ANALYTIC TORSION FOR ARITHMETIC LOCALLY SYMMETRIC MANIFOLDS

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Abstract. In this paper we define a regularized version of the analytic torsion for quotients of a symmetric space of non-positive curvature by arithmetic lattices. The definition is based on the study of the renormalized trace of the corresponding heat operators, which is defined as the geometric side of the Arthur trace formula applied to the heat kernel. Then we study the limiting behavior of the analytic torsion as the lattices run through a sequence of congrunece subgroups of a fixed arithmetic subgroup. Our main result states that for sequences of principal congruence subgroups, which converge to 1 at a fixed finite set of places and strongly acyclic flat bundles, the logarithm of the analytic torsion, devided by the index of the subgroup, converges to the L^2-analytic torsion.

Contents

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1. Introduction

The main purpose of this paper is to define a regularized analytic torsion for locally symmetric manifolds of finite volume which are quotients of a symmetric space of non-positive curvature by an arithmetic group. This is motivated by the recent applications of the Ray-Singer analytic torsion [RS] to the study of the growth of torsion in the cohomology of cocompact arithmetic groups [BV], [MaM], [MP14b]. Since many important arithmetic groups are not cocompact, it is very desirable to extend these results to non-cocompact lattices. There exist some results for hyperbolic manifolds of finite volume [PR], [Ra1], [Ra2], [MR2].

In [MM17] we have defined the regularized analytic torsion for arithmetic quotients of the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. The purpose of the present paper is to extend the definition to all arithmetic quotients of symmetric spaces $\widetilde{X}=G/K$ of non-positive curvature.

To explain the approach we briefly recall the definition of the Ray-Singer analytic torsion for a compact Riemannian manifold X of dimension n. Let $\rho\colon \pi_1(X)\to \mathrm{GL}(V)$ be a finite dimensional representation of the fundamental group of X let $E_\rho\to X$ be the flat vector bundle associated to ρ. Choose a Hermitian fiber metric in E_ρ. Let $\Delta_p(\rho)$ be the Laplace operator on E_ρ-valued p-forms with respect to the metrics on X and in E_ρ. It is an elliptic differential operator, which is formally self-adjoint and non-negative. Let $h_p(\rho):=\dim\ker\Delta_p(\rho)$. Using the trace of the heat operator $e^{-t\Delta_p(\rho)}$, the zeta function $\zeta_p(s;\rho)$ of $\Delta_p(\rho)$ can be defined by

$$\zeta_p(s;\rho):=\frac{1}{\Gamma(s)}\int_0^\infty\left(\operatorname{Tr}\left(e^{-t\Delta_p(\rho)}\right)-h_p(\rho)\right)t^{s-1}\,dt. \tag{1.1}$$

The integral converges for $\operatorname{Re}(s)>n/2$ and admits a meromorphic extension to the whole complex plane, which is holomorphic at $s=0$. Then the Ray-Singer analytic torsion $T_X(\rho)\in\mathbb{R}^+$ is defined by

$$\log T_X(\rho)=\frac{1}{2}\sum_{p=1}^{d}(-1)^p p\frac{d}{ds}\zeta_p(s;\rho)\big|_{s=0}. \tag{1.2}$$

The definition of the analytic torsion depends on the compactness of the underlying manifold. Without this assumption, the heat operator $e^{-t\Delta_p(\rho)}$ is, in general, not a trace class operator.

To generalize the above method to non-cocompact manifolds, the first problem is to define an appropriate regularized trace of the heat operators. For hyperbolic manifolds of finite volume one can follow Melrose [Me] to define the regularized trace by means of the renormalized trace of the heat kernel. This method has been used in [CV], [PR], [MP12], [MP14a], [MP14b]. One uses an appropriate height function to truncate the hyperbolic manifold X at a sufficiently large height $T>0$, that is, one cuts off the cusps sufficiently far out towards infinity. Then one integrates the pointwise trace of the heat kernel over the truncated manifold $X(T)$. This integral has an asymptotic expansion in $\log T$. The constant term is defined to be the renormalized trace of the heat operator. The crucial point is that this definition coincides with the spectral side of the Selberg trace formula applied to the heat kernel.

In higher rank we proceed in the same way as in the case of hyperbolic manifolds. The problem is to define the right truncation. In [MM17] we have dealt with the case $G = \mathrm{GL}(n)$. To define the regularized trace of the heat operators we have used Arthur's truncation operator [Ar78]. The goal of the present paper is to extend this approach to quasi-split reductive groups G.

To this end we pass to the adelic framework. For simplicity assume that G is a connected semisimple algebraic group defined over $\mathbb{Q}$. Assume that $G(\mathbb{R})$ is not compact. Let $\mathbf{K}_\infty$ be a maximal compact subgroup of $G(\mathbb{R})$. Put $\widetilde{X} = G(\mathbb{R})/\mathbf{K}_\infty$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ the ring of finite adeles. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup. We consider the adelic quotient

$$X(K_f) = G(\mathbb{Q})\backslash(\widetilde{X} \times G(\mathbb{A}_f)/K_f). \tag{1.3}$$

This is the adelic version of a locally symmetric space. In fact, $X(K_f)$ is the disjoint union of finitely many locally symmetric spaces $\Gamma_i\backslash\widetilde{X}$, $i = 1, \dots, l$, (see Section 11). If G is simply connected, then by strong approximation we have

$$X(K_f) = \Gamma\backslash\widetilde{X},$$

where $\Gamma = (G(\mathbb{R}) \times K_f) \cap G(\mathbb{Q})$. We will assume that K_f is neat so that $X(K_f)$ is a manifold. Let $\nu\colon \mathbf{K}_\infty \to \mathrm{GL}(V_\nu)$ be a finite dimensional unitary representation. It induces a homogeneous Hermitian vector bundle $\widetilde{E}_\nu$ over $\widetilde{X}$, which is equipped with the canonical connection ∇^ν. Being homogeneous, $\widetilde{E}_\nu$ can be pushed down to a locally homogeneous Hermitian vector bundle over each component $\Gamma_i\backslash\widetilde{X}$ of $X(K_f)$. Their disjoint union is a Hermitian vector bundle E_ν over $X(K_f)$. Let $\widetilde{\Delta}_\nu$ (resp. Δ_ν) be the associated Bochner-Laplace operator acting in the space of smooth section of $\widetilde{E}_\nu$ (resp. E_ν). Let $e^{-t\widetilde{\Delta}_\nu}$ (resp. $e^{-t\Delta_\nu}$), $t > 0$, be the heat semigroup generated by $\widetilde{\Delta}_\nu$ (resp. Δ_ν). Since $\widetilde{\Delta}_\nu$ commutes with the action of $G(\mathbb{R})$, it follows that $e^{-t\widetilde{\Delta}_\nu}$ is a convolution operator with kernel given by a smooth map $H_t^\nu\colon G(\mathbb{R}) \to \mathrm{End}(V_\nu)$. Let $h_t^\nu(g) = \operatorname{tr} H_t^\nu(g)$, $g \in G(\mathbb{R})$. Then h_t^ν belongs to Harish-Chandra's Schwartz space $\mathcal{C}^1(G(\mathbb{R}))$. Let χ_{K_f} be the characteristic function of K_f in $G(\mathbb{A}_f)$. Define the function $\phi_t^\nu \in C^\infty(G(\mathbb{A}))$ by

$$\phi_t^\nu(g_\infty g_f) = h_t^\nu(g_\infty)\chi_{K_f}(g_f), \quad g_\infty \in G(\mathbb{R}),\ g_f \in G(\mathbb{A}_f). \tag{1.4}$$

Then ϕ_t^ν belongs to $\mathcal{C}(G(\mathbb{A}); K_f)$, the adelic version of the Schwartz space (see Section 2 for its definition). Let J_{geom} be the geometric side of the Arthur trace formula introduced in [Ar78]. Then in [MM17] it has been justified to define the regularized trace of $e^{-t\Delta_\nu}$ as

$$\mathrm{Tr}_{\text{reg}}\left(e^{-t\Delta_\nu}\right) := J_{\text{geom}}(\phi_t^\nu) \tag{1.5}$$

(see [MM17, Definition 11.1]). In order to define the zeta function by the analogous formula (1.1) we need to determine the asymptotic behavior of $\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_\nu}\right)$ as $t \to 0$ and $t \to \infty$, respectively. To this end we use the Arthur trace formula.

Our first main result is concerned with the small time behavior of the regularized trace. The general setup is a reductive quasi-split algebraic group G over $\mathbb{Q}$, an open compact subgroup $K_f \subset G(\mathbb{A}_f)$ and the associated adelic quotient $X(K_f)$. Let r denote the split semisimple rank of G. Then we have the following theorem.

Theorem 1.1. *Let $\nu : \mathbf{K}_\infty \longrightarrow \mathrm{GL}(V_\nu)$ be a finite dimensional unitary representation of $\mathbf{K}_\infty$, and let Δ_ν be the associated Bochner-Laplace operator on $X(K_f)$. Suppose that K_f is neat, and let r be the semisimple rank of G. There exist constants $a_j, b_{ij} \in \mathbb{C}$, $j \geq 0$, $0 \leq i \leq r$, depending on ν and K_f such that*

$$\operatorname{Tr}_{reg}(e^{-t\Delta_\nu}) \sim t^{-d/2} \sum_{j=0}^{\infty} a_j t^j + t^{-(d-1)/2} \sum_{j=0}^{\infty} \sum_{i=0}^{r} b_{ij} t^{j/2} (\log t)^i \tag{1.6}$$

as $t \searrow 0$.

For $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$ this result was proved in [MM17], and for hyperbolic manifolds in [Mu17]. The method to prove these results is the same as the one used in the proof of Theorem 1.1. However, the arguments are simplified considerably. In the rank one case, the short time asymptotic expansion of the regularized trace of the heat operators can also be obtained by using methods of microlocal analysis [AR, Theorem A.1]. In fact, this methods works for the more general class of manifolds with cusps. It is a challenging problem to see if in the higher rank case the asymptotic expansion (1.6) can also be derived by methods of microlocal analysis.

To study the large time behavior we restrict attention to twisted Laplace operators, which are relevant for our purpose. As before, let $\tau\colon G(\mathbb{R}) \to \mathrm{GL}(V_\tau)$ be a finite dimensional complex representation. Let $\Gamma_i \backslash \widetilde{X}$, $i = 1, \dots, l$, be the components of $X(K_f)$. The restriction of τ to Γ_i induces a flat vector bundle $E_{\tau,i}$ over $\Gamma_i \backslash \widetilde{X}$. The disjoint union is a flat vector bundle E_τ over $X(K_f)$. By [MM] it is isomorphic to the locally homogeneous vector bundle associated to $\tau|_{\mathbf{K}_\infty}$. It can be equipped with a fiber metric induced from the homogeneous bundle. Let $\Delta_p(\tau)$ be the corresponding twisted Laplace operator on p-forms with values in E_τ. Let $\mathrm{Ad}_{\mathfrak{p}}\colon \mathbf{K}_\infty \to \mathrm{GL}(\mathfrak{p})$ be the adjoint representation of $\mathbf{K}_\infty$ on $\mathfrak{p}$, where $\mathfrak{p} = \mathfrak{k}^\perp$, and $\nu_p(\tau) = \Lambda^p \mathrm{Ad}_{\mathfrak{p}}^* \otimes \tau$. Up to a vector bundle endomorphism, $\Delta_p(\tau)$ equals the Bochner-Laplace operator $\Delta_{\nu_p(\tau)}$. So $\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ is well defined. The large time behavior of the regularized trace is described by the following proposition.

Theorem 1.2. *Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup. Assume that K_f is neat. Let $\tau \in \mathrm{Rep}(G(\mathbb{R}))$. Assume that $\tau \not\cong \tau_\theta$. Then we have*

$$\operatorname{Tr}_{reg}\left(e^{-t\Delta_p(\tau)}\right) = O(e^{-ct}) \tag{1.7}$$

as $t \to \infty$ for all $p = 0, \dots, d$.

The proof of this theorem is an immediate consequence of Proposition 11.3 together with an application of the trace formula.

By Theorems 1.1 and 1.2 we can define the zeta function $\zeta_p(s;\tau)$ of $\Delta_p(\tau)$ as in (11.39), using the regularized trace of $e^{-t\Delta_p(\tau)}$ in place of the usual trace in (1.1). The corresponding Mellin transform converges absolutely and uniformly on compact subsets of the half-plane $\operatorname{Re}(s) > d/2$ and admits a meromorphic extension to the whole complex plane by Theorems 1.1 and 1.2. Because of the presence of the log-terms in the expansion (1.6), the zeta function may have a pole at $s = 0$ so that we need to use the finite part of $\zeta_p(s;\tau)$ in the definition of the analytic torsion of $X(K_f)$. More precisely, for a meromorphic function $f(s)$ on $\mathbb{C}$ and $s_0 \in \mathbb{C}$, let $f(s) = \sum_{k\geq k_0} a_k(s-s_0)^k$ be the Laurent expansion of f at s_0, and put $\operatorname{FP}_{s=s_0} f := a_0$. Now we define the analytic torsion $T_{X(K_f)}(\tau) \in \mathbb{C}\setminus\{0\}$ by

$$\log T_{X(K_f)}(\tau) = \frac{1}{2}\sum_{p=0}^{d}(-1)^p p\left(\operatorname{FP}_{s=0}\frac{\zeta_p(s;\tau)}{s}\right). \tag{1.8}$$

In the case of $G = \mathrm{GL}(3)$ we have determined the coefficients of the log-terms in [MM17]. The calculation shows that the zeta functions definitely have a pole at $s = 0$. However, the combination $\sum_{p=1}^{5}(-1)^p p\zeta_p(s;\tau)$ turns out to be holomorphic at $s = 0$ and we can instead define the logarithm of the analytic torsion by

$$\log T_{X(K_f)}(\tau) = \frac{d}{ds}\left(\frac{1}{2}\sum_{p=1}^{5}(-1)^p p\zeta_p(s;\tau)\right)\bigg|_{s=0}.$$

Remark 1.3. *Let* $\delta(\widetilde{X}) = \operatorname{rank} G(\mathbb{R})^1 - \operatorname{rank}\mathbf{K}_\infty$. *We recall that in the co-compact case, the analytic torsion is trivial, unless* $\delta(\widetilde{X}) = 1$. *As the example of a hyperbolic manifold of even dimension shows* [MP12], *this does not need to be so in the non co–compact case.*

The next problem is to study the limiting behavior of $\log T_{X_N}(\tau)/\operatorname{vol}(X_N)$ as $N\to\infty$, where $X_N = X(K_{N,f})$ is a sequence of congruence quotients with $\operatorname{vol}(X_N)\to\infty$ as $N\to\infty$. The main goal is to generalize the results of [MM2] to other reductive groups. For an adelic quotient $X := X(K_f)$ let $T_X^{(2)}(\tau)$ be the L^2-torsion [Lo], [Mat]. Since the heat kernels on $\widetilde{X}$ are $G(\mathbb{R})$-invariant, one has

$$\log T_X^{(2)}(\tau) = \operatorname{vol}(X)t_{\widetilde{X}}^{(2)}(\tau), \tag{1.9}$$

where $t_{\widetilde{X}}^{(2)}(\tau)$ depends only on $\widetilde{X}$ and τ. Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of open compact subgroups of $G(\mathbb{A}_f)$. We say that K_j converges to 1, denoted by $K_j\to 1$, as $j\to\infty$, if for every open compact subgroup U of $G(\mathbb{A}_f)$ there exists $N_0\in\mathbb{N}$ such that $K_j\subset U$ for all $j\geq N_0$. Based on the known results in the compact case [BV], we make the following conjecture.

Conjecture 1.4. *Let $\tau \in \mathrm{Rep}(G(\mathbb{R}))$ and assume that $\tau \not\cong \tau_\theta$. Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of open compact subgroups of $G(\mathbb{A}_f)$ with $K_j \to 1$ as $j \to \infty$. Then*

$$\lim_{j\to\infty} \frac{\log T_{X(K_j)}(\tau)}{\mathrm{vol}(X(K_j))} = t^{(2)}_{\widetilde{X}}(\tau). \tag{1.10}$$

In [MM2] we established this conjecture for principal congruence subgroups of $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. Let $\delta(\widetilde{X}) := \mathrm{rank}_{\mathbb{C}}(G(\mathbb{R})^1) - \mathrm{rank}_{\mathbb{C}}(K_\infty)$. The constant $t^{(2)}_{\widetilde{X}}(\tau)$ has been computed in [BV, Proposition 5.2]. It is shown that $t^{(2)}_{\widetilde{X}}(\tau) \neq 0$ if and only if $\delta(\widetilde{X}) = 1$.

We are unable to prove Conjecture 1.4 in all generality. Due to problems related to the spectral side of the trace formula, we have to restrict the reductive groups. We will consider the following class of reductive groups.

Definition 1.5. *A reductive group G is called admissible, if G is an inner form of $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$, or a quasi-split classical group.*

Due to problems related to the geometric side of the trace formula, we also have to make restrictions on the sequences of congruence groups that we will consider. The problem is concerned with the global coefficients in the Arthur's fine expansion of the geometric side. For $\mathrm{GL}(n)$ estimations of the global coefficients are known [Ma1]. For groups other than $\mathrm{GL}(n)$ very little is known about these coeffcients. That is why we need to restrict our sequences of congruence groups for which we follow [Cl, Sect. 2]. Let S be a finite set of primes. Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of open compact subgroups of $G(\mathbb{A}_f)$. We say that $\{K_j\}$ converges to 1 at S, denote by $K_j \underset{S}{\to} 1$, if there exists an open compact subgroup $K^S = \prod_{p\notin S} K_p$ of $G(\mathbb{A}^S)$ such that

$$K_j = K_{S,j} \times K^S, \quad \text{with} \quad K_{S,j} = \prod_{p\in S} K_{p,j}, \quad \text{and} \quad K_{S,j} \xrightarrow[j\to\infty]{} 1, \tag{1.11}$$

where the latter condition means that for every open compact subgroup U of $G_S := \prod_{p\in S} G(\mathbb{Q}_p)$ there exists $N_0 \in \mathbb{N}$ such that $K_{S,j} \subset U$ for $j \geq N_0$. An example are the principal congruence subgroups $\Gamma_j := \Gamma(p^j)$ of $\mathrm{SL}(n,\mathbb{R})$, where p is a fixed prime. In the present paper we will only consider principal congruence subgroups. The main result is the following theorem.

Theorem 1.6. *Let G be an admissible reductive algebraic group over $\mathbb{Q}$. Let $\tau \in \mathrm{Rep}(G(\mathbb{R})^1)$ and assume that $\tau \not\cong \tau_\theta$. Let S be a finite set of primes with $2 \notin S$. Let $\{K(N_j)\}_{j\in\mathbb{N}}$ be a sequence of principal congruence subgroups of $G(\mathbb{A}_f)$ with $K(N_j) \underset{S}{\to} 1$ as $j \to \infty$. Let $X(N_j) := X(K(N_j))$. Then*

$$\lim_{j\to\infty} \frac{\log T_{X(N_j)}(\tau)}{\mathrm{vol}(X(N_j))} = t^{(2)}_{\widetilde{X}}(\tau).$$

The more general case of arbitrary congruence subgroups which converge to 1 at a fixed set of primes will be treated in a forthcoming paper.

Another problem is the question if there is a combinatorial counterpart of $T_{X(K_f)}(\tau)$, as there is in the compact case. For hyperbolic manifolds of finite volume there is such a combinatorial counterpart, which is not equal to the analytic torsion, but differs by a rather simple term [MR1]. For applications it is important to extend this result to other locally symmetric spaces. We hope to return to this problems in the future.

Now we briefly explain our method to prove Theorems 1.1 and 1.2. Overall we follow the approach [MM17] but our treatment of the geometric and spectral side of the trace formula will differ in some crucial places. To determine the asymptotic behavior of the regularized trace as $t \to +0$, we start with its definition (1.5) as the geometric side of trace formula. The first step is to show that ϕ_t^ν can be replaced by a compactly supported function $\widetilde{\phi}_t^\nu \in C_c^\infty(G(\mathbb{A})^1)$ without changing the asymptotic behavior. Next we use the coarse geometric expansion of the geometric side, which expresses $J_{\text{geom}}(f)$, $f \in C_c^\infty(G(\mathbb{A})^1)$, as a sum of distributions $J_{\mathfrak{o}}(f)$ associated to semisimple conjugacy classes of $G(\mathbb{Q})$. Let $J_{\text{unip}}(f)$ be the distribution associated to the class of 1. If the support of $\widetilde{\phi}_t^\nu$ is a sufficiently small neighborhood of 1, it follows that

$$\operatorname{Tr}_{\text{reg}}\left(e^{-t\Delta_\nu}\right) := J_{\text{unip}}(\widetilde{\phi}_t^\nu) + O\left(e^{-c/t}\right) \tag{1.12}$$

as $t \to +0$. To analyze $J_{\text{unip}}(\widetilde{\phi}_t^\nu)$, we use the fine geometric expansion [Ar85] which expresses $J_{\text{unip}}(\widetilde{\phi}_t^\nu)$ in terms of weighted orbital integrals. This reduces the proof of Theorem 1.1 to the study of weighted orbital integrals. In [MM17] we dealt with this problem for the group $\mathrm{GL}(n)$. In this case all unipotent orbits are Richardson, which simplifies the analysis considerably. In fact, each local weighted orbital integral can be written as an integral over the unipotent radical of an appropriate parabolic subgroup of $G(\mathbb{R})$ against a weight function that behaves logarithmically in a certain sense. This is the key result for proving Theorem 1.1 in the case of $\mathrm{GL}(n)$.

To deal with the weighted orbital integrals for an arbitrary quasi-split reductive group G we rely on [Ar88]. Using the proof of [Ar88, Corollary 6.2], we obtain an appropriate integral expression for the weighted orbital integrals. Again the main issue is the analysis of the weight function and the proof that it has a certain logarithmic scaling behavior. Then, as in the case of $\mathrm{GL}(n)$ we insert a standard parametrix for the heat kernel into the weighted integral (5.10) and determine its asymptotic behavior as $t \to 0$. Finally, this leads to the proof of Theorem 1.1.

To prove Theorem 1.2, we use the spectral side of the trace formula. Let $\phi_t^{\tau,p}$ be the function in $\mathcal{C}(G(\mathbb{A})^1; K_f)$, which is defined in the same way as ϕ_t^ν in terms of the kernel of the heat operator on the universal covering. Then by the trace formula

$$\operatorname{Tr}_{\text{reg}}\left(e^{-t\Delta_p(\tau)}\right) = J_{\text{spec}}(\phi_t^{\tau,p}).$$

The key input to deal with the spectral side is the refinement of the spectral expansion of the Arthur trace formula established in [FLM11] (see Theorem 9.1). For $f \in \mathcal{C}(G(\mathbb{A})^1)$ we have

$$J_{\mathrm{spec}}(f)=\sum_{[M]} J_{\mathrm{spec},M}(f),$$

where $[M]$ runs over the conjugacy classes of Levi subgroups of G and $J_{\mathrm{spec},M}(f)$ is a distribution associated to M. The distribution associated to G is $\operatorname{Tr} R_{\mathrm{dis}}(f)$, where R_{dis} denotes the restriction of the regular representation of $G(\mathbb{A})^1$ in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ to the discrete subspace. Using our assumption that $\tau\neq\tau_\theta$, we obtain $\dim\ker\Delta_p(\tau)=0$. Then it follows as in the compact case that there exists $c>0$ such that

$$\operatorname{Tr} R_{\mathrm{dis}}(\phi_t^{\tau,p})=O(e^{-ct}),\quad \text{as } \ t\to\infty.$$

For a proper Levi subgroup M of G, $J_{\mathrm{spec},M}(f)$ is an integral whose main ingredient are logarithmic derivatives of intertwining operators. The determination of the asymptotic behavior of $J_{\mathrm{spec},M}(\phi_t^{\tau,p})$ as $t\to\infty$ relies on two properties, one global and one local, of the intertwining operators. The global property is a uniform bound on the winding number of the normalizing factors of the intertwining operators in the co-rank one case. The bound that we need is (11.35), which was established in [Mu02, Theorem 5.3]. The local property is concerned with the estimation of integrals of logarithmic derivatives of normalized local intertwining operators $R_{\overline{P},P}(\pi_v,s)$, which are uniform in π_v. For $\mathrm{GL}(n)$ the pertinent estimations exist for the logarithmic derivatives itself [MS04, Proposition 0.2]. In general, the key ingredient for the estimation of the integrals is a generalization of the classical Bernstein inequality due to Borwein and Erdelélyi [FLM15, Corollary 5.18]. The application of this result involves the estimation of the order at ∞ of matrix coefficients of local normalized intertwining operators $R_{\overline{P},P}(\pi_v,s)$. Using the standard properties of local normalized intertwining operators, the problem can be reduced to the case of square integrable representations π_v, see Proposition 10.2 and its proof for details.

