On the analytic continuation of rank one Eisenstein series

by

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0. Introduction

Let X be a hyperbolic surface of finite area and let $\Delta : C^\infty(X) \to C^\infty(X)$ be the Laplacian of X. Regarded as linear operator in $L^2(X)$ with domain $C_0^\infty(X)$, Δ is essentially self-adjoint. Let $\overline{\Delta}$ be the unique self-adjoint extension of Δ. The continuous spectrum of $\overline{\Delta}$ can be described explicitly by Eisenstein series (cf. [S], [H]). Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a discrete cofinite subgroup such that $X = \Gamma\backslash H$ where H denotes the upper half–plane. The Eisenstein series are attached to the parabolic fixed points of Γ. If $a \in \mathbb{R} \cup \{\infty\}$ is a given parabolic fixed point of Γ, denote by Γ_a the stabilizer of a in Γ. Pick $\sigma \in \mathrm{PSL}(2,\mathbb{R})$ such that $\sigma a = \infty$ and $\sigma\Gamma_a\sigma^{-1} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$. Then the Eisenstein series attached to a is defined by

$$E_a(z,s) = \sum_{\gamma \in \Gamma_a\backslash\Gamma} \mathrm{Im}(\sigma\gamma z)^s, \quad \mathrm{Re}(s) > 1.$$

This series is absolutely convergent in the half–plane $\mathrm{Re}(s) > 1$ and in this half-plane, it satisfies

$$\Delta E_a(z,s) = s(1-s)E_a(z,s). \tag{1}$$

One of the main features of the Eisenstein series is the fact that as functions of s, they admit meromorphic continuations to $\mathbb{C}$ and satisfy a system of functional equations. Furthermore, the functions obtained by analytic continuation also satisfy (1) and, by restriction of the spectral parameter to the line $\mathrm{Re}(s) = 1/2$, one obtains generalized eigenfunctions of the Laplacian. There exist various methods to achieve the analytic continuation of Eisenstein series [S], [F], [C]. The purpose of this paper is to present a new method which may also be used in other cases to construct generalized eigenfunctions.

We shall work in the setting of complete surfaces X of finite area with finitely many cusps. For simplicity, we suppose that X has a single cusp. Thus

$$X = X_0 \cup Z$$

where X_0 is a compact surface with smooth boundary $\partial X_0 = S^1$ and

$$Z \cong [1,\infty) \times S^1, \ \text{ equipped with the metric } ds^2 = (dy^2 + d\theta^2)/y^2.$$

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As above, the Laplacian Δ of X is essentially self-adjoint. The starting point for our method is the same as in [C]. As observed by Colin de Verdiere [C], for $\mathrm{Re}(s) > 1/2$, $s \notin (1/2, 1]$, there is a unique solution of the equation $\Delta E(z,s) = s(1-s)E(z,s)$ such that $y^s - E((y,\theta),s)$ is square integrable on Z. This solution can be constructed as follows. Pick $f \in C^\infty(\mathbb{R})$ such that $f(y) = 0$ for $y \leq 1$ and $f(y) = 1$ for $y \geq 2$. Set $h(s) = (\Delta - s(1-s))(f(y)y^s)$. Then we have $h(s) \in C_0^\infty(X)$ and

$$E(z,s) = f(y)y^s - (\overline{\Delta} - s(1-s))^{-1}\big(h(s)\big), \quad \mathrm{Re}(s) > 1/2,\ s \notin (1/2, 1]. \tag{2}$$

Since the continuous spectrum of $\overline{\Delta}$ equals $[1/4, \infty)$ (see e.g. [Mü1], p.253), (2) is a meromorphic function in the half–plane $\mathrm{Re}(s) > 1/2$. If X is hyperbolic, then $E(z,s)$ coincides in $\mathrm{Re}(s) > 1$ with the Eisenstein series attached to the unique cusp of X. In particular, (2) provides the analytic continuation of the Eisenstein series up to the critical line $\mathrm{Re}(s) = 1/2$. To continue $E(z,s)$ across the line $\mathrm{Re}(s) = 1/2$, it is sufficient to continue the resolvent $(\overline{\Delta} - s(1-s))^{-1}$ to a meromorphic function on $\mathbb{C}$. This can not be achieved in $L^2(X)$ and, therefore, we replace $L^2(X)$ by certain weighted L^2 spaces $L^2_\delta(X)$, $\delta \in \mathbb{R}$. For each $\delta > 0$, we may regard $(\overline{\Delta} - s(1-s))^{-1}$ as operator from $L^2_\delta(X)$ into $L^2_{-\delta}(X)$ and as operator in these spaces, the resolvent admits a meromorphic continuation to $\mathbb{C}$.

We shall now briefly describe the content of the paper. In §1 we construct an appropriate parametrix $Q(s)$ for the resolvent in the half–plane $\mathrm{Re}(s) > 1/2$ and then we continue analytically $Q(s)$ to a meromorphic function with values in the bounded operators from $L^2_\delta(X)$ into $L^2_{-\delta}(X)$. In §2 we use the parametrix to continue the resolvent to a meromorphic operator valued function on $\mathbb{C}$. In §3 we construct and study the corresponding generalized eigenfunctions. In §4 we discuss $\mathbb{Q}$–rank one locally symmetric spaces of finite volume and we show how our method can be extended to this case.

1. The parametrix

In this § we shall construct a parametrix for $(\overline{\Delta} - \lambda)^{-1}$. Let $X_1 = X_0 \cup ([1,2] \times S^1)$ and let M be a closed Riemannian surface such that X_1 is isometrically embedded into M. Let Δ^M be the Laplacian of M. Since M is closed, the resolvent $(\Delta^M - \lambda)^{-1}$ is a meromorphic function of $\lambda \in \mathbb{C}$ with simple poles at the eigenvalues of Δ^M. Let $Q_1(x, x', s)$ be the restriction of the kernel of the resolvent $(\Delta^M - s(1-s))^{-1}$ to $X_1 \times X_1$. This is the interior part of the parametrix. For the exterior part consider the Laplacian on the cusp Z as linear operator in $L^2(Z)$ with domain $C_0^\infty(Z)$. Impose Dirichlet boundary conditions at ∂Z and let Δ^Z be the corresponding self-adjoint extension. Put $Q_2(s) = (\Delta^Z - s(1-s))^{-1}$. This will be our parametrix at infinity.

