THE ASYMPTOTICS OF THE RAY-SINGER ANALYTIC TORSION FOR COMPACT HYPERBOLIC MANIFOLDS

WERNER MÜLLER AND JONATHAN PFAFF

ABSTRACT. In this paper we study the asymptotic behavior of the analytic torsion for compact oriented hyperbolic manifolds with respect to certain rays of representations obtained by restriction of irreducible representations of the group of isometries of the hyperbolic space.

1. INTRODUCTION

Let $G = \mathrm{Spin}(d,1)$ and $K = \mathrm{Spin}(d)$. Then K is a maximal compact subgroup of G and $\widetilde{X} := G/K$ can be identified with the hyperbolic space of dimension d. Let $\Gamma \subset G$ be a discrete, torsion free co-compact subgroup. Then $X = \Gamma\backslash\widetilde{X}$ is a compact oriented hyperbolic manifold of dimension d and every such manifold is of this form. Let ρ be a finite-dimensional representation of Γ on a complex vector space V_ρ. Let $E_\rho \to X$ be the associated flat vector bundle. Pick a Hermitian fibre metric h in E_ρ. Let $\Delta_p(\rho)$ denote the Laplacian on E_ρ-valued p-forms on X. Let $\zeta_p(s;\rho)$ be the zeta function of $\Delta_p(\rho)$ (see [Shu]). It is a meromorphic function of $s \in \mathbb{C}$, which is holomorphic at $s = 0$. Then the Ray-Singer analytic torsion $T_X(\rho;h) \in \mathbb{R}^+$ is defined by

$$(1.1) \qquad \log T_X(\rho;h) := \frac{1}{2}\sum_{p=1}^{d}(-1)^p p \frac{d}{ds}\zeta_p(s;\rho)\big|_{s=0}$$

(see [RS], [Mu1]). In general, $T_X(\rho;h)$ depends on h. If ρ is unitary and $\dim X$ is even, then $T_X(\rho) = 1$ (see [RS, Theorem 2.3]). Furthermore, if $H^*(X, E_\rho) = 0$ and $\dim X$ is odd, then by [Mu1, Corollary 2.7], $T_X(\rho;h)$ is independent of h.

In this paper we consider the special class of representations of Γ which are obtained as restriction to Γ of finite-dimensional irreducible representations of G. Let τ be a finite-dimensional irreducible representation of G and let E_τ denote the flat bundle over X associated to $\tau|_\Gamma$. By [MM, Lemma 3.1], E_τ can be equipped with a distinguished metric, called admissible, which is unique up to scaling. We choose an admissible fibre metric on E_τ and denote the analytic torsion with respect to this metric and the hyperbolic metric by $T_X(\tau)$. We also consider the L^2-torsion $T_X^{(2)}(\tau)$ which is defined as in [Lo].

Assume that $\dim X = 2n + 1$. Then we study the asymptotic behavior of the analytic torsion for special sequences of representations of G. These representations are defined

as follows. Fix natural numbers $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_{n+1}$. For $m \in \mathbb{N}$ let $\tau(m)$ be the finite-dimensional irreducible representation of G with highest weight

$$\Lambda_{\tau(m)} = (\tau_1 + m)e_1 + \cdots + (\tau_{n+1} + m)e_{n+1}$$

as in (2.6). Let θ be the standard Cartan involution of G. For a given representation τ of G, let $\tau_\theta := \tau \circ \theta$. Then the representation $\tau(m)$ satisfies $\tau(m) \neq \tau(m)_\theta$. By the vanishing theorem of Borel/Wallach [BW, Theorem 6.7] it follows that $H^*(X, E_{\tau(m)}) = 0$. Thus by [Mu1, Corollary 2.7] the Ray-Singer analytic torsion is independent of the choice of a Hermitian fibre metric in $E_{\tau(m)}$ and the Riemannian metric on X.

The purpose of this paper is to study the asymptotic behavior of $T_X(\tau(m))$ as $m \to \infty$. Let $\tau := (\tau_1, \ldots, \tau_{n+1})$ and let $\mathrm{vol}(X)$ denote the hyperbolic volume of X. Then our main result is the following theorem.

Theorem 1.1. *Let* $\dim X = 2n + 1$. *There exist* $c > 0$ *and a polynomial* $P_\tau(m)$ *of degree* $n(n+1)/2 + 1$ *whose coefficients depend only on* n *and* τ, *such that*

$$\log T_X(\tau(m)) = \mathrm{vol}(X)P_\tau(m) + O(e^{-cm})$$

as $m \to \infty$.

To prove this theorem we first show that the asymptotic behavior of the analytic torsion is determined by the asymptotic behavior of the L^2-torsion. More precisely we have

Proposition 1.2. *Let* $\dim X = 2n + 1$. *There exists* $c > 0$ *such that*

$$\log T_X(\tau(m)) = \log T_X^{(2)}(\tau(m)) + O(e^{-cm})$$

as $m \to \infty$.

The L^2-torsion is obtained from the contribution of the identity to the Selberg trace formula. It can be computed using the Plancherel formula. We have

Proposition 1.3. *Let* $\dim X = 2n + 1$. *There exists a polynomial* $P_\tau(m)$ *of degree* $n(n+1)/2 + 1$ *whose coefficients depend only on* n *and* τ, *such that*

$$\log T_X^{(2)}(\tau(m)) = \mathrm{vol}(X)P_\tau(m).$$

Combining Propositions 1.2 and 1.3, we obtain Theorem 1.1.

We note that for a unitary representation ρ of Γ one has $T_X^{(2)}(\rho) = \dim(\rho) \cdot T_X^{(2)}$, where $T_X^{(2)}$ is the L^2-torsion with respect to the trivial representation, which equals $C(n) \cdot \mathrm{vol}(X)$. This is not true for the representations τ which arise by restriction of representations of G. Indeed by Weyl's dimension formula (see (2.10)) there exists a constant $C > 0$ such that

$$(1.2) \qquad \dim(\tau(m)) = Cm^{\frac{n(n+1)}{2}} + O(m^{\frac{n(n+1)}{2}-1}), \quad m \to \infty.$$

But the polynomial $P_\tau(m)$ has degree $n(n+1)/2 + 1$.

The coefficient of the leading term of $P_\tau(m)$ can be determined explicitly. Combined with (1.2) we obtain

Corollary 1.4. *Let* $\dim X = 2n+1$. *There exists a constant* $C = C(n) > 0$ *which depends only on* n, *such that we have*

$$-\log T_X(\tau(m)) = C(n) \operatorname{vol}(X) m \cdot \dim(\tau(m)) + O(m^{\frac{n(n+1)}{2}})$$

as $m \longrightarrow \infty$.

The constant $C(n)$ can be computed explicitly from the Plancherel polynomials. The case $n = 1$ was established in [Mu2]. Corollary 1.4 is the extension of Theorem 1.1 of [Mu2] to higher dimensions and to rays of highest weights which exhaust the space of weights in the positive Weyl chamber.

Using the equality of analytic and Reidemeister torsion [Mu1], we get corresponding statements for the Reidemeister torsion. Especially we have

Corollary 1.5. *Let* $\dim X = 2n + 1$. *Let* $\tau_X(\tau(m))$ *be the Reidemeister torsion of* $(X, \tau(m))$. *Then we have*

$$-\log \tau_X(\tau(m)) = C(n) \operatorname{vol}(X) m \cdot \dim(\tau(m)) + O(m^{\frac{n(n+1)}{2}})$$

as $m \longrightarrow \infty$.

It follows from Corollary 1.5 and (1.2) that there is a constant $C_1(n) < 0$ such that

$$(1.3) \qquad \lim_{m \to \infty} \frac{\log \tau_X(\tau(m))}{m^{n(n+1)/2+1}} = C_1(n) \operatorname{vol}(X).$$

Now recall that the Reidemeister torsion of an acyclic representation is defined combinatorially in terms of a smooth triangulation of X. Thus the volume appears as the limit of a sequence of numbers which are defined pure combinatorially.

Again, for hyperbolic 3-manifolds Corollary 1.5 was proved in [Mu2] and it has been used in [MaM] to study the growth of the torsion in the cohomology of arithmetic hyperbolic 3-manifolds. In the same way, Corollary 1.5 can be used to study the torsion in the cohomology of arithmetic hyperbolic manifolds of odd dimension.

Another immediate application is the following corollary which extends [Mu2, Corollary 1.4] to higher dimensions.

Corollary 1.6. *Let* X *be a closed, oriented hyperbolic manifold of odd dimension. Then the volume of* X *is determined by the set* $\{\tau_X(\tau(m)): m \in \mathbb{N}\}$ *of Reidemeister torsion invariants.*

Finally we have the following result about the analytic torsion in even dimensions.

Proposition 1.7. *Assume that* $\dim X$ *is even. Then* $T_X(\tau) = 1$ *for all irreducible finite-dimensional representations* τ *of* G.

Since $\tau|_\Gamma$ is non-unitary, this is not obvious and may be false if we choose other fibre metrics in E_τ (see [Mu1]).

Next we explain our method to prove Theorem 1.1. The approach used in [Mu2] was based on the expression of the analytic torsion in terms of the value at zero of the corresponding twisted Ruelle zeta function. Such a relation between the twisted Ruelle zeta function and

the analytic torsion continues to hold in the higher dimensional case (see [Br], [Wo]) and could be used to prove Corollary 1.4 in the same way. However, we choose a more direct approach which gives the stronger result of Theorem 1.1. Moreover we expect that this method can be used to deal with groups of higher rank.

For the moment let τ be any irreducible finite-dimensional representation of G and denote by $E_\tau \to X$ the flat vector bundle associated to the restriction of τ to Γ. By [MM, Proposition 3.1], E_τ is isomorphic to the locally homogeneous vector bundle defined by the restriction of τ to K. Using this isomorphism, E_τ can be equipped with a canonical Hermitian fibre metric, which is induced from an invariant metric on the corresponding homogeneous vector bundle [MM, Lemma 3.1]. Let $\Delta_p(\tau)$ be the Laplace operator on E_τ-valued p-forms with respect to an admissible metric on E_τ and the hyperbolic metric of X. Let

$$K(t,\tau) = \sum_{p=0}^{d} (-1)^p p \operatorname{Tr}\left(e^{-t\Delta_p(\tau)}\right).$$

Assume that $\tau|_\Gamma$ is acyclic, that is $H^*(X, E_\tau) = 0$. Then the analytic torsion is given by

$$(1.4) \qquad \log T_X(\tau) := \frac{1}{2} \frac{d}{ds}\left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(t,\tau)\, dt\right)\Bigg|_{s=0}.$$

Now we turn to the representations $\tau(m)$, $m \in \mathbb{N}$, defined above. As pointed out above, each $\tau(m)$ is acyclic. Even more is true. There exists $m_0 \in \mathbb{N}$ such that for $m \geq m_0$ we have

$$(1.5) \qquad \Delta_p(\tau(m)) \geq \frac{m^2}{2}$$

(see Corollary 5.2). Let $m \geq m_0$. Since $\tau(m)$ is acyclic, $T_X(\tau(m))$ is metric independent [Mu1]. This means that we can replace $\Delta_p(\tau(m))$ by $\frac{1}{m}\Delta_p(\tau(m))$. If we split the t-integral into the integral over $[0,1)$ and the integral over $[1,\infty]$, we get

$$(1.6) \qquad \begin{aligned} \log T_X(\tau(m)) =&\frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1} K\left(\frac{t}{m}, \tau(m)\right) dt\right)\Bigg|_{s=0} \\ &+ \frac{1}{2}\int_1^\infty t^{-1} K\left(\frac{t}{m}, \tau(m)\right) dt. \end{aligned}$$

It follows from (1.5) and standard estimations of the heat kernel that the second term on the right is $O(e^{-\frac{m}{8}})$ as $m \to \infty$. To deal with the first term, we use a preliminary form of the Selberg trace formula.