To prove Theorem 1.6 we follow the approach used in [MM2] in the case of $\mathrm{SL}(n)$. The new ingredients are the results of [FL17], [FL18] and [FL19] concerning the spectral side of the trace formula. These are estimations of the global normalizing factors of the intertwining operators and bounds on the degrees of coefficients of local intertwing operators. Based on these estimations we can extend the main result of [MM2] about the large time estimation of the regularized traces of the heat operators to admissible reductive groups. On the other hand, we can treat the geometric side in all generality, because estimations of the global coefficients are not available for reductive groups other than $\mathrm{GL}(n)$. So we need to make restrictions on the sequence of congruence subgroups.

The paper is organized as follows. In Section 2 we fix notations and recall some basic facts. In Section 3 we begin with the study of the asymptotic expansion of the regularized trace of the heat operator. We show that for the derivation of the asymptotic expansion one can replace the geometric side of the trace formula by the unipotent contribution. Sections 4, 5 and 6 contain some preparatory material related to weighted unipotent orbital integrals. In Section 7 we show that the weighted unipotent orbital integrals with respect to test functions derived from the heat kernel admit an asymptotic expansion as $t\to 0$. In Section 8 we use this result combined with Arthur's fine geometric expansion to prove Theorem 1.1.

In Section 9 we recall the the refined spectral expansion of Arthur's trace formula. Section 10 is concerned with the study of logarithmic derivatives of local intertwining operators. In the final Section 11 we use the spectral side of the Arthur trace formula to prove Theorem 1.2 which concerns the large time asymptotic behavior of the regularized trace of the heat operator. Together with Theorem 1.1 this enables us to define the regularized analytic torsion. In the final section 12 we study the limiting behavior of the renormalized logarithm of the analytic torsion and prove Theorem 1.6.

2. PRELIMINARIES

Let G be a reductive algebraic group defined over $\mathbb{Q}$. We fix a minimal parabolic subgroup P_0 of G defined over $\mathbb{Q}$ and a Levi decomposition $P_0 = M_0U_0$, both defined over $\mathbb{Q}$. Let $\mathcal{F}$ be the set of parabolic subgroups of G which contain M_0 and are defined over $\mathbb{Q}$. Let $\mathcal{L}$ be the set of subgroups of G which contain M_0 and are Levi components of groups in $\mathcal{F}$. For any $P \in \mathcal{F}$ we write

$$P = M_P N_P,$$

where N_P is the unipotent radical of P and M_P belongs to $\mathcal{L}$.

Let $M \in \mathcal{L}$. Denote by A_M the $\mathbb{Q}$-split component of the center of M. Put $A_P = A_{M_P}$. Let $L \in \mathcal{L}$ and assume that L contains M. Then L is a reductive group defined over $\mathbb{Q}$ and M is a Levi subgroup of L. We shall denote the set of Levi subgroups of L which contain M by $\mathcal{L}^L(M)$. We also write $\mathcal{F}^L(M)$ for the set of parabolic subgroups of L, defined over $\mathbb{Q}$, which contain M, and $\mathcal{P}^L(M)$ for the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. Each of these three sets is finite. If $L = G$, we shall usually denote these sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$.

Let $X(M)_{\mathbb{Q}}$ be the group of characters of M which are defined over $\mathbb{Q}$. Put

$$\mathfrak{a}_M := \operatorname{Hom}(X(M)_{\mathbb{Q}}, \mathbb{R}). \tag{2.13}$$

This is a real vector space whose dimension equals that of A_M. Its dual space is

$$\mathfrak{a}_M^* = X(M)_{\mathbb{Q}} \otimes \mathbb{R}.$$

We shall write,

$$\mathfrak{a}_P = \mathfrak{a}_{M_P},\ A_0 = A_{M_0} \quad \text{and} \quad \mathfrak{a}_0 = \mathfrak{a}_{M_0}. \tag{2.14}$$

For $M \in \mathcal{L}$ let $A_M(\mathbb{R})^0$ be the connected component of the identity of the group $A_M(\mathbb{R})$. Let $W_0 = N_{G(\mathbb{Q})}(A_0)/M_0$ be the Weyl group of (G, A_0), where $N_{G(\mathbb{Q})}(H)$ is the normalizer of H in $G(\mathbb{Q})$. For any $s \in W_0$ we choose a representative $w_s \in G(\mathbb{Q})$. Note that W_0 acts on $\mathcal{L}$ by $sM = w_s M w_s^{-1}$. For $M \in \mathcal{L}$ let $W(M) = N_{G(\mathbb{Q})}(M)/M$, which can be identified with a subgroup of W_0.

For any $L \in \mathcal{L}(M)$ we identify $\mathfrak{a}_L^*$ with a subspace of $\mathfrak{a}_M^*$. We denote by $\mathfrak{a}_M^L$ the annihilator of $\mathfrak{a}_L^*$ in $\mathfrak{a}_M$. Then $r = \dim \mathfrak{a}_0^G$ is the semisimple rank of G. We set

$$\mathcal{L}_1(M) = \{L \in \mathcal{L}(M) : \dim \mathfrak{a}_M^L = 1\}$$

and

$$\mathcal{F}_1(M) = \bigcup_{L \in \mathcal{L}_1(M)} \mathcal{P}(L). \tag{2.15}$$

We shall denote the simple roots of (P, A_P) by Δ_P. They are elements of $X(A_P)_{\mathbb{Q}}$ and are canonically embedded in $\mathfrak{a}_P^*$. Let $\Sigma_P \subset \mathfrak{a}_P^*$ be the set of reduced roots of A_P on the Lie algebra of G. For any $\alpha \in \Sigma_M$ we denote by $\alpha^\vee \in \mathfrak{a}_M$ the corresponding co–root. Let P_1 and P_2 be parabolic subgroups with $P_1 \subset P_2$. Then $\mathfrak{a}_{P_2}^*$ is embedded into $\mathfrak{a}_{P_1}^*$, while $\mathfrak{a}_{P_2}$ is a natural quotient vector space of $\mathfrak{a}_{P_1}$. The group $M_{P_2} \cap P_1$ is a parabolic subgroup of M_{P_2}. Let $\Delta_{P_1}^{P_2}$ denote the set of simple roots of $(M_{P_2} \cap P_1, A_{P_1})$. It is a subset of Δ_{P_1}. For a parabolic subgroup P with $P_0 \subset P$ we write $\Delta_0^P := \Delta_{P_0}^P$.

Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) be the ring of adeles (resp. finite adeles) of $\mathbb{Q}$. We fix a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v = \mathbf{K}_\infty \cdot \mathbf{K}_f$ of $G(\mathbb{A}) = G(\mathbb{R}) \cdot G(\mathbb{A}_f)$. We assume that the maximal compact subgroup $\mathbf{K} \subset G(\mathbb{A})$ is admissible with respect to M_0 [Ar88, §1]. Let $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ be the homomorphism given by

$$e^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_{\mathbb{A}} = \prod_v |\chi(m_v)|_v \tag{2.16}$$

for any $\chi \in X(M)$ and denote by $M(\mathbb{A})^1 \subset M(\mathbb{A})$ the kernel of H_M.

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G(\mathbb{R})$ and $\mathbf{K}_\infty$, respectively. Let θ be the Cartan involution of $G(\mathbb{R})$ with respect to $\mathbf{K}_\infty$. It induces a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We fix an invariant bi-linear form B on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. This choice defines a Casimir operator Ω on $G(\mathbb{R})$, and we denote the Casimir eigenvalue of any $\pi \in \Pi(G(\mathbb{R}))$ by λ_π. Similarly, we obtain a Casimir operator $\Omega_{\mathbf{K}_\infty}$ on $\mathbf{K}_\infty$ and write λ_τ for the Casimir eigenvalue of a representation $\tau \in \Pi(\mathbf{K}_\infty)$ (cf. [BG, §2.3]). The form B induces a Euclidean scalar product $(X, Y) = -B(X, \theta(Y))$ on $\mathfrak{g}$ and all its subspaces. For $\tau \in \Pi(\mathbf{K}_\infty)$ we define $\|\tau\|$ as in [CD, §2.2]. Note that the restriction of the scalar product $(\cdot, \cdot)$ on $\mathfrak{g}$ to $\mathfrak{a}_0$ gives $\mathfrak{a}_0$ the structure of a Euclidean space. In particular, this fixes Haar measures on the spaces $\mathfrak{a}_M^L$ and their duals $(\mathfrak{a}_M^L)^*$. We follow Arthur in the corresponding normalization of Haar measures on the groups $M(\mathbb{A})$ ([Ar78, §1]).

Let $L^2_{\mathrm{disc}}(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))$ be the discrete part of $L^2(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))$, i.e., the closure of the sum of all irreducible subrepresentations of the regular representation of $M(\mathbb{A})$. We denote by $\Pi_{\mathrm{disc}}(M(\mathbb{A}))$ the countable set of equivalence classes of irreducible unitary representations of $M(\mathbb{A})$ which occur in the decomposition of the discrete subspace $L^2_{\mathrm{disc}}(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))$ into irreducible representations.

Let H be a topological group. We will denote by $\Pi(H)$ the set of equivalence classes of irreducible unitary representations of H.

Next we introduce the space $\mathcal{C}(G(\mathbb{A})^1)$ of Schwartz functions. For any compact open subgroup K_f of $G(\mathbb{A}_f)$ the space $G(\mathbb{A})^1/K_f$ is the countable disjoint union of copies of $G(\mathbb{R})^1 = G(\mathbb{R}) \cap G(\mathbb{A})^1$ and therefore, it is a differentiable manifold. Any element $X \in \mathcal{U}(\mathfrak{g}^1_\infty)$ of the universal enveloping algebra of the Lie algebra $\mathfrak{g}^1_\infty$ of $G(\mathbb{R})^1$ defines a left invariant differential operator $f \mapsto f * X$ on $G(\mathbb{A})^1/K_f$. Let $\mathcal{C}(G(\mathbb{A})^1; K_f)$ be the space of smooth right K_f-invariant functions on $G(\mathbb{A})^1$ which belong, together with all their derivatives, to $L^1(G(\mathbb{A})^1)$. The space $\mathcal{C}(G(\mathbb{A})^1; K_f)$ becomes a Fréchet space under the seminorms

$$\|f * X\|_{L^1(G(\mathbb{A})^1)}, \quad X \in \mathcal{U}(\mathfrak{g}^1_\infty).$$

Denote by $\mathcal{C}(G(\mathbb{A})^1)$ the union of the spaces $\mathcal{C}(G(\mathbb{A})^1; K_f)$ as K_f varies over the compact open subgroups of $G(\mathbb{A}_f)$ and endow $\mathcal{C}(G(\mathbb{A})^1)$ with the inductive limit topology.

3. Asymptotic expansion of the regularized trace

Let G be a reductive quasi-split group over $\mathbb{Q}$. We assume that its center Z_G is $\mathbb{Q}$-split and let A_G be the identity component of $Z_G(\mathbb{R})$. Then $G(\mathbb{A}) = G(\mathbb{A})^1 \times A_G$ and $G(\mathbb{R}) = G(\mathbb{R})^1 \times A_G$ with $G(\mathbb{R})^1 = G(\mathbb{A})^1 \cap G(\mathbb{R})$, a semisimple real Lie group. Let $\theta : G(\mathbb{R}) \longrightarrow G(\mathbb{R})$ be a Cartan involution and $\mathbf{K}_\infty = G(\mathbb{R})^\theta$ its fixed points. For each prime p fix some maximal compact subgroup $\mathbf{K}_p$ of $G(\mathbb{Q}_p)$ and let $\mathbf{K}_f = \prod_p \mathbf{K}_p$. Let K_f be a finite index subgroup of $\mathbf{K}_f$ and $X(K_f) = G(\mathbb{Q})\backslash G(\mathbb{A})^1/\mathbf{K}^0_\infty \cdot K_f$, where $\mathbf{K}^0_\infty$ is the connected identity component of $\mathbf{K}_\infty$. Let r denote the split semisimple rank of G so that $r = \dim \mathfrak{a}^G_0$.

We recall the definition of the regularized trace. For that we adopt the notation from [MM17, §11-12]. Let $\nu\colon \mathbf{K}_\infty \to \mathrm{GL}(V_\nu)$ be a finite dimensional unitary representation. Let $\tilde{\Delta}_\nu$ be the Bochner-Laplace operator attached to ν on the universal covering $\tilde{X} = G(\mathbb{R})^1/\mathbf{K}^0_\infty$ of $X(K_f)$. Let $H^\nu_t : G(\mathbb{R})^1 \longrightarrow \mathrm{GL}(V_\nu)$ be the convolution kernel associated with $\tilde{\Delta}_\nu$, and let $h^\nu_t = \operatorname{tr} H^\nu_t$. We extend h^t_ν to $G(\mathbb{R})$ by $h^\nu_t(ag) = h^\nu_t(g)$ for all $a \in A_G$, $g \in G(\mathbb{R})^1$. Let $\mathbf{1}_{K_f} : G(\mathbb{A}_f) \longrightarrow \mathbb{C}$ be the characteristic function of K_f. Put

$$\chi_{K_f} := \frac{\mathbf{1}_{K_f}}{\mathrm{vol}(K_f)} \tag{3.1}$$

and

$$\phi^\nu_t(g) = h^\nu_t(g_\infty)\chi_{K_f}(g_f)$$

for $g = g_\infty \cdot g_f \in G(\mathbb{A}) = G(\mathbb{R}) \cdot G(\mathbb{A}_f)$. Let J_{geom} denote the geometric side of Arthur's trace formula. The regularized trace of $e^{-t\Delta_\nu}$ is defined by

$$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_\nu}) = J_{\mathrm{geom}}(\phi^\nu_t). \tag{3.2}$$

This is well-defined because ϕ^ν_t and all its derivatives are in $L^1(G(\mathbb{A})^1)$ so that $J_{\mathrm{geom}}(\phi^\nu_t)$ is well-defined by [FLM11].

3.1. **Reduction to unipotent distributions.** The proof of Theorem 1.1 rests on an asymptotic expansion of certain unipotent distributions $J^G_M(\mathcal{O}, \cdot)$, which will be introduced in §5 and which are defined only for compactly supported test functions. To state this result we first need to construct compactly supported test functions from ϕ^ν_t.

Let $d(\cdot,\cdot) : \tilde{X} \times \tilde{X} \longrightarrow [0,\infty)$ be the geodesic distance on $\tilde{X}$, and put $r(g_\infty) = d(g_\infty x_0, x_0)$ where $x_0 = \mathbf{K}_\infty \in \tilde{X}$ is the base points. Let $0 < a < b$ be sufficiently small real numbers and let $\beta : \mathbb{R} \longrightarrow [0,\infty)$ be a smooth function supported in $[-b,b]$ such that $\beta(y)=1$ for $0 \le |y| \le a$, and $0 \le \beta(y) \le 1$ for $|y|>a$. Define

$$\psi_t^\nu(g_\infty) = \beta(r(g_\infty)) h_t^\nu(g_\infty). \tag{3.3}$$

and

$$\tilde{\phi}_t^\nu(g) = \psi_t^\nu(g_\infty)\chi_{K_f}(g_f) \tag{3.4}$$

for $g = g_\infty \cdot g_f \in G(\mathbb{A}) = G(\mathbb{R}) \cdot G(\mathbb{A}_f)$. Then $\tilde{\phi}_t^\nu \in C_c^\infty(G(\mathbb{A})^1)$ and $\psi_t^\nu \in C_c^\infty(G(\mathbb{R})^1)$. By [MM17, Proposition 12.1] there is some $c>0$ such that for every $0<t\le 1$ we have

$$\left| J_{\mathrm{geom}}(\phi_t^\nu) - J_{\mathrm{geom}}(\tilde{\phi}_t^\nu) \right| \ll e^{-c/t}. \tag{3.5}$$

We note that in [MM17, Sect. 12] we made the assumption that $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$. However, the proof of the proposition holds without any restriction on G. The next result reduces the considerations to the unipotent contribution to the geometric side. Before we state it, we recall the coarse geometric expansion of Arthur's trace formula [Ar05, §10]: Two elements $\gamma_1, \gamma_2 \in G(\mathbb{Q})$ are called coarsely equivalent if their semisimple parts (in the Jordan decomposition) are conjugate in $G(\mathbb{Q})$. Then for any $f \in C_c^\infty(G(\mathbb{A})^1)$ we have

$$J_{\mathrm{geom}}(f) = \sum_{\mathfrak{o}} J_{\mathfrak{o}}(f),$$

where $\mathfrak{o}$ runs over the coarse equivalence classes in $G(\mathbb{Q})$, and the distribution $J_{\mathfrak{o}}$ is supported in the set of all $g \in G(\mathbb{A})^1$ whose semisimple part is conjugate in $G(\mathbb{A})$ to some semisimple element in $\mathfrak{o}$. If $\mathfrak{o} \neq \mathfrak{o}'$, the supports of $J_{\mathfrak{o}}$ and $J_{\mathfrak{o}'}$ are disjoint. Note that the set of unipotent elements in $G(\mathbb{Q})$ constitute a single equivalence class $\mathfrak{o}_{\mathrm{unip}}$ and we write $J_{\mathrm{unip}} = J_{\mathfrak{o}_{\mathrm{unip}}}$.

Proposition 3.1. *If K_f is neat and the support of β is sufficiently small, then*

$$J_{\mathrm{geom}}(\tilde{\phi}_t^\nu) = J_{unip}(\tilde{\phi}_t^\nu). \tag{3.6}$$

Proof. Let $\rho : G \longrightarrow \mathrm{GL}(N)$ be a faithful representation of G. For each prime p we can find $\nu_p \ge 0$ such that $\mathbf{K}_p \subseteq \rho^{-1}(\mathrm{GL}_N(p^{-\nu_p}\mathbb{Z}_p))$, and $\nu_p = 0$ for all but finitely many p. Hence $K_f \subseteq \rho^{-1}(\mathrm{GL}_N(M^{-1}\hat{\mathbb{Z}}))$ with $M = \prod p^{\nu_p}$.

Let $\chi : \mathrm{GL}_N(\mathbb{A}) \longrightarrow \mathbb{A}^N$ be defined by mapping elements of $\mathrm{GL}_N(\mathbb{A})$ onto the sequence of coefficients of their characteristic polynomials (omitting the coefficient 1 of the highest degree monomial). Let $f_\infty \in C_c^\infty(G(\mathbb{R})^1)$. Suppose that $g \in G(\mathbb{A})^1$ is in the support of $f_\infty \cdot \mathbf{1}_{K_f}$ and that the semisimple part of g is conjugate in $G(\mathbb{A})$ to some $\sigma \in G(\mathbb{Q})$. Then $\chi(\rho(g)) = \chi(\rho(\sigma)) \in \mathbb{Q}^N$, and further $\chi(\rho(g)) \in p^{-N\nu_p}\mathbb{Z}_p^N$ for every prime p. Hence $\chi(\rho(g)) \in M^{-N}\mathbb{Z}^N$. If we choose f_∞ to be supported in a sufficiently small, bi-$\mathbf{K}_\infty$-invariant neighborhood of the identity of $G(\mathbb{R})^1$ (the support of f_∞ will possibly depend on M), we can arrange that $\chi(\rho(g)) \in \mathbb{Z}^N$ so that the eigenvalues of $\rho(g)$ are all algebraic integers. Shrinking the support of f_∞ even further if necessary, we can conclude the eigenvalues of $\rho(g)$ must all be roots of unity. Otherwise, the group generated by the matrix $\rho(g)$ would not be contained in a compact set. By assumption, K_f is neat so that these eigenvalues in fact need to be equal to 1. The semisimple part of $\rho(g)$ therefore equals I_N (the identity matrix in $\mathrm{GL}_N(\mathbb{Q})$), that is, the semisimple part of g equals the identity in $G(\mathbb{A})^1$ so that g is unipotent.

By the discussion above on the coarse geometric expansion, we can now find a bi-$\mathbf{K}_\infty$-invariant neighborhood Ω of the identity in $G(\mathbb{R})^1$ such that whenever $f_\infty \in C_c^\infty(G(\mathbb{R})^1)$ is supported in Ω we have $J_{\mathrm{geom}}(f) = J_{\mathrm{unip}}(f)$ where $f = f_\infty \cdot \mathbf{1}_{K_f} \in C_c^\infty(G(\mathbb{A})^1)$. Hence if we choose the support of β in the definition of $\tilde{\phi}_t^\nu$ sufficiently small, we obtain the proposition. $\square$

In light of (3.5) and (3.6) we therefore only need to study the asymptotic expansion of $J_{\mathrm{unip}}(\tilde{\phi}_t^\nu)$ as $t \searrow 0$ to prove Theorem 1.1. This will be done in the following sections by using the fine geometric expansion of J_{unip} involving weighted orbital integrals over the unipotent conjugacy classes in $G(\mathbb{R})$.

4. Preliminaries on unipotent conjugacy classes and integrals

Until §8 we will be concerned only with the real Lie group $G(\mathbb{R})^1$ so that we write G_∞ for $G(\mathbb{R})^1$.

4.1. **Notation.** In abuse of our previous notation, we write $P_0 = M_0 U_0 \subseteq G_\infty$ for the minimal parabolic $P_0(\mathbb{R}) \cap G_\infty = (M_0(\mathbb{R}) \cap G_\infty)(U_0(\mathbb{R}) \cap G_\infty)$ in G_∞ until §8.

A parabolic subgroup P of G_∞ is called standard if it contains P_0, and semistandard if it contains M_0. A Levi subgroup M in G_∞ is called semistandard if it equals the Levi component containing M_0 of some semistandard parabolic subgroup. We write $\mathcal{L}$ for the set of semistandard Levi subgroups of G_∞. If $M \in \mathcal{L}$, we write $\mathcal{L}(M)$ for the set of all $L \in \mathcal{L}$ with $M \subseteq L$ so that $\mathcal{L} = \mathcal{L}(M_0)$.

If $L \in \mathcal{L}$, then $P_0^L := P_0 \cap L = M_0(U_0 \cap L)$ is a minimal parabolic subgroup in L, and $P \mapsto P \cap L$ defines a surjective map from standard parabolic subgroups in G_∞ to standard parabolic subgroups in L (with respect to P_0^L). Similarly, we get surjective maps from semistandard parabolic and semistandard Levi subgroups in G_∞ to such subgroups in L (with respect to M_0). If $M, L \in \mathcal{L}$, $M \subseteq L$, we write $\mathcal{L}^L(M)$ for the semistandard Levi subgroups in L containing M. Further, we write $\mathcal{F}^L(M)$ for the set of all semistandard parabolic subgroups in L containing M. If $L = G_\infty$, we write $\mathcal{F}(M) = \mathcal{F}^{G_\infty}(M)$. Though this clashes with our global notation previously used, we hope that it will not lead to any confusion.

4.2. **Unipotent conjugacy classes.** We recall some basic facts on unipotent conjugacy classes, which can for example be found in [Car] or [CoMc]. Let $M \in \mathcal{L}$ and let $\mathcal{O} \subseteq M$ be a unipotent M-conjugacy class in M. If $L \in \mathcal{L}(M)$, we write $\mathcal{O}^L$ for the unipotent conjugacy class induced from $\mathcal{O}$ in M to L along some semistandard parabolic subgroup in L (the induced class is independent of that choice of parabolic).

Let $L \in \mathcal{L}(M)$. Let $P^L \subseteq L$ be a Jacobson–Morozov parabolic associated with $\mathcal{O}^L$ in L (see [CoMc, Remark 3.8.5]). We can choose P^L to be standard, and we write $P^L = M^L U^L$ for its Levi decomposition with $M^L \supseteq M_0$.

Let $\mathfrak{l}$ denote the Lie algebra of L, and let

$$\mathfrak{l} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{l}_i$$

be the grading attached to the standard triple corresponding to our choice of Jacobson–Morozov parabolic P^L. Put $\mathfrak{u}_j^L := \bigoplus_{i>j} \mathfrak{l}_i$. Let $X_0 \in \mathfrak{l}_2$ such that $u_0 := e^{X_0} \in \mathcal{O}^L$.

4.3. **Measures on unipotent classes.** We keep the notation from §4.2. To define distributions on the unipotent conjugacy classes, we need to fix measures. We fix once and for all some normalization of measures on G_∞, on $\mathbf{K}_\infty$, on the semistandard Levi subgroups in G_∞, and on the unipotent radicals of the semistandard parabolic subgroups. We choose those measures such that they are compatible with respect to Iwasawa decomposition. We also fix a normalization of the measures on the vector spaces $\mathfrak{l}_i$.