Let $\phi_2 \in C^\infty(\mathbb{R})$ be such that $\phi_2(u) = 0$ for $u \leq 3/2$ and $\phi_2(u) = 1$ for $u \geq 2$. We regard ϕ_2 as function on $[1,\infty) \times S^1$ and extend it by zero to a smooth function on X. Let $\phi_1 = 1 - \phi_2$. Pick $\psi_1 \in C_0^\infty(X)$ such that $\psi_1 \equiv 1$ on $\operatorname{supp} \phi_1$ and let $\psi_2 \in C^\infty(X)$ be such that $\psi_2 \equiv 1$ on $\operatorname{supp} \phi_2$ and $\psi_2 = 0$ on X_0. Then we define the parametrix for $(\overline{\Delta} - s(1-s))^{-1}$ in $\mathrm{Re}(s) > 1/2$, $s \notin (1/2, 1]$ by

$$Q(s) = \phi_1 Q_1(s) \psi_1 + \phi_2 Q_2(s) \psi_2. \tag{3}$$

It follows immediately from the definition that $Q(s)$ extends to a bounded operator $Q(s) : L^2(X) \to H^2(X)$ where $H^2(X)$ denotes the Sobolev space. To prove that $Q(s)$ is indeed a parametrix for $(\overline{\Delta} - s(1-s))^{-1}$ we have to investigate the resolvent of Δ^Z. First of all, it is easy to determine the spectral resolution of Δ^Z. Let $L^2_0(Z) \subset L^2(Z)$ be the subspace of all cusp functions, that is, $f \in L^2(Z)$ belongs to $L^2_0(Z)$ if and only if

$$\int_{S^1} f(y,\theta)\, d\theta = 0 \quad \text{for almost all} \quad y \in [1,\infty).$$

The orthogonal complement of $L^2_0(Z)$ in $L^2(Z)$ is the space of functions on Z which are independent of θ. Thus $L^2(Z)$ splits as

$$L^2(Z) = L^2_0(Z) \oplus L^2\big([1,\infty); dy/y^2\big). \tag{4}$$

This decomposition is invariant under Δ^Z. Let Δ^Z_0 be the restriction of Δ^Z to $L^2_0(Z)$. Next consider $-y^2\, d^2/dy^2$, regarded as operator in $L^2([1,\infty); dy/y^2)$ with domain $C^\infty_0\big((1,\infty)\big)$, and impose Dirichlet boundary conditions. Let A be the corresponding self-adjoint extension. Then with respect to the splitting (4), we have $\Delta^Z = \Delta^Z_0 \oplus A$. If we proceed as in the proof of Lemma 8.7 in [L-P], it follows that Δ^Z_0 has pure point spectrum and, therefore, $(\Delta^Z_0 - s(1-s))^{-1}$ is a compact operator in $L^2(Z)$. The spectrum of A is absolutely continuous and equals $[1/4,\infty)$. Moreover, the kernel of $(A - s(1-s))^{-1}$ in the half–plane $\mathrm{Re}(s) > 1/2$ can be explicitly determined. It is given by

$$q(y,w,s) = \frac{1}{2s-1}(yw)^{1/2} \begin{cases} (y/w)^{1/2-s} - (yw)^{1/2-s}, & y > w; \\ (w/y)^{1/2-s} - (yw)^{1/2-s}, & w > y. \end{cases} \tag{5}$$

Using the observation above together with (5), it is easy to see that

$$Q(s)(\Delta - s(1-s)) = \mathrm{Id} + K(s), \quad \mathrm{Re}(s) > 1/2, \tag{6}$$

where $K(s)$ is a compact operator in $L^2(X)$. Hence, it is legitimate to call $Q(s)$ a parametrix for $(\overline{\Delta} - s(1-s))^{-1}$.

Now we shall show that $Q(s)$ admits a meromorphic continuation to $\mathbb{C}$. First, consider $Q_2(s)$. To continue this operator analytically, we introduce weighted Sobolev spaces. Given $\delta \in \mathbb{R}$, we define the L^2 space with weight δ by

$$L^2_\delta(Z) = \big\{ f : Z \to \mathbb{C} \mid f \text{ measurable and } \int_Z |f(y,\theta)|^2 e^{2\delta y}\, dy d\theta < \infty \big\}.$$

Let $\delta > 0$. Then the following inclusions hold

$$L^2_\delta(Z) \subset L^2(Z) \subset L^2_{-\delta}(Z). \tag{7}$$

Recall that $(\Delta^Z_0 - s(1-s))^{-1} : L^2(Z) \to L^2(Z)$ is a meromorphic function of $s \in \mathbb{C}$. Hence, by composition with the inclusions (7), we get a meromorphic family of bounded linear operators

$$(\Delta^Z_0 - s(1-s))^{-1} : L^2_\delta(Z) \to L^2_{-\delta}(Z), \quad s \in \mathbb{C}.$$

Next observe that the kernel of $(A - s(1-s))^{-1}$ which is given by (5), has an obvious analytic continuation. Let $f \in L^2_\delta(Z)$. Using (5), it follows that the integral

$$(q(s)f)(y) = \int_1^\infty q(y,w,s) f(w)\, \frac{dw}{w^2} \tag{8}$$

is absolutely convergent and $q(s)f$ belongs to $L^2_{-\delta}(Z)$. In this way we get a meromorphic family, depending on $s \in \mathbb{C}$, of bounded linear operators

$$q(s) : L^2_\delta(Z) \to L^2_{-\delta}(Z).$$

By (4), we obtain

Proposition 1. *Let $\delta > 0$. Then the resolvent $(\Delta^Z - s(1-s))^{-1}$, $\mathrm{Re}(s) > 1/2$, extends to a meromorphic function $s \in \mathbb{C} \mapsto \widetilde{Q}_2(s)$ of bounded linear operators from $L^2_\delta(Z)$ into $L^2_{-\delta}(Z)$.*

