Advanced Global Analysis I
Exercise series 9

January 22, 2016
Due: January 29, 2016

Exercise 33. (2 points)
Let

$$K(t,x,y) = (4\pi t)^{-n/2} e^{-\|x-y\|^2/4t}$$

$t > 0, x, y \in \mathbb{R}^n$. Let $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Show that $K(t,x,y)$ satisfies

1)
$$\Big(\frac{\partial}{\partial t} + \Delta_x\Big) K(t,x,y) = 0.$$

2) For all bounded continuous functions f on $\mathbb{R}^n$ one has

$$\lim_{t \to +0} \int_{\mathbb{R}^n} K(t,x,y) f(y) dy = f(x).$$

Exercise 34. (4 points)
Let (X,g) be a compact Riemannian manifold and $E \to X$ a Hermitian vector bundle. Let

$$\Delta \colon \Gamma(E) \to \Gamma(E)$$

be a formally selfadjoint, non-negative, elliptic differential operator of order $m > 0$. For $\lambda \in \mathbb{C}$ denote by $d(\lambda, \mathbb{R}^+)$ the distance of λ from $\mathbb{R}^+$. Show that for all $\lambda \in \mathbb{C} \setminus [0,\infty)$ one has

$$\| (\bar{\Delta} - \lambda)^{-1} \| \leq d(\lambda, \mathbb{R}^+)^{-1}.$$

Exercise 35 (4 points)
Let $\Gamma \subset \mathbb{C}$ be a path of the form

Show that for $t > 0$ we have

$$e^{-t\Delta} = \frac{i}{2\pi} \int_{\Gamma} e^{-t\lambda} (\bar{\Delta} - \lambda)^{-1} d\lambda$$

Hint: Use Exercise 34 to prove that the integral is absolutely convergent.