Let L_{u_0} be the centralizer of u_0 in L. Then $L_{u_0}\backslash L$ is diffeomorphic to $\mathcal{O}^L$ and L_{u_0} is unimodular being a unipotent group. The quotient measure on $L_{u_0}\backslash L$ (denoted by d^*g) defines an L-invariant measure on $\mathcal{O}^L$ which in fact is a Radon measure on $\mathcal{O}^L$ and has an explicit description:

Proposition 4.1 ([Rao]). *There exists $c > 0$ and a polynomial $\varphi : \mathfrak{l}_2 \longrightarrow \mathbb{C}$ of degree* $\dim \mathfrak{l}_1$ *such that if $f \in C_c^\infty(L)$, then*

$$\int_{L_{u_0}\backslash L} f(g^{-1}u_0 g)\, d^*g = c \int_{V_0} \int_{\mathfrak{u}_2^L} f_{K_\infty^L}(e^{X+Z}) |\varphi(X)|^{1/2}\, dZ\, dX \tag{4.7}$$

where $V_0 \subseteq \mathfrak{l}_2$ is the orbit of X_0 under M^L (a dense subset of $\mathfrak{l}_2$), and $f_{K_\infty^L}$ is defined by $f_{K_\infty^L}(g) := \int_{K_\infty^L} f(k^{-1}gk)\, dk$, $g \in L$.

If we want to emphasize the dependence on L, we write V_0^L, $\mathfrak{l}^L$, φ^L etc. If $Q = LV$ is a semistandard parabolic subgroup, we might also write $V_0^Q := V_0^L$, $\mathfrak{l}^Q := \mathfrak{l}^L$, $\varphi^Q := \varphi^L$ etc.

4.4. **Behavior under induction.** Let $\mathcal{O} \subseteq M$, $L \in \mathcal{F}(M)$ be as before. We can induce in stages, that is, $\mathcal{O}^{G_\infty} = \left(\mathcal{O}^L\right)^{G_\infty}$. The invariant measure on $\mathcal{O}^{G_\infty}$ is then given by a constant multiple of

$$\int_{N_Q} \int_{V_0^L} \int_{\mathfrak{u}_2^L} f_{\mathbf{K}_\infty}(e^{X+Z} n) |\varphi^L(X)|^{1/2}\, dZ\, dX\, dn \tag{4.8}$$

for any $f \in C_c^\infty(G_\infty)$ where $Q \in \mathcal{F}(L)$ is such that $L = L_Q$ is the Levi component of Q, N_Q the unipotent radical of Q, and $f_{\mathbf{K}_\infty}(g) = \int_{\mathbf{K}_\infty} (k^{-1}gk)\, dk$.

Remark 4.2. *The dimension of a unipotent orbit can be computed in terms of the dimensions of the grading coming from the attached standard triple. More precisely,* $\dim \mathcal{O}^L = 2\dim \mathfrak{u}_1^L + \dim \mathfrak{l}_1$, *see* [CoMc, Lemma 4.1.3]. *Taking into account that* $\dim V_0^L = \dim \mathfrak{l}_2$, *and* $\deg \varphi^L = \dim \mathfrak{l}_1$ *we get*

$$\dim \mathcal{O}^L = 2\left(\dim \mathfrak{u}_2^L + \dim V_0^L\right) + \deg \varphi^L,$$

in particular, the dimension is even. Suppose that $Q \in \mathcal{F}(M)$ *is any semistandard parabolic subgroup with Levi component* L *and unipotent radical* N, $Q = LN$. *Then* $\dim \mathcal{O}^{G_\infty} = 2\dim N + \dim \mathcal{O}^L$ *so that*

$$\dim \mathcal{O}^{G_\infty} = 2\left(\dim N + \dim \mathfrak{u}_2^L + \dim V_0^L\right) + \deg \varphi^L.$$

5. Weighted unipotent integrals

5.1. **Introduction.** Arthur's fine geometric expansion and his splitting formula (see §§8.1-8.2) describe J_{unip} as a linear combination of certain products of real and p-adic weighted unipotent integrals. For our purposes we only need to be concerned with the archimedean case for which we follow [Ar88]: Let $f \in C_c^\infty(G_\infty)$ and let $\mathcal{O}$ be a unipotent conjugacy class in M. The archimedean weighted orbital integrals $J_M^{G_\infty}(f, \mathcal{O})$ can be defined as sum of integrals over $\mathcal{O}^{G_\infty}$ against certain non–invariant measure. Those non–invariant measures can be described as follows: Using the proof of [Ar88, Corollary 6.2] we have
(5.9)

$$J_M^{G_\infty}(f, \mathcal{O}) = \sum_{Q \in \mathcal{F}(M)} c(Q, \mathcal{O}) \int_{N_Q} \int_{V_0^Q} \int_{\mathfrak{u}_2^Q} f_{\mathbf{K}_\infty}(e^{X+Z} n) w_{M,\mathcal{O}}^Q(e^{X+Z}) |\varphi^Q(X)|^{1/2} \, dZ \, dX \, dn \, dk,$$

where the notation is as follows:

- $f_{\mathbf{K}_\infty}(g) = \int_{\mathbf{K}_\infty} f(k^{-1} g k) \, dk$
- $w_{M,\mathcal{O}}^Q$ is a certain weight function discussed in [Ar88]. We will study this weight function in more detail below,
- $c(Q, \mathcal{O}) > 0$ are suitable constants coming from the normalization of measures in (4.7) and (4.8),
- for $Q \in \mathcal{F}(M)$, $Q = L_Q N_Q$ denotes its Levi decomposition with L_Q its Levi component, $L_Q \supseteq M$, and N_Q its unipotent radical.

Set $\mathfrak{u}_{Q,1} = \mathfrak{u}_1^{L_Q} \oplus \mathfrak{n}_Q$ with $\mathfrak{n}_Q$ the Lie algebra of N_Q. We extend φ^{L_Q} and $w_{M,\mathcal{O}}^Q(\exp(\cdot))$ to all of $\mathfrak{u}_{Q,1}$ by projecting to $\mathfrak{u}_1^{L_Q}$ along $\mathfrak{n}_Q$. Then we can write the integral above also as

$$J_M^{G_\infty}(f, \mathcal{O}) = \sum_{Q \in \mathcal{F}(M)} c(Q, \mathcal{O}) \int_{\mathfrak{u}_{Q,1}} f_{\mathbf{K}_\infty}(e^Y) w_{M,\mathcal{O}}^Q(e^Y) |\varphi^{L_Q}(Y)|^{1/2} \, dY. \tag{5.10}$$

Note that for each Q, the integral is over the same unipotent orbit $\mathcal{O}^{G_\infty}$ but with different weight functions.

5.2. **An asymptotic expansion of the weights.** We now study the functions $w_{M,\mathcal{O}}^{Q}(\cdot)$ from (5.9) in more detail. For convenience of notation, we only consider the case $Q = G_\infty$. We write $w_{M,\mathcal{O}} = w_{M,\mathcal{O}}^{G_\infty}$.

We fix an embedding $\iota : G_\infty \hookrightarrow \mathrm{GL}_n(\mathbb{R})^1$ that satisfies certain properties. Write $H = \iota(G_\infty)$, $S = \iota(T(\mathbb{R}))$, $N_0 = \iota(U(\mathbb{R}))$, where $T(\mathbb{R})$ is a maximal split torus in G_∞, and $N_0(\mathbb{R})$ the unipotent radical of our fixed minimal parabolic subgroup of G_∞. Then we assume ι to satisfy the following:

- H is self-adjoint;
- S is contained in the group of diagonal matrices $T_0 \subseteq \mathrm{GL}_n(\mathbb{R})^1$;
- N_0 is contained in the group of unipotent upper triangular matrices $U_0 \subseteq \mathrm{GL}_n(\mathbb{R})^1$;
- The restriction of the positive roots of (T_0, U_0) to S are positive roots of (S, N_0); every root of (S, N_0) is obtained this way.

The existence of such an embedding follows from [PlRa, Proposition 3.13]. In the following we will write $\mathfrak{h}$ for the Lie algebra of H, $G_n = \mathrm{GL}_n(\mathbb{R})^1$ and $\mathfrak{g}_n$ for the Lie algebra of G_n.

Let $P = MN$ be a semi-standard parabolic subgroup in H and let $\mathcal{O} \subseteq M$ be a unipotent conjugacy class. Let $P_1 = MN_1$ be another semi-standard parabolic subgroup with the same Levi component M. Let ϖ be a weight on $\mathfrak{a}_M/\mathfrak{a}_H$ that is an extremal weight for an irreducible representation Λ_ϖ of H on a finite dimensional vector space V_ϖ, defined over $\mathbb{R}$, which is also P-dominant. Let $H_P : H \longrightarrow \mathfrak{a}_M$ be the Iwasawa projection.

Then for any $h \in H$, Arthur defines a weight function $v_P(\varpi, h)$ by $v_P(\varpi, h) = e^{-\varpi(H_P(h))}$ and as shown in [Ar88, (3.3)], it satisfies

$$v_P(\varpi, h) = e^{-\varpi(H_P(h))} = \|\Lambda_\varpi(h^{-1})\phi_\varpi\|, \quad h \in H,$$

for ϕ_ϖ a unit vector in the representation space V_ϖ of Λ_ϖ with respect to a fixed norm $\|\cdot\|$ on V_ϖ.

Let $\pi = u\nu \in P_1$ with $u \in \mathcal{O}$ and $\nu \in N_1$. Let $a \in A_M$ be regular. Then there is a unique $n \in N_1$ such that

$$a\pi = n^{-1}aun. \tag{5.11}$$

Therefore, $a \mapsto \Lambda_\varpi(n^{-1})\phi_\varpi$ is a rational function on a dense subset of, and hence on all of A_M/A_H.

By Arthur's construction [Ar88, p. 238–239] there exist unique integers $k_\beta \geq 0$ such that

$$\lim_{a \to 1} \prod_{\beta \in \Sigma_P \cap \Sigma_{\overline{P_1}}} (a^\beta - a^{-\beta})^{k_\beta} v_P(\varpi, n) \tag{5.12}$$

exists and is non-zero on a dense subset of $\mathcal{O}N_1$. Here Σ_P denotes the set of reduced roots of A_M on N, and similarly, $\Sigma_{\overline{P_1}}$ is the set of reduced roots on the opposite parabolic. The limit is in fact of the form $\|W_\varpi(1, \pi)\|$ with $W_\varpi(1, \pi) \in V_\varpi$ a polynomial on $\mathcal{O}N_1$.

The weight functions $w_M(\pi)$ appearing in the weighted unipotent integrals are then of the form

$$w_{M,\mathcal{O}}(\pi) = \sum_{\Omega} c_\Omega \prod_{\varpi\in\Omega} \log \|W_\varpi(1,\pi)\| \tag{5.13}$$

where Ω runs over all finite subsets of extremal weights of $\mathfrak{a}_M$, and $c_\Omega \in \mathbb{C}$ are coefficients which vanish for all but finitely many of the Ωs. Note that $w_{M,\mathcal{O}}(\pi)$ attains a finite value on a dense subset of πs.

5.3. **Extending polynomials.** We recall the notion of a Jacobson–Morozov parabolic subalgebra. Recall that $\mathcal{O} \subseteq M$ denotes a unipotent conjugacy class, and let $\mathcal{N} \subseteq \mathfrak{m}$ be the corresponding nilpotent orbit. By the Jacobson–Morozov Theorem [CoMc, 3.3] we can find an $\mathfrak{sl}_2$-triple (h_0, x_0, y_0) for $\mathcal{N}$ in $\mathfrak{m}$ with h_0 semisimple and x_0, y_0 nilpotent. We decompose $\mathfrak{m}$ into eigenspaces under h_0, that is $\mathfrak{m} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{m}_k$ with $\mathfrak{m}_k = \{X \in \mathfrak{m} \mid [h_0, X] = kX\}$. Let $\tilde{M}$ denote the centralizer of A_M in G_n and let $\tilde{\mathfrak{m}} \subseteq \mathfrak{g}$ be its Lie algebra. Then $\tilde{M}$ is a semi–standard Levi subgroup of G_n with $M \subseteq \tilde{M}$, $\mathfrak{m} \subseteq \tilde{\mathfrak{m}}$. Then h_0 also defines a grading on $\tilde{\mathfrak{m}}$, $\tilde{\mathfrak{m}} = \bigoplus_{k\in\mathbb{Z}} \tilde{\mathfrak{m}}_k$.

Let $\mathfrak{q}^M = \bigoplus_{k\geq 0} \mathfrak{m}_k \subseteq \mathfrak{m}$ be the corresponding Jacobson–Morozov parabolic subalgebra. Let $\mathfrak{v}^M = \bigoplus_{k>0} \mathfrak{m}_k$ and $\mathfrak{v}_1^M = \bigoplus_{k>1} \mathfrak{m}_k$. We also define $\tilde{\mathfrak{v}}^M = \oplus_{k>0} \tilde{\mathfrak{m}}_k$ and $\tilde{\mathfrak{q}}^M = \oplus_{k\geq 0} \tilde{\mathfrak{m}}_k$ so that $\mathfrak{v}^M$ is a sub–vectorspace of $\tilde{\mathfrak{v}}^M$.

Then each $u \in \mathcal{O}$ can be written as $k^{-1}e^X k$ for $k \in K \cap M$ and a unique $X \in \mathfrak{v}_1^M$. Note that $W_\varpi(1,\pi)$ is invariant under the conjugation by elements of $K \cap M$ by [Ar88, (3.7)] so that we can assume that $\pi = e^{X+Y}$ with $X \in \mathfrak{v}_1^M$ and $Y \in \mathfrak{n}_1$ where $\mathfrak{n}_1$ is the Lie algebra of N_1. By [Ar88, p. 253] $W_\varpi(1,\pi)$ is a polynomial in $X+Y$. Similarly, $v_P(\varpi, e^Y)$ is a polynomial in $Y \in \mathfrak{n}_1$.

There are semistandard Levi subgroups $\tilde{P} = \tilde{M}\tilde{N}$ and $\tilde{P}_1 = \tilde{M}\tilde{N}_1$ of G_n such that $N \subseteq \tilde{N}$ and $N_1 \subseteq \tilde{N}_1$. Let $\tilde{\mathfrak{n}}_1$ be the Lie algebra of N_1 so that $\mathfrak{n}_1$ is a sub–vectorspace of $\tilde{\mathfrak{n}}_1$. We have a canonical isomorphism $\tilde{\mathfrak{n}}_1 \simeq \mathbb{R}^{\dim \tilde{\mathfrak{n}}_1}$ via the coordinates given by the matrix entries. This also gives a canonical inner product on $\tilde{\mathfrak{n}}_1$ so that we can find the orthogonal complement of $\mathfrak{n}_1$ in $\tilde{\mathfrak{n}}_1$. We can therefore extend any polynomial on $\mathfrak{n}_1$ trivially to a polynomial on $\tilde{\mathfrak{n}}_1$ along that complement. A similar construction holds for polynomials on $\mathfrak{v}_1^M$ so that they can be extended trivially to polynomials on $\tilde{\mathfrak{v}}_1^M$ as well.

In particular, we can extend $X+Y \mapsto W_\varpi(1, e^{X+Y})$ to a polynomial on all of $\tilde{\mathfrak{v}}_1^M + \tilde{\mathfrak{n}}_1$, and $Y \mapsto v_P(\varpi, e^Y)$ to a polynomial on all of $\tilde{\mathfrak{n}}_1$. Since $X+Y$ is nilpotent, $\log(\mathrm{id}+X+Y)$ and e^{X+Y} are finite series. Hence we can also consider $W_\varpi(1, \mathrm{id}+X+Y) = W_\varpi(1, e^{\log(\mathrm{id}+X+Y)})$ and $v_P(\varpi, \mathrm{id}+Y) = v_P(\varpi, e^{\log(\mathrm{id}+Y)})$ with $X \in \tilde{\mathfrak{v}}_1^M$ and $Y \in \tilde{\mathfrak{n}}_1$ which are again both polynomials.

5.4. **In which the group is $GL(n)$.** In this section we first prove a slightly more general version of [MM17, Lemma 7.2]. We change the notation for this section slightly: We assume that $H = G_n$ so that in particular, $P = MN$ and $P_1 = MN_1$ be semi–standard parabolic of G_n with the same Levi component M. Let $a \in A_M$ be regular. Suppose that $\pi \in P_1$ is a unipotent element and write $\pi = u\nu$ with $u \in M$ and $\nu \in N_1$ unipotent. Then there is a unique $n \in N_1$ such that

$$a\pi = n^{-1}aun, \tag{5.14}$$

that is, we have a well-defined polynomial map $\mathcal{U}_M N_1 \ni \pi \mapsto n \in N_1$ depending on a where $\mathcal{U}_M$ denotes the set of all unipotent elements in M.

Let Φ denote the set of all roots of T_0 on $\mathfrak{g}_n$. Let $\Phi_M \subseteq \Phi$ denote the subset of roots which are not trivial when restricted to A_M, and let $\Phi_1 \subseteq \Phi_M$ be the subset of roots that are positive with respect to N_1. Let Φ^+ denote a choice of positive roots in Φ such that $\Phi_1 \subseteq \Phi^+$. Let $\mathfrak{n}' \subseteq \mathfrak{m}$ denote the nilpotent subalgebra corresponding to $\Phi^+ \smallsetminus \Phi_1$. Then $\mathfrak{n}' \oplus \mathfrak{n}_1$ is the nilpotent radical of the minimal parabolic subalgebra of $\mathfrak{g}_n$ corresponding to Φ^+.

If $Z \in \mathfrak{g}_n$ is any matrix and $\beta \in \Phi$ a root, we write Z_β for the matrix entry of Z corresponding to β. Write $u = \mathrm{id} + X_0$, $\nu = \mathrm{id} + X$, and $n = \mathrm{id} + Y$ with X_0, X, Y suitable nilpotent matrices. Up to conjugation with $K \cap M$ we can assume that $X_0 \in \mathfrak{n}'$ which we will do from now on.

We show the following:

Proposition 5.1. *Let $\beta \in \Phi_1$. Then for each subset $\underline{\alpha} \subseteq \Phi_1$ there is a polynomial $Q_{\beta,\underline{\alpha}}(Z; a^\alpha, \alpha \in \underline{\alpha})$, $Z := \pi - \mathrm{id} = X_0 + X + X_0X$, such that:*

- $$Y_\beta = \sum_{\underline{\alpha} \subseteq \Phi_1} \frac{Q_{\beta,\underline{\alpha}}(aZa^{-1}; a^\alpha, \alpha \in \underline{\alpha})}{\prod_{\alpha \in \underline{\alpha}}(a^\alpha - 1)},$$
  where $\underline{\alpha}$ runs over all subsets of Φ_1;
- *as a function in the matrix entries of Z, $Q_{\beta,\underline{\alpha}}$ is a homogeneous polynomial of degree $\#\underline{\alpha}$;*
- *if $Q_{\beta,\underline{\alpha}}(aZa^{-1}; a^\alpha, \alpha \in \underline{\alpha})$ does not vanish identically, then for X in general position, its limit as $a \to 1$ is non-zero.*

Proof. We introduce a grading on the set Φ_1: We say that $\beta \in \Phi_1$ has degree k, $k \geq 1$, if the β-coordinate of A^k is non-zero, but that of A^{k+1} is zero for a general matrix $A \in \mathfrak{n}_1$. We write $\Phi_1^{(k)}$ for the set of $\beta \in \Phi_1$ of degree k. Note that $\Phi_1^{(k)} = \emptyset$ when $k \geq n$.

We rearrange the relation (5.14) as follows:

$$\begin{aligned} \mathrm{id} + X_0 + a(Z - X_0)a^{-1} &= n^{-1}ana^{-1} + n^{-1}X_0ana^{-1} \\ &= \mathrm{id} + \sum_{k=1}^{n-1}(-1)^{k-1}Y^{k-1}\left[aYa^{-1} - Y\right] + X_0 + \sum_{k=1}^{n-1}(-1)^{k-1}Y^{k-1}\left[X_0aYa^{-1} - YX_0\right] \end{aligned}$$

For $k \geq 1$, the non-zero entries in the matrices $Y^{k-1}\left[aYa^{-1}-Y\right]$ all correspond to β in $\Phi_1^{(l)}$, $l \geq k$. Moreover, the matrix entry in

$$\sum_{k=1}^{n-1}(-1)^{k-1}Y^{k-1}\left[aYa^{-1}-Y\right] \tag{5.15}$$

corresponding to a root β of degree k is of the form

$$(a^{\beta}-1)Y_{\beta}+\sum_{\underline{\alpha}}(a^{\alpha_1}-1)C_{\underline{\alpha}}\prod_{\alpha\in\underline{\alpha}}Y_{\alpha}$$

where the sum runs over all tuples $\underline{\alpha}=(\alpha_1,\ldots)$ of pairwise different elements in $\bigcup_{l<k}\Phi_1^{(l)}$ such that $2 \leq \sum \deg \alpha_i \leq k$, and $C_{\underline{\alpha}} \in \mathbb{R}$ are suitable coefficients. Note that this in particular means that the sum over the $\underline{\alpha}$ in (5.15) contains only monomials of degree ≥ 2, and is in fact empty if $k=1$.

The sum $\sum_{k=1}^{n-1}(-1)^{k-1}Y^{k-1}\left[X_0aYa^{-1}-YX_0\right]$ has a similar structure as (5.15), except that each monomial has exactly one linear factor consisting of a matrix entry of X_0. In particular, as a polynomial in Y and X_0, there is no linear factor, and the matrix entry corresponding to a root of degree k has only factors consisting of Y_β with $\deg\beta < k$ and X_0. Moreover, no matrix entry of

$$\sum_{k=1}^{n-1}(-1)^{k-1}Y^{k-1}\left[X_0aYa^{-1}-YX_0\right]$$

is divisible by any of the factors $a^{\alpha}-1$ (unless it is identically 0), since in $\left[X_0aYa^{-1}-YX_0\right]$ the terms $a^{\beta}Y_{\beta}$ and Y_{β} cannot occur with non–vanishing coefficient in the same matrix entry.

We can now argue inductively in the degree of β. If $\beta \in \Phi_1^{(1)}$, then

$$(a^{\beta}-1)Y_{\beta}=a^{\beta}Z_{\beta}.$$

The assertion then follows by induction from the above description of the matrix entries. □

5.5. **Back to H.** We return to the notation of §5.2 and (5.11). Write $u=\mathrm{id}+X_0$, $\pi=\mathrm{id}+Z$ and $n=\mathrm{id}+Y$ with X_0, Z, and Y nilpotent matrices.

Proposition 5.2. *There exists ν such that*

$$\|W_{\varpi}(1,\mathrm{id}+sZ)\| = s^{\nu}\|W_{\varpi}(1,\pi)\| \tag{5.16}$$

for all $s>0$.

Remark 5.3. *Note that $\mathrm{id}+sZ$ is not necessarily contained in $\mathcal{O}$ (it does not even have to be contained in H), but as we discussed above, we can extend $W_{\varpi}(1,\pi)$ to a polynomial on all of $\mathrm{id}+\tilde{\mathfrak{v}}_1^M+\tilde{\mathfrak{n}}_1$.*

Proof. We want to use Proposition 5.1. The element a in (5.11) defines a semistandard Levi subgroup $\tilde{M}$ in G_n by taking its centralizer in G_n. Then a is a regular element of the center of $\tilde{M}$ and $M \subseteq \tilde{M}$. Moreover, as before, there are semistandard parabolic subgroups $\tilde{P} = \tilde{M}\tilde{N}$ and $\tilde{P}_1 = \tilde{M}\tilde{N}_1$ of G_n such that $N \subseteq \tilde{N}$ and $N_1 \subseteq \tilde{N}_1$. As explained above, we can extend $v_P(\varpi, n)$ to a polynomial in $Y \in \tilde{\mathfrak{n}}_1$, $n = \mathrm{id} + Y$. Each coordinate Y_β of Y can be described according to Proposition 5.1 so that under the change $Z \mapsto sZ$ the coordinate becomes

$$Y_\beta = \sum_{\underline{\alpha} \subseteq \Phi_1} s^{\# \underline{\alpha}} \frac{Q_{\beta, \underline{\alpha}}(aZa^{-1}; a^\alpha, \alpha \in \underline{\alpha})}{\prod_{\alpha \in \underline{\alpha}} (a^\alpha - 1)}.$$

Hence by definition of $\|W_\varpi(1, \pi)\|$ in (5.12) we can find $\nu \geq 0$ such that

$$\|W_\varpi(1, \mathrm{id} + sZ)\| = s^\nu \|W_\varpi(1, \mathrm{id} + Z)\|.$$

□

Together with (5.13) this immediately implies the following:

Corollary 5.4. *There exist*

- *constants* $r, q \geq 0$,
- *polynomials* $p_1, \ldots, p_q : \tilde{\mathfrak{v}}^M \oplus \tilde{\mathfrak{n}}_1 \longrightarrow \mathbb{R}$ *which do not vanish on an open dense subset of* Z *with* $\mathrm{id} + Z \in \mathcal{O}N_1$, *and*
- *complex polynomials* Q_j *in* q*-many variables,* $j = 0, \ldots, r$,

such that for all $Z \in \mathfrak{v}^M \oplus \mathfrak{n}_1$ *and* $s > 0$ *we have*

$$w_{M,\mathcal{O}}(\mathrm{id} + sZ) = \sum_{i=0}^{r} (\log s)^i Q_i(\log |p_1(Z)|, \ldots, \log |p_q(Z)|).$$

6. Test functions

6.1. **Linearizing the metric.** Let

$$r(g) = d(g\mathbf{K}_\infty, \mathbf{K}_\infty), \tag{6.17}$$

where $d(\cdot, \cdot)$ denotes the geodesic distance function on $\tilde{X} = G_\infty / \mathbf{K}_\infty$. We continue to assume that G_∞ is a real semisimple Lie group, and we fix an embedding of G_∞ into $\mathrm{GL}_n(\mathbb{R})^1$ for some $n \geq 1$ as at the beginning of §5.2. From now on, we will identify G_∞ and all its subgroups with their image in $\mathrm{GL}_n(\mathbb{R})^1$ instead of writing H etc., and the Lie algebra $\mathfrak{g}$ will be identified with its corresponding image in $\mathfrak{sl}_n(\mathbb{R})$. In addition to the properties satisfied by G_∞ in §5.2, we can choose the embedding furthermore such that

- $\mathbf{K}_\infty \subseteq \mathrm{O}(n)$.
- $\mathfrak{a}_0^G$ is contained in the subalgebra of diagonal matrices in $\mathfrak{sl}_n(\mathbb{R})$.
- If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is the Cartan decomposition of $\mathfrak{g}$, then $\mathfrak{s}$ is contained in the symmetric matrices in $\mathfrak{sl}_n(\mathbb{R})$, and $\mathfrak{k}$ in the skewsymmetric matrices.