Let $\widetilde{\Delta}_p(\tau)$ be the lift of $\Delta_p(\tau)$ to $\tilde{X}$. Then $e^{-t\widetilde{\Delta}_p(\tau)}$ is an invariant integral operator. Using the kernels of the heat operators $e^{-t\widetilde{\Delta}_p(\tau)}$ we construct a smooth K-finite function $k_t^{\tau(m)}$ on G, which belongs to Harish-Chandra's Schwartz space $\mathcal{C}(G)$, such that

$$K(t, \tau(m)) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} k_t^{\tau(m)}(g^{-1}\gamma g)\, d\dot{g}.$$

We split the integral into the contribution of the identity which equals

$$I(t, \tau(m)) = \text{vol}(X) k_t^{\tau(m)}(1).$$

and the integral $H(t, \tau(m))$ of the sum of the non-trivial elements. Using methods of [Do] and [DL] to estimate the heat kernels, it follows that there exist $C, c_1, c_2 > 0$ such that

$$\left| H\left(\frac{t}{m}, \tau(m)\right) \right| \le C e^{-c_1 m} e^{-c_2/t}$$

for all $m \ge m_0$ and $0 < t \le 1$. This implies that the contribution of $H(t/m, \tau(m))$ to the first term on the right of (1.6) is of order $O(e^{-c_1 m})$ as $m \to \infty$. So we are left with the contribution of the identity. The first observation is that the t-integral over $[0, 1]$ can be replaced by the integral over $[0, \infty)$. The difference is exponentially decaying in m. Furthermore, we can change variables back to t. Then the identity contribution is given by

$$(1.7) \qquad \frac{1}{2} \text{vol}(X) \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} k_t^{\tau(m)}(1)\, dt \right) \Bigg|_{s=0}.$$

The t-integral converges absolutely for $\text{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is regular at $s = 0$. Now it is easy to see that (1.7) equals $\log T_X^{(2)}(\tau(m))$. Putting everything together, the proof of Proposition 1.2 follows. To compute the L^2-torsion we apply the Plancherel formula to $k_t^{\tau(m)}(1)$ and use properties of the Plancherel polynomials. This leads to the proof of Proposition 1.3.

The paper is organized as follows. In section 2 we fix notation and collect a number of facts about representation theory which are needed for this paper. In section 3 we prove some basic estimates for the heat kernel of Bochner-Laplace operators. In section 4 we relate the analytic torsion to the Selberg trace formula and compute the Fourier transform of the corresponding test function. In the final section 5 we prove the main results.

2. Preliminaries

In this section we will establish some notation and recall some basic facts about representations of the involved Lie groups.

2.1. For $d \in \mathbb{N}$, $d > 1$ let $G := \text{Spin}(d, 1)$. Recall that G is the universal covering group of $\text{SO}_0(d, 1)$. Let $K := \text{Spin}(d)$. Then K is a maximal compact subgroup of G. Put $\tilde{X} := G/K$. Let

$$G = NAK$$

be the standard Iwasawa decomposition of G and let M be the centralizer of A in K. Then $M = \text{Spin}(d-1)$. The Lie algebras of G, K, A, M and N will be denoted by $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{m}$ and $\mathfrak{n}$, respectively. Define the standard Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ by

$$\theta(Y) = -Y^t, \quad Y \in \mathfrak{g}.$$

The lift of θ to G will be denoted by the same letter θ. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition of $\mathfrak{g}$ with respect to θ. Let $x_0 = eK \in \tilde{X}$. Then we have a canonical isomorphism

$$(2.1) \qquad\qquad\qquad\qquad T_{x_0}\tilde{X} \cong \mathfrak{p}.$$

Define the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by

$$(2.2) \qquad\qquad \langle Y_1, Y_2 \rangle := \frac{1}{2(d-1)} B(Y_1, Y_2), \quad Y_1, Y_2 \in \mathfrak{g}.$$

By (2.1) the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{p}$ defines an inner product on $T_{x_0}\tilde{X}$ and therefore an invariant metric on $\tilde{X}$. This metric has constant curvature -1. Then $\tilde{X}$, equipped with this metric, is isometric to the hyperbolic space $\mathbb{H}^d$.

Let $\Gamma \subset G$ be a discrete, co-compact torsion free subgroup. Then Γ acts properly discontinuously on $\tilde{X}$ and $X = \Gamma \backslash \tilde{X}$ is a compact, oriented hyperbolic manifold of dimension d. Moreover any such manifold is of this form.

2.2. Let now $d = 2n + 1$. Denote by $E_{i,j}$ the matrix in $\mathfrak{g}$ whose entry at the i-th row and j-th column is equal to 1 and all of its other entries are equal to 0. Let

$$(2.3) \qquad\qquad H_i := \begin{cases} E_{1,2} + E_{2,1}, & i = 1; \\ \sqrt{-1}(E_{2i-1,2i} - E_{2i,2i-1}), & i = 2, \ldots n+1. \end{cases}$$

Then

$$\mathfrak{a} = \mathbb{R}H_1$$

and

$$\mathfrak{b} = \mathbb{R}\sqrt{-1}H_2 + \cdots + \mathbb{R}\sqrt{-1}H_{n+1}$$

is the standard Cartan subalgebra of $\mathfrak{m}$. Moreover $\mathfrak{b}$ is also a Cartan subalgebra of $\mathfrak{k}$, and

$$\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{b}$$

is a Cartan-subalgebra of $\mathfrak{g}$. Define $e_i \in \mathfrak{h}_{\mathbb{C}}^*$, $i = 1, \ldots, n+1$, by

$$e_i(H_j) = \delta_{i,j}, \ 1 \le i, j \le n+1.$$

Then the sets of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ and $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ are given by

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\pm e_i \pm e_j, \ 1 \le i < j \le n+1\}$$
$$\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}) = \{\pm e_i, \ 2 \le i < j \le n+1\} \sqcup \{\pm e_i \pm e_j, \ 2 \le i < j \le n+1\}$$
$$\Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}) = \{\pm e_i \pm e_j, \ 2 \le i < j \le n+1\}$$

(see [Kn1, Section IV,2]). We fix positive systems of roots by

$$\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) := \{e_i + e_j, \ i \ne j\} \sqcup \{e_i - e_j, \ i < j\}$$
$$\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}}) := \{e_i + e_j, \ i \ne j, \ i, j \ge 2\} \sqcup \{e_i - e_j, \ 2 \le i < j\}.$$

For $j = 1, \ldots, n+1$ let

$$\rho_j := n + 1 - j.$$

Then the half-sum of positive roots ρ_G and ρ_M, respectively, are given by

$$(2.4) \qquad \rho_G := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})} \alpha = \sum_{j=1}^{n+1} \rho_j e_j$$

and

$$(2.5) \qquad \rho_M := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})} \alpha = \sum_{j=2}^{n+1} \rho_j e_j.$$

Let W_G be the Weyl-group of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.

2.3. Let $\mathbb{Z}\left[\frac{1}{2}\right]^j$ be the set of all $(k_1, \ldots, k_j) \in \mathbb{Q}^j$ such that either all k_i are integers or all k_i are half integers. Let $d = 2n+1$. Then the finite-dimensional irreducible representations $\tau \in \hat{G}$ of G are parametrized by their highest weights

$$(2.6) \qquad \begin{aligned} \Lambda(\tau) &= k_1(\tau)e_1 + \cdots + k_{n+1}(\tau)e_{n+1}, \quad (k_1(\tau), \ldots k_{n+1}(\tau)) \in \mathbb{Z}\left[\frac{1}{2}\right]^{n+1} \\ k_1(\tau) &\geq k_2(\tau) \geq \cdots \geq k_n(\tau) \geq |k_{n+1}(\tau)|. \end{aligned}$$

Furthermore the finite dimensional representations $\nu \in \hat{K}$ of K are parametrized by their highest weights

$$(2.7) \qquad \begin{aligned} \Lambda(\nu) &= k_2(\nu)e_2 + \cdots + k_{n+1}(\nu)e_{n+1}, \quad (k_2(\nu), \ldots k_{n+1}(\nu)) \in \mathbb{Z}\left[\frac{1}{2}\right]^{n} \\ k_2(\nu) &\geq k_2(\nu) \geq \cdots \geq k_n(\nu) \geq k_{n+1}(\nu) \geq 0. \end{aligned}$$

Finally the finite-dimensional irreducible representations $\sigma \in \hat{M}$ of M are parametrized by their highest weights

$$(2.8) \qquad \begin{aligned} \Lambda(\sigma) &= k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}, \quad (k_2(\sigma), \ldots, k_{n+1}(\sigma)) \in \mathbb{Z}\left[\frac{1}{2}\right]^{n}, \\ k_2(\sigma) &\geq k_3(\sigma) \geq \cdots \geq k_n(\sigma) \geq |k_{n+1}(\sigma)|. \end{aligned}$$

For $\tau \in \hat{G}$ let $\tau_\theta := \tau \circ \theta$. Let $\Lambda(\tau)$ denote the highest weight of τ as in (2.6). Then the highest weight $\Lambda(\tau_\theta)$ of τ_θ is given by

$$(2.9) \qquad \Lambda(\tau_\theta) = k_1(\tau)e_1 + \cdots + k_n(\tau)e_n - k_{n+1}(\tau)e_{n+1}.$$

Moreover, by the Weyl dimension formula [Kn1, Theorem 4.48] we have

$$
\dim(\tau) = \prod_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \frac{\langle \Lambda(\tau) + \rho_G, \alpha \rangle}{\langle \rho_G, \alpha \rangle}
$$

(2.10)

$$
= \prod_{i=1}^{n} \prod_{j=i+1}^{n+1} \frac{(k_i(\tau) + \rho_i)^2 - (k_j(\tau) + \rho_j)^2}{\rho_i^2 - \rho_j^2}.
$$

Similarly, for $\sigma \in \hat{M}$ with highest weight $\Lambda(\sigma) \in \mathfrak{b}_{\mathbb{C}}^*$ as in (2.8) we have

$$
\dim(\sigma) = \prod_{\alpha \in \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})} \frac{\langle \Lambda(\sigma) + \rho_M, \alpha \rangle}{\langle \rho_M, \alpha \rangle}
$$

(2.11)

$$
= \prod_{i=2}^{n} \prod_{j=i+1}^{n+1} \frac{(k_i(\sigma) + \rho_i)^2 - (k_j(\sigma) + \rho_j)^2}{\rho_i^2 - \rho_j^2}.
$$

For $\tau \in \hat{G}$ and $\nu \in \hat{K}$ we will denote by $[\tau : \nu]$ the multiplicity of ν in the restriction of τ to K. These multiplicities are described in the following proposition.

Proposition 2.1. *Let $\tau \in \hat{G}$ be of highest weight $\Lambda(\tau)$ as in (2.6). Then τ decomposes with multiplicity one into representations $\nu \in \hat{K}$ with highest weight $\Lambda(\nu)$ as in (2.7) such that $k_{j-1}(\tau) \geq k_j(\nu) \geq |k_j(\tau)|$ for every $j \in \{2, \ldots, n+1\}$ and such that all $k_j(\nu)$ are integers if all $k_j(\tau)$ are integers resp. such that all $k_j(\nu)$ are half-integers if all $k_j(\tau)$ are half integers.*

Proof. [GW][Theorem 8.1.4] □

Let M' be the normalizer of A in K and let $W(A) = M'/M$ be the restricted Weyl-group. It has order two and it acts on the finite-dimensional representations of M as follows. Let $w_0 \in W(A)$ be the non-trivial element and let $m_0 \in M'$ be a representative of w_0. Given $\sigma \in \hat{M}$, the representation $w_0 \sigma \in \hat{M}$ is defined by

$$
w_0 \sigma(m) = \sigma(m_0 m m_0^{-1}), \quad m \in M.
$$

Let $\Lambda(\sigma) = k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$ be the highest weight of σ as in (2.8). Then the highest weight $\Lambda(w_0\sigma)$ of $w_0\sigma$ is given by

(2.12) $$\Lambda(w_0\sigma) = k_2(\sigma)e_2 + \cdots + k_n(\sigma)e_n - k_{n+1}(\sigma)e_{n+1}.$$

Let $R(K)$ and $R(M)$ be the representation rings of K and M. Let $\iota : M \longrightarrow K$ be the inclusion and let $\iota^* : R(K) \longrightarrow R(M)$ be the induced map. If $R(M)^{W(A)}$ is the subring of $W(A)$-invariant elements of $R(M)$, then clearly ι^* maps $R(K)$ into $R(M)^{W(A)}$.

