# SCATTERING AT LOW ENERGIES ON MANIFOLDS WITH CYLINDRICAL ENDS AND STABLE SYSTOLES

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Abstract. Scattering theory for $p$-forms on manifolds with cylindrical ends has a direct interpretation in terms of cohomology. Using the Hodge isomorphism, the scattering matrix at low energy may be regarded as operator on the cohomology of the boundary. Its value at zero describes the image of the absolute cohomology in the cohomology of the boundary. We show that the so-called scattering length, the Eisenbud-Wigner time delay at zero energy, has a cohomological interpretation as well. Namely, it relates the norm of a cohomology class on the boundary to the norm of its image under the connecting homomorphism in the long exact sequence in cohomology. An interesting consequence of this is that one can estimate the scattering lengths in terms of geometric data like the volumes of certain homological systoles.

## 1. Introduction and main results

Scattering theory for manifolds with cylindrical ends deals with the following geometric situation. Let $M$ be an oriented, connected, compact Riemannian manifold with boundary $Y = \partial M$ such that the metric is a product near the boundary, i.e., there is a tubular neighborhood of $Y$ which is isometric to $(-\epsilon, 0] \times Y$, equipped with the product metric $du^2 + h$, where $h$ is a Riemannian metric on $Y$. The non-compact elongation $X$ of $M$ is then obtained from $M$ by attaching the half-cylinder $Z = \mathbb{R}^+ \times Y$ over the boundary (see figure 1):

$$X = Z \cup_Y M. \tag{1}$$

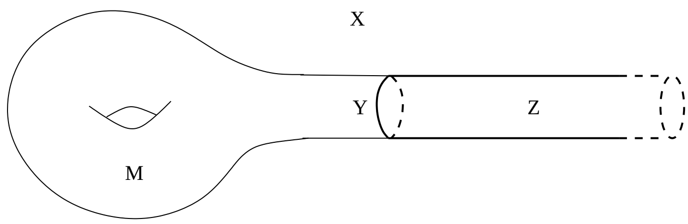


Figure 1. Elongation $X$ of $M$.

The Riemannian metric on $M$ is extended to one on $X$ in the obvious way so that

$$g|_{\mathbb{R}^+ \times Y} = du^2 + h. \tag{2}$$

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Such a manifold is called a *manifold with cylindrical ends.*

Scattering theory on $X$ investigates how wave packets coming in from infinity are scattered in $M$. The scattering of $p$-forms on $X$ is described by the scattering matrix

$$C_p(\lambda) \in \mathrm{End}\left(\bigoplus_{\mu\leq\lambda} \mathrm{Eig}_\mu(\Delta'_p) \oplus \mathrm{Eig}_\mu(\Delta'_{p-1})\right), \tag{3}$$

where $\Delta'_p$ is the Laplace-Beltrami operator acting on $p$-forms on $Y$ and $\mathrm{Eig}_\mu(\Delta'_p)$ the eigenspace of $\Delta'_p$ with eigenvalue $\mu$. In particular for small values of the spectral parameter $\lambda$ we have

$$C_p(\lambda) \in \mathrm{End}\left(\mathcal{H}^p(Y) \oplus \mathcal{H}^{p-1}(Y)\right), \tag{4}$$

where $\mathcal{H}^p(Y) = \ker \Delta'_p$ is the space of harmonic $p$-forms on $Y$. For the purposes of this article one can think of the scattering matrix for small values of the spectral parameter as being defined by the statement of Theorem 2.1. In this case it can be shown that $C(\lambda)$ leaves the direct summands invariant and is of the form

$$C_p(\lambda) = \begin{pmatrix} S_p(\lambda) & 0 \\ 0 & -S_{p-1}(\lambda)\end{pmatrix}, \tag{5}$$

where $S_p(\lambda) \in \mathrm{End}(\mathcal{H}^p(Y))$ is the scattering matrix describing the scattering of coclosed forms. Again $S_p$ can be defined by the statement of Theorem 2.6 as the matrix relating the incoming and outgoing waves. The first observation is that the total scattering matrix for coclosed $p$-forms at energy 0,

$$S(0) = \oplus_p S_p(0) \in \mathrm{End}(\mathcal{H}^*(Y)),$$

is a self-adjoint involution which anti-commutes with the Hodge star operator. The $+1$ eigenspace of $S(0)$ coincides with the space of harmonic forms that represent cohomology classes in $\mathrm{Im}(r\colon H^*(X,\mathbb{R}) \to H^*(Y,\mathbb{R}))$.

The Eisenbud-Wigner time-delay operator $\mathcal{T}_p(\lambda)$ describes the time-delay a $p$-form-wave undergoes when being scattered in $M$ (see Appendix A). It can be calculated from the Eisenbud-Wigner formula (see Appendix A), and for small $\lambda$ it is given by

$$\mathcal{T}_p(\lambda) = -\mathrm{i} C_p(\lambda)^* \frac{d}{d\lambda} C_p(\lambda).$$

Of course

$$\mathcal{T}_p(\lambda) = \begin{pmatrix} T_p(\lambda) & 0 \\ 0 & T_{p-1}(\lambda)\end{pmatrix}, \tag{6}$$

where $T_p(\lambda)$ is the time delay operator for coclosed forms defined by

$$T_p(\lambda) = -\mathrm{i} S_p(\lambda)^{-1} S'_p(\lambda). \tag{7}$$

Its value $T_p(0)$ at zero energy is of particular interest and we call it the *scattering length.* The physical interpretation of the scattering length is as follows. If a coclosed wave packet has very low energy then, by the uncertainty relation, it will be far spread out. In particular it will not be able to “feel” details of the geometry of $M$. The effect of the manifold $M$

in the scattering process for a wave is then close to that of a cylindrical obstacle of length given by the scattering length. It is therefore an interesting question to determine what geometric properties of $M$ have an effect on the scattering length. Since $T(0)$ commutes with the Hodge star operator it is enough to know its restriction to the $-1$ eigenspace of $S(0)$. Denote by $\|\cdot\|_{st}$ the stable norm of a homology class, and by $\|\cdot\|_\infty$ the comass norm on the cohomology groups (see [Gro99]). Let $\nu_1 > 0$ be the smallest positive eigenvalue of $\Delta_p'$. Put

$$\mathrm{Vol}_*(M) = \mathrm{Vol}(M) + \frac{1}{\sqrt{\nu_1}}\mathrm{Vol}(Y).$$

Furthermore, in section 5 we introduce for each $n \in \mathbb{N}$ and $0 \le p \le n$ constants $C(n,p) > 0$, which are related to the estimation of the comass norm on $\Lambda^p\mathbb{R}^n$. They are equal to 1 for $p = 0$ and $p = 1$. One of our main results relates the scattering length to certains norms in homology (Theorem 4.7) and gives rise to the following estimation of the scattering length in terms of geometric data.

**Theorem 1.1.** *Let $0 \le p \le n$. For every $\phi$ in the $-1$-eigenspace of $S_p(0)$ we have*

$$\frac{1}{2}C(n,p+1)^{-1}\mathrm{Vol}_*(M)^{-1}\|[M]\cap\partial\phi\|_{st}^2 \le \langle\phi, T(0)^{-1}\phi\rangle \le \frac{1}{2}C(n,p+1)\mathrm{Vol}(M)\|\partial\phi\|_\infty^2.$$

As an example we treat the case when $Y$ has two connected components $Y_1$ and $Y_2$ and $p = 0$. In this case there is a canonical basis in $H^0(Y,\mathbb{R})$ with respect to which $T_0(0)$ has the form

$$T_0(0) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \tag{8}$$

so that

$$t_1 = 2\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(Y)}, \tag{9}$$

$$C_2 \le t_2 \le C_1, \tag{10}$$

and the constants $C_1$ and $C_2$ are given by

$$C_1 = 2\mathrm{Vol}_*(M)\frac{\mathrm{Vol}(Y_1)\mathrm{Vol}(Y_2)}{\|\iota_*([Y_1])\|_{st}^2(\mathrm{Vol}(Y_1)+\mathrm{Vol}(Y_2))}, \tag{11}$$

$$C_2 = 2\mathrm{Vol}(M)^{-1}\frac{\mathrm{dist}(Y_1,Y_2)^2\mathrm{Vol}(Y_1)\mathrm{Vol}(Y_2)}{\mathrm{Vol}(Y_1)+\mathrm{Vol}(Y_2)}. \tag{12}$$

The map $\iota$ is the inclusion of $Y$ into $M$. So we get an estimate for the scattering length by purely geometric quantities. The physical interpretation of this is as follows. For a wave of low energy the reflection coefficient $r_{11}$ and the transmission coefficient $r_{12}$ for a wave coming in at the boundary component $Y_1$ are approximated by their values at zero, namely (see section 7.3)

$$r_{11} = \frac{\mathrm{Vol}(Y_1)-\mathrm{Vol}(Y_2)}{\mathrm{Vol}(Y_1)+\mathrm{Vol}(Y_2)}, \qquad r_{12} = \frac{2\mathrm{Vol}(Y_1)}{\mathrm{Vol}(Y_1)+\mathrm{Vol}(Y_2)}. \tag{13}$$

The time-delay is then determined by $t_1$ and $t_2$. For example in the case where $\mathrm{Vol}(Y_1) = \mathrm{Vol}(Y_2)$ the reflection coefficient at zero energy is zero and the time delay of the transmitted wave is equal to $\frac{1}{2}(t_1 + t_2)$ (see section 7.3). Another example, the full-torus, is treated in section 7.4.

Given $a > 0$, let $M_a$ be the manifold that is obtained from $M$ by attaching the cylinder $[0, a] \times Y$ to $M$. We also investigate how the $L^2$-norm of a class in $H^p(M_a, \mathbb{R})$ and the $L^2$-norm of its image in $H^{p+1}(M_a, \mathbb{R})$ under the connecting homomorphism are related for the manifold with boundary $M_a$. There is an operator $q_a$ that relates the $L^2$-norm of classes in the complement of the kernel of the connecting homomorphism to the $L^2$-norm of the image of this class under the connecting homomorphism. So $q_a$ measures to what extent the connecting homomorphism is a partial isometry. Theorem 3.3 shows that the operator $q_a$ has an expansion of the form

$$q_a = a\mathbf{1} + \frac{1}{2}T(0) + O(e^{-ca}), \tag{14}$$

as $a \to \infty$. This shows that one can calculate the scattering length by approximating $X$ by the compact manifolds $M_a$ and consider the constant term in the above expansion. The exponential decay of the remainder term may also be useful for numerical computations.

The paper is organized as follows. In sections 2 we review stationary scattering theory for $p$-forms on manifolds with cylindrical ends. Section 3 and section 4 deal with cohomology of $M$ and $X$, and their relation to scattering theory and the continuous spectrum of the Laplacian of $X$. We also derive a cohomological formula for the scattering length. In sections 5 and 6 it is shown that the $L^2$-scalar products on the cohomology groups of $X$ can be estimated in terms of geometric quantities and that these estimates imply estimates on the scattering length. Section 7 treats the case of functions in the case of two boundary components and the case of a full-torus. In this section we demonstrate how our main result can be used to obtain estimates of the scattering length in terms of geometric data. In appendix A we discuss the relation between the stationary and the dynamical approach to scattering theory and we establish the Eisenbud-Wigner formula for manifolds with cylindrical ends.

Whereas for the sake of notational simplicity we restricted ourselves in this paper to manifolds with cylindrical ends most of our analysis carries over in a straightforward manner to waveguides if Neumann boundary conditions are imposed.

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## 2. Stationary scattering theory for manifolds with cylindrical ends

As before let $M$ be an oriented, connected, compact Riemannian manifold with boundary $Y = \partial M$ such that the metric is a product near the boundary. Let $X$ be the elongation of $M$. For any Riemannian manifold $W$ we denote by $\Lambda^p(W)$ (resp. $\Lambda^p_c(W)$, $L^2\Lambda^p(W)$) the space of smooth $p$-forms (resp. smooth $p$-forms with compact supports, $L^2$-forms) on $W$. Let $\Delta_p$ be the Laplace operator on $\Lambda^p(W)$. Throughout this paper a harmonic $p$-form will mean a $p$-form $\phi \in \Lambda^p(W)$ with $\Delta_p \phi = 0$.

Since $X$ is complete, the Laplace-Beltrami operator $\Delta_p$ on $p$-forms is essentially self-adjoint when regarded as operator in $L^2\Lambda^p(X)$ with domain $\Lambda^p_c(X)$ ([Che73]). We continue to denote its self-adjoint extension by $\Delta_p$. In this section we recall some facts concerning the generalized eigenforms of $\Delta_p$ and derive some properties of the scattering matrix for low energy.

Let $\Delta'_p$ denote the Laplacian on $p$-Forms of $Y$. Let $0 \leq \nu_1 < \nu_2 < \cdots$ be the distinct eigenvalues of of $\Delta'_p \oplus \Delta'_{p-1}$. Let $\Sigma \to \mathbb{C}$ the minimal Riemann surface on which $\sqrt{\lambda^2 - \nu_j}$ is a single-valued function for all $j \in \mathbb{N}_0$. As proved by Melrose[Mel93] the resolvent $(\Delta_p - \lambda^2)^{-1}$, regarded as operator $\Lambda^p_c(X) \to L^2_{\text{loc}}\Lambda^p(X)$, admits a meromorphic extension from the half-plane $\text{Im}(\lambda) > 0$ to $\Sigma$. Let $\mu_1^2 > 0$ be the first non-zero eigenvalue of $\Delta'_p \oplus \Delta'_{p-1}$. Then it follows in particular that $(\Delta_p - \lambda^2)^{-1}$ extends to a meromorphic function on the disc $\{\lambda \colon |\lambda| < \mu_1\}$. As a consequence one can define analytic families of generalized eigenforms.

For $a \geq 0$ we denote by $Y_a$ the hypersurface $(a, Y) \subset \mathbb{R}^+_0 \times Y \subset X$. Note that the restriction of the bundle $\Lambda^p T^*X$ to $Y_a$ is canonically isomorphic to the direct sum $\Lambda^p T^*Y \oplus \Lambda^{p-1}T^*Y$ since each vector $f \in \Lambda^p T^*X$ at some point $(u, x)$ can be uniquely decomposed as $f = f_1 + du \wedge f_2$ with $f_1 \in \Lambda^p T^*X$ and $f_2 \in \Lambda^{p-1}T^*X$. Accordingly, the restriction of any $p$-form $\omega \in \Lambda^p(X)$ to the cylinder $\mathbb{R}^+ \times Y$ is of the form

$$\omega = \omega_1 + du \wedge \omega_2, \tag{15}$$

where $\omega_1$ and $\omega_2$ are sections of the pulled back bundles $\pi^*\Lambda^p T^*Y$ and $\pi^*\Lambda^{p-1}T^*Y$, respectively, and $\pi : \mathbb{R}^+ \times Y \to Y$ is the canonical projection. We think of $\omega_1(u)$ and $\omega_2(u)$ as forms on $Y$ that depend smoothly on the additional parameter $u$. The map

$$j_p : \pi^*\Lambda^p T^*Y \oplus \pi^*\Lambda^{p-1}T^*Y \to \Lambda^p T^*Z, \quad (\omega_1, \omega_2) \mapsto \omega_1 + du \wedge \omega_2 \tag{16}$$

is an isomorphism of vector bundles. The exterior differential of such a form $\omega$ is then given by

$$d\omega = d'\omega_1 + du \wedge \partial_u \omega_1 - du \wedge d'\omega_2, \tag{17}$$

where $d'$ denotes the exterior differential on $Y$. In matrix notation this means

$$j^{-1} \circ d \circ j = \begin{pmatrix} d' & 0 \\ \partial_u & -d' \end{pmatrix}, \tag{18}$$

where we denote $j = \oplus j_p$. Since the metric has product structure the decomposition is orthogonal and the formal adjoint $\delta$ of $d$ is therefore

$$j^{-1} \circ \delta \circ j = \begin{pmatrix} \delta' & -\partial_u \\ 0 & -\delta' \end{pmatrix}. \tag{19}$$

Here again we use the notation $\delta'$ for the codifferential on $Y$. We therefore have

$$j^{-1} \circ (d+\delta) \circ j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (d' + \delta') + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_u. \tag{20}$$

and

$$j_p^{-1} \circ \Delta_p \circ j_p = \begin{pmatrix} -\partial_u^2 + \Delta'_p & 0 \\ 0 & -\partial_u^2 + \Delta'_{p-1} \end{pmatrix}$$

for all $p$. This form of an operator allows for a separation of variables on the cylinder $\mathbb{R}^+ \times Y$. Suppose that $(\psi_i)$ is an orthonormal sequence of eigenforms of

$$\begin{pmatrix} \Delta'_p & 0 \\ 0 & \Delta'_{p-1} \end{pmatrix} \tag{21}$$

with eigenvalues $\mu_{\psi_i}^2$. Then for $|\lambda| < \mu_1$, any solution $F$ of the equation $(\Delta_p - \lambda^2)F = 0$ has an expansion of the form

$$F(u) = \sum_{i=0}^{\infty} \left( a_i e^{-\mathrm{i}\sqrt{\lambda^2 - \mu_{\psi_i}^2} u} + b_i e^{\mathrm{i}\sqrt{\lambda^2 - \mu_{\psi_i}^2} u} \right) j_p(\psi_i) + \begin{cases} \sum_{\mu_{\psi_i}=0} c_i u j_p(\psi_i), & \lambda = 0; \\ 0, & \lambda \neq 0. \end{cases} \tag{22}$$

The series converges in the $C^\infty$-topology. The square roots are chosen throughout the article in such a way that $\sqrt{\lambda^2} = \lambda$ if $\mathrm{Im}(\lambda) > 0$. This means that $\sqrt{re^{\mathrm{i}\varphi}} = \sqrt{r} e^{\frac{\mathrm{i}}{2}\varphi}$ if $r > 0$ and $0 \le \varphi < 2\pi$. From the analytic continuation of the resolvent one gets the following result.

