ASYMPTOTICS OF AUTOMORPHIC SPECTRA AND THE TRACE FORMULA

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ABSTRACT. This paper is a survey article on the limiting behavior of the discrete spectrum of the right regular representation in $L^2(\Gamma\backslash G)$ for a lattice Γ in a reductive group G over a number field. We discuss various aspects of the Weyl law, the limit multiplicity problem and the analytic torsion.

CONTENTS

1. INTRODUCTION

Let G be a connected, linear, semisimple algebraic group over $\mathbb{Q}$. Let $\Pi(G(\mathbb{R}))$ denote the set of all equivalence classes of irreducible unitary representations of $G(\mathbb{R})$, equipped with the Fell topology [Di]. We fix a Haar measure on $G(\mathbb{R})$. Let $\Gamma \subset G(\mathbb{R})$ be a lattice, i.e., a discrete subgroup such that $\mathrm{vol}(\Gamma\backslash G(\mathbb{R})) < \infty$. Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$. Let $L^2_{\mathrm{disc}}(\Gamma\backslash G)$ be the span of all irreducible subrepresentations of R_Γ and denote by $R_{\Gamma,\mathrm{disc}}$ the restriction of R_Γ to $L^2_{\mathrm{disc}}(\Gamma\backslash G)$. Then $R_{\Gamma,\mathrm{disc}}$ decomposes discretely as

$$(1.1) \qquad R_{\Gamma,\mathrm{disc}} \cong \widehat{\bigoplus}_{\pi \in \Pi(G)} m_\Gamma(\pi)\pi,$$

where

$$m_\Gamma(\pi) = \dim \mathrm{Hom}_G(\pi, R_\Gamma) = \dim \mathrm{Hom}_G(\pi, R_{\Gamma,\mathrm{disc}})$$

is the multiplicity with which π occurs in R_Γ. The multiplicities are known to be finite under a weak reduction-theoretic assumption on (G, Γ) [OW], which is satisfied if G has no compact factors or if Γ is arithmetic. The study of the multiplicities $m_\Gamma(\pi)$ is one of the main concerns in the theory of automorphic forms. Apart from special cases like discrete series representations, one cannot hope in general to describe the multiplicity function on $\Pi(G)$ explicitly. A more feasible and interesting problem is the study of the asymptotic behavior of the multiplicities with respect to the growth of various parameters such as the level of congruence subgroups or the infinitesimal character of π. This is closely related to the study of families of automorphic forms (see [SST]).

The first problem in this context is the Weyl law. Let K be a maximal compact subgroup of G. Fix an irreducible representation σ of K. Let $\Pi(G; \sigma)$ be the subspace of all $\pi \in \Pi(G)$ such that $[\pi|_K : \sigma] > 0$. Especially, if σ_0 is the trivial representation, then $\Pi(G; \sigma_0)$ is the spherical dual $\Pi(G(\mathbb{R}))_{\mathrm{sph}}$. Given $\pi \in \Pi(G(\mathbb{R}))$, denote by $\lambda_\pi = \pi(\Omega)$ the Casimir eigenvalue of π. For $\lambda \geq 0$ let the counting function be defined by

$$(1.2) \qquad N_\Gamma(\lambda; \sigma) = \sum_{\substack{\pi \in \Pi(G;\sigma) \\ |\lambda_\pi| \leq \lambda}} m_\Gamma(\pi).$$

Then the problem is to determine the behavior of the counting function as $\lambda \to \infty$.

Another basic problem is the limit multiplicity problem, which is the study of the asymptotic behavior of the multiplicities if $\mathrm{vol}(\Gamma \backslash G(\mathbb{R})) \to \infty$. For $G = \mathrm{GL}_n$ this corresponds to the study of harmonic families of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ (see [SST]). More precisely, for a given lattice Γ define the discrete spectral measure μ_Γ on $\Pi(G)$, associated to Γ, by

$$(1.3) \qquad \mu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma \backslash G(\mathbb{R}))} \sum_{\pi \in \Pi(G(\mathbb{R}))} m_\Gamma(\pi)\delta_\pi,$$

where δ_π is the Dirac measure at π. Then the limit multiplicity problem is concerned with the study of the asymptotic behavior of μ_Γ as $\mathrm{vol}(\Gamma \backslash G(\mathbb{R})) \to \infty$. For appropriate sequences of lattices (Γ_n) one expects that the measures μ_{Γ_n} converge to the Plancherel measure μ_{pl} on $\Pi(G(\mathbb{R}))$.

There are closely related problems in topology and spectral theory. One of them concerns Betti numbers. Let K be a maximal compact subgroup of G and put $X = G/K$. Let Γ be a uniform lattice in G and let (Γ_n) be a tower of normal subgroups of Γ. Put $M = \Gamma \backslash X$ and $M_n = \Gamma_n \backslash X$, $n \in \mathbb{N}$. Then $M_n \to M$ is a sequence of finite normal coverings of M. For any topological space Y let $b_k(Y)$ denote the k-th Betti number of Y. Then

$$(1.4) \qquad \lim_{n \to \infty} \frac{b_k(M_n)}{\mathrm{vol}(M_n)} = b_k^{(2)}(X),$$

where $b_k^{(2)}(X)$ is the k-th L^2-Betti number of X. This was proved by Lück [Lu1] in the more general context of CW-complexes. In the case of locally symmetric spaces, it follows

from the results about limit multiplicities. Again, it was extended by Abert et al [AB1] to much more general sequences of uniform lattices.

A more sophisticated spectral invariant is the Ray-Singer analytic torsion $T_X(\rho)$ (see [RS]). It depends on a finite dimensional representation ρ of Γ and is defined in terms of the spectra of the Laplace operators $\Delta_p(\rho)$ on p-forms with coefficients in the flat bundle associated with ρ. Of particular interest are representations of Γ which arise as the restriction of a representation of G. For appropriate representations, called strongly acyclic, Bergeron and Venkatesch [BV] studied the asymptotic behavior of $\log T_{X_n}(\rho)$ as $n \to \infty$. One of their main results is

$$(1.5) \qquad \lim_{n\to\infty} \frac{\log T_{X_n}(\rho)}{\mathrm{vol}(X_n)} = \log T_X^{(2)}(\rho),$$

where $T_X^{(2)}(\rho)$ is the L^2-torsion [Lo], [MV]. Using the equality of analytic torsion and Reidemeister torsion [Ch], [Mu1], (1.5) implies results about the growth of the torsion subgroup in the integer homology of arithmetic groups. Let $\mathbf{G}$ be a semisimple algebraic group over $\mathbb{Q}$, $G = \mathbf{G}(\mathbb{R})$ and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ a co-compact, arithmetic subgroup. As shown in [BV], there are strongly acyclic representations ρ of G on a finite dimensional vector space V such that V contains a Γ-invariant lattice M. Let $\mathcal{M}$ be the local system of free $\mathbb{Z}$-modules over X, attached to M. Then the cohomology $H_*(X, \mathcal{M})$ of X with coefficients in $\mathcal{M}$ is a finite abelian group. Denote by $|H_*(X, \mathcal{M})|$ its order. Assume that $d = \dim(X)$ is odd. Then by [BV] one has

$$\lim_{n\to\infty} \sum_{p=1}^{d} (-1)^{p+\frac{d-1}{2}} \frac{\log |H_p(X_n, \mathcal{M})|}{[\Gamma : \Gamma_n]} = c_{M,G} \, \mathrm{vol}(X),$$

where $c_{M,G}$ is a constant that depends only on G and M. Moreover, if $\delta(G) := \mathrm{rank}\, G - \mathrm{rank}\, K = 1$, then $c_{M,G} > 0$. It is conjectured that the limit

$$(1.6) \qquad \lim_{n\to\infty} \frac{\log |H_j(X_n, \mathcal{M})|}{[\Gamma : \Gamma_n]}$$

always exists and is equal to zero, unless $\delta(G) = 1$ and $j = (d-1)/2$. In the latter case it is equal to $c_{M,G}$ times $\mathrm{vol}(X)$. The conjecture is known to be true for $G = \mathrm{SL}_2(\mathbb{C})$.

An important problem is to extend these results to the non-compact case.

2. The Arthur trace formula

The trace formula is one of the main technical tools to study the kind of spectral problems mentioned in the introduction. For $\mathbb{R}$-rank one groups the Selberg trace formula is available [Wa1]. In the higher rank case the Selberg trace formula is replaced by the Arthur trace formula.

In this section we recall Arthur's trace formula, and in particular the refinement of the spectral expansion obtained in [FLM1].

2.1. **Notation.** We will mostly use the notation of [FLM1]. Let $\mathbf{G}$ be a reductive group defined over $\mathbb{Q}$ and let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. We fix a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v = \mathbf{K}_\infty \cdot \mathbf{K}_{\mathrm{fin}}$ of $\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{R}) \cdot \mathbf{G}(\mathbb{A}_{\mathrm{fin}})$.

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $\mathbf{G}(\mathbb{R})$ and $\mathbf{K}_\infty$, respectively. Let θ be the Cartan involution of $\mathbf{G}(\mathbb{R})$ with respect to $\mathbf{K}_\infty$. It induces a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We fix an invariant bi-linear form B on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. This choice defines a Casimir operator Ω on $\mathbf{G}(\mathbb{R})$, and we denote the Casimir eigenvalue of any $\pi \in \Pi(\mathbf{G}(\mathbb{R}))$ by λ_π. Similarly, we obtain a Casimir operator $\Omega_{\mathbf{K}_\infty}$ on $\mathbf{K}_\infty$ and write λ_τ for the Casimir eigenvalue of a representation $\tau \in \Pi(\mathbf{K}_\infty)$ (cf. [BG, §2.3]). The form B induces a Euclidean scalar product $(X, Y) = -B(X, \theta(Y))$ on $\mathfrak{g}$ and all its subspaces. For $\tau \in \Pi(\mathbf{K}_\infty)$ we define $\|\tau\|$ as in [CD, §2.2].

We fix a maximal $\mathbb{Q}$-split torus $\mathbf{S}_0$ of $\mathbf{G}$ and let $\mathbf{M}_0$ be its centralizer, which is a minimal Levi subgroup defined over $\mathbb{Q}$. We assume that the maximal compact subgroup $\mathbf{K} \subset \mathbf{G}(\mathbb{A})$ is admissible with respect to $\mathbf{M}_0$ [Ar5, §1]. Denote by A_0 the identity component of $\mathbf{S}_0(\mathbb{R})$, which is viewed as a subgroup of $\mathbf{S}_0(\mathbb{A})$. We write $\mathcal{L}$ for the (finite) set of Levi subgroups containing $\mathbf{M}_0$, i.e., the set of centralizers of subtori of $\mathbf{S}_0$. Let $W_0 = N_{\mathbf{G}(\mathbb{Q})}(\mathbf{S}_0)/\mathbf{M}_0$ be the Weyl group of $(\mathbf{G}, \mathbf{S}_0)$, where $N_{\mathbf{G}(\mathbb{Q})}(H)$ is the normalizer of H in $\mathbf{G}(\mathbb{Q})$. For any $s \in W_0$ we choose a representative $w_s \in \mathbf{G}(\mathbb{Q})$. Note that W_0 acts on $\mathcal{L}$ by $s\mathbf{M} = w_s \mathbf{M} w_s^{-1}$.

Let now $\mathbf{M} \in \mathcal{L}$. We write $\mathbf{S}_M$ for the split part of the identity component of the center of $\mathbf{M}$. Set $A_M = A_0 \cap \mathbf{S}_M(\mathbb{R})$ and $W(\mathbf{M}) = N_{\mathbf{G}(\mathbb{Q})}(\mathbf{M})/\mathbf{M}$, which can be identified with a subgroup of W_0. Denote by $\mathfrak{a}_M^*$ the $\mathbb{R}$-vector space spanned by the lattice $X^*(\mathbf{M})$ of $\mathbb{Q}$-rational characters of $\mathbf{M}$ and let $\mathfrak{a}_{M,\mathbb{C}}^* = \mathfrak{a}_M^* \otimes_\mathbb{R} \mathbb{C}$ be its complexification. We write $\mathfrak{a}_M$ for the dual space of $\mathfrak{a}_M^*$, which is spanned by the co-characters of $\mathbf{S}_M$. Let $H_M : \mathbf{M}(\mathbb{A}) \to \mathfrak{a}_M$ be the homomorphism given by

$$e^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_\mathbb{A} = \prod_v |\chi(m_v)|_v$$

for any $\chi \in X^*(\mathbf{M})$ and denote by $\mathbf{M}(\mathbb{A})^1 \subset \mathbf{M}(\mathbb{A})$ the kernel of H_M. Let $\mathcal{L}(\mathbf{M})$ be the set of Levi subgroups containing $\mathbf{M}$ and $\mathcal{P}(\mathbf{M})$ the set of parabolic subgroups of $\mathbf{G}$ with Levi part $\mathbf{M}$. We also write $\mathcal{F}(\mathbf{M}) = \mathcal{F}^G(\mathbf{M}) = \coprod_{\mathbf{L} \in \mathcal{L}(\mathbf{M})} \mathcal{P}(\mathbf{L})$ for the (finite) set of parabolic subgroups of $\mathbf{G}$ containing $\mathbf{M}$. Note that $W(\mathbf{M})$ acts on $\mathcal{P}(\mathbf{M})$ and $\mathcal{F}(\mathbf{M})$ by $s\mathbf{P} = w_s \mathbf{P} w_s^{-1}$. Denote by Σ_M the set of reduced roots of $\mathbf{S}_M$ on the Lie algebra of $\mathbf{G}$. For any $\alpha \in \Sigma_M$ we denote by $\alpha^\vee \in \mathfrak{a}_M$ the corresponding co-root. Let $L^2_{\mathrm{disc}}(A_M \mathbf{M}(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{A}))$ be the discrete part of $L^2(A_M \mathbf{M}(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{A}))$, i.e., the closure of the sum of all irreducible subrepresentations of the regular representation of $\mathbf{M}(\mathbb{A})$. We denote by $\Pi_{\mathrm{disc}}(\mathbf{M}(\mathbb{A}))$ the countable set of equivalence classes of irreducible unitary representations of $\mathbf{M}(\mathbb{A})$ which occur in the decomposition of $L^2_{\mathrm{disc}}(A_M \mathbf{M}(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{A}))$ into irreducible representations.

For any $\mathbf{L} \in \mathcal{L}(\mathbf{M})$ we identify $\mathfrak{a}_L^*$ with a subspace of $\mathfrak{a}_M^*$. We denote by $\mathfrak{a}_M^L$ the annihilator of $\mathfrak{a}_L^*$ in $\mathfrak{a}_M$. We set

$$\mathcal{L}_1(\mathbf{M}) = \{\mathbf{L} \in \mathcal{L}(\mathbf{M}) : \dim \mathfrak{a}_M^L = 1\}$$

and

$$\mathcal{F}_1(\mathbf{M}) = \bigcup_{\mathbf{L} \in \mathcal{L}_1(M)} \mathcal{P}(\mathbf{L}).$$

Note that the restriction of the scalar product $(\cdot, \cdot)$ on $\mathfrak{g}$ defined above gives $\mathfrak{a}_{M_0}$ the structure of a Euclidean space. In particular, this fixes Haar measures on the spaces $\mathfrak{a}_M^L$ and their duals $(\mathfrak{a}_M^L)^*$. We follow Arthur in the corresponding normalization of Haar measures on the groups $\mathbf{M}(\mathbb{A})$ ([Ar1, §1]).

2.2. Intertwining operators.
The main ingredient of the spectral side of the Arthur trace formula are logarithmic derivatives of intertwining operators. We shall now describe the structure of the intertwining operators.

Let $\mathbf{P} \in \mathcal{P}(\mathbf{M})$. We write $\mathfrak{a}_P = \mathfrak{a}_M$. Let $\mathbf{U}_P$ be the unipotent radical of $\mathbf{P}$ and $\mathbf{M}_P$ the unique $\mathbf{L} \in \mathcal{L}(\mathbf{M})$ (in fact the unique $\mathbf{L} \in \mathcal{L}(\mathbf{M}_0)$) such that $\mathbf{P} \in \mathcal{P}(\mathbf{L})$. Denote by $\Sigma_P \subset \mathfrak{a}_P^*$ the set of reduced roots of $\mathbf{S}_M$ on the Lie algebra $\mathfrak{u}_P$ of $\mathbf{U}_P$. Let Δ_P be the subset of simple roots of $\mathbf{P}$, which is a basis for $(\mathfrak{a}_P^G)^*$. Write $\mathfrak{a}_{P,+}^*$ for the closure of the Weyl chamber of $\mathbf{P}$, i.e.

$$\mathfrak{a}_{P,+}^* = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Sigma_P\} = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta_P\}.$$

Denote by δ_P the modulus function of $\mathbf{P}(\mathbb{A})$. Let $\bar{\mathcal{A}}_2(P)$ be the Hilbert space completion of

$$\{\phi \in C^\infty(\mathbf{M}(\mathbb{Q})\mathbf{U}_P(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})) : \delta_P^{-\frac{1}{2}}\phi(\cdot x) \in L^2_{\text{disc}}(A_M\mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A})), \ \forall x \in \mathbf{G}(\mathbb{A})\}$$

with respect to the inner product

$$(\phi_1, \phi_2) = \int_{A_M\mathbf{M}(\mathbb{Q})\mathbf{U}_P(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)} \ dg.$$

Let $\alpha \in \Sigma_M$. We say that two parabolic subgroups $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(\mathbf{M})$ are *adjacent* along α, and write $\mathbf{P}|^\alpha\mathbf{Q}$, if $\Sigma_P \cap -\Sigma_Q = \{\alpha\}$. Alternatively, $\mathbf{P}$ and $\mathbf{Q}$ are adjacent if the closure $\overline{\mathbf{PQ}}$ of $\mathbf{PQ}$ belongs to $\mathcal{F}_1(\mathbf{M})$. Any $\mathbf{R} \in \mathcal{F}_1(\mathbf{M})$ is of the form $\overline{\mathbf{PQ}}$ for a unique unordered pair $\{\mathbf{P}, \mathbf{Q}\}$ of parabolic subgroups in $\mathcal{P}(\mathbf{M})$, namely $\mathbf{P}$ and $\mathbf{Q}$ are the maximal parabolic subgroups of $\mathbf{R}$, and $\mathbf{P}|^\alpha\mathbf{Q}$ with $\alpha^\vee \in \Sigma_P^\vee \cap \mathfrak{a}_M^R$. Switching the order of $\mathbf{P}$ and $\mathbf{Q}$ changes α to $-\alpha$.