Proposition 2.2. *The map ι^* is an isomorphism from $R(K)$ onto $R(M)^{W(A)}$.*

Proof. [BO] , Proposition 1.1. □

2.4. We parametrize the principal series as follows. Given $\sigma \in \hat{M}$ with $(\sigma, V_\sigma) \in \sigma$, let $\mathcal{H}^\sigma$ denote the space of measurable functions $f \colon K \to V_\sigma$ satisfying

$$f(mk) = \sigma(m)f(k), \quad \forall k \in K, \forall m \in M, \quad \text{and} \quad \int_K \| f(k) \|^2 \, dk = \| f \|^2 < \infty.$$

Then for $\lambda \in \mathbb{C}$ and $f \in H^\sigma$ let

$$\pi_{\sigma,\lambda}(g)f(k) := e^{(i\lambda e_1 + \rho) \log H(kg)} f(\kappa(kg)).$$

Recall that the representations $\pi_{\sigma,\lambda}$ are unitary iff $\lambda \in \mathbb{R}$. Moreover, for $\lambda \in \mathbb{R} - \{0\}$ and $\sigma \in \hat{M}$ the representations $\pi_{\sigma,\lambda}$ are irreducible and $\pi_{\sigma,\lambda}$ and $\pi_{\sigma',\lambda'}$, $\lambda, \lambda' \in \mathbb{C}$, are equivalent iff either $\sigma = \sigma'$, $\lambda = \lambda'$ or $\sigma' = w_0\sigma$, $\lambda' = -\lambda$. The restriction of $\pi_{\sigma,\lambda}$ to K coincides with the induced representation $\mathrm{Ind}_M^K(\sigma)$. Hence by Frobenius reciprocity [Kn1, p.208] for every $\nu \in \hat{K}$ one has

$$(2.13) \qquad\qquad\qquad\qquad [\pi_{\sigma,\lambda} : \nu] = [\nu : \sigma] \,.$$

2.5. From now on we assume that $d = 2n+1$. We establish some facts about infinitesimal characters. Let $U(\mathfrak{g}_\mathbb{C})$ denote the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$ and let $Z(U(\mathfrak{g}_\mathbb{C}))$ be its center. Let $\Omega \in Z(U(\mathfrak{g}_\mathbb{C}))$ be the Casimir element with respect to the Killing form normalized as in (2.2). Let $I(\mathfrak{h}_\mathbb{C})$ be the Weyl-group invariant elements of the symmetric algebra $S(\mathfrak{h}_\mathbb{C})$ of $\mathfrak{h}_\mathbb{C}$. Let

$$(2.14) \qquad\qquad\qquad\qquad \gamma : Z(U(\mathfrak{g}_\mathbb{C})) \longrightarrow I(\mathfrak{h}_\mathbb{C})$$

be the Harish-Chandra isomorphism [Kn1, Section VIII,5]. Every $\Lambda \in \mathfrak{h}_\mathbb{C}^*$ defines a homomorphism

$$\chi_\Lambda : Z(U(\mathfrak{g}_\mathbb{C})) \longrightarrow \mathbb{C}$$

by

$$\chi_\Lambda(Z) := \Lambda(\gamma(Z)).$$

Let τ be an irreducible finite-dimensional representation of G with highest weight $\Lambda(\tau)$. Its infinitesimal character will also be denoted by τ, i.e. every $Z \in Z(U(\mathfrak{g}_\mathbb{C}))$ acts by $\tau(Z) \cdot \mathrm{Id}$. It follows from the definition of γ that

$$(2.15) \qquad \tau(Z) = \chi_{\Lambda(\tau)+\rho_G}(Z) = \chi_{w(\Lambda(\tau)+\rho_G)}(Z); \quad Z \in Z(U(\mathfrak{g}_\mathbb{C})), \ w \in W.$$

Moreover, a standard computation gives

$$(2.16) \qquad\qquad\qquad\qquad \gamma(\Omega) = \sum_{j=1}^{n+1} H_j^2 - \sum_{j=1}^{n+1} \rho_j^2,$$

where the H_j are defined by (2.3). Thus, if the highest weight $\Lambda(\tau)$ of τ is as in (2.6) one obtains

$$(2.17) \qquad\qquad\qquad\qquad \tau(\Omega) = \sum_{j=1}^{n+1} \left(k_j(\tau) + \rho_j\right)^2 - \sum_{j=1}^{n+1} \rho_j^2$$

Now let Ω_K be the Casimir operator of $\mathfrak{k}$ with respect to the restriction of the normalized Killing form on $\mathfrak{g}$ to $\mathfrak{k}$. Then Ω_K belongs to $Z(U(\mathfrak{k}_{\mathbb{C}}))$, the center of the universal enveloping algebra of $\mathfrak{k}_{\mathbb{C}}$. If $\nu \in \hat{K}$, we will denote the infinitesimal character of ν by ν too. If the highest weight $\Lambda(\nu)$ of ν given by (2.7), an argument analogous to the one above gives

$$(2.18) \qquad \nu(\Omega_K) = \sum_{j=2}^{n+1} \left(k_j(\nu) + \rho_j + \frac{1}{2} \right)^2 - \sum_{j=2}^{n+1} \left(\rho_j + \frac{1}{2} \right)^2.$$

2.6.   Now we come to the infinitesimal character of $\pi_{\sigma,\lambda}$.

Proposition 2.3. *Let $\sigma \in \hat{M}$ with highest weight $\Lambda(\sigma) \in \mathfrak{b}_{\mathbb{C}}^*$.   Then the infinitesimal character of $\pi_{\sigma,\lambda}$ equals $\chi_{\Lambda(\sigma)+\rho_M+i\lambda e_1}$.*

Proof. [Kn1], Proposition 8.22. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.4. *For $\sigma \in \hat{M}$ with highest weight $\Lambda(\sigma)$ given by (2.8), let*

$$(2.19) \qquad c(\sigma) := \sum_{j=2}^{n+1} (k_j(\sigma) + \rho_j)^2 - \sum_{j=1}^{n+1} \rho_j^2.$$

Then for the Casimir element $\Omega \in Z(\mathfrak{g}_{\mathbb{C}})$ one has

$$(2.20) \qquad \pi_{\sigma,\lambda}(\Omega) = -\lambda^2 + c(\sigma).$$

Proof. This follows form equation (2.16) and Proposition 2.3. $\qquad\qquad\qquad\square$

2.7.   For $\sigma \in \hat{M}$ and $\lambda \in \mathbb{R}$ let $\mu_{\sigma}(\lambda)$ be the Plancherel measure associated to $\pi_{\sigma,\lambda}$. Then, since $\operatorname{rk}(G) > \operatorname{rk}(K)$, $\mu_{\sigma}(\lambda)$ is a polynomial in λ of degree $2n$. Let $\langle \cdot, \cdot \rangle$ be the bilinear form defined by (2.2). Let $\Lambda(\sigma) \in \mathfrak{b}_{\mathbb{C}}^*$ be the highest weight of σ as in (2.8). Then by theorem 13.2 in [Kn1] there exists a constant $c(n)$ such that one has

$$\mu_{\sigma}(\lambda) = -c(n) \prod_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})} \frac{\langle i\lambda e_1 + \Lambda(\sigma) + \rho_M, \alpha \rangle}{\langle \rho_G, \alpha \rangle}..$$

The constant $c(n)$ is computed in [Mi2]. By [Mi2], theorem 3.1, one has $c(n) > 0$. For $z \in \mathbb{C}$ let

$$(2.21) \qquad P_{\sigma}(z) = -c(n) \prod_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})} \frac{\langle z e_1 + \Lambda(\sigma) + \rho_M, \alpha \rangle}{\langle \rho_G, \alpha \rangle}.$$

One easily sees that

$$(2.22) \qquad P_{\sigma}(z) = P_{w_0\sigma}(z).$$

2.8. Let $\tau \in \hat{G}$ and let $\Lambda(\tau) = \tau_1 e_1 + \cdots + \tau_{n+1} e_{n+1}$ be its highest weight. For $w \in W$ let $l(w)$ denote its length with respect to the simple roots which define the positive roots above. Let

$$W^1 := \{w \in W_G \colon w^{-1}\alpha > 0 \; \forall \alpha \in \Delta(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})\}$$

Let V_τ be the representation space of τ. For $k = 0, \ldots, 2n$ let $H^k(\overline{\mathfrak{n}}, V_\tau)$ be the cohomology of $\overline{\mathfrak{n}}$ with coefficients in V_τ. Then $H^k(\overline{\mathfrak{n}}, V_\tau)$ is an MA module. In our case, the theorem of Kostant states:

Proposition 2.5. *In the sense of MA-modules one has*

$$H^k(\overline{\mathfrak{n}}; V_\tau) \cong \sum_{\substack{w \in W^1 \\ l(w) = k}} V_{\tau(w)},$$

where $V_{\tau(w)}$ is the MA module of highest weight $w(\Lambda(\tau) + \rho_G) - \rho_G$.

Proof. for the proof see [BW, Theorem III.3]. □

Corollary 2.6. *As MA-modules we have*

$$\bigoplus_{k=0}^{2n} (-1)^k \Lambda^k \mathfrak{n}^* \otimes V_\tau = \bigoplus_{w \in W^1} (-1)^{l(w)} V_{\tau(w)}.$$

Proof. Note that $\overline{\mathfrak{n}} \cong \mathfrak{n}^*$ as MA-modules. With this remark, the proof follows from proposition 2.5 and the Poincare principle [Ko, (7.2.3)]. □

For $w \in W^1$ let $\sigma_{\tau,w}$ be the representation of M with highest weight

(2.23) $$\Lambda(\sigma_{\tau,w}) := w(\Lambda(\tau) + \rho_G)|_{\mathfrak{b}_{\mathbb{C}}} - \rho_M$$

and let $\lambda_{\tau,w} \in \mathbb{C}$ such that

(2.24) $$w(\Lambda(\tau) + \rho_G)|_{\mathfrak{a}_{\mathbb{C}}} = \lambda_{\tau,w} e_1.$$

For $k = 0, \ldots n$ let

(2.25) $$\lambda_{\tau,k} = \tau_{k+1} + n - k$$

and $\sigma_{\tau,k}$ be the representation of G with highest weight

(2.26) $$\Lambda_{\sigma_{\tau,k}} := (\tau_1 + 1)e_2 + \cdots + (\tau_k + 1)e_{k+1} + \tau_{k+2}e_{k+2} + \cdots + \tau_{n+1}e_{n+1}.$$

Then by the computations in [BW, Chapter VI.3] one has

(2.27) $$\begin{aligned} \{(\lambda_{\tau,w}, \sigma_{\tau,w}, l(w)) \colon w \in W^1\} &= \{(\lambda_{\tau,k}, \sigma_{\tau,k}, k) \colon k = 0, \ldots, n\} \\ &\sqcup \{(-\lambda_{\tau,k}, w_0\sigma_{\tau,k}, 2n - k) \colon k = 0, \ldots, n\}. \end{aligned}$$

Remark 1. Corollary 2.6 was first proved by U. Bröcker[Br] by an elementary but tedious computation without using the theorem of Kostant. Also the convenient notation $\sigma_{\tau,k}$ and $\lambda_{\tau,k}$ is due to him.

We will also need the following proposition.