**Theorem 2.1.** *For each* $\psi$ *in* $\ker(\Delta'_p \oplus \Delta'_{p-1})$ *there exists a* $p$*-form* $\widetilde{F}(\psi, \lambda)$ *which is meromorphic in* $\lambda \in \{z \colon |z| < \mu_1\}$ *such that the following conditions hold*

(i) $\widetilde{F}(\psi, \lambda)$ *is holomorphic in* $\lambda$ *for* $\mathrm{Im}(\lambda) > 0$.

(ii) $\Delta \widetilde{F}(\psi, \lambda) = \lambda^2 \widetilde{F}(\psi, \lambda)$.

(iii) *There exists* $\widetilde{R}_p(\psi, \lambda) \in L^2\Lambda^p(Z)$ *such that on* $Z$ *we have*

$$\widetilde{F}(\psi, \lambda) = e^{-\mathrm{i}\lambda u} j_p(\psi) + e^{\mathrm{i}\lambda u} j_p(C_p(\lambda)\psi) + \widetilde{R}_p(\psi, \lambda).$$

(iv) $C_p(\lambda) : \ker(\Delta'_p \oplus \Delta'_{p-1}) \to \ker(\Delta'_p \oplus \Delta'_{p-1})$ *is a linear operator, and* $C_p(\lambda)$ *and* $\widetilde{R}_p(\psi, \lambda)$ *are meromorphic functions of* $\lambda$.

*Moreover,* $C_p(\lambda)$*,* $\widetilde{R}_p(\psi, \lambda)$ *and* $\widetilde{F}(\psi, \lambda)$ *are uniquely determined by these properties.*

*Proof.* This follows from[Gui89], [Mel93]. For the convenience of the reader we include the details. Let $\Sigma \to \mathbb{C}$ be the minimal Riemann surface to which all the functions $(\lambda^2 - \mu_{\psi_i}^2)^{1/2}$ extend to be holomorphic. By [Gui89, Théorèm 0.2], [Mel95, Theorem7.1] the resolvent $(\Delta_p - \lambda^2)^{-1}$, regarded as operator $\Lambda_c^p(X) \to L^2_{\mathrm{loc}}\Lambda^p(X)$, extends to a meromorphic function of $\lambda \in \Sigma$. Especially $(\Delta_p - \lambda^2)^{-1}$ extends to a meromorphic function of $\lambda \in \{z \in \mathbb{C} \colon |z| < \mu_1\}$ as an operator $\Lambda_c^p(X) \to L^2_{\mathrm{loc}}\Lambda^p(X)$. Let $\chi$ be a smooth function with support in $Z$ which is equal to 1 outside a compact set. Put

$$\widetilde{F}(\psi, \lambda) := \chi e^{-\mathrm{i}\lambda u} j_p(\psi) - (\Delta_p - \lambda^2)^{-1} \left\{ (\Delta_p - \lambda^2)(\chi e^{-\mathrm{i}\lambda u} j_p(\psi)) \right\}. \tag{23}$$

Then $\widetilde{F}(\psi,\lambda)$ is a smooth $p$-form on $X$ which depends meromorphically on $\lambda \in \Sigma$. Moreover, it satisfies

$$(\Delta_p - \lambda^2)\widetilde{F}(\psi,\lambda) = 0. \tag{24}$$

Since $\widetilde{F}(\psi,\lambda) - \chi e^{-\mathrm{i}\lambda u} j_p(\psi)$ is square integrable for $\mathrm{Im}(\lambda) > 0$, the expansion (22) has the form

$$\widetilde{F}(\psi,\lambda) = e^{-\mathrm{i}\lambda u} j_p(\psi) + e^{\mathrm{i}\lambda u} j_p(C_p(\lambda)\psi) + \widetilde{R}_p(\psi,\lambda)$$

where $C_p(\lambda)\psi \in \ker(\Delta'_P \oplus \Delta'_{p-1})$ and

$$\widetilde{R}_p(\psi,\lambda,u) = \sum_{\mu_{\psi_i}\neq 0} \left( a_i(\lambda) e^{-\mathrm{i}\sqrt{\lambda^2-\mu_{\psi_i}^2}\,u} j_p(\psi_i) + b_i(\lambda) e^{+\mathrm{i}\sqrt{\lambda^2-\mu_{\psi_i}^2}\,u} j_p(\psi_i) \right). \tag{25}$$

Moreover, $\widetilde{R}_p(\psi,\lambda)$ is square integrable for $\mathrm{Im}(\lambda) > 0$. This implies that $a_i(\lambda) = 0$ for $\mathrm{Im}(\lambda) > 0$. Since $a_i(\lambda)$ is meromorphic, it follows that $a_i(\lambda) = 0$ for $|\lambda| < \mu_1$. Thus we conclude that $R_p(\psi,\lambda)$ is a meromorphic function in the disc $|\lambda| < \mu_1$ with values in $L^2\Lambda^p(X)$.

The uniqueness is an immediate consequence of the self-adjointness of $\Delta_p$. Namely, if $\widetilde{F}_1(\psi,\lambda)$ and $\widetilde{F}_2(\psi,\lambda)$ both have the above properties then their difference $G(\psi,\lambda)$ is square integrable for $\mathrm{Im}(\lambda) > 0$ and is contained in the kernel of $(\Delta_p - \lambda^2)$. Using that $\Delta_p$ is self-adjoint, we get $G(\psi,\lambda) = 0$ for $\mathrm{Im}(\lambda) > 0$. Since it is meromorphic in $\lambda$ we conclude that $G(\psi,\lambda) = 0$. □

The requirement that $\widetilde{R}_p(\psi,\lambda)$ is in $L^2\Lambda^p(Z)$ in fact implies a much faster decay at infinity.

**Lemma 2.2.** *If $\widetilde{R}(\psi,\lambda)$ is regular at $\lambda$, then all derivatives of $\widetilde{R}(\psi,\lambda)$ in $u$ and $x$ are exponentially decaying as $u \to \infty$. More precisely*

$$|\partial_u^k (\Delta')^l \widetilde{R}(\psi,\lambda)| \le C'_{k,l,\lambda} e^{-C_\lambda u},$$

*for all $k, l \ge 0$ and some positive constants $C'_{k,l,\lambda}$ and $C_\lambda$.*

*Proof.* The proof is already implicitly contained in the proof of the previous theorem. Namely, $\widetilde{R}(\psi,\lambda)$ is smooth by elliptic regularity and thus the expansion (25) converges in $\Lambda^p(Z)$. Since $a_i = 0$, we get the decay for $|\lambda| < \mu_1$ with $C_\lambda = \mathrm{Re}\left(\sqrt{\mu_1^2 - \lambda^2}\right)$. □

Let $\psi \in \ker \Delta'_p$. Put

$$F(\psi,\lambda) := \widetilde{F}\left((\psi,0),\lambda\right), \quad F(du\wedge\psi,\lambda) := \widetilde{F}\left((0,\psi),\lambda\right). \tag{26}$$

Then the expansion (iii) of Theorem 2.1 takes the form

$$\begin{aligned} F(\psi,\lambda) &= e^{-i\lambda u}\psi + e^{i\lambda u} C_p^{11}(\lambda)\psi + e^{i\lambda u} du \wedge C_p^{21}(\lambda)\psi + R_p(\psi,\lambda); \\ F(du\wedge\psi,\lambda) &= e^{-i\lambda u} du\wedge\psi + e^{i\lambda u} du\wedge C_p^{22}(\lambda)\psi + e^{i\lambda u} C_p^{12}(\lambda)\psi + R_p(du\wedge\psi,\lambda), \end{aligned} \tag{27}$$

where

$$C_p(\lambda) = \begin{pmatrix} C_p^{11}(\lambda) & C_p^{12}(\lambda) \\ \\ C_p^{21}(\lambda) & C_p^{22}(\lambda) \end{pmatrix} \tag{28}$$

as endomorphism of $\Delta'_p \oplus \ker\Delta'_{p-1}$. A priory there is no reason why $C_p(\lambda)$ should leave the summands invariant. Nevertheless this is guaranteed by a continuous version of the Hodge decomposition as the proof of the following proposition shows.

**Proposition 2.3.** *$C_p(\lambda)$ leaves the spaces $\ker\Delta'_p$ and $\ker\Delta'_{p-1}$ invariant, i.e., $C_p^{12}(\lambda) = 0$ and $C_p^{21}(\lambda) = 0$.*

*Proof.* Let $\psi \in \Delta'_p$. Using the expansion of 27, it follows that on $\mathbb{R}^+ \times Y$ we have

$$\begin{aligned} \delta dF(\psi,\lambda) &= \lambda^2 e^{-\mathrm{i}\lambda u}\psi + \lambda^2 e^{\mathrm{i}\lambda u} C_p^{11}(\lambda)\psi + \delta d R_p(\psi,\lambda), \\ d\delta F(du\wedge\psi,\lambda) &= \lambda^2 e^{-\mathrm{i}\lambda u} du\wedge\psi + \lambda^2 e^{\mathrm{i}\lambda u} du \wedge C_p^{22}(\lambda)\psi + d\delta R_p(du\wedge\psi,\lambda). \end{aligned} \tag{29}$$

Comparing the leading terms, it follows that

$$\lambda^{-2}\delta dF(\psi,\lambda) = F(\psi,\lambda), \quad \lambda^{-2} d\delta F(du\wedge\psi,\lambda) = F(du\wedge\psi,\lambda).$$

Since all derivatives of $R_p(\psi,\lambda)$ and $R_p(du\wedge\psi,\lambda)$ are exponentially decaying, the uniqueness statement in theorem 2.1 implies immediately $C_p^{12}(\lambda) = 0$ and $C_p^{21}(\lambda) = 0$. □

Note that the splitting $\Lambda^p T^*Z = \pi^*(\Lambda^p T^*Y) \oplus \pi^*(\Lambda^{p-1}T^*Y)$ indeed corresponds to the Hodge decomposition. Let $P_1$ and $P_2$ the projection on the first and second summand, respectively. Then we have

**Proposition 2.4.** *We have $P_2\psi = 0$, i.e. $\psi \in \ker\Delta'_p$ if and only if $\delta\widetilde{F}(\psi,\lambda) = 0$. Similarly $P_1\psi = 0$, i.e. $\psi \in \ker\Delta'_{p-1}$ if and only if $d\widetilde{F}(\psi,\lambda) = 0$.*

*Proof.* Let $\psi = (\psi_1,\psi_2) \in \ker\Delta'_p \oplus \ker\Delta'_{p-1}$. Suppose that $\delta\widetilde{F}(\psi,\lambda) = 0$. Then, in particular, we have $\delta F(du\wedge\psi_2,\lambda) = 0$. Applying $\delta$ to the second equation of (27) and using Proposition 2.3, it follows that $\psi_2 = 0$. For the other direction, observe that by (27), $\delta F(\psi_1,\lambda) = \delta R(\psi_1,\lambda)$ on $Z$. Hence, $\delta F(\psi_1,\lambda)$ is exponentially decaying. In particular it is square integrable for $\mathrm{Im}(\lambda) > 0$. Since $\Delta_p$ is self-adjoint and $(\Delta_p - \lambda^2)\delta F(\psi_1,\lambda) = 0$ we get $\delta F(\psi_1,\lambda) = 0$. Thus, if $\psi_2 = 0$, it follows that $\delta\widetilde{F}(\psi,\lambda) = 0$. The proof of the other case is analogous. □

**Proposition 2.5.** *The following relation holds for $0 \le p < n$.*

$$C_p(\lambda)|_{\ker\Delta'_p} = -C_{p+1}(\lambda)|_{\ker\Delta'_p}, \tag{30}$$

*Proof.* Let $\psi \in \ker\Delta'_p$. Using (27), we get

$$\mathrm{i}\lambda^{-1} dF(\psi,\lambda)|_Z = e^{-\mathrm{i}\lambda u} du\wedge\psi - e^{\mathrm{i}\lambda u} du\wedge C_p^{11}(\lambda)\psi + \mathrm{i}\lambda^{-1} dR_p(\psi,\lambda) \tag{31}$$

Comparing the leading terms, it follows from Theorem 2.1 that

$$\mathrm{i}\lambda^{-1} dF(\psi,\lambda) = F(du\wedge\psi,\lambda).$$

Using (27), we get $C_p^{11}(\lambda) = -C_{p+1}^{22}(\lambda)$, which is equivalent to the statement of the Proposition. $\square$

Let us use the notation $S_p(\lambda)$ for the restriction of $C_p(\lambda)$ to $\ker\Delta'_p$. Then the above proposition shows that

$$C_p(\lambda) = \begin{pmatrix} S_p(\lambda) & 0 \\ 0 & -S_{p-1}(\lambda) \end{pmatrix}. \tag{32}$$

In the following we will suppress the index $p$ and write $S(\lambda)$, meaning that $S(\lambda)$ is acting on the space of harmonic forms, leaving the space of $p$-forms invariant. It is the scattering matrix at low energy for the scattering problem for coclosed forms. Summarizing we have established the following theorem.

**Theorem 2.6.** *For each harmonic $p$-form $\psi \in \ker(\Delta'_p)$ there exists a $p$-form $F(\psi,\lambda)$ on $X$, which is meromorphic in $\lambda$ in the disc $|\lambda| < \mu_1$ such that*

(i) $\delta F(\psi,\lambda) = 0$
(ii) $F(\psi,\lambda)$ *is holomorphic in* $\lambda$ *for* $\mathrm{Im}(\lambda) > 0$.
(iii) $(\Delta_p - \lambda^2)F(\psi,\lambda) = 0$.
(iv) $F(\psi,\lambda) = e^{-\mathrm{i}\lambda u}\psi + e^{\mathrm{i}\lambda u}S(\lambda)\psi + R(\psi,\lambda)$ *on* $\mathbb{R}^+ \times Y$,
(v) $S(\lambda) \in \mathrm{End}(\ker(\Delta'_p))$ *and* $R(\psi,\lambda) \in L^2\Lambda^p(Z)$ *are meromorphic functions of* $\lambda$.

*Moreover, $S(\lambda)$, $R(\psi,\lambda)$ and $F(\psi,\lambda)$ are uniquely determined by these properties.*

The scattering matrix has the following properties

**Theorem 2.7.** *The function $S(\lambda)$ satisfies the following equations*

(i) $S(\lambda)^*S(\overline{\lambda}) = 1$.
(ii) $S(\lambda)S(-\lambda) = 1$.
(iii) $S(\lambda)* = - * S(\lambda)$,

*where $*$ is the Hodge star operator on $Y$.*

*Proof.* For $a > 0$ let $X_a$ be the manifold $([0,a) \times Y) \cup_Y X$ with boundary $Y_a = \{a\} \times Y$. Let $\omega_X$ be the volume form of $X$. By Theorem 2.6, (iii), we have

$$\int_{X_a} \langle F(\psi,\lambda), \Delta_p F(\psi,\overline{\lambda})\rangle \omega_X - \int_{X_a} \langle \Delta_p F(\psi,\lambda), F(\psi,\overline{\lambda})\rangle \omega_X = 0.$$

Using Green's formula, we obtain

$$\int_{Y_a} \langle F(\psi,\lambda), -\partial_u F(\psi,\overline{\lambda})\rangle \omega_Y + \int_{Y_a} \langle \partial_u F(\psi,\lambda), F(\psi,\overline{\lambda})\rangle \omega_Y = 0. \tag{33}$$

In the limit $a \to \infty$ this expression can be evaluated using Theorem 2.6. We obtain

$$\begin{aligned} &\lim_{a\to\infty} \int_{Y_a} (\langle F(\psi,\lambda), -\partial_u F(\psi,\overline{\lambda})\rangle + \langle \partial_u F(\psi,\lambda), F(\psi,\overline{\lambda})\rangle)\omega_Y = \\ &= -2\mathrm{i}\lambda(\|\psi\|^2 - \langle S(\lambda)\psi, S(\overline{\lambda})\psi\rangle) = 0, \end{aligned} \tag{34}$$

which proves the first statement. The second statement follows from the functional equation

$$F(C(\lambda)\psi, -\lambda) = F(\psi, \lambda),$$

which is a simple consequence of the uniqueness statement in theorem 2.1. To show that $C(\lambda)$ anticommutes with the Hodge star operator on $Y$ we note that that Hodge star operator $*_X$ on $X$ commutes with the Laplace operator $\Delta$, i.e. $*_X \Delta_p = \Delta_{n-p} *_X$. Applying this to $F(\psi, \lambda)$ and using the uniqueness statement we obtain immediately

$$*_X C_p = C_{n-p} *_X .$$

For $\psi \in \ker \Delta'_p$ we get $*_X \psi = (-1)^p du \wedge *\psi$ and consequently $*S(\lambda) = -S(\lambda)*$, where we used that $du\wedge$ anticommutes with $C(\lambda)$. □

As an application we obtain the following well known result about the signature $\mathrm{sign}(Y)$ of a closed manifold $Y$.

**Corollary 2.8.** *Let $Y$ be a closed oriented manifold. Assume that $Y$ is the boundary of a compact manifold. Then* $\mathrm{sign}(Y) = 0$.

