $c(\sigma) \geq -1$ for all $\sigma \in \hat{M}$. Since $0 < x < 1$, it follows from (2.14) that

$$-\pi_x^c(\Omega) + c(\sigma) > -1. \tag{7.26}$$

for all $\sigma \in \hat{M}$. Finally, the trivial representation of G occurs in (7.23) only if ν is the trivial representation ν_0. Moreover, by (4.2) we have $m_{\nu_0}(\sigma_l) \neq 0$, only if $l = 0$ or $l = 2$. Thus by (7.25), (7.26), and the definition of $A(\sigma)$ by (4.10), the statement of the Lemma follows. □

We apply this lemma to study the kernel of the operator $\Delta(w)$, $w \in W_G$, which is defined by (6.15).

Lemma 7.2. *Let τ be an irreducible, finite-dimensional representation of G. Assume that $\tau_\theta \ncong \tau$. Then* $\ker\Delta(w) = \{0\}$ *for all $w \in W_G$.*

Proof. Let $\tau = \tau_{m,n}$ with $m \neq n$. We use (6.17) to express $\Delta(w)$ in terms of $A(\sigma_{\tau,w})$. By (3.31) we have $\lambda_{\tau,w} \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$ for all $w \in W_G$. If $|\lambda_{\tau,w}| > 1$, it follows from (6.17) that $\Delta(w) > 0$. It remains to consider the cases $\lambda_{\tau,w} = \pm 1$ and $\lambda_{\tau,w} = \pm 1/2$. In the first case we have $|m-n|=2$. Then it follows from (3.31) that $\sigma_{\tau,w}=\sigma_{2l}$ with $|l|\geq 2$. By Lemma 7.1 we get $\Delta(w)>0$. In the second case we have $|m-n|=1$. By (3.31) it follows that $\sigma_{\tau,w}=\sigma_{2l+1}$ for some $l\in\mathbb{Z}$. Let $\nu\in\hat{K}$ such that $m_\nu(\sigma_{2l+1})\neq 0$. By (4.2) there exists $p\in\mathbb{N}_0$ such that $\nu=\nu_{2p+1}$. Since π_x^c is induced from the trivial representation and $[\nu_{2p+1}|_M:\sigma_0]=0$, Frobenius reciprocity [Kn, p. 208] implies

$$\dim(\mathcal{H}_{\pi_x^c}\otimes V_{\nu_{2p+1}})^K=[\pi_x^c|_K:\nu_{2p+1}]=[\nu_{2p+1}|_M:\sigma_0]=0. \tag{7.27}$$

Thus in this case the complementary series does not occur in (7.23). Also the trivial representation does not occur. By (7.25) it follows that $A(\sigma_{\tau,w})\geq 0$. Using (6.17) we get $\Delta(w)>0$. ∎

Now we can turn to the proof of Theorem 1.5. First assume that $\tau\not\cong\tau_\theta$. Then it follows from [BW, Chapt. VII, Theorem 6.7] that $H^*(X,E_\tau)=0$. Hence $\Delta_p(\tau)>0$ for all p, $0\leq p\leq 3$. By Lemma 7.2 we also have $\Delta(w)>0$, $w\in W_G$. Hence $K(t,\tau)$ and $\operatorname{Tr}(e^{-t\Delta(w)})$, $w\in W_G$, are exponentially decreasing as $t\to\infty$. Therefore we can take the Mellin transform of both sides of (7.21) and we get

$$\frac{1}{\Gamma(s)}\int_0^\infty K(t,\tau)t^{s-1}\,dt=\frac{1}{2}\sum_{w\in W_G}(-1)^{\ell(w)+1}\frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}_s\left(e^{-t\Delta(w)}\right)t^{s-1}\,dt,$$

which holds for $\operatorname{Re}(s)>3/2$. After analytic continuation we compare the derivatives at $s=0$ of both sides. Using (7.2) we get

$$T_X(\tau)^4=\prod_{p=1}^{3}\det\left(\Delta_p(\tau)\right)^{2(-1)^{p+1}p}=\prod_{w\in W_G}\det\nolimits_{\mathrm{gr}}\left(\Delta(w)\right)^{(-1)^{\ell(w)}}. \tag{7.28}$$