- For $X = (X_{ij})_{i,j=1,\dots,n} \in \mathfrak{g}$, put $\|X\|^2 = \sum_{i,j} |X_{ij}|^2$. Then $\|\cdot\|$ coincides with the norm on $\mathfrak{s}$ obtained from the Killing form on $\mathfrak{g}$. (More generally, we define $\|\cdot\|$ similarly on all of $\mathfrak{sl}_n(\mathbb{R})$.)
- The unipotent radical U_0 of the minimal parabolic subgroup P_0 is contained in the group of unipotent upper triangular matrices of $\mathrm{GL}_n(\mathbb{R})^1$.

Let $d_n(\cdot,\cdot)$ denote the geodesic distance function on $\mathrm{GL}_n(\mathbb{R})^1/\mathrm{O}(n)$, and let $r_n(g) = d_n(g\,\mathrm{O}(n), \mathrm{O}(n))$. By our choice of embedding, the Cartan decomposition on G_∞ with respect to K_∞ and on $\mathrm{GL}_n(\mathbb{R})^1$ with respect to $\mathrm{O}(n)$ are compatible, that is, if $g = k_1 a k_2 \in G_\infty = K_\infty A_0^G K_\infty$, then $k_1 a k_2$ is also the Cartan decomposition of g in $\mathrm{GL}_n(\mathbb{R})^1$. Hence $r(g) = r(a)$ and $r_n(g) = r_n(a)$. By [GaVa, (4.6.25)] we have $r(a) = \|\log a\|$ and $r_n(a) = \|\log a\|$ so that r and r_n coincide on $G_\infty \subseteq \mathrm{GL}_n(\mathbb{R})^1$. This allows us to use results on the geodesic distance r_n from [MM17] also for r. Recall from [MM17, Lemma 12.2] that if $g = I_n + X \in \mathrm{GL}_n(\mathbb{R})^1$ with X a nilpotent upper triangular matrix, then

$$r_n(I_n + X) = \frac{1}{4}\|X\|^2 + O(\|X\|^3) \tag{6.18}$$

as $X \to 0$. Here $I_n \in \mathrm{GL}_n(\mathbb{R})^1$ denotes the identity matrix. Hence if X varies over matrices such that $I_n + X \in G_\infty$, then the same is true for $r_n(I_n + X)$ by our above remark.

We also need to understand how $r(g)$ behaves as g varies over unipotent matrices in G_∞ that become unbounded:

Lemma 6.1. *We have*

$$r(g) \ge \frac{1}{2}\log\left(1 + \frac{1}{n}\|g - I_n\|^2\right)$$

for all unipotent $g \in G_\infty$.

Proof. Let $g \in G_\infty$ be unipotent and write $g = e^{Y_0}$ with $Y_0 \in \mathfrak{g}$ nilpotent. There exists $k \in \mathbf{K}_\infty$ such that $Y := \mathrm{Ad}(k)Y_0$ is an upper triangular nilpotent matrix. We can therefore write $e^Y = I_n + N$ for some nilpotent upper triangular $N \in \mathfrak{sl}_n(\mathbb{R})$. Let $X \in \mathfrak{a}_0^G$ and $k_1, k_2 \in \mathbf{K}_\infty$ such that $g = k_1 e^X k_2$. Then $\|X\| = r(g) = r(k e^{Y_0} k^{-1}) = r(e^Y)$, where the first equality follows from [GaVa, (4.6.25)]. Then

$$\operatorname{tr} e^{2X} = \operatorname{tr}(g^t g) = \operatorname{tr} e^{Y_0^t} e^{Y_0} = \operatorname{tr} e^{Y^t} e^Y = \operatorname{tr}(I_n + N)^t (I_n + N) = n + \operatorname{tr} N^t N = n + \|N\|^2.$$

Let $X_1, \dots, X_n$ be the diagonal entries of X. Then

$$\operatorname{tr} e^{2X} = \sum_{j=1}^n e^{2X_j} \le \sum_{j=1}^n e^{2|X_j|} \le n e^{2\|X\|}.$$

Hence

$$e^{\|X\|} \ge (1 + \frac{1}{n}\|N\|)^{1/2}$$

so that

$$r(g) = \|X\| \ge \frac{1}{2}\log(1 + \frac{1}{n}\|N\|^2)$$

as asserted. ☐

Lemma 6.2. *Let $\tilde{\mathfrak{u}}_0$ denote the vector space of all nilpotent upper triangular matrices in $\mathfrak{gl}_n(\mathbb{R})$. Then there exists $x_0 > 0$ such that for all $X \in \tilde{\mathfrak{u}}_0$ with $\|X\| \geq x_0$ we have*

$$\|e^X - I_n\| \geq \frac{1}{\sqrt{2}}\|X\|.$$

Proof. We consider a more abstract situation: Let $P : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a non-negative polynomial satisfying the following two properties:

- $P(x) \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$, where $\|x\|$ denotes the usual Euclidean norm on $\mathbb{R}^N$, and
- $P(x)$ can be written as $P(x) = \|x\|^2 + \sum_\alpha a_\alpha x^\alpha$ with α running over all multiindices $\alpha = (\alpha_1, \ldots, \alpha_N)$ of degree $3 \leq |\alpha| = \sum_i \alpha_i \leq \deg P$.

The function $\tilde{\mathfrak{u}}_0 \ni X \mapsto \|e^X - I_n\|^2$ is a polynomial exactly of this form. It will therefore suffice to show that for such polynomials we have $P(x) \geq \|x\|^2/2$ for $\|x\| \geq x_0$ for some $x_0 > 0$.

Fix $x \in \mathbb{R}^N$ with $\|x\| = 1$ and set $Q_x(t) = P(tx) - t^2$ for $t \in \mathbb{R}$. If Q_x vanishes identically in t, we are done. If Q_x does not vanish identically, then Q_x is a non-trivial polynomial of degree ≥ 3 in t. Since $P(tx) \longrightarrow +\infty$ as $t \to \infty$, we must also have $Q_x(t) \longrightarrow +\infty$ for $t \to \infty$. Hence there exists $t_x > 0$ such that $Q_x(t) \geq 0$ for all $t \geq t_x$, that is, $P(tx) \geq t^2$ for $t \geq t_x$. Since the set of all $x \in \mathbb{R}^N$ with $\|x\| = 1$ is compact, we can find $t_0 > 0$ such that $P(x) \geq \|x\|^2/2$ for all $x \in \mathbb{R}^N$ with $\|x\| \geq t_0$. This finishes the proof. ☐

6.2. **Asymptotic expansion of the heat kernel.** We recall the following asymptotic expansion of h_t^ν which, for example, can be found in [MM17, Corollary 10.4]: If $\psi \in C^\infty(\mathbb{R})$ is non-negative, equals 1 around 0 and has sufficiently small support, then for any sufficiently large N and any $0 < t \leq 1$ we have

$$h_t^\nu(g) = (4\pi t)^{-d/2}\psi(r(g)) \exp\left(-r(g)^2/(4t)\right) \sum_{n=0}^{N} a_n^\nu(g) t^n + O\left(t^{N+1-d/2}\right) \tag{6.19}$$

where $a_n^\nu \in C^\infty(G_\infty)$ are suitable functions and the implied constant depends on the function ψ but not on N. We further note a uniform upper bound for h_t^ν (see, for example, [MM17, Corollary 10.2]): There exists $C > 0$ such that

$$|h_t^\nu(g)| \leq Ct^{-d/2} e^{-r(g)^2/(4t)} \tag{6.20}$$

for all $g \in G_\infty$ and all $0 < t \leq 1$.

7. PROOF OF PROPOSITION 7.1

The unipotent distribution $J_{\text{unip}}(\tilde{\phi}_t^\nu)$ can be written as a linear combination of certain weighted unipotent integrals $J_M^G(\mathcal{O}, \tilde{\phi}_t^\nu)$, see §8.1. It will turn out (see §8.2) that for us only the integrals $J_M^G(\mathcal{O}, \psi_t^\nu)$ will be relevant. For these we have the following:

Proposition 7.1. *Let $P = MU \in \mathcal{F}$, and let $\mathcal{O} \subseteq M(\mathbb{R})$ be a unipotent conjugacy class in $M(\mathbb{R})$. Let $\mathcal{O}^G \subseteq G(\mathbb{R})$ be the unipotent conjugacy class in $G(\mathbb{R})$ induced from $M(\mathbb{R})$ along $P(\mathbb{R})$. Let $d_{\mathcal{O}}^G = \dim \mathcal{O}^G$ and $r_M = \dim \mathfrak{a}_M^G$. Then there exist constants $b_{ij} = b_{ij}(M, \mathcal{O}) \in \mathbb{C}$, $j \geq 0$, $0 \leq i \leq r_M$ such that for every $0 < t < 1$*

$$J_M^G(\mathcal{O}, \psi_t^\nu) \sim t^{-d/2+d_{\mathcal{O}}^G/4} \sum_{j=0}^{\infty} \sum_{i=0}^{r_M} b_{ij} t^{j/2} (\log t)^i.$$

Moreover, the coefficients b_{ij} are uniformly bounded.

We prove a slightly more general version of Proposition 7.1 which will make for a cleaner proof of Theorem 1.1 in the next section. Let $M \in \mathcal{L}$ and let $P_1 = M_1 U_1 \in \mathcal{F}(M)$. If $f \in C^\infty(G_\infty)$ define $f_{P_1} \in C^\infty(M_1)$ by

$$f_{P_1}(m) = \delta_{P_1}(m)^{1/2} \int_{\mathbf{K}_\infty} \int_{U_1} f(k^{-1} muk) \, du \, dk$$

provided the right hand side is finite for any $m \in M_1$. If f has compact support, f_{P_1} is compactly supported as well.

If $\mathcal{O}$ is a unipotent conjugacy class in M, we can define $J_M^{M_1}(\mathcal{O}, f_{P_1})$ as before with G_∞ replaced by M_1.

Proposition 7.2. *Let $M \in \mathcal{L}$, $\mathcal{O} \subseteq M$ a unipotent conjugacy class in M, and $P_1 = M_1 U_1$ a semistandard parabolic subgroup of G with $M \subseteq M_1$. Let $d_{\mathcal{O}}^{G_\infty} = \dim \mathcal{O}^{G_\infty}$ be the dimension of the unipotent orbit in G_∞ induced from M, and let $r_M^{M_1} = \dim \mathfrak{a}_M^{M_1}$. Then there exist constants $b_{ij} = b_{ij}(M, \mathcal{O}) \in \mathbb{C}$, $j \geq 0$, $0 \leq i \leq r_M^{M_1}$ such that for every $0 < t \leq 1$*

$$J_M^{M_1}(\mathcal{O}, (\psi_t^\nu)_{P_1}) \sim t^{-d/2+d_{\mathcal{O}}^G/4} \sum_{j=0}^{\infty} \sum_{i=0}^{r_M^{M_1}} b_{ij} t^{j/2} (\log t)^i. \tag{7.21}$$

Moreover, the coefficients b_{ij} are uniformly bounded.

For the proof of this proposition we follow [MM17, §12] taking into account our results above. Suppose $\tilde{Q} \subseteq M_1$ is a semistandard parabolic which contains M. Let $\tilde{Q} = L_{\tilde{Q}} N_{\tilde{Q}}$ with $M \subseteq L_{\tilde{Q}}$, and let $Q = \tilde{Q} U_1$. Then Q is a semistandard parabolic subgroup of G with Iwasawa decomposition $Q = L_{\tilde{Q}} N_Q$, $N_Q = N_{\tilde{Q}} U_1$, that is, Q and $\tilde{Q}$ have the same Levi component $L_Q = L_{\tilde{Q}}$ containing M. Hence $\varphi^Q = \varphi^{\tilde{Q}}$ and $w_{\mathcal{O},M}^Q = w_{\mathcal{O},M}^{\tilde{Q}}$. Applying (5.10) to $J_M^{M_1}(\mathcal{O}, (\psi_t^\nu)_{P_1})$ and unfolding the definition of $(\psi_t^\nu)_{P_1}$, we obtained

$$J_M^{M_1}(\mathcal{O}, (\psi_t^\nu)_{P_1}) = \sum_Q \tilde{c}(Q, \mathcal{O}) \int_{\mathfrak{u}_{Q,1}} \psi_t^\nu(e^Y) w_{M,\mathcal{O}}^Q(e^Y) |\varphi^{L_Q}(Y)|^{1/2} \, dY \tag{7.22}$$

where the sum runs over all $Q \in \mathcal{F}(M)$ in the image of the map $\tilde{Q} \mapsto Q$, and the constants depend on the normalization of measures. We therefore need to analyze integrals that are of the form as those on the right hand side of (7.22).

Fix $Q \in \mathcal{F}(M)$. We equip $\mathfrak{u}_{Q,1}$ with a Euclidean norm by fixing some isomorphism $\mathfrak{u}_{Q,1} \simeq \mathbb{R}^{\dim \mathfrak{u}_{Q,1}}$ which respects the direct sum decomposition $\mathfrak{u}_{Q,1} = \mathfrak{u}_1^{L_Q} \oplus \mathfrak{n}_Q$. Let $\epsilon > 0$ and let $B(\epsilon) \subseteq \mathfrak{u}_{Q,1}$ be the ball around 0 of radius ϵ. Put $U(\epsilon) = \mathfrak{u}_{Q,1} \smallsetminus B(\epsilon)$. We split the integral on the right hand side of (7.22) into integrals over $B(\epsilon)$ and $U(\epsilon)$.

Recall the function β from the definition (3.3) of ψ_t^ν. We assume that β has sufficiently small support. Let $\epsilon > 0$ be such that $\beta(r(\exp(\cdot)))$ restricted to $B(\epsilon)$ is identically equal to 1. Fix $\psi \in C^\infty(\mathbb{R})$ as in §6.2 (for the expansion (6.19)) such that the support of the restriction of $\psi(r(\exp(\cdot)))$ to $\mathfrak{u}_{Q,1}$ is contained in $B(\epsilon)$.

We first show that the integral over $U(\epsilon)$ decays exponentially in t^{-1} as $t \searrow 0$ and therefore does not contribute to the asymptotic expansion. Indeed, it follows from §5.5 and (6.20) that

$$\left| \int_{U(\epsilon)} h_t^\nu(e^Y) |\varphi^Q(Y)|^{1/2} w_{M,\mathcal{O}}^Q(e^Y)\, dY \right| \tag{7.23}$$

can be bounded by a constant multiple of a finite sum of integrals of the form

$$t^{-d/2} \int_{U(\epsilon)} e^{-r(e^Y)^2/(4t)} |\varphi^Q(Y)|^{1/2} \prod_{j=1,\dots,J} \log \|p_j(e^Y - I)\|\, dY \tag{7.24}$$

where p_j, $j = 1, \dots, J$ are suitable polynomials on the Lie algebra as the ones appearing in Corollary 5.4. In particular, $p_j(e^Y - I)$ does not vanish identically in $Y \in U(\epsilon)$.

The polynomial φ^Q has degree $\dim \mathfrak{l}_1^Q$ so that $|\varphi^Q(Y)|^{1/2} \ll_\epsilon \|Y\|^{\dim \mathfrak{l}_1^Q/2}$. It follows from Lemma 6.1, that there are constants $c_\epsilon, c_1 > 0$ such that (7.24) is bounded by a constant multiple of

$$e^{-c_\epsilon/t} \int_{\mathfrak{u}_{Q,1}} \|Y\|^{\frac{\dim \mathfrak{l}_1^Q}{2}} e^{-c_1 \log^2(1+\|Y\|)/t} \prod_{j=1,\dots,J} \log \|p_j(e^Y - I)\|\, dY$$

provided that ϵ is sufficiently small. Note that $p_j(e^Y - I)$ is again a polynomial in Y. By [MM17, Lemma 7.7] this integral converges. (In fact, [MM17, Lemma 7.7] only treats the case without the power of $\|Y\|$ in the integral, but it is immediate from the proof that a polynomial in $\|Y\|$ does not change the validity of the assertion.) Therefore, (7.24) is bounded by a constant multiple of $e^{-c_\epsilon/t}$, that is, it decays exponentially in t^{-1} as $t \searrow 0$.

For the integral over $B(\epsilon)$ we use (6.19) and the expansion of $w_{M,\mathcal{O}}^Q$ from Corollary 5.4. Using (6.19) we get

$$\begin{aligned} &\int_{B(\epsilon)} h_t^\nu(e^Y) |\varphi^Q(Y)|^{1/2} w_{M,\mathcal{O}}^Q(e^Y)\, dY \\ &\quad = (4\pi t)^{-d/2} \sum_{n=0}^N t^n \int_{B(\epsilon)} \exp\left(-r(e^Y)^2/(4t)\right) \psi(r(e^Y)) a_n^\nu(e^Y) |\varphi^Q(Y)|^{1/2} w_{M,\mathcal{O}}^Q(e^Y)\, dY \\ &\qquad + O\left(t^{N+1-d/2}\right). \end{aligned} \tag{7.25}$$

We now first proceed as in [MM17, §12.2] and use Taylor expansion of the functions $\exp\left(-r(e^Y)^2/(4t)\right)$ and $f(Y) := \psi(r(e^Y))a_n^\nu(e^Y)|\varphi^Q(Y)|^{1/2}$ in $N = e^Y - I$ around 0. Note that except for $\varphi^Q(Y)$ all involved functions depend only on $r(e^Y)$, and can therefore be continued to smooth function on all of $\mathfrak{gl}_n(\mathbb{R})$. Since $\varphi^Q(Y)$ is a polynomial, we can extend it to a polynomial on all of $\mathfrak{gl}_n(\mathbb{R})$ as well. Then, using (6.18), we can write for any $K \geq 1$ and any $0 < t \leq 1$,

$$\exp\left(-r(I + t^{1/2}N)^2/(4t)\right) = \exp\left(-\|N\|^2/4\right)\left(\sum_{k=0}^{K} t^{k/2}q_k(N) + R_K(t, N)\right)$$

where q_k are suitable polynomials of degree $\leq 3nk$, $q_0(N) = 1$, and $R_K(t, N)$ is a remainder term satisfying

$$|R_K(t, N)| \leq c_1 t^{(K-1)/2}(1 + \|N\|)^{3nK}$$

for every $0 < t \leq 1$ with $c_1 > 0$ some suitable constant.

Similarly,

$$f(\log(I + t^{1/2}N)) = \sum_{l \leq L} b_l(N)t^{l/2} + Q_L(t, N)$$

where b_l is a polynomial in N of degree $\leq l$, and $Q_L(t, N)$ is a remainder term satisfying

$$|Q_L(t, N)| \leq c_2 t^{(L+1)/2}(1 + \|N\|)^{L+1}$$

for all $0 < t \leq 1$ and N with $Y = \log(I + N) \in B(\epsilon)$ with $c_2 > 0$ some absolute constant. Note that $b_l(N) = 0$ whenever $l < \dim \mathfrak{l}_1^Q/2$ since $|\varphi^Q|^{1/2}$ is homogeneous in Y of degree $\dim \mathfrak{l}_1^Q/2$.

Hence the left hand side of (7.25) equals after a change of variables

$$\sum_{k \leq K}\sum_{l \leq L} t^{(k+l)/2} t^{\dim \mathfrak{u}_{Q,1}/2} \int_{t^{-1/2}\mathcal{B}(\epsilon)} \exp\left(-\|N\|^2/4\right) q_k(N) b_l(N) w_M^Q(I + t^{1/2}N)\, dN + \Phi_{K,L}(t)$$

where $\mathcal{B}(\epsilon)$ is the image of $B(\epsilon)$ in $\mathfrak{gl}_n(\mathbb{R})$ under $Y \mapsto N = e^Y - I$, and with the remainder $\Phi_{K,L}(t)$ satisfying

$$|\Phi_{K,L}(t)| \leq c_3 t^{(L+K+1)/2}.$$

Note that the Jacobian of the change from Y to N is a polynomial in N with non-vanishing constant term that we absorb into the asymptotic expansion. Using the asymptotic expansion for the weight $w_M^Q(I + t^{1/2}N)$, we can find coefficients $c_{m,j,\Xi}^\nu$ such that the left hand side of (7.25) equals

$$t^{-d/2} t^{\dim \mathfrak{u}_{Q,1}/2} \sum_{\dim \mathfrak{l}_1^Q \leq m \leq N} \sum_{j=0}^{r} t^{m/2}(\log t)^j \sum_{\Xi} \int_{t^{-1/2}\mathcal{B}(\epsilon)} c_{m,j,\Xi}^\nu(Y) \prod_{k \in \Xi} \log\|p_k(Y)\|\, dY + \tilde{\Phi}_N(t)$$

with the polynomials p_k, $k = 1, \ldots, q$, as in Corollary 5.4, Ξ running over (multi-)subsets of $\{1, \ldots, q\}$ whose size is bounded by the polynomials Q_j appearing in Corollary 5.4, and

$\tilde{\Phi}_N(t)$ satisfying

$$\left|\tilde{\Phi}_N(t)\right| \leq c_4 t^{(N-d+\dim \mathfrak{l}_1^Q+\dim \mathfrak{u}_{Q,1}+1)/2}.$$

Now for $0 < t \leq 1$ we have

$$\sum_{\Xi} \int_{t^{-1/2}\mathcal{B}(\epsilon)} c^{\nu}_{m,j,\Xi}(Y) \prod_{k\in\Xi} \log \|p_k(Y)\| \, dY = C_{m,j} + O\left(e^{-c_4/t}\right)$$

which follows as in [MM17, §12]. Hence (7.25) equals

$$t^{(-d+\dim \mathfrak{l}_1^Q+\dim \mathfrak{u}_{Q,1})/2} \sum_{m\leq N} \sum_{j=0}^{r} \tilde{C}_{m,j} t^{m/2} (\log t)^j + O\left(t^{(N-d+\dim \mathfrak{l}_1^Q+1)/2}\right)$$

for $0 < t \leq 1$ where $\tilde{C}_{m,j} = C_{m+\dim \mathfrak{l}_1^Q, j}$. Since $d_{\mathcal{O}}^{G_\infty} = 2(\dim \mathfrak{u}_{Q,1} + \dim \mathfrak{l}_1^Q)$, the assertion of the proposition follows.

8. Proof of Theorem 1.1

The proof of Theorem 1.1 is global so that we return to our global notation. In particular, G denotes again a reductive algebraic group defined over $\mathbb{Q}$. We fix a minimal parabolic subgroup P_0 of G (as an algebraic group over $\mathbb{Q}$), and write $P_0 = M_0 U_0$ with U_0 the unipotent radical of P_0 and M_0 the Levi subgroup of P_0. We call a parabolic subgroup of G standard if it contains P_0, and semistandard if it contains M_0. Let $\mathcal{L}$ denote the set of all semistandard Levi subgroups of G, that is, all $M \subseteq G$ which are Levi components of semistandard parabolic subgroups.