*Proof.* We may assume that $\dim Y = 4k$. Otherwise the signature is zero. Pick a Riemannian metric on $Y$. Let $\mathcal{H}^{2k}_\pm(Y)$ be the $\pm 1$ eigenspaces of $*$ acting in $\mathcal{H}^{2k}(Y)$. Then the signature $\mathrm{sign}(Y)$ of $Y$ is given by

$$\mathrm{sign}(Y) = \dim \mathcal{H}^{2k}_+(Y) - \dim \mathcal{H}^{2k}_-(Y).$$

If $Y$ is the boundary of a compact Riemannian manifold $M$, it follows from Theorem 2.7, that $S(\lambda)$ is regular at $\lambda = 0$, $S(0)^2 = \mathbb{1}$ and $S(0)$ intertwines $\mathcal{H}^{2k}_+(Y)$ and $\mathcal{H}^{2k}_-(Y)$. Hence we get $\mathrm{sign}(Y) = 0$. □

**Remark 2.9.** *The same proof works equally well for Dirac type operators. It implies the cobordism invariance of the index of Dirac operators.*

**Proposition 2.10.** *$S(\lambda)$, $F(\psi, \lambda)$, and $R(\psi, \lambda)$ are regular for real $\lambda$. If $\psi = -S(0)\psi$ then $F(\psi, 0) = 0$.*

*Proof.* For real $\lambda$ it follows from Theorem 2.7 that $S(\lambda)S^*(\lambda) = \mathbb{1}$ and therefore $\|S(\lambda)\| = 1$. In particular, $S$ is bounded on the real line and can not have a pole there. It remains to show that $F$ is regular for real $\lambda$. Suppose that $\phi$ is a square integrable eigensection of $\Delta$ with real eigenvalue $\lambda'$. Then the expansion (22) of $\phi$ on $\mathbb{R}^+ \times Y$ takes the form

$$\phi(u, y) = \sum_{\mu^2_{\psi_i} > \lambda'} a_i e^{-\sqrt{\mu^2_{\psi_i} - \lambda'}\, u} j_p(\psi_i).$$

This implies that $\phi$ is exponentially decaying. Since for real $\lambda$

$$0 = \langle (\Delta - \lambda^2) F(\psi, \lambda), \phi \rangle = -(\lambda^2 - \lambda'^2) \langle F(\psi, \lambda), \phi \rangle, \tag{35}$$

and $\langle F(\psi, \lambda), \phi \rangle$ is a meromorphic function we get $\langle F(\psi, \lambda), \phi \rangle = 0$. Suppose now that $F(\psi, \lambda)$ has a pole of order $k$ at $\lambda'$. Then $G := \lim_{\lambda \to \lambda'} (\lambda - \lambda')^k F(\psi, \lambda)$ is an eigenform with eigenvalue $\lambda'$ and it also is square integrable since $S$ is regular at $\lambda'$. By the above

$\langle G,G\rangle = 0$. It follows that $G = 0$ and therefore $F(\psi,\lambda)$ is regular at $\lambda'$. If $\psi = -S(0)\psi$ then $F(\psi,0)$ is square integrable and harmonic and by the same argument $F(\psi,0) = 0$. $\square$

In particular, it follows that $F(\psi,\lambda)$ is regular at $\lambda = 0$. Therfore, $F(\psi,0)$ and $F'(\psi,0) := \frac{\partial}{\partial\lambda}F(\psi,\lambda)|_{\lambda=0}$ are well defined. From the proof of Proposition 2.10 we obtain the following corollary.

**Corollary 2.11.** *$F(\psi,0)$ and $F'(\psi,0)$ are orthogonal to the space $\mathcal{H}^p_{(2)}(X)$ of square integrable harmonic forms.*

The scattering matrix $S(\lambda)$ is also regular at 0 and it follows from Theorem 2.7 that $S(0)$ is a self-adjoint involution. Hence, $\ker(\Delta'_p)$ decomposes into $+1$ and $-1$ eigenspaces. For $\psi \in \ker(\Delta'_p)$ with $S(0)\psi = \psi$ we get that $F(\psi,0)$ is a smooth coclosed harmonic $p$-form whose restriction to $Z$ equals

$$F(\psi,0) = 2\psi + R(\psi,0),$$

where $R(\psi,0)$ and its derivatives are exponentially decaying. That is, $\psi$ is a limiting value of $\frac{1}{2}F(\psi,0)$ in the sense of Atiyah, Patodi, and Singer ([APS75]). It turns out that the converse is also true.

**Proposition 2.12.** *The $+1$ eigenspace for $S(0)$ is the set of limiting values of coclosed harmonic forms on $X$, i.e, it equals*

$$\{\psi \in \ker\Delta'_p \mid \exists G \in \Lambda^p(X)\colon G|_Z - \psi \in L^2\Lambda^p(Z),\ \Delta_p G = 0,\ \delta G = 0\}.$$

*Furthermore for each $\psi \in \ker\Delta'_p$, $F(\psi,0)$ satisfies $dF(\psi,0) = 0$ and $\delta F(\psi,0) = 0$.*

*Proof.* Suppose that $F \in \Lambda^p(X)$ and $G \in \Lambda^{n-1-p}(X)$ are both coclosed harmonic forms on $X$ with limiting values $\psi$ and $\phi$, respectively. Since $\psi - G$ and $\phi - F$ are both exponentially decaying and $\psi$ and $\phi$ are closed and coclosed on $Y$ the forms $dG$ and $dF$ are exponentially decaying. Using Green's formula, we get

$$0 = \langle \Delta G, G\rangle = \langle \delta dG, G\rangle = \langle dG, dG\rangle.$$

Thus $dG = 0$. Similarly we get $dF = 0$. Using Stokes formula, it follows that

$$0 = \int_{X_a} dG \wedge F = \pm\int_Y \psi\wedge\phi + O(e^{-cu}).$$

Thus, $\langle \psi, *\phi\rangle = 0$. Now suppose that $\phi$ is a limiting value which is in the $-1$ eigenspace to $S(0)$. Since $*$ anticommutes with $S(0)$, it follows that $*\phi$ is in the $+1$ eigenspace. It is therefore a limiting value. Since $*\phi$ and $\phi$ are both limiting values it follows from the above that $\|\phi\|^2 = \pm\langle\phi, **\phi\rangle = 0$ and therefore, $\phi = 0$. Since the set of limiting vectors contains the $+1$ eigenspace it has to coincide with the $+1$ eigenspace. $\square$

Finally we derive some formulas concerning $F'(\psi,0)$ which we are going to use in the next section. Note that the restriction of $F'(\psi,\lambda)$ to the cylinder $Z$ has the form

$$F'(\psi,\lambda)|_Z = -\mathrm{i}u(e^{-\mathrm{i}\lambda u}\psi - e^{+\mathrm{i}\lambda u}S(\lambda)\psi) + e^{\mathrm{i}\lambda u}S'(\lambda)\psi + R'(\psi,\lambda), \tag{36}$$

and for $\lambda = 0$:

$$F'(\psi,0)|_Z = -\mathrm{i}u\,(1-S(0))\,\psi + S'(0)\psi + R'(\psi,0). \tag{37}$$

Differentiating the equation

$$(\Delta - \lambda^2)F(\psi,\lambda) = 0 \tag{38}$$

it follows that

$$\Delta F'(\psi,0) = 0. \tag{39}$$

Hence, $dF'(\psi,0)$ is in the kernel of $\Delta$ and its restriction to the cylinder has the form

$$dF'(\psi,0)|_Z = -\mathrm{i}du\wedge(1-S(0))\psi + dR'(\psi,0). \tag{40}$$

By Theorem 2.7, (iii), we get

$$*_X dF'(\psi,0)|_Z = -\mathrm{i}(1+S(0))*\psi + *_X dR'(\psi,0). \tag{41}$$

Thus, $*_X dF'(\psi,0)$ is an extended harmonic form with limiting value $-\mathrm{i}(1+S(0))*\psi$. Since $*_X dF'(\psi,0)$ is coexact and bounded it is orthogonal to the space $\mathcal{H}^*_{(2)}(X)$ of square integrable harmonic forms. The proof of Prop. 2.10 shows that $\frac{1}{2}F(-\mathrm{i}(1+S(0))*\psi,0)$ is harmonic and orthogonal to $\mathcal{H}^*_{(2)}(X)$. Their difference is therefore square integrable, harmonic and orthogonal to $\mathcal{H}^*_{(2)}(X)$. Thus, it vanishes and we have the following nice formula

$$*_X dF'(\psi,0) = -\frac{\mathrm{i}}{2}F\left((1+S(0))*\psi,0\right). \tag{42}$$

In particular for $\psi\in(\ker\partial)^\perp$ we have

$$*_X dF'(\psi,0) = -\mathrm{i}F(*\psi,0), \tag{43}$$

or equivalently

$$dF'(\psi,0) = -\mathrm{i}F(du\wedge\psi,0). \tag{44}$$

## 3. Cohomology and Hodge theory on $M$

As before let $M$ be a compact manifold with boundary $Y$ and $X = (\mathbb{R}^+\times Y)\cup_Y M$ the associated manifold with a cylindrical end. We consider the long exact sequence

$$\cdots \xrightarrow{\partial} H^k(M,Y,\mathbb{R}) \xrightarrow{e} H^k(M,\mathbb{R}) \xrightarrow{r} H^k(Y,\mathbb{R}) \xrightarrow{\partial} H^{k+1}(M,Y,\mathbb{R}) \xrightarrow{e} \cdots, \tag{45}$$

in de Rham cohomology. Here $e$ is the canonical embedding and $r$ is the restriction homomorphism. There are three cochain complexes which compute the relative de Rham cohomology. Let

$$\Lambda^p(M,Y) := \{\omega\in\Lambda^p(M)\colon i^*\omega = 0\},$$

where $i\colon Y\to M$ is the inclusion. Since $d$ commutes with $i^*$, we get a complex $\Lambda^*(M,Y)$. Its cohomology is denoted by $H^*(M,Y,\mathbb{R})$. There is an exact sequence of complexes

$$0\longrightarrow \Lambda^*(M,Y)\xrightarrow{j}\Lambda^*(M)\xrightarrow{i^*}\Lambda^*(Y)\longrightarrow 0,$$

where $j$ is the inclusion map. It gives rise to the long exact sequence (45). The connecting homomorphism $\partial$ is defined as follows. Let $[\phi] \in H^k(Y, \mathbb{R})$. Extend $\phi$ to a $k$-form $\omega$ on $M$ such that $\omega = \phi$ in a neighborhood of the boundary. Then

$$\partial[\phi] = [d\omega]. \tag{46}$$

For the second description consider the cochain complex $\Lambda^*_{rel}(M, Y)$ of the mapping cone of $i^*$ which is defined by

$$\Lambda^p_{rel}(M, Y) := \Lambda^p(M) \oplus \Lambda^{p-1}(Y)$$

with differential $d$ given by

$$d(\omega, \theta) = (d\omega, i^*\omega - d\theta), \quad \omega \in \Lambda^p(M),\ \theta \in \Lambda^{p-1}(Y).$$

Let

$$\alpha\colon \Lambda^{p-1}(Y) \to \Lambda^p_{rel}(M, Y) \quad \text{and} \quad \beta\colon \Lambda^p_{rel}(M, Y) \to \Lambda^p(M)$$

be defined by $\alpha(\theta) = (0, (-1)^{p-1}\theta)$ and $\beta(\omega, \theta) = \omega$, respectively. Then $\alpha$ and $\beta$ are cochain maps and we get a second exact sequence of cochain complexes

$$0 \longrightarrow \Lambda^{*-1}(Y) \xrightarrow{\alpha} \Lambda^*_{rel}(M, Y) \xrightarrow{\beta} \Lambda^*(M) \longrightarrow 0$$

There is a natural inclusion of cochain complexes

$$\gamma\colon \Lambda^*(M, Y) \to \Lambda^*_{rel}(M, Y), \quad \omega \mapsto (\omega, 0).$$

It follows from the corresponding commutative diagram of long exact sequences that $\gamma$ induces an isomorphism

$$\gamma\colon H^*(M, Y, \mathbb{R}) \cong H^*_{rel}(M, Y, \mathbb{R}).$$

Finally $H^*_{rel}(M, Y, \mathbb{R})$ is also naturally isomorphic to the cohomology with compact supports $H^*_c(X)$. The isomorphism can be described as follows. Let $p\colon Z = \mathbb{R}^+ \times Y \to Y$ be the canonical projection. Integration over the fibre $\mathbb{R}^+$ of $p$ induces a mapping

$$p_*\colon \Lambda^p_c(\mathbb{R}^+ \times Y) \to \Lambda^{p-1}(Y).$$

Define a map

$$\xi\colon \omega \in \Lambda^p_c(X) \mapsto (\omega|_M, -p_*(\omega|_Z)) \in \Lambda^p_{rel}(M, Y)$$

This is a chain map. If the support of $\omega$ is contained in $M \setminus Y$, then $\xi(\omega) = (\omega, 0)$. Since every cohomology class in $H^p_c(X)$ has a representative of this form, it follows that $\xi$ induces an isomorphism

$$\bar{\xi}\colon H^p_c(X) \to H^p_{rel}(M, Y).$$

If we fix a metric on $Y$ we may identify $H^k(Y, \mathbb{R})$ with the space of harmonic forms $\mathcal{H}^k(Y, \mathbb{R})$. In fact, the image of $e$, i.e. the kernel of $r$ can be read off from the scattering matrix at 0.

**Theorem 3.1.** *The* $+1$ *eigenspace of the scattering matrix* $S(0)$ *on* $H^p(Y, \mathbb{R})$ *coincides with* $\operatorname{Im}(H^p(M, \mathbb{R}) \to H^p(Y, \mathbb{R}))$.

*Proof.* Let $\psi \in \mathcal{H}^p(Y,\mathbb{R})$ with $\psi = S(0)\psi$. By Proposition 2.12, $F(\psi,0)$ is closed and coclosed. Therefore the restriction of $F(\psi,0)$ to $M$ defines a cohomology class in $H^p(M)$. Expand $F(\psi,0)$ on $Z$ in terms of an orthonormal basis of $\ker\Delta'_p \oplus \ker\Delta'_{p-1}$. Using that $F(\psi,0)$ is closed and coclosed, it follows that its expansion on $Z$ has the form

$$F(\psi,0) = 2\psi + \sum_{\mu_{\phi_i}>0} a_i e^{-\mu_{\phi_i}u}(d'\phi_i - \mu_{\phi_i} du \wedge \phi_i), \tag{47}$$

where $\{\phi_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $\delta(\Lambda^p(Y))$ consisting of eigenforms of $\Delta'_{p-1}$ with eigenvalues $\mu_{\phi_i}^2$. In particular

$$i_Y^* F(\psi,0) = 2\psi + d'\left(\sum_i a_i\phi_i\right), \tag{48}$$

and therefore the image of the cohomology class $[\frac{1}{2}F(\psi,0)]$ under $r$ is precisely $[\psi]$. Therefore we have shown that the $+1$ eigenspace of $S(0)$ is contained in the image of $H^p(M,\mathbb{R})$ in $H^p(Y,\mathbb{R})$. Now let $\phi$ be an element in the image of $r$, i.e. $\phi$ is a harmonic form that is in the same cohomology class as the restriction of a closed form $f$ on $M$. If $\psi$ is in the $-1$ eigenspace of $S(0)$ then by Theorem 2.7, $*\psi$ is in the $+1$ eigenspace and we have

$$0 = \int_M df \wedge F(*\psi,0) = \int_Y i_Y^*(f)\wedge *\psi = \int_Y \phi\wedge *\psi = \langle\phi,\psi\rangle. \tag{49}$$

Hence, any element in $\mathrm{Im}(H^p(M,\mathbb{R}) \to H^p(Y,\mathbb{R}))$ is in the orthogonal complement to the $-1$ eigenspace of $S(0)$ which is exactly the $+1$ eigenspace. This shows that $\mathrm{Im}(H^p(M,\mathbb{R}) \to H^p(Y,\mathbb{R}))$ is contained in the $+1$ eigenspace and this concludes the proof. $\square$

Hence, the scattering matrix at 0 is determined completely by the metric on the boundary. Namely, it is equal to 1 on the kernel of $\partial$ and equal to $-1$ on the orthogonal complement of the kernel of $\partial$. Recall that by Proposition 2.12, $F(\psi,0)$ is a closed and coclosed $p$-form. Let $\hat{F}: H^k(Y,\mathbb{R}) \to H^k(M,\mathbb{R})$ be the map defined by

$$\hat{F}(\psi) = [\frac{1}{2}F(\psi,0)|_M]. \tag{50}$$

Then, by construction, $r\circ\hat{F}$ is the orthogonal projection onto the kernel of $\partial$.

Hodge theory for manifolds with boundary shows that absolute and relative cohomology classes have unique harmonic representatives that satisfy certain boundary conditions. We recall the definition of the relative and absolute boundary conditions for the Laplace operator. The operator $\Delta_{rel}$ is the closure of the Laplace operator with respect to the relative boundary conditions

$$\omega|_Y = 0, \quad (*\delta\omega)|_Y = 0.$$

The operator $\Delta_{abs}$ is the closure of the Laplace operator with respect to the absolute boundary conditions

$$(*\omega)|_Y = 0, \quad (*d\omega)|_Y = 0.$$

Both operators are self-adjoint and have compact resolvents. Their kernels are the space of harmonic forms satisfying relative and absolute boundary conditions. Equivalently, they are given by

$$\mathcal{H}^p_{rel}(M) = \{\omega \in \Lambda^p(M) \colon d\omega = \delta\omega = 0,\ \omega|_Y = 0\},$$
$$\mathcal{H}^p_{abs}(M) = \{\omega \in \Lambda^p(M) \colon d\omega = \delta\omega = 0,\ (*\omega)|_Y = 0\},$$

Hodge theory for manifolds with boundary shows that the canonical maps

$$\mathcal{H}^p_{rel}(M) \to H^p(M, Y, \mathbb{R}), \quad \mathcal{H}^p_{abs}(M) \to H^p(M, \mathbb{R})$$

are isomorphisms, that is, every absolute/relative cohomology class has a unique harmonic representative satisfying absolute/relative boundary conditions (see e.g. [DS52]).