For any $\mathbf{P} \in \mathcal{P}(\mathbf{M})$ let $H_P \colon \mathbf{G}(\mathbb{A}) \to \mathfrak{a}_P$ be the extension of H_M to a left $\mathbf{U}_P(\mathbb{A})$- and right $\mathbf{K}$-invariant map. Denote by $\mathcal{A}^2(P)$ the dense subspace of $\bar{\mathcal{A}}^2(P)$ consisting of its $\mathbf{K}$- and $\mathfrak{z}$-finite vectors, where $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathfrak{g}\otimes\mathbb{C}$. That is, $\mathcal{A}^2(P)$ is the space of automorphic forms ϕ on $\mathbf{U}_P(\mathbb{A})\mathbf{M}(F)\backslash\mathbf{G}(\mathbb{A})$ such that $\delta_P^{-\frac{1}{2}}\phi(\cdot k)$ is a square-integrable automorphic form on $A_M M(F)\backslash M(\mathbb{A})$ for all $k \in \mathbf{K}$. Let $\rho(P, \lambda)$, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, be the induced representation of $G(\mathbb{A})$ on $\bar{\mathcal{A}}^2(P)$ given by

$$(\rho(P, \lambda, y)\phi)(x) = \phi(xy)e^{\langle\lambda, H_P(xy) - H_P(x)\rangle}.$$

It is isomorphic to $\text{Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}\left(L^2_{\text{disc}}(A_M\mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A})) \otimes e^{\langle\lambda, H_M(\cdot)\rangle}\right).$

For $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(\mathbf{M})$ let

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

be the standard *intertwining operator* [Ar3, §1], which is the meromorphic continuation in λ of the integral

$$[M_{Q|P}(\lambda)\phi](x) = \int_{\mathbf{U}_Q(\mathbb{A}) \cap \mathbf{U}_P(\mathbb{A}) \backslash \mathbf{U}_Q(\mathbb{A})} \phi(nx) e^{\langle \lambda, H_P(nx) - H_Q(x) \rangle} \, dn, \quad \phi \in \mathcal{A}^2(P), \ x \in \mathbf{G}(\mathbb{A}).$$

These operators satisfy the following properties.

(1) $M_{P|P}(\lambda) \equiv \mathrm{Id}$ for all $\mathbf{P} \in \mathcal{P}(\mathbf{M})$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$.
(2) For any $\mathbf{P}, \mathbf{Q}, \mathbf{R} \in \mathcal{P}(\mathbf{M})$ we have $M_{R|P}(\lambda) = M_{R|Q}(\lambda) \circ M_{Q|P}(\lambda)$ for all $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. In particular, $M_{Q|P}(\lambda)^{-1} = M_{P|Q}(\lambda)$.
(3) $M_{Q|P}(\lambda)^* = M_{P|Q}(-\overline{\lambda})$ for any $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(\mathbf{M})$ and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. In particular, $M_{Q|P}(\lambda)$ is unitary for $\lambda \in i\mathfrak{a}^*_M$.
(4) If $\mathbf{P}|^{\alpha}\mathbf{Q}$ then $M_{Q|P}(\lambda)$ depends only on $\langle \lambda, \alpha^\vee \rangle$.

Given $\pi \in \Pi_{\mathrm{disc}}(\mathbf{M}(\mathbb{A}))$, let $\mathcal{A}^2_\pi(P)$ be the space of all $\phi \in \mathcal{A}^2(P)$ for which the function $x \in \mathbf{M}(\mathbb{A}) \mapsto \delta_P^{-\frac{1}{2}}\phi(xg)$, $g \in \mathbf{G}(\mathbb{A})$, belongs to the π-isotypic subspace $L^2(A_M\mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A}))$. For any $\mathbf{P} \in \mathcal{P}(\mathbf{M})$ we have a canonical isomorphism of $\mathbf{G}(\mathbb{A}_f) \times (\mathfrak{g}_\mathbb{C}, K_\infty)$-modules

$$j_P : \mathrm{Hom}(\pi, L^2(A_M\mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A}))) \otimes \mathrm{Ind}^{\mathbf{G}(\mathbb{A})}_{\mathbf{P}(\mathbb{A})}(\pi) \to \mathcal{A}^2_\pi(P).$$

If we fix a unitary structure on π and endow $\mathrm{Hom}(\pi, L^2(A_M\mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A})))$ with the inner product $(A, B) = B^\star A$ (which is a scalar operator on the space of π), the isomorphism j_P becomes an isometry.

Suppose that $\mathbf{P}|^{\alpha}\mathbf{Q}$. The operator $M_{Q|P}(\pi, s) := M_{Q|P}(s\varpi)|_{\mathcal{A}^2_\pi(P)}$, where $\varpi \in \mathfrak{a}\star_M$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$, admits a normalization by a global factor $n_\alpha(\pi, s)$ which is a meromorphic function in s. We may write

$$(2.1) \qquad M_{Q|P}(\pi, s) \circ j_P = n_\alpha(\pi, s) \cdot j_Q \circ (\mathrm{Id} \otimes R_{Q|P}(\pi, s))$$

where $R_{Q|P}(\pi, s) = \otimes_v R_{Q|P}(\pi_v, s)$ is the product of the locally defined normalized intertwining operators and $\pi = \otimes_v \pi_v$ [Ar3, §6], (cf. [Mu6, (2.17)]). In many cases, the normalizing factors can be expressed in terms automorphic L-functions [Sha1], [Sha2]. For example, let $\mathbf{G} = \mathrm{GL}(n)$. Then the global normalizing factors n_α can be expressed in terms of Rankin-Selberg L-functions and the known properties of these functions, which are collected and analyzed in [Mu5, §§4,5]. Write $\mathbf{M} \simeq \prod_{i=1}^r \mathrm{GL}(n_i)$, where the root α is trivial on $\prod_{i \geq 3} \mathrm{GL}(n_i)$, and let $\pi \simeq \otimes \pi_i$ with representations $\pi_i \in \Pi_{\mathrm{disc}}(\mathrm{GL}(n_i, \mathbb{A}))$. Let $L(s, \pi_1 \times \tilde{\pi}_2)$ be the completed Rankin-Selberg L-function associated to π_1 and π_2. It satisfies the functional equation

$$(2.2) \qquad L(s, \pi_1 \times \tilde{\pi}_2) = \epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2) N(\pi_1 \times \tilde{\pi}_2)^{\frac{1}{2}-s} L(1 - s, \tilde{\pi}_1 \times \pi_2)$$

where $|\epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2)| = 1$ and $N(\pi_1 \times \tilde{\pi}_2) \in \mathbb{N}$ is the conductor. Then we have

$$(2.3) \qquad n_\alpha(\pi, s) = \frac{L(s, \pi_1 \times \tilde{\pi}_2)}{\epsilon(\frac{1}{2}, \pi_1 \times \tilde{\pi}_2) N(\pi_1 \times \tilde{\pi}_2)^{\frac{1}{2}-s} L(s+1, \pi_1 \times \tilde{\pi}_2)}.$$

2.3. The trace formula. Arthur's trace formula gives two alternative expressions for a distribution J on $G(\mathbb{A})^1$. Note that this distribution depends on the choice of M_0 and $\mathbf{K}$. For $h \in C_c^\infty(G(\mathbb{A})^1)$, Arthur defines $J(h)$ as the value at the point $T = T_0$ specified in [Ar5, Lemma 1.1] of a polynomial $J^T(h)$ on $\mathfrak{a}_{M_0}$ of degree at most $d_0 = \dim \mathfrak{a}_{M_0}^G$. Here, the polynomial $J^T(h)$ depends in addition on the choice of a parabolic subgroup $P_0 \in \mathcal{P}(M_0)$. Consider the equivalence relation on $G(\mathbb{Q})$ defined by $\gamma \sim \gamma'$ whenever the semisimple parts of γ and γ' are $G(\mathbb{Q})$-conjugate. Let $\mathcal{O}$ be the set of the resulting equivalence classes (which are in bijection with conjugacy classes of semisimple elements). The coarse geometric expansion [Ar1] is

$$(2.4) \qquad J^T(h) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^T(h),$$

where the summands $J_{\mathfrak{o}}^T(h)$ are again polynomials in T of degree at most d_0. Write $J_{\mathfrak{o}}(h) = J_{\mathfrak{o}}^{T_0}(h)$, which depends only on M_0 and $\mathbf{K}$. Then $J_{\mathfrak{o}}(h) = 0$ if the support of h is disjoint from all conjugacy classes of $G(\mathbb{A})$ intersecting $\mathfrak{o}$ (cf. [Ar6, Theorem 8.1]). By [ibid., Lemma 9.1] (together with the descent formula of [Ar5, §2]), for each compact set $\Omega \subset G(\mathbb{A})^1$ there exists a finite subset $\mathcal{O}(\Omega) \subset \mathcal{O}$ such that for h supported in Ω only the terms with $\mathfrak{o} \in \mathcal{O}(\Omega)$ contribute to (2.4). In particular, the sum is always finite. When $\mathfrak{o}$ consists of the unipotent elements of $G(\mathbb{Q})$, we write $J_{\text{unip}}^T(h)$ for $J_{\mathfrak{o}}^T(h)$.

We now turn to the spectral side. Let $L \supset M$ be Levi subgroups in $\mathcal{L}$, $P \in \mathcal{P}(M)$, and let $m = \dim \mathfrak{a}_L^G$ be the co-rank of L in G. Denote by $\mathfrak{B}_{P,L}$ the set of m-tuples $\underline{\beta} = (\beta_1^\vee, \ldots, \beta_m^\vee)$ of elements of $\Sigma_P^\vee$ whose projections to $\mathfrak{a}_L$ form a basis for $\mathfrak{a}_L^G$. For any $\underline{\beta} = (\beta_1^\vee, \ldots, \beta_m^\vee) \in \mathfrak{B}_{P,L}$ let $\text{vol}(\underline{\beta})$ be the co-volume in $\mathfrak{a}_L^G$ of the lattice spanned by $\underline{\beta}$ and let

$$\begin{aligned}
\Xi_L(\underline{\beta}) &= \{(\mathbf{Q}_1, \ldots, \mathbf{Q}_m) \in \mathcal{F}_1(M)^m : \quad \beta_i^\vee \in \mathfrak{a}_M^{Q_i}, \ i = 1, \ldots, m\} \\
&= \{(\overline{P_1 P_1'}, \ldots, \overline{P_m P_m'}) : \quad P_i|^{\beta_i} P_i', \ i = 1, \ldots, m\}.
\end{aligned}$$

For any smooth function f on $\mathfrak{a}_M^*$ and $\mu \in \mathfrak{a}_M^*$ denote by $D_\mu f$ the directional derivative of f along $\mu \in \mathfrak{a}_M^*$. For a pair $P_1|^\alpha P_2$ of adjacent parabolic subgroups in $\mathcal{P}(M)$ write

$$\delta_{P_1|P_2}(\lambda) = M_{P_2|P_1}(\lambda) D_\varpi M_{P_1|P_2}(\lambda) : \mathcal{A}^2(P_2) \to \mathcal{A}^2(P_2),$$

where $\varpi \in \mathfrak{a}_M^*$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$. [1] Equivalently, writing $M_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)$ for a meromorphic function Φ of a single complex variable, we have

$$\delta_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)^{-1} \Phi'(\langle \lambda, \alpha^\vee \rangle).$$

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[1]Note that this definition differs slightly from the definition of $\delta_{P_1|P_2}$ in [FL1].

For any m-tuple $\mathcal{X} = (Q_1, \dots, Q_m) \in \Xi_L(\underline{\beta})$ with $Q_i = \overline{P_i P_i'}$, $P_i|^{\beta_i} P_i'$, denote by $\Delta_{\mathcal{X}}(P, \lambda)$ the expression

$$\frac{\mathrm{vol}(\underline{\beta})}{m!} M_{P_1'|P}(\lambda)^{-1} \delta_{P_1|P_1'}(\lambda) M_{P_1'|P_2'}(\lambda) \cdots \delta_{P_{m-1}|P_{m-1}'}(\lambda) M_{P_{m-1}'|P_m'}(\lambda) \delta_{P_m|P_m'}(\lambda) M_{P_m'|P}(\lambda).$$

In [FLM1, pp. 179-180] we define a (purely combinatorial) map $\mathcal{X}_L : \mathfrak{B}_{P,L} \to \mathcal{F}_1(M)^m$ with the property that $\mathcal{X}_L(\underline{\beta}) \in \Xi_L(\underline{\beta})$ for all $\underline{\beta} \in \mathfrak{B}_{P,L}$.[2]

For any $s \in W(M)$ let L_s be the smallest Levi subgroup in $\mathcal{L}(M)$ containing w_s. We recall that $\mathfrak{a}_{L_s} = \{H \in \mathfrak{a}_M \mid sH = H\}$. Set

$$\iota_s = |\det(s-1)_{\mathfrak{a}_M^{L_s}}|^{-1}.$$

For $P \in \mathcal{F}(M_0)$ and $s \in W(M_P)$ let $M(P, s) : \mathcal{A}^2(P) \to \mathcal{A}^2(P)$ be as in [Ar3, p. 1309]. $M(P, s)$ is a unitary operator which commutes with the operators $\rho(P, \lambda, h)$ for $\lambda \in i\mathfrak{a}_{L_s}^*$. Finally, we can state the refined spectral expansion.

Theorem 1 ([FLM1]). *For any $h \in C_c^\infty(G(\mathbb{A})^1)$ the spectral side of Arthur's trace formula is given by*

$$(2.5) \qquad\qquad\qquad J(h) = \sum_{[M]} J_{\mathrm{spec},M}(h),$$

M ranging over the conjugacy classes of Levi subgroups of G (represented by members of $\mathcal{L}$), where

$$(2.6) \quad J_{\mathrm{spec},M}(h) = \frac{1}{|W(M)|} \sum_{s \in W(M)} \iota_s \sum_{\underline{\beta} \in \mathfrak{B}_{P,L_s}} \int_{i(\mathfrak{a}_{L_s}^G)^*} \mathrm{tr}(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P, \lambda) M(P, s) \rho(P, \lambda, h)) \, d\lambda$$

with $P \in \mathcal{P}(M)$ arbitrary. The operators are of trace class and the integrals are absolutely convergent.

Note that the term corresponding to $M = G$ is $J_{\mathrm{spec},G}(h) = \mathrm{tr}\, R_{\mathrm{disc}}(h)$. Next assume that M is the Levi subgroup of a maximal parabolic subgroup P. Furthermore, let $L = M$. Let $\bar{P}$ be the opposite parabolic subgroup to P. Then up to a constant, the contribution to the spectral side is given by

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)} \int_{i\mathfrak{a}^\star} \mathrm{tr}(M_{\bar{P}|P}(\pi, \lambda)^{-1} \frac{d}{dz} M_{\bar{P}|P}(\pi, \lambda) M(P, s) \rho(P, \pi, \lambda, h)) \, d\lambda.$$

Now assume that $G = \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2)$. Let $h \in \mathrm{SL}(2, \mathbb{R}))$ be bi-K-invariant. Let $C(s)$ be the scattering matrix.

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[2]The map $\mathcal{X}_L$ depends in fact on the additional choice of a vector $\underline{\mu} \in (\mathfrak{a}_M^*)^m$ which does not lie in an explicit finite set of hyperplanes. For our purposes, the precise definition of $\mathcal{X}_L$ is immaterial.

3. The Weyl law

The Weyl law is concerned with the study of the asymptotic behavior of the counting function (1.2) as $\lambda \to \infty$. This is the first problem which needs to be solved in order to be able to pursue a deeper study of the cuspidal automorphic spectrum. For example, the study of statistical properties of the automorphic spectrum requires first of all to know that the spectrum is infinite and has the right asymptotic properties. This, in particular, concerns the study of families of automorphic forms (see [SST]).

The investigation of the asymptotic behavior of the counting function (1.2) is closely related to the study of the counting function of the eigenvalues of the Laplace operator on a compact Riemannian manifold [DG]. Let $\widetilde{X} = G/K$. It can be equipped with a G-invariant metric which is unique up to scaling. Let $X = \Gamma\backslash\widetilde{X}$. Assume that Γ is torsion free. Then X is a complete Riemannian manifold of finite volume. Let $\sigma \in \widehat{K}$ and let $\widetilde{E}_\sigma \to \widetilde{X}$ be the homogeneous vector bundle associated to σ, which is equipped with the invariant Hermitian metric induced by σ. Let $E_\sigma = \Gamma\backslash\widetilde{E}_\sigma$ be the corresponding locally homogeneous vector bundle over X. Let ∇^σ be the connection in E_σ induced by the canonical connection in $\widetilde{E}_\sigma$. Let $\Delta_\sigma = (\nabla^\sigma)^\star\nabla^\sigma$ be the Bochner-Laplace operator, acting in $C^\infty(X, E_\sigma)$. It is an elliptic, second order, formally self-adjoint differential operator of Laplace type, i.e., its principal symbol is given by $\|\xi\|_x^2 \operatorname{Id}_{E_{\sigma,x}}$. The Bochner-Laplace operator is related to the Casimir operator $R_\Gamma(\Omega)$ by

$$(3.1) \qquad \Delta_\sigma = -R_\Gamma(\Omega) + \lambda_\sigma \operatorname{Id},$$

where λ_σ is the Casimir eigenvalue of σ. Assume that X is compact. Then Δ_σ has a pure discrete spectrum consisting of a sequence of eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty$ of finite multiplicities. Let

$$N_{\Gamma,\sigma}(\lambda) = \#\{j \colon \lambda_j \le \lambda\}$$

be the counting function of the eigenvalues, where eigenvalues are counted with there multiplicity. By (3.1) the counting function (1.2) has the same asymptotic behavior as $N_{\Gamma,\sigma}(\lambda)$. The Weyl law for $N_{\Gamma,\sigma}(\lambda)$ can be established by standard methods. For example, for a weak version, which means with no estimation of the remainder term, on can use the asymptotic expansion of the trace of the heat operator $e^{-t\Delta_\sigma}$. Thus if Γ is co-compact we get from these general methods the following formula for the asymptotic behavior of the counting function. Let $d = \dim X$. As $\lambda \to \infty$ we have

$$(3.2) \qquad N_\Gamma(\lambda; \sigma) = \frac{\dim(\sigma)\operatorname{vol}(\Gamma\backslash G/K)}{(4\pi)^{d/2}\mathbf{\Gamma}(d/2+1)}\lambda^{d/2} + o(\lambda^{d/2}),$$

where $\mathbf{\Gamma}(s)$ denotes the Gamma function.