Proposition 2.7. *For every $w \in W^1$ one has*

$$\tau(\Omega) = \lambda_{\tau,w}^2 + c(\sigma_{\tau,w}).$$

Proof. Using (2.15) and (2.16) one gets

$$\tau(\Omega) = \chi_{\Lambda(\tau)+\rho_G}(\Omega) = \chi_{w(\Lambda(\tau)+\rho_G)}(\Omega) = \chi_{\Lambda(\sigma_{\tau,w})+\rho_M+\lambda_\tau(w)e_1}(\Omega) = \lambda_{\tau,w}^2 + c(\sigma_{\tau,w}).$$

$\square$

3. Bochner Laplace operators

Regard G as a principal K-fibre bundle over $\tilde{X}$. By the invariance of $\mathfrak{p}$ under $\mathrm{Ad}(K)$ the assignment

$$T_g^{\mathrm{hor}} := \{\frac{d}{dt}\big|_{t=0} g \exp tX \colon X \in \mathfrak{p}\}$$

defines a horizontal distribution on G. This connection is called the canonical connection. Let ν be a finite-dimensional unitary representation of K on $(V_\nu, \langle \cdot, \cdot \rangle_\nu)$. Let

$$\widetilde{E}_\nu := G \times_\nu V_\nu$$

be the associated homogeneous vector bundle over $\widetilde{X}$. Then $\langle \cdot, \cdot \rangle_\nu$ induces a G-invariant metric $\widetilde{B}_\nu$ on $\widetilde{E}_\nu$. Let $\widetilde{\nabla}^\nu$ be the connection on $\widetilde{E}_\nu$ induced by the canonical connection. Then $\widetilde{\nabla}^\nu$ is G-invariant. Let

$$E_\nu := \Gamma\backslash(G \times_\nu V_\nu)$$

be the associated locally homogeneous bundle over X. Since $\tilde{B}_\nu$ and $\widetilde{\nabla}^\nu$ are G-invariant, they push down to a metric B_ν and a connection ∇^ν on E_ν. Let

$$(3.1) \qquad C^\infty(G,\nu) := \{f \colon G \to V_\nu \colon f \in C^\infty, \ f(gk) = \nu(k^{-1})f(g), \ \forall g \in G, \forall k \in K\}.$$

Let

$$(3.2) \qquad C^\infty(\Gamma\backslash G, \nu) := \{f \in C^\infty(G,\nu) \colon f(\gamma g) = f(g) \, \forall g \in G, \forall \gamma \in \Gamma\}.$$

Let $C^\infty(X, \widetilde{E}_\nu)$ resp. $C^\infty(X, E_\nu)$ denote the space of smooth sections of $\widetilde{E}_\nu$ resp. of E_ν. Then there are canonical isomorphism

$$A \colon C^\infty(X, \widetilde{E}_\nu) \cong C^\infty(G,\nu), \quad A \colon C^\infty(X, E_\nu) \cong C^\infty(\Gamma\backslash G, \nu).$$

There are also a corresponding isometries for the spaces $L^2(X, \widetilde{E}_\nu)$ and $L^2(X, E_\nu)$ of L^2-sections. For every $X \in \mathfrak{g}$, $g \in G$ and every $f \in C^\infty(X, \widetilde{E}_\nu)$ one has

$$A(\widetilde{\nabla}^\nu_{L(g)_*X} f)(g) = \frac{d}{dt}\big|_{t=0} Af(g \exp tX).$$

Let $\widetilde{\Delta}_\nu = \widetilde{\nabla}^{\nu^*}\widetilde{\nabla}^\nu$ be the Bochner-Laplace operator of $\widetilde{E}_\nu$. Since $\widetilde{X}$ is complete, $\widetilde{\Delta}_\nu$ with domain the smooth compactly supported sections is essentially self-adjoint [Ch]. Its self-adjoint extension will be denoted by $\widetilde{\Delta}_\nu$ too. By [Mi1, Proposition 1.1] on $C^\infty(X, \widetilde{E}_\nu)$ one has

$$(3.3) \qquad \widetilde{\Delta}_\nu = -R_\Gamma(\Omega) + \nu(\Omega_K).$$

Let $e^{-t\widetilde{\Delta}_\nu}$ be the corresponding heat semigroup on $L^2(G, \nu)$, where $L^2(G, \nu)$ is defined analogously to (3.1). Then the same arguments as in [CY, section1] imply that there exists a function

$$(3.4) \qquad K_t^\nu \in C^\infty(G \times G, \mathrm{End}(V_\nu)),$$

which is symmetric in the G-variables and for which $g' \mapsto K_t^\nu(g, g')$ belongs to $L^2(G, \mathrm{End}(V_\nu))$ for each $g \in G$ such that

$$K_t^\nu(gk, g'k') = \nu(k^{-1})K_t^\nu(g, g')\nu(k')v, \ \forall g, g' \in G, \ \forall k \in K, \ \forall v \in V_\nu$$

and such that

$$e^{-t\widetilde{\Delta}_\nu}\phi(g) = \int_G K_t^\nu(g, g')\phi(g')dg', \ \phi \in L^2(G, \nu).$$

Since Ω is G-invariant, K_ν is invariant under the diagonal action of G. Hence there exists a function

$$(3.5) \quad H_t^\nu : G \longrightarrow \mathrm{End}(V_\nu); \quad H_t^\nu(k^{-1}gk') = \nu(k)^{-1} \circ H_t^\nu(g) \circ \nu(k'), \ \forall k, k' \in K, \forall g \in G$$

such that

$$(3.6) \qquad K_t^\nu(g, g') = H_t^\nu(g^{-1}g'), \quad \forall g, g' \in G.$$

Thus one has

$$(e^{-t\widetilde{\Delta}_\nu}\phi)(g) = \int_G H_t^\nu(g^{-1}g')\phi(g')dg', \quad \phi \in L^2(G, \nu), \quad g \in G.$$

By the arguments of [BM], Proposition 2.4, H_t^ν belongs to all Harish-Chandra Schwartz spaces $(\mathcal{C}^q(G) \otimes \mathrm{End}(V_\nu))$, $q > 0$. Now let $\|H_t^\nu(g)\|$ be the norm of $H_t^\nu(g)$ in $\mathrm{End}(V_\nu)$. Then by the principle of semigroup domination $\|H_t^\nu(g)\|$ is bounded by the scalar heat kernel.

Proposition 3.1. *Let $\widetilde{\Delta}_0$ be the Laplacian on functions on $\widetilde{X}$ and let H_t^0 be the associated heat-kernel as above. Let $\nu \in \hat{K}$. Then for every $t \in (0, \infty)$ and every $g \in G$ one has*

$$\|H_t^\nu(g)\| \leq H_t^0(g).$$

Proof. First we remark that by (3.6) and [CY, Lemma 1.1] one has $H_t^0(g) > 0$ for every $t \in (0, \infty)$ and every $g \in G$. Now using (3.6) one can adapt the proof of Theorem 4.3 in [DL] to our situation. $\qquad \square$

Now we pass to the quotient $X = \Gamma\backslash\widetilde{X}$. Let $\Delta_\nu = \nabla^{\nu*}\nabla^\nu$ the closure of the Bochner-Laplace operator with domain the smooth sections of E_ν. Then Δ_ν is selfadjoint and on $C^\infty(\Gamma\backslash G, \nu)$ it induces the operator $-R_\Gamma(\Omega) + \nu(\Omega_K)$. Let $e^{-t\Delta_\nu}$ be the heat semigroup of Δ_ν on $L^2(\Gamma\backslash G, \nu)$. Then $e^{-t\Delta_\nu}$ is represented by the smooth kernel

$$(3.7) \qquad H_\nu(t, x, x') := \sum_{\gamma\in\Gamma} H_t^\nu(g^{-1}\gamma g'),$$

where H_t^ν is as above and where $x = \Gamma g$, $x' = \Gamma g'$ with $g, g' \in G$. The convergence of the series in (3.7) can be established for example using proposition 3.1 and the methods from the proof of Proposition 3.2 below. It follows that the trace of the heat operator $e^{-t\Delta_\nu}$ is given by

$$\mathrm{Tr}\left(e^{-t\Delta_\nu}\right) = \int_X \mathrm{tr}\, H_\nu(t, x, x) dx,$$

where $\mathrm{tr}: \mathrm{End}(V_\nu) \to \mathbb{C}$. Thus if for $g \in G$ one lets

$$(3.8) \qquad h_t^\nu(g) := \mathrm{tr}(H_t^\nu(g)),$$

one obtains

$$(3.9) \qquad \mathrm{Tr}(e^{-t\Delta_\nu}) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} h_t^\nu(g^{-1}\gamma g) d\dot{g}.$$

Using results of Donnelly we now prove an estimate for the heat kernel H_t^0 of the Laplacian Δ_0 acting on functions on $\widetilde{X}$.

Proposition 3.2. *There exist constants C_0 and c_0 such that for every $t \in (0, 1]$ and every $g \in G$ one has*

$$\sum_{\substack{\gamma\in\Gamma \\ \gamma\neq 1}} H_t^0(g^{-1}\gamma g) \leq C_0 e^{-c_0/t}.$$

Proof. For $x, y \in \widetilde{X}$ let $\rho(x, y)$ denote the geodesic distance of x, y. Then using (3.6) it follows from [Do, Theorem 3.3] that there exists a constant C_1 such that for every $g \in G$ and every $t \in (0, 1]$ one has

$$(3.10) \qquad H_t^0(g) \leq C_1 t^{-\frac{d}{2}} \exp\left(-\frac{\rho^2(gK, 1K)}{4t}\right).$$

Let $x \in \widetilde{X}$ and let $B_R(x)$ be the metric ball around x of radius R. Then one has

$$(3.11) \qquad \mathrm{vol}\, B_R(x) \leq C_2 e^{2nR}.$$

Since Γ is cocompact and torsion-free, there exists an $\epsilon > 0$ such that $B_\epsilon(x) \cap \gamma B_\epsilon(x) = \emptyset$ for every $\gamma \in \Gamma - \{1\}$ and every $x \in \widetilde{X}$. Thus for every $x \in \widetilde{X}$ the union over all $\gamma B_\epsilon(x)$, where $\gamma \in \Gamma$ is such that $\rho(x, \gamma x) \leq R$ is disjoint and is contained in $B_{R+\epsilon}(x)$. Using (3.11) it follows that there exists a constant C_3 such that for every $x \in \widetilde{X}$ one has

$$\#\{\gamma \in \Gamma: \rho(x, \gamma x) \leq R\} \leq C_3 e^{2nR}.$$

Hence there exists a constant $C_4 > 0$ such that for every $x \in \widetilde{X}$ one has

(3.12)
$$\sum_{\substack{\gamma \in \Gamma \\ \gamma \neq 1}} e^{-\rho^{3/2}(\gamma x, x)} \leq C_4.$$

Now let

$$c := \inf\{\rho(x, \gamma x) \colon \gamma \in \Gamma - \{1\}, \ x \in \widetilde{X}\}.$$

We have $c > 0$ and using (3.10) and (3.12) we get constants c_0 and C_0 such that for every $g \in G$ and $0 < t \leq 1$ we have

$$\sum_{\substack{\gamma \in \Gamma \\ \gamma \neq 1}} H_t^0(g^{-1} \gamma g) \leq C_1 t^{-\frac{d}{2}} e^{-\sqrt{c}/t} \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq 1}} e^{-\rho^{3/2}(\gamma gK, gK)} \leq C_0 e^{-c_0/t}.$$

$\square$

4. The analytic torsion

Let τ be an irreducible finite-dimensional representation of G on V_τ. Let E_τ' be the flat vector bundle over X associated to the restriction of τ to Γ. Then E_τ' is canonically isomorphic to the locally homogeneous vector bundle E_τ associated to $\tau|_K$ (see [MM, Proposition 3.1]. By [MM, Lemma 3.1], there exists an inner product $\langle \cdot, \cdot \rangle$ on V_τ such that

(1) $\langle \tau(Y)u, v \rangle = -\langle u, \tau(Y)v \rangle$ for all $Y \in \mathfrak{k}$, $u, v \in V_\tau$
(2) $\langle \tau(Y)u, v \rangle = \langle u, \tau(Y)v \rangle$ for all $Y \in \mathfrak{p}$, $u, v \in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\tau|_K$ is unitary with respect to this inner product, it induces a metric on E_τ which we also call admissible. Let $\Lambda^p(E_\tau) = \Lambda^p T^*(X) \otimes E_\tau$. Let

(4.1)
$$\nu_p(\tau) := \Lambda^p \operatorname{Ad}^* \otimes \tau \colon K \to \operatorname{GL}(\Lambda^p \mathfrak{p}^* \otimes V_\tau).$$

Then there is a canonical isomorphism

(4.2)
$$\Lambda^p(E_\tau) \cong \Gamma \backslash (G \times_{\nu_p(\tau)} \Lambda^p \mathfrak{p}^* \otimes V_\tau).$$

of locally homogeneous vector bundles. Let $\Lambda^p(X, E_\tau)$ be the space the smooth E_τ-valued p-forms on X. The isomorphism (4.2) induces an isomorphism