The harmonic representative $\phi$ of the cohomology class $[\phi] \in H^p(M, Y, \mathbb{R})$ is the unique minimizer of the functional

$$\omega \mapsto \langle \omega, \omega \rangle_{L^2(M)}$$

in $[\phi]$. Similarly, any harmonic form satisfying absolute boundary conditions minimizes the $L^2$-norm in its absolute cohomology class. Apart from these minimax principles there is another interesting minimizing problem which is described in the following proposition.

**Theorem 3.2.** *Let $\phi \in \Lambda^p(Y)$. Consider the functional $F$ on $\{\omega \in \Lambda^p(M) \colon \omega|_Y = \phi\}$ which is defined by*

$$F(\omega) = \langle d\omega, d\omega \rangle_{L^2}.$$

*Then there exists a unique coclosed harmonic form $\omega_0$ with $\omega_0|_Y = \phi$ such that $\omega_0$ is orthogonal to $\mathcal{H}^p_{rel}(M)$. The minimum of $F$ is attained at $\omega_0$ and $\omega_0$ is the unique coclosed minimizer that is orthogonal to $\mathcal{H}^p_{rel}(M)$. If $\phi$ is closed $d\omega_0$ is the harmonic representative in $\partial[\phi]$.*

*Proof.* We divide the proof into several steps.

**Uniqueness:** If two forms $\omega_0$ and $\omega_0'$ are harmonic, coclosed, and their restrictions to $Y$ coincide, it follows that $\omega_0 - \omega_0'$ is harmonic, coclosed and satisfies relative boundary conditions. Therefore, $\omega_0 - \omega_0' \in \mathcal{H}^p_{rel}(M)$. If both $\omega_0$ and $\omega_0'$ are orthogonal to $\mathcal{H}^p_{rel}(M)$ it follows that $\omega_0 = \omega_0'$.

**Existence:** Choose any extension $\tilde{\psi}$ of $\phi$ to $M$ which in a neighborhood of $Y$ is of the form

$$\phi + u du \wedge \delta\phi. \tag{51}$$

Then $\delta\tilde{\psi}$ vanishes near $Y$. Next we claim that the form $\Delta\tilde{\psi}$ is in the orthogonal complement of the kernel of $\Delta_{rel}$. Indeed, if $\xi \in \mathcal{H}^p_{rel}(M)$, then

$$\begin{aligned}\langle \xi, \Delta\tilde{\psi} \rangle &= \int_M d\delta\tilde{\psi} \wedge *\xi + \int_M \xi \wedge *\delta d\tilde{\psi} = \\ &= \int_Y \delta\tilde{\psi} \wedge *\xi - \int_Y \xi \wedge *d\tilde{\psi} = 0,\end{aligned}$$

where the first integral vanishes because $\tilde{\psi}$ is coclosed near $Y$ and the second integral vanishes because $\xi$ satisfies relative boundary conditions. Let us denote by $p_0$ the orthogonal projection onto the kernel of $\Delta_{rel}$. Since $\Delta\tilde{\psi}$ is in the orthogonal complement of the kernel of $\Delta_{rel}$ the following form is well defined and harmonic

$$\omega_0 = \tilde{\psi} - \left(\Delta_{rel}|_{\mathcal{H}^p_{rel}(M)^\perp}\right)^{-1}(\Delta\tilde{\psi}) - p_0\tilde{\psi}.$$

By construction it is also in the orthogonal complement of $\mathcal{H}^p_{rel}(M)$ and satisfies $\omega_0|_Y = \phi$. Moreover, $\omega_0$ is harmonic and since $\Delta_{rel}$ commutes with $\delta$, we have

$$\delta\omega_0 = \delta\tilde{\psi} - \left(\Delta_{rel}|_{\mathcal{H}^p_{rel}(M)^\perp}\right)^{-1}\Delta\delta\tilde{\psi}$$

Since $\delta\tilde{\psi}$ vansihes near $Y$, it is in the domain of $\Delta_{rel}$ and therefore, the right hand side vanishes. Hence $\delta\omega_0 = 0$.
**Minimizing property:** Suppose that $\psi$ $p$-form with $\psi|_Y = 0$, then

$$\begin{aligned}\langle d(\omega_0+\psi), d(\omega_0+\psi)\rangle &= \langle d\omega_0, d\omega_0\rangle + 2\langle d\omega_0, d\psi\rangle + \langle d\psi, d\psi\rangle = \\ &= \langle d\omega_0, d\omega_0\rangle + \langle d\psi, d\psi\rangle,\end{aligned}$$

because $\langle d\omega_0, d\psi\rangle = \langle \delta d\omega_0, \psi\rangle = 0$. Thus, $\omega_0$ is a minimizer. One can see immediately that the Euler-Lagrange equations for the minimizer are the equations $\delta d\omega_0 = 0$. Thus, any coclosed minimizer has to be harmonic. The uniqueness statement for the minimizer thus follows from the above uniqueness statement. □

Since restriction to the boundary commutes with the differential the map which sends $\phi$ to $\omega_0$ commutes with the differential. Therefore, if $\phi$ is exact or closed, so is $\omega_0$.

Now consider the manifold $M_a$ which is obtained from $M$ by attaching the cylinder $[0,a]\times Y$ to $M$. Then $M_a$ is a manifold with boundary $Y_a$. Let $\phi \in \mathcal{H}^{p-1}(Y)$. We regard it as a harmonic form on $Y_a$. By the Hodge theorem there is a unique harmonic form in $\mathcal{H}^p_{rel}(M_a)$ which represents $\partial[\phi] \in H^p(M_a, \partial M_a)$. Let us denote this form by $\partial_a\phi$, where the notation $\partial_a$ indicates that $\partial_a$ maps $\mathcal{H}^{p-1}(Y)$ to different spaces depending on $a$. For each $a$ we have the $L^2$-inner product on $\mathcal{H}^p_{rel}(M_a)$. It is a natural question to ask whether $\partial$ as a map from one Hilbert space to another one is a partial isometry. The scalar product on $\mathcal{H}^p_{rel}(M_a)$, however, depends on $a$.

**Theorem 3.3.** *Let $Q_a$ be the sesquilinear form on $\ker(\partial)^\perp$ which is defined by*

$$Q_a(\psi,\phi) = \langle \partial_a\psi, \partial_a\phi\rangle_{L^2(M_a)}$$

*and let $q(a)$ be the unique linear operator $\ker(\partial)^\perp \to \ker(\partial)^\perp$ such that*

$$Q_a(\psi,\phi) = \langle \psi, q(a)\phi\rangle.$$

*Then, as $a\to\infty$:*

$$q(a)^{-1} = a\mathbb{1} + \frac{\mathrm{i}}{2}S'(0)|_{\ker(\partial)^\perp} + O(ae^{-\mu_1 a}).$$

*Proof.* Recall that $\ker(\partial)^{\perp} = \ker(\mathrm{I} + S(0))$. Let $\phi \in \ker(\partial)^{\perp}$. Then by (37) the restriction of $F'(\phi, 0)$ to the cylinder $Z$ has the following form

$$F'(\phi,0)|_Z = -2\mathrm{i}u\phi + S'(0)\phi + R'(\phi,0). \tag{52}$$

It follows that $F'(\phi, 0)$ is a coclosed harmonic form. Thus, by theorem 3.2 $F'(\phi, 0)$ is a coclosed minimizer of the functional $\eta \to \langle d\eta, d\eta\rangle_{L^2(M_a)}$ with boundary condition $\eta|_{Y_a} = -2\mathrm{i}a\phi + S'(0)\phi + R'(\phi, 0)|_{Y_a}$. Let $H$ be the minimizer with boundary conditions $H|_Y = a\phi + \frac{\mathrm{i}}{2}S'(0)\phi$ so that $dH$ is the unique harmonic representative of $\partial_a\left((a\mathbb{1} + \frac{\mathrm{i}}{2}S'(0))\phi\right)$. Again by theorem 3.2

$$G_\phi := \frac{i}{2}F'(\phi,0) - H \tag{53}$$

minimizes the functional $\eta \to \langle d\eta, d\eta\rangle_{L^2(M_a)}$ with boundary conditions

$$\eta|_{Y_a} = R'(\phi,0)|_{Y_a}. \tag{54}$$

Then for every $\psi \in \ker(\partial)^{\perp}$ we have

$$Q_a(\psi, (a\mathbb{1} + \frac{\mathrm{i}}{2}S'(0))\phi) = \langle \partial_a\psi, \frac{i}{2}dF'(\phi,0)\rangle_{L^2(M_a)} - \langle \partial_a\psi, dG_\phi\rangle_{L^2(M_a)}. \tag{55}$$

The second term on the right hand side can be estimated using the Cauchy-Schwarz inequality

$$|\langle \partial_a\psi, dG_\phi\rangle_{L^2(M_a)}| \leq \|\partial_a\psi\|_{L^2(M_a)} \cdot \|dG_\phi\|_{L^2(M_a)}. \tag{56}$$

The first term in (55) can be explicitly calculated. Using (44) we get

$$\begin{aligned}\langle \partial_a\psi, \frac{i}{2}dF'(\phi,0)\rangle_{L^2(M_a)} &= \frac{1}{2}\int_Y \psi \wedge *_M F(du\wedge\phi, 0) = \\ &= \int_Y \psi \wedge *\phi = \langle \psi, \phi\rangle,\end{aligned}$$

where we used that $R(du \wedge \phi, 0)|_Y$ is orthogonal to $\psi$. Thus,

$$|Q_a(\psi, (a\mathbb{1} + \frac{\mathrm{i}}{2}S'(0))\phi) - \langle\psi,\phi\rangle| \leq \|\partial_a\psi\|_{L^2(M_a)} \cdot \|dG_\phi\|_{L^2(M_a)}. \tag{57}$$

The terms on the right hand side can be estimated as follows. First note that $\|dG_\phi\|_{L^2(M_a)}$ minimizes $\|d\eta\|_{L^2(M_a)}$ over all forms $\eta$ which restrict to $R'(\phi, 0)|_{Y_a}$ on $Y_a$. Moreover, $\chi_a := R'(\phi, 0)|_{Y_a}$ is exponentially decaying in $a$. Define the form $\eta_a$ by $\eta_a := \frac{u}{a}\chi_a$ on the cylinder and 0 elsewhere. Then,

$$\|d\eta_a\|^2_{L^2(M_a)} = \frac{a}{3}\|d\chi_a\|^2_{L^2(Y)} + \frac{1}{a}\|\chi_a\|^2_{L^2(Y)}. \tag{58}$$

By Lemma 2.2 we have

$$\|d\chi_a\|^2_{L^2(Y)} + \|\chi_a\|^2_{L^2(Y)} \leq C_\phi e^{-2\mu_1 a}, \tag{59}$$

which implies

$$\|dG_\phi\|_{L^2(M_a)} \leq \tilde{C}_\phi e^{-\mu_1 a}. \tag{60}$$

To estimate $\| \partial_a\psi \|_{L^2(M_a)}$, recall that by Theorem 3.2, $\partial_a\psi = d\omega_0$, where $\omega_0$ minimizes the functional $\eta \mapsto \|d\eta\|^2_{L^2(M_a)}$ with boundary conditions $\eta|_{Y_a} = \psi$. Let $f \in C^\infty(\mathbb{R}^-)$ such that $f(u) = 1$ for $-1/4 \le u \le 0$ and $f(u) = 0$ for $u \le -3/4$. For $a \ge 1$ define $\hat{\psi}_a \in \Lambda^p(M_a)$ by

$$\hat{\psi}_a(x) = \begin{cases} f(u-a)\psi(y), & \text{if } x = (u,y) \in [0,a] \times Y; \\ 0, & \text{otherwise.} \end{cases}$$

Then it follows that there exists $C > 0$ such that

$$\|\partial_a\psi\|_{L^2\Lambda^{p+1}(M_a)} \le \|d\hat{\psi}_a\|_{L^2\Lambda^{p+1}(M_a)} \le C\|\psi\|_{L^2\Lambda^p(Y)}.$$

Thus we get

$$|Q_a(\psi, (a\mathbb{1} + \frac{\mathrm{i}}{2}S'(0))\phi) - \langle\psi,\phi\rangle| \le C''_\phi\|\psi\|e^{-\mu_1 a}. \tag{61}$$

By compactness of the unit sphere in $\ker(\partial)^\perp$ the constant can be chosen independent of $\phi$ if we use vectors of norm 1 only. Hence

$$\sup_{\|\psi\|,\|\phi\|=1} \left\langle \psi, (q(a)(a\mathbb{1} + \frac{\mathrm{i}}{2}S'(0)) - \mathbb{1})\phi \right\rangle \le Ce^{-\mu_1 a},$$

which implies

$$\|a\mathbb{1} + \frac{\mathrm{i}}{2}S'(0) - q(a)^{-1}\| \le C\|q(a)^{-1}\|e^{-\mu_1 a}. \tag{62}$$

From this inequality we deduce that $\|q(a)^{-1}\| \le C(1+a)$. Combined with (62) the statement of the theorem follows. ☐

## 4. Hodge theory on $X$ and the scattering length

In this section we give a description of the long exact cohomology sequence of $X$ in terms of harmonic forms and we derive a cohomological formula for the scattering length.

Let $H^*_c(X)$ denote the de Rham cohomology groups with compact supports. It is well known (see [Mel93]) that $H^*(X)$ and $H^*_c(X)$ are canonically isomorphic to certain spaces of extended harmonic forms on $X$. We recall some details.

The space of extended harmonic forms $\mathcal{H}^p_{ext}(X)$ is defined to be the subspace of all (real valued) $\psi \in \Lambda^p(X)$ satifying 1) $\Delta_p\psi = 0$ and 2) there exist $\phi_1 \in \ker\Delta'_p$ and $\phi_2 \in \ker\Delta'_{p-1}$ such that

$$\psi|_Z - \phi_1 - du \wedge \phi_2 \in L^2\Lambda^p(Z).$$

Note that for a given $\psi \in \mathcal{H}^p_{ext}(X)$ the sections $\phi_1$ and $\phi_2$ are uniquely determined. We regard $\phi_1$ (resp. $\phi_2$) as the tangential (resp. normal) boundary value of $\psi$ at infinity and we denote them by $\psi_t$ and $\psi_n$, respectively. The spaces satisfying absolute and relative boundary conditions at infinity are then defined as

$$\mathcal{H}^p_{ext,abs}(X) := \{\psi \in \mathcal{H}^p_{ext}(X) \mid \psi_n = 0\},$$
$$\mathcal{H}^p_{ext,rel}(X) := \{\psi \in \mathcal{H}^p_{ext}(X) \mid \psi_t = 0\}.$$

Since $\psi \in \mathcal{H}^p_{ext}(X)$ is harmonic and the form $\psi - \psi_t - du \wedge \psi_n$ is square integrable, it follows from (22) that there exists $c > 0$ such that

$$(\psi - \psi_t - du \wedge \psi_n)(u, y) \ll e^{-cu}, \quad (u, y) \in Z. \tag{63}$$

Moreover $d\psi$ and $\delta\psi$ are also exponentially decaying. Applying Greens formula to $M_a$, we get

$$0 = \langle \Delta\psi, \psi \rangle_{M_a} = \| d\psi \|^2_{M_a} + \| \delta\psi \|^2_{M_a} + O(e^{-ca}),$$

which implies that

$$d\psi = 0, \ \delta\psi = 0 \quad \text{for all } \psi \in \mathcal{H}^p_{ext}(X). \tag{64}$$

The intersection $\mathcal{H}^p_{ext,abs}(X) \cap \mathcal{H}^p_{ext,rel}(X)$ is the space $\mathcal{H}^p_{(2)}(X)$ of square integrable harmonic forms.