If Γ is not co-compact, then Δ_σ has a nonempty continuous spectrum which consists of a half-line $[c, \infty)$ for some $c \ge 0$. This makes it much more difficult to study the discrete spectrum of this operator, because almost all eigenvalues, if they exist, will be embedded into the continuous spectrum. It is well know from mathematical physics that embedded

eigenvalues are unstable under perturbations. One of the basic tools to study the cuspidal automorphic spectrum is the trace formula.

3.1. Hyperbolic surfaces.

In the non-compact case, a general Weyl law was first derived by Selberg for a hyperbolic surface $X = \Gamma \backslash \mathbb{H}$ of finite area, where $\mathbb{H} = \mathrm{SL}(2, \mathbb{R}) / \mathrm{SO}(2)$ is the upper half-plane. We briefly recall the method which is based on the trace formula. It illustrates the basic idea which is also used in the higher rank case.

Let $\Delta = d^\star d$ be the Laplace operator with respect to the hyperbolic metric. Then Δ, regarded as operator in $L^2(X)$ with domain $C^\infty(X)$, is essentially self-adjoint. The spectrum of Δ is the union of a pure point spectrum and the absolutely continuous spectrum. The pure point spectrum consists of a sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

of finite multiplicities. If X is noncompact then, in general, we only know that λ_0 exists. We slightly change the definition of the counting function by

$$N_\Gamma(\lambda) := \#\{j \colon \sqrt{\lambda_j} \leq \lambda\}.$$

The new terms in the trace formula, which are due to the non-compactness of $\Gamma \backslash \mathbb{H}$ arise from the parabolic conjugacy classes in Γ and the Eisenstein series. Let us recall the definition of Eisenstein series. Let $a_1, ..., a_m \in \mathbb{R} \cup \{\infty\}$ be representatives of the Γ-conjugacy classes of parabolic fixed points of Γ. The a_i's are called *cusps*. For each a_i let Γ_{a_i} be the stabilizer of a_i in Γ. Choose $\sigma_i \in \mathrm{SL}(2, \mathbb{R})$ such that

$$\sigma_i(\infty) = a_i, \quad \sigma_i^{-1} \Gamma_{a_i} \sigma_i = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Then the Eisenstein series $E_i(z, s)$ associated to the cusp a_i is defined as

$$(3.3) \qquad E_i(z, s) = \sum_{\gamma \in \Gamma_{a_i} \backslash \Gamma} \mathrm{Im}(\sigma_i^{-1} \gamma z)^s, \quad \mathrm{Re}(s) > 1.$$

The series converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > 1$ and it satisfies the following properties.

1) $E_i(\gamma z, s) = E_i(z, s)$ for all $\gamma \in \Gamma$.
2) As a function of s, $E_i(z, s)$ admits a meromorphic continuation to $\mathbb{C}$ which is regular on the line $\mathrm{Re}(s) = 1/2$.
3) $E_i(z, s)$ is a smooth function of z and satisfies $\Delta_z E_i(z, s) = s(1-s) E_i(z, s)$.

The contribution of the Eisenstein series to the Selberg trace formula is given by their zeroth Fourier coefficients of the Fourier expansion in the cusps. The zeroth Fourier coefficient of the Eisenstein series $E_k(z, s)$ in the cusp a_l is given by

$$\int_0^1 E_k(\sigma_l(x + iy), s) \, dx = \delta_{kl} y^s + C_{kl}(s) y^{1-s},$$

where δ_{kl} is Kronecker's delta function and $C_{kl}(s)$ is a meromorphic function of $s \in \mathbb{C}$. Put

$$C(s) := (C_{kl}(s))_{k,l=1}^m .$$

This is the so called *scattering matrix*. Let $g \in C_c^\infty(\mathbb{R})$ and let $h = \hat{g}$ be the Fourier transform of g. Let $\phi(s) := \det C(s)$. Denote by $\{\gamma\}$ the hyperbolic Γ-conjugacy classes. For every hyperbolic element γ, denote by γ_0 the primitive hyperbolic element such that $\gamma = \gamma_0^k$ for some $k \in \mathbb{N}$. Every nontrivial hyperbolic conjugacy class $\{\gamma\}$ corresponds to a unique closed geodesic c_γ. Let $l(\gamma)$ denote its length. Write the eigenvalues as

$$\lambda_j = \frac{1}{4} + r_j^2, \quad r_j \in i\mathbb{R} \cup (1/2, 1].$$

Then the trace formula is the following identity.

(3.4)
$$\sum_j h(r_j) - \frac{1}{4\pi} \int_{-\infty}^\infty h(r) \frac{\phi'}{\phi}(1/2 + ir) \, dr + \frac{1}{4}\phi(1/2)h(0)$$
$$= \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{4\pi} \int_{\mathbb{R}} h(r) r \tanh(\pi r) \, dr + \sum_{\{\gamma\}} \frac{l(\gamma_0)}{2\sinh\left(\frac{l(\gamma)}{2}\right)} g(l(\gamma))$$
$$- \frac{m}{2\pi} \int_{-\infty}^\infty h(r) \frac{\mathbf{\Gamma}'}{\mathbf{\Gamma}}(1 + ir) dr + \frac{m}{4}h(0) - m \ln 2 \, g(0).$$

The left hand side is the spectral side, which contains all terms associated with the spectrum and the right hand side is the geometric side. The trace formula holds for every discrete subgroup $\Gamma \subset \text{SL}(2, \mathbb{R})$ with co-finite area. In analogy to the counting function of the eigenvalues we introduce the winding number

(3.5)
$$M_\Gamma(\lambda) = -\frac{1}{4\pi} \int_{-\lambda}^\lambda \frac{\phi'}{\phi}(1/2 + ir) \, dr,$$

which measures the continuous spectrum. Using the cut-off Laplacian of Lax-Phillips [CV] one can deduce the following elementary bounds

(3.6)
$$N_\Gamma(\lambda) \ll \lambda^2, \quad M_\Gamma(\lambda) \ll \lambda^2, \quad \lambda \geq 1.$$

These bounds imply that the the trace formula (3.4) holds for a larger class of functions. In particular, it can be applied to the heat kernel k_t. Its spherical Fourier transform equals $h_t(r) = e^{-t(1/4+r^2)}$, $t > 0$. If we insert h_t into the trace formula we get the following asymptotic expansion as $t \to 0$.

(3.7)
$$\sum_j e^{-t\lambda_j} - \frac{1}{4\pi} \int_{\mathbb{R}} e^{-t(1/4+r^2)} \frac{\phi'}{\phi}(1/2 + ir) \, dr$$
$$= \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{4\pi t} + \frac{a \log t}{\sqrt{t}} + \frac{b}{\sqrt{t}} + O(1)$$

for certain constants $a, b \in \mathbb{R}$. Using [Se1, (8.8), (8.9)] it follows that the winding number $M_\Gamma(\lambda)$ is monotonic increasing for $\lambda \gg 0$. Therefore we can apply a Tauberian theorem to

(3.7) and we get the following Weyl law, established by Selberg [Se1]. As $\lambda \to \infty$ we have

$$(3.8) \qquad N_\Gamma(\lambda) + M_\Gamma(\lambda) \sim \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{4\pi} \lambda^2.$$

In general, we cannot estimate separately the counting function and the winding number. For congruence subgroups, however, the entries of the scattering matrix can be expressed in terms of well-known analytic functions. For $\Gamma(N)$ the determinant of the scattering matrix $\phi(s)$ has been computed by Huxley [Hu]. It has the form

$$(3.9) \qquad \phi(s) = (-1)^l A^{1-2s} \left(\frac{\mathbf{\Gamma}(1-s)}{\mathbf{\Gamma}(s)} \right)^k \prod_\chi \frac{L(2-2s, \bar\chi)}{L(2s, \chi)},$$

where $k, l \in \mathbb{Z}$, $A > 0$, the product runs over Dirichlet characters χ to some modulus dividing N and $L(s, \chi)$ is the Dirichlet L-function with character χ. Especially for $\Gamma(1)$ we have

$$(3.10) \qquad \phi(s) = \sqrt{\pi} \frac{\mathbf{\Gamma}(s-1/2)\zeta(2s-1)}{\mathbf{\Gamma}(s)\zeta(2s)},$$

where $\zeta(s)$ denotes the Riemann zeta function.

Using Stirling's approximation formula to estimate the logarithmic derivative of the Gamma function and standard estimations for the logarithmic derivative of Dirichlet L-functions on the line $\text{Re}(s) = 1$ [Pr, Chapt V, Theormem 7.1], we get

$$(3.11) \qquad \frac{\phi'}{\phi}(1/2 + ir) = O(\log(4 + |r|)), \quad |r| \to \infty.$$

This implies that

$$(3.12) \qquad M_{\Gamma(N)}(\lambda) \ll \lambda \log \lambda.$$

Together with (3.8) we obtain Weyl's law for the point spectrum of the Laplacian on $X(N) = \Gamma(N) \backslash \mathbb{H}$:

$$(3.13) \qquad N_{\Gamma(N)}(\lambda) \sim \frac{\text{Area}(X(N))}{4\pi} \lambda^2, \quad \lambda \to \infty,$$

which is due to Selberg [Se1, p.668]. A similar formula holds for other congruence groups such as $\Gamma_0(N)$. In particular, (3.13) implies that for congruence groups there exist infinitely many linearly independent Maass cusp forms.

By a more sophisticated use of the Selberg trace formula one can estimate the remainder term (see [Mu3]). For congruence subgroups one gets

Theorem 3.1. *For every $N \in \mathbb{N}$ we have*

$$(3.14) \qquad N_{\Gamma(N)}(\lambda) = \frac{\text{Area}(X(N))}{4\pi} \lambda^2 + O(\lambda \log \lambda)$$

as $\lambda \to \infty$.

A finite area hyperbolic surface for which the Weyl law holds is called by Sarnak *essentially cuspidal*. Now it is strongly believed that essential cuspidality is limited to special arithmetic surfaces. This is based on work by Phillips and Sarnak who studied the behavior of the discrete spectrum when Γ is deformed in the corresponding Teichmüller space. We refer to [Sa1] for a detailed discussion of their method. This led Phillips and Sarnak to the following conjectures.

Conjecture 1. *1) The generic Γ in a given Teichmüller space of finite area hyperbolic surfaces is not essentially cuspidal.*

2) Except for the Teichmüller space of the once punctured torus, the generic Γ has only a finite number of discrete eigenvalues.

3.2. **Higher rank.** We turn now to the general case. We assume that $G = \mathbf{G}(\mathbb{R})$, where $\mathbf{G}$ is a connected semisimple algebraic group over $\mathbb{Q}$. Let $X = \Gamma\backslash\widetilde{X} = \Gamma\backslash G/K$ and $E_\sigma \to X$ be as above. Let $\Delta_\sigma\colon C^\infty(X, E_\sigma) \to C^\infty(X, E_\sigma)$ be the Bochner-Laplace operator. As operator in $L^2(X, E_\sigma)$ it is essentially self-adjoint. Let $L^2_{\mathrm{disc}}(X, E_\sigma)$ the subspace of $L^2(X, E_\sigma)$ which is the closure of the span of all L^2-eigensections of Δ_σ. Recall that a cusp form for Γ is a smooth K-finite function $\phi\colon \Gamma\backslash G \to \mathbb{C}$ which is a joint eigenfunction of the center of the universal enveloping algebra $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ and which satisfies

$$\int_{\Gamma \cap N_P \backslash N_P} \phi(nx)\, dn = 0$$

for all unipotent radicals N_P of proper rational parabolic subgroups P of G, i.e., $P = \mathbf{P}(\mathbb{R})$, where $\mathbf{P}$ is a rational parabolic subgroup of $\mathbf{G}$. Put

$$L^2_{\mathrm{cus}}(X, E_\sigma) := (L^2_{\mathrm{cus}}(\Gamma\backslash G) \otimes V_\sigma)^K.$$

Then $L^2_{\mathrm{cus}}(X, E_\sigma)$ is contained in $L^2_{\mathrm{disc}}(X, E_\sigma)$. The orthogonal complement $L^2_{\mathrm{res}}(X, E_\sigma)$ of $L^2_{\mathrm{cus}}(X, E_\sigma)$ in $L^2_{\mathrm{disc}}(X, E_\sigma)$ is called the *residual subspace*. By Langland's theory of Eisenstein series it follows that $L^2_{\mathrm{res}}(X, E_\sigma)$ is spanned by iterated residues of cuspidal Eisenstein series. By definition we have an orthogonal decomposition

$$L^2_{\mathrm{disc}}(X, E_\sigma) = L^2_{\mathrm{cus}}(X, E_\sigma) \oplus L^2_{\mathrm{res}}(X, E_\sigma).$$

Let $N^{\mathrm{disc}}_\Gamma(\lambda, \sigma)$, $N^{\mathrm{cus}}_\Gamma(\lambda, \sigma)$, and $N^{\mathrm{res}}_\Gamma(\lambda, \sigma)$ be the counting function of the eigenvalues with eigensections belonging to the corresponding subspace. The following results about the growth of the counting functions hold for any lattice Γ in a real semisimple Lie group. Let $d = \dim X$. Donnelly [Do] has proved the following bound for the cuspidal spectrum

$$(3.15) \qquad \limsup_{\lambda\to\infty} \frac{N^{\mathrm{cus}}_\Gamma(\lambda, \sigma)}{\lambda^{d/2}} \le \frac{\dim(\sigma)\,\mathrm{vol}(X)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2} + 1\right)}.$$

For the full discrete spectrum, we have at least an upper bound for the growth of the counting function. The main result of [Mu2] states that

$$(3.16) \qquad N^{\mathrm{disc}}_\Gamma(\lambda, \sigma) \ll (1 + \lambda^{2d}).$$

This result implies that invariant integral operators are trace class on the discrete subspace which is the starting point for the trace formula. The proof of (3.16) relies on the description of the residual subspace in terms of iterated residues of Eisenstein series.

Let $N_\Gamma^{\mathrm{cus}}(\lambda)$ be the counting function with respect to the trivial representation σ_0 of K, i.e., the counting function of the cuspidal spectrum of the Laplacian on functions. Then Sarnak [Sa2] conjectured that if $\mathrm{rank}(G/K) > 1$, Weyl's law holds for $N_\Gamma^{\mathrm{cus}}(\lambda)$, which means that equality holds in (3.16). Furthermore, one expects that the growth of the residual spectrum is of lower order than the cuspidal spectrum.

In the meantime Sarnak's conjecture has been verified in quite a number of cases. A. Reznikov proved it for congruence groups in a group G of real rank one, S. Miller [Mi] proved it for $\mathbf{G} = \mathrm{SL}(3)$ and $\Gamma = \mathrm{SL}(3, \mathbb{Z})$, the author [Mu3] established it for $\mathbf{G} = \mathrm{SL}(n)$ and a congruence group Γ. The most general result is due to Lindenstrauss and Venkatesh [LV] who proved the following theorem.

Theorem 3.2. *Let $\mathbf{G}$ be a split adjoint semi-simple group over $\mathbb{Q}$ and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a congruence subgroup. Let $d = \dim S$. Then*

$$(3.17) \qquad N_\Gamma^{\mathrm{cus}}(\lambda) \sim \frac{\mathrm{vol}(\Gamma \backslash \widetilde{X})}{(4\pi)^{d/2} \Gamma\left(\frac{d}{2} + 1\right)} \lambda^{d/2}, \quad \lambda \to \infty.$$

The method used by Lindenstrauss and Venkatesh is based on the construction of convolution operators with pure cuspidal image. It avoids the delicate estimates of the contributions of the Eisenstein series to the trace formula. This proves existence of many cusp forms for these groups.

For an arbitrary K-type, we have the following theorem proved in [Mu3].

Theorem 3.3. *Let $n \geq 2$ and $\widetilde{X} = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$. Let $d = \dim \widetilde{X}$. For every principal congruence subgroup Γ of $\mathrm{SL}(n, \mathbb{Z})$ and every irreducible unitary representation σ of $\mathrm{SO}(n)$ such that $\sigma|_{Z_\gamma} = \mathrm{Id}$, we have*

$$(3.18) \qquad N_\Gamma^{\mathrm{cus}}(\lambda, \sigma) \sim \frac{\dim(\sigma) \, \mathrm{vol}(\Gamma \backslash \widetilde{X})}{(4\pi)^{d/2} \Gamma(d/2 + 1)} \lambda^{d/2}$$

as $\lambda \to \infty$.

The residual spectrum for $\mathrm{SL}(n)$ has been described by Moeglin and Waldspurger [MW]. Combined with (3.15) it follows that for $\mathbf{G} = \mathrm{SL}(n)$ we have

$$(3.19) \qquad N_{\Gamma(N)}^{\mathrm{res}}(\lambda, \sigma) \ll \lambda^{d/2-1},$$

where $d = \dim \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ and $\Gamma(N) \subset \mathrm{SL}(n, \mathbb{Z})$ is the principal congruence subgroup of level N.

The proof of Theorem 3.3 uses the Arthur trace formula combined with the heat equation method similar to the proof of (3.13). The application of the Arthur trace formula requires the adelic reformulation of the problem.