Now we use the determinant formula (6.2) to relate the right hand side to the value at zero of the Ruelle zeta function. Since $\Delta(w)>0$, it follows that $\det_{\mathrm{gr}}(s^2-2s\lambda_{\tau,w}+\Delta(w))$ is regular at $s=0$ and its value at $s=0$ is equal to $\det_{\mathrm{gr}}\left(\Delta(w)\right)\neq 0$. Hence

$$\lim_{s\to 0}\prod_{w\in W_G}\det\nolimits_{\mathrm{gr}}\left(s^2-2s\lambda_{\tau,w}+\Delta(w)\right)^{(-1)^{\ell(w)}}=\prod_{w\in W_G}\det\nolimits_{\mathrm{gr}}\left(\Delta(w)\right)^{(-1)^{\ell(w)}}. \tag{7.29}$$

By (6.20) it follows that $R_\tau(s)R_{\tau_\theta}(s)$ is regular at zero. Now observe that $\overline{\tau}=\tau_\theta$. Furthermore by (3.13) we have

$$\overline{R_\tau(s)}=R_{\tau_\theta}(\overline{s}).$$

This implies that $R_\tau(s)$ is regular at $s=0$ and by (6.20) we get

$$|R_\tau(0)|^2=\prod_{w\in W_G}\det\nolimits_{\mathrm{gr}}\left(\Delta(w)\right)^{(-1)^{\ell(w)}}. \tag{7.30}$$

Combining (7.28) and (7.30), the first statement of Theorem 1.5 follows.

Next assume that $\tau_\theta=\tau$. Then there exists $m\in\mathbb{N}_0$ such that $\tau=\tau_{m,m}$. We use (6.21) to determine the order of $R_\tau(s)$ at $s=0$. For $m\in\mathbb{N}_0$ let

$$h_m=\dim\nolimits_{\mathrm{gr}}\ker(A(\sigma_{2m+2})), \tag{7.31}$$

where $\dim_{\mathrm{gr}}$ denotes the graded dimension of a graded vector space, i.e., if $V = V^+ \oplus V^-$ is a graded finite-dimensional vector space, then $\dim_{\mathrm{gr}} V = \dim V^+ - \dim V^-$. Assume that $m \geq 1$. Then $\det\left((s \pm (m+1))^2 - 1 + \Delta\right)$ is regular and nonzero at $s = 0$. Furthermore, $\det_{\mathrm{gr}}\left(s^2 + A(\sigma_{2m+2})\right)$ has order $2h_m$ at $s = 0$ and

$$\lim_{s \to 0} s^{-2h_m} \det_{\mathrm{gr}}\left(s^2 + A(\sigma_{2m+2})\right) = \det_{\mathrm{gr}}\left(A(\sigma_{2m+2})\right). \tag{7.32}$$

By (6.21), it follows that $R_\tau(s)$ has order $-2h_m$ at $s = 0$ and we have

$$\lim_{s \to 0} s^{2h_m} R_{\tau_{m,m}}(s) = \frac{\det\left((m+1)^2 - 1 + \Delta\right)^2}{\det_{\mathrm{gr}}\left(A(\sigma_{2m+2})\right)}. \tag{7.33}$$

On the other hand, using (3.32), it follows from (7.21) that

$$K(t, \tau) = \operatorname{Tr}_s\left(e^{-tA(\sigma_{2m+2})}\right) - 2e^{-t\left((m+1)^2 - 1\right)} \operatorname{Tr}\left(e^{-t\Delta}\right). \tag{7.34}$$

Taking the limit $t \to \infty$ of both sides of (7.34), we get

$$h_m = \sum_{p=1}^{3} (-1)^p p \dim\left(\ker \Delta_p(\tau)\right). \tag{7.35}$$

Moreover (7.34) also implies

$$T_X(\tau)^2 = \prod_{p=1}^{3} \det\left(\Delta_p(\tau)\right)^{(-1)^{p+1}p} = \frac{\det\left((m+1)^2 - 1 + \Delta\right)^2}{\det_{\mathrm{gr}}\left(A(\sigma_{2m+2})\right)}.$$