8.1. **Arthur's fine geometric expansion.** Let S be a finite set of places of $\mathbb{Q}$, which includes the archimedean place, such that $K_v = \mathbf{K}_v$ for $v \notin S$. Let $G(\mathbb{Q}_S)^1 = G(\mathbb{Q}_S) \cap G(\mathbb{A})^1$.

Let $M \in \mathcal{L}$. Following Arthur, we introduce an equivalence relation on the set of unipotent elements in $M(\mathbb{Q})$ that depends on the set S: Two unipotent elements $u, v \in M(\mathbb{Q})$ are equivalent if and only if u and v are conjugate in $M(\mathbb{Q}_S)$. We denote the equivalence class of u by $[u]_S \subseteq M(\mathbb{Q})$ and let $\mathcal{U}_S^M$ denote the set of all such equivalence classes.

Note that two equivalent unipotent elements define the same unipotent conjugacy class in $M(\mathbb{Q}_S)$, so we can view $\mathcal{U}_S^M$ also as the set of unipotent conjugacy classes in $M(\mathbb{Q}_S)$ which have at least one $\mathbb{Q}$-rational representative, and we denote the corresponding conjugacy class by $[u]_S$ as well. This differs from our notation for unipotent conjugacy classes in $G(\mathbb{R})^1$ from the previous sections, but we now need to keep track of the dependence on S.

Remark 8.1. *(i) If $T \subseteq S$, then we get a well-defined map $\mathcal{U}_S^M \ni [u]_S \mapsto [u]_T \in \mathcal{U}_T^M$.*
(ii) If $G = \mathrm{GL}(n)$, the equivalence relation is independent of S and is the same as conjugation in $M(\mathbb{Q})$.

For $[u]_S \in \mathcal{U}_S^M$ and $f_S \in C_c^\infty(G(\mathbb{Q}_S)^1)$, Arthur associates a weighted orbital integral $J_M^G([u]_S, f_S)$ [Ar88] which is a distribution supported on the $G(\mathbb{Q}_S)$-conjugacy class induced from $[u]_S \subseteq M(\mathbb{Q}_S)$. If $S = \{\infty\}$, these distributions were discussed in §5. Let $\mathbf{1}_{\mathbf{K}^S} \in C_c^\infty(G(\mathbb{A}^S))$ be the characteristic function of $\mathbf{K}^S$, if $f_S \in C_c^\infty(G(\mathbb{Q}_S)^1)$, then we write $f = f_S \mathbf{1}_{\mathbf{K}^S} \in C_c^\infty(G(\mathbb{A})^1)$.

Proposition 8.2 ([Ar85], Corollary 8.3)**.** *There exist unique constants $a^M([u]_S, S) \in \mathbb{C}$, $[u]_S \in \mathcal{U}_S^M$, such that for all $f_S \in C_c^\infty(G(\mathbb{Q}_S)^1)$ we have*

$$J_{unip}(f) = \sum_{M \in \mathcal{L}} \sum_{[u]_S \in \mathcal{U}_S^M} a^M([u]_S, S) J_M^G([u]_S, f_S). \tag{8.26}$$

8.2. **The splitting formula.** To understand the behavior of the distributions $J_M^G([u]_S, f_S)$ for our test functions, we want to apply our asymptotic expansion for the archimedean weighted integral. To that end we need to separate ∞ from the other places in S which we will do by using Arthur's splitting formula [Ar05, (18.7)]: Suppose that $S = S_1 \cup S_2$ with S_1, S_2 non-empty and disjoint, and that f_S is the restriction of a product $f_{S_1} f_{S_2}$ to $G(\mathbb{Q}_S)^1$ with $f_{S_j} \in C^\infty(G(\mathbb{Q}_{S_j}))$, $j = 1, 2$. Then

$$J_M^G([u]_S, f_S) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) J_M^{L_1}([u]_{S_1}, f_{S_1, Q_1}) J_M^{L_2}([u]_{S_2}, f_{S_2, Q_2}), \tag{8.27}$$

where the notation is as follows: The $d_M^G(L_1, L_2) \in \mathbb{R}$ are certain constants which depend only on M, L_1, L_2, G but not on S. In fact, $d_M^G(L_1, L_2)$ is non-zero only if the natural map $\mathfrak{a}_M^{L_1} \oplus \mathfrak{a}_M^{L_2} \longrightarrow \mathfrak{a}_M^G$ is an isomorphism. The Q_j are arbitrary elements in $\mathcal{P}(L_j)$, and

$$f_{S_j, Q_j}(m) = \delta_{Q_j}(m)^{1/2} \int_{\mathbf{K}_{S_j}} \int_{N_j(\mathbb{Q}_{S_j})} f_{S_j}(k^{-1} m n k)\, dn\, dk \tag{8.28}$$

where N_j is the unipotent radical of Q_j. Finally, $J_M^{L_j}([u]_{S_j}, \cdot)$ denotes the S_j-adic distribution supported on the $L_j(\mathbb{Q}_{S_j})$-conjugacy class induced from $[u]_{S_j} \subseteq M(\mathbb{Q}_{S_j})$ and defined as in [Ar88].

8.3. **Completion of the proof of Theorem 1.1.** Let S be as in §8.1 and write $S = \{\infty\} \sqcup S_0$. Then $K_f = K_{S_0} \mathbf{K}^S$. Recall the definition of $\tilde{\phi}_t^\nu$ and ψ_t^ν from (3.4) and (3.3), respectively, so that

$$\tilde{\phi}_t^\nu = \psi_t^\nu \cdot \mathbf{1}_{K_{S_0}} \cdot \mathbf{1}_{\mathbf{K}^S}.$$

By Proposition 8.2 we have

$$J_{\text{unip}}(\tilde{\phi}_t^\nu) = \sum_{M \in \mathcal{L}} \sum_{[u]_S \in \mathcal{U}_S^M} a^M([u]_S, S) J_M^G([u]_S, \psi_t^\nu \cdot \mathbf{1}_{K_{S_0}}).$$

This is a finite sum and the number of summands is independent of t because the support of $\psi_t^\nu \cdot \mathbf{1}_{K_{S_0}}$ is independent of t. To prove Theorem 1.1 it therefore suffices to establish an asymptotic expansion of $J_M^G([u]_S, \psi_t^\nu \cdot \mathbf{1}_{K_{S_0}})$ as $t \searrow 0$. We first apply the splitting formula (8.27) to this integral with $S_1 = \{\infty\}$ and $S_2 = S_0$. We get

$$J_M^G([u]_S, \psi_t^\nu \cdot \mathbf{1}_{K_{S_0}}) = \sum_{L_1,\, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) J_M^{L_1}([u]_\infty, \psi_{t,Q_1}^\nu) J_M^{L_2}([u]_{S_0}, \mathbf{1}_{K_{S_0},Q_2}).$$

Again, this is a finite sum with number of summands independent of t, and $d_M^G(L_1, L_2)$ and $J_M^{L_2}([u]_{S_0}, \mathbf{1}_{K_{S_0},Q_2})$ constant in t. In combination with the asymptotic expansion of $J_M^{L_1}([u]_\infty, \psi_{t,Q_1}^\nu)$ as $t \searrow 0$ from Proposition 7.1 we obtain Theorem 1.1

9. The spectral side of the non-invariant trace formula

For the convenience of the reader we summarize in this section some basic facts about Arthur's non-invariant trace formula. The trace formula is the equality

$$J_{\text{geom}}(f) = J_{\text{spec}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1), \tag{9.1}$$

of the geometric side $J_{\text{geom}}(f)$ and the spectral side $J_{\text{spec}}(f)$ of the trace formula. The geometric side has been described in the previous section. In this section we recall the definition of the spectral side, and in particular the refinement of the spectral expansion obtained in [FLM11].

The main ingredient of the spectral side are logarithmic derivatives of intertwining operators. We briefly recall the structure of the intertwining operators.

Let $P = MU_P \in \mathcal{P}(M)$. Recall that we denote by $\Sigma_P \subset \mathfrak{a}_P^*$ the set of reduced roots of A_M of the Lie algebra $\mathfrak{u}_P$ of U_P. Let Δ_P be the subset of simple roots of P, which is a basis for $(\mathfrak{a}_P^G)^*$. Write $\mathfrak{a}_{P,+}^*$ for the closure of the Weyl chamber of P, i.e.

$$\mathfrak{a}_{P,+}^* = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Sigma_P\} = \{\lambda \in \mathfrak{a}_M^* : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta_P\}.$$

Denote by δ_P the modulus function of $P(\mathbb{A})$. Let $\bar{\mathcal{A}}_2(P)$ be the Hilbert space completion of

$$\{\phi \in C^\infty(M(\mathbb{Q})U_P(\mathbb{A})\backslash G(\mathbb{A})) : \delta_P^{-\frac{1}{2}}\phi(\cdot x) \in L^2_{\text{disc}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A})),\ \forall x \in G(\mathbb{A})\}$$

with respect to the inner product

$$(\phi_1, \phi_2) = \int_{A_M(\mathbb{R})^0 M(\mathbb{Q})\mathbf{U}_P(\mathbb{A})\backslash G(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}\, dg.$$

Let $\alpha \in \Sigma_M$. We say that two parabolic subgroups $P, Q \in \mathcal{P}(M)$ are *adjacent* along α, and write $P|^\alpha Q$, if $\Sigma_P \cap -\Sigma_Q = \{\alpha\}$. Alternatively, P and Q are adjacent if the group $\langle P, Q\rangle$ generated by P and Q belongs to $\mathcal{F}_1(M)$ (see (2.15) for its definition). Any $R \in \mathcal{F}_1(\mathcal{M})$ is of the form $\langle P, Q\rangle$, where P, Q are the elements of $\mathcal{P}(M)$ contained in R. We have $P|^\alpha Q$ with $\alpha^\vee \in \Sigma_P^\vee \cap \mathfrak{a}_M^R$. Interchanging P and Q changes α to $-\alpha$.

For any $P \in \mathcal{P}(M)$ let $H_P \colon G(\mathbb{A}) \to \mathfrak{a}_P$ be the extension of H_M to a left $U_P(\mathbb{A})$-and right $\mathbf{K}$-invariant map. Denote by $\mathcal{A}^2(P)$ the dense subspace of $\bar{\mathcal{A}}^2(P)$ consisting of its $\mathbf{K}$- and $\mathfrak{Z}$-finite vectors, where $\mathfrak{Z}$ is the center of the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}$. That is, $\mathcal{A}^2(P)$ is the space of automorphic forms ϕ on $U_P(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A})$ such that $\delta_P^{-\frac{1}{2}}\phi(\cdot k)$ is a square-integrable automorphic form on $A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A})$ for all $k \in \mathbf{K}$. Let $\rho(P,\lambda)$, $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, be the induced representation of $G(\mathbb{A})$ on $\bar{\mathcal{A}}^2(P)$ given by

$$(\rho(P,\lambda,y)\phi)(x) = \phi(xy)e^{\langle\lambda, H_P(xy)-H_P(x)\rangle}.$$

It is isomorphic to the induced representation

$$\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\left(L^2_{\text{disc}}(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A}))\otimes e^{\langle\lambda,H_M(\cdot)\rangle}\right).$$

For $P, Q \in \mathcal{P}(M)$ let

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}, \tag{9.2}$$

be the standard *intertwining operator* [Ar82, §1], which is the meromorphic continuation in λ of the integral

$$[M_{Q|P}(\lambda)\phi](x) = \int_{U_Q(\mathbb{A})\cap U_P(\mathbb{A})\backslash U_Q(\mathbb{A})} \phi(nx)e^{\left\langle\lambda, H_P(nx)-H_Q(x)\right\rangle}\,dn, \quad \phi\in\mathcal{A}^2(P),\ x\in G(\mathbb{A}).$$

Given $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$, let $\mathcal{A}^2_\pi(P)$ be the space of all $\phi \in \mathcal{A}^2(P)$ for which the function $M(\mathbb{A}) \ni x \mapsto \delta_P^{-\frac{1}{2}}\phi(xg)$, $g \in G(\mathbb{A})$, belongs to the π-isotypic subspace of the space $L^2(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A}))$. For any $P \in \mathcal{P}(\mathcal{M})$ we have a canonical isomorphism of $G(\mathbb{A}_f)\times(\mathfrak{g}_\mathbb{C}, \mathbf{K}_\infty)$-modules

$$j_P : \operatorname{Hom}(\pi, L^2(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A})))\otimes \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi) \to \mathcal{A}^2_\pi(P).$$

If we fix a unitary structure on π and endow $\operatorname{Hom}(\pi, L^2(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A})))$ with the inner product $(A,B) = B^*A$ (which is a scalar operator on the space of π), the isomorphism j_P becomes an isometry.

Suppose that $P|^\alpha Q$. The operator $M_{Q|P}(\pi,s) := M_{Q|P}(s\varpi)|_{\mathcal{A}^2_\pi(P)}$, where $\varpi \in \mathfrak{a}^*_M$ is such that $\langle\varpi,\alpha^\vee\rangle = 1$, admits a normalization by a global factor $n_\alpha(\pi,s)$ which is a meromorphic function in s. We may write

$$M_{Q|P}(\pi,s)\circ j_P = n_\alpha(\pi,s)\cdot j_Q\circ(\operatorname{Id}\otimes R_{Q|P}(\pi,s)) \tag{9.3}$$

where $R_{Q|P}(\pi,s) = \otimes_v R_{Q|P}(\pi_v,s)$ is the product of the locally defined normalized intertwining operators and $\pi = \otimes_v\pi_v$ [Ar82, §6], (cf. [Mu02, (2.17)]). In many cases, the normalizing factors can be expressed in terms automorphic L-functions [Sh81], [Sh88].

We now turn to the spectral side. Let $L \supset M$ be Levi subgroups in $\mathcal{L}$, $P \in \mathcal{P}(M)$, and let $m = \dim\mathfrak{a}^G_L$ be the co-rank of L in G. Denote by $\mathfrak{B}_{P,L}$ the set of m-tuples $\underline{\beta} = (\beta_1^\vee,\dots,\beta_m^\vee)$ of elements of $\Sigma_P^\vee$ whose projections to $\mathfrak{a}_L$ form a basis for $\mathfrak{a}^G_L$. For any $\underline{\beta} = (\beta_1^\vee,\dots,\beta_m^\vee)\in\mathfrak{B}_{P,L}$ let $\operatorname{vol}(\underline{\beta})$ be the co-volume in $\mathfrak{a}^G_L$ of the lattice spanned by $\underline{\beta}$ and let

$$\begin{aligned}\Xi_L(\underline{\beta}) &= \{(Q_1,\dots,Q_m)\in\mathcal{F}_1(M)^m : \quad \beta_i^\vee\in\mathfrak{a}^{Q_i}_M,\ i=1,\dots,m\}\\ &= \{\langle P_1,P_1'\rangle,\dots,\langle P_m,P_m'\rangle) : \quad P_i|^{\beta_i}P_i',\ i=1,\dots,m\}.\end{aligned}$$

For any smooth function f on $\mathfrak{a}_M^*$ and $\mu \in \mathfrak{a}_M^*$ denote by $D_\mu f$ the directional derivative of f along $\mu \in \mathfrak{a}_M^*$. For a pair $P_1|^\alpha P_2$ of adjacent parabolic subgroups in $\mathcal{P}(M)$ write

$$\delta_{P_1|P_2}(\lambda) = M_{P_2|P_1}(\lambda) D_\varpi M_{P_1|P_2}(\lambda) : \mathcal{A}^2(P_2) \to \mathcal{A}^2(P_2), \tag{9.4}$$

where $\varpi \in \mathfrak{a}_M^*$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$. [1] Equivalently, writing $M_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)$ for a meromorphic function Φ of a single complex variable, we have

$$\delta_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha^\vee \rangle)^{-1} \Phi'(\langle \lambda, \alpha^\vee \rangle).$$

For any m-tuple $\mathcal{X} = (Q_1, \dots, Q_m) \in \Xi_L(\underline{\beta})$ with $Q_i = \langle P_i, P_i' \rangle$, $P_i|^{\beta_i} P_i'$, denote by $\Delta_{\mathcal{X}}(P, \lambda)$ the expression

$$\frac{\mathrm{vol}(\underline{\beta})}{m!} M_{P_1'|P}(\lambda)^{-1} \delta_{P_1|P_1'}(\lambda) M_{P_1'|P_2'}(\lambda) \cdots \delta_{P_{m-1}|P_{m-1}'}(\lambda) M_{P_{m-1}'|P_m'}(\lambda) \delta_{P_m|P_m'}(\lambda) M_{P_m'|P}(\lambda). \tag{9.5}$$

Recall the (purely combinatorial) map $\mathcal{X}_L : \mathfrak{B}_{P,L} \to \mathcal{F}_1(M)^m$ with the property that $\mathcal{X}_L(\underline{\beta}) \in \Xi_L(\underline{\beta})$ for all $\underline{\beta} \in \mathfrak{B}_{P,L}$ as defined in [FLM11, pp. 179–180].[2]

For any $s \in W(M)$ let L_s be the smallest Levi subgroup in $\mathcal{L}(M)$ containing w_s. We recall that $\mathfrak{a}_{L_s} = \{H \in \mathfrak{a}_M \mid sH = H\}$. Set

$$\iota_s = |\det(s-1)_{\mathfrak{a}_M^{L_s}}|^{-1}.$$

For $P \in \mathcal{F}(M_0)$ and $s \in W(M_P)$ let $M(P,s) : \mathcal{A}^2(P) \to \mathcal{A}^2(P)$ be as in [Ar81, p. 1309]. $M(P,s)$ is a unitary operator which commutes with the operators $\rho(P, \lambda, h)$ for $\lambda \in \mathrm{i}\mathfrak{a}_{L_s}^*$. Finally, we can state the refined spectral expansion.

Theorem 9.1 ([FLM11]). *For any $h \in C_c^\infty(G(\mathbb{A})^1)$ the spectral side of Arthur's trace formula is given by*

$$J_{spec}(h) = \sum_{[M]} J_{spec,M}(h), \tag{9.6}$$

M ranging over the conjugacy classes of Levi subgroups of G (represented by members of $\mathcal{L}$), where

$$J_{spec,M}(h) = \frac{1}{|W(M)|} \sum_{s \in W(M)} \iota_s \sum_{\underline{\beta} \in \mathfrak{B}_{P,L_s}} \int_{\mathrm{i}(\mathfrak{a}_{L_s}^G)^*} \mathrm{tr}(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P, \lambda) M(P,s) \rho(P, \lambda, h)) \, d\lambda \tag{9.7}$$

with $P \in \mathcal{P}(M)$ arbitrary. The operators are of trace class and the integrals are absolutely convergent with respect to the trace norm and define distributions on $\mathcal{C}(G(\mathbb{A})^1)$.

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[1] Note that this definition differs slightly from the definition of $\delta_{P_1|P_2}$ in [FLM11].

[2] The map $\mathcal{X}_L$ depends in fact on the additional choice of a vector $\underline{\mu} \in (\mathfrak{a}_M^*)^m$ which does not lie in an explicit finite set of hyperplanes. For our purposes, the precise definition of $\mathcal{X}_L$ is immaterial.

10. Logarithmic derivatives of local intertwining operators

In this section we prove some auxiliary results about local intertwining operators. To begin with we recall some facts concerning local intertwining operators and normalizing factors. Let $M \in \mathcal{L}$ and $P, Q \in \mathcal{P}(M)$. Let v be a finite place of $\mathbb{Q}$ and K_v an open compact subgroup of $G(\mathbb{Q}_v)$. Let $\pi_v \in \Pi(M(\mathbb{Q}_v))$. Given $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, let $(I^G_P(\pi_v, \lambda), \mathcal{H}_P(\pi_v))$ denote the induced representation. Let $\mathcal{H}^0_P(\pi_v) \subset \mathcal{H}_P(\pi_v)$ be the subspace of K_v-finite functions. Let

$$J_{Q|P}(\pi_v, \lambda) \colon \mathcal{H}^0_P(\pi_v) \to \mathcal{H}^0_Q(\pi_v)$$

be the local intertwining operator between the induced representations $I^G_P(\pi_v, \lambda)$ and $I^G_Q(\pi_v, \lambda)$ [Sh81]. It is proved in [Ar89], [CLL, Lecture 15] that there exist scalar valued meromorphic functions $r_{Q|P}(\pi_v, \lambda)$ of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ such that the normalized intertwining operators

$$R_{Q|P}(\pi_v, \lambda) = r_{Q|P}(\pi_v, \lambda)^{-1} J_{Q|P}(\pi_v, \lambda) \tag{10.1}$$

satisfy the conditions $(R_1) - (R_8)$ of Theorem 2.1 of [Ar89]. We recall some facts about the local normalizing factors. First assume that v is a finite valuation of $\mathbb{Q}$ with corresponding prime number $q_v \in \mathbb{N}$. Furthermore assume that $\dim(\mathfrak{a}_M/\mathfrak{a}_G) = 1$ and π_v is square integrable. Let $P \in \mathcal{P}(M)$ and let α be the unique simple root of (P, A_M). Then Langlands [CLL, Lecture 15] has shown that there exists a rational function $V_P(\pi_v, z)$ of one variable such that

$$r_{\overline{P}|P}(\pi_v, \lambda) = V_P(\pi_v, q_v^{-\lambda(\widetilde{\alpha})}), \tag{10.2}$$

where $\widetilde{\alpha} \in \mathfrak{a}_M$ is uniquely determined by α. For the construction of V_P see also [Mu02, Sect. 3]. We need the following lemma.

Lemma 10.1. *Let $M \in \mathcal{L}$ be such that $\dim(\mathfrak{a}_M/\mathfrak{a}_G) = 1$. There exists $C > 0$ such that for all $P \in \mathcal{P}(M)$ and all $\pi \in \Pi(M(\mathbb{Q}_v))$ the number of zeros of the rational function $V_P(\pi, z)$ is less than or equal to C.*

Proof. First assume that π is square integrable. Then the corresponding statement for the number of poles was proved in [Mu02, Lemma 3.1]. However, by [Mu02, (3.6)] the number of zeros of $V_P(\pi, z)$ agrees with the number of poles $V_P(\pi, z)$. Now let π be tempered. It is known that π is an irreducible constituent of an induced representation $I^M_R(\sigma)$ where M_R is an admissible Levi subgroup of M and $\sigma \in \Pi(M_R(\mathbb{Q}_v))$ is square integrable modulo A_R. Then by [Ar89, (2.2)] we are reduced to the square integrable case. In general π is a Langlands quotient of of an induced representation $I^M_R(\sigma, \mu)$, where M_R is an admissible Levi subgroup of M, $\sigma \in \Pi_{\text{temp}}(M_R(\mathbb{Q}_v))$, and μ is a point in the chamber of $\mathfrak{a}^*_R/\mathfrak{a}^*_M$ attached to R. Now we use [Ar89, (2.3)] to reduce the proof to the tempered case. □

The main goal of this section is to estimate the logarithmic derivatives of the normalized intertwining operators $R_{Q|P}(\pi, \lambda)$. For $G = \mathrm{GL}(n)$ such estimates were derived in [MS04, Proposition 0.2]. The proof depends on a weak version of the Ramanujan conjecture, which is not available in general. Therefore we will establish only an integrated version of it, which however, is sufficient for our purpose. For $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$ denote by $\mathcal{H}_P(\pi)$ the Hilbert space of the induced representation $I_P^G(\pi,\lambda)$. Furthermore, for an open compact subgroup $K_f \subset G(\mathbb{A}_f)$ and $\nu \in \Pi(\mathbf{K}_\infty)$, denote by $\mathcal{H}_P(\pi)^{K_f}$ the subspace of vectors, which are invariant under K_f and let $\mathcal{H}_P(\pi)^{K_f,\nu}$ denote the ν-isotypical subspace of $\mathcal{H}_P(\pi)^{K_f}$. Let $P, Q \in \mathcal{P}(M)$ be adjacent parabolic subgroups. Then $R_{Q|P}(\pi,\lambda)$ depends on a single variable $s \in \mathbb{C}$ and we will write

$$R'_{Q|P}(\pi, s_0) := \frac{d}{ds} R_{Q|P}(\pi, s)\big|_{s=s_0}$$

for any regular $s_0 \in \mathbb{C}$.

Proposition 10.2. *Let $M \in \mathcal{L}$, and let $P, Q \in \mathcal{P}(M)$ be adjacent parabolic subgroups. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup and let $\nu \in \Pi(\mathbf{K}_\infty)$. Then there exists $C > 0$ such that*

$$\int_{\mathbb{R}} \left\| R_{Q|P}(\pi, it)^{-1} R'_{Q|P}(\pi, it)\big|_{\mathcal{H}_P(\pi)^{K_f,\nu}} \right\| (1+|t|)^{-1} dt \le C \tag{10.3}$$

for all $\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))$ with $\mathcal{H}_P(\pi)^{K_f,\nu} \neq 0$.