We briefly describe the method. For all details we refer to [Mu5]. For simplicity we consider only the trivial K_∞-type, i.e, we consider the counting function $N_\Gamma^{\mathrm{cus}}(\lambda)$. By (3.19) we can replace the counting function $N_\Gamma^{\mathrm{cus}}(\lambda)$ by $N_\Gamma^{\mathrm{disc}}(\lambda)$. Let $\mathbf{G} = \mathrm{GL}(n)$ regarded as an algebraic group over $\mathbb{Q}$. Denote by A_G the split component of the center of $\mathbf{G}$ and let $A_G(\mathbb{R})^0$ be the component of 1 in $A_G(\mathbb{R})$. Let $\Pi_{\mathrm{disc}}(\mathbf{G}(\mathbb{A}), \xi_0)$ be the set of all irreducible subrepresentations of the regular representation of $G(\mathbb{A})$ in $L^2(\mathbf{G}(\mathbb{Q})A_G(\mathbb{R})^0\backslash\mathbf{G}(\mathbb{A}))$. Given a representation $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G}(\mathbb{A}), \xi_0)$, let $m(\pi)$ denote the multiplicity with which π occurs in $L^2(\mathbf{G}(\mathbb{Q})A_G(\mathbb{R})^0\backslash\mathbf{G}(\mathbb{A}))$. For any irreducible representation $\pi = \pi_\infty \otimes \pi_f$ of $\mathbf{G}(\mathbb{A})$, let $\mathcal{H}_{\pi_\infty}$ and $\mathcal{H}_{\pi_f}$ denote the Hilbert space of the representation π_∞ and π_f, respectively. Let K_f be an open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$. Denote by $\mathcal{H}_{\pi_f}^{K_f}$ the subspace of K_f-invariant vectors in $\mathcal{H}_{\pi_f}$ and by $\mathcal{H}_{\pi_\infty}^{K_\infty}$ the subspace of K_∞-invariant vectors in $\mathcal{H}_{\pi_\infty}$. Given $\pi \in \Pi(\mathbf{G}(\mathbb{A}), \xi_0)$, denote by λ_{π_∞} the Casimir eigenvalue of the restriction of π_∞ to $\mathbf{G}(\mathbb{R})^1$. Assume that $-1 \neq K_f$. Then (3.18) for the trivial K_∞-type follows by Karamata's theorem from the existence of an asymptotic expansion of the form

$$(3.20) \qquad \sum_{\pi \in \Pi_{\mathrm{disc}}(\mathbf{G}(\mathbb{A}), \xi_0)} m(\pi) e^{t\lambda_{\pi_\infty}} \dim\big(\mathcal{H}_{\pi_f}^{K_f}\big) \dim\big(\mathcal{H}_{\pi_\infty}^{K_\infty}\big) \sim \frac{\mathrm{vol}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1/K_f)}{(4\pi)^{d/2}} t^{-d/2}$$

as $t \to +0$.

To establish (3.20) we apply the Arthur trace formula as follows. We choose a certain family of test functions $\widetilde{\phi}_t^1 \in C_c^\infty(\mathbf{G}(\mathbb{A})^1)$, depending on $t > 0$, which at the infinite place are given by the heat kernel $h_t \in C^\infty(\mathbf{G}(\mathbb{R})^1)$ of the Laplacian $\widetilde{\Delta}$ on $\widetilde{X}$, multiplied by a certain cutoff function φ_t, and which at the finite places is given by the normalized characteristic function of an open compact subgroup K_f of $\mathbf{G}(\mathbb{A}_f)$. Then by the non-invariant trace formula [Ar1] we have the equality

$$J_{\mathrm{spec}}(\widetilde{\phi}_t^1) = J_{\mathrm{geo}}(\widetilde{\phi}_t^1), \quad t > 0.$$

Then we study asymptotic behavior of the spectral and the geometric side as $t \to 0$. To deal with the geometric side, we use the fine $\mathfrak{o}$-expansion [Ar6]

$$(3.21) \qquad J_{\mathrm{geo}}(f) = \sum_{\mathbf{M} \in \mathcal{L}} \sum_{\gamma \in (\mathbf{M}(\mathbb{Q}_S))_{M,S}} a^M(S, \gamma) J_M(f, \gamma),$$

which expresses the distribution $J_{\mathrm{geo}}(f)$ in terms of weighted orbital integrals $J_M(\gamma, f)$. Here $\mathbf{M}$ runs over the set of Levi subgroups $\mathcal{L}$ containing the Levi component $\mathbf{M}_0$ of the standard minimal parabolic subgroup $\mathbf{P}_0$, S is a finite set of places of $\mathbb{Q}$, and $(\mathbf{M}(\mathbb{Q}_S))_{M,S}$ is a certain set of equivalence classes in $\mathbf{M}(\mathbb{Q}_S)$. This reduces our problem to the investigation of weighted orbital integrals. The key result is that

$$\lim_{t \to 0} t^{d/2} J_M(\widetilde{\phi}_t^1, \gamma) = 0,$$

unless $\mathbf{M} = \mathbf{G}$ and $\gamma = 1$. This follows from the description of the local weighted orbital integrals by [Ar4, Corollary 6.2]. The contributions to (3.21) of the terms where $\mathbf{M} = \mathbf{G}$

and $\gamma = 1$ are easy to determine. Using the behavior of the heat kernel $h_t(1)$ as $t \to 0$, it follows that

$$(3.22) \qquad J_{\text{geo}}(\widetilde{\phi}_t^1) \sim \frac{\text{vol}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1/K_f)}{(4\pi)^{d/2}} t^{-d/2}$$

as $t \to 0$. To deal with the spectral side we use Theorem 1. This theorem allows us to replace $\widetilde{\phi}_t^1$ by a similar function $\phi_t^1 \in \mathcal{C}^1(G(\mathbb{A})^1)$ which is given as the product of the heat kernel h_t at infinity and the normalized characteristic function of K_f. The term in $J_{\text{spec}}(\phi_t^1)$ corresponding to $\mathbf{M} = \mathbf{G}$ is $J_{\text{spec},G}(\phi_t^1) = \text{tr}\, R_{\text{disc}}(\phi_t^1)$, which is equal to the left hand side of (3.20). If $\mathbf{M}$ is a proper Levi subgroup of $\mathbf{G}$, then $J_{\text{spec},M}(\phi_t^1)$ is given by (2.6), which is a finite some of integrals. The main ingredient of the integrals are logarithmic derivatives of intertwining operators and the estimation of these integrals is reduced to the estimation of the logarithmic derivatives. Using (2.1) this problem is reduced to the estimation of the logarithmic derivatives of the normalizing factors and the local intertwining operators. In the case of $\mathbf{G} = \text{GL}(n)$, the normalizing factors are expressed in terms of Ranking-Selberg L-functions (2.3). Using the analytic properties of Rankin-Selberg L-functions, it follows that there exists $C > 0$ such that for $\pi = \pi_1 \otimes \pi_2$, $\pi_i \in \Pi_{\text{disc}}(\text{GL}(n_i, \mathbb{A}))$, and $T > 1$ we have

$$(3.23) \qquad \int_T^{T+1} \left| \frac{n_\alpha'(\pi, i\lambda)}{n_\alpha(\pi, i\lambda)} \right| d\lambda \leq C \log(T + \nu(\pi_1 \times \tilde{\pi}_2)),$$

where $\nu(\pi_1 \times \tilde{\pi}_2) = N(\pi_1 \times \tilde{\pi}_2)(2 + c(\pi_1 \times \tilde{\pi}_2)$, $N(\pi_1 \times \tilde{\pi}_2)$ is the conductor occurring in the functional equation (2.2) and $c(\pi_1 \times \tilde{\pi}_2)$ is the analytic conductor defined in [Mu5, (4.21)]. For the proof of (3.23) see [Mu5, Proposition 5.1]. In the case of $\text{SL}(2, \mathbb{R})$ we have the pointwise estimate (3.11). If we integrate it, we get the analogue of (3.23) which would suffice to derive the Weyl law for the principal congruence subgroups of $\text{SL}(2, \mathbb{Z})$.

Finally we have to deal with normalized intertwining operators

$$R_{Q|P}(\pi, s) = \otimes_v R_{Q|P}(\pi_v, s).$$

Since the open compact subgroup K_{fin} of $\mathbf{G}(\mathbb{A}_{\text{fin}})$ is fixed, there are only finitely many places v for which we have to consider $R_{Q|P}(\pi_v, s)$. The main ingredient for the estimation of the logarithmic derivative of $R_{Q|P}(\pi_v, s)$, which is uniform in π_v, is a weak version of the Ramanujan conjecture (see [MS, Proposition 0.2]).

Combining these estimations, it follows that for every proper Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ we have

$$(3.24) \qquad J_{\text{spec},M}(\phi_t^1) = O(t^{-(d-1)/2})$$

as $t \to +0$. This proves (3.20).

The next problem is to estimate the remainder term in Weyl's law. For $\mathbf{G} = \text{SL}(n)$ this problem has been studied by E. Lapid and the author in [LM]. Actually, we consider not only the cuspidal spectrum of the Laplacian, but the cuspidal spectrum of the whole algebra of invariant differential operators $\mathcal{D}(\widetilde{X})$.

As $\mathcal{D}(\widetilde{X})$ preserves the space of cusp forms, we can proceed as in the compact case and decompose $L^2_{\mathrm{cus}}(\Gamma\backslash\widetilde{X})$ into joint eigenspaces of $\mathcal{D}(\widetilde{X})$. Recall that the characters of $\mathcal{D}(\widetilde{X})$ are parametrized by $\mathfrak{a}^\star_{\mathbb{C}}/W$. Given $\lambda \in \mathfrak{a}^*_{\mathbb{C}}/W$, let

$$\mathcal{E}_{\mathrm{cus}}(\lambda) = \left\{ \varphi \in L^2_{\mathrm{cus}}(\Gamma\backslash\widetilde{X}) \colon D\varphi = \chi_\lambda(D)\varphi \right\}$$

be the associated joint eigenspace. Each eigenspace is finite-dimensional. Let $m(\lambda) = \mathcal{E}_{\mathrm{cus}}(\lambda)$. Define the cuspidal spectrum $\Lambda_{\mathrm{cus}}(\Gamma)$ to be

$$\Lambda_{\mathrm{cus}}(\Gamma) = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}}/W \colon m(\lambda) > 0\}.$$

Then we have an orthogonal direct sum decomposition

$$L^2_{\mathrm{cus}}(\Gamma\backslash\widetilde{X}) = \bigoplus_{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma)} \mathcal{E}_{\mathrm{cus}}(\lambda).$$

Let $\beta(\lambda)$ be the Plancherel measure on $i\mathfrak{a}^\star$. Then in [LM] we established the following extension of main results of [DKV] to congruence quotients of $S = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

Theorem 3.4. *Let $d = \dim\widetilde{X}$. Let $\Omega \subset \mathfrak{a}^*$ be a bounded domain with piecewise smooth boundary. Then for $N \geq 3$ we have*

$$(3.25) \qquad \sum_{\substack{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma(N)) \\ \lambda \in it\Omega}} m(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash\widetilde{X})}{|W|} \int_{t\Omega} \beta(i\lambda)\, d\lambda + O\left(t^{d-1}(\log t)^{\max(n,3)}\right),$$

as $t \to \infty$, and

$$(3.26) \qquad \sum_{\substack{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma(N)) \\ \lambda \in B_t(0)\backslash i\mathfrak{a}^*}} m(\lambda) = O\left(t^{d-2}\right), \quad t \to \infty.$$

If we apply (3.25) and (3.26) to the unit ball in $\mathfrak{a}^*$, we get the following corollary.

Corollary 3.5. *Let $\widetilde{X} = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ and $d = \dim\widetilde{X}$. Let $\Gamma(N)$ be the principal congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$ of level N. Then for $N \geq 3$ we have*

$$N^{\mathrm{cus}}_{\Gamma(N)}(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash\widetilde{X})}{(4\pi)^{d/2}\mathbf{\Gamma}\left(\frac{d}{2}+1\right)}\lambda^{d/2} + O\left(\lambda^{(d-1)/2}(\log\lambda)^{\max(n,3)}\right), \quad \lambda \to \infty.$$

The condition $N \geq 3$ is imposed for technical reasons. It guarantees that the principal congruence subgroup $\Gamma(N)$ is neat in the sense of Borel, and in particular, has no torsion. This simplifies the analysis by eliminating the contributions of the non-unipotent conjugacy classes in the trace formula.

Note that $\Lambda_{\mathrm{cus}}(\Gamma(N))\cap i\mathfrak{a}^*$ is the cuspidal tempered spherical spectrum. The Ramanujan conjecture [Sa3] for $\mathrm{GL}(n)$ at the Archimedean place states that

$$\Lambda_{\mathrm{cus}}(\Gamma(N)) \subset i\mathfrak{a}^*$$

so that (3.26) is empty, if the Ramanujan conjecture is true. However, the Ramanujan conjecture is far from being proved. Moreover, it is known to be false for other groups $\mathbf{G}$ and (3.26) is what one can expect in general.

The method to prove Theorem 3.4 is an extension of the method of [DKV]. The Selberg trace formula, which is one of the basic tools in [DKV], is replaced by the non-invariant Arthur trace formula. Again, one of the main issues in the proof is the estimation of the logarithmic derivatives of the intertwining operators occurring on the spectral side of the trace formula.

3.3. **Upper and lower bounds.** In some cases it suffices to have upper or lower bounds for the counting function. For example, Donnelly's result (3.15) implies that there exists a constant $C > 0$ such that

$$(3.27) \qquad N_\Gamma^{\mathrm{cus}}(\lambda, \sigma) \le C(1 + \lambda^{d/2}), \quad \lambda \ge 0.$$

For the full discrete spectrum we have the bound (3.16). However, the exponent is not the optimal one. For some applications it this necessary to have such a bound which is uniform in Γ. For the cuspidal spectrum this problem has been studied by Deitmar and Hoffmann [DH]. To state the result, we have to introduce some notation. Let $\Gamma_n(N)$ be the principal congruence subgroup of $\mathrm{GL}(n, \mathbb{Z})$ of level N. Let $\mathbf{G}$ be a connected reductive linear algebraic group over $\mathbb{Q}$. Let $\eta \colon \mathbf{G} \to \mathrm{GL}(n)$ be a faithful $\mathbb{Q}$-rational representation. A family $\mathcal{T}$ of subgroups of $\mathbf{G}(\mathbb{Q})$ is called a *family of bounded depth* in $\mathbf{G}(\mathbb{Q})$ if there exists $D \in \mathbb{N}$ which satisfies the following property: For every $\Gamma \in \mathcal{T}$ there exists $N \in \mathbb{N}$ such that $\Gamma_n(N) \cap \eta(\mathbf{G}(\mathbb{Q}))$ is a subgroup of $\eta(\Gamma)$ of index at most D. We note that every $\Gamma \in \mathcal{T}$ is contained in $\Gamma_0 := \Gamma_n(1) \cap \mathbf{G}(\mathbb{Q})$. Then the result of Deitmar and Hoffmann [DH, Corollary 18] is the following theorem.

Theorem 3.6. *Let $\mathcal{T}$ be a family of bounded depth in $\mathbf{G}(\mathbb{Q})$. There exists $C > 0$ such that for all $\Gamma \in \mathcal{T}$ and all $\lambda \ge 0$ we have*

$$(3.28) \qquad N_\Gamma^{\mathrm{cus}}(\lambda; \sigma) \le C[\Gamma_0 : \Gamma](1 + \lambda)^{d/2}.$$

Conjecture 2. *The estimation (3.28) holds for $N_\Gamma^{\mathrm{disc}}(\lambda; \sigma)$.*

Given the description of the residual spectrum for $\mathrm{GL}(n)$ by [MW], it seems to be possible to establish this conjecture for $\mathrm{GL}(n)$.

As for lower bounds there is the weak Weyl law established in [LM]. For $\sigma \in \widehat{K}$ let

$$c_\sigma(\Gamma) = \frac{\dim(\sigma) \operatorname{vol}(\Gamma \backslash \widetilde{X})}{(4\pi)^{d/2} \mathbf{\Gamma}(d/2 + 1)}$$

be the constant in Weyl's law, where $d = \dim(\widetilde{X})$. Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$ and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a congruence subgroup defined by an open compact subgroup $K_{\mathrm{fin}} = \prod_p K_p$ of $\mathbf{G}(\mathbb{A}_{\mathrm{fin}})$. Let S be a finite set of primes. We will say that Γ is deep enough with respect to S, if for every prime $p \in S$, K_p is a subgroup of some minimal parahoric subgroup of $\mathbf{G}(\mathbb{Q}_p)$. Then the main result of [LM] is the following theorem.

Theorem 3.7. *Let **G** be an almost simple connected and simply connected semisimple algebraic group defined over $\mathbb{Q}$ such that $\mathbf{G}(\mathbb{R})$ is non compact. Let S be a finite set of primes containing at least two primes. Then for every congruence subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ there exists a nonnegative constant $c_S(\Gamma) \leq 1$ such that for every $\sigma \in \widehat{K}$ with $\sigma|_{Z_\Gamma} = \mathrm{Id}$ we have*

$$c_\sigma(\Gamma)c_S(\Gamma) \leq \liminf_{\lambda \to \infty} \frac{N_\Gamma^{\mathrm{cus}}(\lambda, \sigma)}{\lambda^{d/2}}.$$

Moreover $c_S(\Gamma) > 0$ if Γ is deep enough with respect to S.

3.4. **Self-dual automorphic representations.** So far, we considered only the family of all cusp forms of $\mathrm{GL}(n, \mathbb{A})$. A nontrivial subfamily is formed by the family of self-dual automorphic representations. They arise as functorial lifts of automorphic representations of classical groups. Functoriality from classical groups to $\mathrm{GL}(N)$ has been established by Cogdell, Kim, Piatetski-Shapiro, and Shahidi for generic automorphic representations and then by Arthur for all representations. In his thesis, V. Kala has studied the counting function of self-dual cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A})$. For $N \in \mathbb{N}$ with prime decomposition $N = \prod_p p^{r(p)}$ let

$$K_p(N) := \left\{ k \in \mathrm{GL}(n, \mathbb{Z}_p) \colon k \equiv 1 \mod p^{r(p)}\mathbb{Z}_p \right\}$$

Let $K(N)$ be the principal congruence subgroup defined by

$$K(N) := O(n) \times \prod_p K_p(N).$$

Let

$$N_{\mathrm{sd}}^{K(N)}(\lambda) := \sum_{\substack{\lambda(\Pi) \leq \lambda \\ \Pi \cong \widetilde{\Pi}}} \dim \Pi^{K(N)},$$

where the sum ranges over all self-dual cuspidal automorphic representations Π of $\mathrm{GL}(n, \mathbb{A})$ with Casimir eigenvalues $\leq \lambda$. Then the main result of [Ka] is the following theorem.

Theorem 3.8. *Let $n = 2m + \varepsilon$ with $\varepsilon = 0, 1$. Put $d = m^2 + m$. For all $N \in \mathbb{N}$ there exist constants $C_1, C_2 > 0$ such that for $\lambda \gg 0$ one has*

$$C_1 \lambda^{d/2} \leq N_{\mathrm{sd}}^{K(N)}(\lambda) \leq C_2 \lambda^{d/2}.$$

By Corollary 3.5, the counting function of all cuspidal representations, counted similarly, is asymptotic to $C\lambda^{d/2}$, where $d = (n^2 + n - 2)/2$. Hence for $n > 2$, the density of self-dual cusp forms is zero.

The main idea of the proof is to consider the descent π of each self-dual cuspidal automorphic representation Π of $\mathrm{GL}(n, \mathbb{A})$ to one of the quasisplit classical groups $\mathbf{G}(\mathbb{A})$ and to use results towards to Weyl's law on $\mathbf{G}(\mathbb{A})$. The key problem is to relate the Casimir eigenvalue and the existence of $K(N)$-fixed vectors for Π and π.