On $\mathcal{H}^p_{ext}(X)$ we introduce an inner product as follows. For $\psi, \phi \in \mathcal{H}^p_{ext}(X)$ let

$$\langle \psi, \phi \rangle = \int_M \psi \wedge *\phi + \int_Z (\psi - \psi_t - du \wedge \psi_n) \wedge *(\phi - \phi_t - du \wedge \phi_n). \tag{65}$$

To verify that this is an inner product, we only need to show that $\| \phi \| = 0$ implies $\phi = 0$. So suppose that $\| \phi \| = 0$. Then, in particular, we have $\phi|_M = 0$ and the unique continuation property for harmonic forms (see [Bae99, Corollary 3]) implies $\phi = 0$. We note that the inner product can be also defined by the following formula:

$$\langle \psi, \phi \rangle = \lim_{a \to \infty} \left( \int_{M_a} \psi \wedge *\phi - a(\langle \psi_t, \phi_t \rangle + \langle \psi_n, \phi_n \rangle) \right). \tag{66}$$

This inner product coincides on the subspace $\mathcal{H}^p_{(2)}(X)$ with the usual inner product on $\mathcal{H}^p_{(2)}(X)$. The orthogonal projections define canonical maps

$$\mathcal{H}^p_{ext,rel}(X) \to \mathcal{H}^p_{(2)}(X), \quad \mathcal{H}^p_{ext,abs}(X) \to \mathcal{H}^p_{(2)}(X).$$

Moreover, we have the maps

$$\begin{aligned} \hat{F} &: \mathcal{H}^p(Y) \to \mathcal{H}^p_{ext,abs}(X), \quad \phi \mapsto \frac{1}{2} F(\phi, 0), \\ \hat{G} &: \mathcal{H}^p(Y) \to \mathcal{H}^p_{ext,rel}(X), \quad \phi \mapsto \frac{\mathrm{i}}{2} dF'(\phi, 0). \end{aligned} \tag{67}$$

Next we define maps into the de Rham cohomology. Let $\phi \in \mathcal{H}^p_{ext,abs}(X)$. By (64), $\phi$ is closed and we get a canonical map

$$R \colon \mathcal{H}^p_{ext,abs}(X) \to H^p(X, \mathbb{R}). \tag{68}$$

Now consider $\psi \in \mathcal{H}^p_{ext,rel}(X)$. By (64) $\psi$ is closed and on the cylinder $\psi$ is of the form

$$\psi|_Z = du \wedge \psi_n + d\theta, \tag{69}$$

where $\theta$ is exponentially decaying. Let $\chi$ be a function with support on the cylinder $Z$ which is equal to 1 outside a compact set. Following [Mel93] we can then define a map

$$R_c \colon \mathcal{H}^p_{ext,rel}(X) \to H^p_c(X, \mathbb{R}), \quad \psi \mapsto [\psi - d(\chi(u\psi_n + \theta))]. \tag{70}$$

This map is well defined and independent of the choice of $\chi$. Indeed changing $\chi$ on a compact subset changes

$$\psi - d(\chi(u\psi_n + \theta))$$

by the differential of a compactly supported form. Let

$$(\cdot,\cdot)\colon H^p_c(X) \times H^{n-p}(X) \to \mathbb{R}$$

be the canonical pairing defined by

$$([\phi],[\psi]) = \int_X \phi \wedge \psi, \quad [\phi] \in H^p_c(X),\ [\psi] \in H^{n-p}(X).$$

Define a pairing

$$(\cdot,\cdot)_{ext}\colon \mathcal{H}^p_{ext,rel}(X) \times \mathcal{H}^{n-p}_{ext,abs}(X) \to \mathbb{R} \tag{71}$$

by taking the constant term in the asymptotic expansion of

$$\int_{M_a} \psi \wedge \phi$$

as $a \to \infty$. Applying Green's formula to $M_a$, it follows that there exists $c > 0$ such that

$$\int_{M_a} \psi \wedge \phi = a \int_Y \psi_n \wedge \phi_t + ([\psi - d(\chi(u \cdot \psi_n + \theta))], [\phi]) + O(e^{-ca})$$

as $a \to \infty$. This implies that the following diagram commutes

$$\begin{tikzcd}
\mathcal{H}^p_{ext,rel}(X) \arrow[d, "R_c"] & \times & \mathcal{H}^{n-p}_{ext,abs}(X) \arrow[d, "R"] \arrow[dr] & \\
 & & & \mathbb{R} \\
H^p_c(X,\mathbb{R}) & \times & H^{n-p}(X,\mathbb{R}) \arrow[ur] &
\end{tikzcd} \tag{72}$$

,

where the horizontal maps are given by the corresponding pairing.

Let $\phi \in \mathcal{H}^p_{ext}(X)$ and $\omega \in \Lambda^{p-1}_c(X)$. Since $\phi$ is co-closed (64), it follows that $\langle d\omega, \phi\rangle = 0$. Therefore, for $\phi \in \mathcal{H}^p_{ext}(X)$, $[\psi] \in H^p_c(X)$, and $\psi' \in [\psi]$, the inner product $\langle \psi', \phi\rangle$ is independent of the representative of the cohomology class $[\psi]$ and will be denoted by $\langle [\psi], \phi\rangle$. This leads to the following alternative description of the inner product in $\mathcal{H}^p_{ext,rel}(X)$.

**Lemma 4.1.** *For all $\phi, \psi \in \mathcal{H}^p_{ext,rel}(X)$ we have*

$$\langle \psi, \phi\rangle = \langle R_c(\psi), \phi\rangle.$$

*Proof.* Applying Stoke's theorem and using that $\theta$ is rapidly decreasing, we get

$$\int_{M_a} d(\chi(u\psi_n + \theta)) \wedge *\phi = a\langle \psi_n, \phi_n\rangle + O(e^{-ca}).$$

By (66) we get

$$\langle \psi, \phi\rangle = \langle \psi - d(\chi(u\psi_n + \theta)), \phi\rangle = \langle R_c(\psi), \phi\rangle.$$

□

Our next goal is to describe the connecting homomorphism $\partial : H^p(Y,\mathbb{R}) \to H_c^{p+1}(X,\mathbb{R})$ on the level of harmonic forms. To this end we need some preapration. Let $\psi$ be in the $-1$ eigenspace of $S(0)$. Then by (37), we have on $Z$

$$\frac{\mathrm{i}}{2}F'(\psi,0)|_Z = u\psi + \frac{\mathrm{i}}{2}S'(0)\psi + \theta,$$

where $\theta$ is exponentially decaying. Let $\chi$ be a smooth function with support in $Z$ which is a equal to 1 outside a compact set. Then

$$\frac{\mathrm{i}}{2}F'(\psi,0) - \chi(u\psi+\theta)$$

is equal to $\frac{\mathrm{i}}{2}S'(0)\psi$ outside a compact set and we conclude that

$$d(\frac{\mathrm{i}}{2}F'(\psi,0) - \chi(u\psi+\theta))$$

represents $\partial[\frac{\mathrm{i}}{2}S'(0)\psi]$ in $H_c^{p+1}(X,\mathbb{R})$. Let $\kappa\colon \mathcal{H}^p(Y) \to H^p(Y)$ be the canonical isomorphism. Then we have shown that for each $\psi \in \mathcal{H}^p(Y)$ we have

$$R_c(\hat{G}(\psi)) = \partial\left[\kappa(\frac{i}{2}S'(0)\psi)\right], \tag{73}$$

where $\hat{G}(\psi)$ is defined by (67).

**Lemma 4.2.** *The operator $S'(0)$ in $\mathcal{H}^*(Y)$ is invertible.*

*Proof.* Differentiating equations (ii) and (iii) of Theorem 2.7, it follows that $S'(0)$ commutes with $S(0)$ and anti-commutes with $*$. Therefore, it suffices to show that the restriction of $S'(0)$ to the $-1$-eigenspace $E_-$ of $S(0)$ is invertible. Let $\psi \in E_-$. Then $S'(0)\psi \in E_-$. By Theorem 3.1 we have $E_- = (\ker\partial)^\perp$. Using (73), it follows that it suffices to show that $R_c(\hat{G}(\psi)) \neq 0$ whenever $\psi \neq 0$. By Lemma 4.1 we have

$$\langle \hat{G}(\psi), \hat{G}(\psi)\rangle = \langle R_c(\hat{G}(\psi)), \hat{G}(\psi)\rangle$$

for all $\psi \in \mathcal{H}^p_{ext,rel}(X)$. Recall that $\hat{G}(\psi)$ is a harmonic form, which is non-zero, if $\psi \neq 0$. Therefore, the left hand side of the above equality is non-zero, if $\psi \neq 0$. $\square$

Now we define maps

$$\tilde{e}\colon \mathcal{H}^p_{ext,rel}(X) \to \mathcal{H}^p_{ext,abs}(X), \quad \tilde{r}\colon \mathcal{H}^p_{ext,abs}(X) \to \mathcal{H}^p(Y), \quad \tilde{\partial}\colon \mathcal{H}^p(Y) \to \mathcal{H}^{p+1}_{ext,rel}(X)$$

as follows. Let $\tilde{e}$ be the composition of the orthogonal projection $\mathcal{H}^p_{ext,rel}(X) \to \mathcal{H}^p_{(2)}(X)$ and the inclusion $\mathcal{H}^p_{(2)}(X) \to \mathcal{H}^p_{ext,abs}(X)$. $\tilde{r}$ assigns to $\phi \in \mathcal{H}^p_{ext,abs}(X)$ its limiting value $\phi_t$. To define $\tilde{\partial}$, we note that by Lemma 4.2, $S'(0)$ is an invertible operator. Put

$$\tilde{\partial} = \hat{G} \circ \left(\frac{\mathrm{i}}{2}S'(0)\right)^{-1}.$$

**Proposition 4.3.** *The sequence*

$$\cdots \xrightarrow{\tilde\partial} \mathcal{H}^p_{ext,rel}(X) \xrightarrow{\tilde e} \mathcal{H}^p_{ext,abs}(X) \xrightarrow{\tilde r} \mathcal{H}^p(Y) \xrightarrow{\tilde\partial} \mathcal{H}^{p+1}_{ext,rel}(X) \xrightarrow{\tilde e} \cdots$$

*is exact.*

*Proof.* Let $E_\pm = \ker(S(0) \mp \mathrm{Id})$. By Proposition 2.12 and (42) it follows that

$$\mathrm{Im}(\tilde r) = E_+ = \ker(\hat G). \tag{74}$$

Since $S'(0)$ preserves $E_\pm$, we get $\mathrm{Im}(\tilde r) = \ker(\tilde\partial)$. By definition we have $\mathrm{Im}(\tilde e) = \mathcal{H}^p(Y)$ and this is also equal to $\ker(\tilde r)$. Finally by Corollary 2.11 it follows that $\mathrm{Im}(\tilde\partial) = \mathcal{H}^{p+1}_{(2)}(X)^\perp$. On the other hand, by definition we have $\ker(\tilde e) = \mathcal{H}^{p+1}_{(2)}(X)^\perp$. Thus $\mathrm{Im}(\tilde\partial) = \ker(\tilde e)$. $\square$

Using the definition of $\tilde\partial$ and (73), it follows that

$$R_c \circ \tilde\partial = \partial \circ \kappa.$$

By [APS75] every element in the image of $H^*_c(X,\mathbb{R})$ in $H^*(X,\mathbb{R})$ can be represented by a unique square integrable harmonic form. Using these facts we obtain the following commutative diagram.

$$\begin{tikzcd}
 & \mathcal{H}^p(Y) \arrow[dl, "\frac{\mathrm{i}}{2}S'(0)"'] \arrow[d, "\hat G"] & \mathcal{H}^p_{(2)}(X) \arrow[dl] \arrow[d] & \ker(S(0)-\mathbb{1}) \arrow[dl, "\hat F"'] \arrow[d] & \mathcal{H}^p(Y) \arrow[dl, "\frac{\mathrm{i}}{2}S'(0)"'] \arrow[d, "\hat G"] & \\
\cdots \arrow[r, "\tilde\partial"] & \mathcal{H}^p_{ext,rel}(X) \arrow[r, "\tilde e"] \arrow[d, "R_c"] & \mathcal{H}^p_{ext,abs}(X) \arrow[r, "\tilde r"] \arrow[d, "R"] & \mathcal{H}^p(Y) \arrow[r, "\tilde\partial"] \arrow[d, "\kappa"] & \mathcal{H}^{p+1}_{ext,rel}(X) \arrow[r, "\tilde e"] \arrow[d, "R_c"] & \cdots \\
\cdots \arrow[r, "\partial"] & H^p_c(X,\mathbb{R}) \arrow[r, "e"] & H^p(X,\mathbb{R}) \arrow[r, "r"] & H^p(Y,\mathbb{R}) \arrow[r, "\partial"] & H^{p+1}_c(X,\mathbb{R}) \arrow[r, "e"] & \cdots
\end{tikzcd}$$

**Proposition 4.4.** *The maps*

$$R\colon \mathcal{H}^p_{ext,abs}(X) \to H^p(X,\mathbb{R}), \quad R_c\colon \mathcal{H}^p_{ext,rel}(X) \to H^p_c(X,\mathbb{R})$$

*are isomorphisms.*

*Proof.* We first consider $R$. Let $H^p_!(X,R) = \mathrm{Im}(e)$. By [APS75], $R$ induces an isomorphism of $\mathcal{H}^p_{(2)}(X)$ onto $H^p_!(X)$. Let $\phi \in \mathcal{H}^p_{ext,abs}(X)$ and suppose that $R(\phi) = 0$. Then it follows that $\tilde r(\phi) = 0$. Hence $\phi \in \mathcal{H}^p_{(2)}(X)$. Since $R$ is an isomorphism on $\phi \in \mathcal{H}^p_{(2)}(X)$, we get $\phi = 0$. This proves injectivity. Let $\psi \in H^p(X,\mathbb{R})$. Using $H^p(X,\mathbb{R}) \cong H^p(M,\mathbb{R})$ and Theorem 3.1, it follows that $\kappa^{-1}(r(\psi)) \in \ker(S(0) - \mathrm{Id})$. Thus by (74) there exists $\phi \in \mathcal{H}^p_{ext,abs}(X)$ such that $\tilde r(\phi) = \kappa^{-1}(r(\psi))$. Then $r(R(\phi) - \psi) = 0$. Hence $R(\phi) - \psi \in H^p_!(X,\mathbb{R})$. By the above remark there is $\omega \in \mathcal{H}^p_{(2)}(X)$ such that $R(\omega) = R(\phi) - \psi$. Thus $R$ is surjectiv and hence an isomorphism. Applying the commutativity of the diagramm and the 5-Lemma, it follows that $R_c$ is an isomorphism too. $\square$

This can also be proved by slightly different methods (e.g. [Mel95, Mel93]).

**Corollary 4.5.** *In every class in $H^p(X,\mathbb{R})$ there is a unique representative in $\mathcal{H}^p_{ext,abs}(X)$.*

**Corollary 4.6.** *For every class $[\psi]$ in $H^p_c(X,\mathbb{R})$ there is a unique element $\hat{\psi}$ in $\mathcal{H}^p_{ext,rel}(X)$ such that for any $\phi \in \mathcal{H}^p_{ext,abs}(X)$:*

$$\langle [\psi], [\phi] \rangle = \langle \hat{\psi}, \phi \rangle.$$

*Moreover, the map*

$$\begin{aligned} H^p_c(X,\mathbb{R}) &\to \mathcal{H}^p_{ext,rel}(X), \\ [\psi] &\mapsto \hat{\psi} \end{aligned}$$

*is an isomorphism.*

We can now consider the scattering length

$$T(0) := -\mathrm{i}S(0)^* S'(0) = -\mathrm{i}S(0)S'(0). \tag{75}$$

Let $\partial\colon H^p(Y,\mathbb{R}) \to H^{p+1}_c(X,\mathbb{R})$ be the connecting homomorphism. We identify $H^p(Y,\mathbb{R})$ with $\mathcal{H}^p(Y)$ and $H^{p+1}_c(X,\mathbb{R})$ with $\mathcal{H}^{p+1}_{ext,rel}(X)$ via Corollary 4.6. Thus we may regard the connecting homomorphism as a map

$$\partial\colon \mathcal{H}^p(Y) \to \mathcal{H}^{p+1}_{ext,rel}(X).$$

Let $(\ker\partial)^\perp$ be the orthogonal complement of $\ker\partial$.

**Theorem 4.7.** *The scattering length $T(0)$ is a positive, invertible operator in $\mathcal{H}^p(Y)$. It is uniquely determined by the following conditions.*

$$\forall \phi,\psi \in (\ker\partial)^\perp : \langle \partial\phi, \partial(T(0)\psi)\rangle = 2\langle \phi,\psi\rangle, \tag{76}$$

$$T(0)* = *T(0). \tag{77}$$

*Proof.* By Theorem 3.1, $\mathrm{Im}(r) = \ker\partial$ equals the $+1$ eigenspace of $S(0)$. Therefore $(\ker\partial)^\perp$ equals the $-1$ eigenspace of $S(0)$. By Theorem 2.7, $*$ anti-commutes with $S(0)$. Hence it interchanges the $\pm1$-eigenspaces. $*$ also anti-commutes with $S'(0)$ and consequently the scattering length $T(0)$ commutes with the Hodge star operator. It is therefore completely determined by its restriction to $(\ker\partial)^\perp$. Let $\phi,\psi \in (\ker\partial)^\perp$. Using (73) and Lemma 4.1, we have

$$\begin{aligned} \langle \partial\phi, \partial T(0)\rangle &= \langle \partial\phi, -i\partial(S'(0)S(0)\psi)\rangle = 2\langle \partial\phi, \partial\left((i/2)S'(0)\psi\right)\rangle \\ &= 2\langle \partial\phi, R_c(\hat{G}(\psi))\rangle = 2\langle \partial\phi, \hat{G}(\psi)\rangle. \end{aligned} \tag{78}$$

Let $\tilde{\phi}$ be the pull-back of $\phi$ to $Z$ and let $\chi \in C^\infty(Z)$ such that $\chi = 0$ on $[0,1]\times Y$ and $\chi = 1$ outside a compact set. Then $d(\chi\tilde{\phi}) \in \Lambda^{p+1}_c(X)$ represents $\partial[\phi] \in H^{p+1}_c(X,\mathbb{R})$. By the same argument as in the proof of Lemma 4.1 we get

$$\langle \partial\phi, \hat{G}(\psi)\rangle = \langle d(\chi\tilde{\phi}), \hat{G}(\psi)\rangle.$$

By (41), $*\hat{G}(\psi) = *\frac{i}{2} dF'(\psi, 0)$ is an extended harmonic form with limiting value $*\psi$. Using Stokes theorem, applied to $M_a$, it follows that

$$\langle d(\chi\tilde{\phi}), \hat{G}(\psi)\rangle = \lim_{a\to\infty} \int_{\partial M_a} \phi \wedge *\hat{G}(\psi) = \int_Y \phi \wedge *\psi = \langle \phi, \psi\rangle. \tag{79}$$

This concludes the proof of the theorem. □

**Corollary 4.8.** *For the scattering length $T(0) = -\mathrm{i}S(0)^* S'(0)$ we have the following formula*

$$T^{-1}(0) = \frac{1}{2}\left(\partial^*\partial + (*)^{-1}\partial^*\partial *\right) = \frac{1}{2}\left(rr^* + (*)^{-1} rr^* *\right).$$

This implies that $\frac{1}{2} r^* T(0) r$ is equal to the orthogonal projection onto the orthogonal complement of $\ker r$.

