Lecture Course: Advanced Global Analysis II (V4B4)

Spectral theory of automorphic forms

The theory of automorphic forms has a (relatively) long history dating back to the time of H. Poincaré and F. Klein. Poincaré named them Fuschsian functions. In the classical sense, an automorphic form is a holomorphic function on the upper half-plane $\mathbb{H}$ which transform in a certain simple way under the action of a discrete subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ such that $\mathrm{vol}(\Gamma\backslash \mathrm{SL}(2,\mathbb{R})) < \infty$. Such groups are called lattices. Of particular interest are is the modular group $\mathrm{SL}(2,\mathbb{Z})$ and its principal congruence subgroups $\Gamma(N)$. Geometrically, an automorphic form is a holomorphic section of some holomorphic line bundle over the Riemann surface $X = \Gamma\backslash\mathbb{H}$. The interest in automorphic forms comes from the relation to number theory.
Since then the concept of automorphic forms has been extended considerably and the theory has undergone a tremendous development. It started with H. Maass, who introduced a new kind of automorpic forms, called *Maass forms*. If the upper half-plane is equipped with the hyperbolic metric, $X = \Gamma\backslash\mathbb{H}$ becomes a hyperbolic surface, which may be non-compact. Then a Maass form is an eigenfunction of the Laplace operator on X which satisfies some growth conditions. Especially a square integrable eigenfunction is a Maass form. A further important step was the the introduction of representation theory into the subject. The modern theory of automorphic uses the concept of *automorphic representations*, which in the case of $\mathrm{SL}(2)$ are irreducible unitary representations of the group of adeles $\mathrm{SL}(2,\mathbb{A})$ which is the restricted product of $\mathrm{SL}(2,\mathbb{Q}_v)$, where v runs over all places of $\mathbb{Q}$.
In the modern theory of automorphic forms, $\mathrm{SL}(2)$ is replaced by any reductive algebraic group over a number field. Correspondingly, $\Gamma\backslash\mathbb{H}$ is replaced by an arbitrary locally symmetric space $X = \Gamma\backslash G/K$. The analytic side of the theory is concerned with spectral theory on X. Here spectral theory referes to the Laplace operator and other invariant differential operators.
Today, the modern theory of automorphic forms is one of the central research areas in mathematics with links to many different fields in mathematics including representation theory of reductive groups over local and global fields, number theory, PDE's, algebraic geometry and differential geometry. It is a repsonse to many different impulses and influences. But so far, the most powerful techniques are the issue, direct or indirect, of the introduction of spectral theory into the subject by Maass and then Selberg.
The course will be an introduction to spectral theory. We will mainly consider the Laplace operator on hyperbolic surfaces $\Gamma\backslash\mathbb{H}$ and its relation to representation theory of $\mathrm{SL}(2,\mathbb{R})$ via the spectral decomposition of the right regular represention of $\mathrm{SL}(2,\mathbb{R})$ in the Hilbert space $L^2(\Gamma\backslash \mathrm{SL}(2,\mathbb{R}))$.

References

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