In a special case Kala's method leads to an exact asymptotic formula. Let $n = 2m$ and $d = m^2 + m$. Let $K = O(n) \times \prod_p K_p$ with $K_p = \mathrm{GL}(n, \mathbb{Z}_p)$. Then there exists $C > 0$ such that

$$(3.29) \qquad\qquad N_{\mathrm{sd}}^K(\lambda) = C\lambda^{d/2} + o(\lambda^{d/2})$$

(see [Ka, Corollary 6.2.2]). One may conjecture that this is true in general.

3.5. Weyl's law for Hecke operators.

One can also study the asymptotic distribution of infinitesimal characters of cuspidal automorphic representations weighted by the eigenvalues of Hecke operators acting on cusp forms of $\mathrm{GL}(n)$. For details we refer to the recent papers by J. Matz [Ma1], J. Matz and N. Templier [MT] and the survey article of J. Matz in these proceedings.

4. THE LIMIT MULTIPLICITY PROBLEM

The limit multiplicity problem is another basic problem which is concerned with the asymptotic behavior of automorphic spectra.

In this section we summarize some of the known results about the limit multiplicity problem. To begin with we recall some facts concerning the Plancherel measure μ_{pl} on $\Pi(G)$. First of all, the support of μ_{pl} is the tempered dual $\Pi(G)_{\mathrm{temp}}$, consisting of the equivalence classes of the irreducible unitary tempered representations. Up to a closed subset of Plancherel measure zero, the topological space $\Pi(G)_{\mathrm{temp}}$ is homeomorphic to a countable union of Euclidean spaces of bounded dimensions. Under this homeomorphism the Plancherel density is given by a continuous function. We call the relatively quasi-compact subsets of $\Pi(G)$ *bounded*. We note that $\mu_\Gamma(A) < \infty$ for bounded sets $A \subset \Pi(G)$ under the reduction-theoretic assumptions on (G, Γ) mentioned above (see [BG]). A bounded subset A of $\Pi(G)_{\mathrm{temp}}$ is called a Jordan measurable subset, if $\mu_{\mathrm{pl}}(\partial A) = 0$, where $\partial A = \bar{A} - \mathrm{int}(A)$ is the boundary of A in $\Pi(G)_{\mathrm{temp}}$. Furthermore, a Riemann integrable function on $\Pi(G)_{\mathrm{temp}}$ is a bounded, compactly supported function which is continuous almost everywhere with respect to the Plancherel measure.

Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of Borel measures on $\Pi(G)$. We say that the sequence $(\mu_n)_{n \in \mathbb{N}}$ has the *limit multiplicity property* (property (LM)), if the following two conditions are satisfied.

1) For every Jordan measurable set $A \subset \Pi(G)_{\mathrm{temp}}$ we have

$$\mu_n(A) \to \mu_{\mathrm{pl}}(A), \quad \text{as } n \to \infty.$$

2) For every bounded subset $A \subset \Pi(G) \setminus \Pi(G)_{\mathrm{temp}}$ we have

$$\mu_n(A) \to 0, \quad \text{as } n \to \infty.$$

We note that condition 1) can be restated as

1a) For every Riemann integrable function f on $\Pi(G)_{\text{temp}}$ one has

$$\lim_{n \to \infty} \mu_n(f) = \mu_{\text{pl}}(f).$$

Now let $(\Gamma_n)_{n \in \mathbb{N}}$ be a sequence of lattices in G. The sequence $(\Gamma_n)_{n \in \mathbb{N}}$ is said to have the limit multiplicity property (LM), if the sequence of measures $(\mu_{\Gamma_n})_{n \in \mathbb{N}}$ has property (LM).

The limit multiplicity problem can be formulated as follows: under which conditions does the sequence of measures μ_{Γ_n} satisfy property (LM)?

The limit multiplicity problem has been studied to a great extent in the case of uniform lattices. In this case, R_Γ decomposes discretely. It started with the work of DeGeorge and Wallach [DW1],[DW2], who considered towers of normal subgroups, i.e., descending sequences of normal subgroups of finite index of a given uniform lattice with trivial intersection. For such sequences they dealt with the case of discrete series representations and the tempered spectrum, if the split rank of G is 1. Subsequently, Delorme [De] solved the limit multiplicity problem affirmatively for normal towers of cocompact lattices. Recently, there has been great progress in proving limit multiplicity for much more general sequences of uniform lattices by Abert et al [AB1],[AB2]. In particular, families of non-commensurable lattices were considered for the first time. The basic idea is the notion of Benjamini-Schramm convergence (BS-convergence), which originally was introduced for sequences of finite graphs of bounded degree and has been adopted by Abert et al to sequences of Riemannian manifolds. For a Riemannian manifold M and $R > 0$ let

$$M_{<R} = \{x \in M \colon \operatorname{injrad}_M(x) < R\}.$$

Let (Γ_n) be a sequence of lattices in G. Then the orbifolds $M_n = \Gamma_n \backslash X$ are said to BS-converge to X, if for every $R > 0$ one has

$$(4.1) \qquad \lim_{n \to +\infty} \frac{\operatorname{vol}((M_n)_{<R})}{\operatorname{vol}(M_n)} = 0.$$

To find examples of sequences (Γ_n) which satisfy this condition, consider a cocompact arithmetic lattice $\Gamma_0 \subset G$. By [AB1, Theorem 5.2] there exist constants $c, \mu > 0$ such that for any congruence subgroup $\Gamma \subset \Gamma_0$ and any $R > 1$ one has

$$(4.2) \qquad \operatorname{vol}((\Gamma \backslash X)_{<R}) \leq e^{cR} \operatorname{vol}(\Gamma \backslash X)^{1-\mu}.$$

Thus any sequence (Γ_n) of congruences subgroups of Γ_0 such that $\operatorname{vol}(\Gamma_n \backslash G) \to \infty$ as $n \to \infty$ satisfies (4.1).

A family of lattices in G is called to be uniformly discrete, if there exists a neighborhood of the identity in G that intersects trivially all of their conjugates. For torsion-free lattices Γ_n this is equivalent to the condition that there is a uniform lower bound of the injectivity radii of the manifolds $\Gamma_n \backslash X$. In particular, any family of normal subgroups (Γ_n) of a fixed uniform lattice Γ is uniformly discrete. Now the following theorem is one of the main results of [AB1, Theorem 1.2].

Theorem 4.1 ([AB1]). *Let (Γ_n) be a uniformly discrete sequence of lattices in G such that the orbifolds $\Gamma_n \backslash X$ BS-converge to X. Then the sequence (Γ_n) has the (LM) property.*

It follows from the discussions above that any sequence of congruence subgroups (Γ_n) of a given cocompact arithmetic lattice Γ_0 of G satisfies the assumptions of the theorem.

A special case of the limit multiplicity property is the case of a singleton $A = \{\pi\}$. Let $\Pi(G)_d \subset \Pi(G)$ be the discrete series and $d(\pi)$ the formal degree of $\pi \in \Pi(G)_d$. If (Γ_n) is a sequence of lattices in G which satisfies the (LM) property, then it follows that

$$(4.3) \qquad \lim_{n \to \infty} \frac{m_{\Gamma_n}(\pi)}{\mathrm{vol}(\Gamma_n \backslash G)} = \begin{cases} d(\pi), & \pi \in \Pi(G)_d, \\ 0, & \text{else.} \end{cases}$$

It was first proved by DeGeorge and Wallach [DW1] that (4.3) holds for any tower of normal subgroups of a given uniform lattice of G.

An important problem is to extend these results to the non-compact case. Then the spectrum contains a continuous part and much less is known. The limit multiplicity problem has been solved for normal towers of arithmetic lattices and discrete series L-packets of representations (with regular parameters) by Rohlfs and Speh [RoS]. Then Savin [Sav] solved the limit multiplicity problem for the discrete series and normal towers of congruence subgroups.

In [FLM2] we dealt with the general case. Let F be a number field and denote by $\mathcal{O}_F$ its ring of integers. For the non-compact lattice $\mathrm{SL}(n, \mathcal{O}_F) \subset \mathrm{SL}(n, F \otimes \mathbb{R})$ we have the following result.

Theorem 4.2. *Let F be a number field. Then the collection of principal congruence subgroups (Γ_N) of $\mathrm{SL}(n, \mathcal{O}_F)$ has the limit multiplicity property.*

In [FL2], T. Finis and E. Lapid extended this result to the collection of all congruence subgroups of $\mathrm{SL}(n, \mathcal{O}_F)$, not containing non-trivial central elements. In [FLM2], we also discussed the case of a general reductive group.

4.1. **The density principle and the trace formula.** A standard approach to the limit multiplicity problem is to use integration against test functions on G and the trace formula. Let K be a maximal compact subgroup of G. Denote by $C^\infty_{c,\mathrm{fin}}(G)$ the space of smooth, compactly supported bi-K-finite functions on G. Given $f \in C^\infty_{c,\mathrm{fin}}(G)$, define $\hat{f}(\pi)$ for $\pi \in \Pi(G)$ by $\hat{f}(\pi) := \mathrm{tr}\,\pi(f)$. The function $\pi \in \Pi(G) \mapsto \hat{f}(\pi)$ on $\Pi(G)$ is the "Fourier transform" of f. Let μ be a Borel measure on $\Pi(G)$. Then $\mu(\hat{f})$ is defined, but might be divergent. Especially, we have

$$(4.4) \qquad \mu_\Gamma(\hat{f}) = \frac{1}{\mathrm{vol}(\Gamma \backslash G)} \, \mathrm{tr}\, R_{\Gamma,\mathrm{disc}}(f).$$

By [Mu2], $R_{\Gamma,\mathrm{disc}}(f)$ is a trace class operator. Thus the right hand side is well defined. Furthermore, by the Plancherel theorem we have $\mu_{\mathrm{pl}}(\hat{f}) = f(1)$. The density principle of Sauvageot [Sau], which is a refinement of the work of Delorme, can be stated as follows.

Theorem 4.3. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of Borel measures on $\Pi(G)$ and assume that for all $f \in C_{c,\mathrm{fin}}^\infty(G)$ we have*

$$(4.5) \qquad \mu_n(\hat{f}) \to \mu_{pl}(\hat{f}) = f(1), \quad \text{as } n \to \infty.$$

Then $(\mu_n)_{n\in\mathbb{N}}$ satisfies (LM).

Now let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of lattices in G. Then by Theorem 4.3 it follows that $(\Gamma_n)_{n\in\mathbb{N}}$ satisfies (LM), if

$$(4.6) \qquad \mu_{\Gamma_n}(\hat{f}) \to f(1), \quad n \to \infty,$$

for all $f \in C_{c,\mathrm{fin}}^\infty(G)$. A standard approach to verify (4.6) is to use the trace formula. In the case of co-compact lattices this is rather simple. Let Γ be a cocompact lattice in G. Then the Selberg trace formula is the following equality

$$\mathrm{vol}(\Gamma\backslash G)\mu_\Gamma(\hat{f}) = \mathrm{tr}\, R_\Gamma(f) = \sum_{\{\gamma\}\in C(\Gamma)} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\, dx,$$

where $C(\Gamma)$ denotes the Γ-conjugacy classes of Γ, and G_γ (resp. Γ_γ) denotes the centralizer of γ in G (resp. Γ). Let $\Gamma_1 \subset \Gamma$ be a finite index subgroup. For $\gamma \in \Gamma$ let

$$(4.7) \qquad c_{\Gamma_1}(\gamma) = |\{\delta \in \Gamma_1\backslash\Gamma \colon \delta\gamma\delta^{-1} \in \Gamma_1\}|.$$

In [Co], Corwin shows that the elements on the right hand side of the trace formula for Γ_1 can be grouped together in a way to give

$$(4.8) \qquad \mu_{\Gamma_1}(\hat{f}) = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\{\gamma\}\in C(\Gamma)} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \frac{c_{\Gamma_1}(\gamma)}{[\Gamma \colon \Gamma_1]} \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\, dx.$$

For a central element γ we obviously have $c_{\Gamma_1}(\gamma) = [\Gamma \colon \Gamma_1]$. Assume that the center of Γ is trivial. Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of finite index subgroups of Γ. Then we have

$$(4.9) \qquad \mu_{\Gamma_n}(\hat{f}) = f(1) + \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\{\gamma\}\in C(\Gamma)\backslash\{1\}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \frac{c_{\Gamma_n}(\gamma)}{[\Gamma \colon \Gamma_n]} \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\, dx.$$

By dominated convergence, it follows that in order to establish (4.5) for the sequence $(\Gamma_n)_{n\in\mathbb{N}}$, it suffices to show that for every $\gamma \in \Gamma$, $\gamma \neq 1$, we have

$$(4.10) \qquad \frac{c_{\Gamma_n}(\gamma)}{[\Gamma \colon \Gamma_n]} \to 0, \quad \text{as } n \to \infty.$$

Now note that if Γ_1 is a normal subgroup of Γ, then $c_{\Gamma_1}(\gamma)/[\Gamma \colon \Gamma_1]$ is the characteristic function of Γ_1. Thus for normal towers of finite index subgroups of Γ the condition (4.10) holds trivially. This implies Delorme's result.

If Γ is not co-compact, the Selberg trace formula is only available in the rank one case. We have to switch to the adelic framework so that we can use the Arthur trace formula.

Thus let now $\mathbf{G}$ be an arbitrary reductive group defined over $\mathbb{Q}$. Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$ be the locally compact adele ring of $\mathbb{Q}$. For every place v of $\mathbb{Q}$ (i.e. $v = \infty$ or $v = p$ a prime)

let $|\cdot|_v$ be the normalized absolute value of $\mathbb{Q}$. As usual. $\mathbf{G}(\mathbb{R})^1$ denotes the intersection of the kernels of the homomorphisms $|\chi|\colon \mathbf{G}(\mathbb{R}) \to \mathbb{R}^+$, where χ runs over the $\mathbb{Q}$-rational characters of $\mathbf{G}$. Similarly we define the normal subgroup $\mathbf{G}(\mathbb{A})^1$ of $\mathbf{G}(\mathbb{A})$. Every $\pi \in \Pi(\mathbf{G}(\mathbb{A})^1)$ can be written as $\pi = \pi_\infty \otimes \pi_{\text{fin}}$, where $\pi_\infty \in \Pi(\mathbf{G}(\mathbb{R})^1)$ and $\pi_{\text{fin}} \in \Pi(\mathbf{G}(\mathbb{A}_{\text{fin}}))$. Fix a Haar measure on $\mathbf{G}(\mathbb{A})$. For any open compact subgroup K_f of $\mathbf{G}(\mathbb{A}_{\text{fin}})$, let $\mu_K = \mu_K^G$ be the measure on $\Pi(\mathbf{G}(\mathbb{R})^1)$ defined by

(4.11)

$$\mu_K = \frac{1}{\text{vol}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1/K)} \sum_{\pi \in \Pi(G(\mathbb{R})^1)} \text{Hom}_{\mathbf{G}(\mathbb{R})^1}(\pi, L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1/K))\delta_\pi$$

$$= \frac{\text{vol}(K)}{\text{vol}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1)} \sum_{\pi \in \Pi(\mathbf{G}(\mathbb{A})^1)} \dim \text{Hom}_{\mathbf{G}(\mathbb{A})^1}(\pi, L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1)) \dim(\pi_{\text{fin}})^K \delta_{\pi_\infty}.$$

We say that a sequence $(K_n)_{n\in\mathbb{N}}$ of open compact subgroups of $\mathbf{G}(\mathbb{A}_{\text{fin}})$ has the limit multiplicity property, if $\mu_{K_n} \to \mu_{\text{pl}}$, $n \to \infty$, in the sense that

(1) For every Jordan measurable subset $A \subset \Pi(\mathbf{G}(\mathbb{R})^1))_{\text{temp}}$ we have $\mu_{K_n}(A) \to \mu_{\text{pl}}(A)$ as $n \to \infty$, and

(2) For every bounded subset $A \subset \Pi(\mathbf{G}(\mathbb{R})^1) \setminus \Pi(\mathbf{G}(\mathbb{R})^1))_{\text{temp}}$, we have $\mu_{K_n}(A) \to 0$ as $n \to \infty$.

Again we can rephrase the first condition by saying that for any Riemann integrable function f on $\Pi(\mathbf{G}(\mathbb{R})^1)_{\text{temp}}$ we have

(4.12) $\mu_{K_n}(f) \to \mu_{\text{pl}}(f), \quad \text{as } n \to \infty.$

Note that when $\mathbf{G}$ satisfies the strong approximation property (which is the case if $\mathbf{G}$ is semisimple, simply connected, and without any $\mathbb{Q}$-simple factor $\mathbf{H}$ for which $\mathbf{H}(\mathbb{R})$ is compact) and K is an open compact subgroup of $\mathbf{G}(\mathbb{A}_{\text{fin}})$, then we have

$$\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K \cong \Gamma_K\backslash\mathbf{G}(\mathbb{R}),$$

where $\Gamma_K = \mathbf{G}(\mathbb{Q}) \cap K$ is a lattice in the connected semisimple Lie group $\mathbf{G}(\mathbb{R})$.

Now for $f \in C^\infty_{c,\text{fin}}(\mathbf{G}(\mathbb{R})^1)$ we have

(4.13) $\mu_K(\hat{f}) = \dfrac{1}{\text{vol}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(A)^1)} \text{tr } R_{\text{disc}}(f \otimes \mathbf{1}_K)$

and

(4.14) $\mu_{\text{pl}}(\hat{f}) = f(1).$

Sauvageot's density principle [Sau] can now be reformulated as follows.

Theorem 4.4. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of open compact subgroups of $\mathbf{G}(\mathbb{A}_{\text{fin}})$. Suppose that for every $f \in C^\infty_{c,\text{fin}}(\mathbf{G}(\mathbb{R})^1)$ we have*

(4.15) $\mu_{K_n}(\hat{f}) \to f(1), \quad n \to \infty.$

Then $(K_n)_{n\in\mathbb{N}}$ has the limit multiplicity property.