(4.3)
$$\Lambda^p(X, E_\tau) \cong C^\infty(\Gamma \backslash G, \nu_p(\tau)),$$

where the latter space is defined as in (3.2). A corresponding isomorphism also holds for the spaces of L^2-sections. Let $\Delta_p(\tau)$ be the Hodge-Laplacian on $\Lambda^p(X, E_\tau)$ with respect to the admissible metric in E_τ. Let R_Γ denote the right regular representation of G in $L^2(\Gamma \backslash G)$. By [MM, (6.9)] it follows that with respect to the isomorphism (4.3) one has

$$\Delta_p(\tau) f = -R_\Gamma(\Omega) f + \tau(\Omega) \operatorname{Id} f, \ f \in C^\infty(\Gamma \backslash G, \nu_p(\tau)).$$

Let

$$(4.4) \qquad K(t,\tau) := \sum_{p=1}^{d} (-1)^p p \operatorname{Tr}(e^{-t\Delta_p(\tau)}).$$

and

$$(4.5) \qquad h(\tau) := \sum_{p=1}^{d} (-1)^p p \dim H^p(X, E_\tau).$$

Then $K(t,\tau) - h(\tau)$ decays exponentially as $t \to \infty$ and it follows from (1.1) that

$$(4.6) \qquad \log T_X(\tau) = \frac{1}{2} \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (K(t,\tau) - h(\tau)) \, dt \right) \bigg|_{s=0},$$

where the right hand side is defined near $s = 0$ by analytic continuation of the Mellin transform. Let $\widetilde{E}_{\nu_p(\tau)} := G \times_{\nu_p(\tau)} \Lambda^p \mathfrak{p}^* \otimes V_\tau$ and let $\widetilde{\Delta}_p(\tau)$ be the lift of $\Delta_p(\tau)$ to $C^\infty(\widetilde{X}, \widetilde{E}_{\nu_p(\tau)})$. Then again it follows from [MM, (6.9)] that on $C^\infty(G, \nu_p(\tau))$ one has

$$(4.7) \qquad \widetilde{\Delta}_p(\tau) = -R_\Gamma(\Omega) + \tau(\Omega) \operatorname{Id}.$$

Let $e^{-t\widetilde{\Delta}_p(\tau)}$, be the corresponding heat semigroup on $L^2(G, \nu_p(\tau))$. This is a smoothing operator which commutes with the action of G. Therefore, it is of the form

$$\left(e^{-t\widetilde{\Delta}_p(\tau)} \phi \right)(g) = \int_G H_t^{\tau,p}(g^{-1} g') \phi(g') \, dg', \quad \phi \in (L^2(G, \nu_p(\tau)), \ g \in G,$$

where the kernel

$$(4.8) \qquad H_t^{\tau,p} \colon G \to \operatorname{End}(\Lambda^p \mathfrak{p}^* \otimes V_\tau)$$

belongs to $C^\infty \cap L^2$ and satisfies the covariance property

$$(4.9) \qquad H_t^{\tau,p}(k^{-1} g k') = \nu_p(\tau)(k)^{-1} H_t^{\tau,p}(g) \nu_p(\tau)(k')$$

with respect to the representation (4.1). Moreover, for all $q > 0$ we have

$$(4.10) \qquad H_t^{\tau,p} \in (\mathcal{C}^q(G) \otimes \operatorname{End}(\Lambda^p \mathfrak{p}^* \otimes V_\tau))^{K \times K},$$

where $\mathcal{C}^q(G)$ denotes Harish-Chandra's L^q-Schwartz space. The proof is similar to the proof of Proposition 2.4 in [BM]. Now we come to the heat kernel of $\Delta_p(\tau)$. First the integral kernel of $e^{-t\Delta_p(\tau)}$ on $L^2(\Gamma\backslash G, \nu_p(\tau)$ is given by

$$(4.11) \qquad H^{\tau,p}(t; x, x') := \sum_{\gamma \in \Gamma} H_t^{\tau,p}(g^{-1} \gamma g'),$$

where $x, x' \in \Gamma\backslash G$, $x = \Gamma g$, $x' = \Gamma g'$. As in section 3 this series converges absolutely and locally uniformly. Therefore the trace of the heat operator $e^{-t\Delta_p(\tau)}$ is given by

$$\operatorname{Tr}\left(e^{-t\Delta_p(\tau)} \right) = \int_X \operatorname{tr} H^{\tau,p}(t; x, x) \, dx,$$

where tr denotes the trace $\mathrm{tr} \colon \mathrm{End}(V_\nu) \to \mathbb{C}$. Let

(4.12)
$$h_t^{\tau,p}(g) := \mathrm{tr}\, H_t^{\tau,p}(g).$$

Using (4.11) we obtain

(4.13)
$$\mathrm{Tr}\left(e^{-t\Delta_p(\tau)}\right) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} h_t^{\tau,p}(g^{-1}\gamma g)\, d\dot{g}.$$

Put

(4.14)
$$k_t^\tau = \sum_{p=1}^{d} (-1)^p p\, h_t^{\tau,p}.$$

Then it follows that

(4.15)
$$K(t,\tau) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} k_t^\tau(g^{-1}\gamma g)\, d\dot{g}.$$

Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$. Then (4.15) can be written as

(4.16)
$$K(t,\tau) = \mathrm{Tr}\, R_\Gamma(k_t^\tau).$$

One can now apply the Selberg trace formula to the right hand side. For this purpose we need to compute the Fourier transform of k_t^τ which we do next.

   To begin with let π be an admissible unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. Set

$$\tilde{\pi}(H_t^{\tau,p}) = \int_G \pi(g) \otimes H_t^{\tau,p}(g)\, dg.$$

This defines a bounded operator on $\mathcal{H}_\pi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau$. As in [BM, pp. 160-161] it follows from (4.9) that relative to the splitting

$$\mathcal{H}_\pi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau = (\mathcal{H}_\pi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau)^K \oplus \left[(\mathcal{H}_\pi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau)^K\right]^\perp,$$

$\tilde{\pi}(H_t^{\tau,p})$ has the form

$$\tilde{\pi}(H_t^{\tau,p}) = \begin{pmatrix} \pi(H_t^{\tau,p}) & 0 \\ 0 & 0 \end{pmatrix}$$

with $\pi(H_t^{\tau,p})$ acting on $(\mathcal{H}_\pi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau)^K$. Using (4.7) it follows as in [BM, Corollary 2.2] that

(4.17)
$$\pi(H_t^{\tau,p}) = e^{t(\pi(\Omega)-\tau(\Omega))}\,\mathrm{Id}$$

on $(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K$. Let $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{e_j\}_{j=1}^m$ be orthonormal bases of $\mathcal{H}_\pi$ and $\Lambda^p \mathfrak{p}^* \otimes V_\tau$, respectively. Then we have

$$
\begin{aligned}
\operatorname{Tr} \pi(H_t^{\tau,p}) &= \sum_{n=1}^\infty \sum_{j=1}^m \langle \pi(H_t^{\tau,p})(\xi_n \otimes e_j), (\xi_n \otimes e_j) \rangle \\
&= \sum_{n=1}^\infty \sum_{j=1}^m \int_G \langle \pi(g)\xi_n, \xi_n \rangle \langle H_t^{\tau,p}(g)e_j, e_j \rangle \, dg \\
&= \sum_{n=1}^\infty \int_G h_t^{\tau,p}(g) \langle \pi(g)\xi_n, \xi_n \rangle \, dg \\
&= \operatorname{Tr} \pi(h_t^{\tau,p}).
\end{aligned}
$$

(4.18)

Let $\pi \in \hat{G}$ and let Θ_π denote its character. Then it follows from (4.14), (4.17) and (4.18) that

$$
(4.19) \qquad \Theta_\pi(k_t^\tau) = e^{t(\pi(\Omega) - \tau(\Omega))} \sum_{p=1}^d (-1)^p p \cdot \dim(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K.
$$

Now we let π be a unitary principal series representation $\pi_{\sigma,\lambda}$, $\lambda \in \mathbb{R}$, $(\sigma, W_\sigma) \in \hat{M}$. Let $c(\sigma)$ be defined by (2.19). By Frobenius reciprocity [Kn1, p.208] and Corollary 2.4 we get

$$
(4.20) \qquad \Theta_{\sigma,\lambda}(k_t^\tau) = e^{-t(\lambda^2 - c(\sigma) + \tau(\Omega))} \sum_{p=1}^d (-1)^p p \cdot \dim(W_\sigma \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^M.
$$

Now observe that as M-modules, $\mathfrak{p}$ and $\mathfrak{a} \oplus \mathfrak{n}$ are equivalent. Thus we get

$$
(4.21) \qquad \sum_{p=1}^d (-1)^p p \, \Lambda^p \mathfrak{p}^* = \sum_{p=1}^d (-1)^p p \left(\Lambda^p \mathfrak{n}^* + \Lambda^{p-1} \mathfrak{n}^* \right) = \sum_{p=0}^{d-1} (-1)^{p+1} \Lambda^p \mathfrak{n}^*.
$$

Therefore we have

$$
(4.22) \qquad \Theta_{\sigma,\lambda}(k_t^\tau) = e^{-t(\lambda^2 - c(\sigma) + \tau(\Omega))} \sum_{p=0}^{d-1} (-1)^{p+1} \dim(W_\sigma \otimes \Lambda^p \mathfrak{n}^* \otimes V_\tau)^M.
$$

We now distinguish two cases. First assume that $d = 2n$ is even. Then for the principal series representations of G we obtain the following lemma.

Lemma 4.1. *Let d be even. Then $\Theta_{\sigma,\lambda}(k_t^\tau) = 0$ for all $\sigma \in \hat{M}$ and $\lambda \in \mathbb{R}$.*

Proof. As M-modules we have $\Lambda^p \mathfrak{n}^* \cong \Lambda^{d-1-p} \mathfrak{n}^*$. Hence, if d is even, we get

$$
\sum_{p=0}^{d-1} (-1)^p \Lambda^p \mathfrak{n}^* = 0.
$$

Combined with (4.22) the lemma follows. □

Next we assume that $d = 2n + 1$ is odd. Then we have the following proposition.

Proposition 4.2. *Assume that $d = 2n + 1$. Then one has*

$$\Theta_{\sigma,\lambda}(k_t^\tau) := \begin{cases} e^{-t(\lambda^2 + \lambda_{\tau,k}^2)}, & \sigma \in \{\sigma_{\tau,k}, \, w_0 \sigma_{\tau,k}\} \\ 0, & otherwise. \end{cases}$$

Here the $\lambda_{\tau,k} \in \mathbb{R}$ and the $\sigma_{\tau,k} \in \hat{M}$ are as in (2.25) and (2.26) .