## 5. Estimates on the norm of extended harmonic forms

By the results of the previous sections we have canonical isomorphisms

$$\begin{aligned} \eta_{abs}\colon \mathcal{H}^p_{ext,abs}(X) &\cong H^p(X,\mathbb{R}) \cong H^p(M,\mathbb{R}), \\ \eta_{rel}\colon \mathcal{H}^p_{ext,rel}(X) &\cong H^p_c(X,\mathbb{R}) \cong H^p(M,Y,\mathbb{R}). \end{aligned} \tag{80}$$

In this section we establish relations between some norms on these spaces. On the cohomology groups $H^p(M,\mathbb{R})$ and $H^p(M,Y,\mathbb{R})$ there is the so-called comass norm (see [Gro99, Ch. 4C]) which is defined as follows. If $V$ is a finite dimensional inner product space, then $\Lambda^p V^*$ has a natural inner product as well and we denote the norm that is induced by this inner product by $\|\cdot\|$. The comass norm $\|\cdot\|_\infty$ on $\Lambda^p V^*$ is defined by

$$\|\omega\|_\infty = \sup\{\omega(e_1,\dots,e_p) \mid \ e_k \in V, \|e_k\| = 1\} \tag{81}$$

Since the norms are equivalent there is a constant $C$ such that

$$\|\omega\|^2 \le C\|\omega\|^2_\infty, \tag{82}$$

and we denote by $C(n,p)$ the optimal such constant. Since all $n$-dimensional inner product spaces are unitarily equivalent the constant depends only on $n$ and $p$. Of course (see also [Fed69]),

$$C(n,0) = C(n,1) = 1, \tag{83}$$

$$C(n,p) \le \binom{n}{p}. \tag{84}$$

Moreover, since the Hodge star operator leaves the space of primitive forms invariant, we have

$$C(n,n-p) = C(n,p). \tag{85}$$

It is also known that

$$C(n,2) = [\frac{n}{2}] \tag{86}$$

(see [GlKo02]). Now let $B$ be a differentiable manifold. Let $\omega \in \Lambda^p(B)$. The comass $\|\omega\|_\infty$ of $\omega$ is defined by

$$\begin{aligned}(87)\qquad \|\omega\|_\infty &= \sup\{\omega_x(e_1,\dots,e_p) \mid x \in B,\ e_i \in T_xB,\ g(e_i,e_i)=1\} = \\ &= \sup\{\|\omega_x\|_\infty \mid x \in B\}.\end{aligned}$$

For a compact manifold $B$ with smooth boundary $\partial B$ this induces a norm on $H^p(B,\partial B,\mathbb{R})$ by

$$(88)\qquad \|\phi\|_\infty = \inf\{\|\omega\|_\infty \mid \phi = [\omega],\ \omega \in \Lambda^p(B,\partial B),\ d\omega = 0\},$$

which we also refer to as the comass norm. To compare the norms on the various cohomology groups, we need some preparation. Let $\psi \in \Lambda^p(M)$. We define an extension $\hat{\psi} \in L^\infty\Lambda^p(X)$ of $\psi$ in the following way. The restriction $\psi|_Y$ can be expanded into eigensections of the Laplace-Beltrami operator on $Y$:

$$\psi|_Y = \phi + \sum_{i=1}^\infty a_i\phi_i,$$

where $\phi$ is harmonic and $\phi_i$ is an orthonormal basis in the orthogonal complement of the space of harmonic forms such that

$$\Delta'\phi_i = \mu_{\phi_i}^2\phi_i.$$

Now define

$$(89)\qquad \hat{\psi}(x) = \begin{cases} \psi(x) & \text{for } x \in M, \\ \phi(y) + \sum_{i=1}^\infty a_i e^{-\mu_{\phi_i}u}\left(\phi_i(y) - \mu_{\phi_i}^{-1} du \wedge \delta'\phi_i(y)\right) & \text{for } x=(u,y) \in Z \end{cases}$$

As usually $x = (u,y)$. The map $\psi \mapsto \hat{\psi}$ is of course linear and maps into the space of bounded sections. Note that in general $\hat{\psi}$ is not continuous. However, it satisfies

$$i_Y^*(\hat{\psi}|_M) = i_Y^*(\hat{\psi}|_Z).$$

Therefore, using Green's formula, it follows that the distributional derivative $d\hat{\psi}$ satisfies

$$\begin{aligned}(90)\qquad d\hat{\psi}(x) &= \sum_{i=1}^\infty a_i e^{-\mu_{\phi_i}u}\left(d'\phi_i - \mu_{\phi_i} du \wedge \phi_i + \mu_{\phi_i}^{-1} du \wedge d'\delta'\phi_i\right) = \\ &= \sum_{i=1}^\infty a_i e^{-\mu_{\phi_i}u}\left(d'\phi_i - \mu_{\phi_i}^{-1} du \wedge \delta' d'\phi_i\right) = \widehat{d\psi}.\end{aligned}$$

In particular $d\hat{\psi}$ is again bounded. If $\psi$ is closed, then $\hat{\psi}$ is also closed. Note that the extension map $\psi \in \Lambda^p(M) \mapsto \hat{\psi} \in L^\infty\Lambda^p(X)$ is chosen so that it inverts the restriction operator on the space $\mathcal{H}^p_{ext,abs}(X)$. Namely, for $F \in \mathcal{H}^p_{ext,abs}(X)$ it follows from (47) that

$$(91)\qquad F = \widehat{F|_M}.$$

We can now establish the comparison results for the norms.

**Lemma 5.1.** *Let $\psi \in \Lambda^p(M)$ be closed and $F \in \mathcal{H}^p_{ext,abs}(X)$ a harmonic form such that $F|_M$ and $\psi$ represent the same element in $H^p(M,\mathbb{R})$. Let $\phi \in \mathcal{H}^p(Y)$ be the unique harmonic representative of the cohomology class of $\psi|_Y$. Then*

$$\|F\|^2 \leq \|\psi\|^2_{L^2\Lambda^p(M)} + \frac{1}{\mu_1}\|\psi|_Y - \phi\|^2_{L^2\Lambda^p(Y)}, \tag{92}$$

*where $\mu_1^2$ is the smallest positive eigenvalue of $\Delta_Y$. In particular*

$$\|F\|^2 \leq C(n,p)\mathrm{Vol}_*(M)\|[F|_M]\|^2_\infty, \tag{93}$$

*where we define the effective volume $\mathrm{Vol}_*(M)$ by*

$$\mathrm{Vol}_*(M) = \mathrm{Vol}(M) + \frac{1}{\mu_1}\mathrm{Vol}(Y). \tag{94}$$

*Proof.* Let $F \in \mathcal{H}^p_{ext,abs}(X)$. Let $\psi \in \Lambda^p(M)$ be a closed form such that

$$F|_M - \psi = dh, \tag{95}$$

for some $h \in \Lambda^{p-1}(M)$. Denote by $\phi$ the unique harmonic representative of the class of $\psi|_Y$. If $\chi_Z$ is the characteristic function of $Z \subset X$, then the norm of $F$ is by definition the $L^2$-norm of $F - \phi\chi_Z$. Therefore,

$$\|\hat{\psi} - \phi\chi_Z\|^2 = \|F - \phi\chi_Z + (\hat{\psi} - F)\|^2 = \tag{96}$$

$$= \|F - \phi\chi_Z\|^2 + \|(\hat{\psi} - F)\|^2 + 2\langle F - \phi\chi_Z, (\hat{\psi} - F)\rangle. \tag{97}$$

Now observe that by (95) and the definition of $\hat{\psi}$, the expansion of $\hat{\psi} - F$ on $Z$ contains no harmonic form. Hence by (63) and (89) the restriction of $\hat{\psi} - F$ to $Z$ is exponentially decreasing. This implies

$$\langle F - \phi\chi_Z, \hat{\psi} - F\rangle = \langle F, \hat{\psi} - F\rangle.$$

By (95) and (90) we have $F - \hat{\psi} = \widehat{dh} = d\hat{h}$, where the latter means the distributional derivative. Since $F$ is closed and coclosed, it follows that

$$\langle F - \phi\chi_Z, F - \hat{\psi}\rangle = \langle F, \widehat{dh}\rangle = \langle F, d\hat{h}\rangle = \langle \delta F, \hat{h}\rangle = 0. \tag{98}$$

Therefore we get

$$\begin{aligned}\|\hat{\psi} - \phi\chi_Z\|^2 &= \|F - \phi\chi_Z + (\hat{\psi} - F)\|^2 = \\ &= \|F - \phi\chi_Z\|^2 + \|(\hat{\psi} - F)\|^2 \geq \|F\|^2.\end{aligned} \tag{99}$$

On the other hand

$$\begin{aligned}\|\hat{\psi} - \phi\chi_Z\|^2 &= \|\psi\|^2_{L^2\Lambda^p(M)} + \int_0^\infty \sum_{i=1}^\infty |a_i|^2 e^{-2\mu_{\phi_i}u}\left(\|\phi_i\|^2 + \mu_{\phi_i}^{-2}\|\delta'\phi_i\|^2\right) du = \\ &= \|\psi\|^2_{L^2\Lambda^p(M)} + \sum_{i=1}^\infty \mu_{\phi_i}^{-1}|a_i|^2\|\phi_i\|^2 = \|\psi\|^2_{L^2\Lambda^p(M)} + \|\Delta_Y^{-\frac{1}{4}}(\psi|_Y - \phi)\|^2_{L^2\Lambda^p(Y)} \leq \\ &\leq \|\psi\|^2_{L^2\Lambda^p(M)} + \frac{1}{\mu_1}\|\psi|_Y - \phi\|^2.\end{aligned} \tag{100}$$

□

**Lemma 5.2.** *Let $\phi \in \mathcal{H}^p_{ext,rel}(X)$. Let $\psi \in \Lambda^p(M,Y)$ be a representative of the class $[\psi] \in H^p(M,Y,\mathbb{R})$ which corresponds to $\phi$ with resepect to the isomorphism* (80). *Then*

$$\|\phi\|^2 \leq \|\psi\|^2_{L^2\Lambda^p(M)}, \tag{101}$$

*and in particular*

$$\|\phi\|^2 \leq C(n,p)\mathrm{Vol}(M)\|[\psi]\|^2_\infty. \tag{102}$$

*Proof.* Let $\phi \in \mathcal{H}^p_{ext,rel}(X)$. Recall the definition of $R_c$ by (70). Choose $\chi$ such that the support of $\phi - d(\chi(u\phi_n+\theta))$ is contained in $M$. Then $[\phi - d(\chi(u\phi_n+\theta))]$ is the image of $\phi$ w.r.t. the isomorphism (80). Since $\psi \in \Lambda^p(M,Y)$ represents this cohomology class, there is $\omega \in \Lambda^{p-1}(M,Y)$ such that

$$\phi - d(\chi(u\phi_n+\theta)) = \psi + d\omega. \tag{103}$$

Let $\tilde\psi$ (resp. $\tilde\omega$) be the differential form on $X$ which is equal to $\psi$ (resp. $\omega$) on $M$ and 0 on $Z$. Then

$$\begin{aligned}\|\psi\|^2 = \|\tilde\psi - \phi + \chi_Z du\wedge\phi_n + (\phi - \chi_Z du\wedge\phi_n)\|^2 = \|\tilde\psi - \phi + \chi_Z du\wedge\phi_n\|^2\\ + \|\phi - \chi_Z du\wedge\phi_n\|^2 + 2\langle\tilde\psi - \phi + \chi_Z du\wedge\phi_n, \phi - \chi_Z du\wedge\phi_n\rangle.\end{aligned} \tag{104}$$

By the definition (65) of the norm in $\mathcal{H}^p_{ext,rel}(X)$, we have

$$\| \phi \| = \| \phi - \chi_Z du\wedge\phi_n \|_{L^2} .$$

By (103) we have

$$\begin{aligned}\langle\tilde\psi - \phi + \chi_Z du\wedge\phi_n, \phi - \chi_Z du\wedge\phi_n\rangle &= \langle\tilde\psi - \phi, \phi - \chi_Z du\wedge\phi_n\rangle\\ &= -\langle d\tilde\omega + d(\chi(u\phi_n+\theta)), \phi - \chi_Z du\wedge\phi_n\rangle.\end{aligned}$$

Since $\omega \in \Lambda^{p-1}(M,Y)$, it follows from Green's formula that

$$\langle d\tilde\omega, \phi - \chi_Z du\wedge\phi_n\rangle = \int_M d\omega\wedge *\phi = \int_Y i_Y^*(\omega)\wedge i_Y^*(*\phi) = 0.$$

Similarly, by Green's formula and (69) we have

$$\begin{aligned}\langle d(\chi(u\phi_n+\theta)), \phi - \chi_Z du\wedge\phi_n\rangle &= \int_M d(\chi(u\phi_n+\theta))\wedge *\phi\\ &+ \int_Z d(\chi(u\phi_n+\theta))\wedge *(\phi - \chi_Z du\wedge\phi_n)\\ &= \int_Y \theta\wedge *(du\wedge\phi_n + d\theta) - \int_Y \theta\wedge *d\theta = 0.\end{aligned}$$

Thus

$$\langle\tilde\psi - \phi + \chi_Z du\wedge\phi_n, \phi - \chi_Z du\wedge\phi_n\rangle = 0$$

and by (104) we get

$$\|\psi\|^2 = \|\phi\|^2 + \|\tilde\psi - \phi + \chi_Z du\wedge\phi_n\|^2 \geq \|\phi\|^2.$$

$\square$

## 6. Estimates on the scattering matrix and stable systoles

We recall some notions from geometric measure theory. Suppose $B$ is a compact oriented Riemannian manifold and let $A$ be a closed submanifold. In our case $A$ will be either the boundary of $B$ or the empty set. For $z \in H_p(B, A, \mathbb{Z})$ let the minimal volume be defined as the infimum of the volumes of all its representatives, i.e.

$$\mathrm{vol}(z) = \inf\{\sum_i |\alpha_i| \mathrm{Vol}(c_i) \mid z = \sum_i \alpha_i [c_i],\ \alpha_i \in \mathbb{Z}\}. \tag{105}$$

where the infimum is over all Lipschitz continuous simplices $c_i$. The stable norm $\|z\|_{st}$ of an element $z \in H_p(B, A, \mathbb{R})$ is defined similarly by

$$\|z\|_{st} := \inf\{\sum_i |\alpha_i| \mathrm{Vol}(c_i) \mid z = \sum_i \alpha_i [c_i],\ \alpha_i \in \mathbb{R}\}. \tag{106}$$

This defines indeed a norm (see [FF60, 9.6, 9.9], and [Fed75, §3], also [Gro99, Ch. 4C], and [Fed69, 5.1.6]). The stable norm of an element in $z \in H_p(B, A, \mathbb{Z})$ is by definition the stable norm of its image in $H_p(B, A, \mathbb{R})$. Clearly,

$$\|z\|_{st} \leq \mathrm{vol}(z) \tag{107}$$

and equality does not hold in general. However, as shown by Federer (see [Fed75, §5]),

$$\|z\|_{st} = \lim_{k \to \infty} \frac{1}{k} \mathrm{vol}(kz). \tag{108}$$

Moreover, for $z \in H_{n-1}(B, A, \mathbb{Z})$ we have equality, i.e. $\|z\|_{st} = \mathrm{vol}(z)$.

By a general result in geometric measure theory the stable norm and the comass are dual to each other, i.e.,

$$\|z\|_{st} = \sup\{|\phi(z)| \mid \phi \in H^p(B, A, \mathbb{R}), \|\phi\|_\infty \leq 1\} \tag{109}$$

(see [Fed75, 4.10], and also [Gro99, 4.35], and [AuB06] for a sketch of the proof in the case without boundary).

### 6.1. Estimate of the $L^2$-norm on $H^*(Y)$.

Now we will apply this result to our problem in the case where $B = Y$ and $A = \emptyset$. We equip $H^p(Y)$ with the norm induced from $\mathcal{H}^p(Y)$ by the de Rham isomorphism. Suppose that $\omega$ is a $p$-form on $Y$. Then using (82) one obtains

$$\|\omega\|_2^2 \leq C(n-1, p) \mathrm{Vol}(Y) \|\omega\|_\infty^2, \tag{110}$$

Now let $\phi$ be a harmonic $p$-form representing an element $[\phi] \in H^p(Y, \mathbb{R})$. Using (110), we get

$$\begin{aligned} \|\phi\|_2 &= \sup\left\{\int \phi \wedge \omega \,|\, \omega \in \Lambda^{n-p-1}(Y), d\omega = 0, \|\omega\|_2 \le 1\right\} \\ &\ge C(n-1,p)^{-1/2}\mathrm{Vol}(Y)^{-1/2} \sup\left\{\int \phi \wedge \omega \,|\, \omega \in \Lambda^{n-p-1}(Y), d\omega = 0, \|\omega\|_\infty \le 1\right\} \\ &= C(n-1,p)^{-1/2}\mathrm{Vol}(Y)^{-1/2} \sup\left\{\langle [\phi] \cup \alpha, [Y]\rangle \mid \alpha \in H^{n-p-1}(Y,\mathbb{R}), \|\alpha\|_\infty \le 1\right\} \end{aligned}$$

Since the stable norm is dual to the comass norm, we finally get

$$\|\phi\|_2 \ge C(n-1,p)^{-1/2}\mathrm{Vol}(Y)^{-1/2}\|[Y] \cap \phi\|_{st}, \tag{111}$$

for any $\phi \in \mathcal{H}^p(Y, \mathbb{R})$.