To try to verify (4.15), it is natural to use Arthur's (non-invariant) trace formula, which is an equality

$$J_{\text{spec}}(h) = J_{\text{geo}}(h), \quad h \in C_c^\infty(\mathbf{G}(\mathbb{A})^1),$$

of two distribution on $\mathbf{G}(\mathbb{A})^1$ [Ar1], [Ar2], [Ar3]. The distribution J_{spec} is expressed in terms of spectral data and J_{geo} in terms of geometric data. The main terms on the geometric side are the elliptic orbital integrals. In particular, the contribution $\text{vol}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1)h(1)$ of the identity element occurs on the geometric side. The main term on the spectral side is $\text{tr}\, R_{\text{disc}}(h)$. By (4.13) it follows that (4.15) can be broken down into the following two statements. For every $f \in C_{c,\text{fin}}^\infty(G(\mathbb{R})^1)$ we have

$$(4.16) \qquad J_{\text{spec}}(f \otimes \mathbf{1}_{K_n}) - \text{tr}\, R_{\text{disc}}(f \otimes \mathbf{1}_{K_n}) \to 0, \quad n \to \infty,$$

and

$$(4.17) \qquad J_{\text{geo}}(f \otimes \mathbf{1}_{K_n}) \to \text{vol}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1)f(1), \quad n \to \infty.$$

We call (4.16) the spectral - and (4.17) the geometric limit property.

4.2. **Bounds on co-rank one intertwining operators.**

In this section we formulate two conditions on the behavior of the intertwining operators $M_{Q|P}$ which imply the spectral limit property for a given $\mathbf{G}$. They also imply Weyl's law for the group $\mathbf{G}$. We call these properties (TWN) (tempered winding number) and (BD) (bounded degree). The first property is global and second local. The first property is connected with analytic problems in the theory of automorphic L-functions.

We will use the notation $A \ll B$ to mean that there exists a constant c (independent of the parameters under consideration) such that $A \le cB$. If c depends on some parameters (say F) and not on others then we will write $A \ll_F B$.

Fix a faithful $\mathbb{Q}$-rational representation $\rho : \mathbf{G} \to \text{GL}(V)$ and an $\mathbb{Z}$-lattice Λ in the representation space V such that the stabilizer of $\hat{\Lambda} = \hat{\mathbb{Z}} \otimes \Lambda \subset \mathbb{A}_{\text{fin}} \otimes V$ in $G(\mathbb{A}_{\text{fin}})$ is the group $\mathbf{K}_{\text{fin}}$. (Since the maximal compact subgroups of $\text{GL}(\mathbb{A}_{\text{fin}} \otimes V)$ are precisely the stabilizers of lattices, it is easy to see that such a lattice exists.) For any $N \in \mathbb{N}$ let

$$(4.18) \qquad \mathbf{K}(N) = \{g \in G(\mathbb{A}_{\text{fin}}) : \rho(g)v \equiv v \pmod{N\hat{\Lambda}}, \quad v \in \hat{\Lambda}\}$$

be the principal congruence subgroup of level N, an open normal subgroup of $\mathbf{K}_{\text{fin}}$. The groups $\mathbf{K}(N)$ form a neighborhood base of the identity element in $\mathbf{G}(\mathbb{A}_{\text{fin}})$. For an open subgroup K of $\mathbf{K}_{\text{fin}}$ let the *level* of K be the smallest integer N such that $\mathbf{K}(N) \subset K$. Analogously, define $\text{level}(K_v)$ for open subgroups $K_v \subset \mathbf{K}_v$.

As in [Mu6], for any $\pi \in \Pi(\mathbf{M}(\mathbb{R}))$ we define $\Lambda_\pi = \sqrt{\lambda_\pi^2 + \lambda_\tau^2}$, where τ is a lowest $\mathbf{K}_\infty$-type of $\text{Ind}_{\mathbf{P}(\mathbb{R})}^{\mathbf{G}(\mathbb{R})}(\pi_\infty)$ and λ_π and λ_τ are the Casimir eigenvalues of π and τ, respectively. Note that this is well-defined, because λ_τ is independent of τ. Roughly speaking, Λ_π measures the size of π. For $\mathbf{M} \in \mathcal{L}$, $\alpha \in \Sigma_M$ and $\pi \in \Pi_{\text{disc}}(\mathbf{M}(\mathbb{A}))$ let $n_\alpha(\pi, s)$ be the global normalizing factor defined by (2.1).

Definition 1. *We say that the group* $\mathbf{G}$ *satisfies the property (TWN) (tempered winding number) if for any* $\mathbf{M} \in \mathcal{L}$, $\mathbf{M} \neq \mathbf{G}$, *and any finite subset* $\mathcal{F} \subset \Pi(\mathbf{K}_{M,\infty})$ *there exists an integer* $k > 1$ *such that for any* $\alpha \in \Sigma_M$ *and any* $\epsilon > 0$ *we have*

$$(4.19) \qquad \int_{i\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-k} \, ds \ll_{\mathcal{F}, \epsilon} (1 + \Lambda_{\pi_\infty})^k \operatorname{level}(K_M)^\epsilon$$

for all open compact subgroups K_M *of* $\mathbf{K}_{M,\mathrm{fin}}$ *and* $\pi \in \Pi_{\mathrm{disc}}(\mathbf{M}(\mathbb{A}))^{\mathcal{F}, K_M}$.

Since the normalizing factors $n_\alpha(\pi, s)$ arise from co-rank one situations, the property (TWN) is hereditary for Levi subgroups.

Remark 4.5. *If we fix an open compact subgroup* K_M, *then the corresponding bound*

$$\int_{i\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-k} \, ds \ll_{K_M} (1 + \Lambda_{\pi_\infty})^k$$

is the content of [Mu6, Theorem 5.3]. *So, the point of (TWN) lies in the dependence of the bound on* K_M.

Remark 4.6. *In fact, we expect that*

$$(4.20) \qquad \int_T^{T+1} \left| \frac{n'_\alpha(\pi, it)}{n_\alpha(\pi, it)} \right| \, dt \ll 1 + \log(1 + T) + \log(1 + \Lambda_{\pi_\infty}) + \log \operatorname{level}(K_M)$$

for all $T \in \mathbb{R}$ *and* $\pi \in \Pi_{\mathrm{disc}}(\mathbf{M}(\mathbb{A}))^{K_M}$. *This would give the following strengthening of (TWN):*

$$\int_{i\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-2} \, ds \ll 1 + \log(1 + \Lambda_{\pi_\infty}) + \log \operatorname{level}(K_M)$$

for any $\pi \in \Pi_{\mathrm{disc}}(\mathbf{M}(\mathbb{A}))^{K_M}$.

Remark 4.7. *If* G' *is simply connected, then by* [Lub, Lemma 1.6] *(cf. also* [FLM2, Proposition 1]*) we can replace* $\operatorname{level}(K_M)$ *by* $\operatorname{vol}(K_M)^{-1}$ *in the definition of (TWN) (as well as in* (4.20)*).*

For $\mathrm{GL}(n)$ the the normalizing factors are expressed in terms of Rankin-Selberg L functions (see (2.3)). The known properties of Rankin-Selberg L-functions lead to the estimation (3.23), which implies the desired estimation. By [FLM2, Lemma 5.4], the case of $\mathrm{SL}(n)$ can be reduced to $\mathrm{GL}(n)$. In this way we get

Theorem 4.8. *The estimate* (4.20) *holds for* $\mathbf{G} = \mathrm{GL}(n)$ *or* $\mathrm{SL}(n)$ *with an implied constant depending only on* n. *In particular, the groups* $\mathrm{GL}(n)$ *and* $\mathrm{SL}(n)$ *satisfy the property (TWN).*

(see [FLM2]).

Remark 4.9. *For general groups* **G** *the normalizing factors are given, at least up to local factors, by quotients of automorphic L-functions associated to the irreducible constituents of the adjoint action of the L-group $^L M$ of* **M** *on the unipotent radical of the corresponding parabolic subgroup of $^L G$ [La]. To argue as above, we would need to know that these L-functions have finitely many poles and satisfy a functional equation with the associated conductor bounded by an arbitrary power of* level(K_M) *for automorphic representations $\pi \in \Pi_{\mathrm{disc}}(\mathbf{M}(\mathbb{A}))^{K_M}$. Unfortunately, finiteness of poles and the expected functional equation are not known in general. It is possible that for classical groups these properties are within reach.*

Now we come to the second condition, which is a condition on the local intertwining operators. Recall that for a finite prime p, the matrix coefficients of the local normalized intertwining operators $R_{Q|P}(\pi_p, s)^{K_p}$ are rational functions of p^s. Moreover, their denominators can be controlled in terms of π_p, and the degrees of these denominators are bounded in terms of **G** only. For any Levi subgroup $\mathbf{M} \in \mathcal{L}$ let $\mathbf{G_M}$ be the closed subgroup of **G** generated by the unipotent radicals $\mathbf{U}_P$, $\mathbf{P} \in \mathcal{P}(\mathbf{M})$. It is a connected semisimple normal subgroup of **G**

Definition 2. *We say that* **G** *satisfies (BD) (bounded degree) if there exists a constant c (depending only on* **G** *and ρ), such that for any $\mathbf{M} \in \mathcal{L}$, $\mathbf{M} \neq \mathbf{G}$, and adjacent parabolic groups $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(\mathbf{M})$, any prime p, any open subgroup $K_p \subset \mathbf{K}_p$ and any smooth irreducible representation π_p of $\mathbf{M}(\mathbb{Q}_p)$, the degrees of the numerators of the linear operators $R_{Q|P}(\pi_p, s)^{K_p}$ are bounded by $c \log_p$ level$^{G_M}(K_P)$ if $\mathbf{K}_p$ is hyperspecial, and by $c(1 + \log_p$ level$^{\mathbf{G_M}}(K_p))$, otherwise.*

Property (BD) has been studied in [FLM3]. By [FLM3, Theorem 1, Proposition 6] we have the following theorem.

Theorem 4.10. *The groups* GL(n) *and* SL(n) *satisfy (BD).*

The property (BD) has the following consequence.

Proposition 4.11. *Suppose that* **G** *satisfies (BD). Let $\mathbf{M} \in \mathcal{L}$ and let $\mathbf{P}, \mathbf{Q} \in \mathcal{P}(\mathbf{M})$ be adjacent parabolic subgroups. Then for all $\pi \in \Pi_{\mathrm{disc}}(\mathbf{M}(\mathbb{A}))$, for all open subgroups $K \subset \mathbf{K}_{\mathrm{fin}}$ and all $\tau \in \Pi(\mathbf{K}_\infty)$ we have*

$$(4.21) \qquad \int_{i\mathbb{R}} \left\| R_{Q|P}(\pi, s)^{-1} \frac{d}{ds} R_{Q|P}(\pi, s) \Big|_{I_P^G(\pi)^{\tau, K}} \right\| (1 + |s|^2)^{-1} \, ds$$
$$\ll 1 + \log(\|\tau\| + \mathrm{level}(K; \mathbf{G_M^+})).$$

The proof of the proposition follows from a generalization of Bernstein's inequality [BE]. Suppose that **G** satisfies (TWN) and (BD). Combining (4.19) and (4.21) we get an appropriate estimate for the corresponding integral involving the logarithmic derivative of the intertwining operators.

4.3. **Application to the limit multiplicity problem.** The limit multiplicity property is a consequence of properties (TWN) and (BD). The proof proceeds by induction over the Levi subgroups of $\mathbf{G}$. The property that is suitable for the induction procedure is not the spectral limit property, but a property that we call *polynomial boundedness* (PB). This is a weaker version of the statement of Conjecture 2.

We write $\mathcal{D}$ for the set of all conjugacy classes of pairs (M, δ) consisting of a Levi subgroup M of $\mathbf{G}(\mathbb{R})^1$ and a discrete series representation δ of M^1, where $M = A_M \times M^1$ and A_M is the largest central subgroup of M isomorphic to a power of $\mathbb{R}^{>0}$. For any $\underline{\delta} \in \mathcal{D}$ let $\Pi(\mathbf{G}(\mathbb{R})^1)_{\underline{\delta}}$ be the set of all irreducible unitary representations which arise by the Langlands quotient construction from the irreducible constituents of $I_M^L(\delta)$ for Levi subgroups $L \supset M$. Here, I_M^L denotes (unitary) induction from an arbitrary parabolic subgroup of L with Levi subgroup M to L.

Definition 3. *Let $\mathfrak{M}$ be a set of Borel measures on $\Pi(\mathbf{G}(\mathbb{R})^1)$. We call $\mathfrak{M}$ polynomially bounded (PB), if for all $\underline{\delta} \in \mathcal{D}$ there exist $N_{\underline{\delta}} > 0$ such that*

$$\mu\left(\{\pi \in \Pi(\mathbf{G}(\mathbb{R})^1)_{\underline{\delta}} \colon |\lambda_\pi| \leq R\}\right) \ll_{\underline{\delta}} (1 + R)^{N_{\underline{\delta}}}$$

for all $\mu \in \mathfrak{M}$ and $R > 0$.

Now consider the measures μ_K defined by (4.11). Let $\mathbf{M} \in \mathcal{L}$ and denote by $\mathbf{K}_M(N)$ the congruence subgroups of $\mathbf{M}(\mathbb{A}_{\mathrm{fin}})$, defined by (4.18). The key result is the following lemma.

Lemma 4.12. *Suppose that $\mathbf{G}$ satisfies (TWN) and (BD). Then for each $\mathbf{M} \in \mathcal{L}$, the collection of measures $\{\mu_{\mathbf{K}_M(N)}^{\mathbf{M}}\}$, $N \in \mathbb{N}$, is polynomially bounded.*

This has the consequence that if $\mathbf{G}$ satisfies (TWN) and (BT), then for every $\mathbf{M} \neq \mathbf{G}$ and $f \in C_{c,\mathrm{fin}}^\infty(\mathbf{G}(\mathbb{R})^1)$ we have

$$J_{\mathrm{spec},M}(f \otimes \mathbf{1}_{\mathbf{K}(N)}) \to 0$$

as $N \to \infty$. Thus by Theorem 1 it follows that if $\mathbf{G}$ satisfies (TWN) and (BT), then for every $f \in C_{c,\mathrm{fin}}^\infty(\mathbf{G}(\mathbb{R})^1)$ we have

$$J_{\mathrm{spec}}(f \otimes \mathbf{1}_{\mathbf{K}(N)}) - \mathrm{tr}\, R_{\mathrm{disc}}(f \otimes \mathbf{1}_{\mathbf{K}(N)}) \to 0$$

for $n \to \infty$. Thus the spectral limit property is satisfied in this case. By Theorems 4.8 and 4.10, the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ satisfy (TWN) and (BD) and therefore, the spectral limit property holds for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$.

To deal with the geometric limit property we use the coarse geometric expansion

$$(4.22) \qquad\qquad J^T(h) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^T(h), \quad h \in C_c^\infty(\mathbf{G}(\mathbb{A})^1),$$

(see (2.4) for the notation). Write $J_{\mathfrak{o}}(f) = J_{\mathfrak{o}}^{T_0}(f)$, which depends only on $\mathbf{M}_0$ and $\mathbf{K}$. Let J_{unip}^T be the contribution of the unipotent elements of $G(\mathbb{Q})$ to the trace formula (2.4), which is a polynomial in $T \in \mathfrak{a}_{M_0}$ of degree at most $d_0 = \dim \mathfrak{a}_{M_0}^G$ [Ar7]. It can be split

into the contributions of the finitely many $G(\bar{\mathbb{Q}})$-conjugacy classes of unipotent elements of $\mathbf{G}(\mathbb{Q})$. It is well known ([ibid., Corollary 4.4]) that the contribution of the unit element is simply the constant polynomial $\mathrm{vol}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1)h(1)$. Write

$$J^T_{\mathrm{unip}-\{1\}}(h) = J^T_{\mathrm{unip}}(h) - \mathrm{vol}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1)h(1), \quad h \in C^\infty_c(\mathbf{G}(\mathbb{A})^1).$$

The distribution J_{unip} is defined as $J^{T_0}_{\mathrm{unip}}$ for a certain vector $T_0 \in \mathfrak{a}_{M_0}$ depending only on $\mathbf{G}$, and analogously for $J_{\mathrm{unip}-\{1\}}$. Since the groups $\mathbf{K}(N)$ form a neighborhood base of the identity element in $\mathbf{G}(\mathbb{A}_{\mathrm{fin}})$, it is easy to see that for a given $h \in C^\infty_c(\mathbf{G}(\mathbb{A})^1)$, for all but finitely many N one has

$$(4.23) \qquad\qquad J(h \otimes \mathbf{1}_{\mathbf{K}(N)}) = J_{\mathrm{unip}}(h \otimes \mathbf{1}_{\mathbf{K}(N)}).$$

For any compact subset $\Omega \subset G(\mathbb{R})^1$ we write $C^\infty_\Omega(\mathbf{G}(\mathbb{R})^1)$ for the Fréchet space of all smooth functions on $G(\mathbb{R})^1$ supported in Ω equipped with the seminorms $\sup_{x\in\Omega}|(Xh)(x)|$, where X ranges over the left-invariant differential operators on $\mathbf{G}(\mathbb{R})$. The key result is the following proposition.

Proposition 4.13. *For any compact subset $\Omega \subset \mathbf{G}(\mathbb{R})^1$ there exists a seminorm $||\cdot||$ on $C^\infty_\Omega(\mathbf{G}(\mathbb{R})^1)$ such that*

$$|J_{\mathrm{unip}-\{1\}}(h_S \otimes \mathbf{1}_{\mathbf{K}(N)})| \leq \frac{(1+\log(N))}{N}||h||$$

for all $h \in C^\infty_\Omega(\mathbf{G}(\mathbb{R})^1)$ and all $N \in \mathbb{N}$.

The proof of Proposition 4.13 consists of a slight extension of Arthur's arguments in [Ar7]. Combining (4.23) and Proposition 4.13 the geometric limit property follows. This completes the proof of Theorem 4.2 for $F = \mathbb{Q}$. The case of a general F is proved similarly. For details see [FLM2].

5. ANALYTIC TORSION AND TORSION IN THE COHOMOLOGY OF ARITHMETIC GROUPS

The theorem of DeGeorge and Wallach on limit multiplicities for discrete series [DW1] implies the statement (1.4) on the approximation of L^2-Betti numbers by normalized Betti numbers of finite covers [AB2]. For towers of normal subgroups of finite index, Lück [Lu1] proved this in the more general context of finite CW complexes. This is part of his study of the approximation of L^2-invariants by their classical counterparts [Lu2]. A more sophisticated spectral invariant is the analytic torsion introduced by Ray and Singer [RS]. The study of the corresponding approximation problem has interesting applications to the torsion in the cohomology of arithmetic groups.