Proof. Let $\sigma \in \hat{M}$. For $w \in W^1$ let $\sigma_{\tau,w}$ be as in (2.23) and let $V_{\sigma_{\tau,w}}$ be its representation space. Then by (4.22) and Corollary 2.6 we have

$$\Theta_{\sigma,\lambda}(k_t^\tau) = e^{-t(\lambda^2 - c(\sigma) + \tau(\Omega))} \sum_{w \in W^1} (-1)^{l(w)+1} \dim(W_\sigma \otimes V_{\sigma_{\tau,w}})^M.$$

Now observe that $\dim(W_\sigma \otimes V_{\sigma_{\tau,w}})^M = 1$ if $\check{\sigma} = \sigma_{\tau,w}$ and that $\dim(W_\sigma \otimes V_{\sigma_{\tau,w}})^M = 0$ otherwise. Here $\check{\sigma}$ denotes the contragredient representation of σ. By [GW, section 3.2.5] if n is even we have $\check{\sigma} = \sigma$ and if n is odd we have $\check{\sigma} = w_0 \sigma$ for every $\sigma \in \hat{M}$. Moreover by (2.12) and (2.19) we have $c(\sigma) = c(w_0 \sigma)$ for every $\sigma \in \hat{M}$. Thus the Proposition follows from (2.27) and proposition 2.7. $\square$

5. Proof of the main results

First assume that $d = 2n$. Let τ be any finite-dimensional irreducible representation of G. Let $K(t, \tau)$ be defined by (4.4). We use (4.16) and apply the Selberg trace formula to $\operatorname{Tr} R_\Gamma(k_t^\tau)$. Since k_t^τ is K-finite and belongs to $\mathcal{C}(G)$, one easily sees that Theorem 6.7 in [Wal] applies also to k_t^τ. Thus using Lemma 4.1 we get

$$(5.1) \qquad\qquad K(t, \tau) = \operatorname{vol}(X) k_t^\tau(1).$$

Now we apply the Plancherel theorem to express $k_t^\tau(1)$ in terms of characters. It follows from the definition of k_t^τ by (4.14) and (4.10) that $k_t^\tau \in \mathcal{C}^q(G)$ for all $q > 0$. Moreover, by (4.9), k_t^τ is left and right K-finite. Therefore k_t^τ is a K-finite function in $\mathcal{C}(G)$. For such functions Harish-Chandra's Plancherel theorem holds [HC, Theorem 3]. For groups of real rank one which have a compact Cartan subgroup it is stated in [Kn1, Theorem 13.5]. By Lemma 4.1 the characters of the principal series vanish on k_t^τ. Therefore we have

$$(5.2) \qquad\qquad k_t^\tau(1) = \sum_{\pi \in \hat{G}_d} a(\pi) \Theta_\pi(k_t^\tau),$$

where $\hat{G}_d$ denotes the discrete series and $a(\pi) \in \mathbb{C}$. Recall that for a given $\nu \in \hat{K}$, there are only finitely many discrete series representations π with $[\pi|_K : \nu] \neq 0$. Since k_t^τ is K-finite, the sum on the right hand side of (5.2) is finite. Now let $\pi \in \hat{G}_d$. Then $\Theta_\pi(k_t^\tau)$ is given by (4.19). To apply this formula, we note that as K-modules we have

$$\Lambda^p \mathfrak{p}^* \cong \Lambda^{d-p} \mathfrak{p}^*, \quad p = 0, ..., d.$$

Since d is even we get

$$\sum_{p=0}^{d}(-1)^p p\Lambda^p\mathfrak{p}^* = -\sum_{p=0}^{d}(-1)^{d-p}(n-p)\Lambda^{d-p}\mathfrak{p}^* + n\sum_{p=0}^{d}(-1)^{d-p}\Lambda^{d-p}\mathfrak{p}^*$$

(5.3)

$$= -\sum_{p=0}^{d}(-1)^p p\Lambda^p\mathfrak{p}^* + n\sum_{p=0}^{d}(-1)^p\Lambda^p\mathfrak{p}^*.$$

Hence we have

(5.4)
$$\sum_{p=0}^{d}(-1)^p p\Lambda^p\mathfrak{p}^* = \frac{n}{2}\sum_{p=0}^{d}(-1)^p\Lambda^p\mathfrak{p}^*.$$

Using (4.19), we get

(5.5)
$$\Theta_\pi(k_t^\tau) = \frac{n}{2}e^{t(\pi(\Omega)-\tau(\Omega))}\sum_{p=0}^{d}(-1)^p\dim(\mathcal{H}_\pi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau)^K.$$

Let $\mathcal{H}_{\pi,K}$ be the subspace of $\mathcal{H}_\pi$ consisting of all smooth K-finite vectors. Then

$$(\mathcal{H}_{\pi,K}\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau)^K = (\mathcal{H}_\pi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau)^K.$$

So the $(\mathfrak{g}, K)$-cohomology $H^*(\mathfrak{g}, K; \mathcal{H}_{\pi,K}\otimes V_\tau)$ is computed from the Lie algebra cohomology complex $([\mathcal{H}_\pi\otimes\Lambda^*\mathfrak{p}^*\otimes V_\tau]^K, d)$ (see [BW]). Thus by the Poincaré principle we get

(5.6)
$$\Theta_\pi(k_t^\tau) = \frac{n}{2}e^{t(\pi(\Omega)-\tau(\Omega))}\sum_{p=0}^{d}(-1)^p\dim H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K}\otimes V_\tau).$$

By [BW, II.3.1, I.5.3] we have

$$H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K}\otimes V_\tau) = \begin{cases} [\mathcal{H}_\pi\otimes\Lambda^p\mathfrak{p}^*\otimes V_\tau]^K, & \pi(\Omega) = \tau(\Omega); \\ 0, & \pi(\Omega) \neq \tau(\Omega). \end{cases}$$

Together with (5.6) this implies that $\Theta_\pi(k_t^\tau)$ is independent of $t > 0$. Hence by (5.2) and (5.1) it follows that $K(t, \tau)$ is independent of $t > 0$. Let $h(\tau)$ be the constant defined by (4.5). Since $\ker\Delta_p(\tau) \cong H^p(X, E_\tau)$, $p = 0, ..., d$, it follows that $\lim_{t\to\infty} K(t, \tau) = h(\tau)$. Thus we get

$$K(t, \tau) - h(\tau) = 0.$$

Together with (4.6) this implies $T_X(\tau) = 1$, which proves Lemma 1.7. $\qquad\square$

Now assume that $d = 2n + 1$. We fix natural numbers $\tau_1, \ldots, \tau_{n+1}$ with $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_{n+1}$. Given $m \in \mathbb{N}$ let $\tau(m)$ be the representation of G with highest weight $(m + \tau_1)e_1 + \cdots + (m + \tau_{n+1})e_{n+1}$. Then by (2.9) one has $\tau(m) \neq \tau(m)_\theta$ for all m. Hence by [BW, Theorem 6.7] we have $H^p(X, E_\tau(m)) = 0$ for all $p \in \{0, \ldots, d\}$. Using (4.6) we obtain

(5.7)
$$\log T_X(\tau(m)) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K(t, \tau(m))\,dt\right)\bigg|_{s=0}.$$

Our goal is now to study the asymptotic behavior of $\log T_X(\tau(m))$ as $m \to \infty$. First we prove an auxiliary result about the spectrum of $\widetilde{\Delta}_p(\tau(m))$. To this end for every $p \in \{0, \ldots, d\}$ and every m we define an endomorphism $E_p(\tau(m))$ on $\Lambda^p \mathfrak{p}^* \otimes V_{\tau(m)}$ by

$$E_p(\tau(m)) := \tau(m)(\Omega) \operatorname{Id} - \nu_p(\tau(m))(\Omega_K).$$

Let $\widetilde{\Delta}_{\nu_p(\tau(m))}$ be the Bochner-Laplace operator on $C^\infty(\widetilde{X}, \widetilde{E}_{\nu_p(\tau(m))})$. Then (3.3) and (4.7) imply that on $C^\infty(G, \nu_p(\tau(m)))$ one has

(5.8) $$\widetilde{\Delta}_p(\tau(m)) = \widetilde{\Delta}_{\nu_p(\tau(m))} + E_p(\tau(m)).$$

The decomposition of $\nu_p(\tau(m))$ into its irreducible components induces a natural decomposition of $C^\infty(G, \nu_p(\tau(m)))$ and of $L^2(G, \nu_p(\tau(m)))$ With respect to this decomposition equation (5.8) can be rewritten as

(5.9) $$\widetilde{\Delta}_p(\tau(m)) = \bigoplus_{\substack{\nu \in \hat{K} \\ [\nu_p(\tau(m)):\nu] \neq 0}} \widetilde{\Delta}_\nu + (\tau(m)(\Omega) - \nu(\Omega_K)) \operatorname{Id},$$

where the sum is a direct sum of unbounded operators. We first study $E_p(\tau(m))$.

Lemma 5.1. *For $p = 0, \ldots, d$ and $m \in \mathbb{N}$ let*

$$C_p(m) := \inf\{\tau(m)(\Omega) - \nu(\Omega_K) \colon \nu \in \hat{K} \colon [\nu_p(\tau(m)) \colon \nu] \neq 0\}.$$

Then we have

$$C_p(m) = m^2 + O(m)$$

as $m \to \infty$.

Proof. Let $\nu_p := \Lambda^p \operatorname{Ad}^* \colon K \to \operatorname{GL}(\Lambda^p \mathfrak{p}^*)$, $p = 0, \ldots, d$. Recall that $\nu_p(\tau(m)) = \tau(m)|_K \otimes \nu_p$. Let $\nu \in \hat{K}$ with $[\nu_p(\tau(m)) : \nu] \neq 0$. Then by [Kn2, Proposition 9.72], there exists a $\nu' \in \hat{K}$ with $[\tau(m) : \nu'] \neq 0$ of highest weight $\Lambda(\nu') \in \mathfrak{b}_\mathbb{C}^*$ and a $\mu \in \mathfrak{b}_\mathbb{C}^*$ which is a weight of ν_p such that the heighest weight $\Lambda(\nu)$ of ν is given by $\mu + \Lambda(\nu')$. Now let $\nu' \in \hat{K}$ be such that $[\tau(m) : \nu'] \neq 0$. Let $\Lambda(\nu')$ be the highest weight of ν' as in (2.7). Then by Proposition 2.1 for every $j = 2, \ldots, n+1$ we have $\tau_{j-1} + m \geq k_j(\nu') \geq \tau_j + m$. Moreover, every weight $\mu \in \mathfrak{b}_\mathbb{C}^*$ of ν_p is given as

$$\mu = \pm e_{j_1} \pm \cdots \pm e_{j_p}, \quad 2 \leq j_1 < \cdots < j_p \leq n+1.$$

Thus, if $\nu \in \hat{K}$ is such that $[\nu_p(\tau(m)) : \nu] \neq 0$, the highest weight $\Lambda(\nu)$ of ν given as in (2.7) satisfies

$$\tau_{j-1} + m + 1 \geq k_j(\nu) \geq \tau_j + m - 1, \quad j \in \{2, \ldots, n+1\}.$$

The lemma follows from equation (2.17) and equation (2.18). $\qquad\square$

Corollary 5.2. *There exists $m_0 \in \mathbb{N}$ such that for all $p = 0, \ldots, d$ and $m \geq m_0$ we have*

$$\Delta_p(\tau(m)) \geq \frac{m^2}{2}.$$

Proof. By (5.8) we have

$$\Delta_p(\tau(m)) = \Delta_{\nu_p(\tau(m))} + E_p(\tau(m)).$$

Now by definition we have $\Delta_{\nu_p(\tau(m))} \geq 0$. Hence the corollary follows from Lemma 5.1. □

Proposition 5.3. *Let* $h_t^{\tau(m),p}$ *be defined by* (4.12) *and let* H_t^0 *be the heat kernel of the Laplacian* $\widetilde{\Delta}_0$ *on* $C^\infty(\widetilde{X})$. *There exist* $m_0 \in \mathbb{N}$ *and* $C_5 > 0$ *such that for all* $m \geq m_0$, $g \in G$, $t \in (0,\infty)$ *and* $p \in \{0,\ldots,d\}$ *one has*

$$\left| h_t^{\tau(m),p}(g) \right| \leq C_5 \dim(\tau(m)) e^{-t\frac{m^2}{2}} H_t^0(g).$$

Proof. Let $p \in \{0,\ldots,n\}$. Let $H_t^{\nu_p(\tau(m))}$ and $H_t^{\tau(m),p}$ be defined by (3.5) and (4.8), respectively. It follows from (5.8) and (5.9) that

$$H_t^{\tau(m),p}(g) = e^{-tE_p(\tau(m))} \circ H_t^{\nu_p(\tau(m))}(g).$$

Thus by proposition 3.1 and lemma 5.1 there exists an m_0 such that for $m \geq m_0$ one has

$$\left\| H_t^{\tau(m),p}(g) \right\| \leq e^{-t\frac{m^2}{2}} H_t^0(g).$$

Taking the trace in $\mathrm{End}(V_\tau(m) \otimes \Lambda^p \mathfrak{p}^*)$ for every $p \in \{0,\ldots,d\}$, the corollary follows. □

Now we can continue with the study of the asymptotic behavior of $T_X(\tau(m))$. From now on we assume that $m \geq m_0$. Since $\tau(m)$ is acyclic, $T_X(\tau(m))$ is metric independent [Mu1]. Especially we can rescale the metric by $\sqrt{m}$ without changing $T_X(\tau(m))$. Equivalently we can replace $\Delta_p(\tau(m))$ by $\frac{1}{m}\Delta_p(\tau(m))$. Using (5.7) we get

$$\log T_X(\tau(m)) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K\left(\frac{t}{m}, \tau(m)\right) dt \right)\Bigg|_{s=0}.$$

