Advanced Global Analysis I
Exercise series 8

January 15, 2016
Due: January 22, 2016

Exercise 29. (4 points)
Let
$$\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$
be the Laplace operator in $\mathbb{R}^n$.
For $t > 0$ let
$$e^{-t\Delta}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$
be the heat operator, defined using the Fourier transform. Show that $e^{-t\Delta}$ is an integral operator with kernel
$$K(t,x,y) = (4\pi t)^{-n/2} e^{-\|x-y\|^2/4t}$$

Exercise 30. (4 points)
Let $I = [a,b]$, $a < b$. Show that for every $f \in C^0([a,b])$ the following problem
$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = 0, \\ u(x,0) = f(x),\ x \in [a,b], \\ u(a,t) = u(b,t) = 0,\ t > 0, \end{cases}$$
has a unique solution $u \in C^\infty((a,b) \times \mathbb{R}^+)$.

Hint: Use Fourier series.

Exercise 31. (2 points)
Let (X,g) be a compact Riemannian manifold, $E \to X$ a Hermitian vector bundle and
$$\Delta\colon \Gamma(E) \to \Gamma(E)$$
a formally selfadjoint, elliptic differential operator of order $m > 0$ with $\Delta \geq 0$.
Let $e^{-t\Delta}\colon L^2(E) \to L^2(E)$ be the heat operator. Show that $e^{-t\Delta}, t > 0$, is a semigroup, i.e.,
$$e^{-(t_1+t_2)\Delta} = e^{-t_1\Delta} \circ e^{-t_2\Delta}$$
for all $t_1, t_2 > 0$.

Exercise 32. (2 points)
Let (X,g) be a compact Riemannian manifold. Let $\Delta\colon C^\infty(X) \to C^\infty(X)$ be the Laplace operator. Let $K(t,x,y)$ be the kernel of the heat operator $e^{-t\Delta}$. Show that K has the following properties

1) $\left(\frac{\partial}{\partial t} + \Delta_x\right) K(t,x,y) = 0.$
2) $\forall f \in C^\infty(X)\colon \lim_{t\to+0} \int_X K(t,x,y) f(y)dy = f(x).$