To extend this result to $Q(s)$, we introduce global weighted L^2 spaces. Let $\rho \in C^\infty(X)$ be such that $\rho(x) = 1$ for $x \in X_0$ and $\rho(y,\theta) = y$ for $(y,\theta) \in [2,\infty) \times S^1$. Given $\delta \in \mathbb{R}$, let

$$L^2_\delta(X) = \big\{ f : X \to \mathbb{C} \mid f \text{ measurable and } \int_X |f(x)|^2 e^{2\delta\rho(x)}\, dx < \infty \big\}.$$

For $\delta > 0$, we have the inclusions

$$L^2_\delta(X) \subset L^2(X) \subset L^2_{-\delta}(X).$$

By construction, $Q_1(s)$ has an obvious extension to a meromorphic function $\widetilde{Q}_1(s)$ on $\mathbb{C}$. Using Proposition 1, we get immediately

Proposition 2. *Let $\delta > 0$. Then $Q(s)$ extends to a meromorphic function $s \in \mathbb{C} \mapsto \widetilde{Q}(s)$ of bounded linear operators $\widetilde{Q}(s) : L^2_\delta(X) \to L^2_{-\delta}(X)$.*

2. Analytic continuation of the resolvent

Now consider the operator $K(s)$ which is defined by the right hand side of (6). Let

$$\Omega = \big\{ s \in \mathbb{C} \mid \mathrm{Re}(s) > 1/2 \big\} - (1/2, 1] \big\}.$$

Recall that $K(s) : L^2(X) \to L^2(X)$ is defined for all $s \in \Omega$ and $s \in \Omega \mapsto K(s)$ is a holomorphic function of compact operators in $L^2(X)$. Let $\mathcal{L}(L^2(X))$ be the space of bounded linear operators in $L^2(X)$.

Proposition 3. *Let $\delta > 0$. Then the holomorphic function $s \in \Omega \mapsto K(s) \in \mathcal{L}\big(L^2(X)\big)$ extends to a meromorphic function $s \in \mathbb{C} \mapsto K(s)$ of compact operators in $L^2_{-\delta}(X)$.*

Proof. Let ψ_i' and ψ_i'' denote the first and second derivative, respectively, of ψ_i, $i = 1, 2$. By definition, we have $\operatorname{supp}\psi_i' \subset (1,2) \times S^1$. Let D and V denote the operators in $C^\infty((1,2) \times S^1)$ which are defined by $(Df)(v,\theta) = v^2(\partial f/\partial v)(v,\theta)$ and $(Vf)(v,\theta) = v^2 f(v,\theta)$, respectively. Then, by definition of $K(s)$, we have

$$K(s) = -\phi_1 Q_1(s) D\,\psi_1' \; - \; \phi_1 Q_1(s) V \psi_1'' \; - \; \phi_2 Q_2(s) D\,\psi_2' \; - \; \phi_2 Q_2(s) V \psi_2''. \tag{9}$$

Now recall that $Q_1(s)$ is the resolvent of the Laplacian Δ^M of the closed surface M. Therefore, by Rellich's theorem, it follows that the first two terms on the right of (9) are meromorphic functions on $\mathbb{C}$ with values in the compact operators on $L^2(X)$. Furthermore, since ψ_1' and ψ_1'' have compact support, multiplication by ψ_1' and ψ_1'', respectively, defines bounded linear operators $L^2_{-\delta}(X) \to L^2(X)$. Using the inclusion $L^2(X) \subset L^2_{-\delta}(X)$, it follows that the first two terms on the right of (9) are meromorphic functions on $\mathbb{C}$ with values in the compact operators in $L^2_{-\delta}(X)$.

To deal with the remaining terms we observe that

$$Q_2(s) = (\Delta_0^Z - s(1-s))^{-1} \oplus q(s) \tag{10}$$

where $q(s)$ is the operator defined by (8). This gives a corresponding splitting of the third and fourth term on the right of (9). Repeating the argument used above, it follows that $\phi_2(\Delta_0^Z - s(1-s))^{-1} D\,\psi_2'$ and $\phi_2(\Delta_0^Z - s(1-s))^{-1} V \psi_2''$ are meromorphic families of compact operators in $L^2_{-\delta}(X)$. Furthermore, by (4), we get

$$\int_0^\infty \int_1^2 |q(u,v,s)|^2 \frac{dv}{v^2}\, e^{-2\delta u} \frac{du}{u^2} < \infty.$$

This implies that $\phi_2 q(s) V \psi_2''$ and $\phi_2 q(s) D \psi_2'$ define both Hilbert–Schmidt operators in $L^2_{-\delta}(X)$ and hence, compact operators in $L^2_{-\delta}(X)$. q.e.d.

Lemma 1. *There exists $C > 0$ such that*

$$\| K(s) \| \leq \frac{C}{\sqrt{\operatorname{Re}(s)}}, \quad \operatorname{Re}(s) > 2.$$

Proof. Let $\operatorname{Re}(s) > 2$. Then it follows that $s(1-s) \in \mathbb{C} - \mathbb{R}^+$ and the first two terms on the right hand side of (9) can be estimated by a constant times

$$\| (\Delta^M - s(1-s))^{-1} (\Delta^M + 1)^{1/2} \| \leq \frac{1}{\operatorname{dist}(s(1-s), \mathbb{R}^+)} \leq \frac{1}{\sqrt{\operatorname{Re}(s)}}.$$

To estimate the two remaining terms on the right of (9), we use (10) to split these terms correspondingly. The contribution of $(\Delta_0^Z - s(1-s))^{-1}$ can be estimated as above. Finally, by (5) it follows that

$$\int_1^\infty \int_1^2 |q(u,v,s)|^2 \frac{dv}{v^2} \frac{du}{u^2} \leq \frac{C}{(2\operatorname{Re}(s) - 1)^2}.$$

This implies $\| \phi_2 q(s) V \psi_2'' \| \leq C \operatorname{Re}(s)^{-1}$. Similarly, we get $\| \phi_2 q(s) D \psi_2'' \| \leq C \operatorname{Re}(s)^{-1/2}$ and the claimed estimate follows. q.e.d.