Proof. We may assume that K_f is factorisable, i.e., $K_f = \prod_v K_v$. Let S be the finite set of finite places such that K_v is not hyperspecial. Since P and Q are adjacent, by standard properties of normalized intertwining operators [Ar89, Theorem 2.1] we may assume that P is a maximal parabolic subgroup and $Q = \overline{P}$, the opposite parabolic subgroup to P. By [Ar89, Theorem 2.1, (R8)], $R_{Q|P}(\pi_v, s)^{K_v}$ is independent of s if v is finite and $v \notin S$. Thus we have

$$\begin{aligned} R_{\overline{P}|P}(\pi, s)^{-1} R'_{\overline{P}|P}(\pi, s)\big|_{H_P(\pi)^{K_f,\nu}} &= R_{\overline{P}|P}(\pi_\infty, s)^{-1} R'_{\overline{P}|P}(\pi_\infty, s)\big|_{\mathcal{H}_P(\pi_\infty)^\nu} \\ &\quad + \sum_{v\in S} R_{\overline{P}|P}(\pi_v, s)^{-1} R'_{\overline{P}|P}(\pi_v, s)\big|_{\mathcal{H}_P(\pi_v)^{K_v}} \end{aligned} \tag{10.4}$$

This reduces our problem to the operators at the local places. We distinguish between the archimedean and the non-archimedean case.

Case 1: $v < \infty$. Define $A_v \colon \mathbb{C} \to \mathrm{End}(\mathcal{H}_P(\pi_v)^{K_v})$ by

$$A_v(q_v^{-s}) := R_{\overline{P}|P}(\pi_v, s)\big|_{\mathcal{H}_P(\pi_v)^{K_v}}.$$

This is a meromorphic function with values in the space of endomorphisms of a finite dimensional vector space. It has the following properties. By the unitarity of $R_{\overline{P}|P}(\pi_v, it)$, $t \in \mathbb{R}$, it follows that $A_v(z)$ is holomorphic for $z \in S^1$ and satisfies $\|A_v(z)\| \le 1$, $|z| = 1$. By [Ar89, Theorem 2.1], the matrix coefficients of $A_v(z)$ are rational functions. As in [FLM15, (14)] we get

$$\int_{\mathbb{R}} \left\| R_{\overline{P}|P}(\pi_v, it)^{-1} R'_{\overline{P}|P}(\pi_v, it)\big|_{\mathcal{H}_P(\pi_v)^{K_v}} \right\| (1+|t|)^{-1} dt \le \left(2 + \frac{1}{2}\log q_v\right) \int_{S^1} \|A'_v(z)\|\ |dz|. \tag{10.5}$$

As explained above, A_v satisfies the assumptions of [FLM15, Corollary 5.18]. Denote by $z_1, \ldots, z_m \in \mathbb{C} \setminus S^1$ be the poles of $A_v(z)$. Then $(z - z_1) \cdots (z - z_m) A_v(z)$ is a polynomial of degree n with coefficients in $\operatorname{End}(\mathcal{H}_{\pi_v}^{K_v})$ and by [FLM15, Corollary 5.18] we get

$$\|A_v'(z)\| \leq \max \left(\max(n-m, 0) + \sum_{j \colon |z_j|>1} \frac{|z_j|^2 - 1}{|z_j - z|^2}, \sum_{j \colon |z_j|<1} \frac{1 - |z_j|^2}{|z_j - z|^2} \right), \quad z \in S^1.$$

To estimate the right hand side, we need the following lemma.

Lemma 10.3. *There exists $C > 0$ such that for all $z_0 \in \mathbb{C} \setminus S^1$*

$$\int_{S^1} \frac{|z_0|^2 - 1}{|z - z_0|^2} |dz| \leq C. \tag{10.6}$$

Proof. First consider the case $|z_0| < 1$. If we change variables by $z = e^{i\theta}$, then up to a constant, the integral equals the integral of the Poisson kernel for the unit disc over the unit circle. From the theory of the Poisson kernel it is well known that, as a function of z_0, this integral is a continuous function on the closed unit disc which is equal to 1 on the unit circle. Thus the lemma holds for $|z_0| < 1$. For $|z_0| > 1$ we use that

$$\int_0^{2\pi} \frac{|z_0|^2 - 1}{|e^{i\theta} - z_0|^2} d\theta = \int_0^{2\pi} \frac{|z_0^{-1}|^2 - 1}{|e^{i\theta} - z_0^{-1}|^2} d\theta,$$

which reduces the problem to the previous case. $\square$

Next we estimate m and n. First consider m. Let $J_{\overline{P}|P}(\pi_v, s)$ be the usual intertwining operator so that

$$R_{\overline{P}|P}(\pi_v, s) = r_{\overline{P}|P}(\pi_v, s)^{-1} J_{\overline{P}|P}(\pi_v, s),$$

where $r_{\overline{P}|P}(\pi_v, s)$ is the normalizing factor [Ar89]. By [Sh81, Theorem 2.2.2] there exists a polynomial $p(z)$ with $p(0) = 1$ whose degree is bounded independently of π_v, such that $p(q_v^{-s}) J_{\overline{P}|P}(\pi_v, s)$ is holomorphic on $\mathbb{C}$. To deal with the normalizing factor we use (10.2) together with Lemma 10.1 to count the number of poles of $r_{\overline{P}|P}(\pi_v, s)^{-1}$. This leads to a bound for m which depends only on G. To estimate n we fix an open compact subgroup K_v of $G(\mathbb{Q}_v)$. Our goal is now to estimate the order at ∞ of any matrix coefficient of $R_{\overline{P}|P}(\pi_v, s)$ regarded as a function of $z = q_v^{-s}$. Write π_v as Langlands quotient $\pi_v = J_R^M(\delta_v, \mu)$ where R is a parabolic subgroup of M, δ_v a square integrable representation of $M_R(\mathbb{Q}_v)$ and $\mu \in (\mathfrak{a}_R^*/\mathfrak{a}_M^*)_{\mathbb{C}}$ with $\operatorname{Re}(\mu)$ in the chamber attached to R. Then by [Ar89, p. 30] we have

$$R_{\overline{P}|P}(\pi_v, s) = R_{\overline{P}(R)|P(R)}(\delta_v, s + \mu)$$

with respect to the identifications described in [Ar89, p. 30]. Here s is identified with a point in $(\mathfrak{a}_R^*/\mathfrak{a}_G^*)_{\mathbb{C}}$ with respect to the canonical embedding $\mathfrak{a}_M^* \subset \mathfrak{a}_G^*$. Using again the factorization of normalized intertwining operators we reduce the problem to the case of a square-integrable representation δ_v. Moreover δ_v has to satisfy $[I_P^G(\delta_v,s)\big|_{K_v}:\mathbf{1}]\geq 1$. By [Si, Lemma 1] we have

$$[I_P^G(\delta_v,s)\big|_{K_v}:\mathbf{1}]\geq 1\Leftrightarrow[\delta_v\big|_{K_v\cap M(\mathbb{Q}_v)}:\mathbf{1}]\geq 1 \tag{10.7}$$

Let $\Pi_2(M(\mathbb{Q}_v))$ be the space of square-integrable representations of $M(\mathbb{Q}_v)$. This space has a manifold structure [HC], [Si]. By [HC, Theorem 10] the set of square-integrable representations $\Pi_2(M(\mathbb{Q}_v),K_v)$ of $M(\mathbb{Q}_v)$ with $[\delta_v\big|_{K_v\cap M(\mathbb{Q}_v)}:\mathbf{1}]\geq 1$ is a compact subset of $\Pi_2(M(\mathbb{Q}_v))$. Under the canonical action of $i\mathfrak{a}_M$, the set $\Pi_2(M(\mathbb{Q}_v),K_v)$ decomposes into a finite number of orbits. For $\mu\in i\mathfrak{a}_M$ and $\delta_v\in\Pi_2(M(\mathbb{Q}_v),K_v)$, let $(\delta_v)_\mu\in\Pi_2(M(\mathbb{Q}_v),K_v)$ be the result of the canonical action. Then it follows that

$$R_{\overline{P}|P}((\delta_v)_\mu,\lambda)=R_{\overline{P}|P}(\delta_v,\lambda+\mu).$$

In this way our problem is finally reduced to the consideration of the matrix coefficients of $R_{\overline{P}|P}(\pi_v,s)\big|_{K_v}$ for a finite number of representations π_v. This implies that n is bounded by a constant which is independent of π_v. Together with Lemma 10.3 it follows that for each finite place v of $\mathbb{Q}$ and each open compact subgroup K_v of $G(\mathbb{Q}_v)$ there exists $C_v>0$ such that

$$\int_{\mathbb{R}}\left\|R_{\overline{P}|P}(\pi_v,it)^{-1}R'_{\overline{P}|P}(\pi_v,it)\big|_{\mathcal{H}_P(\pi_v)^{K_v}}\right\|(1+|t|)^{-1}dt\leq C_v \tag{10.8}$$

for all $\pi_v\in\Pi(M(\mathbb{Q}_v))$ with $I_P^G(\pi_v)\big|_{\mathcal{H}_P(\pi_v)^{K_v}}\neq 0$.

Case 2: $v=\infty$. As above let $M\in\mathcal{L}$ with $\dim(\mathfrak{a}_M/\mathfrak{a}_G)=1$ and $P\in\mathcal{P}(M)$. Let $\pi_\infty\in\Pi(M(\mathbb{R}))$ and $\nu\in\Pi(\mathbf{K}_\infty)$. As explained in [MS04, Appendix], there exist $w_1,\dots,w_r\in\mathbb{C}$ and $m\in\mathbb{N}$ such that the poles of $R_{\overline{P}|P}(\pi_\infty,s)\big|_{\mathcal{H}_P(\pi_\infty)^\nu}$ are contained in $\cup_{j=1}^r\{w_j-k\colon k=1,\dots,m\}$. Moreover, by [MS04, Proposition A.2] there exists $c>0$ which depends only on G, such that

$$r\leq c,\quad m\leq c(1+\|\nu\|). \tag{10.9}$$

Let $A\colon\mathbb{C}\to\mathcal{H}_{\pi_\infty}(\nu)$ be defined by

$$A(z):=R_{\overline{P}|P}(\pi_\infty,z)\big|_{\mathcal{H}_P(\pi_\infty)^\nu}$$

and let $b(z)=\prod_{j=1}^r\prod_{k=1}^m(z-w_j+k)$. Then it follows from (R_6) of [Ar89, Theorem 2.1] that $b(z)A(z)$ is a polynomial function. Moreover, by unitarity of $R_{\overline{P}|P}(\pi_\infty,it)$, $t\in\mathbb{R}$, we have $\|A(it)\|\leq 1$. Thus $A(z)$ satisfies the assumptions of [FLM15, Lemma 5.19]. Thus by [FLM15, Lemma 5.19] and (10.9) we get

(10.10)

$$\begin{aligned}\int_{\mathbb{R}}\left\|R_{\overline{P}|P}(\pi_\infty,it)^{-1}R'_{\overline{P}|P}(\pi_\infty,it)\big|_{\mathcal{H}_P(\pi_\infty)^\nu}\right\|(1+|t|^2)^{-1}dt&=\int_{\mathbb{R}}\|A'(it)\|(1+|t|^2)^{-1}dt\\&\ll\sum_{j=1}^r\sum_{k=1}^m\frac{|\operatorname{Re}(w_j)-k|+1}{(|\operatorname{Re}(w_j)-k|+1)^2+(\operatorname{Im}(w_j))^2}\\&\ll 1+\|\nu\|.\end{aligned}$$

Combining (10.4), (10.8) and (10.10), the proposition follows. ☐

11. The analytic torsion

As before we consider a reductive quasi-split algebraic group G over $\mathbb{Q}$, an open compact subgroup K_f of $G(\mathbb{A}_f)$ and the adelic quotient

$$X := X(K_f),$$

defined by (1.3). For simplicity we assume that G is semisimple and simply connected. Let $\mathbf{K}_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup and let $\widetilde{X} = G(\mathbb{R})/\mathbf{K}_\infty$. Then by strong approximation we have

$$X = \Gamma\backslash\widetilde{X}, \tag{11.1}$$

where $\Gamma = (G(\mathbb{R}) \times K_f) \cap G(\mathbb{Q})$. In general, there are finitely many arithmetic subgroups $\Gamma_j \subset G(\mathbb{R})^1$, $j = 1, ..., l$, such that $X(K_f)$ is the disjoint union of $\Gamma_j\backslash\widetilde{X}$, $j = 1, ..., l$.

11.1. **The Hodge-Laplace operator and heat kernels.**

Let τ be an irreducible finite-dimensional complex representation of $G(\mathbb{R})$ on V_τ. Let E_τ be the flat vector bundle over X associated to the restriction of τ to Γ. Let $\widetilde{E}^\tau$ be the homogeneous vector bundle associated to $\tau|_{\mathbf{K}_\infty}$ and let $E^\tau := \Gamma\backslash\widetilde{E}^\tau$. There is a canonical isomorphism

$$E^\tau \cong E_\tau \tag{11.2}$$

[MM, Proposition 3.1]. By [MM, Lemma 3.1], there exists an inner product $\langle\cdot,\cdot\rangle$ on V_τ, which is unique up to scaling, which satisfies

(1) $\langle\tau(Y)u, v\rangle = -\langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{k}$, $u, v \in V_\tau$
(2) $\langle\tau(Y)u, v\rangle = \langle u, \tau(Y)v\rangle$ for all $Y \in \mathfrak{p}$, $u, v \in V_\tau$.

Such an inner product is called admissible. We fix an admissible inner product. Since $\tau|_{\mathbf{K}_\infty}$ is unitary with respect to this inner product, it induces a metric on E^τ, and by (11.2) on E_τ, which we also call admissible. Let $\Lambda^p(E_\tau) = \Lambda^p T^*(X) \otimes E_\tau$. Let

$$\nu_p(\tau) := \Lambda^p \operatorname{Ad}^* \otimes \tau : \ \mathbf{K}_\infty \to \operatorname{GL}(\Lambda^p\mathfrak{p}^* \otimes V_\tau). \tag{11.3}$$

Then by (11.2) there is a canonical isomorphism

$$\Lambda^p(E_\tau) \cong \Gamma\backslash(G(\mathbb{R}) \times_{\nu_p(\tau)} (\Lambda^p\mathfrak{p}^* \otimes V_\tau)). \tag{11.4}$$

of locally homogeneous vector bundles. Let $\Lambda^p(X, E_\tau)$ be the space the smooth E_τ-valued p-forms on X. Let

$$\begin{aligned} C^\infty(G(\mathbb{R}), \nu_p(\tau)) := \{f : G(\mathbb{R}) \to \Lambda^p\mathfrak{p}^* \otimes V_\tau \colon f \in C^\infty,\ f(gk) = \nu_p(\tau)(k^{-1})f(g),& \\ \forall g \in G(\mathbb{R}),\ \forall k \in \mathbf{K}_\infty\},& \end{aligned} \tag{11.5}$$

and

$$C^\infty(\Gamma\backslash G(\mathbb{R}), \nu_p(\tau)) := \{f \in C^\infty(G(\mathbb{R}), \nu_p(\tau)) \colon f(\gamma g) = f(g),\ \forall g \in G(\mathbb{R}),\ \forall \gamma \in \Gamma\}. \tag{11.6}$$

The isomorphism (11.4) induces an isomorphism

$$\Lambda^p(X,E_\tau)\cong C^\infty(\Gamma\backslash G(\mathbb{R}),\nu_p(\tau)). \tag{11.7}$$

A corresponding isomorphism also holds for the spaces of L^2-sections. Let $\Delta_p(\tau)$ be the Hodge-Laplacian on $\Lambda^p(X,E_\tau)$ with respect to the admissible metric in E_τ. Let R_Γ denote the right regular representation of $G(\mathbb{R})$ in $L^2(\Gamma\backslash G(\mathbb{R}))$. Let $\Omega\in\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the Casimir element. By [MM, (6.9)] it follows that with respect to the isomorphism (11.7) one has

$$\Delta_p(\tau)=-R_\Gamma(\Omega)+\tau(\Omega)\operatorname{Id}. \tag{11.8}$$

Let $\widetilde{E}_\tau\to\widetilde{X}$ be the lift of E_τ to $\widetilde{X}$. There is a canonical isomorphism

$$\Lambda^p(\widetilde{X},\widetilde{E}_\tau)\cong C^\infty(G(\mathbb{R}),\nu_p(\tau)). \tag{11.9}$$

Let $\widetilde{\Delta}_p(\tau)$ be the lift of $\Delta_p(\tau)$ to $\widetilde{X}$. Then again it follows from [MM, (6.9)] that with respect to the isomorphism (11.9) we have

$$\widetilde{\Delta}_p(\tau)=-R(\Omega)+\tau(\Omega)\operatorname{Id}. \tag{11.10}$$

Let $e^{-t\widetilde{\Delta}_p(\tau)}$ be the corresponding heat semigroup. Regarded as an operator in the Hilbert space $L^2(G(\mathbb{R}),\nu_p(\tau))$, it is a convolution operator with kernel

$$H_t^{\tau,p}\colon G(\mathbb{R})\to\operatorname{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau) \tag{11.11}$$

which belongs to $C^\infty\cap L^2$ and satisfies the covariance property

$$H_t^{\tau,p}(k^{-1}gk')=\nu_p(\tau)(k)^{-1}H_t^{\tau,p}(g)\nu_p(\tau)(k') \tag{11.12}$$

with respect to the representation (11.3). Moreover, for all $q>0$ we have

$$H_t^{\tau,p}\in(\mathcal{C}^q(G(\mathbb{R}))\otimes\operatorname{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau))^{\mathbf{K}_\infty\times\mathbf{K}_\infty}, \tag{11.13}$$

where $\mathcal{C}^q(G(\mathbb{R}))$ denotes Harish-Chandra's L^q-Schwartz space (see [MP13, Sect. 4]). Let $h_t^{\tau,p}\in C^\infty(G(\mathbb{R}))$ be defined by

$$h_t^{\tau,p}(g)=\operatorname{tr}H_t^{\tau,p}(g),\quad g\in G(\mathbb{R}). \tag{11.14}$$

Let χ_{K_f} be defined by (3.1). As in (1.4) we define $\phi_t^{\tau,p}\in C^\infty(G(\mathbb{A}))$ by

$$\phi_t^{\tau,p}(g_\infty g_f):=h_t^{\tau,p}(g_\infty)\chi_{K_f}(g_f) \tag{11.15}$$

for $g_\infty\in G(\mathbb{R})$ and $g_f\in G(\mathbb{A}_f)$. Following the definition 1.5, we define the regularized trace of $e^{-t\Delta_p(\tau)}$ by

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right):=J_{\mathrm{geom}}(\phi_t^{\tau,p}). \tag{11.16}$$

11.2. **The asymptotic behavior of the regularized trace.** Our next goal is to determine the asymptotic behavior of $\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ as $t \to 0$ and $t \to \infty$. For $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$ this has been carried out in [MM17]. Concerning the asymptotic behavior as $t \to 0$, we have

Lemma 11.1. *There exist $a_j, b_{ij} \in \mathbb{C}$, $j \in \mathbb{N}_0$, $i = 0, ..., r_j$, such that as $t \to 0$, there is an asymptotic expansion*

$$\operatorname{Tr}_{reg}\left(e^{-t\Delta_p(\tau)}\right) \sim t^{-d/2}\sum_{j=0}^{\infty} a_j t^j + t^{-(d-1)/2}\sum_{j=0}^{\infty}\sum_{i=0}^{r_j} b_{ij} t^{j/2}(\log t)^i.$$

Proof. Let $\Omega_{\mathbf{K}_\infty} \in \mathcal{Z}(\mathfrak{k}_{\mathbb{C}})$ be the Casimir element of $\mathbf{K}_\infty$. For $p = 0, ..., n$ put

$$E_p(\tau) := \nu_p(\tau)(\Omega_{\mathbf{K}_\infty}),$$

which we regard as an endomorphism of $\Lambda^p\mathfrak{p}^* \otimes V_\tau$. It defines an endomorphism of $\Lambda^p T^*(X) \otimes E_\tau$. By [Mia, Proposition 1.1] and (11.10) we have

$$\widetilde{\Delta}_p(\tau) = \widetilde{\Delta}_{\nu_p(\tau)} + \tau(\Omega)\operatorname{Id} - E_p(\tau).$$

Let $\nu_p(\tau) = \oplus_{\nu\in\Pi(\mathbf{K}_\infty)} m(\nu)\nu$ be the decomposition of $\nu_p(\tau)$ into irreducible representations. This induces a corresponding decomposition of the homogeneous vector bundle

$$\widetilde{E}_{\nu_p(\tau)} = \bigoplus_{\nu\in\Pi(\mathbf{K}_\infty)} m(\nu)\widetilde{E}_\nu. \tag{11.17}$$

With respect to this decomposition we have

$$E_p(\tau) = \bigoplus_{\nu\in\Pi(\mathbf{K}_\infty)} m(\nu)\sigma(\Omega_{\mathbf{K}_\infty})\operatorname{Id}_{V_\nu}, \tag{11.18}$$

where $\nu(\Omega_{\mathbf{K}_\infty})$ is the Casimir eigenvalue of ν and V_ν the corresponding representation space. Let $\widetilde{\Delta}_\nu$ be the Bochner-Laplace operator associated to ν. By (11.17) we get a corresponding decomposition of $C^\infty(\widetilde{X}, \widetilde{E}_{\nu_p(\tau)})$ and with respect to this decomposition we have

$$\widetilde{\Delta}_{\nu_p(\tau)} = \bigoplus_{\nu\in\Pi(\mathbf{K}_\infty)} m(\nu)\widetilde{\Delta}_\nu. \tag{11.19}$$

This shows that $\widetilde{\Delta}_{\nu_p(\tau)}$ commutes with $E_p(\tau)$, and therefore we have

$$H_t^{\tau,p} = e^{-t(\tau(\Omega)-E_p(\tau))} \circ H_t^{\nu_p(\tau)}. \tag{11.20}$$

Let H_t^ν be the kernel of $e^{-t\widetilde{\Delta}_\nu}$ and $h_t^\nu := \operatorname{tr}\circ H_t^\nu$. Then it follows from (11.18) and (11.19) that

$$h_t^{\tau,p} = \sum_{\nu\in\Pi(\mathbf{K}_\infty)} m(\nu)e^{-t(\tau(\Omega)-\nu(\Omega_{\mathbf{K}_\infty}))}h_t^\nu, \tag{11.21}$$

Using the definition of $\phi_t^{\tau,p}$ and ϕ_t^{ν}, respectively, we get

$$\begin{aligned}\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)=J_{\mathrm{geom}}(\phi_t^{\tau,p})&=\sum_{\nu\in\Pi(\mathbf{K}_\infty)}m(\nu)e^{-t(\tau(\Omega)-\nu(\Omega_{\mathbf{K}_\infty}))}J_{\mathrm{geom}}(\phi_t^{\nu})\\&=\sum_{\nu\in\Pi(\mathbf{K}_\infty)}m(\nu)e^{-t(\tau(\Omega)-\nu(\Omega_{\mathbf{K}_\infty}))}\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_\nu}\right).\end{aligned}$$

Applying Theorem 1.1 concludes the proof. □

To study the asymptotic behavior as $t\to\infty$ we use the Arthur trace formula (9.1). By [FLM11, Corollary 1], J_{spec} is a distribution on $\mathcal{C}(G(\mathbb{A});K_f)$ (see section 2 for its definition) and by [FL16, Theorem 7.1], J_{geom} is continuous on $\mathcal{C}(G(\mathbb{A});K_f)$. This implies that (9.1) holds for $\phi_t^{\tau,p}$ and we have

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)=J_{\mathrm{spec}}(\phi_t^{\tau,p}). \tag{11.22}$$

Now we apply Theorem 9.1 to study the asymptotic behavior as $t\to\infty$ of the right hand side. First we have

$$J_{\mathrm{spec}}(\phi_t^{\tau,p})=\sum_{[M]}J_{\mathrm{spec},M}(\phi_t^{\tau,p}), \tag{11.23}$$

where the sum ranges over the conjugacy classes of Levi subgroups of G and $J_{\mathrm{spec},M}(\phi_t^{\tau,p})$ is given by (9.7).