To continue, we split the t-integral into the integral over $[0,1]$ and the integral over $[1,\infty)$. This leads to

$$(5.10) \qquad \begin{aligned} \log T_X(\tau(m)) =& \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} K\left(\frac{t}{m}, \tau(m)\right) dt \right)\Bigg|_{s=0} \\ &+ \frac{1}{2}\int_1^\infty t^{-1} K\left(\frac{t}{m}, \tau(m)\right) dt. \end{aligned}$$

We first consider the second term on the right. Using (4.15), Proposition 5.3 and (3.9) we obtain

$$\begin{aligned} \left| K\left(\frac{t}{m}, \tau(m)\right) \right| &\leq C_5 e^{-\frac{m}{2}t} \dim(\tau(m)) \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} H_{t/m}^0(g^{-1}\gamma g)\, d\dot{g} \\ &= C_5 e^{-\frac{m}{2}t} \dim(\tau(m)) \mathrm{Tr}(e^{-\frac{t}{m}\Delta_0}). \end{aligned}$$

Furthermore, by the heat asymptotic we have

$$\mathrm{Tr}(e^{-\frac{1}{m}\Delta_0}) = C_d \mathrm{vol}(X) m^{d/2} + O\left(m^{(d-1)/2}\right)$$

as $m \to \infty$. Hence there exists $C_6 > 0$ such that

$$\left| K\left(\frac{t}{m}, \tau(m)\right) \right| \le C_6 m^{d/2} \dim(\tau(m)) e^{-\frac{m}{2}t}, \quad t \ge 1.$$

Thus we obtain

$$\left| \int_1^\infty t^{-1} K\left(\frac{t}{m}, \tau(m)\right) \, dt \right| \le C_6 m^{d/2} \dim(\tau(m)) e^{-m/4} \int_1^\infty t^{-1} e^{-\frac{m}{4}t} \, dt.$$

Using (1.2), it follows that

$$(5.11) \qquad \int_1^\infty t^{-1} K\left(\frac{t}{m}, \tau(m)\right) \, dt = O\left(e^{-m/8}\right).$$

Now we turn to the first term on the right hand side of (5.10). We need to estimate $K(t, \tau(m))$ for $0 < t \le 1$. To this end we use (4.15) to decompose $K(t, \tau(m))$ into the sum of two terms. The contribution of the identity is given by

$$I(t, \tau(m)) := \mathrm{vol}(X) k_t^{\tau(m)}(1)$$

and

$$H(t, \tau(m)) := \int_{\Gamma \backslash G} \sum_{\substack{\gamma \in \Gamma \\ \gamma \ne 1}} k_t^{\tau(m)}(g^{-1}\gamma g) \, d\dot{g}$$

is the hyperbolic contribution to $K(t, \tau(m))$. First we consider the hyperbolic contribution. Using Proposition 5.3 and Proposition 3.2, it follows that for every $m \ge m_0$ and every $t \in (0, 1]$ we have

$$\sum_{\substack{\gamma \in \Gamma \\ \gamma \ne 1}} \left| k_t^{\tau(m)}(g^{-1}\gamma g) \right| \le C_5 e^{-t\frac{m^2}{2}} \dim(\tau(m)) \sum_{\substack{\gamma \in \Gamma \\ \gamma \ne 1}} H_t^0(g^{-1}\gamma g)$$

$$\le C_6 \dim(\tau(m)) e^{-t\frac{m^2}{2}} C_0 e^{-c_0/t}.$$

Hence using (1.2) we get

$$\left| H\left(\frac{t}{m}, \tau(m)\right) \right| \le C_7 e^{-c_1 m} e^{-c_1/t}, \quad 0 < t \le 1.$$

This implies that there is $c_2 > 0$ such that

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} H\left(\frac{t}{m}, \tau(m)\right) \, dt \right)\Big|_{s=0} = \int_0^1 t^{-1} H\left(\frac{t}{m}, \tau(m)\right) \, dt = O\left(e^{-c_2 m}\right)$$

as $m \to \infty$.

It remains to consider the contribution of the identity. Again k_t^τ is a K-finite function in $\mathcal{C}(G)$ and thus by [HC, Theorem 3] Harish-Chandra's Plancherel theorem holds for k_t^τ. For groups of real rank one which do not possess a compact Cartan subgroup it is stated in [Kn1, Theorem 13.2]. Let the $\sigma_{\tau(m),k}$ and $\lambda_{\tau(m),k}$, $k = 0, \ldots, n$, be defined by (2.25) and

(2.26), respectively. Then for every k we have $\sigma_{\tau(m),k} \neq w_0 \sigma_{\tau(m),k}$. Thus using (2.22) and Proposition 4.2 we obtain

$$(5.12) \qquad I(t, \tau(m)) = 2 \operatorname{vol}(X) \sum_{k=0}^{n} (-1)^{k+1} e^{-t\lambda^2_{\tau(m),k}} \int_{\mathbb{R}} e^{-t\lambda^2} P_{\sigma_{\tau(m),k}}(i\lambda) d\lambda.$$

Here the $P_{\sigma_{\tau(m),k}}$ are the polynomials defined in (2.21). The polynomials are given explicitly as follows.

Lemma 5.4. *The Plancherel polynomial* $P_{\sigma_{\tau(m),k}}(t)$ *is given by*

$$P_{\sigma_{\tau(m),k}}(t) = -c(n)(-1)^k \dim(\tau(m)) \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{t^2 - \lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k} - \lambda^2_{\tau(m),j}},$$

where $c(n)$ *is the constant occurring in the description of the Plancherel polynomial by* (2.21).

Proof. This is proved in Bröcker's thesis [Br, p. 60]. For convenience, we recall the proof. To safe notation, put

$$\lambda_i := \lambda_{\tau(m),i}, \quad i = 0, \ldots, n.$$

By (2.26) we have

$$\Lambda(\sigma_{\tau(m),k}) + \rho_M = \sum_{i=2}^{k+1} (\tau_{i-1} + m + 2 + n - i)e_i + \sum_{i=k+2}^{n+1} (\tau_i + m + n + 1 - i)e_i$$

$$= \sum_{i=2}^{k+1} \lambda_{i-2} e_i + \sum_{i=k+2}^{n+1} \lambda_{i-1} e_i.$$

Hence by (2.21) and (2.10) we have

$$
\begin{aligned}
P_{\sigma_{\tau(m),k}}(t) = &- c(n) \prod_{j=1}^{n} \prod_{q=j+1}^{n+1} \frac{\left\langle te_1 + \sum_{i=2}^{k+1} \lambda_{i-2}e_i + \sum_{i=k+2}^{n+1} \lambda_{i-1}e_i, e_j + e_q \right\rangle}{\left\langle \sum_{l=1}^{n+1} \rho_l e_l, e_j + e_q \right\rangle} \\
&\cdot \prod_{j=1}^{n} \prod_{q=j+1}^{n+1} \frac{\left\langle te_1 + \sum_{i=2}^{k+1} \lambda_{i-2}e_i + \sum_{i=k+2}^{n+1} \lambda_{i-1}e_i, e_j - e_q \right\rangle}{\left\langle \sum_{l=1}^{n+1} \rho_l e_l, e_j - e_q \right\rangle}
\end{aligned}
$$

$$(5.13) \qquad = - c(n) \prod_{\substack{0 \le i \le n \\ i \neq k}} \left(t^2 - \lambda_i^2\right) \prod_{\substack{0 \le j < i \le n \\ i,j \neq k}} \left(\lambda_j^2 - \lambda_i^2\right) \prod_{1 \le j < i \le n+1} (\rho_j^2 - \rho_i^2)^{-1}$$

$$= - c(n)(-1)^k \prod_{0 \le j < i \le n} \frac{\lambda_j^2 - \lambda_i^2}{\rho_{j+1}^2 - \rho_{i+1}^2} \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{t^2 - \lambda_j^2}{\lambda_k^2 - \lambda_j^2}$$

$$= - c(n)(-1)^k \dim(\tau(m)) \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{t^2 - \lambda_j^2}{\lambda_k^2 - \lambda_j^2}.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Now recall that by (2.25) we have

$$\lambda_{\tau(m),i} = m + \tau_{i+1} + n - i.$$

Since $\tau_i \ge \tau_j$ for $i < j$, it follows that

$$|\lambda_{\tau(m),i}^2 - \lambda_{\tau(m),j}^2| \ge 1, \quad \forall i \neq j.$$

Using Lemma 5.4, it follows that

$$P_{\sigma_{\tau(m),k}}(t) = \sum_{i=0}^{n} a_{k,i}(m) t^{2i}$$

and there exists $C > 0$ such that

$$|a_{k,i}(m)| \le Cm^{2n+n(n+1)/2}$$

for all $k, i = 0, \dots, n$ and $m \in \mathbb{N}$. Furthermore $\lambda_{\tau(m),i} \ge m$ for $i = 0, ..., n$. Together with Lemma 5.4 it follows that there exist $C, c > 0$ such that

$$\left| I\left(\frac{t}{m}, \tau(m)\right) \right| \le C e^{-cm} e^{-ct}, \quad t \ge 1.$$

Hence we get

$$\int_1^{\infty} t^{-1} I\left(\frac{t}{m}, \tau(m)\right) dt = O\left(e^{-cm}\right).$$

This implies that we can replace the integral over $[0,1]$ by the integral over $[0,\infty)$. We need the following auxiliary proposition.

Proposition 5.5. *Let $c > 0$ and $\sigma \in \hat{M}$. For $\mathrm{Re}(s) \geq n + 1$ let*

$$E(s) := \int_0^\infty t^{s-1} e^{-tc^2} \left(\int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) \, d\lambda \right) dt.$$

Then $E(s)$ has a meromorphic continuation to $\mathbb{C}$. Moreover $E(s)$ is regular at zero and

$$E(0) = -2\pi \int_0^c P_\sigma(t) dt.$$

Proof. By (2.21) every $P_\sigma(i\lambda)$ is an even polynomial in λ. The proposition is obtained using integration by parts (see [Fr] Lemma 2 and Lemma 3). $\qquad\square$

Changing variables by $t \mapsto t \cdot m$, it follows from the proposition that

$$\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} I\left(\frac{t}{m}, \tau(m) \right) \, dt \right) \bigg|_{s=0} = \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} I\left(t, \tau(m) \right) \, dt \right) \bigg|_{s=0}.$$

By Proposition 5.5 the Mellin transform

$$\int_0^\infty t^{s-1} I(t, \tau(m)) \, dt$$

of $I(t, \tau(m))$ is a meromorphic function of $s \in \mathbb{C}$, which is regular at $s = 0$. Let $\mathcal{M}I(\tau(m))$ denote its value at $s = 0$. By (5.7), $\frac{1}{2}\mathcal{M}I(\tau(m))$ is the contribution of the identity to $\log T_X(\tau(m))$. Combining our estimates, we have proved

$$(5.14) \qquad\qquad \log T_X(\tau(m)) = \frac{1}{2}\mathcal{M}I(\tau(m)) + O\left(e^{-cm} \right)$$

as $m \to \infty$.