5.1. **Analytic torsion and L^2-torsion.** Let X be a compact Riemannian manifold of dimension n and let $\rho\colon \pi_1(X) \to \mathrm{GL}(V)$ a finite dimensional representation of its fundamental group. Let $E_\rho \to X$ be the flat vector bundle associated with ρ. Choose a Hermitian fiber metric in E_ρ. Let $\Delta_p(\rho)$ be the Laplace operator on E_ρ-valued p-forms with respect to the metrics on X and in E_ρ. It is an elliptic differential operator, which is formally self-adjoint and non-negative. Since X is compact, $\Delta_p(\rho)$ has a pure discrete spectrum consisting of sequence of eigenvalues $0 \le \lambda_0 \le \lambda_1 \le \cdots \to \infty$ of finite multiplicity. Let

$$(5.1) \qquad \zeta_p(s;\rho) := \sum_{\lambda_j > 0} \lambda_j^{-s}$$

be the zeta function of $\Delta_p(\rho)$. The series converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > n/2$ and admits a meromorphic extension to $s \in \mathbb{C}$, which is holomorphic at $s = 0$. Then the Ray-Singer analytic torsion $T_X(\rho) \in \mathbb{R}^+$ is defined by

$$(5.2) \qquad T_X(\rho) := \exp\left(\frac{1}{2} \sum_{p=1}^n (-1)^p p \frac{d}{ds} \zeta_p(s;\rho)\big|_{s=0} \right).$$

It depends on the metrics on X and E_ρ. However, if $\dim(X)$ is odd and ρ acyclic, which means that $H^*(X, E_\rho) = 0$, then $T_X(\rho)$ is independent of the metrics [Mu3]. The analytic torsion has a topological counterpart. This is the Reidemeister torsion $T_X^{\mathrm{top}}(\rho)$ (usually it is denoted by $\tau_X(\rho)$), which is defined in terms of a smooth triangulation of X [RS], [Mu1]. It is known that for unimodular representations ρ (meaning that $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(X)$) one has the equality

$$(5.3) \qquad T_X(\rho) = T_X^{\mathrm{top}}(\rho)$$

[Ch], [Mu1]. The in general case of a non-unimodular representation the equality does not hold, but the defect can be described [BMZ].

Let $X_i \to X$, $i \in \mathbb{N}$, be sequence of finite coverings of X. Let $\inf(X_j)$ denote the injectivity radius of X_j and assume that $\mathrm{inj}(X_j) \to \infty$ as $j \to \infty$. Then the question is: Does

$$(5.4) \qquad \frac{\log T_{X_j}(\rho)}{\mathrm{vol}(X_j)}$$

converge as $j \to \infty$ and if so, what is the limit? For a tower of normal coverings and the trivial representation ρ_0 a conjecture of Lück [Lu2, Conjecture 7.4] states that the sequence (5.4) converges and the limit is the L^2-torsion, first introduced by Lott [Lo] and Mathei [MV]. The L^2-torsion is defined as follows. Recall that the zeta function $\zeta_p(s)$ can be expressed in terms of the heat operator

$$\zeta_p(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\mathrm{Tr}\left(e^{-t\Delta_p} \right) - b_p \right) t^{s-1}\, dt,$$

where b_p is the p-th Betti number and $\mathrm{Re}(s) > n/2$. Let $e^{-t\widetilde{\Delta}_p}$ be the heat operator of the Laplace operator $\widetilde{\Delta}_p$ on p-forms on the universal covering $\widetilde{X}$ of X. Let $\widetilde{K}_p(t, x, y)$ be the

kernel of $e^{-t\widetilde{\Delta}_p}$. Note that $\widetilde{K}_p(t, x, y)$ is a homomorphism of $\Lambda^p T_y^\star(X)$ to $\Lambda^p T_x^\star(X)$. Let $F \subset \widetilde{X}$ be a fundamental domain for the action of $\Gamma := \pi_1(X)$ on $\widetilde{X}$. Then the Γ-trace of $e^{-t\widetilde{\Delta}_p(\rho)}$ is defined as

$$(5.5) \qquad \mathrm{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p}\right) := \int_F \mathrm{tr}\,\widetilde{K}_p(t, x, x)\,dx.$$

The L^2-Betti number $b_p^{(2)}$ is defined as

$$b_p^{(2)} := \lim_{t \to \infty} \mathrm{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p}\right).$$

In order to be able to define the Mellin transform of the Γ-trace one needs to know the asymptotic behavior of $\mathrm{Tr}_\Gamma(e^{-t\widetilde{\Delta}_p})$ as $t \to 0$ and $t \to \infty$. Using a parametrix for the heat kernel which is pulled back from a parametrix on X, one can show that for $t \to 0$, $\mathrm{Tr}_\Gamma(e^{-t\widetilde{\Delta}_p})$ has an asymptotic expansion similar to the compact case [Lo]. For the large time behavior we need to introduce the Novikov-Shubin invariants

$$(5.6) \qquad \widetilde{\alpha}_p = \sup\left\{\beta_p \in [0, \infty)\colon \mathrm{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p}\right) - b_p^{(2)} = O(t^{-\beta_p/2}) \ \text{as}\ t \to \infty\right\}$$

Assume that $\widetilde{\alpha}_p > 0$ for all $p = 1, \dots, n$. Then the L^2- torsion $T_X^{(2)} \in \mathbb{R}^+$ can be defined by

$$(5.7) \qquad \log T_X^{(2)} = \frac{1}{2}\sum_{p=1}^n (-1)^p p \left[\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^1 \mathrm{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}'_p}\right)t^{s-1}\,dt\right)\bigg|_{s=0} \right.$$
$$\left. + \int_1^\infty t^{-1}\,\mathrm{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}'_p}\right)\,dt\right],$$

where $\widetilde{\Delta}'_p$ denotes the restriction of $\widetilde{\Delta}_p$ to the orthogonal complement of $\ker \widetilde{\Delta}_p$ and the first integral is defined near $s = 0$ by analytic continuation. This definition can be generalized to all finite dimensional representations ρ of Γ, if the corresponding Novikov-Shubin invariants are all positive. Then the L^2-torsion $T_X^{(2)}(\rho)$ is defined as in (5.7). If there exists $c > 0$ such that the spectrum of $\Delta_p(\rho)$ is bounded from below by c, then the integral

$$\int_0^\infty \mathrm{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\rho)}\right)t^{s-1}\,dt$$

converges for $\mathrm{Re}(s) > n/2$ and admits a meromorphic continuation to $\mathbb{C}$ which is holomorphic at $s = 0$. Thus, if there is a positive lower bound of the spectrum of all $\Delta_p(\rho)$, $p = 1, \dots, n$, then $T_X^{(2)}(\rho)$ can be defined in the usual way by

$$\log T_X^{(2)}(\rho) = \frac{1}{2}\sum_{p=1}^n (-1)^p p\,\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\rho)}\right)t^{s-1}\,dt\right)\bigg|_{s=0}.$$

Let $\Gamma = \pi_1(X, x_0)$ and let $(\Gamma_i)_{i \in \mathbb{N}_0}$ be a tower of normal subgroups of finite index of $\Gamma = \Gamma_0$. Let $X_i = \Gamma_i \backslash \widetilde{X}$, $i \in \mathbb{N}_0$, be the corresponding covering of X. Let T_X and $T_X^{(2)}$ denote the analytic torsion and L^2-torsion with respect to the trivial representation. W. Lück [Lu2, Conjecture 7.4] has made the following conjecture.

Conjecture 3. *For every closed Riemannian manifold X the L^2-torsion $T_X^{(2)}$ exists and for a sequence of coverings $(X_i \to X)_{i \in \mathbb{N}}$ as above one has*

$$\lim_{i \to \infty} \frac{\log T_{X_i}}{[\Gamma : \Gamma_i]} = \log T_X^{(2)}.$$

One is tempted to make this conjecture for any finite dimensional representation ρ.

5.2. **Compact locally symmetric spaces.** Now we turn to the locally symmetric case. Let $X = \Gamma \backslash \widetilde{X}$, where $\widetilde{X} = G/K$ is a Riemannian symmetric space of non-positive curvature and $\Gamma \subset G$ is a discrete, torsion free, cocompact subgroup. Let τ be an irreducible finite dimensional complex representation of **G**. Let $E_\tau \to X$ be the flat vector bundle associated to the representation $\tau|_\Gamma$ of Γ. By [MM], E_τ can be equipped with a canonical Hermitian fiber metric, called admissible, which is unique up to scaling. Let $\Delta_p(\tau)$ be the Laplace operator on p-forms with values in E_τ, with respect to the choice of any admissible fiber metric in E_τ. Let $T_X(\tau)$ be the corresponding analytic torsion. Let $\widetilde{\Delta}_p(\tau)$ be the Laplace operator on $\widetilde{E}_\tau$-valued p-forms on $\widetilde{X}$. Let $\widetilde{E}_\tau \to \widetilde{X}$ be the homogeneous vector bundle defined by $\tau|_K$. By [MM] there is a canonical isomorphism

$$E_\tau \cong \Gamma \backslash \widetilde{E}_\tau$$

and the metric on E_τ is induced by the homogeneous metric on $\widetilde{E}_\tau$. Thus

$$(5.8) \qquad\qquad C^\infty(\widetilde{X}, \widetilde{E}_\tau) \cong (C^\infty(G) \otimes V_\tau)^K.$$

Let R be the right regular representation of G in $C^\infty(G)$ and let let $R(\Omega)$ be the operator in $(C^\infty(G) \otimes V_\tau)^K$ induced by the Casimir element. Then with respect to the isomorphism (5.8) we have

$$\widetilde{\Delta}_p(\tau) = -R(\Omega) + \lambda_\tau \operatorname{Id}$$

(see [MM]). This implies that the heat operator $e^{-t\widetilde{\Delta}_p(\tau)}$ is a convolution operator given by a kernel

$$H_t^{p,\tau} \colon G \to \operatorname{End}(\Lambda^p \mathfrak{p}^\star \otimes V_\tau).$$

Let $h_t^{p,\tau} \in C^\infty(G)$ be defined by $h_t^{p,\tau}(g) = \operatorname{tr} H_t^{p,\tau}(g)$, $g \in G$. Then it follows from (5.5) that

$$(5.9) \qquad\qquad \operatorname{Tr}_\Gamma \left(e^{-t\widetilde{\Delta}_p(\tau)} \right) = \operatorname{vol}(X) h_t^{p,\tau}(1).$$

Now one can use the Plancherel theorem to compute $h_t^{p,\tau}(1)$ and determine its asymptotic bahavior as $t \to 0$ and $t \to \infty$. For the trivial representation this was carried out in [Ol] and for strongly acyclic τ in [BV]. So let $\widetilde{\Delta}_p(\tau)'$ be the restriction of $\widetilde{\Delta}_p(\tau)$ to the orthogonal complement of the kernel of $\widetilde{\Delta}_p(\tau)$. Now let

$$(5.10) \qquad \widetilde{\alpha}_p(X, \tau) := \sup \left\{ \beta_p \in [0, \infty) \colon \operatorname{Tr}_\Gamma \left(e^{-t\widetilde{\Delta}_p(\tau)'} \right) = O(t^{-\beta_p/2}) \text{ as } t \to \infty \right\},$$

$p = 0, \ldots, n$, be the twisted Novikov-Shubin invariants. Assume that $\widetilde{\alpha}_p(X, \tau) > 0$, $p = 0, \ldots, n$. Then the L^2-torsion $T_X^{(2)}(\tau)$ is defined. By [Ol, Theorem 1.1] this is the case for

the trivial representation. Furthermore, if τ is strongly acyclic, then $\widetilde{\alpha}_p(X, \tau) = \infty$ for all p. Using the definition of the L^2-torsion, it follows that

$$\log T_X^{(2)}(\tau) = \operatorname{vol}(X) t_{\widetilde{X}}^{(2)}(\tau), \tag{5.11}$$

where $t_{\widetilde{X}}^{(2)}(\tau)$ is a constant that depends only on $\widetilde{X}$ and τ.

Now let (Γ_j) be sequence of torsion free cocompact lattices in G. Let $X_j = \Gamma_j \backslash \widetilde{X}$ and assume that $\operatorname{inj}(X_j) \to \infty$ if $j \to \infty$. A representation $\tau \colon G \to \operatorname{GL}(V)$ is called *strongly acyclic*, if there is $c > 0$ such that the spectrum of $\Delta_{X_j,p}(\tau)$ is contained in $[c, \infty)$ for all $j \in \mathbb{N}$ and $p = 0, \ldots, n$.

Now let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group. Let $G = \mathbf{G}(\mathbb{R})$. Then it is proved in [BV] that strongly acylic representations exist. For such representations Bergeron and Venkatesh [BV, Theorem 4.5] established the following theorem.

Theorem 5.1. *Let* $\tau \colon G \to \operatorname{GL}(V)$ *be strongly acyclic. Then*

$$\lim_{j \to \infty} \frac{\log(T_{X_j}(\tau))}{\operatorname{vol}(X_j)} = t_X^{(2)}(\tau), \tag{5.12}$$

where $X_j = \Gamma_j \backslash \widetilde{X}$ *and* $\operatorname{inj}(X_j) \to \infty$ *as* $j \to \infty$.

The number $t_X^{(2)}(\tau)$ can be computed using the Plancherel theorem. Let $\delta(G) = \operatorname{rank}(G) - \operatorname{rank}(K)$ be the fundamental rank or "deficiency" of G. By [BV, Proposition 5.2] one has

Proposition 5.2. *If* $\delta(G) \neq 1$, *then* $t_X^{(2)}(\tau) = 0$. *For* $\delta(G) = 1$ *one has*

$$(-1)^{\frac{\dim \widetilde{X} - 1}{2}} t_X^{(2)}(\tau) > 0.$$

We note that the simple Lie groups G with $\delta(G) = 1$ are $\operatorname{SL}_3(\mathbb{R})$ and $\operatorname{SO}(p, q)$ with pq odd, especially $G = \operatorname{SO}^0(2m + 1, 1)$ is a group with fundamental rank 1.

Next we briefly recall the main steps of the proof of Theorem 5.1. To indicate the dependence of the heat operator and other quantities on the covering X_j, we use the subscript X_j. The uniform spectral gap at 0 implies that there exist constants $C, c > 0$ such that for all $p = 0, \ldots, n$, $j \in \mathbb{N}$ and $t \geq 1$ one has

$$\operatorname{Tr}\left(e^{-t\Delta_{X_j,p}(\tau)}\right) \leq Ce^{-tc}\operatorname{vol}(X_j) \tag{5.13}$$

(see [BV]). This is the key result that makes the method to work. Let

$$K_{X_j}(t, \tau) := \frac{1}{2}\sum_{p=1}^{n}(-1)^p p \operatorname{Tr}\left(e^{-t\Delta_{X_j,p}(\tau)}\right). \tag{5.14}$$

Using (5.13) it follows that the analytic torsion can be defined by

$$\log T_{X_j}(\tau) = \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty K_{X_j}(t, \tau)t^{s-1}\,dt\right)\Big|_{s=0}. \tag{5.15}$$

Let $T > 0$. Then we can split the integral and rewrite the right hand side as

$$\log T_{X_j}(\tau) = \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T K_{X_j}(t,\tau)t^{s-1}\,dt\right)\bigg|_{s=0} + \int_T^\infty K_{X_j}(t,\tau)t^{-1}\,dt.$$

By (5.13) there exist $C, c > 0$ such that

(5.16) $$\frac{1}{\text{vol}(X_j)}\left|\int_T^\infty K_{X_j}(t,\tau)t^{-1}\,dt\right| \le Ce^{-cT}$$

for all $j \in \mathbb{N}_0$ and $T > 1$. To deal with the first term one can use the Selberg trace formula. Put

$$k_t^\tau := \frac{1}{2}\sum_{p=1}^n (-1)^p p h_t^{p,\tau}.$$

Then the Selberg trace formula gives

$$K_{X_j}(t,\tau) = \text{vol}(X_j)k_t^\tau(1) + H_{X_j}(k_t^\tau),$$

where $H_{X_j}(k_t^\tau)$ is the contribution of the hyperbolic conjugacy classes. Using (5.9) and the definition of k_t^τ, it follows that

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^T k_t^\tau(1)t^{s-1}\,dt\right)\bigg|_{s=0} = t_{\widetilde{X}}^{(2)}(\tau) + O\left(e^{-cT}\right)$$

as $T \to \infty$. Regrouping the terms of the hyperbolic contribution $H_{X_j}(k_t^\tau)$ as in (4.9) it follows that the corresponding integral divided by $\text{vol}(X_j)$ converges to 0 as $j \to \infty$. This proves the theorem.

One expects Theorem 5.1 to be true in general. However, if there is no spectral gap at zero, one cannot argue as above. The key problem is to control the small eigenvalues as $j \to \infty$. Sufficient conditions on the behavior of the small eigenvalues are discussed in [Lu2] and in the 3-dimensional case also in [BSV].

In view of the potential applications to the cohomology of arithmetic groups, discussed in the next section, it is very desirable to extend Theorem 5.1 to the non-compact case. The first problem one faces is that the corresponding Laplace operators have a nonempty continuous spectrum and therefore, the heat operators are not trace class and the analytic torsion can not be defined as above. This problem has been studied by Raimbault [Ra1] for hyperbolic 3-manifolds and in [MP2] for hyperbolic manifolds of any dimension.

So let $G = \text{SO}^0(n,1)$, $K = \text{SO}(n)$ and $\widetilde{X} = G/K$. Equipped with a suitably normalized G-invariant metric, $\widetilde{X}$ becomes isometric to the n-dimensional hyperbolic space $\mathbb{H}^n$. Let $\Gamma \subset G$ be a torsion free lattice. Then $X = \Gamma\backslash\widetilde{X}$ is an oriented n-dimensional hyperbolic manifold of finite volume. As above, let $\tau\colon G \to \text{GL}(V)$ be a finite dimensional complex representation of G. The first step is to define a regularized trace of the heat operators $e^{-t\Delta_p(\tau)}$. To this end one uses an appropriate height function to truncate X at sufficient high level $Y > Y_0$ to get a compact manifold $X(Y) \subset X$ with boundary $\partial X(Y)$, which consists of a disjoint union of $n-1$-dimensional tori. Let $K^{p,\tau}(t,x,y)$ be the kernel of the

heat operator $e^{-t\Delta_p(\tau)}$. Using the spectral resolution of $\Delta_p(\tau)$, it follows that there exist $\alpha(t) \in \mathbb{R}$ such that $\int_{X(Y)} \operatorname{tr} K^{p,\tau}(t,x,x)\ dx - \alpha(t) \log Y$ has a limit as $Y \to \infty$. Then we define the regularized trace as

$$(5.17) \qquad \operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) := \lim_{Y \to \infty}\left(\int_{X(Y)} \operatorname{tr} K^{p,\tau}(t,x,x)\ dx - \alpha(t)\log Y\right).$$

We note that the regularized trace is not uniquely defined. It depends on the choice of truncation parameters on the manifold X. However, if $X_0 = \Gamma_0\backslash\mathbb{H}^n$ is given and if truncation parameters on X_0 are fixed, then every finite covering X of X_0 is canonically equipped with truncation parameters, namely one simply pulls back the height function on X_0 to a height function on X via the covering map.