To analyze these terms, we proceed as in [MM17, Section 13]. Let $M\in\mathcal{L}$ and $P\in\mathcal{P}(M)$. We use the notation introduced in section 9. Recall that the discrete subspace $L^2_{\mathrm{dis}}(A_M(\mathbb{R})^0M(\mathbb{Q})\backslash M(\mathbb{A}))$ splits as the completed direct sum of its π-isotypic components for $\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))$. We have a corresponding decomposition of $\bar{\mathcal{A}}^2(P)$ as a direct sum of Hilbert spaces $\hat{\oplus}_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))}\bar{\mathcal{A}}^2_\pi(P)$. Similarly, we have the algebraic direct sum decomposition

$$\mathcal{A}^2(P)=\bigoplus_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))}\mathcal{A}^2_\pi(P),$$

where $\mathcal{A}^2_\pi(P)$ is the $\mathbf{K}$-finite part of $\bar{\mathcal{A}}^2_\pi(P)$. Let $\mathcal{A}^2_\pi(P)^{K_f}$ be the subspace of K_f-invariant functions in $\mathcal{A}^2_\pi(P)$, and for any $\sigma\in\Pi(\mathbf{K}_\infty)$ let $\mathcal{A}^2_\pi(P)^{K_f,\sigma}$ be the σ-isotypic subspace of $\mathcal{A}^2_\pi(P)^{K_f}$. Recall that $\mathcal{A}^2_\pi(P)^{K_f,\sigma}$ is finite dimensional [Mu07, Prop. 3.5].

For $P,Q\in\mathcal{P}(M)$ let $M_{Q|P}(\lambda)$ be the intertwining operator (9.2). Denote by $M_{Q|P}(\pi,\lambda)$ the restriction of $M_{Q|P}(\lambda)$ to $\mathcal{A}^2_\pi(P)$. Recall that the operator $\Delta_{\mathcal{X}}(P,\lambda)$, which appears in the formula (9.7), is defined by (9.5). Its definition involves the intertwining operators $M_{Q|P}(\lambda)$. If we replace $M_{Q|P}(\lambda)$ by its restriction $M_{Q|P}(\pi,\lambda)$ to $\mathcal{A}^2_\pi(P)$, we obtain the restriction $\Delta_{\mathcal{X}}(P,\pi,\lambda)$ of $\Delta_{\mathcal{X}}(P,\lambda)$ to $\mathcal{A}^2_\pi(P)$. Similarly, let $\rho_\pi(P,\lambda)$ be the induced representation in $\bar{\mathcal{A}}^2_\pi(P)$. Fix $\beta\in\mathfrak{B}_{P,L_s}$ and $s\in W(M)$. Then for the integral on the right of (9.7) with $h=\phi_t^{\tau,p}$ we get

$$\sum_{\pi\in\Pi_{\mathrm{dis}}(M(\mathbb{A}))}\int_{i(\mathfrak{a}^G_{L_s})^*}\operatorname{Tr}\left(\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\lambda)M(P,\pi,s)\rho_\pi(P,\lambda,\phi_t^{\tau,p})\right)\,d\lambda. \tag{11.24}$$

In order to deal with the integrand, we need the following result. Let π be a unitary admissible representation of $G(\mathbb{R})$. Let $A\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ be a bounded operator which is an intertwining operator for $\pi|_{\mathbf{K}_\infty}$. Then $A \circ \pi(h_t^\nu)$ is a finite rank operator. Define an operator $\tilde{A}$ on $\mathcal{H}_\pi \otimes V_\nu$ by $\tilde{A} := A \otimes \mathrm{Id}$. Then by [MM17, (9.13)] we have

$$\operatorname{Tr}(A \circ \pi(h_t^\nu)) = e^{t(\pi(\Omega)-\nu(\Omega_{\mathbf{K}_\infty}))} \operatorname{Tr}\left(\tilde{A}|_{(\mathcal{H}_\pi \otimes V_\nu)^{\mathbf{K}_\infty}}\right). \tag{11.25}$$

We will apply this to the induced representation $\rho_\pi(P,\lambda)$. Let $P, Q \in \mathcal{P}(M)$ and $\nu \in \Pi(\mathbf{K}_\infty)$. Assume that $(\mathcal{A}^2_\pi(P)^{K_f} \otimes V_\nu)^{\mathbf{K}_\infty} \neq 0$. Denote by $\widetilde{M_{Q|P}}(\pi,\nu,\lambda)$ the restriction of

$$M_{Q|P}(\pi,\lambda) \otimes \mathrm{Id}\colon \mathcal{A}^2_\pi(P) \otimes V_\nu \to \mathcal{A}^2_\pi(P) \otimes V_\nu$$

to $(\mathcal{A}^2_\pi(P)^{K_f} \otimes V_\nu)^{\mathbf{K}_\infty}$. Denote by $\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)$ and $\widetilde{M}(P,\pi,\nu,s)$ the corresponding restrictions. Let $m(\pi)$ denote the multiplicity with which π occurs in the regular representation of $M(\mathbb{A})$ in $L^2_{\mathrm{dis}}(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A}))$. Then

$$\rho_\pi(P,\lambda) \cong \oplus_{i=1}^{m(\pi)} \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi,\lambda). \tag{11.26}$$

Fix positive restricted roots of $\mathfrak{a}_P$ and let $\rho_{\mathfrak{a}_P}$ denote the corresponding half-sum of these roots. For $\xi \in \Pi(M(\mathbb{R}))$ and $\lambda \in \mathfrak{a}_P^*$ let

$$\pi_{\xi,\lambda} := \operatorname{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\xi \otimes e^{i\lambda})$$

be the unitary induced representation. Let $\xi(\Omega_M)$ be the Casimir eigenvalue of ξ. Define a constant $c(\xi)$ by

$$c(\xi) := -\langle \rho_{\mathfrak{a}_P}, \rho_{\mathfrak{a}_P} \rangle + \xi(\Omega_M). \tag{11.27}$$

Then for $\lambda \in \mathfrak{a}_P^*$ one has

$$\pi_{\xi,\lambda}(\Omega) = -\|\lambda\|^2 + c(\xi) \tag{11.28}$$

(see [Kn, Theorem 8.22]). Let

$$\mathcal{T} := \{\nu \in \Pi(\mathbf{K}_\infty)\colon [\nu_p(\tau)\colon \nu] \neq 0\}. \tag{11.29}$$

Using (11.21), (11.26) and (11.25), it follows that (11.24) is equal to

$$\begin{aligned} &\sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))} \sum_{\nu \in \mathcal{T}} m(\pi) e^{-t(\tau(\Omega)-c(\pi_\infty))} \\ &\qquad \cdot \int_{i(\mathfrak{a}^G_{L_s})^*} e^{-t\|\lambda\|^2} \operatorname{Tr}\left(\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)\widetilde{M}(P,\pi,\nu,s)\right) d\lambda. \end{aligned} \tag{11.30}$$

Since $M(P,\pi,s)$ is unitary, (11.30) can be estimated by

$$\begin{aligned} &\sum_{\pi \in \Pi_{\mathrm{dis}}(M(\mathbb{A}))} \sum_{\nu \in \mathcal{T}} m(\pi) \dim\left(\mathcal{A}^2_\pi(P)^{K_f,\nu}\right) \\ &\qquad \cdot e^{-t(\tau(\Omega)-c(\pi_\infty))} \int_{i(\mathfrak{a}^G_{L_s})^*} e^{-t\|\lambda\|^2} \|\widetilde{\Delta}_{\mathcal{X}_{L_s}(\underline{\beta})}(P,\pi,\nu,\lambda)\| \, d\lambda. \end{aligned} \tag{11.31}$$

For $\pi \in \Pi(M(\mathbb{A}))$ denote by λ_{π_∞} the Casimir eigenvalue of the restriction of π_∞ to $M(\mathbb{R})^1$. Given $\lambda > 0$, let

$$\Pi_{\text{dis}}(M(\mathbb{A});\lambda) := \{\pi \in \Pi(M(\mathbb{A}))\colon |\lambda_{\pi_\infty}| \le \lambda\}.$$

Let $d = \dim M(\mathbb{R})^1/\mathbf{K}_\infty^M$. If we use [Mu89, Theorem 0.1] and argue in the same way as in the proof of [Mu07, Proposition 3.5] it follows that for every $\nu \in \Pi(\mathbf{K}_\infty)$ there exists $C > 0$ such that

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A});\lambda)} m(\pi) \dim \mathcal{A}^2_\pi(P)^{K_f,\nu} \le C(1+\lambda^{2d}) \tag{11.32}$$

for all $\lambda \ge 0$. Next we estimate the integral in (11.31). We use the notation of section 9. Let $\underline{\beta} = (\beta_1^\vee, \dots, \beta_m^\vee)$ and $\mathcal{X}_{L_s}(\underline{\beta}) = (Q_1, \dots, Q_m) \in \Xi_{L_s}(\underline{\beta})$ with with $Q_i = \langle P_i, P_i' \rangle$, $P_i|^{\beta_i} P_i'$, $i = 1, \dots, m$. Using the definition (9.5) of $\Delta_{\mathcal{X}_{L_s}(\underline{\beta})}(P, \pi, \nu, \lambda)$, it follows that we can bound the integral by a constant multiple of

$$\dim(\nu) \int_{i(\mathfrak{a}^G_{L_s})^*} e^{-t\|\lambda\|^2} \prod_{i=1}^m \left\| \delta_{P_i|P_i'}(\lambda)\Big|_{\mathcal{A}^2_\pi(P_i')^{K_f,\nu}} \right\| \, d\lambda, \tag{11.33}$$

where $\delta_{P_i|P_i'}(\lambda)$ is defined by (9.4). We introduce new coordinates $s_i := \langle \lambda, \beta_i^\vee \rangle$, $i = 1, \dots, m$, on $(\mathfrak{a}^G_{L_s,\mathbb{C}})^*$. Using (9.3) and (9.4), we can write

$$\delta_{P_i|P_i'}(\lambda) = \frac{n'_{\beta_i}(\pi, s_i)}{n_{\beta_i}(\pi, s_i)} + j_{P_i'} \circ (\operatorname{Id} \otimes R_{P_i|P_i'}(\pi, s_i)^{-1} R'_{P_i|P_i'}(\pi, s_i)) \circ j_{P_i'}^{-1}. \tag{11.34}$$

Put

$$\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}} = \bigoplus_{\nu \in \mathcal{T}} \mathcal{A}^2_\pi(P)^{K_f,\nu},$$

where $\mathcal{T}$ is defined by (11.29). It follows from [Mu02, Theorem 5.3] that there exist $N, k \in \mathbb{N}$ and $C > 0$ such that

$$\int_{i\mathbb{R}} \left| \frac{n'_{\beta_i}(\pi, s)}{n_{\beta_i}(\pi, s)} \right| (1+|s|^2)^{-k} \, ds \le C(1+\lambda^2_{\pi_\infty})^N, \; i = 1, \dots, m, \tag{11.35}$$

for all $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$ with $\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}} \neq 0$. Combining (11.34), (11.35) and Proposition 10.2, it follows that we have

$$\int_{i(\mathfrak{a}^G_{L_s})^*} e^{-t\|\lambda\|^2} \prod_{i=1}^m \left\| \delta_{P_i|P_i'}(\lambda)\Big|_{\mathcal{A}^2_\pi(P_i')^{K_f,\nu}} \right\| \, d\lambda \ll (1+\lambda^2_{\pi_\infty})^{mN}$$

for all $t \ge 1$, and $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$ with $\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}} \neq 0$. Thus (11.31) can estimated by a constant multiple of

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))} \sum_{\nu \in \mathcal{T}} m(\pi) \dim \left(\mathcal{A}^2_\pi(P)^{K_f,\nu} \right) (1+\lambda^2_{\pi_\infty})^N e^{-t(\tau(\Omega) - c(\pi_\infty))}. \tag{11.36}$$

Now we can proceed as in [MM17]. For the convenience of the reader we recall the arguments. By (11.27) we have

$$\tau(\Omega) - c(\pi_\infty) = \tau(\Omega) + \|\rho_{\mathfrak{a}}\|^2 - \lambda_{\pi_\infty}. \tag{11.37}$$

Together with [MM17, Lemma 13.2], it follows that there exists $\lambda_0 > 0$ such that

$$\tau(\Omega) - c(\pi_\infty) \geq |\lambda_{\pi_\infty}|/2$$

for all $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$ with $\mathcal{A}^2_\pi(P)^{K_f,\mathcal{T}} \neq 0$ and $|\lambda_{\pi_\infty}| \geq \lambda_0$. We decompose the sum over π in (11.36) in two summands $\Sigma_1(t)$ and $\Sigma_2(t)$, where in $\Sigma_1(t)$ the summation runs over all π with $|\lambda_{\pi_\infty}| \leq \lambda_0$. Using (11.32), it follows that for $t \geq 1$

$$\Sigma_2(t) \ll e^{-t|\lambda_0|/2}.$$

Since $\Sigma_1(t)$ is a finite sum, both in π and ν, it follows from [MM17, Lemma 13.1] that there exists $c > 0$ such that

$$\Sigma_1(t) \ll e^{-ct}$$

for $t \geq 1$. Recall that [MM17, Lemma 13.1] requires that $P = MAN$ is a proper parabolic subgroup of G. Putting everything together we obtain the following result.

Lemma 11.2. *Let $\tau \in \operatorname{Rep}(G(\mathbb{R}))$. Assume that $\tau \not\cong \tau_\theta$. Let M be a proper Levi subgroup of G. There exists $c > 0$ such that*

$$J_{spec,M}(\phi_t^{\tau,p}) = O(e^{-ct})$$

for $t \geq 1$.

It remains to deal with the case $M = G$. This has been done already in [MM17] for an arbitrary reductive group G. Using [MM17, (13.34)] and the considerations following this equality, we get

$$J_{\text{spec},G}(\phi_t^{\tau,p}) = O(e^{-ct}) \tag{11.38}$$

for some $c > 0$. Combined with Lemma 11.2 we obtain

Proposition 11.3. *There exists $c > 0$ such that*

$$J_{spec}(\phi_t^{\tau,p}) = O\left(e^{-ct}\right)$$

for all $t \geq 1$ and $p = 0, \dots, n$.

Applying the trace formula (9.1), we get

$$\operatorname{Tr}_{\text{reg}}\left(e^{-t\Delta_p(\tau)}\right) = O(e^{-ct}), \quad \text{as } t \to \infty,$$

which is the proof of Theorem 1.2. The asymptotic expansion as $t \to +0$ is provided by Lemma 11.1.

Thus the corresponding zeta function $\zeta_p(s;\tau)$, defined by the Mellin transform

$$\zeta_p(s;\tau) := \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}_{\text{reg}}\left(e^{-t\Delta_p(\tau)}\right) t^{s-1}\, dt. \tag{11.39}$$

is holomorphic in the half-plane $\operatorname{Re}(s) > d/2$ and admits a meromorphic extension to the whole complex plane. It may have a pole at $s = 0$. Let $f(s)$ be a meromorphic function on $\mathbb{C}$. For $s_0 \in \mathbb{C}$ let

$$f(s) = \sum_{k \geq k_0} a_k (s - s_0)^k$$

be the Laurent expansion of f at s_0. Put $\mathrm{FP}_{s=s_0} := a_0$. Now we define the analytic torsion $T_X(\tau) \in \mathbb{R}^+$ by

$$\log T_X(\tau) = \frac{1}{2}\sum_{p=0}^{d}(-1)^p p\left(\mathrm{FP}_{s=0}\frac{\zeta_p(s;\tau)}{s}\right). \tag{11.40}$$

Put

$$K(t,\tau) := \sum_{p=1}^{d}(-1)^p p\,\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right). \tag{11.41}$$

Then $K(t,\tau) = O(e^{-ct})$ as $t \to \infty$ and the Mellin transform

$$\int_0^\infty K(t,\tau)t^{s-1}dt$$

converges absolutely and uniformly on compact subsets of $\mathrm{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$. Moreover, by (11.40) we have

$$\log T_X(\tau) = \mathrm{FP}_{s=0}\left(\frac{1}{\Gamma(s)}\int_0^\infty K(t,\tau)t^{s-1}dt\right). \tag{11.42}$$

Let $\delta(\widetilde{X}) := \mathrm{rank}_{\mathbb{C}}(G(\mathbb{R})^1) - \mathrm{rank}_{\mathbb{C}}(\mathbf{K}_\infty)$. Let $\Gamma \subset G(\mathbb{R})^1$ be a torsion free, co-compact lattice. Let $X = \Gamma\backslash\widetilde{X}$. If $\dim X$ is even or $\delta(\widetilde{X}) \geq 2$, then $T_X(\tau) = 1$ for every $\tau \in \mathrm{Rep}(G(\mathbb{R})^1)$ [MP13, Prop. 4.2]. We note that the proof of [MP13, Prop. 4.2] for the case $\delta(\widetilde{X}) \geq 2$ contains a mistake, which is the claim that the Grothendieck group of admissible representations is generated by induced representations. This is only true for $\mathrm{GL}(n,\mathbb{R})$. However, the proof can be fixed by proceeding as in the proof of [MoS, Corollary 2.2]. As the example in [MP12] shows, in the non-compact case the analytic torsion need not be trivial if $\dim X$ is even. Hence one may guess that this is also true if $\delta(\widetilde{X}) \geq 2$. We consider here the contribution of the discrete spectrum to the analytic torsion and study when it vanishes. Let

$$\phi_t^\tau := \sum_{p=1}^{d}(-1)^p p\phi_t^{\tau,p} \quad \text{and} \quad k_t^\tau := \sum_{p=1}^{d}(-1)^p p h_t^{\tau,p}.$$

Then by (11.16) we have

$$K(t,\tau) = J_{\mathrm{spec}}(\phi_t^\tau). \tag{11.43}$$

For $\pi \in \Pi(G(\mathbb{R}))$ let Θ_π be the global character. Then the contribution of the discrete spectrum is given by

$$J_{\mathrm{spec},G}(\phi_t^\tau) = \sum_{\pi\in\Pi_{\mathrm{dis}}(G(\mathbb{A}))} m(\pi)\dim\left(\mathcal{H}_{\pi_f}^{K_f}\right)\Theta_{\pi_\infty}(k_t^\tau). \tag{11.44}$$

If $\dim X$ is even, we can follow the proof of [MP13, Prop. 4.2] to show that $J_{\mathrm{spec},G}(\phi_t^\tau) = 0$. We conjecture that $J_{\mathrm{spec},G}(\phi_t^\tau) = 0$, if $\delta(\widetilde{X}) \geq 2$. In [MM17, Lemma 13.5] it was shown that this holds for $G = \mathrm{GL}(n)$, i.e., $J_{\mathrm{spec},\mathrm{GL}(n)}(\phi_t^\tau) = 0$, if $n \geq 5$.

12. Approximation of L^2-analytic torsion

In this section we prove the Theorem 1.6. To begin with we introduce some notation. Let G be a quasi-split reductive algebraic group over $\mathbb{Q}$. Fix a faithful $\mathbb{Q}$-rational representation $\rho\colon G \to \mathrm{GL}(V)$ and a lattice Λ in the representation space V such that the stabilizer of $\widehat{\Lambda} = \widehat{\mathbb{Z}} \otimes \Lambda \subset \mathbb{A}_f \otimes V$ in $G(\mathbb{A}_f)$ is the group K_f. Since the maximal compact subgroups of $\mathrm{GL}(\mathbb{A}_f \otimes V)$ are precisely the stabilizers of lattices , it is easy to see that such a lattice exists. For $N \in \mathbb{N}$ let

$$K(N) = \{g \in G(\mathbb{A}_f)\colon \rho(g)v \equiv v \mod N\widehat{\Lambda},\ v \in V\} \tag{12.1}$$

be the principal congruence subgroup of level N, which is a factorizable normal open subgroup of $\mathbf{K}_f$. Let

$$X(N) := G(\mathbb{Q})\backslash(\widetilde{X} \times G(\mathbb{A}_f)/K(N)) \tag{12.2}$$

be the adelic quotient. Let $\tau \in \mathrm{Rep}(G(\mathbb{R})^1)$ and assume that τ satisfies $\tau \not\cong \tau \circ \theta$. Let $E_{\tau,N} \to X(N)$ be the flat vector bundle over $X(N)$ associated to τ. Let $T_{X(N)}(\tau)$ be the regularized analytic torsion.

In [FLM15, Definition 5.9] two conditions, called (TWN) and (BD), for an arbitrary reductive group have been formulated. These conditions imply appropriate estimations for the logarithmic derivatives of the intertwining operators, which occur on the spectral side of the trace formula. Property (TWN) is a global condition concerning the scalar-valued normalizing factors of the intertwining operators, while (BD) is a condition for the local intertwining operators. For (BD) we need additional assumptions introduced in [FL19, Definition 2]. Let S be a set of primes. We say that G satisfies property (BD) for S, if the local groups $G(\mathbb{Q}_p)$, $p \in S$, satisfy (BD) with a uniform value of C (see [FL19] for details).

Proposition 12.1. *Let G be admissible. Then G satisfies property (TWN), and property (BD) with respect to $S_{\mathrm{fin}} \setminus \{2\}$.*

Proof. In [FL17, Theorem 3.11] it this proved that an admissible G satisfies property (TWN+) which is even stronger than (TWN). It follows from [FL19, Corollary 1] that G satisfies property (BD) with respect to $S_{\mathrm{fin}} \setminus \{2\}$. See [FL19, Remark 3] for additional explanations. $\square$

To establish Theorem 1.6, we follow the approach of [MM2]. Let $\Delta_{p,N}(\tau)$ the Laplace operator acting in the space of $E_{\tau,N}$-valued p-forms on $X(N)$. We write

$$\begin{aligned}\int_0^\infty \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,N}(\tau)}\right) t^{s-1}dt &= \int_0^T \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,N}(\tau)}\right) t^{s-1}dt \\ &\quad + \int_T^\infty \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,N}(\tau)}\right) t^{s-1}dt.\end{aligned} \tag{12.3}$$

To begin with we estimate the second integral on the right hand side. The first auxiliary result is the following proposition.

Proposition 12.2. *Let G be admissible. Then there exist $C, c > 0$ such that*

$$\frac{1}{\operatorname{vol}(X(N))}\left|\operatorname{Tr}_{reg}\left(e^{-t\Delta_{p,N}(\tau)}\right)\right| \leq Ce^{-ct} \tag{12.4}$$

for all $t \geq 1$, $p = 0, ..., d$, and $N \in \mathbb{N}$ with $2 \nmid N$.

Proof. The proof follows from [MM2, Prop. 6.6] together with Proposition 12.1. □

Using Proposition 12.2 it follows that there exist $C, c > 0$ such that

$$\frac{1}{\operatorname{vol}(X(N))}\left|\int_T^\infty \operatorname{Tr}_{\text{reg}}\left(e^{-t\Delta_{p,N}(\tau)}\right) t^{-1} dt\right| \leq Ce^{-cT} \tag{12.5}$$

for all $T \geq 1$, $p = 0, \dots, d$, and $N \in \mathbb{N}$ with $2 \nmid N$.

Now we turn to the estimation of the first integral on the right hand side of (12.3). In order to study the short time behavior of the regularized trace of the heat operator with the help of the trace formula, we need to show that we can replace $h_t^{\tau,p}$ by an appropriate compactly supported test function without changing the asymptotic behavior as $t \to 0$. Let $r(g)$ be the function on G_∞, which is defined by (6.17). For $R > 0$ let

$$B_R := \{g \in G_\infty \colon r(g) < R\}.$$

We need the following auxiliary lemma.

Lemma 12.3. *There exist $C, c > 0$ such that*

$$\int_{G_\infty} e^{-r^2(g)/t} dg \leq Ce^{ct}$$

for all $t > 0$.

Proof. For $G = \operatorname{GL}_n$ this was proved in [MM2, Lemma 7.1]. Let $d_n(\cdot, \cdot)$ denote the geodesic distance function on $GL_n(\mathbb{R})^1/\operatorname{O}(n)$, and let $r_n(g) = d_n(g\operatorname{O}(n), \operatorname{O}(n))$. At the beginning of section 6 we have shown that with respect to our choice of the embedding $G_\infty \subset \operatorname{GL}_n(\mathbb{R})^1$, r and r_n coincide on G_∞. Thus the lemma follows from the case of GL_n. □

Let $f \in C^\infty(\mathbb{R})$ such that $f(u) = 1$, if $|u| \leq 1/2$, and $f(u) = 0$, if $|u| \geq 1$. Let $\varphi_R \in C_c^\infty(G_\infty)$ be defined by

$$\varphi_R(g) := f\left(\frac{r(g)}{R}\right). \tag{12.6}$$

Then we have $\operatorname{supp}\varphi_R \subset B_R$. Extend φ_R to $G(\mathbb{R})$ by

$$\varphi_R(g_\infty z) = \varphi_R(g_\infty), \quad g_\infty \in G_\infty,\ z \in A_G(\mathbb{R})^0.$$

Define $\widetilde{h}_{t,R}^{\tau,p} \in C^\infty(G(\mathbb{R}))$ by

$$\widetilde{h}_{t,R}^{\tau,p}(g_\infty) := \varphi_R(g_\infty) h_t^{\tau,p}(g_\infty), \quad g_\infty \in G(\mathbb{R}). \tag{12.7}$$

Then the restriction of $\widetilde{h}_{t,R}^{\tau,p} \otimes \chi_{K(N)}$ to $G(\mathbb{A})^1$ belongs to $C_c^\infty(G(\mathbb{A})^1)$.