Lemma 1 implies that for $\operatorname{Re}(s) \gg 0$, $\mathrm{Id}+K(s)$ is invertible as operator in $L^2(X)$. Hence, by (6) we get

$$(\overline{\Delta} - s(1-s))^{-1} = (\mathrm{Id} + K(s))^{-1} \circ Q(s), \quad \operatorname{Re}(s) > C, \tag{11}$$

for some constant $C > 0$. Here all operators act in $L^2(X)$. Now pick $\delta > 0$. It follows from (9) and Lemma 1 that $\| K(s) \|_{-\delta} \leq C_1 \operatorname{Re}(s)^{-1/2}$, $\operatorname{Re}(s) > 2$. Thus, for $\operatorname{Re}(s) \gg 0$, $\mathrm{Id}+K(s)$ is invertible as operator in $L^2_{-\delta}(X)$. Now, Theorem 1 of [St] together with Proposition 3 imply that $s \in \mathbb{C} \mapsto (\mathrm{Id} + K(s))^{-1}$ is a meromorphic function of bounded operators in $L^2_{-\delta}(X)$. Combining this result with Proposition 2, it follows that the right hand side of (11), regarded as operator of $L^2_{\delta}(X)$ into $L^2_{-\delta}(X)$, has an analytic continuation to $\mathbb{C}$. Put $R(s) = (\mathrm{Id}+K(s))^{-1} \circ \widetilde{Q}(s)$ and let $\mathcal{L}(L^2_{\delta}(X), L^2_{-\delta}(X))$ be the space of bounded linear operators of $L^2_{\delta}(X)$ into $L^2_{-\delta}(X)$. Summarizing, we have proved

Theorem 1. *Let $\delta > 0$. Then the resolvent of the Laplacian $\overline{\Delta}$ extends to a meromorphic function $s \in \mathbb{C} \mapsto R(s) \in \mathcal{L}(L^2_{\delta}(X), L^2_{-\delta}(X))$.*

3. Generalized eigenfunctions

Let $f \in C^\infty(\mathbb{R})$ and suppose that $f(y) = 0$ for $y \leq 1$ and $f(y) = 1$ for $y \geq 2$. Let $s \in \mathbb{C}$. We consider $f(y)y^s$ as a function on $[1, \infty) \times S^1 = Z$ in the obvious way and then extend it by zero to a C^∞ function on X. Let

$$h(s) = (\Delta - s(1-s))(f(y)y^s).$$

Then we have $h(s) \in C_0^\infty(X)$. In particular, $h(s)$ belongs to every weighted L^2 space $L^2_\delta(X)$, $\delta \in \mathbb{R}$. Let $\delta > 0$ and put

$$E(z,s) = f(y)y^s \; - \; R(s)\big(h(s)\big)(z), \quad s \in \mathbb{C}. \tag{12}$$

By Theorem 1, we have $E(s) \in L^2_{-\delta}(X)$, for all $s \in \mathbb{C}$ which are not poles of $R(s)$. In particular, $E(.,s)$ is locally integrable. Moreover, $E(s)$ is a meromorphic function of $s \in \mathbb{C}$ and for $s \in \Omega$, $E(z,s)$ is given by (2). Since $E(z,s)$ is locally integrable, we can apply Δ in the distributional sense. From (2) follows that $(\Delta - s(1-s))E(s) = 0$ for $s \in \Omega$. Since $(\Delta - s(1-s))E(s)$ is a meromorphic function, it vanishes for all $s \in \mathbb{C}$. By elliptic regularity, it follows that $E(s) \in C^\infty(X)$ and we have proved

Theorem 2. *The function $E(z,s)$, defined by (12), has the following properties*

1) *$E(z,s)$ is smooth in $z \in X$ and meromorphic in $s \in \mathbb{C}$.*
2) *$(\Delta - s(1-s))E(z,s) = 0$, $s \in \mathbb{C}$.*

3) For $s \in \Omega$, $f(y)y^s - E(z,s)$ is square integrable.

Since $\overline{\Delta}$ is self-adjoint, it follows that $E(z,s)$ is uniquely determined by properties 1) - 3). Furthermore, 2) and 3) imply that on $[1,\infty) \times S^1$, we have the following expansion

$$E((y,\theta),s) = y^s + C(s)y^{1-s} + \sum_{n\neq 0} a_n(s)\sqrt{y}\,K_{s-1/2}(2\pi|y|y)e^{2\pi in\theta}. \tag{13}$$

Here $K_w(y)$ denotes the modified Bessel function which decays exponentially in y. Hence, if $\mathrm{Re}(s) < 1/2$, then $E(z,s) - C(s)\,f(y)y^{1-s}$ is square integrable. Put $\omega(s) = C(s)E(1-s) - E(s)$ and let $\mathrm{Re}(s) > 1/2$. Then we have $\omega(s) \in L^2(X)$ and $(\Delta - s(1-s))\omega(s) = 0$. This implies $\omega(s) = 0$, that is, $C(s)E(1-s) = E(s)$. Comparing the expansions (13) of the left and right hand side, we get

Theorem 3. *For all $s \in \mathbb{C}$, the following functional equations hold*

$$C(s)E(1-s) = E(s) \quad \text{and} \quad C(s)C(1-s) = 1.$$

By (2), we get $\overline{E(z,s)} = E(z,\overline{s})$, $s \in \mathbb{C}$, and therefore

$$\overline{C(s)} = C(\overline{s}). \tag{14}$$

Together with the functional equation, (14) implies that

$$|C(1/2+i\lambda)|^2 = C(1/2+i\lambda)C(1/2-i\lambda) = 1.$$

Hence, $C(s)$ is regular on the line $\mathrm{Re}(s) = 1/2$.

Remark. Our method has an obvious extension to the case of several cusps. A parametrix can be constructed as above by patching an interior parametrix to the exterior parametrix which is now given by a sum with contributions from each cusp. To obtain the analytic continuation of the resolvent, we may proceed in exactly the same way as in the case of a single cusp. If we carry out the construction of section 3 for each cusp individually, we obtain a set of generalized eigenfunctions attached to the various cusps.