Let θ be the Cartan involution of G with respect to $K = \mathrm{SO}(n)$. Let $\tau_\theta = \tau \circ \theta$. If $\tau \not\cong \tau_\theta$, it can be shown that $\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ is exponentially decreasing as $t \to \infty$ and admits an asymptotic expansion as $t \to 0$. Therefore, the regularized zeta function $\zeta_{\mathrm{reg},p}(s;\tau)$ of $\Delta_p(\tau)$ can be defined as in the compact case by

$$(5.18) \qquad \zeta_{\mathrm{reg},p}(s;\tau) := \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) t^{s-1}\ dt.$$

The integral converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > n/2$ and admits a meromorphic extension to the whole complex plane, which is holomorphic at $s = 0$. So in analogy with the compact case, the regularized analytic torsion $T_X(\tau) \in \mathbb{R}^+$ can be defined by the same formula (5.2).

In even dimension the analytic torsion is rather trivial. Therefore, we assume that $n = 2m + 1$. Furthermore, for technical reasons we assume that every lattice $\Gamma \subset G$ satisfies the following condition: For every Γ-cuspidal parabolic subgroup P of G one has

$$(5.19) \qquad \Gamma \cap P = \Gamma \cap N_P,$$

where N_P denotes the unipotent radical of P. Let Γ_0 be a fixed lattice in G and let $X_0 = \Gamma_0\backslash\widetilde{X}$. Let Γ_j, $j \in \mathbb{N}$, be a sequence of finite index torsion free subgroups of Γ_0. This sequence is called to be *cusp uniform*, if the tori which arise as cross sections of the cusps of the manifolds $X_J := \Gamma_j\backslash\widetilde{X}$ satisfy some uniformity condition (see [MP2, Definition 8.2]).

The following theorem and its corollaries are established in [MP2]. One of the main results of [MP2] is the following theorem which may be regarded as an analog of Theorem 5.1 for oriented finite volume hyperbolic manifolds.

Theorem 5.3. *Let Γ_0 be a lattice in G and let Γ_i, $i \in \mathbb{N}$, be a sequence of finite-index normal subgroups which is cusp uniform and such that each Γ_i, $i \geq 1$, is torsion-free and satisfies (5.19). If $\lim_{i\to\infty}[\Gamma_0 : \Gamma_i] = \infty$ and if each $\gamma_0 \in \Gamma_0 - \{1\}$ only belongs to finitely many Γ_i, then for each τ with $\tau \neq \tau_\theta$ one has*

$$(5.20) \qquad \lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{[\Gamma : \Gamma_i]} = t^{(2)}_{\mathbb{H}^n}(\tau)\operatorname{vol}(X_0).$$

In particular, if under the same assumptions Γ_i *is a tower of normal subgroups, i.e.* $\Gamma_{i+1} \subset \Gamma_i$ *for each i and* $\cap_i \Gamma_i = \{1\}$, *then* (5.20) *holds.*

For hyperbolic 3-manifolds, Theorem 5.3 was proved by J. Raimbault [Ra1] under additional assumptions on the intertwining operators. We emphasize that the above theorem holds without any additional assumptions.

Now we specialize to arithmetic groups. First consider $\Gamma_0 := \mathrm{SO}^0(n,1)(\mathbb{Z})$. Then Γ_0 is a lattice in $\mathrm{SO}^0(n,1)$. For $q \in \mathbb{N}$ let $\Gamma(q)$ be the principal congruence subgroup of Γ_0 of level q. Using a result of Deitmar and Hoffmann [DH], it follows that the family of principal congruence subgroups $\Gamma(q)$ is cusp uniform [MP2, Lemma 10.1]. Thus Theorem 5.3 implies the following corollary (see [MP2, Corollary 1.3]).

Corollary 5.4. *For any finite dimensional irreducible representation* τ *of* $\mathrm{SO}^0(n,1)$ *with* $\tau \not\cong \tau_\theta$ *the principal congruence subgroups* $\Gamma(q)$, $q \geq 3$, *of* $\Gamma_0 := \mathrm{SO}^0(n,1)(\mathbb{Z})$ *satisfy*

$$\lim_{q \to \infty} \frac{\log T_{X_q}(\tau)}{[\Gamma : \Gamma(q)]} = t_{\mathbb{H}^n}^{(2)}(\tau) \, \mathrm{vol}(X_0),$$

where $X_q := \Gamma(q) \backslash \mathbb{H}^n$ *and* $X_0 := \Gamma_0 \backslash \mathbb{H}^n$.

We recall that by Proposition 5.2 we have $(-1)^{\frac{n-1}{2}} t_{\mathbb{H}^n}^{(2)}(\tau) > 0$.

Next we consider the 3-dimensional case. We note that every lattice $\Gamma \subset \mathrm{SO}^0(3,1)$ can be lifted to a lattice $\Gamma' \subset \mathrm{Spin}(3,1)$. Moreover, recall that there is a natural isomorphism $\mathrm{Spin}(3,1) \cong \mathrm{SL}_2(\mathbb{C})$. If ρ is the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$, then the finite dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ are given by $\mathrm{Sym}^p \rho \otimes \mathrm{Sym}^q \bar\rho$, $p, q \in \mathbb{N}$, where Sym^k denotes the k-th symmetric power and $\bar\rho$ denotes the complex conjugate representation to ρ. One has $(\mathrm{Sym}^p \rho \otimes \mathrm{Sym}^q \bar\rho)_\theta = \mathrm{Sym}^q \rho \otimes \mathrm{Sym}^p \bar\rho$. For $D \in \mathbb{N}$ square free let $\mathcal{O}_D$ be the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ and let $\Gamma(D) := \mathrm{SL}_2(\mathcal{O}_D)$. Then $\Gamma(D)$ is a lattice in $\mathrm{SL}_2(\mathbb{C})$. If $\mathfrak{a}$ is a non-zero ideal in $\mathcal{O}_D$, let $\Gamma(\mathfrak{a})$ be the associated principal congruence subgroup of level $\mathfrak{a}$. Then Theorem 5.1 implies the following corollary (see [MP2, Corollary 1.4].

Corollary 5.5. *Let* $D \in \mathbb{N}$ *be square free. Let* $\mathfrak{a}_i$ *be a sequence of non-zero ideals in* $\mathcal{O}_D$ *such that each* $N(\mathfrak{a}_i)$ *is sufficiently large and such that* $\lim_{i \to \infty} N(\mathfrak{a}_i) = \infty$. *Put* $X_D := \Gamma(D) \backslash \mathbb{H}^3$ *and* $X_i := \Gamma(\mathfrak{a}_i) \backslash \mathbb{H}^3$. *Let* $\tau = \mathrm{Sym}^p \rho \otimes \mathrm{Sym}^q \bar\rho$ *with* $p \neq q$. *Then one has*

$$\lim_{i \to \infty} \frac{\log T_{X_i}(\tau)}{[\Gamma(D) : \Gamma(\mathfrak{a}_i)]} = t_{\mathbb{H}^3}^{(2)}(\tau) \, \mathrm{vol}(X_D).$$

5.3. Applications to the cohomology of arithmetic groups - the cocompact case.
Theorem 5.1 has interesting consequences for the cohomology of arithmetic groups. Let $\Gamma \subset G$ be a discrete, torsion free, cocompact subgroup. Let $\tau \colon G \to \mathrm{GL}(V)$ be a finite dimensional real representation and let $E \to X$ be the associated vector bundle. Choose a fiber metric h in E. Assume that there exist a Γ-invariant lattice $M \subset V$. Let $\mathcal{M}$ be the associated local system of free $\mathbb{Z}$-modules over X. Then we have $E = \mathcal{M} \otimes \mathbb{R}$. Let

$H^\star(X, \mathcal{M})$ be the cohomology of X with coefficients in $\mathcal{M}$. Each $H^q(X, \mathcal{M})$ is a finitely generated $\mathbb{Z}$-module. Let $H^q(X, \mathcal{M})_{\text{tors}}$ be the torsion subgroup and

$$H^q(X; \mathcal{M})_{\text{free}} = H^q(X, \mathcal{M})/H^q(X, \mathcal{M})_{\text{tors}}.$$

We identify $H^q(X, \mathcal{M})_{\text{free}}$ with a subgroup of $H^q(X, E)$. Let $\langle \cdot, \cdot \rangle_q$ be the inner product in $H^q(X, E)$ induced by the L^2-metric on $\mathcal{H}^q(X, E)$. Let $e_1, ..., e_{r_q}$ be a basis of $H^q(X, \mathcal{M})_{\text{free}}$ and let G_q be the Gram matrix with entries $\langle e_k, e_l \rangle$. Put

$$R_q(\tau, h) = \sqrt{|\det G_q|}, \quad q = 0, ..., n.$$

Define the "regulator" $R(\tau, h)$ by

$$(5.21) \qquad\qquad R(\tau, h) = \prod_{q=0}^{n} R_q(\tau, h)^{(-1)^q}.$$

Recall that the Reidemeister torsion $T_X^{\text{top}}(\tau, h)$ depends on the metric h through the choice of an orthonormal basis in the cohomology $H^\star(X, E)$, where the inner product in $H^\star(X, E_\tau)$ is defined as above. The key result relating Reidemeister torsion and cohomology is the following proposition.

Proposition 5.6. *Let τ be a unimodular representation of Γ on a finite-dimensional $\mathbb{R}$-vector space V. Let $M \subset V$ be a Γ-invariant lattice and let $\mathcal{M}$ be the associated local system of finitely generated free $\mathbb{Z}$-modules on X. Let h be a fiber metric in the flat vector bundle $E = \mathcal{M} \otimes \mathbb{R}$. Then we have*

$$(5.22) \qquad\qquad T_X^{\text{top}}(\tau, h) = R(\tau, h) \cdot \prod_{q=0}^{n} |H^q(X, \mathcal{M})_{\text{tors}}|^{(-1)^{q+1}}.$$

Especially, if $\tau|_\Gamma$ is acyclic, i.e., if $H^\star(X, E) = 0$, then $T_X^{\text{top}}(\tau, h)$ is independent of h and we denote it by $T_X^{\text{top}}(\tau)$. Moreover, $R(\tau, h) = 1$. Then $H^\star(X, \mathcal{M})$ is a torsion group and one has

$$T_X^{\text{top}}(\tau) = \prod_{q=0}^{n} |H^q(X, \mathcal{M})|^{(-1)^{q+1}}.$$

Representations τ of G which admit a Γ-invariant lattice arise in the following arithmetic situation. Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$ and let $G = \mathbf{G}(\mathbb{R})$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Let V_0 be a $\mathbb{Q}$-vector space and let $\rho\colon \mathbf{G} \to \mathrm{GL}(V_0)$ be a rational representation. Then there exists a lattice $M \subset V_0$ which is invariant under Γ and $V_0 = M \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $V = V_0 \otimes_{\mathbb{Q}} \mathbb{R}$ and let $\tau\colon G \to \mathrm{GL}(V)$ be the representation induced by ρ. Then $M \subset V$ is a Γ-invariant lattice.

Assume that $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is cocompact in G (equivalently assume that $\mathbf{G}$ is anisotropic). Then it is proved in [BV] that strongly acyclic arithmetic Γ-modules M exist. Assume that $\delta(G) = 1$. Let M be a strongly acyclic arithmetic Γ-module. Then by (5.3), Theorem

5.1 and Proposition 5.2 it follows that there exists a constant $C > 0$, which depends on G and M, such that

$$(5.23) \qquad \lim_{j \to \infty} \sum_{k=0}^{n} (-1)^{k + \frac{\dim(\widetilde{X}) - 1}{2}} \frac{\log |H_k(X_j, \mathcal{M})|}{[\Gamma : \Gamma_j]} = C \operatorname{vol}(\Gamma \backslash \widetilde{X})$$

(see [BV, (1.4.2)]). This implies the following theorem of Bergeron and Venkatesh. [BV, Theorem 1.4].

Theorem 5.7. *Suppose that $\delta(\widetilde{X}) = 1$. Then strongly acyclic arithmetic Γ-modules exist. For any such*

$$\liminf_{j} \sum_{k \equiv a \,(\mathrm{mod}\, 2)} \frac{\log |H_k(X_j, \mathcal{M})|}{[\Gamma : \Gamma_j]} \geq C \operatorname{vol}(\Gamma \backslash \widetilde{X})$$

where $a = (\dim(\widetilde{X}) - 1)/2$ and $C > 0$ depends only on G and M.

In Theorem 5.7, one cannot in general isolate the degree which produces torsion. A conjecture of Bergeron and Venkatesh [BV, Conjecture 1.3] claims the following.

Conjecture 4. *The limit*

$$\lim_{j \to \infty} \frac{\log |H_k(X_j, \mathcal{M})_{\mathrm{tors}}|}{[\Gamma : \Gamma_j]}$$

exists for each k and is zero unless $\delta(G) = 1$ and $k = \frac{\dim(\widetilde{X}) - 1}{2}$. In that case, it is always positive and equal to a positive constant $C_{G,M}$, which can be explicitly described, times $\operatorname{vol}(\Gamma \backslash \widetilde{X})$.

An example, for which this conjecture can be verified is $G = \mathrm{SL}(2, \mathbb{C})$.

If the representation τ of G is not acyclic, various difficulties occur. First of all, the spectrum of the Laplace operators has no positive lower bound which causes the problem with the small eigenvalues discussed above in the context of analytic torsion. Secondly the regulator $R(\tau, h)$ is in general nontrivial. It turns out to be rather difficult to control the growth of the regulator. Of particular interest is the case of the trivial representation, i.e., the integer homology $H_k(X_j, \mathbb{Z})$. The 3-dimensional case has been studied in [BSV]. In this paper the authors discuss conditions which imply that the results of [BV] on strongly acyclic local systems can be extended to the case of the trivial local system. There are conditions on the cohomology and the spectrum of the Laplace operator on 1-Forms. The conditions on the spectrum are as follows. Let $(\Gamma_i)_{i \in \mathbb{N}}$ be a sequence of cocompact congruence subgroups of a fixed arithmetic subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$. Let $X_i = \Gamma_i \backslash \mathbb{H}^3$ and put $V_i := \operatorname{vol}(X_i)$. Let $\lambda_j^{(i)}$ $j \in \mathbb{N}$, be the eigenvalues of the Laplace operator on 1-forms of X_i. Assume:

1) For every $\varepsilon > 0$ there exists $c > 0$ such that

$$\limsup_{i \to \infty} \frac{1}{V_i} \sum_{0 < \lambda_j^{(i)} \leq c} |\log \lambda_j^{(i)}| \leq \varepsilon.$$

2) $b_1(X_i, \mathbb{Q}) = o(\frac{V_i}{\log V_i})$.

Let T_{X_i} be the analytic torsion with respect to the trivial local system. As shown in [BSV], conditions 1) and 2) imply that

$$\frac{\log T_{X_i}}{V_i} \longrightarrow t^{(2)}_{\mathbb{H}^3} = -\frac{1}{6\pi}, \quad i \to \infty.$$

Unfortunately, it seems to be difficult to verify 1) and 2). The other problem is to estimate the growth of the regulator (see [BSV]).

5.4. **The finite volume case.** Many important arithmetic groups are not cocompact. So it is desirable to extend the results of the previous section to the finite volume case. In order to achieve this one has to deal with the following problems.

1) Define an appropriate regularized version $T^{\mathrm{reg}}_X(\rho)$ of the analytic torsion for a finite volume locally symmetric space $X = \Gamma \backslash \widetilde{X}$ and establish the analog of (5.12). So let $\Gamma_j \subset \Gamma$ be a sequence of subgroups of finite index and $X_j := \Gamma_j \backslash \widetilde{X}$, $j \in \mathbb{N}$. Assume that $\mathrm{vol}(V_j) \to \infty$. Under appropriate additional assumptions on the sequence $(\Gamma_j)_{j \in \mathbb{N}}$ one has to show that

$$\lim_{j \to \infty} \frac{\log T^{\mathrm{reg}}_{X_j}(\rho)}{\mathrm{vol}(X_j)} = t^{(2)}_{\widetilde{X}}(\rho).$$

2) Show that $T^{\mathrm{reg}}_X(\rho)$ has a topological counterpart $T^{\mathrm{top}}_X(\rho)$, possibly the Reidemeister torsion of an intersection complex.

3) If E_ρ is arithmetic, i.e., if there is a local system of finite rank free $\mathbb{Z}$-modules $\mathcal{M}$ over X such that $E_\rho = \mathcal{M} \otimes \mathbb{R}$, establish an analog of (5.22).

4) Estimate the growth of the regulator.

For hyperbolic manifolds 1) has been proved in [Ra1] in the 3-dimensional case and in [MP1] and [MP2] in general. It would be very interesting to extend these results to the higher rank case. $\mathrm{SL}(3, \mathbb{R})$ seems to be doable.

Raimbault [Ra2] has studied 2) in the 3-dimensional case and established a kind of asymptotic equality of analytic and Reidemeister torsion, which is sufficient for the present purpose. Of course, the goal is to prove an exact equality. For hyperbolic manifolds there is some recent progress [AR]. Unfortunately, this paper does not cover the relevant flat bundles. The method requires that the flat bundle can extended to the boundary at infinity. This is not the case for the flat bundles which arise from representations of G by restriction to Γ. J. Pfaff [Pf] has established a gluing formula for the regularized analytic torsion of a hyperbolic manifold, which reduces the problem to the case of a cusp.

4) has been studied by Raimbault [Ra2] for 3-dimensional hyperbolic manifolds. It turns out to be very difficult. The real cohomology never vanishes. There is always the part of the cohomology coming from the boundary. This is the Eisenstein cohomology introduced by Harder [Ha]. These cohomology classes are represented by Eisenstein classes, which are

rational cohomology classes. The problem is to estimate the denominators of the Eisenstein classes which seems to be a hard problem.

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