Proposition 12.4. *There exist constants* $C_1, C_2, C_3 > 0$ *such that*

$$\frac{1}{\operatorname{vol}(Y(N))}\left|J_{spec}(h_t^{\tau,p}\otimes\chi_{K(N)}) - J_{spec}(\widetilde{h}_{t,R}^{\tau,p}\otimes\chi_{K(N)})\right| \le C_1 e^{-C_2R^2/t+C_3t}$$

for all $N\in\mathbb{N}$, $p=0,\dots,d$, $t>0$ *and* $R\ge 1$.

Proposition 12.4 allows us to replace $h_t^{\tau,p}$ by a compactly supported function.

Proof. Let $\psi_R := 1-\varphi_R$. Then

$$J_{\mathrm{spec}}(h_t^{\tau,p}\otimes\chi_{K(N)}) - J_{\mathrm{spec}}(\widetilde{h}_{t,R}^{\tau,p}\otimes\chi_{K(N)}) = J_{\mathrm{spec}}(\psi_R h_t^{\tau,p}\otimes\chi_{K(N)}).$$

Now we use the refined spectral expansion (9.7). Let $M\in\mathcal{L}$ and let $J_{\mathrm{spec},M}$ be the distribution on the right hand side of (9.7), which corresponds to M. Let

$$\Delta_G = -\Omega + 2\Omega_{K_\infty},$$

where Ω (resp. Ω_{K_∞}) denotes the Casimir operator of $G(\mathbb{R})^1$(resp. K_∞). Observe that $\psi_R h_t^{\tau,p}\otimes\chi_{K(N)}$ belongs to $\mathcal{C}(G(\mathbb{A})^1)$ and the proof of Lemma 7.2 and Corollary 7.4 in [FLM11] extends to $h\in\mathcal{C}(G(\mathbb{A})^1)$. For an open compact subgroup $K_{M,f}\subset M(\mathbb{A}_f)$ let $\mu_{K_f}^M$ be the measure on $\Pi(M(\mathbb{R})^1)$ defined by

$$\begin{aligned}\mu_{K_f}^M =& \frac{\operatorname{vol}(K_{M,f})}{\operatorname{vol}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)}\\ &\cdot\sum_{\pi\in\Pi(M(\mathbb{A})^1)}\dim\operatorname{Hom}_{M(\mathbb{A})^1}(\pi, L^2(M(\mathbb{Q})\backslash M(\mathbb{A})^1))\dim\pi_f^{K_{M,f}}\delta_{\pi_\infty}.\end{aligned}\tag{12.8}$$

In the notation of [FLM15] this is the measure $\mu_{K_f}^{M,S_\infty}$, where $S_\infty=\{\infty\}$. It follows from [FL18, Prop. 4.7] and Proposition 12.1 that the collection of measures $\{\mu^M_{K_M(N)}\}_{N\in\mathbb{N}}$ is polynomial bounded. Then by [FLM15, Corollary 7.4] it follows that there exists $k\ge 1$ such that for any $\varepsilon>0$ we have

$$\begin{aligned}\frac{1}{\operatorname{vol}(X(N))}J_{\mathrm{spec},M}(\psi_R h_t^{\tau,p}\otimes\chi_{K(N)}) &= \frac{1}{\operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1)}J_{\mathrm{spec},M}(\psi_R h_t^{\tau,p}\otimes\mathbf{1}_{K(N)})\\ &\ll_{\mathcal{T},\varepsilon}\|(\operatorname{Id}+\Delta_G)^k(\psi_R h_t^{\tau,p})\|_{L^1(G_\infty)}N^{(\dim M-\dim G)/2+\varepsilon}\end{aligned}\tag{12.9}$$

for all $N\in\mathbb{N}$, $p=0,\dots,d$, $t>0$, and $R>0$, where $\mathcal{T}$ is defined by (11.29). Using Lemma 12.3 we can proceed exactly as in the proof of Proposition 7.2 in [MM2] and estimate $\|(\operatorname{Id}+\Delta_G)^k(\psi_R h_t^{\tau,p})\|_{L^1(G_\infty)}$. Combined with (12.9) it follows that for every $\varepsilon>0$ we have

$$\frac{1}{\operatorname{vol}(X(N))}J_{\mathrm{spec},M}(\psi_R h_t^{\tau,p}\otimes\chi_{K(N)}) \ll_\varepsilon e^{-cR^2/t}N^{(\dim M-\dim G)/2+\varepsilon}$$

for all $N\in\mathbb{N}$, $p=0,\dots,d$, and $t>0$. and $R\ge 1$. Let $M\ne G$. Then there exist $C_1, C_2, C_3>0$ such that

$$\frac{1}{\operatorname{vol}(X(N))}|J_{\mathrm{spec},M}(\psi_R h_t^{\tau,p}\otimes\chi_{K(N)})| \le C_1e^{-C_2R^2/t+C_3t}\tag{12.10}$$

for all $N \in \mathbb{N}$, $p = 0, \dots, d$, and $t > 0$, and $R \geq 1$.

It remains to deal with the case $M = G$. In this case we have

$$\begin{aligned} J_{\mathrm{spec},G}(\psi_R h_t^{\tau,p} \otimes \chi_{K(N)}) &= \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A})^1)} m_\pi \operatorname{Tr} \pi(\psi_R h_t^{\tau,p} \otimes \chi_{K(N)}) \\ &= \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A})^1)} m_\pi \dim(\pi_f^{K(N)}) \operatorname{Tr} \pi_\infty(\psi_R h_t^{\tau,p}). \end{aligned}$$

For $\nu \in \Pi(K_\infty)$ denote by $\mathcal{H}_{\pi_\infty}(\nu)$ the ν-isotypic subspace. Let

$$\Pi_{\mathrm{dis}}(G(\mathbb{A})^1)^{\mathcal{T}} := \{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A})^1) \colon \exists\, \nu \in \mathcal{T} \text{ such that } \mathcal{H}_{\pi_\infty}(\nu) \neq 0\}.$$

If we argue as in the proof of Propostion 7.2 in [MM2, pp. 335], we get

$$\begin{aligned} &\frac{1}{\operatorname{vol}(Y(N))} |J_{\mathrm{spec},G}(\psi_R h_t^{\tau,p} \otimes \chi_{K(N)})| \\ &\qquad \leq C_k e^{-C_2 R^2/t + C_3 t} \operatorname{vol}(K(N)) \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A})^1)^{\mathcal{T}}} m_\pi \dim(\pi_f^{K(N)})(1 + |\lambda_{\pi_\infty}|)^{-k}. \end{aligned}$$

Now recall that, as observed above, the collection of measures $\{\mu_{K(N)}^G\}_{N \in \mathbb{N}}$ is polynomially bounded. This is equivalent to condition (1) of [FLM15, Proposition 6.1]. For $l \in \mathbb{N}$ let $g_{l,\mathcal{T}}$ be the non-negative function on $\Pi(G_\infty)$ defined by

$$g_{l,\mathcal{T}}(\pi) := \begin{cases} (1 + |\lambda_\pi|)^{-l}, & \text{if } \pi \in \Pi(G(\mathbb{R}))^{\mathcal{T}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then it follows from [FLM15, Proposition 6.1, (4)] that there exists $l \in \mathbb{N}$, which depends only on $\mathcal{T}$, such that

$$\mu_{K(N)}^G(g_{l,\mathcal{T}}) = \frac{\operatorname{vol}(K(N))}{\operatorname{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})^1)} \sum_{\pi \in \Pi_{\mathrm{dis}}(G(\mathbb{A})^1)^{\mathcal{T}}} m_\pi \dim(\pi_f^{K(N)})(1 + |\lambda_{\pi_\infty}|)^{-l} \tag{12.11}$$

is bounded independently of $N \in \mathbb{N}$. Hence there exist $C_1, C_2, C_3 > 0$ such that

$$\frac{1}{\operatorname{vol}(Y(N))} |J_{\mathrm{spec},G}(\psi_R h_t^{\tau,p} \otimes \chi_{K(N)})| \leq C_1 e^{-C_2 R^2/t + C_3 t}$$

for all $t > 0$, $p = 0, \dots, d$, $N \in \mathbb{N}$, and $R \geq 1$. This completes the proof of the proposition. □

Now recall that

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) = J_{\mathrm{spec}}(h_t^{\tau,p} \otimes \chi_{K(N)}).$$

For $R > 0$ let $\varphi_R \in C_c^\infty(G(\mathbb{R})^1)$ be the function defined by (12.6). By Proposition 12.4 we have

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,Y(N)}(\tau)}\right) = J_{\mathrm{spec}}(\varphi_R h_t^{\tau,p} \otimes \chi_{K(N)}) + r_R(t), \tag{12.12}$$

where $r_R(t)$ is a function of $t \in [0,T]$ which satisfies

$$\frac{1}{\operatorname{vol}(Y(N))}|r_R(t)| \le C_1 e^{-C_2R^2/t+C_3t} \tag{12.13}$$

for $0 \le t \le T$. This implies that $\int_0^T r_R(t)t^{s-1}dt$ is holomorphic in $s \in \mathbb{C}$ and

$$FP_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T r_R(t)t^{s-1}dt\right) = \int_0^T r_R(t)t^{-1}dt.$$

Moreover

$$\begin{aligned}\frac{1}{\operatorname{vol}(Y(N))}\left|\int_0^T r_R(t)t^{-1}dt\right| &\le C_1\int_0^T e^{-C_2R^2/t+C_3t}t^{-1}dt\\ &\le C_1 e^{-C_4R^2/T+C_3T}\int_0^{T/R^2} e^{-C_4/t}t^{-1}dt.\end{aligned} \tag{12.14}$$

Now put $R = T^2$ and let

$$h^{\tau,p}_{t,T} := \varphi_{T^2}h^{\tau,p}_t. \tag{12.15}$$

Then it follows from (12.12) and (12.14) that there exist $C, c > 0$ such that

$$\begin{aligned}\frac{1}{\operatorname{vol}(X(N))}\Bigg|&\operatorname{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T \operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{p,X(N)}(\tau)}\right)t^{s-1}dt\right)\\ &-\operatorname{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_{\mathrm{spec}}(h^{\tau,p}_{t,T}\otimes\chi_{K(N)})t^{s-1}dt\right)\Bigg| \le Ce^{-cT}\end{aligned} \tag{12.16}$$

for $T \ge 1$, $p = 0,\dots,d$, and $N \in \mathbb{N}$. Using the trace formula, we are reduced to deal with

$$\operatorname{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_{\mathrm{geom}}(h^{\tau,p}_{t,T}\otimes\chi_{K(N)})t^{s-1}dt\right).$$

For the following considerations we need to restrict the sequences of principal congruence subgroups by imposing additional conditions on the levels. Let S_1 be a finite set of primes. We say that a sequence $\{N_j\}_{j\in\mathbb{N}}$ of natural numbers converges to ∞ at S_1, denoted by

$$N_j \xrightarrow[S_1]{} \infty, \quad j \to \infty, \tag{12.17}$$

if there exists $N_0 \in \mathbb{N}$ such that

$$N_j = N_0 \cdot N_{j,S_1}, \quad N_{j,S_1} = \prod_{p\in S_1} p^{r_p(j)}, \quad \text{and} \quad N_{j,S_1} \to \infty. \tag{12.18}$$

The corresponding sequence of principal congruence subgroups $\{K(N_j)\}_{j\in\mathbb{N}}$ converges to 1 at S_1 in the sense of (1.11), i.e.,

$$K(N_j) \xrightarrow[S_1]{} 1, \quad j \to \infty.$$

Let $\varphi \in C_c^\infty(G(\mathbb{R})^1)$ be such that $\varphi(g) = 1$ in a neighborhood of $1 \in G(\mathbb{R})^1$. Put

$$\widetilde{h}^{\tau,p}_t = \varphi h^{\tau,p}_t.$$

We consider test functions with $\widetilde{h}_t^{\tau,p}$ at the infinite place and $\chi_{K(N_j)}$ at the finite places. By Proposition (3.1) we have

$$J_{\text{geom}}(\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N_j)}) = J_{\text{unip}}(\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N_j)}) \tag{12.19}$$

for all N_j provided we choose the support of $\widetilde{h}_t^{\tau,p}$ sufficiently small (this choice depends on $K(1)$ only). Put

$$S_0 := \{p\colon p \text{ prime },\ p|N_0\},\quad S := \{\infty\}\cup S_0\cup S_1.$$

Note that $K(N_j)=\prod_v K_v$ with $K_v=\mathbf{K}_v$ for $v\notin S$. Hence by the fine geometric expansion (8.26) we have

$$\begin{aligned}J_{\text{unip}}(\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N_j)}) &= \sum_{M\in\mathcal{L}}\sum_{[u]_S\in\mathcal{U}_S^M} a^M([u]_S,S)J_M^G([u]_S,\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N_j)})\\ &= \operatorname{vol}(G(\mathbb{Q})\backslash G(\mathbb{A})^1/K(N_j))\widetilde{h}_t^{\tau,p}(1)\\ &\quad+\sum_{(M,[u]_S)\neq(G,\{1\})} a^M([u]_S,S)J_M^G([u]_S,\widetilde{h}_t^{\tau,p}\otimes\chi_{K(N_j)}).\end{aligned}\tag{12.20}$$

Concerning the volume factor in the first summand on the right hand side, we used that $\chi_{K(N_j)}=\mathbf{1}_{K(N_j)}/\operatorname{vol}(K(N_j))$. To deal with the first term on the right hand side, we note that $\widetilde{h}_t^{\tau,p}(1)=h_t^{\tau,p}(1)$. Furthermore, by [MP13, (5.11)] there is an asymptotic expansion

$$h_t^{\tau,p}(1)\sim\sum_{j=0}^{\infty}a_jt^{-d/2+j} \tag{12.21}$$

as $t\to 0$, and by [MP13, (5.16)] there exists $c>0$ such that

$$h_t^{\tau,p}(1)=O(e^{-ct}) \tag{12.22}$$

as $t\to\infty$. From (12.21) and (12.22) follows that the integral

$$\int_0^\infty h_t^{\tau,p}(1)t^{s-1}dt \tag{12.23}$$

converges in the half-plane $\operatorname{Re}(s)>d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s=0$. The same is true for the integral over $[0,T]$ and we get

$$\operatorname{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T h_t^{\tau,p}(1)t^{s-1}\right)=\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty h_t^{\tau,p}(1)t^{s-1}\right)\Bigg|_{s=0}+O(e^{-cT}). \tag{12.24}$$

Nw recall the definition of the $L^{(2)}$-analytic torsion [Lo], [Mat]. For $t>0$ let

$$K^{(2)}(t,\tau):=\sum_{p=1}^d(-1)^p p h_t^{\tau,p}(1).$$

Put

$$t_{\widetilde{X}}^{(2)}(\tau):=\frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty K^{(2)}(t,\tau)t^{s-1}dt\right)\Bigg|_{s=0}.$$

Then by [MP13, (5.20)], the $L^{(2)}$-analytic torsion $T^{(2)}_{X(N_j)}(\tau) \in \mathbb{R}^+$ is given by

$$\log T^{(2)}_{X(N_j)}(\tau) = \operatorname{vol}(X(N_j)) \cdot t^{(2)}_{\widetilde{X}}(\tau).$$

To summarize, we get

$$\frac{1}{2}\sum_{p=1}^{d}(-1)^p p \operatorname{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T h_t^{\tau,p}(1)t^{s-1}dt\right) = t^{(2)}_{\widetilde{X}}(\tau) + O(e^{-cT}) \tag{12.25}$$

for $T \geq 1$.

Next we consider the weighted orbital integrals on the right hand side of (12.20). Note that by definition of $\chi_{K(N_j)}$ we have

$$J^G_M([u]_S, \widetilde{h}^{\tau,p}_t \otimes \chi_{K(N_j)}) = \frac{1}{\operatorname{vol}(K(N_j))} J^G_M([u]_S, \widetilde{h}^{\tau,p}_t \otimes \mathbf{1}_{K(N_j)}).$$

To deal with the integral on the right hand side, we use the splitting formula (8.27). Let $S_f := S \setminus \{\infty\} = S_0 \cup S_1$. Then we get

$$J^G_M([u]_S, \widetilde{h}^{\tau,p}_t \otimes \chi_{K(N_j)}) = \sum_{L_1,\, L_2 \in \mathcal{L}(M)} d^G_M(L_1, L_2) J^{L_1}_M([u]_\infty, \widetilde{h}^{\tau,p}_{t,Q_1}) J^{L_2}_M([u]_{S_f}, \mathbf{1}_{K(N_j)_{S_f},Q_2}). \tag{12.26}$$

Using (11.21) and Proposition (7.2), it follows that as $t \to 0$, $J^{L_1}_M([u]_\infty, \widetilde{h}^{\tau,p}_{t,Q_1})$ has an asymptotic expansion of the form (7.21). This implies that the integral

$$\int_0^T J^{L_1}_M([u]_\infty, \widetilde{h}^{\tau,p}_{t,Q_1}) t^{s-1} dt \tag{12.27}$$

converges absolutely and uniformly on compact subsets of the half plane $\operatorname{Re}(s) > d/2$ and admits a meromorphic extension to $s \in \mathbb{C}$. Put

$$A^{L_1}_M([u]_\infty, T) := \operatorname{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J^{L_1}_M([u]_\infty, \widetilde{h}^{\tau,p}_{t,Q_1}) t^{s-1} dt\right). \tag{12.28}$$

By (12.26) it follows that the Mellin transform of $J^G_M([u]_S, \widetilde{h}^{\tau,p}_t \otimes \chi_{K(N_j)})$ as a function of t is a meromorphic function on $\mathbb{C}$, and we get

$$\begin{aligned}\operatorname{FP}_{s=0}&\left(\frac{1}{s\Gamma(s)}\int_0^T J^G_M([u]_S, \widetilde{h}^{\tau,p}_t \otimes \chi_{K(N_j)}) t^{s-1} dt\right)\\ &= \sum_{L_1,\, L_2 \in \mathcal{L}(M)} d^G_M(L_1, L_2) A^{L_1}_M([u]_\infty, T) J^{L_2}_M([u]_{S_f}, \mathbf{1}_{K(N_j)_{S_f},Q_2}),\end{aligned} \tag{12.29}$$

where $\mathbf{1}_{K(N_j)_{S_f},Q_2}$ is defined by (8.28). Next we deal with the orbital integral on the right hand side. Let $L \in \mathcal{L}$ and let $Q = LV$ be a semistandard parabolic subgroup. Consider any congruence subgroup $K(N)$. Using the definition (8.28) and the fact that $K(N)$ is a normal subgroup of $\mathbf{K}_f$, we have

$$\mathbf{1}_{K(N)_{S_f},Q}(m) = \delta_Q(m)^{1/2} \operatorname{vol}(\mathbf{K}_{S_f}) \int_{V(\mathbb{Q}_{S_f})} \mathbf{1}_{K(N)_{S_f}}(mv)dv$$

for any $m \in L(\mathbb{A}_f)$. Hence $\mathbf{1}_{K(N)_{S_f},Q}(m) = 0$ unless $m \in K^L(N)_{S_f} = K(N)_{S_f} \cap L(\mathbb{A}_f)$. Now if $m \in K^L(N)_{S_f}$, we have $mv \in K(N)_{S_f}$ if and only if $v \in K(N)_{S_f}$. Hence

$$\mathbf{1}_{K(N)_{S_f},Q}(m) = \operatorname{vol}(\mathbf{K}_{S_f})\mathbf{1}_{K^L(N)_{S_f}}(m).$$

Therefore, it suffices to consider the orbital integral $J_M^L([u]_{S_f}, \mathbf{1}_{K^L(N)_{S_f}})$. We can assume that $L = G$. Now we argue as in [Cl, Lemma 7] to conclude that for $[u]_{S_f} \neq 1$,

$$\lim_{j\to\infty} J_M^G([u]_{S_f}, \mathbf{1}_{K(N_j)_{S_f}}) = 0. \tag{12.30}$$

Denote by $J_{\mathrm{unip}-\{1\}}(\widetilde{h}_t^{\tau,p} \otimes \chi_{K(N_j)})$ the sum on the right hand side of (12.20). Combining our results, we have proved

Lemma 12.5. *For every $T \geq 1$ we have*

$$\frac{1}{\operatorname{vol}(X(N_j))} \operatorname{FP}_{s=0}\left(\frac{1}{s\Gamma(s)} \int_0^T J_{\mathrm{unip}-\{1\}}(\widetilde{h}_t^{\tau,p} \otimes \chi_{K(N_j)})t^{s-1}dt\right) \to 0$$

as $j \to \infty$.

Now we can complete the proof of Theorem 1.6. Let

$$K_N(t,\tau) := \frac{1}{2}\sum_{k=1}^{d}(-1)^k k \operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_{k,N}(\tau)}\right).$$

Let $T > 0$. By (1.8) and (11.39) we have

$$\log T_{X(N)}(\tau) = \operatorname{FP}_{s=0}\left(\frac{1}{s\Gamma(s)} \int_0^T K_N(t,\tau)t^{s-1}dt\right) + \int_T^\infty K_N(t,\tau)t^{-1}dt. \tag{12.31}$$

By (12.5) there exist $C, c > 0$ such that

$$\frac{1}{\operatorname{vol}(X(N)}\left|\int_T^\infty K_N(t,\tau)t^{-1}dt\right| \leq Ce^{-cT} \tag{12.32}$$

for all $T \geq 1$ and $n \in \mathbb{N}$.

Now we turn to the first term on the right hand side. Let $h_{t,T}^{\tau,p} \in C_c^\infty(G(\mathbb{R})^1)$ be defined by (12.15). Put

$$K_N(t,\tau;T) := \frac{1}{2}\sum_{k=1}^{d}(-1)^k k J_{\mathrm{geom}}(h_{t,T}^{\tau,k} \otimes \chi_{K(N)}).$$

Using (12.16) and the trace formula, it follows that, up to a term of order $O(e^{-cT})$, we can replace $K_N(t,\tau)$ by $K_N(t,\tau,T)$ in the first term on the right hand side of (12.31). Let

$$K_{\mathrm{unip}-\{1\},N}(t,\tau;T) := \frac{1}{2}\sum_{p=1}^{d}(-1)^p p J_{\mathrm{unip}-\{1\}}(h_{t,T}^{\tau,p}\otimes\chi_{K(N)}). \tag{12.33}$$

By [Cl, Lemma 5] and (12.20) it follows that for every $T\geq 1$ there exists $N_0(T)\in\mathbb{N}$ such that

$$K_N(t,\tau;T) = \frac{\mathrm{vol}(X(N))}{2}\sum_{k=1}^{d}(-1)^k k h_{t,T}^{\tau,k}(1) + K_{\mathrm{unip}-\{1\},N}(t,\tau;T)$$

for $N\geq N_0(T)$. Using (12.25) it follows that for every $T\geq 1$ there exists $N_0(T)\in\mathbb{N}$ such that

$$\begin{aligned}\frac{1}{\mathrm{vol}(X(N))}&\mathrm{FP}_{s=0}\biggl(\frac{1}{s\Gamma(s)}\int_0^T K_N(t,\tau)t^{s-1}dt\biggr)\\ &= t_{\widetilde{X}}^{(2)}(\tau) + \frac{1}{\mathrm{vol}(X(N))}\mathrm{FP}_{s=0}\biggl(\frac{1}{s\Gamma(s)}\int_0^T K_{\mathrm{unip}-\{1\},N}(t,\tau)t^{s-1}dt\biggr)\\ &\quad + O(e^{-cT})\end{aligned} \tag{12.34}$$

for $N\geq N_0(T)$. For the last step we have to specialize to the principal congruence groups $\{K(N_j)\}_{j\in\mathbb{N}}$ which converge to 1 at S_1. Applying Lemma 12.5 we get that for every $T\geq 1$ and $\varepsilon>0$ there exist constants $C_1(T), C_2, c>0$ and $N_0(T)\in\mathbb{N}$ such that

$$\left|\frac{1}{\mathrm{vol}(X(N_j))}\mathrm{FP}_{s=0}\biggl(\frac{1}{s\Gamma(s)}\int_0^T K_{N_j}(t,\tau)t^{s-1}dt\biggr) - t_{\widetilde{X}}^{(2)}(\tau)\right| \leq C_1(T)\varepsilon + C_2e^{-cT} \tag{12.35}$$

for $j\geq N_0(T)$. Combined with (12.31), (12.32) and (12.5), Theorem 1.6 follows.

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