4. Locally symmetric spaces of Q-rank one

In this section we shall describe how our method can be used to obtain the analytic continuation of Eisenstein series for all locally symmetric spaces of $\mathbb{Q}$–rank one. Let G be a real reductive group of noncompact type, K a maximal compact subgroup of G, and Γ a rank one lattice in G such that the pair (G,Γ) satisfies the conditions of [OW, pp. 62-63]. Then $X = \Gamma\backslash G/K$ is a rank one locally symmetric space of finite volume. Examples are Hilbert modular varieties where $G = \mathrm{SL}(2,\mathbb{R})^m$ and Γ is commensurable with Hilbert modular group $\mathrm{SL}(2,\mathcal{O}_F)$ of some totally real number field F of degree m over $\mathbb{Q}$ [Hi]. As in the case of surfaces, we may work in the more general setting of manifolds X with cusps of rank one [Mü2]. Then X has a decomposition

$$X = X_0 \cup_i W_i$$

where X_0 is a compact manifold with boundary and W_i, $i = 1, ..., m$, are the ends of X. Moreover, each end W_i is isometric to a cusp in some locally symmetric space of rational rank one, which may depend on i. Thus, each cusp W_i is diffeomorphic to $\mathbb{R}^+ \times Y_i$ where Y_i, $i = 1, ..., k$, are the components of ∂X_0. The metric on the cusp is described in [Bo, p.247]. We change coordinates by $y = e^r$. Then the structure is as follows: There is a fibre bundle

$$\begin{array}{ccc} \Gamma_i \cap N_i \backslash N_i & \rightarrow & Y_i \\ & & \downarrow \\ & & Z_i \end{array} \tag{15}$$

where $\Gamma_i \cap N_i \backslash N_i$ is a compact nilmanifold and Z_i is a compact locally symmetric space. Associated with this fibering is a local decomposition of the metric on $\mathbb{R}^+ \times Y_i$

$$ds^2 = dr^2 \; + \; dz_i^2 \; + \; e^{-2b_i r} dn_1^2(z_i) \; + \; e^{-4b_i r} dn_2^2(z_i) \tag{16}$$

where $b_i > 0$, dz_i^2 is the metric on Z_i and $dn_1^2(z_i)$, $dn_2^2(z_i)$ are supported along the fibre $\Gamma_i \cap N_i \backslash N_i$ over z_i. The volume element is given by

$$\mathrm{dvol} = e^{-a_i r} dr \; \mathrm{dvol}(z) \; \mathrm{dvol}(n). \tag{17}$$

Here $\mathrm{dvol}(z)$ and $\mathrm{dvol}(n)$ are r–independent volume elements along the base and fibre, and $a_i > 0$.

Let Δ be the Laplacian of X. We may now proceed exactly as in §1 and construct a paramatrix $Q(\lambda)$ for $(\overline{\Delta} - \lambda)^{-1}$. For simplicity, we again assume that X has a single cusp $W = \mathbb{R}^+ \times Y$ with associated fibration $\pi : Y \to Z$. The metric on W determines via (16) a number $b > 0$ and the volume element (17) involves $a > 0$. Let Δ_W be the Dirichlet Laplacian of the cusp W and set $Q_2(\lambda) = (\Delta_W - \lambda)^{-1}$, $\lambda \in \mathbb{C} - \mathbb{R}^+$, $i = 1, ..., m$. Furthermore, let

$$X_1 = X_0 \cup \big([1, \infty] \times Y\big)$$

and let $Q_1(\lambda)$ be the restriction to X_1 of the resolvent of the Laplacian of a closed Riemannian manifold containing X_1 as isometric submanifold. Picking appropriate patching functions φ_i, ψ_i, $i = 1, 2$, we put

$$Q(\lambda) = \sum_{i=1}^{2} \varphi_i Q_i(\lambda) \psi_i.$$

To study $Q(\lambda)$, we have to analyze the Laplacian on the cusp W. Let φ be a function on $\mathbb{R}^+ \times Y$ which is constant along the fibres of (15). We may identify φ with a function on $\mathbb{R}^+ \times Z$. Set $d\rho = e^{-ar} dr \; \mathrm{dvol}(z)$ and let $L^2(\mathbb{R}^+ \times Z; d\rho)$ be the space of measurable functions on $\mathbb{R}^+ \times Y$ which are square integrable with respect to the measure $d\rho$. It follows from (17) that we may identify $L^2(\mathbb{R}^+ \times Z; d\rho)$ with a subspace of $L^2(\mathbb{R}^+ \times Y)$. The orthogonal complement of this subspace consists of the space of cuspidal functions $L_0^2(\mathbb{R}^+ \times Y)$. Given $\varphi \in L^2(\mathbb{R}^+ \times Y)$, set

$$\varphi^0(x) = \int_{\Gamma\cap N\backslash N} \varphi(nx)\, dn. \tag{18}$$

Then φ belongs to $L_0^2(\mathbb{R}^+ \times Y)$ if and only if $\varphi^0(x) = 0$ for almost all $x \in \mathbb{R}^+ \times Y$. From this description we immediately obtain the orthogonal splitting

$$L^2(\mathbb{R}^+ \times Y) = L^2(\mathbb{R}^+ \times Z; d\rho) \oplus L_0^2(\mathbb{R}^+ \times Y). \tag{19}$$

Let Δ_Z denote the Laplacian of Z. If $\varphi \in C_0^\infty(\mathbb{R}^+ \times Z)$, then the structure of the metric (16) implies that

$$\Delta_W(\varphi) = \left(-\frac{\partial^2}{\partial r^2} + a\frac{\partial}{\partial r} + \Delta_Z\right)\varphi \tag{20}$$

where $a > 0$ is determined by (17). Now one can easily see that (19) is an orthogonal splitting into invariant subspaces of Δ_W, that is,

$$\Delta_W = \Delta_{W,0} \oplus \Delta_{W,1}$$

where $\Delta_{W,0}$ is Friedrichs' extension of the positive symmetric operator

$$-e^{ar}\frac{\partial}{\partial r}\left(e^{-ar}\frac{\partial}{\partial r}\right) + \Delta_Z : C_0^\infty(\mathbb{R}^+ \times Z) \to L^2(\mathbb{R}^+ \times Z; d\rho)$$

and $\Delta_{W,1}$ is the restriction of Δ_W to the subspace $L_0^2(\mathbb{R}^+ \times Y)$.