On the other hand the inequality (110) may also be used directly. Since $\|\phi\|_2$ is the infimum of the $L^2$-norms of all representatives of the cohomology class $[\phi]$, we have

**Proposition 6.1.** *The Hilbert space norm on $H^p(Y, \mathbb{R})$, induced from the harmonic forms, satisfies*

$$C(n-1,p)^{-1/2}\mathrm{Vol}(Y)^{-1/2}\|[Y] \cap \phi\|_{st} \le \|\phi\|_2 \le C(n-1,p)^{1/2}\mathrm{Vol}(Y)^{1/2}\|\phi\|_\infty. \tag{112}$$

6.2. **Estimate of the norms on $\mathcal{H}_{ext}(X)$.** We can use Lemmas 5.1 and 5.2 in the same way as above to get similar estimates of the norms of elements in $\mathcal{H}^p_{ext,rel}(X)$. First note that $*$ induces an isomorphism

$$*\colon \mathcal{H}^p_{ext,rel}(X) \to \mathcal{H}^{n-p}_{ext,abs}(X).$$

Let $(\cdot,\cdot)_{ext}$ be the pairing (71). Then we have

$$\langle \phi, \psi \rangle = (\phi, *\psi)_{ext}, \quad \phi, \psi \in \mathcal{H}^p_{ext,rel}(X).$$

Let $F \in \mathcal{H}^p_{ext,rel}(X)$. Then we get

$$\begin{aligned} \|F\| &= \sup\{\langle F, \omega\rangle \mid \omega \in \mathcal{H}^p_{ext,rel}(X), \|\omega\| \le 1\} \\ &= \sup\left\{(F,\psi)_{ext} \mid \psi \in \mathcal{H}^{n-p}_{ext,abs}(X),\ \|\psi\| \le 1\right\}. \end{aligned} \tag{113}$$

Choose a representative $\phi$ of the cohomology class $R_c(F) \in H^p_c(X)$ with $\mathrm{supp}(\phi) \subset M$. Then $\eta_{rel}(F) = [\phi]$. Since the diagramm (72) commutes, we get

$$\langle \phi, \psi \rangle = (R_c(\phi), R(*\psi)) = ([\phi], [*\psi|_M]), \quad \phi, \psi \in H^p_{ext,rel}(X).$$

Using Lemma 5.1, it follows from (113) that

$$\begin{aligned} \|F\|_2 &= \sup\{([\phi], [\psi|_M]) \mid \psi \in \mathcal{H}^{n-p}_{ext,abs}(X),\ \|\psi\| \le 1\} \\ &\ge C(n,p)^{-1/2}\mathrm{Vol}_*(M)^{-1/2} \sup\{\langle [\phi] \cup \alpha, [M]\rangle \mid \alpha \in H^{n-p}(M,\mathbb{R}), \|\alpha\|_\infty \le 1\}. \end{aligned}$$

Since the stable norm is dual to the comass norm, we finally get

$$\|F\| \ge C(n,p)^{-1/2}\mathrm{Vol}_*(M)^{-1/2}\|[M] \cap [\phi]\|_{st}, \tag{114}$$

Note that the Poincare-Lefschetz dual $[M]\cap[\phi]$ of $[\phi]$ is in $H_p(M,\mathbb{R})$ and its stable norm equals its stable norm as an element of $H_p(M,\mathbb{R})$. Lemma 5.2 gives an upper bound for $\|F\|$ and we get

**Proposition 6.2.** *Let $F \in \mathcal{H}^p_{ext,rel}(X)$ and $\phi = \eta_{rel}(F) \in H^p(M,Y,\mathbb{R})$. Then we have*

$$C(n,p)^{-1/2}\mathrm{Vol}_*(M)^{-1/2}\|[M]\cap\phi\|_{st} \le \|F\| \le C(n,p)^{1/2}\mathrm{Vol}(M)^{1/2}\|\phi\|_\infty. \tag{115}$$

Using Theorem 4.7, we obtain Theorem 1.1.

If we denote by $\iota$ the inclusion map $Y \to M$ and by $\iota_* : H_p(Y,\mathbb{R}) \to H_p(M,\mathbb{R})$ the induced map in homology, then

$$[M]\cap\partial\phi = \iota_*\left([Y]\cap\phi\right). \tag{116}$$

That is $[M]\cap\partial\phi$ coincides with the image of the Poincare dual of the class $\phi$ in $H_{n-p-1}(M,\mathbb{R})$. The stable norm of $\partial\phi$ can be calculated in many cases explicitly in terms of geometric data using the fact that the stable mass is dual to the comass norm. In order to make statements about the spectrum of the map $T(0)$ one can combine these estimates with the estimates of the $L^2$-norms of the cohomology classes on $Y$ (see Proposition 6.1).

## 7. Examples

7.1. **The scattering length for functions.** Note that the extended harmonic functions are exactly the constant functions. Therefore, $\mathcal{H}^0(Y)\cap\ker\partial$ is spanned by the function equal to 1 on $Y$. Thus, the $+1$ eigenspace to $S_0(0)$ is spanned by 1. Moreover, since $S$ anticommutes with the Hodge $*$ operator we have $S_{n-1}(0)*1 = -*1$. By Stokes formula

$$(\partial[*1])[M] = [*1][Y] \tag{117}$$

and therefore

$$\partial[*1] = \frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(M)}[*_M 1] \tag{118}$$

and by the above equation we immediately obtain

$$T_{n-1}(0)|_{(\ker\partial)^\perp} = 2\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(Y)}. \tag{119}$$

This in turn implies that

$$T_0(0)|_{\ker\partial} = 2\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(Y)}. \tag{120}$$

7.2. $Y$ **has only one connected component.** In this case $\mathcal{H}^0(Y)$ consists of the constant functions only. By the above we have

$$T_{n-1}(0) = 2\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(Y)}. \tag{121}$$

This in turn implies that

$$T_0(0) = 2\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(Y)}. \tag{122}$$

Note that if $Y = S^{n-1}$ the only non-vanishing cohomology groups are $H^0(Y)$ and $H^{n-1}(Y)$. Thus, in this case the above formulas determine $T(0)$ completely.

7.3. $Y = Y_1 \cup Y_2$ **has two boundary components.** Now we have $\mathcal{H}^0(Y) \cong \mathbb{R}^2$ and therefore on $\mathcal{H}^0(Y)$ is a direct sum of the one dimensional spaces $\ker \partial$ and $\ker \partial^\perp$. Under the splitting $\mathcal{H}^0(Y) = \ker \partial \oplus \ker \partial^\perp$ the operator $T_0(0)$ is of the form

$$T_0(0) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \tag{123}$$

where we have already seen, that

$$t_1 = 2\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(Y)}. \tag{124}$$

Our formula now allows us to give an estimate of $t_2$ in purely geometric terms. The harmonic functions in $\ker \partial$ are multiples of the function 1 on $Y$. The complement $\ker \partial^\perp$ has dimension 1 and is spanned by the function

$$\chi(x) = \begin{cases} \mathrm{Vol}(Y_2) \text{ for } x \in Y_1 \\ -\mathrm{Vol}(Y_1) \text{ for } x \in Y_2 \end{cases} \tag{125}$$

Clearly,

$$\|\chi\|_2^2 = \mathrm{Vol}(Y_2)^2\mathrm{Vol}(Y_1) + \mathrm{Vol}(Y_1)^2\mathrm{Vol}(Y_2). \tag{126}$$

It remains to estimate the $L^2$-norm of the class $\partial\chi$ in $H^1(M, Y, \mathbb{R})$. The comass norm of $\partial\chi$ may be calculated using the duality between the stable norm and the comass norm. Let $c$ be a Lipschitz continuous chain whose boundary is homologuous to $y_2 - y_1$, where $y_1 \in Y_1$ and $y_2 \in Y_2$. Then $c$ defines a relative cycle $[c]$ in $H_1(M, Y, \mathbb{R})$ and we have

$$\partial\chi([c]) = \int_c \partial\chi = \mathrm{Vol}(Y_1) + \mathrm{Vol}(Y_2) = \mathrm{Vol}(Y). \tag{127}$$

We observe that the cycle $c$ can be written as a linear combination of curves which are either closed or whose end points are in the boundary. This implies

$$\|[c]\|_{st} \geq \mathrm{dist}(Y_1, Y_2), \tag{128}$$

with equality if for example $[c]$ is in the same homology class as a shortest curve connecting $Y_1$ with $Y_2$.

Duality implies

$$\|\partial\chi\|_\infty = \mathrm{dist}(Y_1, Y_2)^{-1}\mathrm{Vol}(Y), \tag{129}$$

and therefore

$$\|\partial\chi\|_2 \leq \mathrm{Vol}(M)^{1/2}\mathrm{dist}(Y_1, Y_2)^{-1}\mathrm{Vol}(Y). \tag{130}$$

To get the estimate from above we need to look at $\|M \cap \partial\chi\|_{st}$ that is the stable norm of the Poincare dual of the class $\partial\chi$. By Equation (116) we have $M \cap \partial\chi = \iota_*([Y] \cap \chi)$. Of course,

$$\iota_*([Y] \cap \chi) = \iota_*(\mathrm{Vol}(Y_2)[Y_1] - \mathrm{Vol}(Y_1)[Y_2]) = \mathrm{Vol}(Y)\iota_*([Y_1]), \tag{131}$$

where we have used that $\iota_*([Y_1]) + \iota_*([Y_2]) = 0$. Therefore,

$$\|M \cap \partial\chi\|_{st} = \mathrm{Vol}(Y)\|\iota_*([Y_1])\|_{st}. \tag{132}$$

Combining these two estimates we obtain

$$C_2 \leq t_2 \leq C_1 \tag{133}$$

with

$$C_1 = 2\mathrm{Vol}_*(M)\frac{\mathrm{Vol}(Y_1)\mathrm{Vol}(Y_2)}{\|\iota_*([Y_1])\|^2_{st}(\mathrm{Vol}(Y_1) + \mathrm{Vol}(Y_2))}, \tag{134}$$

$$C_2 = 2\mathrm{Vol}(M)^{-1}\frac{\mathrm{dist}(Y_1, Y_2)^2\mathrm{Vol}(Y_1)\mathrm{Vol}(Y_2)}{\mathrm{Vol}(Y_1) + \mathrm{Vol}(Y_2)}. \tag{135}$$

Note that with respect to this basis, the scattering matrix at zero has the form

$$S_0(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{136}$$

In order to interpret this in terms of reflection and transmission coefficients we choose another basis $(\chi_1, \chi_2)$ of $\mathcal{H}^0(Y)$, where $\chi_i$ is constant equal to 1 on $Y_i$ and equal to zero on the other boundary component. In this basis we get

$$S_0(0) = \frac{1}{\mathrm{Vol}(Y_1) + \mathrm{Vol}(Y_2)}\begin{pmatrix} \mathrm{Vol}(Y_1) - \mathrm{Vol}(Y_2) & 2\mathrm{Vol}(Y_2) \\ 2\mathrm{Vol}(Y_1) & \mathrm{Vol}(Y_2) - \mathrm{Vol}(Y_1) \end{pmatrix}. \tag{137}$$

The reflection coefficient $r_{11}$ and the transmission coefficient $r_{12}$ of a wave of low energy that comes in at the end $Y_1$ and is scattered in $M$ therefore are $r_{11} = \frac{\mathrm{Vol}(Y_1)-\mathrm{Vol}(Y_2)}{\mathrm{Vol}(Y_1)+\mathrm{Vol}(Y_2)}$ and $r_{12} = \frac{2\mathrm{Vol}(Y_1)}{\mathrm{Vol}(Y_1)+\mathrm{Vol}(Y_2)}$. Transforming $T_0(0)$ into this basis gives

$$T_0(0) = \frac{1}{\mathrm{Vol}(Y_1) + \mathrm{Vol}(Y_2)}\begin{pmatrix} t_1\mathrm{Vol}(Y_1) + t_2\mathrm{Vol}(Y_2) & (t_1 - t_2)\mathrm{Vol}(Y_2) \\ (t_1 - t_2)\mathrm{Vol}(Y_1) & t_1\mathrm{Vol}(Y_2) + t_2\mathrm{Vol}(Y_1) \end{pmatrix}. \tag{138}$$

As a remark we would like to add here that in physics the scattering matrix for one dimensional scattering problems has the transmission coefficients on the diagonal and the reflection coefficients off the diagonal. Our notation differs here as we consider the operator with Neumann boundary conditions at each end as the unperturbed operator.

7.4. **The full-torus.** Let $M$ be the full torus $D \times S^1$ with boundary $Y = T^2 = S^1 \times S^1$. We view both $D$ and $S^1$ as subsets of the complex planes and use coordinates $z = re^{\mathrm{i}y}$ on $D$ and $z = e^{\mathrm{i}x}$ on $S^1$. We assume that we are given a metric on $M$ which has product structure in a small neighborhood of the boundary and that the metric on the boundary is equal to the product metric

$$\ell_1^2 dx^2 + \ell_2^2 dy^2, \tag{139}$$

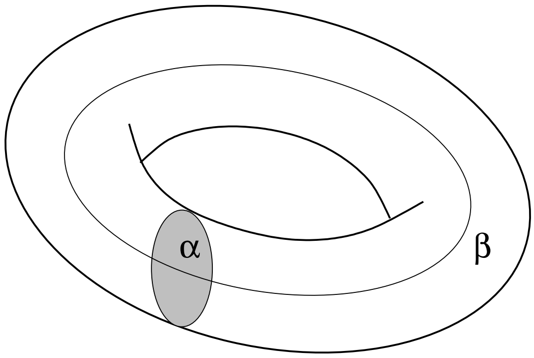


FIGURE 2. The cycles $\alpha \in H_2(M,Y)$ and $\beta \in H_1(M)$ on $M$.

with positive real numbers $\ell_1, \ell_2$. Then, the volume is $\mathrm{Vol}(Y) = 4\pi^2\ell_1\ell_2$. Moreover, $H^1(M,\mathbb{R}) \cong \mathbb{R}$ and it is gerated by the class of the one form $dx$. The group $H^1(Y,\mathbb{R})$ is isomorphic to $\mathbb{R}^2$ and is generated by the classes of the two harmonic 1-forms $dx$ and $dy$. The $L^2$-norms of these forms is easily calculated.

$$||dx||_{L^2}^2 = \int_Y dx \wedge *dx = 4\pi^2\frac{\ell_2}{\ell_1}, \tag{140}$$

$$||dy||_{L^2}^2 = \int_Y dy \wedge *dy = 4\pi^2\frac{\ell_1}{\ell_2}. \tag{141}$$

and, moreover, $dx$ and $dy$ are orthogonal to each other. The restriction of the form $dx$ on $M$ to $Y$ is the form $dx$ regarded as a form on $Y$. Therefore, the kernel of the connecting homomorphism $\partial$ is spanned by $[dx]$. Hence, $(\ker\partial)^\perp$ is spanned by $[dy]$. Since the Hodge star operator commutes with $T_2(0)$ the map $T_2(0)$ has the form

$$T_2(0) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \tag{142}$$

with respect to the decomposition $H^1(T^2,\mathbb{R}) = \ker\partial \oplus (\ker\partial)^\perp$. In order to get an estimate for $t$ we need to calculate $\|\partial[dy]\|_\infty$ and $\|[M] \cap \partial[dy]|_{st}$. The stable norm of $\|\partial[dy]\|_\infty$ can again be calculated by duality. For this note that $H_2(M,Y,\mathbb{R}) \cong \mathbb{R}$ by Alexander duality and it is generated by the relative cycle $\alpha = D \times \{1\} \subset M$. Now, of course

$$\langle \partial[dy], [\alpha] \rangle = \int_{\partial D \times \{1\}} dy = 2\pi. \tag{143}$$

Duality of comass norm and stable norm now implies that

$$\|\partial[dy]\|_\infty = 2\pi \|[D \times \{1\}]\|_{st}^{-1}. \tag{144}$$

The class $[M] \cap \partial[dy]$ is given by $-2\pi\iota_*(\{1\} \times S^1)$. So $\frac{1}{2\pi}\|[M] \cap \partial[dy]\|_{st}$ is equal to the infimum of the lengths of all representatives of the cycle $\beta = \{1\} \times S^1$ in $H_1(M,\mathbb{R})$. The geometric picture is described in Fig. 2. Theorem 1.1 now gives an estimate of $t$:

$$2\frac{\ell_1}{\ell_2}\frac{\|\alpha\|_{st}^2}{\mathrm{Vol}(M)} \leq t \leq 2\frac{\ell_1}{\ell_2}\frac{\mathrm{Vol}_*(M)}{\|\beta\|_{st}^2}. \tag{145}$$

Since $\mu_1 = \min\{\ell_1^{-1}, \ell_2^{-1}\}$ we have $\mathrm{Vol}_*(M) = \mathrm{Vol}(M) + 4\pi^2\ell_1\ell_2 \max\{\ell_1, \ell_2\}$.

## Appendix A. Dynamical approach and spectral decomposition

In this appendix we discuss the relation between the stationary and the dynamical approach to scattering theory and we establish the Eisenbud-Wigner formula for manifolds with cylindrical ends. For details concerning scattering theory we refer to [Ya92]. For original papers on the Eisenbud-Wigner formula see [Eis48] and [Wi55]. In order to simplify the relation of the scattering length to the time-delay operator we consider scattering theory for the square root of the Laplace Beltrami operator, i.e. we consider scattering theory and the time-delay in relativistic quantum mechanics.