Next we will identify $\frac{1}{2}\mathcal{M}I(\tau(m))$ with the L^2-torsion. Recall its definition [Lo]. For $p = 0, \ldots, d$ let $\mathrm{Tr}_\Gamma(e^{-t\tilde{\Delta}_p(\tau(m))})$ denote the Γ-trace of the heat operator $e^{-t\tilde{\Delta}_p(\tau(m))}$ on $\tilde{X}$ (see [Lo]). Recall that $e^{-t\tilde{\Delta}_p(\tau(m))}$ is a convolution operator whose kernel is given by the function

$$H_t^{\tau(m),p} \colon G \to \mathrm{End}(\Lambda^p \mathfrak{p}^* \otimes V_{\tau(m)})$$

which satisfies (4.9). Let $h_t^{\tau(m),p} = \mathrm{tr}\, H_t^{\tau(m),p}$. Then it follows that

$$(5.15) \qquad\qquad \mathrm{Tr}_\Gamma \left(e^{-t\tilde{\Delta}_p(\tau(m))} \right) = \mathrm{vol}(X) h_t^{\tau(m),p}(1).$$

Let $k_t^{\tau(m)}$ be defined by (4.14). Then it follows that

$$(5.16) \qquad\qquad \sum_{p=1}^d (-1)^p p \, \mathrm{Tr}_\Gamma \left(e^{-t\tilde{\Delta}_p(\tau(m))} \right) = \mathrm{vol}(X) k_t^{\tau(m)}(1) = I(t, \tau(m)).$$

Using Proposition 5.3 it follows that for $m \geq m_0$ we have

$$\mathrm{Tr}_\Gamma(e^{-t\tilde{\Delta}_p(\tau(m))}) = O\left(e^{-t\frac{m^2}{2}} \right)$$

as $t \to \infty$. Furthermore, applying the Plancherel theorem to $h_t^{\tau(m),p}(1)$ and using (5.15), it follows that as $t \to 0$, there is an asymptotic expansion

$$\operatorname{Tr}_\Gamma \left(e^{-t\widetilde{\Delta}_p(\tau(m))} \right) \sim \sum_{j \geq 0} a_j t^{-d/2+j}.$$

This implies that

$$\int_0^\infty t^{s-1} \sum_{p=1}^d (-1)^p p \operatorname{Tr}_\Gamma (e^{-t\widetilde{\Delta}_p(\tau(m))}) \, dt$$

converges absolutely for $\operatorname{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$. Hence for $m \geq m_0$ the L^2-torsion is defined by

$$\log T_X^{(2)}(\tau(m)) = \frac{1}{2} \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{p=1}^d (-1)^p p \operatorname{Tr}_\Gamma (e^{-t\widetilde{\Delta}_p(\tau(m))}) \, dt \right) \Bigg|_{s=0}.$$

Using (5.16) it follows that

(5.17) $$\log T_X^{(2)}(\tau(m)) = \frac{1}{2} \mathcal{M}I(\tau(m)).$$

Combined with (5.14), we obtain the proof of Proposition 1.2.

To compute the L^2-torsion, we observe that by (5.12) and Proposition 5.5 we have

(5.18) $$\mathcal{M}I(\tau(m)) = 4\pi \operatorname{vol}(X) \sum_{k=0}^n (-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(\lambda) \, d\lambda.$$

Using the explicit form of the Plancherel polynomial as in the first equality of (5.13) together with $\lambda_i = \tau_{i+1} + m + n - i$, it follows that

(5.19) $$P_\tau(m) := 2\pi \sum_{k=0}^n (-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(\lambda) \, d\lambda$$

is a polynomial in m of degree $n(n+1)/2 + 1$ whose coefficients depend only on n and τ. Moreover by (5.17) we get

$$\log T_X^{(2)}(\tau(m)) = \operatorname{vol}(X) P_\tau(m),$$

which proves Proposition 1.3.

It remains to determine the leading coefficient of $P_\tau(m)$. To this end we need some additional facts about the Plancherel polynomials.

Lemma 5.6. *For every sequence* $s_0, \ldots, s_n$, $s_i \neq s_j$ *for* $i \neq j$, *one has*

$$\sum_{k=0}^n \prod_{\substack{j=0 \\ j \neq k}}^n \frac{t - s_j}{s_k - s_j} = 1.$$

Proof. The expression is a polynomial in t of order n and is equal to 1 at the $n+1$ points $s_0, \ldots, s_n$. $\square$

Corollary 5.7. *One has*

$$\sum_{k=0}^{n}(-1)^k P_{\sigma_{\tau(m),k}}(t) = -c(n)\dim(\tau(m)).$$

Proof. This follows from lemma 5.4 and lemma 5.6. □

Now we are ready to determine the leading term. By (2.25) we have

$$\lambda_{\tau(m),0} > \lambda_{\tau(m),1} > \cdots > \lambda_{\tau(m),n}.$$

Using 5.4 and Corollary 5.7, we get

(5.20)

$$\sum_{k=0}^{n}(-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)dt = \int_0^{\lambda_{\tau(m),n}} \sum_{k=0}^{n}(-1)^k P_{\sigma_{\tau(m),k}}(t)dt$$

$$- c(n)\dim(\tau(m)) \sum_{k=0}^{n-1} \int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{t^2 - \lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k} - \lambda^2_{\tau(m),j}}dt$$

$$= -c(n)\lambda_{\tau(m),n} \dim\tau(m)$$

$$- c(n)\dim\tau(m) \sum_{k=0}^{n-1} \int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{t^2 - \lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k} - \lambda^2_{\tau(m),j}}dt.$$

Now recall that by (2.25) we have

(5.21) $$\lambda_{\tau(m),k} = \tau_{k+1} + m + n - k.$$

Using (1.2) it follows that the first term on the right hand side of (5.20) equals

$$-c(n)m\dim\tau(m) + O(m^{\frac{n(n+1)}{2}}).$$

Furthermore we have

(5.22) $$\int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{t^2 - \lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k} - \lambda^2_{\tau(m),j}}\, dt = \int_{\tau_{n+1}}^{\tau_{k+1}+n-k} \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{(t+m)^2 - \lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k} - \lambda^2_{\tau(m),j}}\, dt.$$

Using (5.21), a direct computation shows that the integrand on the right hand side is bounded as $m \to \infty$. Hence we get

(5.23) $$\sum_{k=0}^{n-1} \int_{\lambda_{\tau(m),n}}^{\lambda_{\tau(m),k}} \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{t^2 - \lambda^2_{\tau(m),j}}{\lambda^2_{\tau(m),k} - \lambda^2_{\tau(m),j}} = O(1), \text{ as } m \to \infty.$$

Using (1.2), we can estimate the second term on the right hand side of (5.20) by $O(m^{\frac{n(n+1)}{2}})$. Together with (5.19) we get

$$P_\tau(m) = 2\pi \sum_{k=0}^{n} (-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)dt = -2\pi c(n)m \dim(\tau(m)) + O(m^{\frac{n(n+1)}{2}}),$$

as $m \to \infty$.

Now the proof of Corollary 1.4 follows from (5.14) with

$$(5.24) \qquad\qquad\qquad C(n) := 2\pi c(n).$$

Remark 2. We have assumed that $\tau_1, \ldots, \tau_{n+1}$ are natural numbers and that $m \in \mathbb{N}$. Clearly, we can also assume that $\tau_1, \ldots, \tau_{n+1}$ and m are in $\frac{1}{2}\mathbb{N}$. Then the obvious modifications give Theorem 1.1 and Corollary 1.4 also in this case.

Remark 3. At the end of this section we compute the Polynomial $P_\tau(m)$ in the three-dimensional case explicitly. In this case the group G can be realized as $\mathrm{SL}_2(\mathbb{C})$, K can be realized as $\mathrm{SU}(2)$ and M can be realized as $\mathrm{SO}(2)$. If $c(n)$ is the constant in (2.21), it follows from [Kn1], Theorem 11.2, and a minor correction that

$$c(n) = \frac{1}{4\pi^2}.$$

For $l \in \mathbb{N}$ write σ_l for the representation of M with highest weight le_2 as in (2.8). Then by (2.21) one has

$$P_{\sigma_l}(z) = -\frac{1}{4\pi^2}(z^2 - l^2).$$

Let $\tau_1 \in \mathbb{N}$. Then for $\tau(m) \in \hat{G}$ with highest weight $(m + \tau_1)e_1 + me_2$ as in (2.6) it follows that

$$\sum_{k=0}^{1} (-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(t)dt$$

$$= -\frac{1}{4\pi^2} \left(\int_0^{\tau_1+m+1} (t^2 - m^2)dt - \int_0^m (t^2 - (m + \tau_1 + 1)^2)dt \right)$$

$$= -\frac{1}{4\pi^2} \left(2m^2(\tau_1 + 1) + 2m(\tau_1 + 1)^2 + \frac{1}{3}(\tau_1 + 1)^3 \right).$$

Together with (5.24) this gives

$$(5.25) \qquad P_\tau(m) = -\frac{\mathrm{vol}(X)}{2\pi} \left(2m^2(\tau_1 + 1) + 2m(\tau_1 + 1)^2 + \frac{1}{3}(\tau_1 + 1)^3 \right).$$

Now if $\tau_1 = 0$, in the notation of [Mu2] the representation $\tau(m)$ corresponds to the representation τ_{2m}. Hence (5.25) is consistent with Theorem 1.1 in [Mu2].

References

[BM]   D. Barbasch, H. Moscovici, *L^2-index and the trace formula*, J. Funct. An. **53** (1983), no.2, 151-201.

[BW]   A. Borel, N. Wallach *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Princeton University Press, Princeton, 1980.

[BO]   U.Bunke, M. Olbrich, *Selberg zeta and theta functions*, Akademie Verlag, Berlin (1995)

[Br]   U. Bröcker, *Die Ruellesche Zetafunktion für G-induzierte Anosov-Fluesse*, Ph.D. thesis, Humboldt-Universität Berlin, Berlin, 1998.

[CY]   J. Cheeger, S.T. Yau, *A lower bound for the heat kernel*, Comm. Pure Appl. Math. **34** (1981), no. 4, 465-480.

[Ch]   P. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, J. Funct. An. **12** (1973), 401-414.

[Do]   H. Donnelly, *Asymptotic expansions for the compact quotients of properly discontinuous group actions*, Illinois J. Math. **23** (1979), no. 3, 485?496.

[DL]   H. Donnelly, P. Li, *Lower bounds for the eigenvalues of Riemannian manifolds*, Michigan math. J. **29** (1982), 149 - 161.

[Fr]   D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. Math. **84** (1986), no. 3, 523-540

[GW]   R. Goodman, N. Wallach *Representations and invariants of the classical groups*, Encyclopedia of Mathematics and its Applications, 68. Cambridge University Press, Cambridge, 1998

[HC]   Harish-Chandra *Harmonic analysis on real reductive groups. III. The Maass-Selberg relations and the Plancherel formula.* Ann. of Math. (2) **104** (1976), no. 1, 117-201.

[Kn1]   A. Knapp, *Representation theory of semisimple groups*, Princeton University Press, Princeton, 2001.

[Kn2]   A. Knapp, *Lie Groups Beyond and introduction*, Second Edition, Birkhäuser, Boston, 2002

[Ko]   B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Annals of Math. (2) **74** (1961), 329-378.

[Lo]   J. Lott, *Heat kernels on covering spaces and topological invariants*, J. Diff. Geom. **35** (1992), 471-510.

[MM]   Y.Matsushima and S. Murakami, *On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds*, Ann. of Math. (2) **78** (1963), 365-416.

[Mi1]   R. J. Miatello, *The Minakshisundaram-Pleijel coefficients for the vector-valued heat kernel on compact locally symmetric spaces of negative curvature.* Trans. Amer. Math. Soc. **260** (1980), no. 1, 1 - 33.

[Mi2]   R. J. Miatello, *On the Plancherel measure for linear Lie groups of rank one*, Manuscripta Math. **29** (1979), no. 2-4, 249-276.

[Mu1]   W. Müller, *Analytic torsion and R-torsion for unimodular representations*, J. Amer. Math. Soc. **6** (1993), 721-753.

[Mu2]   W. Müller, *The asymptotics of the Ray-Singer analytic torsion of hyperbolic 3 manifolds*, Preprint 2010, arXiv:1003.5168.

[MaM]   S. Marshall and W. Müller, *On the torsion in the cohomology of arithmetic hyperbolic 3-manifolds*, Preprint 2011, arXiv:1103.2262.

[RS]   D. B. Ray, I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds.* Advances in Math. **7** (1971), 145–210.

[Shu]   M.A. Shubin, *Pseudodifferential operators and spectral theory.* Springer-Verlag, Berlin, 1987.

[Wal]   N. Wallach, *On the Selberg trace formula in the case of compact quotient*, Bull. Amer. Math. Soc. **82** (1976), no.2, 171-195.

[Wo]   A. Wotzke, *Die Ruellesche Zetafunktion und die analytische Torsion hyperbolischer Mannigfaltigkeiten*, Bonner Mathematische Schriften **389** (2008).

UNIVERSITÄT BONN, MATHEMATISCHES INSTITUT, ENDENICHER ALLEE 60, D – 53115 BONN, GERMANY

E-mail address: mueller@math.uni-bonn.de

UNIVERSITÄT BONN, MATHEMATISCHES INSTITUT, ENDENICHER ALLE 60, D – 53115 BONN, GERMANY

E-mail address: pfaff@math.uni-bonn.de