Let $E_{W,1}(x, y, t)$ be the kernel of the heat operator $\exp(-t\Delta_{W,1})$. This kernel can be studied in the same way as in [Do]. The only difference is that we work with Dirichlet boundary conditions instead of Neumann boundary conditions. The estimates, however, remain the same. In particular, Theorem 7.2 of [Do] holds for $E_{W,1}(x, y, t)$ as well. This implies that the heat operator $\exp(-t\Delta_{W,1})$ is trace class for $t > 0$. Thus, we proved

Lemma 2. *The operator $\Delta_{W,1}$ has pure point spectrum.*

By Lemma 2, $(\Delta_{W,1} - \lambda)^{-1}$ is a meromorphic function of $\lambda \in \mathbb{C}$. The nature of the spectrum of $\Delta_{W,0}$ is described by

Lemma 3. *The operator $\Delta_{W,0}$ has pure absolutely continuous spectrum.*

Proof. We define an isometry $B : L^2(\mathbb{R}^+ \times Z; d\rho) \to L^2(\mathbb{R}^+ \times Z)$ by $(B\varphi)(r, z) = e^{-ar/2}\varphi(r, z)$. Then we have

$$\Delta_{W,0} = B^{-1} \circ \left(-\frac{\partial^2}{\partial r^2} + \Delta_Z\right) \circ B + \frac{a^2}{4}\,\mathrm{Id}, \tag{21}$$

that is, B intertwines $\Delta_{W,0}$ and the Dirichlet Laplacian of $\mathbb{R}^+ \times Z$, shifted by $a^2/4$. The latter has pure absolutely continuous spectrum. q.e.d.

By (21), it is easy to determine the kernel of the resolvent of $\Delta_{W,0}$. Let $0 = \mu_1 \leq \mu_2 \leq \cdots$ be the sequence of eigenvalues, counted with multiplicity, of Δ_Z. Let $\{\phi_j\}_{j\in\mathbb{N}}$ be a corresponding orthonormalized sequence of eigenfunctions. For each $j \in \mathbb{N}$, let $\lambda \mapsto \sqrt{\lambda - \mu_j - a^2/4}$ be the branch of the square root whose imaginary part is negative on $\mathbb{C} - \mathbb{R}^+$. Then for $r \neq r'$, the kernel of $(\Delta_{W,0} - \lambda)^{-1}$ is given by

$$Q_{W,0}\big((r,z),(r',z'),\lambda\big) = e^{a(r+r')} \sum_{j\in\mathbb{N}} \frac{e^{-i\sqrt{\lambda-\mu_j-a^2/4}\,|r-r'|}}{\sqrt{\lambda-\mu_j-a^2/4}}\, \phi_j(z) \otimes \phi_j(z'). \tag{22}$$

The series is absolutly convergent for $r \neq r'$. As in §1, this explicit description of the resolvent kernel can be used to show that $Q(\lambda)(\Delta - \lambda) = \mathrm{Id} + K(\lambda), \quad \lambda \in \mathbb{C} - \mathbb{R}^+$ where $K(\lambda)$ is a compact operator in $L^2(X)$.

To obtain an analytic continuation of $Q(\lambda)$, we introduce again weighted L^2 spaces. Since we use different coordinates at the cusps, we have to modify the definition of the weighted L^2 spaces. Given $\delta \in \mathbb{R}$, we set now

$$L^2_\delta(W) = \Big\{ f : W \to \mathbb{C} \mid \ f \text{ measurable and } \int_W |f(r,y)|^2\, e^{2\delta r^2}\, dr dy < \infty \Big\}. \tag{23}$$

Furthermore, we have to introduce the Riemann surface Σ to which the square roots $\lambda \mapsto \sqrt{\lambda - \mu_j - a^2/4}$, $j \in \mathbb{N}$, extend holomorphically. Let $0 = \widetilde{\mu}_0 < \widetilde{\mu}_1 < \cdots$ be the strictly increasing sequence of eigenvalues of Δ_Z. To construct Σ we start with the Riemann surface Σ_0 which corresponds to the square root $\sqrt{\lambda - a^2/4}$. This is a double covering $\pi_0 : \Sigma_0 \to \mathbb{C}$ with ramification at $a^2/4$. Next observe that $\pm\sqrt{\lambda - \widetilde{\mu}_1 - a^2/4}$ lifts from $\mathbb{C} - [\widetilde{\mu}_1 + a^2/4, \infty)$ to $\Sigma_0 - \pi_0^{-1}([\widetilde{\mu}_1 + a^2/4, \infty))$. Taking two copies of Σ_0, cut along $\pi_0^{-1}([\widetilde{\mu}_1 + a^2/4, \infty))$, and identifying pairs of cuts appropriately gives a four fold covering Σ_1 of $\mathbb{C}$ to which $\pm\sqrt{\lambda - a^2/4}$ and $\pm\sqrt{\lambda - \widetilde{\mu}_1 - a^2/4}$ extend holomorphically. In this way we construct successively Riemann surfaces Σ_j. This construction leaves an increasingly large compact subset of the corresponding Riemann surface unchanged and, therefore, in the limit we get the desired Riemann surface Σ. Using (22) and (23), we obtain

Proposition 4. *Let $\delta > 0$ be given. Then the resolvent $(\Delta_W - \lambda)^{-1}$ extends from $\mathbb{C} - \mathbb{R}^+$ to a meromorphic function $\lambda \in \Sigma \mapsto \widetilde{Q}_2(\lambda)$ of bounded linear operators from $L^2_\delta(W)$ into $L^2_{-\delta}(W)$.*

The rest is now completely analogous to §1. Let $L^2_\delta(X)$ be the global weighted L^2 space defined in the obvious way. Then we obtain

Theorem 5. *Let $\delta > 0$. Then the resolvent $(\overline{\Delta} - \lambda)^{-1}$ extends from $\mathbb{C} - \mathbb{R}^+$ to a meromorphic function $\lambda \in \Sigma \to R(\lambda)$ of bounded linear operators from $L^2_\delta(X)$ into $L^2_{-\delta}(X)$.*

We can now use Theorem 5 to construct generalized eigenfunctions for Δ. Let $\phi \in \mathbb{C}^\infty(Z)$ be an eigenfunction of Δ_Z with eigenvalue μ. Let $\sqrt{\lambda - \mu - a^2/4}$ be the branch of the square root whose imaginary part is positive on $\mathbb{C}-\mathbb{R}^+$. Furthermore, let $f \in \mathbb{C}^\infty(\mathbb{R})$ such that $f(r)=0$ for $r \leq 1$ and $f(r)=1$ for $r \geq 2$. Set

$$h(\phi,\lambda,x)=\begin{cases} f(r)e^{\left(a/2-i\sqrt{\lambda-\mu-a^2/4}\right)r}\,\phi(\pi(y)), & \text{if } x=(r,y)\in\mathbb{R}^+\times Y;\\ 0, & \text{otherwise.}\end{cases}$$