Combining this equality with (7.33) and (7.35), we obtain the second statement of Theorem 1.5 for $m \geq 1$. Finally consider the case $\tau = 1$. In this case we need to take into account the simple zero of $\det(s^2 \pm 2s + \Delta)$ at $s = 0$. Thus we get

$$\lim_{s \to 0} s^{2h_0 - 2} R_1(s) = \frac{\det(\Delta)^2}{\det_{\mathrm{gr}}\left(A(\sigma_2)\right)}. \tag{7.36}$$

Furthermore, (7.34) gives.

$$h_0 = \sum_{p=1}^{3} (-1)^p p \dim H^p(X, \mathbb{R}) + 2 = \dim H^1(X, \mathbb{R}) - 1.$$

As above, this implies that the order of $R_1(s)$ at $s = 0$ equals $4 - 2\dim H^1(X, \mathbb{R})$.

8. Proof of Theorem 1.1

We are now ready to prove our main result. We consider the representation τ_m. Using (2.9), it follows from (3.14) that

$$R_{\tau_m}(s) = \prod_{k=0}^{m} R\left(s - (m/2 - k), \sigma_{m-2k}\right). \tag{8.1}$$

We distinguish the cases where m is odd and even. Let $m \geq 3$. Then we get

$$
\begin{aligned}
R_{\tau_{2m}}(s) &= \prod_{k=0}^{4} R(s-(2-k), \sigma_{4-2k}) \prod_{k=3}^{m} R(s-k, \sigma_{2k}) R(s+k, \sigma_{-2k}) \\
&= R_{\tau_4}(s) \prod_{k=3}^{m} R(s-k, \sigma_{2k}) R(s+k, \sigma_{-2k}).
\end{aligned}
\tag{8.2}
$$

Similarly, for $m \geq 2$ we get

$$
R_{\tau_{2m+1}}(s) = R_{\tau_3}(s) \prod_{k=2}^{m} R(s-k-1/2, \sigma_{2k+1}) R(s+k+1/2, \sigma_{-(2k+1)}).
\tag{8.3}
$$

Now recall that by Proposition 3.2, 1), each $R(s, \sigma_l)$, $l \in \mathbb{Z}$, is regular in the half-plane $\operatorname{Re}(s) > 2$ and does not vanish in this half-plane. By the functional equation (3.11) the same holds in the half-plane $\operatorname{Re}(s) < -2$. Therefore the products on the right hand side of (8.2) and (8.3) are regular at $s = 0$. Furthermore it follows from (3.6) that

$$
|R(s, \sigma_l)| = |R(\overline{s}, \sigma_{-l})|.
\tag{8.4}
$$

Using (8.2), (8.3) and Theorem (1.5) we get

$$
T_X(\tau_{2m})^2 = T_X(\tau_4)^2 \prod_{k=3}^{m} |R(k, \sigma_{2k})| \cdot |R(-k, \sigma_{2k})|, \quad m \geq 3.
\tag{8.5}
$$

and

$$
T_X(\tau_{2m+1})^2 = T_X(\tau_3)^2 \prod_{k=2}^{m} |R(k+1/2, \sigma_{2k+1})| \cdot |R(-k-1/2, \sigma_{2k+1})|, \quad m \geq 2.
\tag{8.6}
$$

By the functional equation (3.11) and (8.4) we get

$$
|R(-k, \sigma_{2k})| = \exp\left(-\frac{4}{\pi} \operatorname{vol}(\Gamma\backslash\mathbb{H}^3) k\right) |R(k, \sigma_{2k})|
$$

Together with (8.5) this leads to

$$
T_X(\tau_{2m}) = T_X(\tau_4) \prod_{k=3}^{m} \exp\left(-\frac{2}{\pi} \operatorname{vol}(\Gamma\backslash\mathbb{H}^3) k\right) |R(k, \sigma_{2k})|.
\tag{8.7}
$$

Similarly

$$
T_X(\tau_{2m+1}) = T_X(\tau_3) \prod_{k=2}^{m} \exp\left(-\frac{2}{\pi} \operatorname{vol}(\Gamma\backslash\mathbb{H}^3)(k+1/2)\right) |R(k+1/2, \sigma_{2k+1})|.
\tag{8.8}
$$

To continue we need the following estimation.