Let $\overline{\Delta_p}$ be the closure in $L^2$ of the operator $\Delta_p$ with domain $\Lambda^p_c(X)$. Denote by $\Delta_{p,0}$ the self-adjoint extension of $\Delta_p$ with respect to Neumann boundary conditions along $Y$. Note that $\Delta_{p,0}$ is the self-adjoint operator associated to the quadratic form

$$\phi \mapsto \int_M q_x(\phi)dx + \int_Z q_x(\phi),$$
$$q_x(\phi) = d\phi(x) \wedge \overline{*d\phi(x)} + \delta\phi(x) \wedge \overline{*\delta\phi(x)}$$

on $H^1(M, \Lambda^pT^*M) \oplus H^1(Z, \Lambda^pT^*Z) \subset L^2\Lambda^p(X)$. With respect to the decomposition

$$L^2\Lambda^p(X) = L^2\Lambda^p(M) \oplus L^2\Lambda^p(Z)$$

we have

$$\Delta_{p,0} = \Delta_{p,M} \oplus \Delta_{p,Z},$$

where $\Delta_{p,M}$ and $\Delta_{p,Z}$ are the corresponding self-adjoint extensions of $\Delta_p|_M$ and $\Delta_p|_Z$, respectively, with Neumann boundary conditions imposed. Let $H$ be the square root of $\overline{\Delta_p}$ defined by spectral calculus and similarly define $H_0 = (\Delta_{p,0})^{1/2}$, $H_M = \Delta_{p,M}^{1/2}$, and $H_Z = \Delta_{p,Z}^{1/2}$. Since $M$ is compact, $H_M$ has purely discrete spectrum. The spectral resolution of $H_Z$ can be determined, using separation of variables. The spectrum is absolutely continuous. A complete set of generalized eigensections is given by

$$F_0(\phi, \lambda) = \left(e^{+\mathrm{i}\sqrt{\lambda^2-\mu_\phi^2}u} + e^{-\mathrm{i}\sqrt{\lambda^2-\mu_\phi^2}u}\right) j_p(\phi), \tag{146}$$

where $\phi$ runs through an orthonormal basis of eigensection of the operator $\Delta'_p \oplus \Delta'_{p-1}$ with eigenvalue $\mu_\phi^2$.

Denote by $P_{ac}$ (resp. $P^0_{ac}$) the orthogonal projection onto the absolutely continuous subspace of $H$ (resp. $H_0$). Denote by $H_{ac}$ and $H_{0,ac}$ the restrictions of $H$ and $H_0$, respectively, to the absolutely continuous subspaces. Furthermore, for an open interval $(a, b)$ we denote by $P_{ac}(a, b)$ (resp. $P^0_{ac}(a, b)$) the projection onto the continuous part of the spectral subspace of the interval. As explained above, the absolutely continuous subspace for $H_0$ is $L^2\Lambda^p(Z)$. Thus $P^0_{ac}$ is the orthogonal projection of $L^2\Lambda^p(X)$ onto the subspace $L^2\Lambda^p(Z)$.

By [Gui89, Théorèm 3.6] and the Birman-Kato invariance principle the wave operators

$$W_{\pm} = \mathrm{s} - \lim_{t \to \pm\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} P^0_{ac} \tag{147}$$

exist and are complete. This means that the strong limit exists and and the operators $W_{\pm}$ define isometries of $P^0_{ac} L^2\Lambda^p(X)$ onto $P_{ac} L^2\Lambda^p(X)$, intertwining $H_{0,ac}$ and $H_{ac}$. In this context the scattering operator $S$ is defined as

$$\mathcal{S} = W_+^* W_-. \tag{148}$$

This is a unitary operator in $P^0_{ac} L^2\Lambda^p(X)$ which commutes with $H_{0,ac}$. Let $\sigma_0 = \sigma_{ac}(H_0)$ be the absolutely continuous spectrum of $H_0$. It equals $[\mu, \infty)$ with $\mu \geq 0$. Let $\{E_0(\lambda)\}_{\lambda \in \sigma_0}$ be the spectral family of $H_{0,ac}$. Since $S$ commutes with $H_{0,ac}$, we have

$$S = \int_{\sigma_0} S(\lambda)\, dE_0(\lambda),$$

where $S(\lambda) = S(\lambda; H, H_0)$ acts in the finite-dimensional Hilbert space

$$\mathcal{H}(\lambda) = \bigoplus_{\mu \leq \lambda^2} \mathrm{Eig}_\mu(\Delta'_p) \oplus \mathrm{Eig}_\mu(\Delta'_{p-1}). \tag{149}$$

For the purposes of this paper we need only to investigate the structure of the continuous spectrum of $H$ in the interval $[0, \mu_1)$ in the situations when $\mu = 0$. The generalized eigensections $F(\phi, \lambda)$ constructed in Theorem 2.1 are well defined as distributions on $(0, \mu_1)$ with values in $L^2\Lambda^p(X)$. This follows easily from the properties of the Fourier transform and the expansion of $F(\phi, \lambda)$ on $Z$. Of course, in the weak topology on the space of distributions we have

$$\langle F(\phi, \lambda), F(\psi, \lambda')\rangle = \lim_{a \to \infty} \int_{M_a} F(\phi, \lambda)(x) \wedge \overline{*F(\psi, \lambda')(x)}. \tag{150}$$

The smooth function

$$\int_{M_a} F(\phi, \lambda)(x) \wedge \overline{*F(\psi, \lambda')(x)} \tag{151}$$

can be determined as the continuous extension of

$$\frac{1}{\lambda^2 - (\lambda')^2} \int_{M_a} \Delta_p F(\phi, \lambda) \wedge \overline{*F(\psi, \lambda')(x)} - F(\phi, \lambda) \wedge \overline{*\Delta_p F(\psi, \lambda')(x)}. \tag{152}$$

This integral can be simplified using Green's formula and the limit $a \to \infty$ can be explicitly evaluated (see Equ. (172)). A simple exercise in distribution theory shows that indeed we have the orthogonality relations

$$\langle F(\phi, \lambda), F(\psi, \lambda')\rangle = 2\pi \langle \phi, \psi\rangle \delta(\lambda - \lambda') \tag{153}$$

as distributions on $(0, \mu_1) \times (0, \mu_1)$. Moreover, in the same way as in [Gui89], Theorem 6.2, one shows that

$$W_+ F_0(\phi, \lambda) = F(C^*(\lambda)\phi, \lambda), \tag{154}$$

$$W_- F_0(\phi, \lambda) = F(\phi, \lambda). \tag{155}$$

Therefore, in the distributional sense for $\lambda \in (0,\mu_1)$

$$\mathcal{S}F_0(\phi,\lambda) = F_0(C_p(\lambda),\lambda) \tag{156}$$

which shows that the dynamical and the stationary scattering matrices coincide

$$\mathcal{S}(\lambda) = C_p(\lambda). \tag{157}$$

The time delay operator $\mathcal{T}$ is defined in the following way. If $\phi \in P_{ac}L^2\Lambda^p(X)$ then, according to the laws of quantum mechanics, the probability of finding the particle with wave function $\phi$ in $M_a$ at time $t$ is given by

$$\int_{M_a} \|e^{-\mathrm{i}Ht}\phi\|_x^2 dx = \|\chi_{M_a} e^{-\mathrm{i}Ht}\phi\|^2. \tag{158}$$

The total time spent in $M_a$ is then given by

$$\int_{-\infty}^{\infty} \|\chi_{M_a} e^{-\mathrm{i}Ht}\phi\|^2 dt. \tag{159}$$

This expression is not necessarily finite for all $\phi$. Now, according to scattering theory, for $\phi \in P^0_{ac}L^2\Lambda^p(X)$ the states $e^{-\mathrm{i}tH}W_-\phi$ and $e^{-\mathrm{i}H_0 t}\phi$ are asymptotically the same for $t \to -\infty$. Thus, the time excess due to the interaction (the presence of M) is

$$\int_{-\infty}^{\infty} \left(\|\chi_{M_a} e^{-\mathrm{i}Ht}W_-\phi\|^2 - \|\chi_{M_a} e^{-\mathrm{i}H_0 t}\phi\|^2\right) dt. \tag{160}$$

The Eisenbud-Wigner time-delay operator $\mathcal{T}$ is formally defined by

$$\langle \phi, \mathcal{T}\phi\rangle = \lim_{a\to\infty} \int_{-\infty}^{\infty} \left(\|\chi_{M_a} e^{-\mathrm{i}Ht}W_- P^0_{ac}\phi\|^2 - \|\chi_{M_a} e^{-\mathrm{i}H_0 t}P^0_{ac}\phi\|^2\right) dt. \tag{161}$$

In many situations in potential scattering it can be shown that the above defines a closable quadratic form and $\mathcal{T}$ is a self-adjoint operator that commutes with $H_0$ and the Eisenbud-Wigner formula

$$\mathcal{T} = \int_{\sigma_{ac}(H_0)} \mathcal{T}(\lambda)\; dE_0(\lambda), \tag{162}$$

$$\mathcal{T}(\lambda) = -\mathrm{i}\mathcal{S}(\lambda)^{-1}\frac{d\mathcal{S}}{d\lambda}(\lambda) \tag{163}$$

holds. Since we are only interested in the spectrum near 0, we prove this formula only for elements in $P^0_{ac}(0,\mu_1)$.

**Proposition A.1.** *Suppose that $g \in C_0^\infty(0,\mu_1)$, $\phi \in \ker\Delta'_p \oplus \Delta'_{p-1}$. Let $F_0(\phi,g) := \int_{\mathbb{R}} F_0(\phi,\lambda)g(\lambda)d\lambda$. Then,*

$$\int_{-\infty}^{\infty} \|\chi_{M_a} e^{-\mathrm{i}Ht}W_- F_0(\phi,g)\|^2 dt < \infty,$$
$$\int_{-\infty}^{\infty} \|\chi_{M_a} e^{-\mathrm{i}H_0 t}F_0(\phi,g)\|^2 dt < \infty.$$

*Moreover,*

$$\langle F_0(\phi, g), \mathcal{T} F_0(\phi, g)\rangle = 2\pi \int_{\sigma_0} \mathcal{T}(\lambda) g(\lambda)^2 \, dE_0(\lambda), \tag{164}$$

*where*

$$\mathcal{T}(\lambda) = -\mathrm{i}\mathcal{S}(\lambda)^{-1}\mathcal{S}'(\lambda). \tag{165}$$

*Hence, $\mathcal{T}$ is a self-adjoint operator on $P^0_{ac}(0, \mu_1)L^2\Lambda^p(X)$ which has the form*

$$\mathcal{T} = \int_{\sigma_0} \mathcal{T}(\lambda) \, dE_0(\lambda) \tag{166}$$

*with respect to the spectral family $\{E_0(\lambda)\}_{\lambda\in\sigma_0}$ of $H_{0,ac}$.*

*Proof.* We can do this calculation in the explicit spectral decomposition.

$$\begin{aligned} &\int_{-\infty}^{\infty} \|\chi_{M_a} e^{-\mathrm{i}Ht} F_0(\phi, g)\|^2 d\lambda = \\ &= \int_{-\infty}^{\infty} \int \int \int_{M_a} e^{-\mathrm{i}(\lambda-\lambda')t} \langle F_0(\phi, \lambda), F_0(\phi, \lambda')\rangle_x g(\lambda) g(\lambda') dx d\lambda d\lambda' dt = \\ &= 2\pi \int \langle F_0(\phi, \lambda), \chi_{M_a} F_0(\phi, \lambda)\rangle g(\lambda)^2 d\lambda. \end{aligned} \tag{167}$$

and similarly,

$$\begin{aligned} &\int_{-\infty}^{\infty} \|\chi_{M_a} e^{-\mathrm{i}Ht} F(\phi, g)\|^2 d\lambda = \\ &= 2\pi \int \langle\langle F(\phi, \lambda), \chi_{M_a} F(\phi, \lambda)\rangle g(\lambda)^2 d\lambda. \end{aligned} \tag{168}$$

In the limit $a \to \infty$ both integrals can be computed up to exponentially small errors since we have the expansions on $Z$:

$$F(\phi, \lambda)|_Z = e^{-\mathrm{i}\lambda u} j_p(\phi) + e^{+\mathrm{i}\lambda u} j_p(S(\lambda)\phi) + R, \tag{169}$$

$$F_0(\phi, \lambda)|_Z = e^{-\mathrm{i}\lambda u} j_p(\phi) + e^{+\mathrm{i}\lambda u} j_p(\phi). \tag{170}$$

Therefore, integration by parts yields

$$\begin{aligned} &\langle F_0(\phi, \lambda), \chi_{M_a} F_0(\phi', \lambda')\rangle = \\ &= \frac{1}{\lambda^2 - (\lambda')^2} \int_{M_a} \langle \Delta_p F_0(\phi, \lambda), F_0(\phi', \lambda')\rangle_x - \langle F_0(\phi, \lambda), \Delta_p F_0(\phi', \lambda')\rangle_x = \\ &= \frac{2}{\lambda - \lambda'} \sin\left((\lambda - \lambda')a\right) \langle \phi, \phi'\rangle + \frac{2}{\lambda + \lambda'} \sin\left((\lambda + \lambda')a\right) \langle \phi, \phi'\rangle. \end{aligned} \tag{171}$$

Similarly, we have

$$\langle F(\phi,\lambda), \chi_{M_a} F(\phi',\lambda')\rangle =$$
$$= \frac{1}{\lambda^2-(\lambda')^2}\int_{M_a} \langle \Delta_p F(\phi,\lambda), F(\phi',\lambda')\rangle_x - \langle F(\phi,\lambda), \Delta_p F(\phi',\lambda')\rangle_x =$$
$$= \frac{2}{\lambda-\lambda'}\sin\left((\lambda-\lambda')a\right)\langle\phi,\phi'\rangle + \left(\frac{\mathrm{i}}{\lambda-\lambda'}e^{\mathrm{i}(\lambda-\lambda')a}\right)\langle\phi,(\mathbb{1}-\mathcal{S}^*(\lambda)\mathcal{S}(\lambda'))\phi'\rangle - \tag{172}$$
$$-\frac{\mathrm{i}}{\lambda+\lambda'}\left(e^{\mathrm{i}(\lambda+\lambda')a}\langle\phi,\mathcal{S}^*(\lambda)\phi'\rangle - e^{-\mathrm{i}(\lambda+\lambda')a}\langle\phi,\mathcal{S}(\lambda')\phi'\rangle\right) + O(e^{-\mu_1 a}).$$

Therefore,

$$\langle F(\phi,\lambda), \chi_{M_a} F(\phi',\lambda')\rangle - \langle F_0(\phi,\lambda), \chi_{M_a} F_0(\phi',\lambda')\rangle =$$
$$= \left(\frac{\mathrm{i}}{\lambda-\lambda'}e^{\mathrm{i}(\lambda-\lambda')a}\right)\langle\phi,(\mathbb{1}-\mathcal{S}^*(\lambda)\mathcal{S}(\lambda'))\phi'\rangle - \frac{2}{\lambda+\lambda'}\sin\left((\lambda+\lambda')a\right)\langle\phi,\phi'\rangle -$$
$$-\frac{\mathrm{i}}{\lambda+\lambda'}\left(e^{\mathrm{i}(\lambda+\lambda')a}\langle\phi,\mathcal{S}^*(\lambda)\phi'\rangle - e^{-\mathrm{i}(\lambda+\lambda')a}\langle\phi,\mathcal{S}(\lambda')\phi'\rangle\right) + O(e^{-\mu_1 a}),$$

and one obtains

$$\lim_{a\to\infty}\lim_{\lambda'\to\lambda}\langle F(\phi,\lambda), \chi_{M_a} F(\phi',\lambda)\rangle - \langle F_0(\phi,\lambda), \chi_{M_a} F_0(\phi',\lambda)\rangle = \langle\phi, \mathcal{T}(\lambda)\phi'\rangle, \tag{173}$$

where the second limit is in the distributional sense and

$$\mathcal{T}(\lambda) = -\mathrm{i}\mathcal{S}(\lambda)^{-1}\mathcal{S}'(\lambda). \tag{174}$$

□

By the properties of the scattering matrix $\mathcal{T}(\lambda)$ commutes with the Hodge star operator and leaves the summands in $\ker\Delta_p' \oplus \Delta_{p-1}'$ invariant (see (32)). We therefore have

$$\mathcal{T}_p(\lambda) = \begin{pmatrix} T_p(\lambda) & 0 \\ 0 & T_{p-1}(\lambda)\end{pmatrix}, \tag{175}$$

where $T_p(\lambda)$ is the time delay operator for coclosed forms defined by

$$T_p(\lambda) = -\mathrm{i}S_p(\lambda)^{-1}S_{p-1}'(\lambda). \tag{176}$$

This operator describes the time-delay of coclosed forms of energy $\lambda < \mu_1$ scattered in $M$. In the physics literature the time delay of the $\ell = 0$ partial wave for potential scattering in $\mathbb{R}^3$ is called the scattering length (e.g. [RS79]). As the $\ell = 0$ partial wave corresponds to the constant function on the sphere at infinity we call $T_p(0)$ in analogy with this the scattering length. Note however, that in the physics literature the time-delay is usually considered for non-relativistic Schrödinger mechanics so that it differs from the relativistic time-delay by an energy dependent factor. A simple relation that equates time-delay and scattering length can for dimensional reasons only hold in relativistic theories.

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