It follows from the definition that $(\Delta-\lambda)h(\phi,\lambda)\in C_0^\infty(X)$ and, therefore, belongs to every weighted L^2 space $L^2_\delta(X)$, $\delta\in\mathbb{R}$. Let $\delta>0$ and put

$$E(\phi,\lambda)=h(\phi,\lambda)\ -\ R(\lambda)\big((\Delta-\lambda)(h(\phi,\lambda)\big).$$

By Theorem 5, $E(\phi,\lambda)$ is a meromorphic function of $\lambda\in\Sigma$ with values in $L^2_{-\delta}(X)$. Furthermore, for $\lambda\in\mathbb{C}-\mathbb{R}^+$, which is identified with one sheet of the Riemann surface Σ, $R(\lambda)=(\overline{\Delta}-\lambda)^{-1}$. Hence, it follows that

$$\Delta E(\phi,\lambda)=\lambda E(\phi,\lambda). \tag{24}$$

Since both sides are meromorphic functions, this equality holds for all $\lambda\in\Sigma$ which are not poles. Moreover, for $\lambda\in\mathbb{C}-\mathbb{R}^+$, it follows that $h(\phi,\lambda)-E(\phi,\lambda)$ is square integrable. Let $E^0(\phi,\lambda)\in\mathbb{C}^\infty(\mathbb{R}^+\times Y)$ be the constant term of $E(\phi,\lambda)$ defined by (18). Moreover, let $\mathcal{E}(\mu')$ be the eigenspace of Δ_Z corresponding to the eigenvalue $\mu'\in\operatorname{Spec}(\Delta_Z)$. Then for $\lambda\in\mathbb{C}-\mathbb{R}^+$, we get the following expansion

$$\begin{aligned} E^0(\phi,\lambda,(r,y)) =& e^{\left(a/2-i\sqrt{\lambda-\mu-a^2/4}\right)r}\phi(\pi(y))\\ &+\sum_{\mu'\in\operatorname{Spec}(\Delta_Z)} e^{\left(a/2+i\sqrt{\lambda-\mu'-a^2/4}\right)r}S_{\mu\mu'}(\lambda)\big(\phi\big)(\pi(y))\end{aligned} \tag{25}$$

where $S_{\mu\mu'}(\lambda):\mathcal{E}(\mu)\to\mathcal{E}(\mu')$ is a linear operator which depends holomorphically on $\lambda\in\mathbb{C}-\mathbb{R}^+$. Obviously, each $S_{\mu\mu'}$ extends to a meromorphic function on Σ with values in $\operatorname{Hom}(\mathcal{E}(\mu),\mathcal{E}(\mu'))$. Let $\lambda\in\mathbb{R}^+$. Let χ_W be the characteristic function of W. Since $E(\phi,\lambda)-\chi_W E^0(\phi,\lambda)$ is square integrable, it follows that

$$\begin{aligned} E(\phi,\lambda,(r,y)) =& e^{\left(a/2-i\sqrt{\lambda-\mu-a^2/4}\right)r}\phi(\pi(y))\\ &+\sum_{\mu'\leq\lambda-a^2/4} e^{\left(a/2+i\sqrt{\lambda-\mu'-a^2/4}\right)r}S_{\mu\mu'}(\lambda)\big(\phi\big)(\pi(y))\\ &+\ F(\psi,\lambda,(r,y))\end{aligned} \tag{26}$$

where $F(\phi,\lambda)$ is square integrable on $\mathbb{R}^+\times Y$. Lemma 2 implies that the wave operators $W_\pm(\overline{\Delta},\Delta_{W,0})$ exist and are complete. Moreover, they can be expressed in terms of the generalized eigenfunctions (cf.Ch.8 of [Mü2]). Put

$$\Psi_{\mu\mu'}(\lambda)=\left(\frac{\lambda-\mu-a^2/4}{\lambda-\mu'-a^2/4}\right)^{1/2}S_{\mu\mu'}(\lambda),\quad \mu,\mu'\in\operatorname{Spec}(\Delta_Z).$$

Then it follows from the expression of the wave operators in terms of the generalized eigenfunctions that the scattering matrix is given by

$$\Psi(\lambda) = \big(\Psi_{\mu\mu'}(\lambda)\big), \quad \mu, \mu' \leq \lambda - a^2/4.$$

Now consider the case where $X = \Gamma \setminus G/K$ is a finite volume locally symmetric space of rational rank one (or, more generally, we may assume that Γ is a rank one lattice in G). We shall briefly discuss Eisenstein series. For details see [HCh].

For simplicity, we assume again that X has a single cusp. Then there is a unique Γ–conjugacy class of rational rank one parabolic subgroups of G. Let P be a representative of this unique conjugacy class. Then P admits a Langlands decomposition

$$P = N \cdot A \cdot M$$

where N is nilpotent, M is reductive and $A \cong \mathbb{R}^+$ (cf. [Bo]). Moreover, P acts transitively on the symmetric space $\tilde{X} = G/K$ so that $G = P \cdot K$. Therefore, each $g \in G$ can be written as

$$g = namk,\ n \in N,\ a \in A,\ m \in M,\ k \in K. \tag{27}$$

The discrete group $\Gamma \cap P$ is contained in NM and, under the canonical map $NM \longrightarrow M$, $\Gamma \cap P$ projects to a co–compact discrete subgroup Γ_M of M. Furthermore, we have $P \cap K = M \cap K$ which we denote by K_M. Then K_M is a maximal compact subgroup of M and

$$Y = \Gamma \cap P \setminus NM/K_M \longrightarrow \Gamma_M \setminus M/K_M = Z$$

is the fibration (15).