Lemma 8.1. *There exists $C > 0$ such that for all $m \in \mathbb{N}$, $m \geq 3$, we have*

$$
\sum_{k=3}^{m} \big| \log |R(k, \sigma_{2k})| \big| \leq C, \quad \sum_{k=2}^{m} \big| \log |R(k+1/2, \sigma_{2k+1})| \big| \leq C.
$$

Proof. We consider the first case. Since $|\sigma_{2k}(m_\gamma)| = 1$, we have

$$1 - e^{-k\ell(\gamma)} \le \left|1 - \sigma_{2k}(m_\gamma) e^{-k\ell(\gamma)}\right| \le 1 + e^{-k\ell(\gamma)}.$$

Let $k \ge 3$. Using that the infinite product (3.6) is absolutely convergent for $\operatorname{Re}(s) > 2$, we get

$$\sum_{\substack{[\gamma]\neq e\\ \text{prime}}} \log\left(1 - e^{-k\ell(\gamma)}\right) \le \log|R(k,\sigma_{2k})| \le \sum_{\substack{[\gamma]\neq e\\ \text{prime}}} \log\left(1 + e^{-k\ell(\gamma)}\right),$$

which implies

$$\big|\log|R(k,\sigma_{2k})|\big| \le \sum_{\substack{[\gamma]\neq e\\ \text{prime}}} \sum_{n=1}^{\infty} \frac{1}{n} e^{-nk\ell(\gamma)}. \tag{8.9}$$

Let $\delta = \inf\{\ell(\gamma) \colon \gamma \in \Gamma \setminus \{e\}\}$. Put $C_1 = (1 - e^{-\delta})^{-1}$. Using (8.9) we get

$$\begin{aligned}
\sum_{k=3}^{m} \big|\log|R(k,\sigma_{2k})|\big| &\le \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{[\gamma]\neq e\\ \text{prime}}} \sum_{k=3}^{m} e^{-\ell(\gamma)nk}\\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{[\gamma]\neq e\\ \text{prime}}} \left(\frac{1 - e^{-n(m+1)\ell(\gamma)}}{1 - e^{-n\ell(\gamma)}} - \left(1 + e^{-n\ell(\gamma)} + e^{-2n\ell(\gamma)}\right)\right)\\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{[\gamma]\neq e\\ \text{prime}}} \frac{e^{-3n\ell(\gamma)} - e^{-(m+1)n\ell(\gamma)}}{1 - e^{-n\ell(\gamma)}}\\
&\le C_1 \sum_{\substack{[\gamma]\neq e\\ \text{prime}}} \sum_{n=1}^{\infty} \frac{e^{-3n\ell(\gamma)}}{n}\\
&= C_1 \log R(3,\sigma_0)^{-1} = C.
\end{aligned}$$

The other case is similar. ∎

Taking the logarithm of both sides of (8.7) and (8.8), respectively, we obtain

$$\log T_X(\tau_{2m}) = \log T_X(\tau_4) + \sum_{k=3}^{m} \log|R_{2k}(k)| - \frac{1}{\pi} \operatorname{vol}(\Gamma\backslash\mathbb{H}^3)\,(m(m+1) - 6)\,.$$

and

$$\log T_X(\tau_{2m+1}) = \log T_X(\tau_3) + \sum_{k=2}^{m} \log\left|R_{2k+1}\left(k + \frac{1}{2}\right)\right| - \frac{1}{\pi} \operatorname{vol}(\Gamma\backslash\mathbb{H}^3)(m(m+2) - 3).$$

Applying Lemma 8.1 we get

$$-\log T_X(\tau_m) = \frac{1}{4\pi} \operatorname{vol}(\Gamma\backslash\mathbb{H}^3) m^2 + O(m)$$

as $m \to \infty$. This completes the proof of Theorem 1.1.

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