Let $\mathfrak{g}$ and $\mathfrak{m}$ be the Lie algebras of G and M, respectively, and let $\mathfrak{Z}(\mathfrak{g})$ and $\mathfrak{Z}(\mathfrak{m})$ be the centers of the universal enveloping algebras of $\mathfrak{g} \otimes \mathbb{C}$ and $\mathfrak{m} \otimes \mathbb{C}$, respectively.

Recall that each $D \in \mathfrak{Z}(\mathfrak{m})$ defines an invariant differential operator on M/K_M which we also denote by D. Similarly, each element of $\mathfrak{Z}(\mathfrak{g})$ defines an invariant differential operator on G/K. For a given character $\chi\colon \mathfrak{Z}(\mathfrak{m}) \longrightarrow \mathbb{C}$, set

$$\mathcal{A}(\chi) = \{\phi \in C^\infty(Z) | D\phi = \chi(D)\phi \text{ for all } D \in \mathfrak{Z}(\mathfrak{m})\}.$$

This is a space of Γ_M–automorphic forms for M. Let $\phi \in \mathcal{A}(\chi)$. We identify ϕ with a C^∞–function on M which satisfies $\phi(\gamma mk) = \phi(m)$ for $\gamma \in \Gamma_M,\ m \in M,$ and $k \in K_M$. Let $s \in \mathbb{C}$. Using (27), we define a function

$$\varphi_s \in C^\infty(\Gamma \cap P \setminus G/K)$$

by

$$\varphi_s(na_r mk) = e^{(s+a/2)r}\phi(m),$$

where a is determined by (17). Then φ_s is well–defined and satisfies

$$\varphi_s(\gamma gk) = \varphi_s(g) \text{ for } \gamma \in \Gamma \cap P,\ g \in G,\ k \in K.$$

Hence, φ_s can be identified with some function in $C^\infty(\Gamma \cap P \setminus G/K)$. Put

$$\mathcal{E}(\phi, s, x) = \sum_{\gamma \in \Gamma \cap P \setminus \Gamma} \varphi_s(\gamma x). \tag{28}$$

The series is absolutely convergent for $\mathrm{Re}(s) > a/2$ and it is a common eigenfunction of $\mathfrak{Z}(\mathfrak{g})$. In particular, $\mathcal{E}(\phi, s)$ is an eigenfunction of Δ_X. Suppose that $\Delta_Z \phi = \mu\phi$. Then we have

$$\Delta_X \mathcal{E}(\phi, s) = (-s^2 + a^2/4 + \mu)\mathcal{E}(\phi, s)$$

Furthermore, the constant term $\mathcal{E}^\circ(\phi, s)$ of $\mathcal{E}(\phi, s)$ along the cusp takes the form

$$\mathcal{E}^\circ(\phi, s(r, m)) = e^{(s+a/2)r}\phi(m) + e^{(-s+a/2)r}(C_\chi(s)\phi)(m). \tag{29}$$

where $C_\chi(s)\colon \mathcal{A}(\chi) \longrightarrow \mathcal{A}(\chi)$ is the intertwining operator. Moreover, $\mathcal{E}(\phi, s) - \chi_W \mathcal{E}^\circ(\phi, s)$ is square integrable. Put

$$u(\phi, s) = E(\phi, -s^2 + a^2/4 + \mu) - \mathcal{E}(\phi, s), \ \mathrm{Re}(s) > a/2.$$

Then it follows from (26) and (29) that $u(\phi, s)$ is square integrable. Since $\Delta u(\phi, s) = (-s^2 + a^2/4 + \mu)u(\phi, s)$, we obtain

$$\mathcal{E}(\phi, s) = E(\phi, -s^2 + a^2/4 + \mu), \ \mathrm{Re}(s) > a/2. \tag{30}$$

Let $\pi_1\colon \sum_1 \longrightarrow \mathbb{C}$ be the Riemann surface with ramification at $\mu' + a^2/4$, $\mu' \in \mathrm{Spec}\,(\Delta_Z) - \{\mu\}$, which is constructed as above. Note that $\sum$ covers $\sum_1$. Then the right hand side of (30) provides the analytic continuation of the Eisenstein series $\mathcal{E}(\phi, s)$ to a meromorphic function $\mathcal{E}(\phi, \Lambda)$ on $\sum_1$. Comparing (25) and (29), it follows that

$$S_{\mu,\mu'} = 0 \text{ for } \mu' \neq \mu, \text{ and } S_{\mu\mu} = C_\chi \text{ on } \mathcal{A}(\chi).$$

For $\mu' \in \mathrm{Spec}(\Delta_Z)$, let $\gamma_{\mu'}$ be the automorphism of order 2 of the covering $\pi_1\colon \sum_1 \longrightarrow \mathbb{C}$, generated by a simple closed curve surrounding $\mu' + a^2/4$. Suppose that $\mathrm{Re}(s) > a/2$. We indentify s wiht a point in the physical sheet of $\sum_1 \longrightarrow \mathbb{C}$. If $\mu' \neq \mu$, then the constant term of $\mathcal{E}^\circ\big(\phi, \gamma_{\mu'}(s)\big)$ is given by

$$\mathcal{E}^\circ(\phi, \gamma_{\mu'}(s), (r, m)) = e^{(s+a/2)r}\phi(m) + e^{(-s+a/2)r}\big(C_\chi\big(\gamma_{\mu'}(s)\big)\phi\big)(m).$$

Arguing as above, it follows that $\mathcal{E}\big(\phi, \gamma_{\mu'}(s)\big) = \mathcal{E}(\phi, s)$ for all $\mu' \in \mathrm{Spec}(\Delta_Z) - \{\mu\}$. Hence, $\mathcal{E}(\phi, s)$ is a meromorhic function on $\mathbb{C}$. Now observe that $\gamma_\mu s = -s$. If we proceed as above, we get the functional equations satisfied by the Eisenstein series. Summarizing, we have proved

Theorem 6. *Let $\phi \in C^{\infty}(Z)$ be an eigenfunction of $\mathfrak{Z}(\mathfrak{m})$ with character χ. Then the Eisenstein series $\mathcal{E}(\phi, s)$, defined by (28), has a meromorphic continuation to $\mathbb{C}$. Moreover, the following functional equations hold*

$$C_{\chi}(s)C_{\chi}(-s) = Id$$
$$\mathcal{E}(C_{\chi}(s)\phi, -s) = \mathcal{E}(\phi, s).$$

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